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Asymptotic behavior of integral functionals for a two-parameter singularly perturbed nonlinear traction problem

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Abstract

We consider a nonlinear traction boundary value problem for the Lamé equations in an unbounded periodically perforated domain. The edges lengths of the periodicity cell are proportional to a positive parameter δ, whereas the relative size of the holes is determined by a second positive parameter ε. Under suitable assumptions on the nonlinearity, there exists a family of solutions $\{u(\varepsilon,\delta,\cdot)\}_{(\varepsilon,\delta)\in]0,\varepsilon'[\times]0,\delta'[}$. We analyze the asymptotic behavior of two integral functionals associated to such a family of solutions when the perturbation parameter pair (ε,δ) is close to the degenerate value $(0,0)$.

KEYWORDS:

nonlinear traction boundary value problems, integral equation methods, linearized elastostatics, periodically perforated domains, singularly perturbed domains, real analytic continuation in Banach spaces, integral functionals

1 | INTRODUCTION

In this article we consider a nonlinear traction boundary value problem for linearized elastostatics in a singularly perturbed periodically perforated domain. The domain depends upon two singular perturbation parameters, ε and δ, the first modeling the (relative) radius of the holes and the second describing the size of the periodic structure. Before illustrating in details the problem, we need to introduce the geometric setting. We fix once for all

$$n\in\mathbb{N}\setminus\{0,1\},\qquad q_{11},\ldots,q_{nn}\in\,]0,+\infty[,$$

where $\mathbb{N}$ denotes the set of natural numbers including 0. We set

$$q\equiv\begin{pmatrix} q_{11} & 0 & \cdots & 0\\ 0 & q_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & q_{nn}\end{pmatrix},\qquad Q\equiv\prod_{j=1}^{n}\,]0,q_{jj}[\,\subseteq\mathbb{R}^n. \tag{1}$$

The set Q is the fundamental periodicity cell and the diagonal matrix q is the periodicity matrix associated with the cell Q. Clearly, $q\mathbb{Z}^n\equiv\{qz\,:\,z\in\mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to Q. Let $m\in\mathbb{N}\setminus\{0\}$ and $\alpha\in\,]0,1[$. We consider a set $\Omega\subseteq\mathbb{R}^n$ satisfying the following assumption:

$$\begin{gathered}\Omega \text{ is a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{m,\alpha}\\ \text{such that } \mathbb{R}^n\setminus\overline{\Omega} \text{ is connected and } 0\in\Omega,\end{gathered} \tag{2}$$

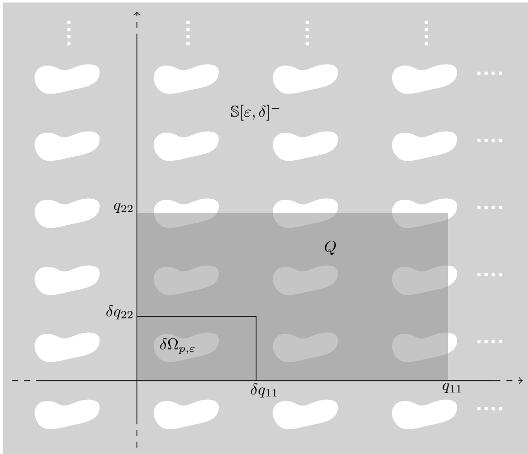

FIGURE 1 An example, in dimension $n = 2$, of the geometric setting. The intersection of the shaded gray sets is the set $Q \cap \mathbb{S}[\varepsilon,\delta]^-$.

where the symbol $\bar{\cdot}$ denotes the closure of a set. Next we fix $p \in Q$. Clearly, there exists a real number $\varepsilon_0 \in]0,+\infty[$ such that

$$p + \varepsilon\overline{\Omega} \subseteq Q \quad \forall \varepsilon \in]-\varepsilon_0, \varepsilon_0[. \tag{3}$$

To shorten our notation we set

$$\Omega_{p,\varepsilon} \equiv p + \varepsilon\Omega \quad \forall \varepsilon \in \mathbb{R}.$$

Then we introduce the periodic domains

$$\mathbb{S}[\Omega_{p,\varepsilon}] \equiv \bigcup_{z\in\mathbb{Z}^n}(qz + \Omega_{p,\varepsilon}), \quad \mathbb{S}[\Omega_{p,\varepsilon}]^- \equiv \mathbb{R}^n \setminus \overline{\mathbb{S}[\Omega_{p,\varepsilon}]} \qquad \forall \varepsilon \in]-\varepsilon_0, \varepsilon_0[.$$

The set $\mathbb{S}[\Omega_{p,\varepsilon}]^-$ represents a periodically perforated domain obtained removing from $\mathbb{R}^n$ the periodic set of holes $\overline{\mathbb{S}[\Omega_{p,\varepsilon}]}$. Next, we rescale the set $\mathbb{S}[\Omega_{p,\varepsilon}]^-$ by a positive real number δ, that is we define

$$\mathbb{S}[\varepsilon,\delta]^- \equiv \delta\mathbb{S}[\Omega_{p,\varepsilon}]^- \qquad \forall(\varepsilon,\delta) \in]-\varepsilon_0, \varepsilon_0[\times]0,+\infty[.$$

Clearly, $\mathbb{S}[\varepsilon,\delta]^-$ is a periodically perforated set with periodicity cell δQ and whose holes are $\delta\overline{\mathbb{S}[\Omega_{p,\varepsilon}]}$. When ε approaches zero, the relative size of the holes (proportional to ε) goes to zero. Instead, when δ tends to zero each periodicity cell shrinks to a point and the whole periodicity structure degenerates (see Figure 1).

We are now ready to introduce the nonlinear traction boundary value problem in $\mathbb{S}[\varepsilon,\delta]^-$. We denote by $\mathcal{T}$ the function from $]1-\frac{2}{n},+\infty[\times M_n(\mathbb{R})$ to $M_n(\mathbb{R})$ defined by

$$\mathcal{T}(\omega, A) \equiv (\omega-1)(\mathrm{tr}A)\mathbb{I}_n + (A + A^T) \quad \forall \omega \in \left]1-\frac{2}{n},+\infty\right[, \ \ A \in M_n(\mathbb{R}).$$

Here, $M_n(\mathbb{R})$ denotes the space of $n\times n$ matrices with real entries, $\mathbb{I}_n$ denotes the $n \times n$ identity matrix, $\mathrm{tr}A$ and A^T denote the trace and the transpose matrix of A, respectively. We note that $(\omega-1)$ plays the role of the ratio between the first and the second Lamé constants and that the classical linearization of the Piola-Kirchoff tensor equals the second Lamé constant times $\mathcal{T}(\omega,\cdot)$ (cf., *e.g.*, Kupradze, Gegelia, Basheleishvili, and Burchuladze[1]). Let G be a (possibly) nonlinear function from $\partial\Omega \times \mathbb{R}^n$ to $\mathbb{R}^n$. Let $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$, $\delta \in]0,+\infty[$ and $\omega \in]1-\frac{2}{n},+\infty[$. Then, we consider the following periodic problem in $\mathbb{S}[\varepsilon,\delta]^-$:

$$\begin{cases} \mathrm{div}\mathcal{T}(\omega, Du(x)) = 0 & \forall x \in \mathbb{S}[\varepsilon,\delta]^-, \\ u(x + \delta q_{kk}e_k) = u(x) & \forall x \in \overline{\mathbb{S}[\varepsilon,\delta]^-}, \ \forall k \in \{1,\ldots,n\}, \\ \mathcal{T}(\omega, Du(x))\nu_{\delta\Omega_{p,\varepsilon}}(x) = G\left(\frac{x-\delta p}{\delta\varepsilon}, u(x)\right) & \forall x \in \delta\partial\Omega_{p,\varepsilon}. \end{cases} \tag{4}$$

Here above, $\{e_1,\ldots,e_n\}$ denotes the canonical basis of $\mathbb{R}^n$ and $\nu_{\delta\Omega_{p,\varepsilon}}$ denotes the outward unit normal field to $\delta\partial\Omega_{p,\varepsilon}$. We note that due to the presence of the nonlinearity G in the boundary condition of problem (4), the existence of a solution is not obivious and has to be considered. Under suitable assumptions we will prove that there exist $\varepsilon' \in]0,\varepsilon_0[$ and $\delta' \in]0,+\infty[$ such that problem (4) has a solution

$$u(\varepsilon,\delta,\cdot)\in C^{m,\alpha}(\overline{\mathbb{S}[\varepsilon,\delta]^-},\mathbb{R}^n)\qquad \forall(\varepsilon,\delta)\in\,]0,\varepsilon'[\times]0,\delta'[.$$

We can then introduce two integral functionals associated with the solution $u(\varepsilon,\delta,\cdot)$ for all $(\varepsilon,\delta)\in\,]0,\varepsilon'[\times]0,\delta'[$. The energy integral of $u(\varepsilon,\delta,\cdot)$ in the cell Q is defined as

$$\mathrm{En}(\omega,u(\varepsilon,\delta,\cdot))\equiv\frac{1}{2}\int_{Q\cap\mathbb{S}[\varepsilon,\delta]^-}\mathrm{tr}(\mathcal{T}(\omega,D_xu(\varepsilon,\delta,x))(D_xu(\varepsilon,\delta,x))^T)\,dx.$$

Similarly, the integral of $u(\varepsilon,\delta,\cdot)$ in the cell Q is defined as

$$\mathrm{Int}(u(\varepsilon,\delta,\cdot))\equiv\int_{Q\cap\mathbb{S}[\varepsilon,\delta]^-}u(\varepsilon,\delta,x)\,dx.$$

The main goal of the present paper is to investigate the behavior of the family of solutions $\{u(\varepsilon,\delta,\cdot)\}_{(\varepsilon,\delta)\in]0,\varepsilon'[\times]0,\delta'[}$ and of the integral functionals $\mathrm{En}(\omega,u(\varepsilon,\delta,\cdot))$ and $\mathrm{Int}(u(\varepsilon,\delta,\cdot))$ associated with such a family as (ε,δ) tends to $(0,0)$. Namely, our main aim is to answer the following question:

- What can be said on the functions $(\varepsilon,\delta)\mapsto\mathrm{En}(\omega,u(\varepsilon,\delta,\cdot))$ and $(\varepsilon,\delta)\mapsto\mathrm{Int}(u(\varepsilon,\delta,\cdot))$ as (ε,δ) tends to $(0,0)$ in $]0,\varepsilon'[\times]0,\delta'[$?

