EFT Strings, Quantum Gravity Bounds and Wormholes

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Abstract

The subject of Quantum Gravity and its effects on physical observables and effective field theories remains one of the biggest challenges in theoretical physics. String Theory provides a controlled framework in which we can hope to gather some evidence of general features of Quantum Gravity, even if its large landscape of vacua seems to represent an obstacle to the ability of formulating universal statements. The Swampland Program has emerged to try and address such questions: by using consistency conditions coming from general considerations about Quantum Gravity we can distinguish between effective field theories of gravity that admit a UV completion and those that do not. In the first part of this thesis, we study the conditions coming from the coupling of $N = 1$ effective theories of gravity in 4 dimensions to a class of 2-dimensional extended object called EFT strings and derive non-trivial constraints on the possible values of the (s)axionic couplings in the effective Lagrangian, in addition to upper bounds on the relevant gauge group of the theory. In the second part, we study the problem of the possible non-perturbative corrections from wormhole solutions and their relevance in the light of the previously derived consistency bounds. In doing so, we stress the interplay between EFT strings and the energy scale of validity of such solutions, represented by the species scale.

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Chapter 1

Introduction

Despite having being object of thorough research for the past 80 years, the topic of Quantum Gravity remains one of the biggest challenges in modern Theoretical Physics. The questions surrounding the topic are both conceptual and phenomenological in nature. From the conceptual point of view, it would be interesting to understand the necessary conditions that an effective quantum field theory coupled to gravity, i.e. General Relativity, has to satisfy in order to admit a consistent embedding in a UV complete theory of Quantum Gravity. From the phenomenological point of view, it would be thrilling if the very fact of coming from a fully complete theory of Quantum Gravity implied the presence of some universal features imprinted into the effective theory which could be observed experimentally in the near future, bypassing the well-known difficulties of probing directly Planckian scales for signatures of quantum behavior of gravity. The Swampland Program [1] began exactly to try and address those questions (excellent reviews can be found in [2], [3], [4]). This line of research enjoyed a significant growth in the last decade and resulted in a large and thriving field of study which touches manifold branches of String Theory, Theoretical Physics and even pure Mathematics. The typical modus operandi of research in Swampland is represented by trying and formulating general consistency conditions coming from aspects of Quantum Gravity which are expected to be universal in all its low energy effective realizations, then putting those under severe tests and checks in explicit settings of effective theories of Quantum Gravity. The conditions take the form of Swampland Conjectures for which evidence can be gathered, counterexamples found or, in very specific scenarios, even formal proof be accomplished. The hope is that these conditions may help us in distinguishing the effective field theories which can be consistently uplifted to full quantum theories coupled to gravity, colloquially those belonging to the "Landscape" of Quantum Gravity, from those which cannot, populating the so-called "Swampland". It is not surprising that for this purpose considerations about semiclassical gravity, black hole physics and holography are crucial in order to identify putative general features shared by all the effective models of Quantum Gravity, in addition to requirements of unitarity and low energy causality common also in the framework of ordinary Quantum Field Theory. In all of this, String Theory plays a central role in being a perturbatively controlled theoretical playground from which we can both take inspiration in formulating the consistency conditions, thanks to universal patterns observed in the large set of String Theory low energy realizations, and providing concrete examples upon which to test the various Swampland Conjectures. Among the most important Swampland Conjectures enjoying a large amount of evidence we must recall the absence of global symmetries [5–11] (and the closely related triviality of the Cobordism group of Quantum Gravity [12, 13]), the Weak Gravity Conjecture [14–20], the Completeness of the spectrum of charged states under gauge symmetries [21] and the Distance Conjecture [22]. A wide sector of the research in the Swampland Program is also focused on some less understood conjectures, among which we can recall the instability of non-supersymmetric AdS vacua [23], the absence of scale separation [24] and the dS/TransPlanckian Censorship Conjecture [25–27]. Those enjoy a less or more prescriptive degree of evidence but are nevertheless interesting subjects on their own and capable, in theory, to give greater insights into phenomenological imprints of Quantum Gravity effects in realistic effective models. In recent years, a greater and greater attention has been dedicated by the Swampland community to a more phenomenological direction. Indeed, the second soul of the Program is the hope that, by understanding the general features of quantum systems coupled to gravity, we may predict their universal empirical signatures which should be observable also in our Universe, allowing for indirect experimental detection of Quantum Gravity consequences without relying on Planckian scale center of mass energies in particle accelerators. It is promising for this hope that one of the traits of Quantum Gravity seems to be the mixing of UV with IR energy scales, leading to possible upper bounds on the scale of new Physics beyond the Standard Model and even putative explanations for the hierarchy problem of particle physics. Recent developments in this direction are represented for example by the so-called Dark Dimension Scenario, for which relevant references can be found in [28–31], by attempts to constrain the parameter space of new Physics based on Swampland considerations [32] and by new approaches to cosmological models informed by the same principles [33].

As the Swampland Program follows this twofold path, so this thesis too is organised in a twofold way, based upon two works which can be seen as representing each one of the two souls of the Swampland research line. The *fil rouge* connecting those two works is the tool which we chose to study our problems: a peculiar class of axionic string-like extended objects dubbed *EFT strings*. These are a privileged and interesting set of BPS objects because they cannot be resolved by the field theory, hence they are natural candidates to probe Quantum Gravity aspects of the theories, but at the same time their presence does not spoil the perturbative regime in which we would like to remain in order to maintain a certain degree of control over our calculations. This occurs because EFT strings carry axionic charges such that the non-perturbative effects are highly suppressed along the flow towards their core. Such a flow can also be interpreted as heading towards an infinite distance point in moduli space, opening possibilities of studying topics related to the Swampland Distance Conjecture in these settings. In addition, it was conjectured in [34] that this correspondence between infinite distance limits and EFT strings goes both ways, with recently [35] exploring the connections to the Tameness Conjecture and the intriguing possibility of its use to completely classify such limits.

In Chapter 3 our main focus will be a more formal one and the question which we would like to answer is the following: which consistency conditions can be derived from the very coupling of these objects to four-dimensional effective theories of gravity with minimal supersymmetry? The technique of extended objects as probes for the consistency of a gravity background is a well-known and fruitful one which has already been exploited in several other cases in higher dimensions and/or with higher number of supercharges. Among the many results of this method we cannot forget to mention the proof of the uniqueness of the set of consistent ten-dimensional heterotic supergravity [36], the constraints to six-dimensional supergravity with abelian gauge factors [37], the upper bound of the rank of the gauge group for theories with 16 supercharges in various dimensions [38] and the bounds on effective five-dimensional supergravity theories [39]. The arguments for the correctness of this procedure come both from top-down and bottom-up considerations. From the top-down point of view, we know String Theory provides for a plethora of extended objects which have to be consistently coupled to the effective gravitational backgrounds for the whole framework to make sense. But, even ignoring such top-down motivations, the presence of charged extended objects is to be expected in a theory capable of consistent uplift to Quantum Gravity on the ground of the Swampland condition on Completeness of the spectrum. This is one of the most solid Swampland Conjectures whose relationships with the absence of global symmetries (both invertible and non-invertible ones), the triviality of the Cobordism group of Quantum Gravity and the Weak Gravity Conjecture have been appreciated and explored thoroughly in the literature [21]. Therefore, being the absence of global symmetries or equivalently the Cobordism Conjecture deeply rooted in the holographic principle and hence ultimately in the background independence which Quantum Gravity observables should display, we will be more than willing to take this assumption as verified and analyse its consequences in the cases under our inquiry.

As we stated above, the setting for our exploration will be given by four-dimensional effective theories of supergravity with minimal supersymmetry, i.e. four supercharges, for which a basic introduction will be given in chapter 2 together with the definition of EFT string. Those represent the most realistic models for which we retain an acceptable degree of control thanks to the minimal supersymmetry. They also bring with them a series of technical complications unknown to the cases in higher dimensions or with more supersymmetries. Just to begin, since string-like objects in four dimensions have codimension equal to two, their role has weakly coupled probes of the gravity background might be in jeopardy. Furthermore, we can no longer ignore the effects of those higher derivative corrections to the effective action which are identically vanishing in examples with larger supersymmetry. This adds up to the fact that already at the 2-derivative level the constraints imposed by less supersymmetry are clearly weaker, for example the kinetic term of the chiral multiplets is controlled by a non-holomorphic potential. EFT strings reveal themselves as the suitable candidates to address the former issue: even if in general we can no longer apply a conformal fixed point argument for their worldsheet theory, we will provide evidence that they can be described by a weakly coupled non-linear sigma model in the perturbative asymptotic regime in which we are interested. In such limits, an approximate axionic symmetry emerges, signalling their infinite distance nature, and the presence of the string can be detected by the gauge and gravitational sector of the effective theory through standard (s)axionic couplings. The form of those couplings can be seen as a working assumption, being them the minimal ones, but it was also argued that they might be necessary for the purpose of gauging otherwise global -1-form symmetries in the setting [40]. The mechanism with which those couplings can enforce consistency conditions is the anomaly inflow. The mere presence of the string implies a tree level anomaly localised completely on its worldsheet and hence, being it associated with local symmetries of the bulk effective theory, it must be cancelled by an appropriate 1-loop anomaly generated by chiral matter living on the worldsheet itself. Thanks to the residual supersymmetry on the string, and modulo some reasonable assumptions, we were able to derive from this cancellation condition non-trivial constraints for the parameters of the effective action, in particular discreteness and positive bounds for (s)axionic coupling constants which confirm and extend already known results on the topic. In addition, upper bounds on the gauge group of the gauge sector coupled to the string can be calculated, all in agreement or strengthening expected conclusions.

After the derivation of these new consistency conditions, we proceeded to their extensive check in various explicit String Theory constructions. Those include type IIB/F-theory on smooth bases, $SO(32)$ and $E_8 \times E_8$ heterotic backgrounds and M-theory on G_2 manifolds. In all cases we found perfect agreement and often these tests allowed for new unexpected insights. For example, in the type IIB/F-theory setting with $O3$-planes and no $O7$-planes we were able to obtain from our physical considerations a known mathematical theorem about the number of $O3$-planes. Also the other type IIB/F-theory models unravelled a surprising interplay between the geometrical nature of the chiral matter on the worldsheet of the EFT strings and our bounds, showing the conditions under which a stronger version of our results may apply. The heterotic compactifications on the other hand provided a rich playground in which the importance of each subtle contribution and correction was manifest for arriving at satisfying our constraints correctly.

