

SBV FUNCTIONS IN CARNOT-CARATHÉODORY SPACES

MARCO DI MARCO, SEBASTIANO DON, AND DAVIDE VITTONÉ

ABSTRACT. We introduce the space SBV_X of special functions with bounded X -variation in Carnot-Carathéodory spaces and study its main properties. Our main outcome is an approximation result, with respect to the BV_X topology, for SBV_X functions.

1. INTRODUCTION

Functions with Bounded Variation (BV), and in particular their subclass of *special* functions with Bounded Variation (SBV), provide a natural framework for studying problems involving discontinuities, such as image processing, signal analysis, and variational problems. Over recent years, a considerable effort was put into the study of BV functions in Carnot-Carathéodory (CC) spaces. The aim of this paper is to contribute to this area of research by introducing the space SBV_X of special functions with bounded X -variation and studying its properties. In particular, we extend to the setting of CC spaces the following approximation result for classical SBV functions proved by G. de Philippis, N. Fusco and A. Pratelli in [19].

Theorem 1.1 ([19, Theorem A]). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in SBV(\Omega)$. Then, there exists a sequence of functions $u_k \in SBV(\Omega)$ and of compact C^1 -manifolds $M_k \subset\subset \Omega$ such that $\mathcal{J}_{u_k} \subseteq M_k \cap \mathcal{J}_u$, $\mathcal{H}^{n-1}(\overline{\mathcal{J}_{u_k}} \setminus \mathcal{J}_{u_k}) = 0$ and*

$$\|u_k - u\|_{BV(\Omega)} \xrightarrow{k \rightarrow +\infty} 0, \quad u_k \in C^\infty(\Omega \setminus \overline{\mathcal{J}_{u_k}}).$$

Recall that smooth functions are not dense in BV with respect to the BV topology¹, as their closure in BV is the Sobolev space $W^{1,1}$, i.e., BV functions whose derivatives admits no singular part (not even “nice” jumps) with respect to the Lebesgue measure. In this sense, Theorem 1.1 provides a class of “nice” (although, clearly, not smooth) BV functions that are dense in SBV with respect to the BV topology. As explained in [19], this result is sharp and, besides being interesting *per se*, it led to the proof of a conjecture by L. Ambrosio, J. Bourgain, H. Brezis and A. Figalli [4] (see also [33]) about a BMO-type characterization of BV functions.

Before stating our main result we need to briefly introduce the notion of special function of bounded variation in CC spaces. A *Carnot-Carathéodory space* (see Definition 2.1) is the space \mathbb{R}^n endowed with a distance arising from a collection $X = (X_1, \dots, X_m)$ of smooth and linearly independent vector fields satisfying the Hörmander condition. In this paper, we will deal with *equi-regular* CC spaces, where a homogeneous dimension Q , usually larger than the topological dimension n , can be defined. The space $BV_X(\Omega)$ [13, 28] of functions with bounded X -variation (see Definition 2.2) consists of those functions u on an open set $\Omega \subset \mathbb{R}^n$ whose derivatives X_1u, \dots, X_mu in the sense of distributions are represented by a vector-valued measure D_Xu with finite total variation $|D_Xu|$. The space BV_X has been the subject of intensive studies: see

2020 *Mathematics Subject Classification.* 26B30, 53C17, 49Q15.

Key words and phrases. Special functions with bounded variation, Carnot-Carathéodory spaces, sub-Riemannian geometry, rectifiable sets.

M. D. M. and D. V. are supported by University of Padova and GNAMPA of INdAM. D. V. is also supported by PRIN 2022PJ9EFL project *Geometric Measure Theory: Structure of Singular Measures, Regularity Theory and Applications in the Calculus of Variations* funded by the European Union - Next Generation EU, Mission 4, component 2 - CUP:E53D23005860006.

¹Smooth functions are dense in BV only with respect to the so-called strict topology in BV, see e.g. [5, Theorem 3.9].

[11, 14, 16, 17, 27, 29, 30, 34, 35, 45] and the more recent [6, 7, 9, 12, 15, 21, 23, 24, 25, 39, 43, 46, 47].

In the classical Euclidean setting the space of SBV functions, first introduced in [18], naturally arises in the study of free discontinuity problems. The first contribution of this paper is the introduction of special functions with Bounded Variation in CC spaces (SBV_X functions). Recall ([25]) that, if $u \in \text{BV}_X(\Omega)$, then $D_X u$ can be decomposed as

$$D_X u = D_X^{\text{ap}} u \mathcal{L}^n + D_X^s u = D_X^{\text{ap}} u \mathcal{L}^n + D_X^j u + D_X^c u,$$

where $D_X^{\text{ap}} u$ is the approximate X -gradient of u , \mathcal{L}^n is the usual Lebesgue measure, $D_X^s u$ is the singular part of $D_X u$, $D_X^j u$ is the jump part of $D_X u$, and $D_X^c u$ is the Cantor part of $D_X u$. See Section 2 for precise definitions.

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be an open subset of an equiregular Carnot-Carathéodory space (\mathbb{R}^n, X) and let $u \in \text{BV}_X(\Omega)$. We say that u is a *special function of bounded X -variation*, and we write $u \in \text{SBV}_X(\Omega)$, if

- (i) $D_X^c u = 0$, and
- (ii) the jump set \mathcal{J}_u of u is a countably X -rectifiable set.

A set is said to be countably X -rectifiable (see Definition 2.12) if it can be covered, up to a set which is negligible with respect to the Hausdorff measure \mathcal{H}^{Q-1} , by a countable family of C_X^1 -hypersurfaces (Definition 2.11), that provide the intrinsic counterpart in CC spaces of classical C^1 -hypersurfaces. Recall that, for classical BV functions, the jump set is always countably rectifiable; on the contrary, in CC spaces this – i.e., the validity of condition (ii) above for every BV_X function u – is an important open problem. Let us however recall that, if the CC space satisfies the so-called *property \mathcal{R}* (“rectifiability”, see Definition 2.17), then condition (ii) in Definition 1.2 is automatically satisfied for every $u \in \text{BV}_X$; see [25, Theorem 1.5]. There is a multitude of examples of CC spaces which satisfy property \mathcal{R} , such as Heisenberg groups, step 2 Carnot groups and Carnot groups of type \star , see [25, Theorem 4.3]. In this paper we tried to work in the widest possible generality, hence the extra requirement (ii) in Definition 1.2. For this reason, let us also stress that our definition might be a priori different from the definition of SBV function in metric measure spaces. We refer to [8] for a general overview of SBV functions in metric measure spaces and to [37] for an approximation result for BV functions via SBV functions in this context.

In Section 3 we study several properties of SBV_X (or locally SBV_X) functions: we collect the main ones in the following theorem, which summarizes (some of) the results stated in Proposition 3.2, Theorem 3.3, Lemma 3.4, Theorem 3.6 and Theorem 3.8.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be an open subset of an equiregular Carnot-Carathéodory space (\mathbb{R}^n, X) ; then, the following statements hold:*

- (i) $u \in \text{SBV}_{X,\text{loc}}(\Omega)$ if and only if $D_X^s u = f \nu_R \mathcal{H}^{Q-1} \llcorner R$ for some countably X -rectifiable set $R \subset \Omega$ with horizontal normal ν_R and some $f \in L^1_{\text{loc}}(R, \mathcal{H}^{Q-1})$;
- (ii) $\text{SBV}_X(\Omega)$ is a closed subspace of $\text{BV}_X(\Omega)$;
- (iii) the space $\text{SBV}_{\text{loc}}(\Omega)$ of special function of (Euclidean) locally bounded variation is contained in $\text{SBV}_{X,\text{loc}}(\Omega)$;
- (iv) for every $w \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$ there exists $u \in \text{SBV}_{X,\text{loc}}(\Omega)$ such that $D_X^{\text{ap}} u = w$ a.e. in Ω ;
- (v) for every countably X -rectifiable set $R \subseteq \Omega$ oriented by ν_R , every $\theta \in L^1(\mathcal{H}^{Q-1} \llcorner R)$ and every $\delta > 0$ there exists $u \in \text{SBV}_X(\Omega)$ such that

$$D_X^j u \equiv \theta \nu_R \mathcal{H}^{Q-1} \llcorner R, \quad \|u\|_{L^1(\Omega)} < \delta, \quad \text{and} \quad |D_X u|(\Omega) \leq (1 + \delta) \|\theta\|_{L^1(\mathcal{H}^{Q-1} \llcorner R)}.$$

Statements (iii), (iv) and (v), in particular, provide meaningful subclasses or examples of special functions of bounded X -variation which, in particular, turn out to form a quite large and interesting space.

We can now to state our main result.

Theorem 1.4. *Let Ω be an open subset of an equiregular Carnot-Carathéodory space (\mathbb{R}^n, X) and let $u \in \text{SBV}_X(\Omega)$. Then, there exists a sequence of functions $(u_k)_{k \in \mathbb{N}} \subset \text{SBV}_X(\Omega)$ and of C_X^1 -hypersurfaces $(M_k)_{k \in \mathbb{N}} \subset \Omega$ such that, for every $k \in \mathbb{N}$, $\mathcal{J}_{u_k} \subseteq M_k \cap \mathcal{J}_u$, \mathcal{J}_{u_k} is compact, and*

$$\|u - u_k\|_{\text{BV}_X(\Omega)} \xrightarrow{k \rightarrow +\infty} 0, \quad u_k \in C^\infty(\Omega \setminus \mathcal{J}_{u_k}).$$

Our proof of Theorem 1.4 differs from the one of Theorem 1.1 in that, rather than using mollifications with variable kernel as in [19], we exploit a partition-of-the-unity argument (reminiscent of [10, 20, 28, 47, 48]) that allows to approximate u out of a fixed compact set C_k . *A posteriori*, the set C_k coincides with the jump set \mathcal{J}_{u_k} , which in particular turns out to be compact itself, thus providing a slight improvement in Theorem 1.1.

We believe that Theorem 1.4 will play a role in a possible, future BMO-type characterization of BV_X functions *à la* Ambrosio-Bourgain-Brezis-Figalli, [4]: this, in fact, was one of the original motivations of our work, and will be the subject of further investigations.

2. DEFINITIONS AND PRELIMINARY RESULTS

Definition 2.1. Let $1 \leq m \leq n$ be integers and let $X = (X_1, \dots, X_m)$ be a m -tuple of smooth and linearly independent vector fields on \mathbb{R}^n . We say that an absolutely continuous curve $\gamma: [0, T] \rightarrow \mathbb{R}^n$ is an X -subunit path joining p and q if $\gamma(0) = p$, $\gamma(T) = q$ and there exist $h_1, \dots, h_m \in L^\infty([0, T])$ such that $\sum_{j=1}^m h_j^2 \leq 1$ and

$$\gamma'(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)), \quad \text{for a.e. } t \in [0, T].$$

For every $p, q \in \mathbb{R}^n$ we define

$$d(p, q) := \inf\{T > 0 : \text{there exists an } X\text{-subunit path } \gamma \text{ joining } p \text{ and } q\},$$

where we agree that $\inf \emptyset := +\infty$.

By the Chow–Rashevskii Theorem (see for instance [1, Subsection 3.2.1]), if for every $p \in \mathbb{R}^n$ the linear span of all iterated commutators of the vector fields X_1, \dots, X_m computed at p has dimension n (i.e. X_1, \dots, X_m satisfy the *Hörmander condition*), then d is a distance: the latter means that for every couple of points of \mathbb{R}^n there always exists a X -subunit path joining them. In this case we say that (\mathbb{R}^n, X) is a *Carnot-Carathéodory space* of rank m and d is the associated *Carnot-Carathéodory distance*.

For every $p \in \mathbb{R}^n$ and for every $i \in \mathbb{N}$ we denote by $\mathfrak{L}^i(p)$ the linear span of all the commutators of X_1, \dots, X_m up to order i computed at p . We say that a Carnot-Carathéodory space (\mathbb{R}^n, X) is *equiregular* if there exist natural numbers n_0, n_1, \dots, n_s such that

$$0 = n_0 < n_1 < \dots < n_s = n \text{ and } \dim \mathfrak{L}^i(p) = n_i, \quad \forall p \in \mathbb{R}^n, \forall i \in \{1, \dots, s\}.$$

The natural number s is called *step* of the Carnot-Carathéodory space. If (\mathbb{R}^n, X) is equiregular, then the *homogeneous dimension* is $Q := \sum_{i=1}^s i(n_i - n_{i-1})$.

Notation. In the following, $(\mathbb{R}^n, \mathbf{X})$ denotes an equiregular Carnot-Carathéodory space associated with the family $\mathbf{X} = (X_1, \dots, X_m)$. We use d to denote the Carnot-Carathéodory distance associated with X , $B(\cdot, \cdot)$ to denote the associated open balls, \mathcal{H}^k to denote the associated Hausdorff k -measure and \mathcal{S}^k to denote the associated spherical Hausdorff k -measure; on the other hand we will denote by d_E the usual Euclidean distance, by $B_E(\cdot, \cdot)$ the associated open balls, by \mathcal{H}_E^k the associated Hausdorff k -measure and by \mathcal{S}_E^k the associated spherical Hausdorff k -measure. By $\Omega \subseteq (\mathbb{R}^n, X)$ we denote a fixed open set and by Q we denote the homogeneous dimension of (\mathbb{R}^n, X) . Later we will also use the following notation:

- for every $1 \leq i \leq m$ and $x \in \mathbb{R}^n$ we write

$$X_i(x) = (a_{i,1}(x), \dots, a_{i,n}(x))$$

where $a_{i,t} \in C^\infty(\mathbb{R}^n)$ for $1 \leq t \leq n$;

- for every $1 \leq i \leq m$ and $x \in \mathbb{R}^n$ we write

$$(\operatorname{div} X_i)(x) := \sum_{t=1}^n \frac{\partial a_{i,t}}{\partial x_t}(x)$$

- for every $1 \leq i \leq m$, $\varphi \in C^1(\Omega)$ and $x \in \Omega$ we write

$$(X_i \varphi)(x) := \sum_{t=1}^n a_{i,t}(x) \frac{\partial \varphi}{\partial x_t}(x);$$

- for every $1 \leq i \leq m$ we denote by X_i^* the formal adjoint of X_i , i.e., for every $\varphi \in C^1(\Omega)$, $x \in \Omega$ we write

$$(X_i^* \varphi)(x) := \sum_{t=1}^n \frac{\partial (a_{i,t} \varphi)}{\partial x_t}(x);$$

- given a Radon measure μ , we use the notation

$$\int_A u d\mu := \frac{1}{\mu(A)} \int_A u d\mu,$$

to denote the average integral of a measurable function u on a μ -measurable set A with $\mu(A) > 0$;

- we will consider a Riemannian metric, namely a smoothly varying family of scalar products $\langle \cdot, \cdot \rangle_p = \langle \cdot, \cdot \rangle: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ which makes the horizontal vectors X_1, \dots, X_m orthonormal at every point $p \in \mathbb{R}^n$.