The asymptotic behavior of the solution of problems in singularly perturbed perforated domains has been long investigated in many different frameworks and from several points of view. Many authors have applied asymptotic analysis to compute 'approximations' of the solutions or of related functionals as the singular perturbation parameter approaches a degenerate value. Here we mention, for example, Kozlov, Maz'ya, and Movchan[2], Maz'ya and Movchan[3], Maz'ya, Movchan, and Nieves[4,5], Maz'ya, Nazarov, and Plamenewskij[6], Novotny and Sokołowski[7], and Novotny, Sokołowski, and Żochowski[8]. Problems in periodically perforated domains have also been investigated in the frame of homogenization theory. Here we mention, *e.g.*, the seminal contributions by Cioranescu and Murat[9] and Marčenko and Khruslov[10]. We also mention the recent work[11] by Jing, where the author exploits periodic layer potentials for the Lamé system to obtain quantitative homogenization results for the Lamé systems in periodically perforated domains.

In this paper we follow a different approach. Namely, taking advantage of the tools of potential theory and of some abstract results of functional analysis, we wish to represent the integral functionals $(\varepsilon,\delta)\mapsto\mathrm{En}(\omega,u(\varepsilon,\delta,\cdot))$ and $(\varepsilon,\delta)\mapsto\mathrm{Int}(\omega,u(\varepsilon,\delta,\cdot))$ in terms of real analytic maps defined in a whole neighborhood of $(0,0)$. Such an approach, which is now known as the Functional Analytic Approach (FAA), has been proposed by Lanza de Cristoforis (cf. [12,13]) and it has been applied to singularly perturbed problems of linearized elastostatics in Dalla Riva and Lanza de Cristoforis[14,15,16,17]. The present paper represents an extension to the Lamé equations of the results of[18,19], where the asymptotic behavior of a nonlinear Robin problem for the Poisson equation in the set $\mathbb{S}[\varepsilon,\delta]^-$ has been analyzed, and relies on[20] where the authors studied the (δ-independent) nonlinear traction problem in $\mathbb{S}[\Omega_{p,\varepsilon}]^-$. One of the main advantages of the FAA is that a real analytic continuation result around the singularity implies the possibility to expand the quantity under consideration into a convergent power series of the singular perturbation parameter.

The paper is organized as follows. In Section 2 we introduce some basic notation. Section 3 is the toolbox of the paper. That is, here we introduce the periodic elastic layer potentials. In Section 4 we first transform the problem into an equivalent auxiliary problem defined in a δ-independent domain. Then, by means of layer potentials, we convert such an auxiliary problem into an integral equation. Analyzing this integral equation, we are able to show the existence of a family of solutions $\{u^\sharp(\varepsilon,\delta,\cdot)\}_{(\varepsilon,\delta)\in]0,\varepsilon'[\times]0,\delta'[}$ of the auxiliary problem and to study the behavior of $u^\sharp(\varepsilon,\delta,\cdot)$ and of corresponding integrals near $(\varepsilon,\delta)=(0,0)$. In Section 5, exploiting the equivalence of the auxiliary problem and problem (4), we study the behavior of the integral functionals of $u(\varepsilon,\delta,\cdot)$ near $(\varepsilon,\delta)=(0,0)$.

2 | NOTATION

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X}\times\mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}}\equiv\|x\|_{\mathcal{X}}+\|y\|_{\mathcal{Y}}$ for all $(x,y)\in\mathcal{X}\times\mathcal{Y}$, whereas we use the Euclidean norm for $\mathbb{R}^n$. We denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of linear and continuous maps from $\mathcal{X}$ to $\mathcal{Y}$, equipped with its usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a real-valued function g, or the inverse of a matrix B, which are denoted g^{-1} and B^{-1}, respectively.

For all $R > 0$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$ and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m-times continuously differentiable real valued functions on Ω is denoted by $C^m(\Omega, \mathbb{R})$, or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$ and $f \in (C^m(\Omega))^r$. Then, Df denotes the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{(i,j)\in\{1,\dots,r\}\times\{1,\dots,n\}}$. Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \dots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\overline{\Omega}$ is denoted $C^m(\overline{\Omega})$. The subspace of $C^m(\overline{\Omega})$ whose functions have m-th order derivatives that are uniformly Hölder continuous in $\overline{\Omega}$ with exponent $\alpha \in]0, 1]$ is denoted $C^{m,\alpha}(\overline{\Omega})$. If $f \in C^{m,\alpha}(\overline{\Omega})$, then its α-Hölder constant is denoted $|f : \overline{\Omega}|_\alpha$. The subspace of $C^m(\overline{\Omega})$ of those functions f such that $f_{|\overline{\Omega \cap \mathbb{B}_n(0,R)}}$ belongs to $C^{m,\alpha}(\overline{\Omega \cap \mathbb{B}_n(0, R)})$ for all $R > 0$ is denoted $C^{m,\alpha}_{loc}(\overline{\Omega})$. If $\mathbb{D} \subseteq \mathbb{R}^r$, we set

$$C^{m,\alpha}(\overline{\Omega}, \mathbb{D}) \equiv \{f \in (C^{m,\alpha}(\overline{\Omega}))^r : f(\overline{\Omega}) \subseteq \mathbb{D}\}.$$

Then, we set

$$C^m_b(\overline{\Omega}, \mathbb{R}^n) \equiv \{u \in C^m(\overline{\Omega}, \mathbb{R}^n) : D^\eta u \text{ is bounded for all } \eta \in \mathbb{N}^n \text{ with } |\eta| \leq m\}$$

and we endow $C^m_b(\overline{\Omega}, \mathbb{R}^n)$ with its usual norm

$$\|u\|_{C^m_b(\overline{\Omega},\mathbb{R}^n)} \equiv \sum_{\substack{\eta\in\mathbb{N}^n \\ |\eta|\leq m}} \sup_{x\in\overline{\Omega}} |D^\eta u(x)|.$$

We define

$$C^{m,\alpha}_b(\overline{\Omega}, \mathbb{R}^n) \equiv \{u \in C^{m,\alpha}(\overline{\Omega}, \mathbb{R}^n) : D^\eta u \text{ is bounded for all } \eta \in \mathbb{N}^n \text{ with } |\eta| \leq m\}$$

and we endow $C^{m,\alpha}_b(\overline{\Omega}, \mathbb{R}^n)$ with its usual norm

$$\|u\|_{C^{m,\alpha}_b(\overline{\Omega},\mathbb{R}^n)} \equiv \sum_{\substack{\eta\in\mathbb{N}^n \\ |\eta|\leq m}} \sup_{x\in\overline{\Omega}} |D^\eta u(x)| + \sum_{\substack{\eta\in\mathbb{N}^n \\ |\eta|= m}} |D^\eta u : \overline{\Omega}|_\alpha.$$

If Ω is a bounded open subset of $\mathbb{R}^n$, then $C^m(\overline{\Omega})$ and $C^{m,\alpha}(\overline{\Omega})$ endowed with their usual norms are well known to be Banach spaces. We say that a bounded open subset Ω of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if its closure is a manifold with boundary embedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively. If Ω is a bounded open set of class $C^{m,\alpha}$ with $m \geq 1$, then one can define the spaces $C^{k,\alpha}(\partial\Omega)$ and $C^{k,\alpha}(\partial\Omega, \mathbb{R}^n)$ on $\partial\Omega$ with $k \in \{0, \dots, m\}$ by means of the local parametrization. We will also need the following space:

$$C^{k,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \equiv \left\{ f \in C^{k,\alpha}(\partial\Omega, \mathbb{R}^n) : \int_{\partial\Omega} f \, d\sigma = 0 \right\}.$$

Here above and throughout the paper $d\sigma$ denotes the area element of $\partial\Omega$. For standard properties of functions in Schauder spaces, we refer to Gilbarg and Trudinger [21]. Next we turn to periodic domains. Let Q be as in (1). If Ω_Q is an open subset of $\mathbb{R}^n$ such that $\overline{\Omega_Q} \subseteq Q$, then we set

$$\mathbb{S}[\Omega_Q] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega_Q) = q\mathbb{Z}^n + \Omega_Q, \qquad \mathbb{S}[\Omega_Q]^- \equiv \mathbb{R}^n \setminus \overline{\mathbb{S}[\Omega_Q]}.$$

Then a function u from $\overline{\mathbb{S}[\Omega_Q]}$ or from $\overline{\mathbb{S}[\Omega_Q]^-}$ is q-periodic if $u(x + q_{hh}e_h) = u(x)$ for all x in the domain of definition of u and for all $h \in \{1, \dots, n\}$. If $m \in \mathbb{N}$, $\alpha \in]0, 1]$, then we denote by $C^m_q(\overline{\mathbb{S}[\Omega_Q]}, \mathbb{R}^n)$, $C^{m,\alpha}_q(\overline{\mathbb{S}[\Omega_Q]}, \mathbb{R}^n)$, $C^m_q(\overline{\mathbb{S}[\Omega_Q]^-}, \mathbb{R}^n)$, and $C^{m,\alpha}_q(\overline{\mathbb{S}[\Omega_Q]^-}, \mathbb{R}^n)$ the subsets of the q-periodic functions belonging to the spaces $C^m_b(\overline{\mathbb{S}[\Omega_Q]}, \mathbb{R}^n)$, $C^{m,\alpha}_b(\overline{\mathbb{S}[\Omega_Q]}, \mathbb{R}^n)$, $C^m_b(\overline{\mathbb{S}[\Omega_Q]^-}, \mathbb{R}^n)$, and $C^{m,\alpha}_b(\overline{\mathbb{S}[\Omega_Q]^-}, \mathbb{R}^n)$, respectively, and we regard them as Banach subspaces of the corresponding space.

Throughout the paper, analytic means real analytic. For standard definitions of calculus in normed spaces and for the definition and properties of real analytic maps in Banach spaces, we refer to Deimling [22].