This work also opened a lot of unexplored avenues which are worthy of future investigation. The sector of complex structure moduli was left mainly unchecked. Similarly the questions of what happens to our conclusions when the (s)axions gain mass and/or supersymmetry is completely broken, even if extremely interesting on their own, have not been faced. We hope to be able to delve into these and related topics in future work.

In Chapter 4 we depart partially from the topic of constraints on the Quantum Gravity Landscape to enter the realm of the Swampland Program more adjacent to a phenomenological direction. It is well known that String Theory compactifications often lead to effective field theories boasting a large number of axion fields. This is due to the large number of real moduli fields, or equivalently the large Hodge numbers of the chosen compact manifold, which have axionic fields as partners in complex supersymmetric multiplets. The addition of an axionic field to the matter content of the Standard Model has been explored at long in the phenomenological literature as a possible solution to the so-called strong CP problem of strong interactions [41, 42]. In addition, it has also been subject of thorough study for its role in inflationary models. At a first glance, it would seem that a huge number of axions, when historically experiments have not found empirical evidence of any of them so far, is phenomenologically not viable or at least not favored. Nevertheless phenomenological models with many axions after moduli stabilization, the so-called Axiverse Scenario [43], have seen a rising amount of interest recently, both from the phenomenological and the formal community of Theoretical Physics. Among the possible issues of this scenario there is what is known as the Peccei-Quinn quality problem, a result of the presence of multiple interacting axions and instantons which can easily spoil the nice features of the PQ mechanism. Quite surprisingly, recent works seem to suggest that this is not the case in a large class of String Theory compactifications, with the quality problem becoming less and less relevant as the number of species increase thanks to the decreasing of magnitude of non-perturbative corrections at the same time [44–46]. Intrigued by this results, we undertook a careful study of gravitational non-perturbative effects in Axiverse-like string-inspired models.

We focused our effort mainly on (multi)saxionic Euclidean wormholes in four dimensions and the non-perturbative corrections which they can generate in the effective action. In particular we described a simple but wide class of homogeneous multi-saxionic wormholes supported by axionic charges being dual to the ones which characterize EFT strings. For such solutions, which carry intrinsically a length scale in them in the form of the parameter governing the size of the throat, considerations about control are entwined with those about the energy scale where gravity becomes strongly coupled. It is natural then that this exploration led us to delving into the notion of species scale [47]. We dedicated some time to elucidate the connection between the species scale and the energy scale defined by the tension of the EFT strings in their asymptotic limits, providing examples from explicit String Theory constructions coming from both type IIB/F-theory and heterotic models, and to discuss its role in the suppression of the non-perturbative corrections generated by on-shell higher derivative terms, in particular the Gauss-Bonnet contribution. In particular we argued for the species scale being always equal to or bounded from above by the energy scale associated to the EFT string tensions, the latter on the ground that such tension represents either that of a fundamental string or that of an extended object whose excitations cannot be quantized within the ordinary framework of perturbative String Theory.

To enlarge the ensemble of evidence, we did not limit ourselves to this special class of solutions, but we also performed numerical scans for finding similar wormholes in the neighbourhood of the analytically known homogeneous ones. Whenever we were successful in finding them numerically, their features confirmed a series of expectations which we were able to formulate through perturbative expansion around the homogeneous wormholes. Another numerical analysis was instrumental in gathering evidence in support of our arguments about the relationship between the species scale and the EFT string tensions. For such a purpose, another numerical scan was carried out over the Kreuzer-Skarke database thanks to the CYTools package [48] and the results were all in agreement with our with our proposals.

Finally, we explored the limit case of extremal/marginally degenerate wormholes and their link to extremal BPS instantons. We supported the idea that such configurations might represent "ensembles" of extremal instantons due to the analogous corrections they give to the effective action. We concluded with the remarks on such corrections and the physical consequences which they imply to a large set of effective field theories with minimal supersymmetry.

1.1 About this Thesis

This is the thesis of *Nicolò Risso*, submitted as part of the requirements for the degree of PhD in Physics at the Dipartimento di Fisica e Astronomia "Galileo Galilei", University of Padova, Italy.

This thesis is based upon the content of the following publications:

- L. Martucci, N. Risso, and T. Weigand, *"Quantum Gravity Bounds on N=1 Effective Theories in Four Dimensions"*, JHEP 03 (2023) 197, arXiv:2210.10797 [hep-th];
- L. Martucci, N. Risso, A. Valenti and L. Vecchi, *"Wormholes in the axiverse and the species scale"*, To appear soon.

During my PhD, I also contributed to the following papers:

- M.B. Fröb, C. Imbimbo and N. Risso, *"Deformations of supergravity and supersymmetry anomalies"*, JHEP 12 (2021) 009, arXiv:2107.03401 [hep-th];
- I. Basile and N. Risso, *"Runaway behavior in scale separated Casimir vacua"*, To appear.

However, we have eventually chosen not to cover such works in the content of this thesis.

1.2 Chapter List

The content of this thesis is organised as follows:

Chapter 1 Introduction. Here we present a general introduction to the motivations behind and the content of this thesis.

Chapter 2 The Setting: EFTs and EFT strings. Here we present a brief review of the fundamentals of the general setting of our study, that is $N = 1$ four dimensional effective theories of gravity, and of the class of extended object which we call EFT strings.

Chapter 3 Quantum Gravity Bounds from EFT Strings. Here we present the results of our study of consistency conditions coming from the coupling of EFT strings with effective minimal supergravity models in four dimensions. We include both the derivation of the bounds, the discussion of the subtleties present in such derivation and their check on various explicit String Theory constructions.

Chapter 4 Wormholes in the axiverse and the Species Scale. Here we present the results of our study of Euclidean wormhole contributions to String Theory inspired effective models. We include an introduction to and the derivation of the wormhole solutions in which we are interested, the discussion about the conditions for their existence and the relevant energy scales of the problem, together with their links with the formalism of EFT string limits, and the numerical evidence supporting our statements.

Chapter 5 Conclusion. Here the conclusions of this thesis are presented and summarised.

Chapter 2

The Setting: EFTs and EFT strings

In this beginning chapter we will briefly recall the main relevant features of the setting that we have chosen in our study, that is $N = 1$ supergravity theories in 4 dimensions, and of the class of extended objects with codimension 2, called EFT strings, that are suitable to our purposes. In particular, we will focus on the (s)axionic couplings in the theory to the gauge and higher derivative gravitational sectors, in addition to provide motivations for why the EFT strings represent a good candidate of probes for the consistency of effective theories of gravity. The reasons for having chosen such framework are twofold: first, because the study of extended objects as probes of the consistency of supergravity theories has proven fruitful in cases with more dimensions and more supersymmetries in order to constrain, and sometimes get all, the set of allowed effective theories (look for example at [36], [39] and [49]), it is then natural to ask whether it is possible to extend such methods to other less symmetrical situation; and second because it is a setting that is close to a realistic empirical model, with the remaining unbroken supersymmetry to have at least a certain degree of control. A pedagogical review of supergravity can be found in the evergreen [50].

2.1 Perturbative bulk EFT structure

In the standard Wilsonian interpretation, any effective field theory (EFT) is associated with a given ultraviolet (UV) cut-off energy scale Λ. We will consider four-dimensional EFTs which preserve minimal $\mathcal{N} = 1$ supersymmetry for a sufficiently high cut-off energy scale Λ. This minimal supersymmetry may be spontaneously broken at lower energy scales $\Lambda_{\rm SB} \ll \Lambda$, but this will not affect our conclusions as these regard the structure of the EFT defined at the scale Λ.

We will be particularly interested in extracting some general constraints on the (massless) EFT gauge sector that is weakly coupled at the UV cut-off scale Λ. The gauge couplings will be regarded as determined by the vacuum expectation values (VEVs) of the scalar fields. This is common in string theory realizations and also expected from more general quantum gravity principles, which forbid freely tunable parameters. It is then natural to associate any weakly-coupled gauge sector with a certain region of the field space which identifies a given perturbative EFT regime. In the sequel we will confirm this expectation and make it more precise.

In order to identify the possible perturbative EFT regimes, we will adopt the general prescription provided in [34, 51], which was proposed to be valid for any four-dimensional $\mathcal{N}=1$ EFT consistent with quantum gravity and tested in large classes of string theory models. Some of its key ingredients will be reviewed below. Combined with additional quantum gravity criteria, this framework will allow us to extract non-trivial information on the gauge sector and on some higher curvature terms.

2.1.1 Gauge sector

Following [34, 51, 52], the perturbative regime of an $\mathcal{N}=1$ supersymmetric EFT in four dimensions is characterized by the presence of a set of axions a^i, with fixed periodicity

$$a^i \simeq a^i + 1\,, \tag{2.1.1}$$

and a corresponding set of saxions s^i. These pair up into a set of complex scalar fields

$$t^i = a^i + \mathrm{i} s^i \tag{2.1.2}$$

forming the bosonic components of $\mathcal{N}=1$ chiral superfields. Together with additional chiral fields ϕ^α, they parametrize the Kähler field space $\mathcal{M}$ of the EFT.

The saxions in particular determine the exponential suppression factors of the BPS instantons in the theory. In a more precise definition, a perturbative EFT regime is associated with a set $\mathcal{C}_{\rm I}$ of non-vanishing BPS instanton charge vectors $\mathbf{m}=\{m_i\}$, and a *saxionic cone*

$$\Delta = \{\boldsymbol{s} \in \mathbb{R}^{\#\text{axions}} | \langle \mathbf{m}, \boldsymbol{s}\rangle > 0\,, \forall \mathbf{m} \in \mathcal{C}_{\rm I}\}\,. \tag{2.1.3}$$

Here

$$\langle \mathbf{m}, \boldsymbol{s}\rangle \equiv m_i s^i \tag{2.1.4}$$

is the natural pairing between instanton charges and saxions. In terms of this pairing, a BPS instanton with charge vector $\mathbf{m}\in\mathcal{C}_{\rm I}$ is suppressed as $|e^{-2\pi\langle \mathbf{m}, \boldsymbol{s}\rangle}|$.