Definition 2.2. We say that $u \in L^1_{\text{loc}}(\Omega)$ is a *function of locally bounded X -variation*, and we write $u \in \text{BV}_{X,\text{loc}}(\Omega)$, if there exists a \mathbb{R}^m -valued Radon measure $D_X u = (D_{X_1} u, \dots, D_{X_m} u)$ on Ω such that, for every open set $A \subset\subset \Omega$, for every $1 \leq i \leq m$ and for every $\varphi \in C_c^1(A)$ one has

$$\int_A \varphi d(D_{X_i} u) = - \int_A u X_i^* \varphi d\mathcal{L}^n.$$

Moreover, if $u \in L^1(\Omega)$ and $D_X u$ has bounded total variation $|D_X u|$, then we say that u has *bounded X -variation* and we write $u \in \text{BV}_X(\Omega)$.

Definition 2.3. For every $u \in \text{BV}_X(\Omega)$ we define the norm

$$\|u\|_{\text{BV}_X(\Omega)} := \|u\|_{L^1(\Omega)} + |D_X u|(\Omega).$$

The space $\text{BV}_X(\Omega)$ equipped with the above norm is a Banach space.

Definition 2.4. For every $u \in \text{BV}_X(\Omega)$ we decompose

$$D_X u = D_X^a u + D_X^s u$$

where $D_X^a u$ denotes the *absolutely continuous part* of $D_X u$ (with respect to the usual Lebesgue measure \mathcal{L}^n) and $D_X^s u$ denotes the *singular part* of $D_X u$.

Definition 2.5. We say that a measurable set $E \subseteq \mathbb{R}^n$ has *locally finite X -perimeter* (respectively, *finite X -perimeter*) in Ω if its characteristic function χ_E belongs to $\text{BV}_{X,\text{loc}}(\Omega)$ (respectively, $\chi_E \in \text{BV}_X(\Omega)$). In such a case we define the *X -perimeter measure* P_E^X of E as $P_E^X := |D_X \chi_E|$.

Definition 2.6. Let $\Omega \subseteq (\mathbb{R}^n, X)$ be an open set and $f: \Omega \rightarrow \mathbb{R}$. We say that $f \in C_X^1(\Omega)$ if f is continuous and its *horizontal gradient* $Xf := (X_1 f, \dots, X_m f)$, in the sense of distributions, is represented by a continuous function.

Definition 2.7. Let $u \in L^1_{\text{loc}}(\Omega)$, $z \in \mathbb{R}$ and $p \in \Omega$. We say that z is the *approximate limit* of u at p if

$$\lim_{r \rightarrow 0} \int_{B(p,r)} |u - z| d\mathcal{L}^n = 0.$$

If the approximate limit of u at p exists, it is also unique (see [25, Definition 2.19]). We hence denote by $u^*(p)$ the approximate limit of u at p and by \mathcal{S}_u the subset of points in Ω where u does not admit an approximate limit.

Definition 2.8. Let $u \in L^1_{\text{loc}}(\Omega)$ and $p \in \Omega \setminus \mathcal{S}_u$. We say that u is *approximately X -differentiable* at p if there exist a neighbourhood $U \subset \Omega$ of p and $f \in C^1_X(U)$ such that $f(p) = 0$ and

$$\lim_{r \rightarrow 0} \int_{B(p,r)} \frac{|u - u^*(p) - f|}{r} d\mathcal{L}^n = 0.$$

The set of points in Ω where u is approximately X -differentiable is denoted by \mathcal{D}_u . The vector $Xf(p) \in \mathbb{R}^m$ is uniquely determined (see [25, Proposition 2.30]): we call it *approximate X -gradient* of u at p and we denote it by $D_X^{\text{ap}}u(p)$. Similarly, we write $D_{X_i}^{\text{ap}}u(p) := X_i f(p)$ for every $i \in \{1, \dots, m\}$.

The next two results collect some of the “fine” properties of BV_X functions proved in [25].

Theorem 2.9 ([25, Theorem 1.1]). *Let $u \in BV_X(\Omega)$. Then u is approximately X -differentiable at \mathcal{L}^n -almost every point of Ω . Moreover, the approximate X -gradient coincides \mathcal{L}^n -almost everywhere with the density of $D_X^a u$ with respect to \mathcal{L}^n .*

Theorem 2.10 ([25, Theorem 1.3]). *There exists $\lambda: \mathbb{R}^n \rightarrow (0, +\infty)$ locally bounded away from 0 such that, for every open set $\Omega \subset \mathbb{R}^n$ and every $u \in BV_X(\Omega)$*

$$|D_X u| \geq \lambda |u^+ - u^-| \mathcal{H}^{Q-1} \llcorner \mathcal{J}_u.$$

Moreover, for every Borel set $B \subseteq \Omega$ the following implications hold:

- (1) $\mathcal{H}^{Q-1}(B) = 0 \quad \Rightarrow \quad |D_X u|(B) = 0;$
- (2) $\mathcal{H}^{Q-1}(B) < +\infty$ and $B \cap \mathcal{S}_u = \emptyset \quad \Rightarrow \quad |D_X u|(B) = 0.$

We now spend a few words about intrinsically C^1 (or C^1_X) hypersurfaces and the notion of X -rectifiability.

Definition 2.11. We say that $S \subseteq (\mathbb{R}^n, X)$ is a C^1_X -hypersurface if for every $p \in S$ there exist $r > 0$ and $f \in C^1_X(B(p, r))$ such that the following facts hold:

- (i) $S \cap B(p, r) = \{q \in B(p, r) : f(q) = 0\},$
- (ii) $Xf \neq 0$ on $B(p, r).$

We define the *horizontal normal* to S at $p \in S$ as

$$\nu_S(p) := \frac{Xf(p)}{|Xf(p)|}.$$

Notice that $\nu_S(p)$ is well defined up to a sign and, in particular, it does not depend on the choice of f , see [25, Corollary 2.14].

Definition 2.12. Let $S \subseteq (\mathbb{R}^n, X)$. We say that S is *countably X -rectifiable* if there exists a family $\{S_h : h \in \mathbb{N}\}$ of C^1_X -hypersurfaces such that

$$\mathcal{H}^{Q-1} \left(S \setminus \bigcup_{h \in \mathbb{N}} S_h \right) = 0.$$

Moreover, if $\mathcal{H}^{Q-1}(S) < +\infty$, we say that S is *X -rectifiable*. We define the *horizontal normal* of a countably X -rectifiable set S at $p \in S$ as

$$\nu_S(p) := \nu_{S_h}(p) \text{ if } p \in S_h \setminus \bigcup_{k < h} S_k.$$

Notice that ν_S is well defined, up to a sign, \mathcal{H}^{Q-1} -a.e., see [25, Proposition 2.18].

The following Lemma provides an equivalent definition of X -rectifiability; although not difficult and probably quite known (see [36, Lemma 2.4] for a proof in Heisenberg groups), we include a proof for the sake of completeness.

Lemma 2.13. *A set $R \subseteq (\mathbb{R}^n, X)$ is X -rectifiable if and only if, for every $\varepsilon > 0$, there exists a C_X^1 -hypersurface $S_\varepsilon \subseteq (\mathbb{R}^n, X)$ such that $\mathcal{H}^{Q-1}(S_\varepsilon) < \infty$ and*

$$\mathcal{H}^{Q-1}(R \setminus S_\varepsilon) < \varepsilon.$$

Proof. Since R is X -rectifiable we can write

$$R \subseteq S_0 \cup \bigcup_{i \in \mathbb{N}} S_i$$

where S_0 is a \mathcal{H}^{Q-1} -negligible set and, for every $i \in \mathbb{N}$, S_i is a C_X^1 -hypersurface. It is not restrictive to assume $\mathcal{H}^{Q-1}(S_i) < \infty$ for every $i \in \mathbb{N}$. For every $\varepsilon > 0$ there exists a positive integer M such that

$$\mathcal{H}^{Q-1}\left(R \setminus \bigcup_{i \leq M} S_i\right) < \frac{\varepsilon}{2}.$$

We define the C_X^1 -hypersurface $S'_1 := \{p \in S_1 : d(p, \partial S_1) > r_1\}$, where $\partial S'_1 := \overline{S'_1} \setminus S'_1$ and r_1 is chosen so that

$$\mathcal{H}^{Q-1}(R \cap \partial S'_1) = 0 \quad \text{and} \quad \mathcal{H}^{Q-1}((R \cap S_1) \setminus S'_1) < \frac{\varepsilon}{4}.$$

Let us prove that such r_1 exists: for $r > 0$ we define the set

$$S'_1(r) := \{p \in S_1 : d(p, \partial S_1) > r\}.$$

Since $\{R \cap \partial S'_1(r) : r > 0\}$ is a family of uncountably many pairwise disjoint subsets of R and $\mathcal{H}^{Q-1}(R) < \infty$, then $\mathcal{H}^{Q-1}(R \cap \partial S'_1(r)) = 0$ for arbitrarily small $r > 0$. Moreover, since $R \cap S_1$ is the union of the nested sets $(R \cap S_1) \setminus S'_1(r)$, we have, by the continuity of measure, that

$$\mathcal{H}^{Q-1}((R \cap S_1) \setminus S'_1(r)) \xrightarrow{r \rightarrow 0} 0.$$

Reasoning by induction, for every $i = 2, \dots, M$ we can define the C_X^1 -hypersurfaces

$$S'_i = \left\{ p \in S_i \setminus \bigcup_{j < i} \overline{S'_j} : d\left(p, \partial\left(S_i \setminus \bigcup_{j < i} \overline{S'_j}\right)\right) > r_j \right\},$$

where we used the fact that $S_i \setminus \bigcup_{j < i} \overline{S'_j}$ is a C_X^1 -hypersurface and $r_i > 0$ is chosen so that

$$\mathcal{H}^{Q-1}(R \cap \partial S'_i) = 0 \quad \text{and} \quad \mathcal{H}^{Q-1}\left(R \cap \left(S_i \setminus \bigcup_{j < i} \overline{S'_j}\right) \setminus S'_i\right) < \frac{\varepsilon}{2^{i+2}}.$$

Now consider $S_\varepsilon := \bigcup_{i=1}^M S'_i$, which is a C_X^1 -hypersurface because it is union of finitely many C_X^1 -hypersurface at positive distance from each other. Then

$$\begin{aligned} \mathcal{H}^{Q-1}(R \setminus S_\varepsilon) &\leq \mathcal{H}^{Q-1}\left(R \setminus \bigcup_{i \leq M} S_i\right) + \mathcal{H}^{Q-1}\left(R \cap \left(\bigcup_{i \leq M} S_i\right) \setminus \left(\bigcup_{j \leq M} S'_j\right)\right) \\ &< \frac{\varepsilon}{2} + \mathcal{H}^{Q-1}\left(\bigcup_{i \leq M} \left((R \cap S_i) \setminus \bigcup_{j \leq M} S'_j\right)\right) \\ &\leq \frac{\varepsilon}{2} + \mathcal{H}^{Q-1}\left(\bigcup_{i \leq M} \left(R \cap \left(S_i \setminus \bigcup_{j \leq i} S'_j\right)\right)\right) \\ &= \frac{\varepsilon}{2} + \mathcal{H}^{Q-1}\left(\bigcup_{i \leq M} \left(R \cap \left(S_i \setminus \bigcup_{j < i} \overline{S'_j}\right) \setminus S'_i\right)\right) \\ &< \varepsilon, \end{aligned}$$

where we used that $\mathcal{H}^{Q-1}(R \cap \partial S'_j) = 0$. Finally we observe that

$$\mathcal{H}^{Q-1}(S_\varepsilon) = \sum_{i \leq M} \mathcal{H}^{Q-1}(S'_i) \leq \sum_{i \leq M} \mathcal{H}^{Q-1}(S_i) < \infty.$$

This proves one implication, the other one is trivial. \square

Definition 2.14. Fix $p \in (\mathbb{R}^n, X)$, $R > 0$ and $\nu \in \mathbb{S}^{m-1}$. Let $f \in C_X^1(B(p, R))$ be such that $f(p) = 0$ and $\frac{Xf(p)}{|Xf(p)|} = \nu$. For every $r \in (0, R)$ we set

$$\begin{aligned} B_\nu^+(p, r) &:= B(p, r) \cap \{f > 0\}, \\ B_\nu^-(p, r) &:= B(p, r) \cap \{f < 0\}. \end{aligned}$$

Definition 2.15. Let $u \in L_{loc}^1(\Omega)$ and $p \in \Omega$. We say that u has an *approximate X -jump* at p if there exist $u^+, u^- \in \mathbb{R}$ with $u^+ \neq u^-$ and $\nu \in \mathbb{S}^{m-1}$ such that

$$(3) \quad \lim_{r \rightarrow 0} \int_{B_\nu^+(p, r)} |u - u^+| d\mathcal{L}^n = \lim_{r \rightarrow 0} \int_{B_\nu^-(p, r)} |u - u^-| d\mathcal{L}^n = 0.$$

The *jump set* \mathcal{J}_u is defined as the set of points where u has an approximate X -jump. Notice that condition (3) does not depend on the choice of the function f used to construct the sets $B_\nu^+(p, r)$ and $B_\nu^-(p, r)$, see [25, Proposition 2.26 and Remark 2.27].