3 | PERIODIC ELASTIC LAYER POTENTIALS

This section is devoted to introduce the main tools that we will use: the periodic analog of the elastic layer potentials. Before doing this, we need to introduce the classical elastic layer potentials. We denote by S_n the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n} \log|x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n \geq 3, \end{cases}$$

where s_n denotes the $(n-1)$-dimensional measure of $\partial\mathbb{B}_n(0,1)$. S_n is well known to be the fundamental solution of the Laplace operator. Let $\omega \in]1 - \frac{2}{n}, +\infty[$. We denote by $\Gamma_{n,\omega}$ the matrix-valued function from $\mathbb{R}^n \setminus \{0\}$ to $M_n(\mathbb{R})$ which takes x to the matrix $\Gamma_{n,\omega}(x)$ with (i,j)-entry defined by

$$\Gamma^j_{n,\omega,i}(x) \equiv \frac{\omega+2}{2(\omega+1)}\delta_{i,j}S_n(x) - \frac{\omega}{2(\omega+1)}\frac{1}{s_n}\frac{x_i x_j}{|x|^n} \qquad \forall (i,j) \in \{1,\dots,n\}^2.$$

As is well known, $\Gamma_{n,\omega}$ is the fundamental solution of the operator

$$L[\omega] \equiv \Delta + \omega\nabla\text{div}.$$

We note that the classical operator of linearized homogeneous isotropic elastostatics equals $L[\omega]$ times the second constant of Lamé, that

$$L[\omega]u = \text{div}\mathcal{T}(\omega, Du)$$

for all regular vector valued functions u, and that the classical fundamental solution of the operator of linearized homogeneous isotropic elastostatics equals $\Gamma_{n,\omega}$ times the reciprocal of the second constant of Lamé (cf., *e.g.*, Kupradze, Gegelia, Basheleishvili, and Burchuladze [1]). We find also convenient to set $\Gamma^j_{n,\omega} \equiv (\Gamma^j_{n,\omega,i})_{i\in\{1,\dots,n\}}$, which we think as a column vector for all $j \in \{1,\dots,n\}$. Now, let $\alpha \in]0,1]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \equiv (\mu_1,\dots,\mu_n) \in C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)$. The (classical) elastic single layer potential is defined by:

$$\mathrm{v}[\omega,\mu](x) \equiv \int_{\partial\Omega} \Gamma_{n,\omega}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n.$$

As is well known, the elastic single layer potential $\mathrm{v}[\omega,\mu]$ is continuous in $\mathbb{R}^n$, and we set

$$\mathrm{v}^+[\omega,\mu] \equiv \mathrm{v}[\omega,\mu]_{|\overline{\Omega}}, \qquad \mathrm{v}^-[\omega,\mu] \equiv \mathrm{v}[\omega,\mu]_{|\mathbb{R}^n\setminus\Omega}.$$

We further set

$$\mathrm{w}^*[\omega,\mu](x) \equiv \int_{\partial\Omega} \sum_{l=1}^n \mu_l(y)\mathcal{T}(\omega, D\Gamma^l_{n,\omega}(x-y))\nu_\Omega(x)d\sigma_y \quad \forall x \in \partial\Omega.$$

For the properties of elastic layer potentials, we refer to Dalla Riva and Lanza de Cristoforis [14, Theorem A.2]. Our method is based on a periodic version of classical potential theory. In order to construct periodic elastic layer potentials, we replace the classical fundamental solution by a $n \times n$ matrix of q-periodic tempered distributions $\Gamma^q_{n,\omega} \equiv (\Gamma^{q,k}_{n,\omega,j})_{(j,k)\in\{1,\dots,n\}^2}$ defined by

$$\Gamma^{q,k}_{n,\omega,j} \equiv \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{4\pi^2|Q||q^{-1}z|^2}\left[-\delta_{j,k} + \frac{\omega}{\omega+1}\frac{(q^{-1}z)_j(q^{-1}z)_k}{|q^{-1}z|^2}\right]E_{2\pi\mathbf{i}q^{-1}z} \qquad \forall (j,k) \in \{1,\dots,n\}^2$$

in the sense of distributions in $\mathbb{R}^n$, where $E_{2\pi\mathbf{i}q^{-1}z}$ is the function from $\mathbb{R}^n$ to $\mathbb{C}$ defined by $E_{2\pi\mathbf{i}q^{-1}z}(x) \equiv e^{2\pi\mathbf{i}(q^{-1}z)\cdot x}$ for all $x \in \mathbb{R}^n$ and for all $z \in \mathbb{Z}^n$. Since $\Gamma^q_{n,\omega}$ satisfies

$$L[\omega]\Gamma^q_{n,\omega} = \sum_{z\in\mathbb{Z}^n} \delta_{qz}\mathbb{I}_n - \frac{1}{|Q|}\mathbb{I}_n$$

in the sense of distributions, where δ_{qz} denotes the Dirac measure with mass in qz, it can be effectively exploited as a periodic analog of $\Gamma_{n,\omega}$. Moreover, $\Gamma^q_{n,\omega}$ is even and real analytic from $\mathbb{R}^n \setminus q\mathbb{Z}^n$ to $M_n(\mathbb{R})$, and $\Gamma^{q,k}_{n,\omega,j}$ is locally integrable in $\mathbb{R}^n$ for all $(j,k) \in \{1,\dots,n\}^2$. Finally, the difference $\Gamma^q_{n,\omega} - \Gamma_{n,\omega}$ can be extended to a real analytic function from $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$ to $M_n(\mathbb{R})$, which we denote by $R^q_{n,\omega}$, such that

$$L[\omega]R^q_{n,\omega} = \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \delta_{qz}\mathbb{I}_n - \frac{1}{|Q|}\mathbb{I}_n$$

in the sense of distributions. For the proof of all the above properties of $\Gamma^q_{n,\omega}$ we refer to Ammari and Kang[23, Lemma 9.21], Ammari, Kang, and Lim[24, Lemma 3.2], and[20, Theorem 3.1]. We find convenient to set

$$\Gamma^{q,j}_{n,\omega} \equiv \left(\Gamma^{q,j}_{n,\omega,i}\right)_{i\in\{1,\dots,n\}}, \quad R^{q,j}_{n,\omega} \equiv \left(R^{q,j}_{n,\omega,i}\right)_{i\in\{1,\dots,n\}}$$

which we think as column vectors for all $j \in \{1,\dots,n\}$. Let $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω_Q be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\overline{\Omega_Q} \subseteq Q$. Let $\mu \in C^{0,\alpha}(\partial\Omega_Q, \mathbb{R}^n)$. We denote by $\mathrm{v}_q[\omega,\mu]$ the periodic elastic single layer potential, *i.e.* the function from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$\mathrm{v}_q[\omega,\mu](x) \equiv \int_{\partial\Omega_Q} \Gamma^q_{n,\omega}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n.$$

The single layer potential $\mathrm{v}_q[\omega,\mu]$ is q-periodic and continuous in $\mathbb{R}^n$, and we set

$$\mathrm{v}^+_q[\omega,\mu] \equiv \mathrm{v}_q[\omega,\mu]_{|\overline{\mathbb{S}[\Omega_Q]}}, \qquad \mathrm{v}^-_q[\omega,\mu] \equiv \mathrm{v}_q[\omega,\mu]_{|\mathbb{R}^n\setminus\mathbb{S}[\Omega_Q]}.$$

We also set

$$\mathrm{w}^*_q[\omega,\mu](x) \equiv \int_{\partial\Omega_Q} \sum_{l=1}^n \mu_l(y)\mathcal{T}(\omega, D\Gamma^{q,l}_{n,\omega}(x-y))\nu_{\Omega_Q}(x)\,d\sigma_y \qquad \forall x \in \partial\Omega_Q.$$

For the properties of $\mathrm{v}_q[\omega,\mu]$ we refer to[20, Theorem 3.2]. Here we recall that

$$L[\omega]\mathrm{v}_q[\omega,\mu] = -\frac{1}{|Q|}\int_{\partial\Omega_Q} \mu\,d\sigma \qquad \text{in } \mathbb{R}^n \setminus \partial\mathbb{S}[\Omega_Q],$$

that the operator which takes μ to $\mathrm{v}^-_q[\omega,\mu]$ is linear and continuous from $C^{m-1,\alpha}(\partial\Omega_Q,\mathbb{R}^n)$ to $C^{m,\alpha}_q(\overline{\mathbb{S}[\Omega_Q]^-},\mathbb{R}^n)$, and that the following jump formula holds:

$$T\left(\omega, D\mathrm{v}^-_q[\omega,\mu]\right)\nu_{\Omega_Q} = \frac{1}{2}\mu + \mathrm{w}^*_q[\omega,\mu] \qquad \text{on } \partial\Omega_Q. \tag{5}$$

4 | AN AUXILIARY PROBLEM AND ITS ASYMPTOTIC BEHAVIOR

The work plan of the present section is the following. The main issue we face is that our boundary value problem is defined on a domain which depends both on ε and on δ. Therefore, we wish to get rid of such a dependence and to formulate a problem on a domain which is independent of these two perturbation parameters. We provide such a formulation in two steps: one for the parameter δ and another one for ε. We first convert problem (4) into the equivalent (δ-dependent) auxiliary boundary value problem (10) defined in a δ-independent domain (which, however, still depends on ε). Then we exploit layer potential techniques and a suitable change of variables and we shall provide a formulation of this auxiliary problem in terms of an integral equation which depends on the singular perturbation parameter pair (ε,δ), but which is defined on a fixed set. In other words, we move the dependence upon the pair (ε,δ) from the domain to the equation. By means of the implicit function theorem we deduce the existence of a family of solutions of the integral equation that depends real analytically upon (ε,δ) around $(0,0)$. Finally, by using the representation of the solutions in terms of layer potentials, we show that the associated solutions of the auxiliary problem (10) and their integral functionals depend real analytically on (ε,δ) near $(0,0)$.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let Ω be as in (2). If $G \in C^0(\partial\Omega\times\mathbb{R}^n,\mathbb{R}^n)$, then we denote by F_G the (nonlinear nonautonomous) composition operator from $C^0(\partial\Omega,\mathbb{R}^n)$ to itself, which takes $f \equiv (f_1,\dots,f_n)$ in $C^0(\partial\Omega,\mathbb{R}^n)$ to the function $F_G[f]$ from $\partial\Omega$ to $\mathbb{R}^n$, defined by

$$F_G[f](t) \equiv G(t, f(t)) \quad \forall t \in \partial\Omega.$$

We consider the following assumptions on the function G:

$$G \in C^0(\partial\Omega\times\mathbb{R}^n,\mathbb{R}^n), \tag{6}$$

$$F_G \text{ maps } C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n) \text{ to itself.} \tag{7}$$

We note here that if $G \in C^0(\partial\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ is such that F_G is real analytic from $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ to itself, then one can prove that the gradient matrix D_uG of G with respect to the variable in $\mathbb{R}^n$ exists. If $\tilde{f} \in C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ and $dF_G[\tilde{f}]$ denotes the Fréchet differential of F_G at $\tilde{f}$, then we have

$$dF_G[\tilde{f}](v) = \sum_{l=1}^{n} F_{\partial_{u_l} G}[\tilde{f}]v_l \qquad \forall v \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n) \tag{8}$$

(cf. Lanza de Cristoforis[13, Proposition 6.3]). Moreover,

$$D_u G(\cdot, \xi) \in C^{m-1,\alpha}(\partial\Omega, M_n(\mathbb{R})) \quad \forall \xi \in \mathbb{R}^n, \tag{9}$$

where $C^{m-1,\alpha}(\partial\Omega, M_n(\mathbb{R}))$ denotes the space of functions of class $C^{m-1,\alpha}$ from $\partial\Omega$ to $M_n(\mathbb{R})$. We are ready to transform problem (4) so as to remove the parameter δ from the domain. We have the following proposition which is a straightforward consequence of a change of variable.