We then say that the EFT is in the perturbative regime associated with $\mathcal{C}_{\rm I}$, or equivalently Δ, if the saxions lie sufficiently deep inside the saxionic cone Δ. In this regime the axionic shift symmetries are broken only by exponentially suppressed non-perturbative corrections dominated by the BPS instantons, which have the form $\sim |e^{2\pi \mathrm{i}\langle \mathbf{m}, \boldsymbol{t}\rangle}| = |e^{-2\pi\langle \mathbf{m}, \boldsymbol{s}\rangle}| \ll 1$ (with $\langle \mathbf{m}, \boldsymbol{t}\rangle \equiv m_i t^i$). This identifies the perturbative regime with a field space region $\mathcal{M}^{\Delta}_{\rm pert}$. Note that for increasing saxionic values inside the saxionic cone, the axionic shift symmetries are better and better preserved. The expected absence of (non-accidental) global symmetries in quantum gravity implies that $\mathcal{M}^{\Delta}_{\text{pert}}$ can be identified with the neighborhood of a field space boundary component $\partial\mathcal{M}^{\Delta}_{\infty} \subset \partial\mathcal{M}$ which is at infinite distance. The by now well-tested Swampland Distance Conjecture (SDC) [22] implies that one cannot really reach $\partial\mathcal{M}^{\Delta}_{\infty}$ within the four-dimensional EFT because of the appearance of infinite towers of new microscopic states which become light exponentially fast in the field distance. This causes the EFT to break down as soon as the corresponding tower mass scale m_* becomes smaller than the cutoff Λ. As we will recall in the sequel, [34, 51] identify a physically distinguished way to reach the infinite distance points of $\partial\mathcal{M}^{\Delta}_{\infty}$ and to realize the SDC.

Consider now the gauge theory sector, with gauge group

$$G = \prod_A U(1)_A \times \prod_I G_I \,, \tag{2.1.5}$$

where G_I denote simple group factors. In the superspace conventions of [50], the associated terms in the two-derivative effective action are

$$\frac{1}{8\pi \mathrm{i}} \int \mathrm{d}^4x \mathrm{d}^2\theta \, 2\mathcal{E} \, f^{AB}(t,\phi)\mathcal{W}_A\mathcal{W}_B + \frac{1}{16\pi \mathrm{i}} \int \mathrm{d}^4x \mathrm{d}^2\theta \, 2\mathcal{E} \, f^I(t,\phi) \, \mathrm{tr}(\mathcal{W}\mathcal{W})_I + \text{c.c.} \,, \tag{2.1.6}$$

which includes the bosonic terms

$$-\frac{1}{4\pi} \int \left(\mathrm{Im} \, f^{AB} F_A \wedge *F_B + \mathrm{Re} \, f^{AB} F_A \wedge F_B\right) - \frac{1}{8\pi} \int \left[\mathrm{Im} \, f^I \, \mathrm{tr}(F \wedge *F)_I + \mathrm{Re} \, f^I \, \mathrm{tr}(F \wedge F)_I\right] \,. \tag{2.1.7}$$

Here $f^{AB}(t,\phi)$ and $f^I(t,\phi)$ are holomorphic functions and we denote chiral superfields and their bottom scalar components by the same symbols. In (2.1.7), the trace tr on the algebra $\mathfrak{g}$ of a simple group G is defined by $\mathrm{tr} \equiv \frac{2}{\ell(\mathbf{r})}\mathrm{tr}_{\mathbf{r}}$, where $\mathrm{tr}_{\mathbf{r}}$ is the standard trace in any unitary representation $\mathbf{r}$, and $\ell(\mathbf{r})$ is the Dynkin index.[1]

In (2.1.7) we assume that

$$\mathrm{Im} \, f^{AB} \gg 1 \,, \qquad \mathrm{Im} \, f^I \gg 1 \,, \tag{2.1.8}$$

so that the gauge sectors can be considered weakly coupled at the cut-off scale Λ and the EFT hence admits a sensible perturbative expansion in the gauge couplings. Since we would like to focus on the asymptotic field space region $\mathcal{M}^{\Delta}_{\text{pert}}$ defined above, we furthermore assume that

[1] We define the Dynkin index by $\mathrm{tr}_{\mathbf{r}} \, t_a t_b = \frac{\ell(\mathbf{r})}{2h(\mathfrak{g})} \, \mathrm{tr}_{\mathbf{adj}} \, t_a t_b$, where $h(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$. For instance, $\ell(\mathbf{fund}) = 1$ and $\ell(\mathbf{fund}) = 2$ if $\mathfrak{g} = \mathfrak{su}(n)/\mathfrak{sp}(n)$ and $\mathfrak{g} = \mathfrak{so}(n)$, respectively, or $\ell(\mathbf{adj}) = 60$ for $\mathfrak{g} = \mathfrak{e}_8$. In this thesis we use *hermitian* gauge fields $A = A^a t_a$ and field strengths $F = F^a t_a$, so that $A^\dagger = A$ and $F^\dagger = F$. The hermitian generators t_a of the gauge algebra $\mathfrak{g}$ can be normalised so that $\mathrm{tr} \, t_a t_b = 2\delta_{ab}$. The instanton number can be identified with the integer $n = -\frac{1}{16\pi^2} \int \mathrm{tr}(F \wedge F) \in \mathbb{Z}$, see for instance [53], and $n = \frac{1}{16\pi^2} \int \mathrm{tr}(F \wedge *F) > 0$ for (Euclidean) anti-self-dual ($F = - * F$) instanton configurations. Note that in the literature characteristic classes and anomaly polynomials are often expressed in terms of the *anti*-hermitian field strength $F_{\text{AH}} \equiv -\mathrm{i}F$.

the holomorphic gauge functions f^{AB}, f^I can be expanded as

$$f^{AB}(t,\phi) = \langle \mathbf{C}^{AB}, \boldsymbol{t}\rangle + \Delta f^{AB}(\phi) + \dots \quad , \quad f^I(t,\phi) = \langle \mathbf{C}^{I}, \boldsymbol{t}\rangle + \Delta f^{I}(\phi) + \dots \, . \tag{2.1.9}$$

Here we are omitting exponentially suppressed non-perturbative terms $\sim \mathcal{O}(|e^{2\pi \mathrm{i}\langle \mathbf{m},\boldsymbol{t}\rangle}|)$ and we are employing the same index-free notation as in (2.1.3) for $t^i = a^i + \mathrm{i}s^i$, e.g.

$$\langle \mathbf{C}^{AB}, \boldsymbol{t}\rangle \equiv C_i^{AB} t^i \, . \tag{2.1.10}$$

From the expansion (2.1.9) we see that (2.1.7) contains in particular the (s)axionic couplings

$$-\frac{1}{4\pi} C_i^{AB} \int \left(s^i\, F_A \wedge *F_B + a^i\, F_A \wedge F_B\right) - \frac{1}{8\pi} C_i^I \int \left[s^i\, \mathrm{tr}(F\wedge *F)_I + a^i\, \mathrm{tr}(F\wedge F)_I\right] \, , \tag{2.1.11}$$

from which one can extract C_i^{AB} and C_i^I. Note that the form (2.1.11) for the (s)axionic couplings is a non-trivial assumption which need not hold for every gauge sector. We will come back to this caveat in the paragraph after (2.2.4).

Assuming that the gauge instanton configurations are defined on a Euclidean spin manifold, as is natural in supergravity, the compatibility of the gauge instanton corrections with the axion periodicity (2.1.1) requires the quantization conditions

$$C_i^{AB},\ C_i^I \in \mathbb{Z} \, . \tag{2.1.12}$$

In the following we will provide complementary evidence in favor of (2.1.12), which holds directly in Lorentzian signature.

For generic values of the fields ϕ^α, we expect $\Delta f^{AB}(\phi), \Delta f^I(\phi) \sim \mathcal{O}(1)$ in (2.1.9). Hence (2.1.8) suggests that the constants (2.1.12) should obey the constraints

$$\{\langle \mathbf{C}^{AB}, \boldsymbol{s}\rangle\} \geq 0 \, , \qquad \langle \mathbf{C}^{I}, \boldsymbol{s}\rangle \geq 0 \qquad\quad \forall \boldsymbol{s} \in \Delta \, . \tag{2.1.13}$$

The first condition means that $\langle \mathbf{C}^{AB}, \boldsymbol{s}\rangle$ is a positive definite matrix, and we are again using the index-free notation introduced above, e.g. $\langle \mathbf{C}^{AB}, \boldsymbol{s}\rangle \equiv C_i^{AB} s^i$ and $t^i = a^i + \mathrm{i}s^i$. In other words, it is natural to require that $\mathbf{C}^I \in \mathcal{C}_{\rm I}$, where we recall from the discussion before (2.1.3) that $\mathcal{C}_{\rm I}$ is the cone of BPS instanton charges dual to the saxionic cone Δ, and that $\{\mathbf{C}^{AB}\} \in \mathcal{C}_{\rm I}$ in a matrix sense. This is also consistent with the requirement that the perturbative gauge interactions preserve the axionic shift symmetries, while gauge instantons break them by exponentially suppressed non-perturbative corrections. To our knowledge the condition (2.1.13) is realised in all string theory models and in the following we will assume that it should hold in any EFT compatible with quantum gravity.