It is worth remarking that, by [25, Theorem 1.2 and Remark 2.25], the Hausdorff measure \mathcal{H}^{Q-1} on the jump set is σ -finite.

Remark 2.16. It was proved in [25, Theorem 1.5] that, if (\mathbb{R}^n, X) satisfies the additional property \mathcal{R} (see Definition 2.17 below), then the jump set \mathcal{J}_u is countably X -rectifiable. It is worth recalling that Heisenberg groups, Carnot groups of step 2 and Carnot groups of type \star all satisfy property \mathcal{R} , see [25, Theorem 4.3].

Definition 2.17. We say that (\mathbb{R}^n, X) satisfies the *property \mathcal{R}* if for every open set $\Omega \subseteq \mathbb{R}^n$ and every $E \subseteq \mathbb{R}^n$ with locally finite X -perimeter in Ω , the essential boundary $\partial^* E \cap \Omega$ is countably X -rectifiable. Let us recall for completeness that the *essential boundary* of a measurable set E is $\partial^* E := \mathbb{R}^n \setminus (E_0 \cup E_1)$, where for $\lambda \in [0, 1]$ we denote by E_λ the set of points $p \in \mathbb{R}^n$ where E has density λ , i.e.,

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(E \cap B(p, r))}{\mathcal{L}^n(B(p, r))} = \lambda.$$

We are ready to introduce the jump part of the derivative $D_X u$.

Definition 2.18. Let $u \in \text{BV}_X(\Omega)$. We define the *jump part* of $D_X u$ as

$$D_X^j u := D_X^s u \llcorner \mathcal{J}_u$$

and the *Cantor part* of $D_X u$ as

$$D_X^c u := D_X^s u \llcorner (\Omega \setminus \mathcal{J}_u).$$

Remark 2.19. If (\mathbb{R}^n, X) satisfies both properties \mathcal{R} and \mathcal{D} (see Definition 2.20 below), then the jump part has the representation

$$D_X^j u = \sigma(\cdot, \nu_{\mathcal{J}_u})(u^+ - u^-) \nu_{\mathcal{J}_u} \mathcal{S}^{Q-1} \llcorner \mathcal{J}_u.$$

for a suitable function $\sigma: \mathbb{R}^n \times \mathbb{S}^{m-1} \rightarrow (0, +\infty)$; see [25, Theorem 1.7]. Again, Heisenberg groups, Carnot groups of step 2 and Carnot groups of type \star all satisfy both properties, see [25, Theorem 4.3].

Definition 2.20. We say that (\mathbb{R}^n, X) satisfies the *property \mathcal{D}* if there exists a function $\sigma: \mathbb{R}^n \times \mathbb{S}^{m-1} \rightarrow (0, +\infty)$ such that, for every C_X^1 -hypersurface $S \subseteq \mathbb{R}^n$ and every $p \in S$, one has

$$\lim_{r \rightarrow 0} \frac{\mathcal{S}^{Q-1}(S \cap B(p, r))}{r^{Q-1}} = \sigma(p, \nu_S(p)).$$

We mention that the validity of property \mathcal{D} is related to the broader problem of computing the Federer density for the perimeter measure of surfaces; see, e.g. [31, 38, 40, 41]. If property \mathcal{D} holds, then the function σ of Definition 2.20, is actually explicit in many cases, see e.g. [22, 42].

We conclude this section with a couple of technical results that will be useful in the sequel.

Proposition 2.21. Let $u \in \text{BV}_X(\Omega)$.

(i) Let $\xi \in C_c^\infty(\Omega)$. Then $u\xi \in \text{BV}_X(\Omega)$ and

$$D_{X_i}(u\xi) = \xi D_{X_i} u + u X_i \xi \mathcal{L}^n.$$

(ii) Let $K \in C_c^\infty(B_E(0, r))$ be spherically symmetric. Then $u * K \in C^\infty(\Omega)$ and for any $y \in \Omega$ such that $d_E(y, \partial\Omega) > r$ one has

$$X_i(u * K)(y) = (D_{X_i} u * K)(y) + R_i(u, K; y)$$

where

$$(4) \quad R_i(u, K; y) := \int_{\Omega} u(x) ((\text{div } X_i)(x) K(x - y) - \langle X_i(y) - X_i(x), \nabla K(x - y) \rangle) dx.$$

Proof. Let us first prove (i). Let $\varphi \in C_c^1(\Omega)$. We have

$$\begin{aligned} - \int_{\Omega} \varphi \xi d(D_{X_i} u) &= \int_{\Omega} u X_i^*(\varphi \xi) d\mathcal{L}^n = \sum_{t=1}^n \int_{\Omega} u \frac{\partial(a_{i,t} \varphi \xi)}{\partial x_t} d\mathcal{L}^n = \\ &= \sum_{t=1}^n \int_{\Omega} u \xi \frac{\partial(a_{i,t} \varphi)}{\partial x_j} d\mathcal{L}^n + \sum_{t=1}^n \int_{\Omega} u \varphi a_{i,t} \frac{\partial \xi}{\partial x_t} d\mathcal{L}^n = \\ &= \int_{\Omega} u \xi X_i^* \varphi d\mathcal{L}^n + \int_{\Omega} u \varphi X_i \xi d\mathcal{L}^n. \end{aligned}$$

Rearranging the equation we get

$$\int_{\Omega} u \xi X_i^* \varphi d\mathcal{L}^n = - \int_{\Omega} \varphi d(u X_i \xi \mathcal{L}^n) - \int_{\Omega} \varphi d(\xi D_{X_i} u)$$

which implies

$$D_{X_i}(u\xi) = u X_i \xi \mathcal{L}^n + \xi D_{X_i} u.$$

For a proof of (ii) see [35, Lemma 2.6] and [32]. \square

Lemma 2.22 ([28, Lemma 1.2.1 (i)]). *Let $W_\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}$ be, for $\varepsilon > 0$, a family of measurable functions supported in $B_E(0, \varepsilon)$, satisfying $|W_\varepsilon(x)| \leq C\varepsilon^{-n}$ for some positive constant C and $\int_{B_E(0, \varepsilon)} W_\varepsilon(x) dx = 0$. Then for every $u \in L^1(\Omega)$ we have*

$$\lim_{\varepsilon \rightarrow 0} \|W_\varepsilon * u\|_{L^1(\Omega)} = 0.$$

3. SPECIAL FUNCTIONS OF BOUNDED X -VARIATION

We start by introducing the main object of this paper. From now on, $\Omega \subset \mathbb{R}^n$ is a fixed open set.

Definition 3.1. Let $u \in \text{BV}_X(\Omega)$. We say that u is a *special function of bounded X -variation*, and we write $u \in \text{SBV}_X(\Omega)$, if

- (i) $D_X^s u = 0$,
- (ii) \mathcal{J}_u is a countably X -rectifiable set.

If u is in $\text{BV}_{X, \text{loc}}(\Omega)$ only, we say that u is a *special function of locally bounded X -variation*, and we write $u \in \text{SBV}_{X, \text{loc}}(\Omega)$.

As explained in Remark 2.16, when (\mathbb{R}^n, X) satisfies property \mathcal{R} then condition (ii) in Definition 3.1 is always automatically satisfied by any $u \in \text{BV}_{X, \text{loc}}(\Omega)$.

We now provide an equivalent definition for special functions of bounded X -variation that will be useful in the sequel.

Proposition 3.2. *The following statements are equivalent:*

- (i) $u \in \text{SBV}_{X, \text{loc}}(\Omega)$;
- (ii) $u \in \text{BV}_{X, \text{loc}}(\Omega)$ and there exist a countably X -rectifiable set $R \subset \Omega$ and a function² $f: R \rightarrow \mathbb{R}$ such that

$$D_X^s u = f \nu_R \mathcal{H}^{Q-1} \llcorner R,$$

where ν_R denotes the horizontal normal to R .

Moreover, the jump set \mathcal{J}_u coincides with $R_0 := \{p \in R : f(p) \neq 0\}$ up to \mathcal{H}^{Q-1} -negligible sets.

Proof. Assume (i); then, the jump set \mathcal{J}_u can be covered, up to a \mathcal{H}^{Q-1} -negligible set, by countably many C_X^1 -hypersurfaces $(S_j)_{j \in \mathbb{N}}$, that we may assume to be pairwise disjoint. Since $|D_X u|(S_j \setminus \mathcal{J}_u) = 0$ for every j , we obtain

$$D_X^s u = D_X^j u = D_X u \llcorner \mathcal{J}_u = \sum_{j \in \mathbb{N}} D_X u \llcorner (\mathcal{J}_u \cap S_j).$$

Observe that, locally, each hypersurface S_j separates the space \mathbb{R}^n into two X -regular open sets (see [47]): denoting by P_j^X the measure on S_j defined locally as the X -perimeter measure of (each of) these two components, using [47, Theorems 1.4 and 1.6] (see also [25, Proposition 3.7]) one finds

$$D_X^s u = \sum_{j \in \mathbb{N}} (u^+ - u^-) \nu_{S_j} P_j^X \llcorner \mathcal{J}_u,$$

where u^\pm are the *traces* (see [47]) of u on S_j . By [3, Theorem 4.2] we further obtain

$$D_X^s u = \sum_{j \in \mathbb{N}} (u^+ - u^-) \lambda \nu_{S_j} \mathcal{H}^{Q-1} \llcorner (\mathcal{J}_u \cap S_j)$$

for a suitable $\lambda: \bigcup_j S_j \rightarrow (0, +\infty)$. Up to changing the sign of $\nu_{\mathcal{J}_u}$, we can write $\nu_{\mathcal{J}_u} = \nu_{S_j} \mathcal{H}^{Q-1}$ -a.e. on $\mathcal{J}_u \cap S_j$, hence concluding that

$$D_X^s u = \sum_{j \in \mathbb{N}} (u^+ - u^-) \lambda \nu_{\mathcal{J}_u} \mathcal{H}^{Q-1} \llcorner (\mathcal{J}_u \cap S_j),$$

which proves (ii) with $R := \mathcal{J}_u$ and $f := (u^+ - u^-) \lambda$.

²Observe that necessarily $f \in L^1_{\text{loc}}(R, \mathcal{H}^{Q-1})$, for otherwise $u \notin \text{BV}_{X, \text{loc}}(\Omega)$.

Concerning the opposite implication, assume (ii); clearly, it is not restrictive to assume that $R = R_0$. Cover R , up to a \mathcal{H}^{Q-1} -negligible set, by countably many C_X^1 -hypersurfaces $(S_j)_{j \in \mathbb{N}}$, that we may assume to be pairwise disjoint. Using again [47, Theorems 1.4 and 1.6] (see also [25, Proposition 3.7]) and [3, Theorem 4.2], for every fixed j we have

$$D_X^s u \llcorner S_j = (u^+ - u^-) \lambda \nu_{S_j} \mathcal{H}^{Q-1} \llcorner S_j,$$

where again u^\pm are the traces of u on S_j and $\lambda: S_j \rightarrow (0, +\infty)$. On the other hand, by assumption, we have

$$D_X^s u \llcorner S_j = f \nu_R \mathcal{H}^{Q-1} \llcorner (R \cap S_j),$$

which implies that (up to a change of sign for ν_R and f)

$$\begin{aligned} u^+ - u^- &= 0 \quad \mathcal{H}^{Q-1}\text{-a.e. on } S_j \setminus R, \\ (u^+ - u^-) \lambda &= f \neq 0 \quad \text{and } \nu_{S_j} = \nu_R \quad \mathcal{H}^{Q-1}\text{-a.e. on } S_j \setminus R. \end{aligned}$$

Therefore, \mathcal{H}^{Q-1} -a.e. point of $R \cap S_j$ is an approximate X -jump point and, in particular, $\mathcal{H}^{Q-1}(R \setminus \mathcal{J}_u) = 0$. The proof will be accomplished if we show that $\mathcal{H}^{Q-1}(\mathcal{J}_u \setminus R) = 0$; if, instead, $\mathcal{H}^{Q-1}(\mathcal{J}_u \setminus R) > 0$, then by [25, Theorem 1.3 (i)] we would obtain

$$|D_X^s u|(\mathcal{J}_u \setminus R) = |D_X u|(\mathcal{J}_u \setminus R) > 0,$$

which clearly contradicts assumption (ii). This concludes the proof. \square

The following theorem provides our first main result about SBV_X functions.

Theorem 3.3. *The subspace $\text{SBV}_X(\Omega)$ is closed in $\text{BV}_X(\Omega)$.*

Proof. If I is finite or countable, $u_i \in \text{SBV}_X(\Omega)$ for any $i \in I$ and $\sum_{i \in I} u_i$ converges to $u \in \text{BV}_X(\Omega)$ in the BV_X norm, then $D_X u = \sum_{i \in I} D_X u_i$. Since $\sum_i D_X^a u_i$ is absolutely continuous with respect to \mathcal{L}^n and $\sum_i D_X^s u_i$ is singular, we have

$$D_X^a u = \sum_{i \in I} D_X^a u_i, \quad D_X^s u = \sum_{i \in I} D_X^s u_i.$$

Proposition 3.2 implies that $u \in \text{SBV}_X(\Omega)$, hence $\text{SBV}_X(\Omega)$ is closed in $\text{BV}_X(\Omega)$. \square

In the following lemma we denote by $Du = (D_1 u, \dots, D_n u)$ the derivatives of u in the sense of distribution; moreover, when $\mu = (\mu_1, \dots, \mu_n)$ is a vector-valued Radon measure and $X(x) = (a_1(x), \dots, a_n(x))$ is a smooth vector field, we denote by $\langle \mu, X \rangle$ the (scalar) Radon measure $\sum_{t=1}^n a_t \mu_t$.