Proposition 1. Let $\omega \in]1 - \frac{2}{n}, +\infty[$, $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be as in assumption (2). Let $p \in Q$ and ε_0 be as in assumption (3). Let G be such that assumption (6) holds. Let $(\varepsilon, \delta) \in]0, \varepsilon_0[\times]0, +\infty[$. Then a function $u \in C^{m,\alpha}(\overline{\mathbb{S}[\varepsilon,\delta]^-}, \mathbb{R}^n)$ is a solution of problem (4) if and only if the function $u^\sharp \in C^{m,\alpha}_q(\overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-}, \mathbb{R}^n)$ defined by

$$u^\sharp(x) \equiv u(\delta x) \qquad \forall x \in \overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-}$$

solves the auxiliary problem

$$\begin{cases} L[\omega]u^\sharp(x) = 0 & \forall x \in \mathbb{S}[\Omega_{p,\varepsilon}]^-, \\ u^\sharp(x + q_{kk}e_k) = u^\sharp(x) & \forall x \in \overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-}, \\ \mathcal{T}(\omega, Du^\sharp(x))\nu_{\Omega_{p,\varepsilon}}(x) = \delta G(\frac{x-p}{\varepsilon}, u^\sharp(x)) & \forall x \in \partial\Omega_{p,\varepsilon}. \end{cases} \tag{10}$$

We shall now transform the auxiliary problem (10) into an integral equation. To do this, we need the following representation formula of[20, Lemma 5.1].

Lemma 2. Let $\omega \in]1 - \frac{2}{n}, +\infty[$, $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be as in assumption (2). Let $p \in Q$, let ε_0 be as in assumption (3) and $\varepsilon \in]0, \varepsilon_0[$. Let $u^* \in C^{m,\alpha}_q(\overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-}, \mathbb{R}^n)$ be such that

$$L[\omega]u^*(x) = 0 \qquad \forall x \in \mathbb{S}[\Omega_{p,\varepsilon}]^-.$$

Then there exists a unique pair $(\theta, \xi) \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ such that

$$u^*(x) = \varepsilon^{n-1} \int_{\partial\Omega} \Gamma^q_{n,\omega}(x - p - \varepsilon s)\theta(s)d\sigma_s + \xi \quad \forall x \in \overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-}.$$

We are now ready to transform problem (10) into an integral equation.

Theorem 3. Let $\omega \in]1 - \frac{2}{n}, +\infty[$, $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be as in assumption (2). Let $p \in Q$ and ε_0 be as in assumption (3). Let G be as in assumptions (6) and (7). Let Λ be the map from $]-\varepsilon_0, \varepsilon_0[\times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ defined by

$$\begin{aligned} \Lambda[\varepsilon, \delta, \theta, \xi](t) \equiv\ & \frac{1}{2}\theta(t) + \mathrm{w}^*[\omega, \theta](t) \\ & + \varepsilon^{n-1} \int_{\partial\Omega} \sum_{l=1}^{n} \theta_l(s)\mathcal{T}(\omega, DR^{q,l}_{n,\omega}(\varepsilon(t-s)))\nu_\Omega(t)d\sigma_s \\ & - G\left(t, \delta\varepsilon \mathrm{v}[\omega, \theta](t) + \delta\varepsilon^{n-1} \int_{\partial\Omega} R^q_{n,\omega}(\varepsilon(t-s))\theta(s)d\sigma_s + \xi\right) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{11}$$

for all $(\varepsilon, \delta, \theta, \xi) \in]-\varepsilon_0, \varepsilon_0[\times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$. If $(\varepsilon, \delta) \in]0, \varepsilon_0[\times]0, +\infty[$, then the map $u^\sharp[\varepsilon, \delta, \cdot, \cdot]$ from the set of pairs $(\theta, \xi) \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ that solve equation

$$\Lambda[\varepsilon, \delta, \theta, \xi] = 0 \tag{12}$$

to the set of functions in $C^{m,\alpha}_q(\overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-}, \mathbb{R}^n)$ that solve problem (10), which takes (θ, ξ) to

$$u^\sharp[\varepsilon, \delta, \theta, \xi](x) \equiv \delta\varepsilon^{n-1} \int_{\partial\Omega} \Gamma^q_{n,\omega}(x - p - \varepsilon s)\theta(s)d\sigma_s + \xi \quad \forall x \in \overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-} \tag{13}$$

is a bijection.

Proof. Let $(\varepsilon, \delta) \in]0, \varepsilon_0[\times]0, +\infty[$. Assume that a function $u^\sharp \in C_q^{m,\alpha}(\overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-}, \mathbb{R}^n)$ solves problem (10). Then Lemma 2 implies that there exists a unique pair $(\theta, \xi) \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ such that $u^\sharp$ equals the right-hand side of (13). A simple computation based on the jump formula (5) and on a change of variable shows that the pair (θ, ξ) is a solution of equation (12). Conversely, one can read backward the above argument and prove that if a pair $(\theta, \xi) \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ solves equation (12), then the function $u^\sharp[\varepsilon, \delta, \theta, \xi]$ defined by (13) is a solution of problem (10). □

As we have seen, we have first converted our problem (4) into the auxiliary problem (10) which, in turn, has been reduced to the integral equation (12) which is defined on the fixed domain $\partial\Omega$. We note that equation (12) is not singular around $(\varepsilon, \delta) = (0, 0)$. For $(\varepsilon, \delta) = (0, 0)$ we obtain the following equation, that we address to as the *limiting equation*:

$$\frac{1}{2}\theta(t) + \mathrm{w}^*[\omega, \theta](t) - G(t, \xi) = 0 \quad \forall t \in \partial\Omega. \tag{14}$$

Under suitable assumptions the limiting equation (14) is solvable, as showed in the following proposition (see[20, Proposition 5.3]).

Proposition 4. Let $\omega \in]1 - \frac{2}{n}, +\infty[$, $\alpha \in]0, 1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be as in assumption (2). Let G be as in assumptions (6) and (7). Assume that there exists $\tilde{\xi} \in \mathbb{R}^n$ such that

$$\int_{\partial\Omega} G(t, \tilde{\xi}) d\sigma_t = 0.$$

Then the integral equation

$$\frac{1}{2}\theta(t) + \mathrm{w}^*[\omega, \theta](t) - G(t, \tilde{\xi}) = 0 \quad \forall t \in \partial\Omega \tag{15}$$

has a unique solution $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0$, which we denote by $\tilde{\theta}$. As a consequence, the pair $(\tilde{\theta}, \tilde{\xi})$ is a solution in $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ of the limiting equation (14).

Under suitable assumptions, we can prove the following real analyticity result for a family of solutions of equation (12).

Theorem 5. Let $\omega \in]1 - \frac{2}{n}, +\infty[$, $\alpha \in]0, 1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be as in assumption (2). Let $p \in Q$ and ε_0 be as in assumption (3). Let G be as in assumption (6). Assume that

$$F_G \text{ is real analytic from } C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n) \text{ to itself.} \tag{16}$$

Assume that there exists $\tilde{\xi} \in \mathbb{R}^n$ such that

$$\int_{\partial\Omega} G(t, \tilde{\xi})\, d\sigma_t = 0 \quad \text{and} \quad \det \int_{\partial\Omega} D_u G(t, \tilde{\xi})\, d\sigma_t \neq 0. \tag{17}$$

Let Λ be as in Theorem 3. Let $\tilde{\theta}$ be the unique solution in $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0$ of (15) given by Proposition 4. Then there exist $(\varepsilon', \delta') \in]0, \varepsilon_0[\times]0, +\infty[$, an open neighborhood $\mathcal{U}$ of $(\tilde{\theta}, \tilde{\xi})$ in $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$, and a real analytic map (Θ, Ξ) from $]-\varepsilon', \varepsilon'[\times]-\delta', \delta'[$ to $\mathcal{U}$ such that the zeros of the map Λ in $]-\varepsilon', \varepsilon'[\times]-\delta', \delta'[\times \mathcal{U}$ coincides with the graph of (Θ, Ξ). In particular, $(\Theta[0, 0], \Xi[0, 0]) = (\tilde{\theta}, \tilde{\xi})$.

Proof. We plan to apply the implicit function theorem for real analytic maps to the map Λ from $]-\varepsilon_0, \varepsilon_0[\times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ defined by (11). Standard properties of integral operators with real analytic kernels (cf., *e.g.*,[25]), classical potential theory for linearized elastostatics (cf., *e.g.*, Dalla Riva and Lanza de Cristoforis[14, Theorem A.2]), and assumption (16), imply that Λ is real analytic from $] - \varepsilon_0, \varepsilon_0[\times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$. By standard calculus in Banach spaces, the differential of Λ at $(0, 0, \tilde{\theta}, \tilde{\xi})$ with respect to the variables (θ, ξ) is delivered by the following formula:

$$\partial_{(\theta,\xi)}\Lambda[0, 0, \tilde{\theta}, \tilde{\xi}](\psi, \eta)(t) = \frac{1}{2}\psi(t) + \mathrm{w}^*[\omega, \psi](t) - D_u G(t, \tilde{\xi})\eta \quad \forall t \in \partial\Omega,$$

for all $(\psi, \eta) \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ (see also formula (8)). By assumption (17) and[20, Proposition 5.4], we deduce that $\partial_{(\theta,\xi)}\Lambda[0, 0, \tilde{\theta}, \tilde{\xi}]$ is a linear homeomorphism from $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)_0 \times \mathbb{R}^n$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ (see also (9)). Then the statement follows by the implicit function theorem for real analytic maps in Banach spaces (cf., *e.g.*, Deimling[22, Theorem 15.3]). □

Under the assumption of Theorem 5, if $u^\sharp[\cdot, \cdot, \cdot, \cdot]$ is defined as in (13), we set

$$u^\sharp(\varepsilon, \delta, x) \equiv u^\sharp[\varepsilon, \delta, \Theta[\varepsilon, \delta], \Xi[\varepsilon, \delta]](x) \qquad \forall x \in \overline{\mathbb{S}[\Omega_{p,\varepsilon}]^-}, \quad \forall (\varepsilon, \delta) \in]0, \varepsilon'[\times]0, \delta'[.$$

Clearly $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ is a family of solutions of problem (10). Moreover, if we set

$$u(\epsilon,\delta,x)\equiv u^\sharp\left(\epsilon,\delta,\frac{x}{\delta}\right) \qquad \forall x\in\overline{\mathbb{S}[\epsilon,\delta]^-}, \quad \forall(\epsilon,\delta)\in\,]0,\epsilon'[\times]0,\delta'[, \tag{18}$$

Proposition 1 implies that $\{u(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ is a family of solutions of problem (4).