2.1.2 Higher curvature terms

One can generically write down higher-derivative corrections to the leading two-derivative supergravity. For our purposes it is sufficient to focus on the contribution

$$\frac{1}{24\pi\mathrm{i}}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\, 2\mathcal{E}\,\tilde{f}(t,\phi)Y + \text{c.c.}\,, \tag{2.1.14}$$

where $\tilde{f}(t,\phi)$ is another holomorphic function of the chiral multiplets and

$$Y \equiv \mathcal{W}^{\alpha\beta\gamma}\mathcal{W}_{\alpha\beta\gamma} - \frac{1}{4}\left(\overline{\mathcal{D}}^2 - 8\mathcal{R}\right)\left(a\mathcal{R}\overline{\mathcal{R}} + bG^aG_a\right) \tag{2.1.15}$$

is a composite chiral superfield [54–56] constructed out of the chiral superfields $\mathcal{W}_{\alpha\beta\gamma}$, $\mathcal{R}$ and the real superfield G_a of minimal $\mathcal{N}=1$ supergravity [50]. The superspace contribution (2.1.14) includes in particular the curvature-squared terms [55, 56]

$$-\frac{1}{96\pi}\int \mathrm{Im}\,\tilde{f}\;\mathrm{tr}(R\wedge *R) - \frac{1}{96\pi}\int \mathrm{Re}\,\tilde{f}\,\mathrm{tr}(R\wedge R) + \dots\,. \tag{2.1.16}$$

Here tr denotes the standard trace on the free indices of the Riemann two-form $R^m{}_n = \frac{1}{2}R^m{}_{npq}\,\mathrm{d}x^p\wedge\mathrm{d}x^q$ – see also [57] and [58] for some useful details on the necessary superspace manipulations.[2] In (2.1.16) we have omitted $R_{mn}R^{mn}$ and R^2 terms, since their coefficients depend on the constants a and b and then are not uniquely fixed by the Pontryagin term containing $\mathrm{tr}(R\wedge R)$. On the other hand, the arbitrariness is uniquely fixed to $a=2$ and $b=1$ by requiring that the omitted $R_{mn}R^{mn}$ and R^2 terms combine with the first term in (2.1.16) to give the Gauss-Bonnet term $\frac{1}{192\pi}\int \mathrm{Im}\,\tilde{f}\,E_{\mathrm{GB}} * 1$, with

$$E_{\mathrm{GB}} \equiv R_{mnpq}R^{mnpq} - 4R_{mn}R^{mn} + R^2\,, \tag{2.1.17}$$

which is not affected by ghost issues [59].

As for the gauge functions (2.1.9), under our general assumptions in the asymptotic field space region $\mathcal{M}^{\Delta}_{\mathrm{pert}}$ the holomorphic function $\tilde{f}$ necessarily takes the form

$$\tilde{f}(t,\phi) = \tilde{C}_i\,t^i + \Delta\tilde{f}(\phi) + \dots\,, \tag{2.1.18}$$

where again we are omitting exponentially suppressed non-perturbative terms. In particular, (2.1.16) contains the couplings

$$-\frac{1}{96\pi}\tilde{C}_i\int\left[s^i\;\mathrm{tr}(R\wedge *R) + a^i\;\mathrm{tr}(R\wedge R)\right]\,, \tag{2.1.19}$$

from which one can extract $\tilde{C}_i$.

[2]We thank Fotis Farakos for useful discussions about the necessary superspace gymnastics.

According to the normalization of (2.1.14), the axion periodicity (2.1.1) is not broken by possible gravitational instantons if we require that

$$2\tilde{C}_i \in \mathbb{Z}\,. \tag{2.1.20}$$

This can be understood by recalling that the integral of the first Pontryagin class $p_1(M) = -\frac{1}{8\pi^2}\text{tr}(R\wedge R)$ over a Euclidean spin four-manifold M is always a multiple of 48. In the sequel we will see that quantum gravity constraints require $\tilde{C}_i$ to be integral, rather than half-integral as in (2.1.20).

Unlike for the analogous quantities discussed in the previous subsection, there is no obvious reason to expect any definite sign of $\text{Im}\,\tilde{f}$ and $\langle\tilde{\mathbf{C}},\boldsymbol{s}\rangle \equiv \tilde{C}_i s^i$. On the other hand, various results suggest that the positivity of the coefficient of the Gauss-Bonnet term may be a general feature of EFTs consistent with quantum gravity. For instance this is necessary in order to suppress problematic wormhole effects [7]. More recently, [60] provides an argument for positivity based on unitarity in pure in pure $d > 4$ gravity, [61] discusses the implications of the sign on the non-perturbative (in)stability of simple dilatonic models, [62] shows how Gauss-Bonnet positivity follows from the WGC for certain black holes and [63] analyses constraints based on holography. In our context this would mean that $\text{Im}\,\tilde{f} > 0$ and then, as for the gauge theory sector, it would be natural to require that

$$\langle\tilde{\mathbf{C}},\boldsymbol{s}\rangle > 0\,, \tag{2.1.21}$$

or equivalently $\tilde{\mathbf{C}} \in \mathcal{C}_{\text{I}}$. (Here we are already using the integrality of $\tilde{C}_i$, anticipated above but not proven yet.) We will find that (2.1.21) follows, under certain additional mild restrictions, from the quantum gravity arguments of the following sections.

2.2 EFT strings as quantum gravity probes

The perturbative EFT regimes of Section 2.1 can be characterised in terms of a specific class of BPS axionic strings, called *EFT strings* in [34, 51] – see also [52]. In this section, after recalling the main properties of such EFT strings, we will describe the anomaly inflow mechanism from the four-dimensional bulk theory to the string worldsheet. We will then argue that the EFT string worldsheet theory can be treated as a weakly coupled non-linear sigma model (NLSM) and characterise the spectrum of its massless fields. This will form the basis for the derivation of quantum gravity constraints for the four-dimensional field theory in Section 4.2.2.

2.2.1 Perturbative EFT regimes, dual formulation and EFT strings

The presence of the perturbative axionic shift symmetries (2.1.1) naturally leads one to consider axionic strings in four dimensions, around which the axions undergo integral shifts

$$a^i \to a^i + e^i\,. \tag{2.2.1}$$

Here we assume that the EFT $U(1)$ gauge fields do not acquire a Stückelberg mass by gauging the axions a^i. In this way we exclude axionic strings which can break by nucleation of monopole pairs, leaving the investigation of this interesting generalization for future work.

In order to make contact with QG structures highlighted in [51, 64, 65], it is convenient to recall the basic features of the dual formulation, in which the axions are traded for 2-form potentials $\mathcal{B}_{2,i}$ with corresponding field strengths $\mathcal{H}_{3,i} = \mathrm{d}\mathcal{B}_{2,i} = -M_{\rm P}^2 \mathcal{G}_{ij} * \mathrm{d}a^i$. This duality transformation can be completed into a full supersymmetric duality which trades the t^i chiral multiplets for corresponding linear multiplets [66].

The integers $e^i \in \mathbb{Z}$ can be regarded as the magnetic axionic charges of the string, or as the electric charges under the dual two-form potentials $\mathcal{B}_{2,i}$ – see [34, 67] for more details on the dualization. In the dual formulation, the strings contribute to the EFT by a localized term

$$e^i \int_W \mathcal{B}_{2,i} \,, \tag{2.2.2}$$

where W denotes the string world-sheet. Furthermore, imposing that these strings are compatible with the bulk supersymmetry completely fixes [67] the additional contribution $-\int_W \mathcal{T}_{\mathbf{e}}\, \mathrm{vol}_W$ to the effective action: The tension $\mathcal{T}_{\mathbf{e}}$ takes the form

$$\mathcal{T}_{\mathbf{e}} \equiv M_{\rm P}^2\, e^i \ell_i \,, \tag{2.2.3}$$

in terms of the *dual saxions* ℓ_i. Together with $\mathcal{B}_{2,i}$, these form the bosonic components of the linear multiplets dual [66] to the chiral multiplets t^i and are defined by

$$\ell_i \equiv -\frac{1}{2}\frac{\partial K}{\partial s^i} \,. \tag{2.2.4}$$

Here K is the EFT Kähler potential K, which is assumed to be invariant under the axionic shift symmetries. Note that the dual formulation in terms of $\mathcal{B}_{i,e}$ and ℓ_i, as well as the localised terms (2.2.2) and (2.2.3), really make sense only if the axionic shift symmetries are preserved at the perturbative level, as we are assuming.[3]

The kinetic terms are specified by the kinetic function

$$\mathcal{F} = K + 2\ell_i s^i \tag{2.2.5}$$

which must be considered as a function of the dual saxions ℓ_i (and of the spectator fields). The leading order action can be equivalently re-written as

$$S^{(\mathrm{M})} = \frac{1}{2} M_{\rm P}^2 \int R * 1 - \frac{1}{2} M_{\rm P}^2 \int \mathcal{G}^{ij} \mathrm{d}\ell_i \wedge * \mathrm{d}\ell_j - \frac{1}{2M_{\rm P}^2} \int \mathcal{G}^{ij} \mathcal{H}_{3,i} \wedge * \mathcal{H}_{3,j} + \mathrm{BT} \tag{2.2.6}$$

[3] In this dual formulation the non-perturbative corrections can be generated by the mechanism described in [7].

where

$$\mathcal{G}^{ij} \equiv -\frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial \ell_i \partial \ell_j} \tag{2.2.7}$$

is the inverse matrix of the saxion kinetic matrix. Furthermore the inverse of the relation (2.2.4) is given by

$$s^i = \frac{1}{2}\frac{\partial \mathcal{F}}{\partial \ell_i}\,. \tag{2.2.8}$$

At this stage it is important to stress that we are excluding possible monodromy transformations of the $U(1)$ vectors under the axionic integral shifts (2.2.1). For instance, if we focus on two $U(1)$ field strengths and one chiral field $t = a + \mathrm{i}s$, we could impose that the discrete identification $t \simeq t + 1$ involves also a shift $F_1 \simeq F_1 + F_2$. F_1 can then enter the effective Lagrangian only through the monodromy invariant combination $\hat{F}_1 = F_1 - aF_2$, which can be completed into the super-field strength $\hat{\mathcal{W}}_1^\alpha = \mathcal{W}_1^\alpha - t\mathcal{W}_2^\alpha$. These monodromy effects can be immediately generalized to a larger number of $U(1)$s and axions, and allow for the appearance of quadratic and cubic axion couplings to $F_A \wedge F_B$, rather than the standard linear coupling (2.1.11). As an example of such non-standard couplings to the axions, Kaluza-Klein $U(1)$s have been discussed in [68]. Such couplings *obstruct* the dualization of the axions to $\mathcal{B}_{2,i}$ (as well as its supersymmetric completion). Then the EFT contribution of the corresponding axionic strings cannot be described as in (2.2.2) and (2.2.3). While these monodromy effects naturally appear in extended four-dimensional supergravities, they look more exotic in a minimally supersymmetric context, and we will henceforth only consider effective theories not exhibiting such subtle effects.