Lemma 3.4. *The following statements hold:*

- (i) $\text{BV}_{\text{loc}}(\Omega) \subseteq \text{BV}_{X,\text{loc}}(\Omega)$ and $D_{X_i} u = \langle Du, X_i \rangle$ for every $i = 1, \dots, m$.
- (ii) $\text{SBV}_{\text{loc}}(\Omega) \subseteq \text{SBV}_{X,\text{loc}}(\Omega)$.

Proof. (i) Let $u \in \text{BV}_{\text{loc}}(\Omega)$. For every open set $A \subset\subset \Omega$, for every $1 \leq i \leq m$ and for every $\varphi \in C_c^1(A)$ we have

$$\int_A u X_i^* \varphi d\mathcal{L}^n = \sum_{t=1}^n \int_A u \frac{\partial(a_{i,t} \varphi)}{\partial x_t} d\mathcal{L}^n = - \int_A \varphi d \left(\sum_{t=1}^n (a_{i,t} D_t u) \right),$$

as claimed.

(ii) For every $u \in \text{SBV}_{\text{loc}}(\Omega)$ we write $Du = D^{a,E} u \mathcal{L}^n + (u_E^+ - u_E^-) \nu^E \mathcal{H}_E^{n-1} \llcorner \mathcal{J}_u^E$, where $D^{a,E} u$ denotes the (Euclidean) approximate gradient of u , \mathcal{J}_u^E is the (Euclidean) jump set of u oriented by its (Euclidean) unit normal $\nu^E = (\nu_1^E, \dots, \nu_n^E)$ and u_E^+ and u_E^- are the (Euclidean) traces of u on \mathcal{J}_u^E . By statement (i) we know that $u \in \text{BV}_{X,\text{loc}}(\Omega)$ and

$$D_{X_i} u = \langle D^{a,E} u, X_i \rangle \mathcal{L}^n + (u_E^+ - u_E^-) \langle \nu^E, X_i \rangle \mathcal{H}_E^{n-1} \llcorner \mathcal{J}_u^E.$$

We know that \mathcal{J}_u^E is countably rectifiable in the Euclidean sense, hence, up to modifying \mathcal{J}_u^E on a \mathcal{H}_E^{n-1} -negligible set, there exists a countable collection of C^1 -hypersurfaces $(S_j)_{j \in \mathbb{N}}$ such that

$$\mathcal{J}_u^E \subset \bigcup_{j \in \mathbb{N}} S_j.$$

Without loss of generality we can assume that the C^1 -hypersurfaces $(S_j)_{j \in \mathbb{N}}$ are pairwise disjoint; in this way ν^E coincides with the Euclidean unit normal $\nu_{S_j}^E$ to S_j on $\mathcal{J}_u^E \cap S_j$. For every $j \in \mathbb{N}$ we introduce the characteristic set $S_j^{\text{ch}} \subset S_j$ as

$$S_j^{\text{ch}} := \{p \in S_j : \text{span}(X_1(p), \dots, X_m(p)) \subseteq T_p S_j\}.$$

For $p \in \mathcal{J}_u^E$ let $k \in \mathbb{N}$ be such that $p \in S_k$; if $p \in S_k^{\text{ch}}$, then $\langle X_i(p), \nu^E(p) \rangle = \langle X_i(p), \nu_{S_k}^E(p) \rangle = 0$ for every $i \in \{1, \dots, m\}$, hence we can rewrite $D_{X_i} u$ as

$$(5) \quad D_{X_i} u = \langle X_i, D^{a,E} u \rangle \mathcal{L}^n + (u_E^+ - u_E^-) \langle X_i, \nu^E \rangle \mathcal{H}_E^{n-1} \llcorner R,$$

where

$$R := \mathcal{J}_u^E \setminus \bigcup_{j \in \mathbb{N}} S_j^{\text{ch}}.$$

The set R is countably X -rectifiable because $R \subset \bigcup_{j \in \mathbb{N}} (S_j \setminus S_j^{\text{ch}})$ and each $S_j \setminus S_j^{\text{ch}}$ is a C_X^1 -hypersurface. For $p \in R$, let $k \in \mathbb{N}$ be the unique integer such that $p \in S_k \setminus S_k^{\text{ch}}$ and let f be a C^1 defining function for S_k in a neighborhood of p ; then

$$(\nu_R)_i(p) = \frac{X_i f(p)}{|X f(p)|} = \frac{\langle X_i(p), \nabla f(p) \rangle}{|X f(p)|} = \frac{\langle X_i(p), \nu^E(p) \rangle}{|X f(p)|} |\nabla f(p)|, \quad \forall i = 1, \dots, m,$$

i.e.,

$$(6) \quad \langle X_i, \nu^E \rangle = \sigma_1 (\nu_R)_i, \quad \forall i = 1, \dots, m$$

for a suitable function $\sigma_1 : R \rightarrow (0, +\infty)$. As in the proof of Proposition 3.2 we observe that, locally, each C^1 - and C_X^1 -hypersurface $S_j \setminus S_j^{\text{ch}}$ separates the space into two connected open components of locally finite X -perimeter; combining [47, Propositions 4.1 and 4.5, Remark 4.7 and Corollary 4.14] (see also [22]) it can be shown that these X -perimeter measures have an integral representation with respect to both $\mathcal{H}^{Q-1} \llcorner (S_j \setminus S_j^{\text{ch}})$ and $\mathcal{H}_E^{n-1} \llcorner (S_j \setminus S_j^{\text{ch}})$. Ultimately, this gives

$$(7) \quad \mathcal{H}_E^{n-1} \llcorner R = \sigma_2 \mathcal{H}^{Q-1} \llcorner R$$

for a suitable $\sigma_2 : R \rightarrow (0, +\infty)$. Combining (5), (6) and (7) we obtain

$$D_{X_i} u = \langle X_i, D^{a,E} u \rangle \mathcal{L}^n + (u_E^+ - u_E^-) \sigma_1 \sigma_2 (\nu_R)_i \mathcal{H}^{Q-1} \llcorner R,$$

i.e.,

$$D_X^s u = (u_E^+ - u_E^-) \sigma_1 \sigma_2 \nu_R \mathcal{H}^{Q-1} \llcorner R.$$

Proposition 3.2 implies that $u \in \text{SBV}_{X,\text{loc}}(\Omega)$, as claimed. \square

Remark 3.5. A deeper inspection of the proof of Lemma 3.4 (ii) and, in particular, of the results from [47] that were used reveals that, if $u \in \text{SBV}_{\text{loc}}(\Omega)$ and R is as in the proof, then the traces u^\pm equal the Euclidean ones u_E^\pm .

The following result is an easy consequence of the celebrated Lusin-type theorem for gradients by G. Alberti [2].

Theorem 3.6. *For every $w \in L_{\text{loc}}^1(\Omega; \mathbb{R}^m)$ there exists $u \in \text{SBV}_{X,\text{loc}}(\Omega)$ such that $D_X^{\text{ap}} u = w \mathcal{L}^n$ -a.e. on Ω .*

Moreover, if Ω is bounded, then there exists $C = C(\Omega) > 0$ such that, for every $w \in L^1(\Omega; \mathbb{R}^m)$, the function u can be chosen in such a way that $|D_X u|(\Omega) \leq C \|w\|_{L^1(\Omega)}$.

Proof. Consider the horizontal vector field $X_w := w_1 X_1 + \dots + w_m X_m$. By [2, Theorem 3] there exists $u \in \text{SBV}_{\text{loc}}(\Omega)$ whose (Euclidean) approximate gradient is X_w and this, together with Lemma 3.4, proves the first part of the statement. The second part is a consequence of the estimate stated in [2, Theorem 3]. \square

We conclude with a result, Theorem 3.8, where we provide a recipe to produce lots of SBV_X functions: in fact, *any* L^1 function on *any* countably X -rectifiable set can appear as the jump part of an SBV_X function.

Lemma 3.7. *Let $S \subseteq \Omega$ be a C_X^1 -hypersurface oriented by a normal ν , let $\theta \in L^1(\mathcal{H}^{Q-1} \llcorner S)$ and $\delta > 0$. Then there exists $u \in \text{SBV}_X(\Omega)$ such that*

$$D_X^j u \equiv \theta \nu \mathcal{H}^{Q-1} \llcorner S, \quad \|u\|_{L^1(\Omega)} < \delta, \quad \text{and} \quad |D_X u|(\Omega) \leq (1 + \delta) \|\theta\|_{L^1(\mathcal{H}^{Q-1} \llcorner S)}.$$

Proof. Fix a countable family $(B_j)_{j \in \mathbb{N}}$ of balls, contained in Ω and with centers on S , and functions $f_j \in C_X^1(B_j)$ such that, for every $j \in \mathbb{N}$,

$$S \cap B_j = \{q \in B_j : f_j(q) = 0\}, \quad X f_j \neq 0 \text{ on } B_j, \quad S \subset \bigcup_{j \in \mathbb{N}} B_j.$$

We can also assume that $\langle X f_j, \nu \rangle > 0$ on $S \cap B_j$. Without loss of generality, we can assume that each ball B_j intersects only a finite number of other balls of the collection. Now consider a partition of the unity associated with $(B_j)_{j \in \mathbb{N}}$, i.e., a collection of functions $(\zeta_j)_{j \in \mathbb{N}}$ such that, for every $j \in \mathbb{N}$,

$$\zeta_j \in C_c^\infty(B_j), \quad 0 \leq \zeta_j \leq 1, \quad \text{and} \quad \sum_{j \in \mathbb{N}} \zeta_j \equiv 1 \text{ on } S.$$

Fix $j \in \mathbb{N}$, we define

$$(8) \quad B_j^+ := \{q \in B_j : f_j(q) > 0\}, \quad B_j^- := \{q \in B_j : f_j(q) < 0\}.$$

For every j , let $\theta_j := \theta / \sigma_j \in L^1(S \cap B_j)$, where σ_j is a function on $S \cap B_j$ with $\inf \sigma_j > 0$ that will be introduced later. Using [47, Theorem 1.5] we can find $\tilde{u}_j \in C^\infty(B_j^+) \cap W_X^{1,1}(B_j^+)$ ³ such that $\text{spt } \tilde{u}_j \subset\subset B_j$ and

$$\|\tilde{u}_j\|_{L^1(B_j^+)} \leq \frac{\delta}{2^j}, \quad \|X \tilde{u}_j\|_{L^1(B_j^+)} \leq \left(1 + \frac{\delta}{2^j}\right) \|\zeta_j \theta_j\|_{L^1(\mathcal{H}^{Q-1} \llcorner S \cap B_j)},$$

and, for \mathcal{H}^{Q-1} -almost every $q \in S \cap B_j$, we have

$$\lim_{r \rightarrow 0} \int_{B_\nu^+(q,r)} |\tilde{u}_j - \zeta_j \theta_j| d\mathcal{L}^n = 0.$$

where $B_\nu^+(q, r)$ is defined as in Definition 2.14. We define u_j on Ω as

$$u_j := \begin{cases} \tilde{u}_j & \text{on } B_j^+ \\ 0 & \text{on } \mathbb{R}^n \setminus B_j^+. \end{cases}$$

By [47, Theorem 5.3 and Theorem 1.3], $u_j \in \text{BV}_X(\Omega)$ and, for some $C > 1$ and for some \mathcal{H}^{Q-1} -measurable function $\sigma_j : S \cap B_j \rightarrow [1/C, C]$ we have

$$\begin{aligned} D_X u_j &= (X \tilde{u}_j) \mathcal{L}^n \llcorner B_j^+ + \zeta_j \theta_j \sigma_j \nu \mathcal{H}^{Q-1} \llcorner (B_j \cap S) \\ &= (X \tilde{u}_j) \mathcal{L}^n \llcorner B_j^+ + \zeta_j \theta \nu \mathcal{H}^{Q-1} \llcorner (B_j \cap S), \end{aligned}$$

the latter implying that $u_j \in \text{SBV}_X(\Omega)$. The function $u := \sum_{j \in \mathbb{N}} u_j$ satisfies the statement of the Lemma. \square

³Remember that $W_X^{1,1}$ is the space of functions u such that both u and Xu belong to L^1 .