The rest of this section is devoted to analyze the behavior near $(\epsilon,\delta)=(0,0)$ of the family of solutions $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ of problem (10) and the behavior of the corresponding integral functionals.

Theorem 6. Let the assumptions of Theorem 5 hold. Let $\tilde{\theta},\epsilon',\delta',\Theta,\Xi$ be as in Theorem 5. Then, the following statements hold.

(i) Let $R>0$ be such that $\overline{\Omega}\subseteq\mathbb{B}_n(0,R)$. Then there exist $\tilde{\epsilon}_R\in\,]0,\epsilon'[$ such that

$$p+\epsilon\overline{\mathbb{B}_n(0,R)}\subseteq Q \qquad \forall\epsilon\in\,]-\tilde{\epsilon}_R,\tilde{\epsilon}_R[\tag{19}$$

and a real analytic map $U^\sharp_R$ from $]-\tilde{\epsilon}_R,\tilde{\epsilon}_R[\times]-\delta',\delta'[$ to $C^{m,\alpha}(\overline{\mathbb{B}_n(0,R)}\setminus\Omega,\mathbb{R}^n)$ such that

$$u^\sharp(\epsilon,\delta,p+\epsilon t)=\delta\epsilon U^\sharp_R[\epsilon,\delta](t)+\Xi[\epsilon,\delta] \quad \forall t\in\overline{\mathbb{B}_n(0,R)}\setminus\Omega, \tag{20}$$

for all $(\epsilon,\delta)\in\,]0,\tilde{\epsilon}_R[\times]0,\delta'[$. Moreover,

$$\begin{cases} U^\sharp_R[0,0](t)=\mathrm{v}^-[\omega,\tilde{\theta}](t) & \forall t\in\overline{\mathbb{B}_n(0,R)}\setminus\Omega,\\ \Xi[0,0]=\tilde{\xi}. \end{cases} \tag{21}$$

(ii) Let $R>0$ be such that $\overline{\Omega}\subseteq\mathbb{B}_n(0,R)$ and let $\tilde{\epsilon}_R$ be as in statement (i). Then there exists a real analytic map $W^\sharp_R$ from $]-\tilde{\epsilon}_R,\tilde{\epsilon}_R[\times]-\delta',\delta'[$ to $C^{m-1,\alpha}(\overline{\mathbb{B}_n(0,R)}\setminus\Omega,M_n(\mathbb{R}))$ such that

$$D_t(u^\sharp(\epsilon,\delta,p+\epsilon t))=\delta\epsilon W^\sharp_R[\epsilon,\delta](t) \qquad \forall t\in\overline{\mathbb{B}_n(0,R)}\setminus\Omega, \tag{22}$$

for all $(\epsilon,\delta)\in\,]0,\tilde{\epsilon}_R[\times]0,\delta'[$. Moreover,

$$W^\sharp_R[0,0](t)=D\mathrm{v}^-[\omega,\tilde{\theta}](t) \qquad \forall t\in\mathbb{B}_n(0,R)\setminus\Omega\,. \tag{23}$$

Proof. We start by proving (i). Clearly, there exists $\tilde{\epsilon}_R$ such that (19) holds. If $\epsilon\in\,]0,\tilde{\epsilon}_R[$ and $\delta\in\,]0,\delta'[$, then

$$\begin{aligned} &u^\sharp(\epsilon,\delta,p+\epsilon t)\\ &=\delta\epsilon^{n-1}\int_{\partial\Omega}\Gamma^q_{n,\omega}(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s+\Xi[\epsilon,\delta]\\ &=\delta\epsilon\left(\int_{\partial\Omega}\Gamma_{n,\omega}(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s+\epsilon^{n-2}\int_{\partial\Omega}R^q_{n,\omega}(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\right)+\Xi[\epsilon,\delta] \quad \forall t\in\overline{\mathbb{B}_n(0,R)}\setminus\Omega \end{aligned}$$

(for $n=2$ we recall that $\int_{\partial\Omega}\Theta[\epsilon,\delta]\,d\sigma=0$). Thus it is natural to set

$$\begin{aligned} U^\sharp_R[\epsilon,\delta](t)&\equiv\int_{\partial\Omega}\Gamma_{n,\omega}(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s+\epsilon^{n-2}\int_{\partial\Omega}R^q_{n,\omega}(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\\ &=\mathrm{v}^-[\omega,\Theta[\epsilon,\delta]](t)+\epsilon^{n-2}\int_{\partial\Omega}R^q_{n,\omega}(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s \quad \forall t\in\overline{\mathbb{B}_n(0,R)}\setminus\Omega \end{aligned}$$

for all $(\epsilon,\delta)\in\,]-\tilde{\epsilon}_R,\tilde{\epsilon}_R[\times]-\delta',\delta'[$. Standard properties of integral operators with real analytic kernels (cf., *e.g.*,[25]), and the analyticity of Θ imply that the map from $]-\tilde{\epsilon}_R,\tilde{\epsilon}_R[\times]-\delta',\delta'[$ to $C^{m,\alpha}(\overline{\mathbb{B}_n(0,R)}\setminus\Omega,\mathbb{R}^n)$ which takes (ϵ,δ) to

$$\epsilon^{n-2}\int_{\partial\Omega}R^q_{n,\omega}(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s$$

of the variable $t\in\overline{\mathbb{B}_n(0,R)}\setminus\Omega$ is analytic. By classical results of potential theory and by the analyticity of Θ, the map from $]-\tilde{\epsilon}_R,\tilde{\epsilon}_R[\times]-\delta',\delta'[$ to $C^{m,\alpha}(\overline{\mathbb{B}_n(0,R)}\setminus\Omega,\mathbb{R}^n)$, which takes (ϵ,δ) to $\mathrm{v}^-[\omega,\Theta[\epsilon,\delta]]_{|\overline{\mathbb{B}_n(0,R)}\setminus\Omega}$ is analytic (cf., *e.g.*, Dalla Riva and Lanza de Cristoforis[14, Theorem A.2]). So we deduce that $U^\sharp_R$ is analytic from $]-\tilde{\epsilon}_R,\tilde{\epsilon}_R[\times]-\delta',\delta'[$ to $C^{m,\alpha}(\overline{\mathbb{B}_n(0,R)}\setminus\Omega,\mathbb{R}^n)$ and satisfies equalities (20) and (21). To prove (ii), we set $W^\sharp_R[\epsilon,\delta](t)\equiv D_tU^\sharp_R[\epsilon,\delta](t)$ for all $t\in\overline{\mathbb{B}_n(0,R)}\setminus\Omega$ and for all

$(\varepsilon,\delta)\in\,]-\tilde{\varepsilon}_R,\tilde{\varepsilon}_R[\times]-\delta',\delta'[$. Then the continuity of the operator D_t implies that $W_R^\sharp$ is analytic from $]-\tilde{\varepsilon}_R,\tilde{\varepsilon}_R[\times]-\delta',\delta'[$ to $C^{m-1,\alpha}(\overline{\mathbb{B}_n(0,R)}\setminus\Omega, M_n(\mathbb{R}^n))$ and that (22) and (23) hold. Hence, the proof is complete. □

Next we turn to consider the behavior of the energy integral of $u^\sharp(\varepsilon,\delta,\cdot)$, that is the function from $]0,\varepsilon'[\times]0,\delta'[$ to $\mathbb{R}$ which takes (ε,δ) to

$$\mathrm{En}^\sharp(\omega,u^\sharp(\varepsilon,\delta,\cdot))\equiv\frac{1}{2}\int_{Q\setminus\overline{\Omega_{p,\varepsilon}}}\mathrm{tr}(\mathcal{T}(\omega,D_x u^\sharp(\varepsilon,\delta,x))(D_x u^\sharp(\varepsilon,\delta,x))^T)\,dx \tag{24}$$

and we prove the following.

Theorem 7. Let the assumptions of Theorem 5 hold. Let $\tilde{\theta},\varepsilon',\delta',\Theta,\Xi$ be as in Theorem 5. Then there exist $\varepsilon_e\in\,]0,\varepsilon'[$ and a real analytic map $E^\sharp$ from $]-\varepsilon_e,\varepsilon_e[\times]-\delta',\delta'[$ to $\mathbb{R}$ such that

$$\mathrm{En}^\sharp(\omega,u^\sharp(\varepsilon,\delta,\cdot))=\delta^2\varepsilon^n E^\sharp[\varepsilon,\delta]\qquad\forall(\varepsilon,\delta)\in\,]0,\varepsilon_e[\times]0,\delta'[. \tag{25}$$

Moreover,

$$E^\sharp[0,0]=\frac{1}{2}\int_{\mathbb{R}^n\setminus\overline{\Omega}}\mathrm{tr}(\mathcal{T}(\omega,D\mathrm{v}^-[\omega,\tilde{\theta}])(D\mathrm{v}^-[\omega,\tilde{\theta}])^T)dx.$$