Let us come back to the dualization (2.2.4) and the fact that this procedure is possible only in presence of a perturbative shift symmetry. This becomes crucial once combined with the observation that the strings coupling to the two-form, being codimension-two objects, have a strong backreaction. The backreaction may force the surrounding bulk scalar fields to flow away from the asymptotic region $\mathcal{M}^{\Delta}_{\text{pert}}$ associated with the perturbative regime. However, as emphasised in [34, 51], this does not pose any problem at the EFT level: What one really needs for consistency of the dualization and the description of the strings is that the bulk sector is in the weakly coupled region in a small neighborhood of the string, of minimal radius of order $r_\Lambda \equiv \Lambda^{-1}$. It is then sufficient to require that the string backreaction is under control in this neighborhood. At such short distances, the string can be well approximated by a straight $\frac{1}{2}$-BPS string and its backreaction on the saxions is given by

$$\boldsymbol{s} = \boldsymbol{s}_0 + \mathbf{e}\,\sigma\,, \qquad \sigma \equiv \frac{1}{2\pi}\log\frac{r_0}{r}\,. \tag{2.2.9}$$

Here r is the radial distance from the string and $\boldsymbol{s}_0 = \{s_0^i\}$ represents the initial saxionic values at $r = r_0$. Note that the additional chiral fields ϕ^α do not flow. Hence, if the charge vector $\mathbf{e}$ belongs to

$$\mathcal{C}_{\text{S}}^{\text{EFT}} \equiv \overline{\Delta}|_{\mathbb{Z}} = \{\mathbf{e} \in \mathbb{Z}^{\#\,\text{axions}} \mid \langle \mathbf{m}, \mathbf{e}\rangle \geq 0\,,\ \forall \mathbf{m} \in \mathcal{C}_{\text{I}}\}\,, \tag{2.2.10}$$

a flow (2.2.9) starting from any $\boldsymbol{s}_0 \in \Delta$ never exits the saxionic cone Δ as one approaches the string. Actually, the scalars are driven more and more inside the asymptotic weakly coupled region $\mathcal{M}^{\Delta}_{\rm pert}$ as $\sigma \to \infty$. This can be taken as the defining property of the EFT strings, which hence admit a controlled weakly-coupled EFT description. Note that the EFT strings must be considered as fundamental strings[4], in the sense that they cannot be completed into smooth solitonic objects within a four-dimensional EFT, say by adding a finite number of new degrees of freedom.

In [34] it was shown how the validity of 'weak gravity bounds' [69] for instantons and strings requires that in $\mathcal{M}^{\Delta}_{\rm pert}$ the Kähler potential receives a leading saxionic contribution of the form

$$K = -\log P(\boldsymbol{s}) + \dots\,, \tag{2.2.11}$$

where $P(\boldsymbol{s})$ is a homogeneous function of the saxions. This asymptotic structure of the Kähler potential is indeed universally realised in all string theory models, in which the homogeneity degree of $P(\boldsymbol{s})$ turns out to be integral. In the following, whenever we will need it, we will implicitly assume this asymptotic form of the Kähler potential. This in particular implies that the EFT string flows (2.2.9) can be regarded as infinite field distance limits, and the Distant Axionic String Conjecture (DASC) of [34] proposes that all infinite distance points of $\partial\mathcal{M}^{\Delta}_{\infty}$ can actually be reached by an EFT string flow.

Another important property of the EFT strings is the following. It is not difficult to see that the field-dependent tension $\mathcal{T}_{\mathbf{e}}$ of an EFT string decreases along its own saxionic flow (2.2.9), as $\sigma \to \infty$. On the other hand, since this is an infinite field distance limit the Swampland Distance Conjecture implies that along the flow an infinite tower of microscopic massive modes appears, at a characteristic mass scale m_*, which exponentially vanishes with the field distance. By inspecting a large class of string theory models, in [34] it was found that along the EFT flows m_*^2 scales to zero as an integral power of $\mathcal{T}_{\mathbf{e}}$, in Planck units. This 'experimental' observation was promoted to a possible universal property of EFT strings. In its stronger form proposed in [52], this is the content of the Integral Weight Conjecture (IWC):

$$m_*^2 \sim M_{\rm P}^2 \left(\frac{\mathcal{T}_{\mathbf{e}}}{M_{\rm P}^2}\right)^w \qquad w \in \{1,2,3\}\,, \tag{2.2.12}$$

where w is the 'scaling weight' associated with the EFT string. Note also that, according to the Emergent String Conjecture (ESC) [70], in some duality frame an EFT string with $w = 1$ should coincide with a *critical* string, while EFT string flows with $w = 2, 3$ should correspond to decompactification limits, as further characterised in [71].

In order to motivate our definition of (2.2.10) we used purely EFT arguments. However, as we already mentioned, EFT strings are 'fundamental' and their existence depends on the UV completion of the theory. In the following discussion we will need to assume that the EFT string

[4]This notion is not to be confused with that of a *critical* string such as the heterotic or Type II string.

lattice $\mathcal{C}_{\rm S}^{\rm EFT}$ is actually fully populated. This non-trivial assumption is certainly realized in the large classes of string theory models considered in [34], and can be more generically motivated by invoking an EFT string version of the Completeness Conjecture [72], which is one of the better tested quantum gravity criteria.[5]

Note that the EFT strings make sense only in a gravitational context. Indeed, the analysis of [34] implies that the vanishing of an EFT string tension $\mathcal{T}_{\mathbf{e}} = M_{\rm P}^2 e^i \ell_i$ identifies a component of the infinite field distance boundary $\partial\mathcal{M}_\infty^\Delta$. Hence there is no (finite distance) point in field space at which $\mathcal{T}_{\mathbf{e}}/M_{\rm P}^2 \to 0$ and one cannot decouple the string dynamics from the gravitational sector. Furthermore, in all the string theory realizations considered so far, the EFT strings uplift to brane configurations which can be continuously deformed through the entire internal compactification space. In this sense, the EFT strings can 'probe' the entire UV completion of the EFT. We will indeed see that their quantum consistency provides additional non-trivial constraints on the EFT.

Note also that by consistency any BPS string tension must be positive in the interior of $\mathcal{P} \equiv \Delta^\vee$, and the condition $\langle \boldsymbol{\ell}, \mathbf{e}\rangle \geq 0$ for any $\boldsymbol{\ell} \in \mathcal{P}$ can be taken as defining condition of the BPS string charges $\mathbf{e} \in \mathcal{C}_{\rm S}$, that is: $\mathcal{C}_{\rm S} \in \mathcal{P}^\vee \cap V_{\mathbb{Z}}^*$. This implies that different boundary components of $\partial\mathcal{P}$ can be associated with the vanishing of different BPS string tensions [64]. In particular, components of $\partial\mathcal{P}$ which are at infinite field distance can be detected by the vanishing of some EFT string tensions, i.e. corresponding to some $\mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}$. On the other hand, finite distance components of $\partial\mathcal{P}$ can be detected by tensionless non-EFT BPS strings, i.e. with $\mathbf{e} \in \mathcal{C}_{\rm S} - \mathcal{C}_{\rm S}^{\rm EFT}$. In this latter case, one should keep in mind that these finite distance boundaries are not so sharply defined, since around them both perturbative and non-perturbative corrections become relevant and can for instance generate strong corrections to the 'bare' formula (2.2.3). Other strongly coupled regions are reached by radially moving away from the tip of $\mathcal{P}$ along different directions. If one insists in using the 'bare' kinetic potential (4.1.4), these are at infinite field distance and not associated with any tensionless string. Rather, in this limit all the BPS tensions (2.2.3) diverge, and then the above description breaks down since it assumes that $\mathcal{T}_{\mathbf{e}} \lesssim 2\pi M_{\rm P}^2$ in order for the string to have a weak gravitational backreaction [51]. This is again not the end of the story, since one again expects strong corrections which may completely change the nature of these limits. In any case, we see how the behavior of the BPS tension (2.2.3) can be a useful proxy to qualitative characterize the different boundary components of $\mathcal{P}$.

[5]Recently [73] has found examples of effective field theories in higher dimensions which violate the Completeness Conjecture for *BPS* strings; at the same time, there always exists another theory differing only at the massive level which does satisfy the BPS Completeness Conjecture. For our purpose of constraining the massless EFT spectra this would in fact be sufficient.

Chapter 3

Quantum Gravity Bounds from EFT Strings

An increasing amount of evidence suggests that to couple a gauge theory to gravity, severe constraints have to be fulfilled at the quantum level, many more than one would expect from those coming from ordinary quantum field theory considerations. For example, while in quantum field theory, any anomaly free spectrum, no matter how contrived, leads to a consistent quantum gauge theory, it is widely believed that when quantum gravity is taken into account, the number of degrees of freedom should remain finite, due to general expectations on black hole entropy. Determining more generally which constraints a quantum gauge theory has to fulfill in presence of gravity is one of the objectives of the Swampland program [1] as reviewed, for instance, in [4, 18, 74].

A particularly fruitful idea that has emerged is to derive consistency conditions on an effective field theory by examining the inclusion of higher dimensional defects as probes in the theory.[1] For instance, if a quantum gravity theory contains higher p-form gauge fields, the Completeness Conjecture [72] implies that the theory must contain all the p-dimensional objects charged under this symmetry whose charges are compatible with Dirac quantization. Consistency of the worldvolume theory of these objects poses novel constraints on the bulk effective field theory which might go far beyond the usual cancellation of gauge and gravitational anomalies. This approach has been successfully applied to minimally supersymmetric gauge-gravity systems in ten [36], eight [49, 77], six [36, 37, 78–80], and five [39] dimensions, as well as to theories with sixteen supercharges in various dimensions [38]. In this thesis we will present quantum gravity bounds in four-dimensional theories with minimal $\mathcal{N} = 1$ supersymmetry.

An $\mathcal{N} = 1$ supergravity theory in four dimensions typically contains light complex chiral scalar

[1] As one of the earlier incarnations of this idea, [75] derived for instance the K-theory tadpole cancellation conditions in D-brane models by requiring absence of Witten anomalies on probe branes. See furthermore [76] and references therein for an analysis of the correspondence between spacetime and worldvolume physics.

fields parametrizing the field space of the theory. In the weak coupling regime, a shift symmetry arises in an emergent axionic sector of the theory, which is only broken by non-perturbative effects. In the limit where the latter are sufficiently suppressed, the axions can be dualized into two-form fields. By the Completeness Conjecture of quantum gravity, these must couple to string-like objects, which in fact can be half-BPS in four dimensions. This places the theory precisely into a context, similar to that of [36–39, 49, 77–79], in which consistency conditions of these axionic strings can be turned into new constraints on the four-dimensional effective field theory.