Theorem 3.8. *Let $S \subseteq \Omega$ be a countably X -rectifiable set oriented by ν ; let $\theta \in L^1(\mathcal{H}^{Q-1} \llcorner S)$ and $\delta > 0$ be fixed. Then there exists $u \in \text{SBV}_X(\Omega)$ such that*

$$D_X^j u \equiv \theta \nu \mathcal{H}^{Q-1} \llcorner S, \quad \|u\|_{L^1(\Omega)} < \delta, \quad \text{and} \quad |D_X u|(\Omega) \leq (1 + \delta) \|\theta\|_{L^1(\mathcal{H}^{Q-1} \llcorner S)}.$$

Proof. Since S is countably X -rectifiable there exists a countable collection $(S_i)_{i \in \mathbb{N}}$ of C_X^1 -hypersurfaces such that

$$\mathcal{H}^{Q-1} \left(S \setminus \bigcup_{i \in \mathbb{N}} S_i \right) = 0.$$

Without loss of generality we can assume that the C_X^1 -hypersurfaces $(S_i)_{i \in \mathbb{N}}$ are pairwise disjoint. We extend θ to 0 outside S and, for every $i \in \mathbb{N}$, we define

$$\theta_i := \theta|_{S_i}.$$

For every $i \in \mathbb{N}$ we use Lemma 3.7 to obtain a function $u_i \in \text{SBV}_{X,\text{loc}}(\Omega)$ such that

$$D_X^j u_i \equiv \theta_i \nu \mathcal{H}^{Q-1} \llcorner S_i, \quad \|u_i\|_{L^1(\Omega)} < \frac{\delta}{2^i}, \quad \text{and} \quad |D_X u_i|(\Omega) \leq \left(1 + \frac{\delta}{2^i}\right) \|\theta_i\|_{L^1(\mathcal{H}^{Q-1} \llcorner S_i)}.$$

The function $u := \sum_{i \in \mathbb{N}} u_i$ satisfies the statement of the Theorem. \square

4. PROOF OF THEOREM 1.4

4.1. Construction of the approximating sequence. This section is devoted to the construction of the approximating sequence $(u_k)_{k \in \mathbb{N}}$ of SBV_X functions that will be used to prove Theorem 1.4. In the following Construction 4.1 we start by proving that, if $u \in \text{BV}_X(\Omega)$ and \mathcal{J}_u is countably X -rectifiable, then it is possible to approximate \mathcal{J}_u with a X -rectifiable set that can be in turn approximated with a C_X^1 -hypersurface. We underline that the following construction is valid, as a particular case, for functions $u \in \text{SBV}_X(\Omega)$.

Construction 4.1. Fix $u \in \text{BV}_X(\Omega)$ with a countably X -rectifiable \mathcal{J}_u . For every $\eta > 0$ we define the set

$$(9) \quad \mathcal{J}_{u,\eta} := \left\{ x \in \mathcal{J}_u : |u^+(x) - u^-(x)| \geq \frac{1}{\eta} \right\} \cap B(0, \eta).$$

Since the jump set \mathcal{J}_u is countably X -rectifiable, also $\mathcal{J}_{u,\eta}$ is countably X -rectifiable for every $\eta > 0$. Let us moreover observe that the set $\mathcal{J}_{u,\eta}$ is X -rectifiable, i.e., that $\mathcal{H}^{Q-1}(\mathcal{J}_{u,\eta}) < \infty$. In fact, thanks to Theorem 2.10, there exists a positive constant $C > 0$, only depending on η such that

$$\mathcal{H}^{Q-1}(\mathcal{J}_{u,\eta}) \leq C\eta |D_X u|(\Omega).$$

Since the family $\mathcal{J}_{u,\eta}$ is increasing and invades \mathcal{J}_u when $\eta \rightarrow +\infty$ we also have

$$|D_X u|(\mathcal{J}_u \setminus \mathcal{J}_{u,\eta}) \xrightarrow{\eta \rightarrow +\infty} 0$$

so that, for every $k \in \mathbb{N}$, we can choose an $\eta_k > 0$ such that $(\eta_k)_{k \in \mathbb{N}}$ is increasing and

$$(10) \quad |D_X u|(\mathcal{J}_u \setminus \mathcal{J}_{u,\eta_k}) < \frac{1}{k}.$$

For the sake of brevity let us write $\mathcal{J}_u^k := \mathcal{J}_{u,\eta_k}$. Now, for every $\delta > 0$, using Lemma 2.13, we can find a C_X^1 -hypersurface M_δ such that

$$\mathcal{H}^{Q-1}(\mathcal{J}_u^k \setminus M_\delta) < \delta.$$

By Theorem 2.10 one has $|D_X u| \llcorner \mathcal{J}_u \ll \mathcal{H}^{Q-1} \llcorner \mathcal{J}_u$, so for every $k \in \mathbb{N}$ we can find a C_X^1 -hypersurface M_k such that

$$(11) \quad |D_X u|(\mathcal{J}_u^k \setminus M_k) < \frac{1}{k}.$$

Before starting the construction of the approximating sequence we need the following Lemma.

Lemma 4.2. *Let $u \in \text{BV}_X(\Omega)$ be such that \mathcal{J}_u is countably X -rectifiable and consider the function*

$$\begin{aligned} \mathbf{j}_u : \mathcal{J}_u &\rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{m-1} \\ x &\rightarrow (u^+(x), u^-(x), \nu_{\mathcal{J}_u}(x)). \end{aligned}$$

Then for every $k \in \mathbb{N}$ there exist a compact set $C_k \subseteq \mathcal{J}_u^k \cap M_k$ (where \mathcal{J}_u^k and M_k are defined as in Construction 4.1) and a representative of \mathbf{j}_u such that $\mathbf{j}_u|_{C_k}$ is continuous and

$$(12) \quad |D_X u|((\mathcal{J}_u^k \cap M_k) \setminus C_k) < \frac{1}{k}.$$

Proof. By [25, Proposition 2.28] we can choose a Borel representative of \mathbf{j}_u so it suffices to use Lusin's Theorem [26, Theorem 2.3.5]. \square

Using the previous Lemma we can construct the required approximating sequence. Recall that we want to obtain a sequence of functions $(u_k)_{k \in \mathbb{N}}$ such that $u_k \in \text{SBV}_X(\Omega)$, $u_k \in C^\infty(\Omega \setminus \mathcal{J}_{u_k})$ and u_k converges to u in the BV_X norm. Fix a representative of $u \in \text{SBV}_X(\Omega)$ and $k \in \mathbb{N}$ and consider the compact set C_k given by Lemma 4.2. For $\ell \in \mathbb{N}$ we define the sets

$$\begin{aligned} A_k^1 &:= \left\{ x \in \Omega : d_E(x, C_k) > \frac{1}{2} \right\}, \\ A_k^\ell &:= \left\{ x \in \Omega : \frac{1}{\ell+1} < d_E(x, C_k) < \frac{1}{\ell-1} \right\} \text{ if } \ell > 1. \end{aligned}$$

We observe that

$$\bigcup_{\ell \in \mathbb{N}} A_k^\ell = \Omega \setminus C_k$$

and that, for every $\bar{\ell} \in \mathbb{N}$, $A_k^{\bar{\ell}}$ intersects at most two of the sets of the family $(A_k^i)_i$, namely $A_k^{\bar{\ell}+1}$ and $A_k^{\bar{\ell}-1}$. Now for $s \in \mathbb{N}$ we define the bounded open sets

$$\begin{aligned} A_k^{\ell,1} &:= A_k^\ell \cap \{|x|_{\mathbb{R}^n} < 2\}, \\ A_k^{\ell,s} &:= A_k^\ell \cap \{s-1 < |x|_{\mathbb{R}^n} < s+1\} \text{ if } s > 1. \end{aligned}$$

We observe that

$$\bigcup_{s \in \mathbb{N}} A_k^{\ell,s} = A_k^\ell$$

and that, for every $\bar{s} \in \mathbb{N}$, $A_k^{\ell,\bar{s}}$ intersects at most two of the sets of the family $(A_k^{\ell,i})_i$, namely $A_k^{\ell,\bar{s}+1}$ and $A_k^{\ell,\bar{s}-1}$. Consider a partition of unity on $\Omega \setminus C_k$ associated with $(A_k^{\ell,s})_{\ell,s \in \mathbb{N}}$, that is, functions $\xi_k^{\ell,s} \in C_c^\infty(A_k^{\ell,s})$ such that $0 \leq \xi_k^{\ell,s} \leq 1$ and $\sum_{\ell,s \in \mathbb{N}} \xi_k^{\ell,s} \equiv 1$ on $\Omega \setminus C_k$. Let us also define

$$(13) \quad Z_k^{\ell,s} := \left\{ x \in \mathbb{R}^n : d_E(x, \text{spt}(\xi_k^{\ell,s})) \leq \frac{d_E(\partial A_k^{\ell,s}, \text{spt}(\xi_k^{\ell,s}))}{5} \right\}.$$

Notice that $Z_k^{\ell,s}$ is compact and $Z_k^{\ell,s} \subset A_k^{\ell,s}$. Fix a mollification kernel, i.e., a spherically symmetric non-negative function $K \in C_c^\infty(B_E(0,1))$ such that $\int_{\mathbb{R}^n} K d\mathcal{L}^n = 1$. For $\varepsilon > 0$ we define $K_\varepsilon(x) = \varepsilon^{-n} K(x/\varepsilon)$. For $k \in \mathbb{N}$ we finally define

$$(14) \quad u_k := \sum_{\ell,s \in \mathbb{N}} (\xi_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}} \text{ on } \Omega \setminus C_k$$

where the $\varepsilon_k^{\ell,s}$'s are chosen so small that, for every $1 \leq i \leq m$, $1 \leq h \leq n$, $1 \leq t \leq n$, $h \neq t$, we have⁶

⁴For convenience we also define $A_k^\ell := \emptyset$ if $\ell < 1$.

⁵For convenience we also define $A_k^{\ell,s} := \emptyset$ if either $\ell < 1$ or $s < 1$.

⁶Notice that conditions (18) and (19) can be requested because of Lemma 2.22.

$$(15) \quad \varepsilon_k^{\ell,s} < \frac{1}{2^{\ell+s}k},$$

$$(16) \quad \varepsilon_k^{\ell,s} < \frac{d_E(\partial A_k^{\ell,s}, \text{spt}(\xi_k^{\ell,s}))}{10},$$

$$(17) \quad \left\| K_{\varepsilon_k^{\ell,s}} * (u X_i \xi_k^{\ell,s}) - u X_i \xi_k^{\ell,s} \right\|_{L^1(\Omega)} < \frac{1}{2^{\ell+s}k},$$

$$(18) \quad \left\| \left(u \xi_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_t} \right) * W_{\varepsilon_k^{\ell,s}}^t \right\|_{L^1(\Omega)} < \frac{1}{n 2^{\ell+s}k},$$

$$(19) \quad \left\| \left(u \xi_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_h} \right) * W_{\varepsilon_k^{\ell,s}}^{t,h} \right\|_{L^1(\Omega)} < \frac{1}{n^2 2^{\ell+s}k},$$

$$(20) \quad \left\| (\xi_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}} - \xi_k^{\ell,s} u \right\|_{L^1(\Omega)} < \frac{1}{2^{\ell+s}k},$$

$$(21) \quad \left\| (\xi_k^{\ell,s} D_{X_i}^{ap} u) * K_{\varepsilon_k^{\ell,s}} - \xi_k^{\ell,s} D_{X_i}^{ap} u \right\|_{L^1(\Omega)} < \frac{1}{2^{\ell+s}k},$$

$$(22) \quad \varepsilon_k^{\ell,s} < \frac{1}{100(\ell+1)(\ell+2)},$$

$$(23) \quad \varepsilon_k^{\ell,s} < \frac{1}{C \|\nabla K\|_{L^\infty} \|u\|_{L^1(Z_k^{\ell,s})} 2^{\ell+s}k}.$$

The number $C > 0$ appearing in (23) is a constant that will be chosen in Proposition 4.7 below, and $W_{\varepsilon_k^{\ell,s}}^t$ and $W_{\varepsilon_k^{\ell,s}}^{t,h}$ appearing in (18) and (19) are defined as

$$(24) \quad W_{\varepsilon_k^{\ell,s}}^t(x) := \left(K_{\varepsilon_k^{\ell,s}}(x) + x_t \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x) \right),$$

$$(25) \quad W_{\varepsilon_k^{\ell,s}}^{t,h}(x) := x_h \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x),$$

being x_v the v -th component of x , $v = 1, \dots, n$.

Remark 4.3. Fix $\ell, s, k \in \mathbb{N}$. Then for any $x \in A_k^{\ell,s}$ we have, using condition (22), that

$$B_E(x, \varepsilon_k^{\ell,s}) \subseteq A_k^{\ell,s} \cup A_k^{\ell,s+1} \cup A_k^{\ell,s-1} \cup A_k^{\ell-1,s} \cup A_k^{\ell+1,s}.$$

Let us also observe that, thanks to condition (16), we have that $\text{spt}[(\xi_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}}] \subseteq A_k^{\ell,s}$, the latter implying that the sum in (14) is locally finite, hence $u_k \in C^\infty(\Omega \setminus C_k)$. Moreover, u_k is defined out of a \mathcal{L}^n -negligible set C_k and, from (20), $u_k \in L^1(\Omega)$ and $\|u_k - u\|_{L^1(\Omega)} \xrightarrow{k \rightarrow +\infty} 0$. Later, using Lemma 4.6 and Proposition 4.7, we will prove in Proposition 4.8 that $u_k \in \text{SBV}_X(\Omega)$.