Proof. Let $(\varepsilon,\delta)\in\,]0,\varepsilon'[\times]0,\delta'[$. Since $\Xi[\cdot,\cdot]$ is a constant vector valued function with respect to the variable $x\in Q\setminus\overline{\Omega_{p,\varepsilon}}$, we can rewrite (24) as

$$\mathrm{En}^\sharp(\omega,u^\sharp(\varepsilon,\delta,\cdot))=\frac{1}{2}\int_{Q\setminus\overline{\Omega_{p,\varepsilon}}}\mathrm{tr}(\mathcal{T}(\omega,D_x u^\sharp(\varepsilon,\delta,x))(D_x(u^\sharp(\varepsilon,\delta,x)-\Xi[\varepsilon,\delta]))^T)\,dx$$

for all $(\varepsilon,\delta)\in\,]0,\varepsilon'[\times]0,\delta'[$. By the periodicity of $u^\sharp$ and applying the divergence theorem we have

$$\begin{aligned}
&\frac{1}{2}\int_{Q\setminus\overline{\Omega_{p,\varepsilon}}}\mathrm{tr}(\mathcal{T}(\omega,D_x u^\sharp(\varepsilon,\delta,x))(D_x(u^\sharp(\varepsilon,\delta,x)-\Xi[\varepsilon,\delta]))^T)\,dx\\
&\quad=\frac{1}{2}\int_{\partial Q}(u^\sharp(\varepsilon,\delta,x)-\Xi[\varepsilon,\delta])^T\mathcal{T}(\omega,D_x u^\sharp(\varepsilon,\delta,x))\nu_Q(x)\,d\sigma_x\\
&\quad\quad-\frac{1}{2}\int_{\partial\Omega_{p,\varepsilon}}(u^\sharp(\varepsilon,\delta,x)-\Xi[\varepsilon,\delta])^T\mathcal{T}(\omega,D_x u^\sharp(\varepsilon,\delta,x))\nu_{\Omega_{p,\varepsilon}}(x)\,d\sigma_x\\
&\quad=-\frac{\varepsilon^{n-1}}{2}\int_{\partial\Omega}(u^\sharp(\varepsilon,\delta,p+\varepsilon t)-\Xi[\varepsilon,\delta])^T\mathcal{T}(\omega,D_x u^\sharp(\varepsilon,\delta,p+\varepsilon t))\nu_\Omega(t)\,d\sigma_t.
\end{aligned}$$

Let $R>0$ be such that $\overline{\Omega}\subseteq\mathbb{B}_n(0,R)$ and $\varepsilon_e\in\,]0,\varepsilon'[$ such that (19) holds. By equalities (20) and (22) we have

$$\begin{aligned}
&-\frac{\varepsilon^{n-1}}{2}\int_{\partial\Omega}(u^\sharp(\varepsilon,\delta,p+\varepsilon t)-\Xi[\varepsilon,\delta])^T\mathcal{T}(\omega,D_x u^\sharp(\varepsilon,\delta,p+\varepsilon t))\nu_\Omega(t)\,d\sigma_t\\
&\quad=-\frac{\varepsilon^{n-2}}{2}\int_{\partial\Omega}(\varepsilon\delta U_R^\sharp[\varepsilon,\delta](t))^T\mathcal{T}(\omega,\varepsilon\delta W_R^\sharp[\varepsilon,\delta](t))\nu_\Omega(t)\,d\sigma_t\\
&\quad=-\frac{\delta^2\varepsilon^n}{2}\int_{\partial\Omega}(U_R^\sharp[\varepsilon,\delta](t))^T\mathcal{T}(\omega,W_R^\sharp[\varepsilon,\delta](t))\nu_\Omega(t)\,d\sigma_t
\end{aligned}$$

for all $(\varepsilon,\delta)\in\,]0,\varepsilon_e[\times]0,\delta'[$. Thus it is natural to set

$$E^\sharp[\varepsilon,\delta]\equiv-\frac{1}{2}\int_{\partial\Omega}(U_R^\sharp[\varepsilon,\delta](t))^T\mathcal{T}(\omega,W_R^\sharp[\varepsilon,\delta](t))\nu_\Omega(t)\,d\sigma_t\qquad\forall(\varepsilon,\delta)\in\,]-\varepsilon_e,\varepsilon_e[\times]-\delta',\delta'[.$$

Then $E^\sharp$ is a real analytic map from $]-\varepsilon_e, \varepsilon_e[\times]-\delta', \delta'[$ to $\mathbb{R}$ by Theorem 6 and standard calculus in Banach spaces. Thus equality (25) holds. Moreover, (21), (23) and the divergence theorem imply that

$$\begin{aligned} E^\sharp[0,0] &= -\frac{1}{2}\int_{\partial\Omega} (U^\sharp_R[0,0](t))^T \mathcal{T}(\omega, W^\sharp_R[0,0](t))\nu_\Omega(t)\,d\sigma_t \\ &= -\frac{1}{2}\int_{\partial\Omega} (\mathrm{v}^-[\omega,\tilde\theta](t))^T \mathcal{T}(\omega, D\mathrm{v}^-[\omega,\tilde\theta](t))\nu_\Omega(t)\,d\sigma_t \\ &= \frac{1}{2}\int_{\mathbb{R}^n\setminus\overline{\Omega}} \mathrm{tr}(\mathcal{T}(\omega, D\mathrm{v}^-[\omega,\tilde\theta](t))(D\mathrm{v}^-[\omega,\tilde\theta](t))^T)\,dt. \end{aligned}$$

Hence the proof is complete. □

Then we consider the behavior of the integral of $u^\sharp(\varepsilon,\delta,\cdot)$, that is the function from $]0,\varepsilon'[\times]0,\delta'[$ to $\mathbb{R}^n$ which takes (ε,δ) to

$$\mathrm{Int}^\sharp(u^\sharp(\varepsilon,\delta,\cdot)) \equiv \int_{Q\setminus\overline{\Omega_{p,\varepsilon}}} u^\sharp(\varepsilon,\delta,x)\,dx$$

and we prove the following.

Theorem 8. Let the assumptions of Theorem 5 hold. Let $\tilde\theta, \varepsilon', \delta', \Theta, \Xi$ be as in Theorem 5. Then there exists a real analytic map $I^\sharp$ from $]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$ to $\mathbb{R}^n$ such that

$$\mathrm{Int}^\sharp(u^\sharp(\varepsilon,\delta,\cdot)) = I^\sharp[\varepsilon,\delta] \qquad \forall(\varepsilon,\delta)\in]0,\varepsilon'[\times]0,\delta'[. \tag{26}$$

Moreover,

$$I^\sharp[0,0] = \tilde\xi|Q|\,. \tag{27}$$

Proof. If $(\varepsilon,\delta)\in]0,\varepsilon'[\times]0,\delta'[$, we clearly have

$$\begin{aligned} \mathrm{Int}^\sharp(u^\sharp(\varepsilon,\delta,\cdot)) &= \int_{Q\setminus\overline{\Omega_{p,\varepsilon}}} \left(\delta\varepsilon^{n-1}\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s + \Xi[\varepsilon,\delta]\right)dx \\ &= \delta\varepsilon^{n-1}\int_{Q\setminus\overline{\Omega_{p,\varepsilon}}}\left(\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx + \Xi[\varepsilon,\delta]\Big(|Q|-\varepsilon^n|\Omega|\Big). \end{aligned}$$

On the other hand, for all $(\varepsilon,\delta)\in]0,\varepsilon'[\times]0,\delta'[$, we have

$$\begin{aligned} \int_{Q\setminus\overline{\Omega_{p,\varepsilon}}}\left(\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx &= \int_{Q}\left(\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx \\ &\quad -\int_{\overline{\Omega_{p,\varepsilon}}}\left(\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx\,. \end{aligned}$$

Since $\int_{\partial\Omega}\Theta[\varepsilon,\delta]\,d\sigma = 0$, we have

$$\begin{aligned} \int_{\overline{\Omega_{p,\varepsilon}}}\left(\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx &= \varepsilon^n\int_{\overline{\Omega}}\left(\int_{\partial\Omega}\Gamma^q_{n,\omega}(\varepsilon(t-s))\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dt \\ &= \varepsilon^n\int_{\overline{\Omega}}\left(\int_{\partial\Omega}\Gamma_{n,\omega}(\varepsilon(t-s))\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dt + \varepsilon^n\int_{\overline{\Omega}}\left(\int_{\partial\Omega}R^q_{n,\omega}(\varepsilon(t-s))\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dt \\ &= \varepsilon^2\left(\int_{\overline{\Omega}}\left(\int_{\partial\Omega}\Gamma_{n,\omega}(t-s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dt + \varepsilon^{n-2}\int_{\overline{\Omega}}\left(\int_{\partial\Omega}R^q_{n,\omega}(\varepsilon(t-s))\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dt\right), \end{aligned}$$

for all $(\varepsilon,\delta)\in\,]0,\varepsilon'[\times]0,\delta'[$. Therefore, we set

$$I_1[\varepsilon,\delta]\equiv\int_{\overline{\Omega}}\left(\int_{\partial\Omega}\Gamma_{n,\omega}(t-s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dt+\varepsilon^{n-2}\int_{\overline{\Omega}}\left(\int_{\partial\Omega}R^q_{n,\omega}(\varepsilon(t-s))\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dt\,,$$

for all $(\varepsilon,\delta)\in\,]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$. Then by mapping properties of elastic layer potentials (cf.,*e.g.*, Dalla Riva and Lanza de Cristoforis[14, Theorem A.2]), by standard properties of integral operators with real analytic kernels (cf., *e.g.*,[25]), by the analyticity of the map Θ, and by standard calculus in Banach spaces, we deduce that I_1 is a real analytic map from $]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$ to $\mathbb{R}^n$. We now turn to the analysis of $\int_Q\left(\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx$ and we note that

$$\int_Q\left(\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx=\int_Q\left(\int_{\partial\Omega}\Gamma_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx+\int_Q\left(\int_{\partial\Omega}R^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx\,,$$

for all $(\varepsilon,\delta)\in\,]0,\varepsilon'[\times]0,\delta'[$. Then we set

$$I_2[\varepsilon,\delta]\equiv\int_Q\left(\int_{\partial\Omega}R^q_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx\,,$$

for all $(\varepsilon,\delta)\in\,]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$. Then by standard properties of integral operators with real analytic kernels (cf., *e.g.*,[25]), by the analyticity of the map Θ, and by standard calculus in Banach spaces, we deduce that I_2 is a real analytic map from $]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$ to $\mathbb{R}^n$. It now remains to consider $\int_Q\left(\int_{\partial\Omega}\Gamma_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx$ and we note that by the Fubini's Theorem we have

$$\int_Q\left(\int_{\partial\Omega}\Gamma_{n,\omega}(x-p-\varepsilon s)\Theta[\varepsilon,\delta](s)\,d\sigma_s\right)dx=\int_{\partial\Omega}\left(\int_Q\Gamma_{n,\omega}(x-p-\varepsilon s)\,dx\right)\Theta[\varepsilon,\delta](s)\,d\sigma_s\,,$$