As a complication specific to four dimensions, however, string-like objects induce a non-negligible backreaction on the fields in the two directions normal to the string. This is owed to the fact that the string worldsheet is of real codimension two in spacetime [81]. For general strings, the backreaction questions the validity of applying the probe approximation to the string and of viewing it as weakly coupled. Interestingly, in [34, 51] it was understood that for a certain class of half-BPS strings a perturbative treatment of the string worldsheet theory is nonetheless justified. The strings in question were called 'EFT strings' and have the property that their backreaction on the moduli of the theory is precisely such that close to the string core, the effective theory becomes weakly coupled: More precisely, asymptotically close to the string core, the axionic shift symmetries which are needed to dualize the axions to two-form fields become exact because the instantons dual to the string become suppressed. As it turns out, the strings with this property are precisely of the form that they cannot decouple from the gravitational sector, making them ideal candidates to probe the quantum gravity nature of the theory in those asymptotic limits. The original motivation in studying such strings was the conjecture of [34, 51] that any infinite distance limit in four dimensions can be obtained as the endpoint of an RG flow induced by the backreaction of an EFT string. This idea has been investigated further in [35, 71, 82] from various perspectives.

In this chapter of the thesis, we will explore the role of EFT strings as weakly coupled probes of the four-dimensional $\mathcal{N} = 1$ supersymmetric gauge-gravity sector. In this way, we will be able to constrain the input data of an effective $\mathcal{N} = 1$ supergravity theory beyond the consistency conditions from anomaly cancellation in the bulk alone.

The constraints which we will find should be valid for all such theories with a standard coupling to the axionic sector: By this we mean theories in which the gauge field strength and the curvature two-form couple to the axions a^i via terms of the form[2]

$$S_{\text{EFT}} \supset C_i \int a^i \, \text{tr} F \wedge F + \tilde{C}_i \int a^i \, \text{tr} R \wedge R \,. \tag{3.0.1}$$

As we will see, such couplings induce an anomaly inflow from the four-dimensional bulk to the string worldsheet. As in [83] and in the higher-dimensional gravitational theories treated in

[2] The precise normalisation factors will be given in Section 2.1.

[36, 37, 39, 78], this anomaly inflow must be cancelled by the two-dimensional anomalies on the string worldsheet resulting from the presence of chiral matter on it. Importantly, for EFT strings in the sense of [34, 51], we can reliably compute this worldsheet anomaly in terms of the 1-loop perturbative anomaly generated by the modes along the string because the theory flows to weak coupling close to the core of the string. This eventually leads to constraints on the four-dimensional bulk theory. A notable difference to the analysis in [36, 37, 39, 78] is that we do not assume that the string worldsheet theory flows to a conformal field theory in the infra-red. Indeed, in four dimensions this may a priori not be justified due to the peculiarities of the string, but is also not required as long as the string worldsheet is weakly coupled. The EFT strings are precisely of this type.

Our main results are two types of constraints: First, (3.2.3) and (3.2.4) constrain the quantization and the signs of the axionic curvature couplings in (3.0.1). In particular, (3.2.3) fixes the saxionic Gauss-Bonnet term in a large class of gravitational $\mathcal{N} = 1$ supersymmetric effective actions to be positive. Our EFT string analysis therefore complements previous arguments for the positivity of these higher-derivative terms [7, 60–63]. Second, (3.2.26) bounds the possible ranks of the gauge groups which can be coupled to an axionic sector as in (3.0.1).

Our derivation of these bounds is subject to certain assumptions on the spectrum of the worldsheet theory of the EFT strings, which we motivate in Section 3.1.3. These are manifestly realised in explicit string theoretic settings and moreover appear natural more generally. Modulo these assumptions, whenever the EFT string derived constraints, in particular the bounds (3.2.26) on the ranks of a gauge sector, are violated in a consistent quantum gravity with minimal supersymmetry, our analysis implies that the theory cannot exhibit standard axionic couplings of the form (3.0.1). This is a rather non-trivial prediction from a purely effective field theoretic point of view.

Our results can be compared with the concrete constraints imposed on effective field theories in string theory compactifications. In the context of F-theory compactifications, we will confront the EFT string bounds (3.2.26) with the geometric bounds from the construction of explicit Weierstrass models. The latter have been analysed in a series of works [84–87] in various dimensions to constrain the ranks and matter content of gauge-gravity theories in F-theory.[3] In fact, based on our knowledge of the worldsheet theory of the EFT strings in F-theory following from [94], we propose (3.3.36) as a stronger bound on the rank of the gauge group, which should be valid in geometric F-theory compactifications with a smooth base (and minimal supersymmetry).

To test the EFT string bounds in the heterotic context, we will first have to extend the analysis in [34] of EFT strings in such setups by including also higher curvature corrections to the effective action. This part of our analysis is in fact interesting by itself and reveals an intriguing

[3] More recent examples of the rich corpus of studies investigating the constraints on gauge data imposed by string theory in various dimensions include [88–93].

modification of the structure of the cone of EFT strings due to said correction terms. We will compare the structure of the heterotic cone of EFT strings with its F-theory dual and discuss the bounds on the theory. Furthermore, we give a preliminary analysis of the EFT string derived bounds in M-theory on G_2 manifolds, which can serve as a starting point for a more detailed investigation in the future.

The content of this chapter, based upon [65], is organized according to the following structure: In Section 2.1 we have already reviewed and specified the structure of the four-dimensional $\mathcal{N} = 1$ supersymmetric gauge-gravity theories for which we will derive quantum gravity constraints. In particular, we will explain the assumption that the effective field theory enjoys axionic couplings of the form (3.0.1). In Section 2.2 the reader can find the review of the main concepts underlying the ideas of EFT strings from [34, 51]. We then derive the anomaly inflow from the bulk to the string worldsheet induced solely by the axionic couplings (3.0.1), up to an interesting subtlety discussed in Section 3.1.2. Finally, we specify the main ingredients of the weakly coupled non-linear sigma-model governing the $\mathcal{N} = (0, 2)$ supersymmetry worldsheet theory. Of key importance for us is a discussion of the worldsheet modes in Section 3.1.3, including our assumptions on their charges. Section 4.2.2 contains our main results: We derive the EFT string consistency conditions by demanding that the anomaly inflow be cancelled by the anomalies (reviewed in Appendix A.1) of the weakly coupled worldsheet spectrum. In Section 3.3.1 we show how these constraints rule out simple otherwise consistent supergravities as theories with a quantum gravity completion.

In the second part of the chapter, we test and apply our general results in explicit string theoretic frameworks. In Section 3.3.2, we apply our bounds to F-theory compactifications to four dimensions. We will illustrate the validity of the assumptions made in the general setting in Section 3.1.3, and in addition motivate a sharpened bound on the rank of the gauge group, given by (3.3.36) in F-theory on smooth three-fold bases. As another application, we will derive the constraint that in any Type IIB orientifold with O3-planes (and no O7-planes), the number of O3-planes is quantized in units of 16, which represents on its own a physics post-diction of a mathematical theorem. Section 3.3.3 analyses the intriguing structure of the cone of EFT strings in heterotic string theory including higher derivative corrections. In the interest of readability, we have relegated some of the technicalities to Appendix A.3 and A.4. In Appendix A.2 we corroborate our claims that the EFT string can be described by a weakly coupled non-linear sigma model, by analysing the EFT strings in F-theory and the heterotic theory in more detail. In Section 3.3.5 we apply our EFT string constraints in the context of M-theory compactifications on G_2 manifolds. Our conclusions and a list of open questions are presented in the chapter 5.

3.1 Anomaly inflow

In this section, we will show how the mechanism of anomaly inflow in our setting leads to a tree-level worldsheet anomaly on the probe EFT string, that must be compensated by a suitable 1-loop 't Hooft anomaly from the chiral matter supported on the worldsheet itself. We will also argue and provide motivations for the specific form of the worldsheet matter content in the asymptotic regime, that should be given by a weakly coupled non-linear sigma model. In doing so, we will underline some subtleties that arise, like the contribution from the non-vanishing Euler class of the worldsheet normal bundle, that will be crucial for the successive discussion about concrete checks in string theory examples.

3.1.1 World-sheet anomaly from anomaly inflow

It is well known that axionic strings can support chiral fermions whose anomaly must be cancelled by a bulk anomaly inflow [83]. As in this reference, the axion couplings appearing in (2.1.7) and (2.1.16) produce an anomaly inflow to the EFT strings. Taking into account (2.1.9) and (2.1.18), the four-dimensional couplings can be written in the form

$$2\pi \int a^i I_{4,i} = -2\pi \int h_1^i \wedge I_{3,i}^{(0)} \,. \tag{3.1.1}$$

Here we have introduced the one-forms $h_1^i = \mathrm{d}a^i$, which are globally defined even in presence of axionic strings, and

$$I_{4,i} \equiv \mathrm{d}I_{3,i}^{(0)} \equiv -\frac{1}{8\pi^2} C_i^{AB} F_A \wedge F_B - \frac{1}{16\pi^2} C_i^I \,\mathrm{tr}(F \wedge F)_I - \frac{1}{192\pi^2} \tilde{C}_i \,\mathrm{tr}(R \wedge R) \,. \tag{3.1.2}$$

The anomaly inflow is generated by the term (3.1.1) since, in presence of an axionic string of charges e^i and world-sheet W, the one-forms h_1^i are not closed, but rather satisfy

$$\mathrm{d}h_1^i = e^i \delta_2(W) \,. \tag{3.1.3}$$

Under general gauge transformations and local Lorentz transformations we can write $\delta I_{3,i}^{(0)} = \mathrm{d}I_{2,i}^{(1)}$, and then (3.1.1) produces the following localized contribution to the corresponding variation of the bulk action:

$$\delta S_{\mathrm{bulk}} = -2\pi e^i \int_W I_{2,i}^{(1)} \,. \tag{3.1.4}$$

The same result can be obtained in the dual formulation, by taking into account that the couplings (3.1.1) modify the dual field strengths into $\mathcal{H}_{3,i} = \mathrm{d}\mathcal{B}_{2,i} + 2\pi I_{3,i}^{(0)}$. Since $\mathcal{H}_{3,i}$ must be gauge invariant, under gauge and local Lorentz transformations $\mathcal{B}_{2,i}$ must transform non-trivially:

$$\delta \mathcal{B}_{2,i} = -2\pi I_{2,i}^{(1)} \,. \tag{3.1.5}$$

Hence (2.2.2) produces precisely the same localized contribution to the gauge variation of the action found above.