Lemma 4.4. *Let u, u_k and C_k be defined as above. Then for every $M > 0$ and every $y \in C_k$ one has*

$$\lim_{r \rightarrow 0} r^{-M} \int_{B_E(y,r) \cap \Omega} |u_k - u| d\mathcal{L}^n = 0$$

Proof. Fix $r > 0$ and $y \in C_k$. From the fact that $\xi_k^{\ell,s} \in C_c^\infty(A_k^{\ell,s})$ and (16) we have that

$$u_k(x) - u(x) = \sum_{s \in \mathbb{N}} \sum_{\ell=\ell_0}^{\infty} \left((\xi_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}}(x) - \xi_k^{\ell,s} u(x) \right), \quad \forall x \in B_E(y, r) \cap \Omega$$

where $\ell_0 \in \mathbb{N}$ is defined as $\ell_0 := [1/r]$ and $[\cdot]$ denotes the floor function. From (20) we obtain

$$\|u_k - u\|_{L^1(B_E(y,r) \cap \Omega)} \leq \sum_{s \in \mathbb{N}} \sum_{\ell=\ell_0}^{\infty} \frac{1}{2^{\ell+s} k}.$$

From the definition of ℓ_0 and the fact that for every $M > 0$ one has $\lim_{r \rightarrow 0^+} r^{-M} 2^{-1/r} = 0$ we obtain the thesis. \square

Lemma 4.5. *Let u, u_k and C_k be defined as above. Then for every $M > 0$ and for every $y \in C_k$ one has*

$$\lim_{r \rightarrow 0} r^{-M} \int_{B(y,r) \cap \Omega} |u_k - u| d\mathcal{L}^n = 0$$

Proof. For any $p, q \in \overline{(B_E(y,1))}$ one has $d_E(p, q) \leq Cd(p, q)$ where $C \geq 1$ is a constant only depending on the vector fields X_i 's and on a compact set $K \supset \supset (B_E(y,1))$ (see [44]). Hence for any sufficiently small $r > 0$, we have $B(y, r) \subseteq B_E(y, Cr)$. The latter implies that for any such $r > 0$ we have

$$r^{-M} \int_{B(y,r) \cap \Omega} |u_k - u| d\mathcal{L}^n \leq r^{-M} \int_{B_E(y, Cr) \cap \Omega} |u_k - u| d\mathcal{L}^n.$$

The result then follows upon letting $r \rightarrow 0$ and using Lemma 4.4. \square

Lemma 4.6. *Let u, u_k and C_k be defined as above. Then for every $y \in C_k$*

$$\lim_{r \rightarrow 0} \int_{B_{\nu_{\mathcal{J}_u}(y)}^+(y,r)} |u_k(x) - u^+(y)| dx = 0, \quad \lim_{r \rightarrow 0} \int_{B_{\nu_{\mathcal{J}_u}(y)}^-(y,r)} |u_k(x) - u^-(y)| dx = 0.$$

Proof. We will prove only that $\lim_{r \rightarrow 0} \int_{B_{\nu_{\mathcal{J}_u}(y)}^+(y,r)} |u_k(x) - u^+(y)| dx = 0$, the other limit is analogous. For the sake of brevity we write $B_r^+ := B_{\nu_{\mathcal{J}_u}(y)}^+(y, r)$ and $u^+ := u^+(y)$. Since

$$\int_{B_r^+} |u_k(x) - u^+| dx \leq \int_{B_r^+} |u_k(x) - u(x)| dx + \int_{B_r^+} |u(x) - u^+| dx,$$

and $\int_{B_r^+} |u(x) - u^+| dx \xrightarrow{r \rightarrow 0} 0$, it suffices to prove that

$$\lim_{r \rightarrow 0} \int_{B_r^+} |u_k(x) - u(x)| dx = 0.$$

We observe that

$$\begin{aligned} \int_{B_r^+} |u_k(x) - u(x)| dx &= \frac{1}{\mathcal{L}^n(B_r^+)} \int_{B_r^+} |u_k(x) - u(x)| dx \leq \frac{1}{\mathcal{L}^n(B_r^+)} \int_{B(p,r) \cap \Omega} |u_k(x) - u(x)| dx \\ &= \frac{\mathcal{L}^n(B(y, r))}{\mathcal{L}^n(B_r^+)} \frac{1}{\mathcal{L}^n(B(y, r))} \int_{B(p,r) \cap \Omega} |u_k(x) - u(x)| dx. \end{aligned}$$

From [21, Proposition 2.1.5] we have $\frac{\mathcal{L}^n(B(p,r))}{\mathcal{L}^n(B_r^+)} \xrightarrow{r \rightarrow 0} 2$ so the result follows upon letting $r \rightarrow 0$, using Lemma 4.5 and the fact that there exists a positive constant $C > 0$ such that $\mathcal{L}^n(B(y, r)) \geq Cr^Q$ (see for instance [44]). \square

4.2. Estimates on the total variation. Fix $k \in \mathbb{N}$, $i \in \{1, \dots, m\}$ and $y \in \Omega \setminus C_k$. By Proposition 2.21 we have

$$\begin{aligned}
(X_i u_k)(y) &= X_i \left(\sum_{\ell, s \in \mathbb{N}} (\xi_k^{\ell, s} u) * K_{\varepsilon_k^{\ell, s}} \right) (y) = \sum_{\ell, s \in \mathbb{N}} X_i [(\xi_k^{\ell, s} u) * K_{\varepsilon_k^{\ell, s}}] (y) = \\
&= \sum_{\ell, s \in \mathbb{N}} \left[(D_{X_i} (\xi_k^{\ell, s} u) * K_{\varepsilon_k^{\ell, s}})(y) + R_i(\xi_k^{\ell, s} u, K_{\varepsilon_k^{\ell, s}}; y) \right] = \\
&= \sum_{\ell, s \in \mathbb{N}} \left\{ \left[\left(u X_i \xi_k^{\ell, s} \mathcal{L}^n + \xi_k^{\ell, s} D_{X_i} u \right) * K_{\varepsilon_k^{\ell, s}} \right] (y) + R_i(\xi_k^{\ell, s} u, K_{\varepsilon_k^{\ell, s}}; y) \right\} = \\
&= \sum_{\ell, s \in \mathbb{N}} \left[(\xi_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}}(y) + \int_{\mathbb{R}^n} K_{\varepsilon_k^{\ell, s}}(y-x) u(x) (X_i \xi_k^{\ell, s})(x) dx + R_i(\xi_k^{\ell, s} u, K_{\varepsilon_k^{\ell, s}}; y) \right].
\end{aligned}$$

For the sake of brevity let us define

$$(26) \quad S_i^{k, \ell, s}(y) := \int_{\mathbb{R}^n} K_{\varepsilon_k^{\ell, s}}(y-x) u(x) (X_i \xi_k^{\ell, s})(x) dx, \quad R_i^{k, \ell, s}(y) := R_i(\xi_k^{\ell, s} u, K_{\varepsilon_k^{\ell, s}}; y)$$

so that

$$(27) \quad (X_i u_k)(y) = \sum_{\ell, s \in \mathbb{N}} \left((\xi_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}}(y) + S_i^{k, \ell, s}(y) + R_i^{k, \ell, s}(y) \right).$$

Now we want to estimate the L^1 -norm of the two remainders $R_i^{k, \ell, s}$ and $S_i^{k, \ell, s}$: part of the following Proposition is a rewriting of [28, Lemma 2.1.1] in a language more useful for our purposes.

Proposition 4.7. *If $S_i^{k, \ell, s}$ and $R_i^{k, \ell, s}$ are defined as above (see (4) and (26)), then for any $i = 1, \dots, m$*

$$\left\| \sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s} \right\|_{L^1(\Omega)} \leq \frac{1}{k} \quad \text{and} \quad \left\| \sum_{\ell, s \in \mathbb{N}} R_i^{k, \ell, s} \right\|_{L^1(\Omega)} \leq \frac{3}{k}.$$

Proof. We start by estimating $S_i^{k, \ell, s}$. Fix $y \in \Omega \setminus C_k$: since $\sum_{\ell, s \in \mathbb{N}} X_i \xi_k^{\ell, s} \equiv 0$, then

$$\begin{aligned}
\sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s}(y) &= \sum_{\ell, s \in \mathbb{N}} K_{\varepsilon_k^{\ell, s}} * (u X_i \xi_k^{\ell, s})(y) \\
&= \sum_{\ell, s \in \mathbb{N}} K_{\varepsilon_k^{\ell, s}} * (u X_i \xi_k^{\ell, s})(y) - \sum_{\ell, s \in \mathbb{N}} (u X_i \xi_k^{\ell, s})(y) \\
&= \sum_{\ell, s \in \mathbb{N}} \left(K_{\varepsilon_k^{\ell, s}} * (u X_i \xi_k^{\ell, s}) - u X_i \xi_k^{\ell, s} \right)(y).
\end{aligned}$$

Using (17) and the fact that $\mathcal{L}^n(C_k) = 0$, we immediately obtain $\left\| \sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s} \right\|_{L^1(\Omega)} \leq 1/k$.

To estimate $R_i^{k, \ell, s}$ we start by observing that for every $y \in \Omega \setminus C_k$ one has

$$(28) \quad R_i^{k, \ell, s}(y) = \int_{\Omega} \xi_k^{\ell, s}(x) u(x) \left((\operatorname{div} X_i)(x) K_{\varepsilon_k^{\ell, s}}(x-y) - \langle X_i(y) - X_i(x), \nabla K_{\varepsilon_k^{\ell, s}}(x-y) \rangle \right) dx$$

$$(29) \quad = \sum_{t=1}^n \int_{\Omega} \xi_k^{\ell, s}(x) u(x) \left(\frac{\partial a_{i,t}}{\partial x_t}(x) K_{\varepsilon_k^{\ell, s}}(x-y) - (a_{i,t}(y) - a_{i,t}(x)) \frac{\partial K_{\varepsilon_k^{\ell, s}}}{\partial x_t}(x-y) \right) dx$$

Now, for every $y, x \in Z_k^{\ell, s}$ (see (13)) we can expand each $a_{i,t}$ with a Taylor's expansion

$$(30) \quad a_{i,t}(y) - a_{i,t}(x) = \sum_{h=1}^n \frac{\partial a_{i,t}}{\partial x_h}(x) (y-x)_h + T_{i,t}(y, x)$$

where $(y - x)_h$ denotes the h -component of the vector $y - x \in \mathbb{R}^n$ and

$$(31) \quad |T_{i,t}(y, x)| \leq C_{i,t}^{k,\ell,s} |y - x|^2$$

where $C_{i,t}^{k,\ell,s}$ is a positive constant only depending on the L^∞ -norm of the second derivatives of $a_{i,t}$ on $Z_k^{\ell,s}$. Replacing (30) into (28) we obtain:

$$\begin{aligned} R_i^{k,\ell,s}(y) &= \sum_{t=1}^n \int_{\Omega} \xi_k^{\ell,s}(x) u(x) \frac{\partial a_{i,t}}{\partial x_t}(x) K_{\varepsilon_k^{\ell,s}}(x - y) dx \\ &\quad - \sum_{t,h=1}^n \int_{\Omega} \xi_k^{\ell,s}(x) u(x) \frac{\partial a_{i,t}}{\partial x_h}(x) (y - x)_h \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x - y) dx \\ &\quad - \sum_{t=1}^n \int_{\Omega} \xi_k^{\ell,s}(x) u(x) T_{i,t}(y, x) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x - y) dx \\ &= \sum_{t=1}^n \int_{\Omega} \xi_k^{\ell,s}(x) u(x) \frac{\partial a_{i,t}}{\partial x_t}(x) \left(K_{\varepsilon_k^{\ell,s}}(x - y) - (y - x)_t \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x - y) \right) dx \\ &\quad - \sum_{t,h=1, t \neq h}^n \int_{\Omega} \xi_k^{\ell,s}(x) u(x) \frac{\partial a_{i,t}}{\partial x_h}(x) (y - x)_h \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x - y) dx \\ &\quad - \sum_{t=1}^n \int_{\Omega} \xi_k^{\ell,s}(x) u(x) T_{i,t}(y, x) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x - y) dx. \end{aligned}$$

Recall that K (and therefore $K_{\varepsilon_k^{\ell,s}}$) is spherically symmetric so that we can write

$$\begin{aligned} R_i^{k,\ell,s}(y) &= \sum_{t=1}^n \left(u \xi_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_t} \right) * W_{\varepsilon_k^{\ell,s}}^t(y) + \sum_{t,h=1, t \neq h}^n \left(u \xi_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_h} \right) * W_{\varepsilon_k^{\ell,s}}^{t,h}(y) \\ &\quad - \sum_{t=1}^n \int_{\Omega} \xi_k^{\ell,s}(x) u(x) T_{i,t}(y, x) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x - y) dx \end{aligned}$$

where $W_{\varepsilon_k^{\ell,s}}^t$ and $W_{\varepsilon_k^{\ell,s}}^{t,h}$ are defined as in (24). Then we have

$$\begin{aligned} \left\| \sum_{\ell,s \in \mathbb{N}} R_i^{k,\ell,s} \right\|_{L^1(\Omega)} &\leq \underbrace{\sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n \left\| \left(u \xi_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_t} \right) * W_{\varepsilon_k^{\ell,s}}^t \right\|_{L^1(\Omega)}}_{(A)} + \underbrace{\sum_{\ell,s \in \mathbb{N}} \sum_{\substack{t,h=1 \\ t \neq h}}^n \left\| \left(u \xi_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_h} \right) * W_{\varepsilon_k^{\ell,s}}^{t,h} \right\|_{L^1(\Omega)}}_{(B)} \\ &\quad + \underbrace{\sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n \int_{Z_k^{\ell,s}} \int_{B_E(x, \varepsilon_k^{\ell,s})} \left| \xi_k^{\ell,s}(x) u(x) T_{i,t}(y, x) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x - y) \right| dy dx}_{(C)} \end{aligned}$$

From (18) and (19) we obtain that $(A) \leq 1/k$ and $(B) \leq 1/k$. Concerning (C) we have, using (31) and the fact that $\left| \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t} \right| < \|\nabla K\|_{L^\infty} (\varepsilon_k^{\ell,s})^{-n-1}$ and $|x - y| < \varepsilon_k^{\ell,s}$, that

$$\begin{aligned} (C) &\leq \|\nabla K\|_{L^\infty} \sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n \int_{Z_k^{\ell,s}} \int_{B_E(x, \varepsilon_k^{\ell,s})} (\varepsilon_k^{\ell,s})^{-n-1} |\xi_k^{\ell,s}(x)u(x)| |T_{i,t}(y, x)| dy dx \\ &\leq \|\nabla K\|_{L^\infty} \sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n \int_{Z_k^{\ell,s}} (\varepsilon_k^{\ell,s})^{-n-1} |\xi_k^{\ell,s}(x)u(x)| C_{i,t}^{k,\ell,s} \int_{B_E(x, \varepsilon_k^{\ell,s})} |y - x|^2 dy dx \\ &\leq \|\nabla K\|_{L^\infty} \sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n C_{i,t}^{k,\ell,s} \varepsilon_k^{\ell,s} \int_{Z_k^{\ell,s}} |\xi_k^{\ell,s}(x)u(x)| dx \end{aligned}$$

and using (23) with the specific choice of

$$C := \sum_{t=1}^n C_{i,t}^{k,\ell,s},$$

we obtain $(C) \leq 1/k$, concluding the proof. \square

Proposition 4.8. *For every $k \in \mathbb{N}$ the function u_k defined in (14) satisfies the following properties:*

- (i) $u_k \in \text{BV}_X(\Omega \setminus C_k)$,
- (ii) $u_k \in \text{BV}_X(\Omega)$,
- (iii) $u_k \in \text{SBV}_X(\Omega)$.