for all $(\varepsilon,\delta)\in\,]0,\varepsilon'[\times]0,\delta'[$. We now observe that, since $\Gamma_{n,\omega}$ is even, we have

$$\int_Q\Gamma_{n,\omega}(x-p-\varepsilon s)\,dx=\int_Q\Gamma_{n,\omega}(p+\varepsilon s-x)\,dx\,,$$

for all $\varepsilon\in\,]0,\varepsilon'[$. We now set

$$\mathcal{P}(y)\equiv\int_Q\Gamma_{n,\omega}(y-x)\,dx\qquad\forall y\in Q\,.$$

By classical results on the regularity of the volume potential associated with the fundamental solution of the Lamé equations (cf., *e.g.*, Dalla Riva[26, §2.4]), we verify that $\mathcal{P}$ is an analytic function from Q to $\mathbb{R}^n$. By analyticity results on the composition operator in Schauder spaces (cf. Böhme and Tomi[27, p. 10], Henry[28, p. 29], Valent[29, Theorem 5.2, p. 44]), we have that the map which takes $\varepsilon\in\,]-\varepsilon',\varepsilon'[$ to the function

$$\int_Q\Gamma_{n,\omega}(p+\varepsilon s-x)\,dx\qquad s\in\overline{\Omega}$$

in $C^{0,\alpha}(\overline{\Omega},M_n(\mathbb{R}))$ is real analytic. Then, by the continuity of the restriction operator from $C^{0,\alpha}(\overline{\Omega},M_n(\mathbb{R}))$ to $C^{0,\alpha}(\partial\Omega,M_n(\mathbb{R}))$, by the analyticity of the map Θ, and by standard calculus in Banach spaces, we deduce that the map I_3 from $]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$ to $\mathbb{R}^n$ defined by

$$I_3[\varepsilon,\delta]\equiv\int_{\partial\Omega}\left(\int_Q\Gamma_{n,\omega}(x-p-\varepsilon s)\,dx\right)\Theta[\varepsilon,\delta](s)\,d\sigma_s\qquad\forall(\varepsilon,\delta)\in\,]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$$

is real analytic. Therefore, if we set

$$I^\sharp[\varepsilon,\delta]\equiv\Xi[\varepsilon,\delta]\Big(|Q|-\varepsilon^n|\Omega|\Big)+\delta\varepsilon^{n-1}\Big(-\varepsilon^2I_1[\varepsilon,\delta]+I_2[\varepsilon,\delta]+I_3[\varepsilon,\delta]\Big)$$

for all $(\varepsilon,\delta)\in\,]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$, by the analyticity of Ξ, I_1, I_2, and I_3, we easily deduce that $I^\sharp$ is a real analytic map from $]-\varepsilon',\varepsilon'[\times]-\delta',\delta'[$ to $\mathbb{R}^n$ such that equalities (26) and (27) hold. Hence the proof is complete. □

By its definition and by Theorem 5 it is easily seen that the solution $u^\sharp(\epsilon,\delta,\cdot)$ converges almost everywhere to $\tilde{\xi}$ as $(\epsilon,\delta)\to(0,0)$. However, it is meaningful asking whether $u^\sharp(\epsilon,\delta,\cdot)$ converges in some function space to $\tilde{\xi}$. Later, we will use this result to recover the corresponding convergence result for $u(\epsilon,\delta,\cdot)$. To investigate such a possibility, we need to introduce an extension operator. Let $\alpha\in\,]0,1[$, $m\in\mathbb{N}\setminus\{0\}$. Let Ω be as in assumption (2). Let $p\in Q$ and ϵ_0 be as in assumption (3). Let $(\epsilon,\delta)\in\,]0,\epsilon_0[\times]0,+\infty[$. If v is a function from $\overline{\mathbb{S}[\epsilon,\delta]^-}$ to $\mathbb{R}^n$, then we denote by $\mathbf{E}_{(\epsilon,\delta)}[v]$ the extension by zero of v, *i.e.* the function from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$\mathbf{E}_{(\epsilon,\delta)}[v](x)\equiv\begin{cases}v(x) & \forall x\in\overline{\mathbb{S}[\epsilon,\delta]^-},\\ 0 & \forall x\in\mathbb{R}^n\setminus\overline{\mathbb{S}[\epsilon,\delta]^-}.\end{cases}$$

Then we have the following convergence result, where, in contrast with the proof of[18, Proposition 7.1], we do not make use of the maximum principle.

Proposition 9. Let the assumptions of Theorem 5 hold. Let $\tilde{\xi},\epsilon',\delta',\Theta,\Xi$ be as in Theorem 5. Let $\{(\epsilon_j,\delta_j)\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon'[\times]0,\delta'[$ which converges to $(0,0)$. Let $r\in[1,+\infty[$. Then,

$$\lim_{j\to+\infty}\mathbf{E}_{(\epsilon_j,1)}[u^\sharp(\epsilon_j,\delta_j,\cdot)]=\tilde{\xi}\qquad\text{in } (L^r(Q))^n. \tag{28}$$

Proof. In order to prove the statement, we show that there exists a constant $M>0$ such that

$$\left|\mathbf{E}_{(\epsilon_j,1)}[u^\sharp(\epsilon_j,\delta_j,\cdot)](x)\right|\le M\qquad\forall x\in Q,\ \forall j\in\mathbb{N}. \tag{29}$$

Indeed, if (29) holds, then the pointwise convergence of $u^\sharp(\epsilon_j,\delta_j,\cdot)$ to $\tilde{\xi}$ and the dominated convergence theorem imply the validity of (28). Accordingly, we turn to prove (29). The definition of $u^\sharp(\epsilon_j,\delta_j,\cdot)$ implies that

$$\left|\mathbf{E}_{(\epsilon_j,1)}[u^\sharp(\epsilon_j,\delta_j,\cdot)](x)\right|\le\delta_j\epsilon_j^{n-1}\left|\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\epsilon_j s)\Theta[\epsilon_j,\delta_j](s)\,d\sigma_s\right|+\left|\Xi[\epsilon_j,\delta_j]\right|\qquad\forall x\in Q,\ \forall j\in\mathbb{N}.$$

Clearly, by the analyticity of Ξ, there exists $M_1>0$ such that $\left|\Xi[\epsilon_j,\delta_j]\right|\le M_1$ for all $j\in\mathbb{N}$. Next, we note that

$$\begin{aligned}\delta_j\epsilon_j^{n-1}\left|\int_{\partial\Omega}\Gamma^q_{n,\omega}(x-p-\epsilon_j s)\Theta[\epsilon_j,\delta_j](s)\,d\sigma_s\right|\le{}&\delta_j\epsilon_j^{n-1}\left|\int_{\partial\Omega}\Gamma_{n,\omega}(x-p-\epsilon_j s)\Theta[\epsilon_j,\delta_j](s)\,d\sigma_s\right|\\&+\delta_j\epsilon_j^{n-1}\left|\int_{\partial\Omega}R^q_{n,\omega}(x-p-\epsilon_j s)\Theta[\epsilon_j,\delta_j](s)\,d\sigma_s\right|\qquad\forall x\in Q,\ \forall j\in\mathbb{N}.\end{aligned} \tag{30}$$

We consider the first term in the right hand side of (30). Since Θ is analytic, since $\Theta[\epsilon_j,\delta_j]\in C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0$ for all $j\in\mathbb{N}$ and by the mapping properties of the elastic single layer potential (cf., *e.g.*, Dalla Riva and Lanza de Cristoforis[14, Theorem A.2] and Dalla Riva[26, §2.1.2]), there exists $M_2>0$ such that

$$\begin{aligned}\delta_j\epsilon_j^{n-1}\left|\int_{\partial\Omega}\Gamma_{n,\omega}(x-p-\epsilon_j s)\Theta[\epsilon_j,\delta_j](s)\,d\sigma_s\right|&=\delta_j\epsilon_j^{n-1}\left|\int_{\partial\Omega}\Gamma_{n,\omega}\left(\epsilon_j\left(\frac{x-p}{\epsilon_j}-s\right)\right)\Theta[\epsilon_j,\delta_j](s)\,d\sigma_s\right|\\&=\delta_j\epsilon_j\left|\mathrm{v}[\omega,\Theta[\epsilon_j,\delta_j]]\left(\frac{x-p}{\epsilon_j}\right)\right|\\&\le M_2\delta_j\epsilon_j\qquad\forall x\in Q,\ \forall j\in\mathbb{N}.\end{aligned}$$

Finally, since $R^q_{n,\omega}$ is non-singular around zero, there exists $M_4>0$ such that

$$\delta_j\epsilon_j^{n-1}\left|\int_{\partial\Omega}R^q_{n,\omega}(x-p-\epsilon_j s)\Theta[\epsilon_j,\delta_j](s)\,d\sigma_s\right|\le M_4\qquad\forall x\in Q,\ \forall j\in\mathbb{N}.$$

Thus (29) holds and the statement follows. □

5 | BEHAVIOR OF THE INTEGRAL FUNCTIONALS FOR THE SOLUTION OF THE ORIGINAL PROBLEM

In this section we exploit what we have proved for the auxiliary problem (10) in order to analyze the behavior of the family of solutions and the integral functionals of problem (4). First, we show the validity of the following convergence theorem in Lebesgue spaces.

Proposition 10. Let $\omega \in]1-\frac{2}{n}, +\infty[$, $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be as in assumption (2). Let $p \in Q$ and ϵ_0 be as in assumption (3). Let G be as in assumption (6) and assume that (16) holds. Let $\tilde{\xi} \in \mathbb{R}^n$ and let assumption (17) hold. Let ϵ', δ' be as in Theorem 5. Let $\{(\epsilon_j, \delta_j)\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon'[\times]0,\delta'[$ which converges to $(0,0)$. Let $r \in [1,+\infty[$. Then

$$\lim_{j\to\infty} \mathbf{E}_{(\epsilon_j,\delta_j)}[u(\epsilon_j,\delta_j,\cdot)] = \tilde{\xi} \qquad \text{in } (L^r(V))^n.$$

for all bounded open subsets V of $\mathbb{R}^n$.