Since a consistent EFT must be anomaly free, the localized contribution (3.1.4) must be cancelled by a string world-sheet anomaly $\delta S_W^{\rm quantum} = +2\pi e^i \int_W I^{(1)}_{2,i}$. Via the usual descent equation, this worldsheet anomaly corresponds to the anomaly polynomial

$$e^i I_{4,i} = -\frac{\langle \mathbf{C}^{AB}, \mathbf{e}\rangle}{8\pi^2} F_A \wedge F_B - \frac{\langle \mathbf{C}^I, \mathbf{e}\rangle}{16\pi^2}\,\mathrm{tr}(F\wedge F)_I - \frac{\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle}{192\pi^2}\,\mathrm{tr}(R\wedge R)\,. \tag{3.1.6}$$

The last term is proportional to the bulk first Pontryagin class $p_1(M) = -\frac{1}{8\pi^2}\,\mathrm{tr}(R\wedge R)$. Since $TM|_W = TW \oplus N_W$, where N_W is the normal bundle of W, $p_1(M)$ can be decomposed into

$$p_1(M) = p_1(W) + c_1(N_W)^2 = -\frac{1}{8\pi^2}\,\mathrm{tr}(R_W\wedge R_W) + \frac{1}{4\pi^2} F_{\rm N}\wedge F_{\rm N}\,, \tag{3.1.7}$$

where $F_{\rm N} = \mathrm{d}A_{\rm N}$ is the field strength of the $U(1)_{\rm N}$ connection $A_{\rm N}$ induced on the normal bundle by the bulk Riemannian connection. The last term of (3.1.6) then democratically contributes to the gravitational and $U(1)_{\rm N}$ anomaly of the world-sheet theory.

There can occur an additional, and more subtle, contribution to the $U(1)_{\rm N}$ anomaly inflow. It is analogous to the additional contribution associated with the type IIA NS5-brane normal bundle, discussed in [95]. Adapted to our context – see also [96] – the key observation is that the pull-back to the string world-sheet W of the singular right-hand side of (3.1.3) gives a non-vanishing finite term. Indeed, in cohomology

$$\delta_2(W)|_W = \chi(N_W)\,, \tag{3.1.8}$$

where $\chi(N_W)$ is the Euler class of the normal bundle N_W. Since we have a distinguished connection $A_{\rm N}$ on N_W, it is natural to promote (3.1.8) to an equation for (distributional) differential forms, by identifying

$$\chi(N_W) = \frac{1}{2\pi} F_{\rm N}\,. \tag{3.1.9}$$

The restriction of (3.1.3) to W then contains a finite contribution coming from the delta-source:

$$\mathrm{d}h_1^i|_W = \frac{e^i}{2\pi} F_{\rm N}\,. \tag{3.1.10}$$

Topologically, this equation implies that the normal bundle N_W must be trivial. Hence, we know that we can globally write $F_{\rm N} = \mathrm{d}A_{\rm N}$. Following [95] we will consider (3.1.10) as part of the definition of our axionic strings.

By adopting the 'magnetic' axionic formulation, we can now write down the following additional

EFT term localised on the string:

$$S_{\rm N} = -\frac{1}{24}\hat{C}_i(\mathbf{e})\int_W h_1^i \wedge A_{\rm N}\,. \tag{3.1.11}$$

Here the normalization is chosen for later convenience and we assume that the constants $\hat{C}_i(\mathbf{e})$ necessarily depend on $\mathbf{e}$ since otherwise the term (3.1.11) would be independent of the EFT string charge, which does not seem physically reasonable. By recalling (3.1.10) it is easy to see that (3.1.11) is not invariant under $U(1)_{\rm N}$ a gauge transformations $\delta A_{\rm N} = {\rm d}\lambda_{\rm N}$:

$$\delta S_{\rm N} = -\frac{1}{48\pi}\hat{C}_i(\mathbf{e})e^i\int_W \lambda_{\rm N}\, F_{\rm N}\,. \tag{3.1.12}$$

Again, we may obtain this anomalous contribution in the dual description, in which the term (3.1.11) does not appear and is encoded in a modification of the $\mathcal{H}_{3,i}$ Bianchi identities:

$${\rm d}\mathcal{H}_{3,i} = 2\pi I_{4,i} + \frac{1}{24}\hat{C}_i(\mathbf{e})F_{\rm N}\wedge\delta_2(W)\,. \tag{3.1.13}$$

This can be solved by setting

$$\mathcal{H}_{3,i} = {\rm d}\mathcal{B}_{2,i} + 2\pi I^{(0)}_{3,i} + \frac{1}{24}\hat{C}_i(\mathbf{e})A_{\rm N}\wedge\delta_2(W)\,. \tag{3.1.14}$$

It follows that, in addition to (3.1.5), under local Lorentz transformations the variation of $\mathcal{B}_{2,i}$ receives also the localised contribution

$$\delta_{\rm N}\mathcal{B}_{2,i} = -\frac{1}{24}\lambda_{\rm N}\,\hat{C}_i(\mathbf{e})\delta_2(W)\,, \tag{3.1.15}$$

which, applied to (2.2.2), precisely reproduces (3.1.12).

The anomalous contribution (3.1.12) must be cancelled by a contribution $2\pi\int_W \hat{I}^{(1)}_{2,{\rm N}}$ to the world-sheet anomaly, corresponding to the anomaly polynomial

$$\hat{I}_{4,{\rm N}} = \frac{\langle\hat{\mathbf{C}}(\mathbf{e}),\mathbf{e}\rangle}{96\pi^2}F_{\rm N}\wedge F_{\rm N}\,, \tag{3.1.16}$$

where as usual $\langle\hat{\mathbf{C}}(\mathbf{e}),\mathbf{e}\rangle \equiv e^i\hat{C}_i(\mathbf{e})$. Hence, to sum up, the cancellation of the anomaly inflow requires that the world-sheet anomaly polynomial must be given by

$$\begin{aligned} I^{\rm ws}_{4\,\mathbf{e}} &= e^i I_{4\,i} + \hat{I}_{4,{\rm N}} \\ &= -\frac{\langle\mathbf{C}^{AB},\mathbf{e}\rangle}{8\pi^2}F_A\wedge F_B - \frac{\langle\mathbf{C}^{I},\mathbf{e}\rangle}{16\pi^2}\,{\rm tr}(F\wedge F)_I \\ &\quad -\frac{\langle\tilde{\mathbf{C}},\mathbf{e}\rangle}{192\pi^2}\,{\rm tr}(R_W\wedge R_W) + \frac{\langle\tilde{\mathbf{C}},\mathbf{e}\rangle + \langle\hat{\mathbf{C}}(\mathbf{e}),\mathbf{e}\rangle}{96\pi^2}F_{\rm N}\wedge F_{\rm N}\,. \end{aligned} \tag{3.1.17}$$

Note that this anomaly polynomial does not include possible mixed $U(1)_{\rm N}$-$U(1)_A$ anomalies. Indeed, they can be assumed to be cancelled by terms of the form (3.1.11) involving the $U(1)_A$ gauge fields A_A instead of $A_{\rm N}$ and can then be ignored.

With the exception of the additional terms in (3.1.16), the anomalies encoded in (3.1.17) are all proportional to the pairing of the couplings C_i^{AB}, C_i^I and $\tilde{C}_i$ with the charges e^i characterising the EFT string. Recall that the presence of such a string induces a flow of the form (2.2.9) in the saxionic moduli space. In view of the relations (2.1.9), we therefore conclude that precisely those gauge sectors $U(1)_A$ and G_I that become weakly coupled as a consequence of the backreaction of an EFT string, as encoded in the flow (2.2.9), can be detected by its world-sheet anomalies. Gauge sectors which stay strongly coupled in presence of an EFT string, on the other hand, never induce an anomaly on its worldsheet, and more generally the EFT string in question does not detect them via its world-sheet. In particular this applies to gauge sectors with non-standard axionic monodromies of the form mentioned after (2.2.4), which fall outside the class of theories which can be constrained by the analysis of EFT string anomalies as studied in the present work.

The conditions (2.1.13) can be rewritten in terms of EFT string charges as

$$\{\langle \mathbf{C}^{AB}, \mathbf{e}\rangle\} \geq 0\,, \quad \langle \mathbf{C}^{I}, \mathbf{e}\rangle \geq 0\,, \qquad \forall \mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}\,, \tag{3.1.18}$$

where the first inequality means that the matrix $\langle \mathbf{C}^{AB}, \mathbf{e}\rangle$ is positive semi-definite. This implies that the coefficients of the gauge world-sheet anomaly polynomial (3.1.17) have semi-definite sign.

3.1.2 UV origin of the additional world-sheet contribution

As will be made more explicit in the next subsection, the constants $\hat{C}_i(\mathbf{e})$ introduced in (3.1.11) must satisfy appropriate quantization conditions and should then be associated with some additional discrete structure defining the EFT. In a generic $\mathcal{N}=1$ four-dimensional EFT there is no natural candidate, but the more constrained higher dimensional supersymmetric theories can indeed be characterized by additional discrete data. This suggests that the presence of a term (3.1.11) may be interpreted as the manifestation of some hidden higher dimensional structure which can be detected by the EFT string.