Proof.

(i) We know from Proposition 2.21 that for every $\ell, s \in \mathbb{N}$ one has $u_k^{\ell,s} \in \text{BV}_X(\Omega \setminus C_k)$ where, for the sake of brevity, we defined $u_k^{\ell,s} := (\xi_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}}$. To prove $u_k \in \text{BV}_X(\Omega \setminus C_k)$, it is enough to show that

$$(32) \quad \left\| \sum_{\ell,s \in \mathbb{N}} u_k^{\ell,s} \right\|_{\text{BV}_X(\Omega \setminus C_k)} = \left\| \sum_{\ell,s \in \mathbb{N}} u_k^{\ell,s} \right\|_{L^1(\Omega)} + \left| D_X \left(\sum_{\ell,s \in \mathbb{N}} u_k^{\ell,s} \right) \right|(\Omega \setminus C_k) < \infty.$$

As we mentioned before, thanks to (20), $u_k \in L^1(\Omega)$ so that the first addend on the right hand side of (32) is finite. We are left to estimate the second term. Since $u_k \in C^\infty(\Omega \setminus C_k)$ for every $i \in \{1, \dots, m\}$, we have that

$$D_{X_i} u_k = X_i u_k \mathcal{L}^n \text{ on } \Omega \setminus C_k,$$

meaning that

$$\left| D_{X_i} \left(\sum_{\ell,s \in \mathbb{N}} u_k^{\ell,s} \right) \right|(\Omega \setminus C_k) = \left\| \sum_{\ell,s \in \mathbb{N}} X_i u_k^{\ell,s} \right\|_{L^1(\Omega)}.$$

Hence, by the decomposition (27), we write

$$\|X_i u_k\|_{L^1(\Omega)} \leq \left\| \sum_{\ell,s \in \mathbb{N}} (\xi_k^{\ell,s} D_{X_i} u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell,s \in \mathbb{N}} S_i^{k,\ell,s} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell,s \in \mathbb{N}} R_i^{k,\ell,s} \right\|_{L^1(\Omega)}.$$

Thanks to Proposition 4.7, we are only left to prove the boundedness of the first term of the right hand side of the inequality above. Using [5, Theorem 2.2 (b)] we have

$$\left\| \sum_{\ell,s \in \mathbb{N}} (\xi_k^{\ell,s} D_{X_i} u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(\Omega)} \leq \sum_{\ell,s \in \mathbb{N}} \left\| (\xi_k^{\ell,s} D_{X_i} u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(A_k^{\ell,s})} \leq \sum_{\ell,s \in \mathbb{N}} |D_{X_i} u|(A_k^{\ell,s} + \varepsilon_k^{\ell,s}).$$

where we have written $A_k^{\ell,s} + \varepsilon_k^{\ell,s}$ to denote

$$A_k^{\ell,s} + \varepsilon_k^{\ell,s} := \bigcup_{x \in A_k^{\ell,s}} B_E(x, \varepsilon_k^{\ell,s}).$$

By Remark 4.3 we have

$$A_k^{\ell,s} + \varepsilon_k^{\ell,s} \subseteq A_k^{\ell,s} \cup A_k^{\ell,s+1} \cup A_k^{\ell,s-1} \cup A_k^{\ell-1,s} \cup A_k^{\ell+1,s}$$

so that

$$|D_{X_i} u|(A_k^{\ell,s} + \varepsilon_k^{\ell,s}) \leq |D_{X_i} u|(A_k^{\ell,s} \cup A_k^{\ell,s+1} \cup A_k^{\ell,s-1} \cup A_k^{\ell-1,s} \cup A_k^{\ell+1,s}).$$

and, finally,

$$\sum_{\ell,s \in \mathbb{N}} |D_{X_i} u|(A_k^{\ell,s} + \varepsilon_k^{\ell,s}) \leq 5 \sum_{\ell,s \in \mathbb{N}} |D_{X_i} u|(A_k^{\ell,s}) \leq 15 \sum_{\ell \in \mathbb{N}} |D_{X_i} u|(A_k^\ell) \leq 45 |D_{X_i} u|(\Omega)$$

which, together with the fact that $u \in \text{BV}_X(\Omega)$, allows us to conclude the boundedness of $\|u_k\|_{\text{BV}_X(\Omega \setminus C_k)}$.

(ii) We aim to prove that $u_k \in \text{BV}_X(\Omega)$. Suppose first that, in addition to our previous assumptions, we have $u \in C^\infty(\Omega)$. Then $C_k = \emptyset$ and clearly $u_k \in C^\infty(\Omega) \subset C_X^1(\Omega)$ for any $k \in \mathbb{N}$, hence

$$\begin{aligned} |D_{X_i} u_k|(\Omega) &= \|X_i u_k\|_{L^1(\Omega)} \\ &\leq \left\| \sum_{\ell,s \in \mathbb{N}} (\xi_k^{\ell,s} D_{X_i} u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell,s \in \mathbb{N}} S_i^{k,\ell,s} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell,s \in \mathbb{N}} R_i^{k,\ell,s} \right\|_{L^1(\Omega)} \\ &\leq 45 |D_{X_i} u|(\Omega) + \frac{4}{k}, \end{aligned}$$

the latter implying that $u_k \in \text{BV}_X(\Omega)$.

Now we drop the smoothness assumption on u and we just assume that $u \in \text{BV}_X(\Omega)$. We know, by [28, Theorem 2.2.2], that there exists a sequence $(u^t)_{t \in \mathbb{N}}$ such that, for every $t \in \mathbb{N}$ we have $u^t \in C^\infty(\Omega) \cap \text{BV}_X(\Omega)$ and

$$\|u^t - u\|_{L^1(\Omega)} \xrightarrow{t \rightarrow +\infty} 0, \quad |D_{X_i} u^t|(\Omega) \xrightarrow{t \rightarrow +\infty} |D_{X_i} u|(\Omega), \quad |D_{X_i} u^t|(\Omega) \leq |D_{X_i} u|(\Omega) + \frac{1}{t}$$

for every $i \in \{1, \dots, m\}$. Now for every $t \in \mathbb{N}$ consider the approximation sequence $(u_k^t)_{k \in \mathbb{N}}$ constructed as in (14). For the observations we just made on the approximation of smooth functions we know that $u_k^t \in C_X^1(\Omega) \cap \text{BV}_X(\Omega)$. Let us prove that $\|u_k^t - u_k\|_{L^1(\Omega)} \xrightarrow{t \rightarrow +\infty} 0$. First we observe that, thanks to Remark 4.3, we have $\|u_k\|_{L^1(\Omega)} \leq 45 \|u\|_{L^1(\Omega)}$ so that

$$\|u_k^t - u_k\|_{L^1(\Omega)} = \|(u^t - u)_k\|_{L^1(\Omega)} \leq 45 \|u^t - u\|_{L^1(\Omega)}.$$

The inequality above shows that $\|u_k^t - u_k\|_{L^1(\Omega)} \xrightarrow{t \rightarrow +\infty} 0$. Then we observe

$$|D_{X_i} u_k^t|(\Omega) \leq 45 |D_{X_i} u^t|(\Omega) + \frac{4}{k} \leq 45 |D_{X_i} u|(\Omega) + \frac{45}{t} + \frac{4}{k} \leq 45 |D_{X_i} u|(\Omega) + 49.$$

Passing to the lim inf for $t \rightarrow +\infty$ in the above inequality and using the lower semicontinuity of the total variation is enough to obtain $u_k \in \text{BV}_X(\Omega)$.

(iii) From Lemma 4.6, the construction of C_k in Lemma 4.2 and the fact that $u_k \in \text{BV}_X(\Omega) \cap C^\infty(\Omega \setminus C_k)$ we obtain that $u_k \in \text{SBV}_X(\Omega)$. \square

The following Lemma will be the last step needed to prove our main result.

Lemma 4.9. *Let $v, w \in \text{SBV}_X(\Omega)$ and $R \subseteq \mathcal{J}_v \cap \mathcal{J}_w$. Let j_v and j_w be defined as in Lemma 4.2 and such that $j_v = j_w \mathcal{H}^{Q-1}$ -a.e. on R . Then*

$$|D_X(v - w)|(R) = 0.$$

Proof. Let us observe that $v - w \in \text{SBV}_X(\Omega)$ and $\mathcal{H}^{Q-1}(R \cap \mathcal{S}_{v-w}) = 0$. In fact, for \mathcal{H}^{Q-1} -a.e. $p \in R$ one has $j_v(p) = j_w(p)$ and, letting $\nu := \nu_{\mathcal{J}_v}(p) = \nu_{\mathcal{J}_w}(p)$, we notice that

$$\begin{aligned} \lim_{r \rightarrow 0} \int_{B(p,r)} |v - w| d\mathcal{L}^n &\leq \lim_{r \rightarrow 0} \int_{B_v^+(p,r)} |v - v^+| d\mathcal{L}^n + \lim_{r \rightarrow 0} \int_{B_v^+(p,r)} |w - w^+| d\mathcal{L}^n + \\ &\quad + \lim_{r \rightarrow 0} \int_{B_v^-(p,r)} |v - v^-| d\mathcal{L}^n + \lim_{r \rightarrow 0} \int_{B_v^-(p,r)} |w - w^-| d\mathcal{L}^n = 0. \end{aligned}$$

The latter implies that $v - w$ has approximate limit 0 at p , i.e., $p \in R \setminus \mathcal{S}_{v-w}$. This proves that $\mathcal{H}^{Q-1}(R \cap \mathcal{S}_{v-w}) = 0$ and by Theorem 2.10 (1)

$$(33) \quad |D_X(v - w)|(R \cap \mathcal{S}_{v-w}) = 0.$$

Moreover, the measure $\mathcal{H}^{Q-1} \llcorner (R \setminus \mathcal{S}_{v-w})$ is σ -finite and Theorem 2.10 (2) implies that

$$(34) \quad |D_X(v - w)|(R \setminus \mathcal{S}_{v-w}) = 0.$$

The desired equality $|D_X(v - w)|(R) = 0$ follows from (33) and (34). \square

4.3. Proof of Theorem 1.4. We are ready to prove our main result, Theorem 1.4, that we restate for the reader's convenience.

Theorem 4.10. *Let $u \in \text{SBV}_X(\Omega)$. Then there exists a sequence of functions $(u_k)_{k \in \mathbb{N}} \subset \text{SBV}_X(\Omega)$ and of C_X^1 -hypersurfaces $(M_k)_{k \in \mathbb{N}} \subset \Omega$ such that, for every $k \in \mathbb{N}$, $\mathcal{J}_{u_k} \subseteq M_k \cap \mathcal{J}_u$, \mathcal{J}_{u_k} is compact, and*

$$\|u - u_k\|_{\text{BV}_X(\Omega)} \xrightarrow{k \rightarrow +\infty} 0, \quad u_k \in C^\infty(\Omega \setminus \mathcal{J}_{u_k}).$$

Proof. Let C_k and $(u_k)_{k \in \mathbb{N}}$ be defined as in Lemma 4.2 and (14). By definition of BV_X -norm we have

$$(35) \quad \|u - u_k\|_{\text{BV}_X(\Omega)} = \|u - u_k\|_{L^1(\Omega)} + |D_X(u - u_k)|(\Omega).$$

Thanks to (20) we estimate

$$(36) \quad \|u - u_k\|_{L^1(\Omega)} < \frac{1}{k}.$$

Concerning the other summand, we estimate

$$(37) \quad \begin{aligned} &|D_X(u - u_k)|(\Omega) \\ &\leq |D_X(u - u_k)|(\Omega \setminus \mathcal{J}_u) + |D_X(u - u_k)|(\mathcal{J}_u \setminus \mathcal{J}_u^k) + |D_X(u - u_k)|(\mathcal{J}_u^k \setminus M_k) \\ &\quad + |D_X(u - u_k)|((\mathcal{J}_u \cap M_k) \setminus C_k) + |D_X(u - u_k)|(C_k). \end{aligned}$$

Because of Lemma 4.6, Proposition 4.8 and Lemma 4.9 we have

$$(38) \quad |D_X(u - u_k)|(C_k) = 0.$$

Then, since $u_k \in C^\infty(\Omega \setminus C_k)$, (10), (11) and (12) imply that

$$(39) \quad \begin{aligned} |D_X(u - u_k)|((\mathcal{J}_u \cap M_k) \setminus C_k) &= |D_X u|((\mathcal{J}_u \cap M_k) \setminus C_k) < \frac{1}{k}, \\ |D_X(u - u_k)|(\mathcal{J}_u \setminus \mathcal{J}_u^k) &= |D_X u|(\mathcal{J}_u \setminus \mathcal{J}_u^k) < \frac{1}{k}, \\ |D_X(u - u_k)|(\mathcal{J}_u^k \setminus M_k) &= |D_X u|(\mathcal{J}_u^k \setminus M_k) < \frac{1}{k}. \end{aligned}$$