Proof. By [18, Lemma A.6], there exists $C \in]0,+\infty[$ such that

$$\begin{aligned}\|\mathbf{E}_{(\epsilon_j,\delta_j)}[u(\epsilon_j,\delta_j,\cdot)] - \tilde{\xi}\|_{(L^r(V))^n} &= \|\mathbf{E}_{(\epsilon_j,\delta_j)}[u^\sharp(\epsilon_j,\delta_j,\cdot/\delta)] - \tilde{\xi}\|_{(L^r(V))^n} \\ &= \|\mathbf{E}_{(\epsilon_j,1)}[u^\sharp(\epsilon_j,\delta_j,\cdot)](\cdot/\delta) - \tilde{\xi}\|_{(L^r(V))^n} \\ &\leq C\|\mathbf{E}_{(\epsilon_j,1)}[u^\sharp(\epsilon_j,\delta_j,\cdot)] - \tilde{\xi}\|_{(L^r(Q))^n} \quad \forall j \in \mathbb{N}.\end{aligned}$$

Then the statement follows by Theorem 9. □

We now recall that the integral functionals associated with the family of solutions $\{u(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ defined in (18) that we will consider are the map from $]0,\epsilon'[\times]0,\delta'[$ to $\mathbb{R}$ defined by

$$(\epsilon,\delta) \mapsto \mathrm{En}(\omega, u(\epsilon,\delta,\cdot)) \equiv \frac{1}{2}\int_{Q\cap\mathbb{S}[\epsilon,\delta]^-} \mathrm{tr}(\mathcal{T}(\omega, D_x u(\epsilon,\delta,x))(D_x u(\epsilon,\delta,x))^T)\,dx$$

and the map from $]0,\epsilon'[\times]0,\delta'[$ to $\mathbb{R}^n$ defined by

$$(\epsilon,\delta) \mapsto \mathrm{Int}(u(\epsilon,\delta,\cdot)) \equiv \int_{Q\cap\mathbb{S}[\epsilon,\delta]^-} u(\epsilon,\delta,x)\,dx.$$

We start by analyzing the behavior of $\mathrm{En}(\omega, u(\epsilon,\delta,\cdot))$ in the cell Q as (ϵ,δ) approaches $(0,0)$. In the following theorem, we represent $\mathrm{En}(\omega, u(\epsilon,\delta,\cdot))$ in terms of analytic maps evaluated at (ϵ,δ) when δ equals the reciprocal of some integer number $l \in \mathbb{N}\setminus\{0\}$.

Theorem 11. Let $\omega \in]1-\frac{2}{n}, +\infty[$, $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be as in assumption (2). Let $p \in Q$ and ϵ_0 be as in assumption (3). Let G be as in assumption (6) and assume that (16) holds. Let $\tilde{\xi} \in \mathbb{R}^n$ and let assumption (17) hold. Let δ', ϵ_e, $E^\sharp$ be as in Theorem 7. Then

$$\mathrm{En}(\omega, u(\epsilon, l^{-1},\cdot)) = \epsilon^n E^\sharp[\epsilon, l^{-1}]$$

for all $\epsilon \in]0,\epsilon_e[$ and $l \in \mathbb{N}\setminus\{0\}$ such that $l > 1/\delta'$.

Proof. We first note that if $(\epsilon,\delta) \in]0,\epsilon_e[\times]0,\delta'[$, then

$$\begin{aligned}\frac{1}{2}\int_{\delta(Q\setminus\overline{\Omega_{p,\epsilon}})} \mathrm{tr}(\mathcal{T}(\omega, D_x u(\epsilon,\delta,x))(D_x u(\epsilon,\delta,x))^T)\,dx &= \frac{1}{2}\delta^{-2}\int_{\delta(Q\setminus\overline{\Omega_{p,\epsilon}})} \mathrm{tr}\left(\mathcal{T}(\omega, D_x u^\sharp\left(\epsilon,\delta,\frac{x}{\delta}\right))(D_x u^\sharp\left(\epsilon,\delta,\frac{x}{\delta}\right)^T\right)\,dx \\ &= \frac{1}{2}\delta^{n-2}\int_{Q\setminus\overline{\Omega_{p,\epsilon}}} \mathrm{tr}(\mathcal{T}(\omega, D_x u^\sharp(\epsilon,\delta,y))(D_x u^\sharp(\epsilon,\delta,y))^T)\,dy.\end{aligned} \tag{31}$$

Next we note that if $\delta = l^{-1}$, then

$$Q\cap\mathbb{S}[\epsilon,l^{-1}]^- = Q\cap\left(\mathbb{R}^n \setminus \bigcup_{z\in\mathbb{Z}^n} l^{-1}(qz+\overline{\Omega_{p,\epsilon}})\right),$$

and the set in the right hand side differs by a set of measure zero from the set

$$Q \cap \left(\bigcup_{z \in \mathbb{Z}^n} l^{-1}(qz + (Q \setminus \overline{\Omega_{p,\epsilon}})) \right) = \bigcup_{\substack{z \in \mathbb{Z}^n \\ 0 \le z_j \le l-1}} (l^{-1}qz + l^{-1}(Q \setminus \overline{\Omega_{p,\epsilon}})),$$

which is the union of a family of l^n sets, all of which are translations of $l^{-1}(Q \setminus \overline{\Omega_{p,\epsilon}})$. Hence, formula (31) implies that

$$\begin{aligned} \frac{1}{2} \int_{Q \cap \mathbb{S}[\epsilon, l^{-1}]^-} \mathrm{tr}(\mathcal{T}(\omega, D_x u(\epsilon, l^{-1}, x))(D_x u(\epsilon, l^{-1}, x))^T)\, dx &= l^n \frac{1}{2} \int_{l^{-1}(Q \setminus \overline{\Omega_{p,\epsilon}})} \mathrm{tr}(\mathcal{T}(\omega, D_x u(\epsilon, l^{-1}, x))(D_x u(\epsilon, l^{-1}, x))^T)\, dx \\ &= \frac{l^2}{2} \int_{Q \setminus \overline{\Omega_{p,\epsilon}}} \mathrm{tr}(\mathcal{T}(\omega, D_x u^\sharp(\epsilon, l^{-1}, y))(D_x u^\sharp(\epsilon, l^{-1}, y))^T)\, dy \end{aligned}$$

for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > 1/\delta'$. Then, Theorem 7 implies that

$$\begin{aligned} \mathrm{En}(\omega, u(\epsilon, l^{-1}, \cdot)) &= \frac{l^2}{2} \int_{Q \setminus \overline{\Omega_{p,\epsilon}}} \mathrm{tr}(\mathcal{T}(\omega, D_x u^\sharp(\epsilon, l^{-1}, y))(D_x u^\sharp(\epsilon, l^{-1}, y))^T)\, dy \\ &= l^2 (l^{-2} \epsilon^n E^\sharp[\epsilon, l^{-1}]) = \epsilon^n E^\sharp[\epsilon, l^{-1}] \end{aligned}$$

for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > 1/\delta'$ and for all $\epsilon \in]0, \epsilon_e[$. □

Next, we want to show an asymptotic estimate for the energy integral which holds not only for $\delta = l^{-1}$ but for any small δ. If x is a real number, we denote by $[x]$ its integer part. Moreover we set $[x]^- \equiv [x]$ if $x \in \mathbb{R} \setminus \mathbb{Z}$ and $[x]^- \equiv [x] - 1$ if $x \in \mathbb{Z}$. By a straightforward modification of the proof of[19, Proposition 5.2], we deduce the validity of the following.

Proposition 12. Let $\omega \in]1 - \frac{2}{n}, +\infty[$, $\alpha \in]0, 1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be as in assumption (2). Let $p \in Q$ and ϵ_0 be as in assumption (3). Let G be as in assumption (6) and assume that (16) holds. Let $\tilde{\xi} \in \mathbb{R}^n$ and let assumption (17) hold. Let δ', ϵ_e, $E^\sharp$ be as in Theorem 7. Then

$$[\delta^{-1}]^n \delta^n (\epsilon^n E^\sharp[\epsilon, \delta]) \le \mathrm{En}(\omega, u(\epsilon, \delta, \cdot)) \le ([\delta^{-1}]^- + 1)^n \delta^n (\epsilon^n E^\sharp[\epsilon, \delta])$$

for all $(\epsilon, \delta) \in]0, \epsilon_e[\times]0, \delta'[$. Moreover,

$$\mathrm{En}(\omega, u(\epsilon, \delta, \cdot)) \sim \epsilon^n E^\sharp[\epsilon, \delta] \quad \text{as } (\epsilon, \delta) \to (0, 0).$$

Finally, we consider the behavior of $\mathrm{Int}(u(\epsilon, \delta, \cdot))$ as (ϵ, δ) approaches $(0, 0)$ and we represent $\mathrm{Int}(u(\epsilon, \delta, \cdot))$ in terms of an analytic map when δ is equal to the inverse of a positive natural number.

Theorem 13. Let $\omega \in]1 - \frac{2}{n}, +\infty[$, $\alpha \in]0, 1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be as in assumption (2). Let $p \in Q$ and ϵ_0 be as in assumption (3). Let G be as in assumption (6) and assume that (16) holds. Let $\tilde{\xi} \in \mathbb{R}^n$ and let assumption (17) hold. Let ϵ', δ' be as in Theorem 5. Let $I^\sharp$ be as in Theorem 8. Then

$$\mathrm{Int}(u(\epsilon, l^{-1}, \cdot)) = I^\sharp[\epsilon, l^{-1}],$$

for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > 1/\delta'$ and for all $\epsilon \in]0, \epsilon'[$.

Proof. Let $l \in \mathbb{N} \setminus \{0\}$ be such that $l > 1/\delta'$. Since $u(\epsilon, l^{-1}, \cdot)$ is $l^{-1}q$-periodic, it is also q-periodic. Next we observe that

$$\bigcup_{0 \le z_j \le l-1} (qz + l^{-1}Q) \subseteq Q, \qquad \left| Q \setminus \bigcup_{0 \le z_j \le l-1} (qz + l^{-1}Q) \right| = 0.$$

Accordingly, the $l^{-1}q$-periodicity of $u(\epsilon, l^{-1}, \cdot)$ implies that

$$\mathrm{Int}(u(\epsilon, l^{-1}, \cdot)) = l^n \int_{l^{-1}(Q \setminus \overline{\Omega_{p,\epsilon}})} u(\epsilon, l^{-1}, x)\, dx = l^n \int_{l^{-1}(Q \setminus \overline{\Omega_{p,\epsilon}})} u^\sharp(\epsilon, l^{-1}, lx)\, dx = \int_{Q \setminus \overline{\Omega_{p,\epsilon}}} u^\sharp(\epsilon, l^{-1}, y)\, dy = I^\sharp[\epsilon, l^{-1}],$$

for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > 1/\delta'$ and for all $\epsilon \in]0, \epsilon'[$. Hence, the statement holds true. □

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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