By Theorem 2.9, one has $D_X u = D_X^{\text{ap}} u \mathcal{L}^n + D_X^j u$ so that, for every $i \in \{1, \dots, m\}$,

$$\begin{aligned} &|D_{X_i}(u - u_k)|(\Omega \setminus \mathcal{J}_u) = \|D_{X_i}^{\text{ap}} u - X_i u_k\|_{L^1(\Omega)} \\ &\leq \left\| D_{X_i}^{\text{ap}} u - \sum_{\ell, s \in \mathbb{N}} [(\xi_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}}] \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} R_i^{k, \ell, s} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s} \right\|_{L^1(\Omega)}. \end{aligned}$$

By Proposition 4.7

$$(40) \quad \left\| \sum_{\ell, s \in \mathbb{N}} R_i^{k, \ell, s} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s} \right\|_{L^1(\Omega)} \leq \frac{4}{k},$$

while

$$\begin{aligned} \left\| \sum_{\ell, s \in \mathbb{N}} [(\xi_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}}] - D_{X_i}^{\text{ap}} u \right\|_{L^1(\Omega)} &= \left\| \sum_{\ell, s \in \mathbb{N}} [(\xi_k^{\ell, s} (D_{X_i}^{\text{ap}} u \mathcal{L}^n + D_{X_i}^j u)) * K_{\varepsilon_k^{\ell, s}}] - D_{X_i}^{\text{ap}} u \right\|_{L^1(\Omega)} \\ &\leq \left\| \sum_{\ell, s \in \mathbb{N}} [(\xi_k^{\ell, s} D_{X_i}^{\text{ap}} u) * K_{\varepsilon_k^{\ell, s}}] - \sum_{\ell, s \in \mathbb{N}} \xi_k^{\ell, s} D_{X_i}^{\text{ap}} u \right\|_{L^1(\Omega)} \\ &\quad + \left\| \sum_{\ell, s \in \mathbb{N}} (\xi_k^{\ell, s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(\Omega)}. \end{aligned}$$

Thanks to (21) we have

$$(41) \quad \left\| \sum_{\ell, s \in \mathbb{N}} [(\xi_k^{\ell, s} D_{X_i}^{\text{ap}} u) * K_{\varepsilon_k^{\ell, s}}] - \sum_{\ell, s \in \mathbb{N}} \xi_k^{\ell, s} D_{X_i}^{\text{ap}} u \right\|_{L^1(\Omega)} \leq \frac{1}{k},$$

while

$$\left\| \sum_{\ell, s \in \mathbb{N}} (\xi_k^{\ell, s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(\Omega)} \leq \sum_{\ell, s \in \mathbb{N}} \left\| (\xi_k^{\ell, s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(A_k^{\ell, s})}.$$

Fix $\ell, s \in \mathbb{N}$. By [5, Theorem 2.2 (b)] we can write

$$\left\| (\xi_k^{\ell, s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(A_k^{\ell, s})} \leq |D_{X_i}^j u|(A_k^{\ell, s} + \varepsilon_k^{\ell, s}),$$

which in turn, by Remark 4.3, satisfies the following inclusion

$$A_k^{\ell, s} + \varepsilon_k^{\ell, s} \subseteq A_k^{\ell, s} \cup A_k^{\ell, s+1} \cup A_k^{\ell, s-1} \cup A_k^{\ell-1, s} \cup A_k^{\ell+1, s}$$

Hence

$$|D_{X_i}^j u|(A_k^{\ell, s} + \varepsilon_k^{\ell, s}) \leq |D_{X_i}^j u|(A_k^{\ell, s} \cup A_k^{\ell, s+1} \cup A_k^{\ell, s-1} \cup A_k^{\ell-1, s} \cup A_k^{\ell+1, s}).$$

Finally, we obtain

$$(42) \quad \begin{aligned} \sum_{\ell, s \in \mathbb{N}} \left\| (\xi_k^{\ell, s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(A_k^{\ell, s})} &\leq \sum_{\ell, s \in \mathbb{N}} |D_{X_i}^j u|(A_k^{\ell, s} \cup A_k^{\ell, s+1} \cup A_k^{\ell, s-1} \cup A_k^{\ell-1, s} \cup A_k^{\ell+1, s}) \\ &\leq 45 |D_{X_i}^j u|(\Omega \setminus C_k) \leq \frac{45}{k}. \end{aligned}$$

Combining (36),(37),(38),(39),(40),(41),(42) with (35) and letting $k \rightarrow +\infty$ one achieves the desired conclusion. \square

REFERENCES

- [1] AGRACHEV, A., BARILARI, D., AND BOSCAIN, U. *A comprehensive introduction to sub-Riemannian geometry*, vol. 181 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020. From the Hamiltonian viewpoint, With an appendix by Igor Zelenko.
- [2] ALBERTI, G. A Lusin type theorem for gradients. *J. Funct. Anal.* *100*, 1 (1991), 110–118.
- [3] AMBROSIO, L. Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces. *Adv. Math.* *159*, 1 (2001), 51–67.
- [4] AMBROSIO, L., BOURGAIN, J., BREZIS, H., AND FIGALLI, A. BMO-type norms related to the perimeter of sets. *Comm. Pure Appl. Math.* *69*, 6 (2016), 1062–1086.

- [5] AMBROSIO, L., FUSCO, N., AND PALLARA, D. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [6] AMBROSIO, L., GHEZZI, R., AND MAGNANI, V. BV functions and sets of finite perimeter in sub-Riemannian manifolds. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 32, 3 (2015), 489–517.
- [7] AMBROSIO, L., AND MAGNANI, V. Weak differentiability of BV functions on stratified groups. *Math. Z.* 245, 1 (2003), 123–153.
- [8] AMBROSIO, L., MIRANDA, JR., M., AND PALLARA, D. Special functions of bounded variation in doubling metric measure spaces. In *Calculus of variations: topics from the mathematical heritage of E. De Giorgi*, vol. 14 of *Quad. Mat.* Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 1–45.
- [9] AMBROSIO, L., AND SCIENZA, M. Locality of the perimeter in Carnot groups and chain rule. *Ann. Mat. Pura Appl. (4)* 189, 4 (2010), 661–678.
- [10] ANZELLOTTI, G., AND GIAQUINTA, M. BV functions and traces. *Rend. Sem. Mat. Univ. Padova* 60 (1978), 1–21.
- [11] BIROLI, M., AND MOSCO, U. Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 6, 1 (1995), 37–44.
- [12] BRAMANTI, M., MIRANDA, JR., M., AND PALLARA, D. Two characterization of BV functions on Carnot groups via the heat semigroup. *Int. Math. Res. Not. IMRN*, 17 (2012), 3845–3876.
- [13] CAPOGNA, L., DANIELLI, D., AND GAROFALO, N. The geometric Sobolev embedding for vector fields and the isoperimetric inequality. *Comm. Anal. Geom.* 2, 2 (1994), 203–215.
- [14] CAPOGNA, L., AND GAROFALO, N. Ahlfors type estimates for perimeter measures in Carnot-Carathéodory spaces. *J. Geom. Anal.* 16, 3 (2006), 455–497.
- [15] COMI, G. E., AND MAGNANI, V. The Gauss-Green theorem in stratified groups. *Adv. Math.* 360 (2020), 106916, 85.
- [16] DANIELLI, D., GAROFALO, N., AND NHIEU, D.-M. Trace inequalities for Carnot-Carathéodory spaces and applications. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 27, 2 (1998), 195–252.
- [17] DANIELLI, D., GAROFALO, N., AND NHIEU, D.-M. Non-doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot-Carathéodory spaces. *Mem. Amer. Math. Soc.* 182, 857 (2006), x+119.
- [18] DE GIORGI, E., AND AMBROSIO, L. New functionals in the calculus of variations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* 82, 2 (1988), 199–210.
- [19] DE PHILIPPIS, G., FUSCO, N., AND PRATELLI, A. On the approximation of SBV functions. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* 28, 2 (2017), 369–413.
- [20] DI MARCO, M., JULIA, A., NICOLUSSI GOLO, S., AND VITTONI, D. Submanifolds with boundary and Stokes’ Theorem in Heisenberg groups, 2024. To appear on Transactions of the American Mathematical Society, preprint available online at <https://arxiv.org/abs/2403.18675>.
- [21] DON, S. Functions of bounded variation in Carnot-Carathéodory spaces, 2019. PhD Thesis available online at https://dottorato.math.unipd.it/sites/default/files/tesi_definitiva_Sebastiano_Don.pdf.
- [22] DON, S., AND MAGNANI, V. Surface measure on, and the local geometry of, sub-Riemannian manifolds. *Calc. Var. Partial Differential Equations* 62, 9 (2023), Paper No. 254, 42.
- [23] DON, S., MASSACCESI, A., AND VITTONI, D. Rank-one theorem and subgraphs of BV functions in Carnot groups. *J. Funct. Anal.* 276, 3 (2019), 687–715.
- [24] DON, S., AND VITTONI, D. A compactness result for BV functions in metric spaces. *Ann. Acad. Sci. Fenn. Math.* 44, 1 (2019), 329–339.

- [25] DON, S., AND VITTONI, D. Fine properties of functions with bounded variation in Carnot-Carathéodory spaces. *J. Math. Anal. Appl.* 479, 1 (2019), 482–530.
- [26] FEDERER, H. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York, Inc., New York, 1969.
- [27] FRANCHI, B., GALLOT, S., AND WHEEDEN, R. L. Sobolev and isoperimetric inequalities for degenerate metrics. *Math. Ann.* 300, 4 (1994), 557–571.
- [28] FRANCHI, B., SERAPIONI, R., AND SERRA CASSANO, F. Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields. *Houston J. Math.* 22, 4 (1996), 859–890.
- [29] FRANCHI, B., SERAPIONI, R., AND SERRA CASSANO, F. Rectifiability and perimeter in the Heisenberg group. *Math. Ann.* 321, 3 (2001), 479–531.
- [30] FRANCHI, B., SERAPIONI, R., AND SERRA CASSANO, F. On the structure of finite perimeter sets in step 2 Carnot groups. *J. Geom. Anal.* 13, 3 (2003), 421–466.
- [31] FRANCHI, B., SERAPIONI, R. P., AND SERRA CASSANO, F. Area formula for centered Hausdorff measures in metric spaces. *Nonlinear Anal.* 126 (2015), 218–233.
- [32] FRIEDRICHS, K. O. The identity of weak and strong extensions of differential operators. *Trans. Amer. Math. Soc.* 55 (1944), 132–151.
- [33] FUSCO, N., MOSCARIELLO, G., AND SBORDONE, C. A formula for the total variation of SBV functions. *J. Funct. Anal.* 270, 1 (2016), 419–446.
- [34] GAROFALO, N., AND NHIEU, D.-M. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.* 49, 10 (1996), 1081–1144.
- [35] GAROFALO, N., AND NHIEU, D.-M. Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces. *J. Anal. Math.* 74 (1998), 67–97.
- [36] JULIA, A., NICOLUSSI GOLO, S., AND VITTONI, D. Lipschitz functions on submanifolds of Heisenberg groups. *Int. Math. Res. Not. IMRN*, 9 (2023), 7399–7422.
- [37] LAHTI, P. Approximation of BV by SBV functions in metric spaces. *Journal of Functional Analysis* 279, 11 (2020), 108763.
- [38] LECCESE, G., AND MAGNANI, V. A study of measure-theoretic area formulas. *Ann. Mat. Pura Appl. (4)* (2021), online.
- [39] MAGNANI, V. *Elements of geometric measure theory on sub-Riemannian groups*. Scuola Normale Superiore, Pisa, 2002.
- [40] MAGNANI, V. On a measure-theoretic area formula. *Proc. Roy. Soc. Edinburgh Sect. A* 145 (2015), 885–891.
- [41] MAGNANI, V. A new differentiation, shape of the unit ball, and perimeter measure. *Indiana Univ. Math. J.* 66, 1 (2017), 183–204.
- [42] MAGNANI, V. Rotational symmetries and spherical measure in homogeneous groups. *J. Geom. Anal.* 32, 4 (2022), Paper No. 119, 31.
- [43] MARCHI, M. Regularity of sets with constant intrinsic normal in a class of Carnot groups. *Ann. Inst. Fourier (Grenoble)* 64, 2 (2014), 429–455.
- [44] NAGEL, A., STEIN, E. M., AND WAINGER, S. Balls and metrics defined by vector fields. I. Basic properties. *Acta Math.* 155, 1-2 (1985), 103–147.
- [45] SELBY, C. An extension and trace theorem for functions of H-bounded variation in Carnot groups of step 2. *Houston J. Math.* 33, 2 (2007), 593–616.
- [46] SONG, Y. Q., AND YANG, X. P. BV functions in the Heisenberg group H^n . *Chinese Ann. Math. Ser. A* 24, 5 (2003), 541–554.
- [47] VITTONI, D. Lipschitz surfaces, perimeter and trace theorems for BV functions in Carnot-Carathéodory spaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 11, 4 (2012), 939–998.
- [48] VITTONI, D. Lipschitz graphs and currents in Heisenberg groups. *Forum Math. Sigma* 10 (2022), Paper No. e6, 104.

(M. Di Marco) DIPARTIMENTO DI MATEMATICA “T. LEVI-CIVITA”, UNIVERSITÀ DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY.

Email address: marco.dimarco@phd.unipd.it

(S. Don) DIPARTIMENTO DI INGEGNERIA CIVILE, ARCHITETTURA, TERRITORIO, AMBIENTE E DI MATEMATICA (DICATAM), VIA BRANZE, 38, 25123 BRESCIA, ITALY

Email address: sebastiano.don@unibs.it

(D. Vittone) DIPARTIMENTO DI MATEMATICA “T. LEVI-CIVITA”, UNIVERSITÀ DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY.

Email address: davide.vittone@unipd.it