More concretely, consider a situation in which the axions a^i come from the dimensional reduction of the $U(1)$ vectors A^i of an $\mathcal{N}=1$ five-dimensional supergravity. The latter is characterized by a set of quantized constants [97]

$$\hat{C}_{ijk} \in \mathbb{Z}\,, \tag{3.1.19}$$

which are totally symmetric in the indices and in particular define the five-dimensional Chern-Simons term

$$S_{\rm 5d} = \frac{1}{6(2\pi)^2}\hat{C}_{ijk}\int A^i \wedge F^j \wedge F^k\,. \tag{3.1.20}$$

The four-dimensional axionic strings uplift to five-dimensional monopole strings. For such monopole strings of charge vector $\mathbf{e}$, the term (3.1.20) generates the inflow contribution [39, 96, 98, 99]

$$-\frac{2\pi}{24}\hat{C}_{ijk}e^i e^j e^k \int_W p_1^{(1)}(\hat{N}_W) \tag{3.1.21}$$

to the $SO(3)$ normal bundle anomaly of the monopole, where $p_1^{(1)}(\hat{N}_W)$ is the descent two-form associated with the first Pontryagin class of the normal bundle $\hat{N}_W$. Under dimensional reduction to four dimensions the normal bundle splits as $\hat{N}_W = N_W \oplus L$, where L is a trivial real line bundle, and then (3.1.21) precisely takes the form (3.1.12) with

$$\hat{C}_i(\mathbf{e}) = \hat{C}_{ijk}e^j e^k \,. \tag{3.1.22}$$

Hence, with this choice, (3.1.11) encodes the information on a microscopic five-dimensional structure that can be detected by the EFT strings. One can also more directly reproduce (3.1.11) from the dimensional reduction of (3.1.20), as in [96]. Since this effect is local, all that is required is a five-dimensional EFT term of the form (3.1.20), without the need to assume that the five-dimensional configuration globally preserves eight supercharges.[4] Furthermore, the reduction from five to four dimensions would imply a leading contribution to the Kähler potential of the form (2.2.11) with $P(\boldsymbol{s}) = \hat{C}_{ijk}s^i s^j s^k$, inherited by the five-dimensional supersymmetric structure, which would correlate a world-sheet term of the form (3.1.21) to the bulk EFT structure. In the sequel we will more explicitly illustrate these ideas in concrete string theory realizations.

Let us now investigate if the term (3.1.11) can instead detect a six-dimensional minimally supersymmetric structure. As reviewed for instance in [36], in six dimensions the $\mathcal{N} = (1,0)$ tensor multiplet sector is characterized by a symmetric matrix $\hat{C}_{ij}$ and the strings act as both electric and magnetic sources for the $\mathcal{B}_{2,i}$ fields. The Green-Schwarz term produces an anomaly inflow polynomial proportional to $\hat{C}_{ij}e^i e^j \chi(\hat{N}_W)$, where $\chi(\hat{N}_W)$ is the Euler class of the string normal bundle $\hat{N}_W$ which has $SO(4) \simeq SU(2)_r \times SU(2)_l$ structure group. It would then be natural to guess that in four dimensions this effect can again be captured by (3.1.11), by choosing $\hat{C}_{\mathbf{e}} \propto \hat{C}_{ij}e^j$. However, this naive guess turns out to be wrong. Indeed, consider a compactification from six to four dimensions. Since the string is point-like in the compact two dimensions, its normal bundle splits into $\hat{N}_W = N_W \oplus L \oplus L$ where L is again trivial real line bundle. This implies that $\chi(\hat{N}_W) = \chi(N_W)\chi(L \oplus L) = 0$. Hence, even though $\hat{C}_i(\mathbf{e}) \propto \hat{C}_{ij}e^j$ has the right structure to be associated with a six-dimensional supergravity, apparently no anomaly survives in the reduction to four dimensions. We are then led to exclude this possibility as being physically irrelevant.

Supergravity theories in $d > 6$ dimensions are too rigid to allow for a choice of some discrete

[4] Note that a simple circle compactification from five to four dimensions would preserve eight supercharges and contain a KK $U(1)$ field. This KK $U(1)$ would be afflicted by monodromy effects of the kind discussed in the paragraph after (2.2.4) – see for instance [68] – which are not covered by our analysis.

data, which could then enter $\hat{C}_i(\mathbf{e})$. For these reasons, in the rest of this thesis we will assume (3.1.22) whenever we will need a more explicit form of $\hat{C}_i(\mathbf{e})$, otherwise keeping it generic.

3.1.3 EFT string as weakly coupled NLSM

The world-sheet anomaly polynomial (3.1.6) has been fixed by anomaly inflow arguments, a procedure which basically uses only the axionic nature of the EFT strings. While EFT strings naturally allow for a coupling to a weakly-coupled bulk sector at the EFT cut-off scale Λ, they generically lead to infra-red divergences due to their large backreaction. In particular, one cannot a priori assume that their world-sheet theory flows to a CFT, as was for instance possible in the higher-dimensional context of [36, 39].

However, the analysis of explicit UV complete models carried out in Appendix A.2 provides evidence that also the world-sheet sector supported by EFT strings can be considered weakly-coupled at the EFT cut-off scale, and can then be described by a weakly-coupled non-linear sigma model (NLSM). This can be understood as follows. In string theory models EFT strings are microscopically associated to branes which can freely propagate in the internal compactification space, which then determines the geometry of the effective two-dimensional NLSM. One may be worried that the string backreaction could obstruct the possibility to make its NLSM weakly coupled. The key point is that for EFT strings this does not occur, and actually the EFT string flow tends to render the NLSM more weakly coupled.

Recall the observation made after (2.2.12) and suppose first that $w = 2$ or 3. The Emergent String Conjecture [70] implies that in some duality frame the EFT flow involves the decompactification of some internal direction, as has been made more precise for the EFT string limits in [71]. The string theory models of Appendix A.2 clearly indicate that this dynamical decompactification makes the effective two-dimensional NLSM more and more weakly coupled.

If instead $w = 1$, then according to the Emergent String Conjecture there should exist a duality frame in which the EFT string is a critical string whose flow drives the dilaton to infinity (i.e. weak coupling) [34]. The remaining moduli do not flow and we may rescale them in order to make the compactification space arbitrarily large, at least if the dual description admits a geometric phase. In this regime the string certainly admits a weakly-coupled NLSM description. In the non-generic cases in which the large volume regime does not exist, as for instance the rigid Landau-Ginzberg-like theories of the type considered in [100], the string is critical in the deep UV and then it should - and can - instead be directly treated as a CFT.

Motivated by these observations, we propose that the world-sheet sector supported by EFT strings generically admits a weakly-coupled NLSM description and we will proceed with this working assumption.[5] Precisely in the non-generic cases in which this is not possible, one can rephrase the following discussion in CFT terms (along the lines of [38]) and arrive at the same

[5]More precisely, some EFT string may host a 'spectator' strongly coupled subsector, which does not participate in the cancellation of the anomaly inflow and does not interfere with the weakly coupled NLSM dynamics.

results. Hence in the following we will focus on the NLSM description.

The NLSM includes the universal 'center of mass' sector, which is described by the Green-Schwarz (GS) formulation of [67]. We can use this formulation to make it clear why these strings locally preserve $\mathcal{N} = (0,2)$ supersymmetry. Besides the world-sheet embedding coordinates, the GS-string [67] supports a GS fermion Θ^α, transforming as a four-dimensional chiral spinor. The preserved supersymmetry is determined by the kappa-symmetry $\delta\Theta_\alpha = \kappa_\alpha$ with $\kappa_\alpha = \Gamma_\alpha{}^\beta \kappa_\beta$. Here we are using the index notation of [50] and $\Gamma_\alpha{}^\beta$ is the kappa symmetry operator, which can be identified with a two-dimensional chiral operator. In a locally adapted reference frame in which the string is locally stretched along the (x^0, x^3)-directions, we have that $\Gamma_\alpha{}^\beta = (\sigma_3)_\alpha{}^\beta$. As in [101], we can relabel the bulk spinor components as two-dimensional chiral components, e.g. $\Theta_\alpha = (\Theta_-, \Theta_+)$ and $\Theta^\alpha = (\Theta^-, \Theta^+) = (\Theta_+, -\Theta_-)$. The projection condition on $\kappa_\alpha = (\kappa_-, \kappa_+)$ imposes that $\kappa_+ = 0$ and the kappa-symmetry then implies that $\rho_+ \equiv \Theta_+$ is the physical right-moving component of Θ_α, while Θ_- is pure gauge and can be set to zero. We can also use local static gauge and combine the transversal embedding coordinates into $u = x^1 + \mathrm{i}x^2$. Then u and ρ_+ can be reorganized into the 'universal' $\mathcal{N} = (0,2)$ two-dimensional chiral superfield

$$U = u + \sqrt{2}\theta^+ \rho_+ - 2\mathrm{i}\theta^+ \bar{\theta}^+ \partial_{++} u\,. \tag{3.1.23}$$

We follow the conventions of [101, 102] – see also [103] for an extensive introduction to $\mathcal{N} = (0,2)$ two-dimensional models. The components u and ρ_+ have charges 1 and $\frac{1}{2}$, respectively, under a $U(1)_{\rm N}$ rotation. We can then interpret $U(1)_{\rm N}$ as an R-symmetry which acts also on θ^+ with charge $J_{\rm N}[\theta^+] = \frac{1}{2}$, so that $J_{\rm N}[U] = 1$.

The $\mathcal{N} = (0,2)$ NLSM [104, 105] supported by an EFT string generically includes also an 'internal' sector. In particular, it can include $n_{\rm C}$ additional 'non-universal' scalar chiral multiplets

$$\Phi = \varphi + \sqrt{2}\theta^+ \chi_+ - 2\mathrm{i}\theta^+ \bar{\theta}^+ \partial_{++} \varphi\,, \tag{3.1.24}$$

where φ is a complex scalar and χ_+ is a right-moving fermion. (For simplicity, in (3.1.24) we are suppressing indices running from 1 to $n_{\rm C}$, since we will not need them.) The bosonic components φ parametrize the 'internal' NLSM target space $\mathcal{M}_{\rm NLSM}$ (while (3.1.23) has the 'external' four-dimensional spacetime as its target space), and the fermions χ_+ take values in the corresponding tangent bundle $T\mathcal{M}_{\rm NLSM}$. Representing internal degrees of freedom, the chiral superfields Φ are neutral under the normal $U(1)_{\rm N}$ rotational symmetry: $J_{\rm N}[\Phi] = 0$. Since $J_{\rm N}[\theta^+] = \frac{1}{2}$, the right-moving fermions χ_+ therefore have $U(1)_{\rm N}$ charge $J_{\rm N}[\chi_+] = -\frac{1}{2}$.

Due to the minimal amount of supersymmetry, some of the directions in the NLSM moduli space may be obstructed at higher order.[6] The obstructed directions should be specified by the vanishing of $n_{\rm N}$ holomorphic superpotentials $J_a(\phi)$ in the $\mathcal{N} = (0,2)$ theory. In order to

[6] In the presence of such obstructions, it is clear that only (some of) the unobstructed directions may admit a gauged axionic shift symmetry, see again Section 3.2.2 for details.

implement these constraints, we must include a corresponding number $n_{\rm N}$ of neutral Fermi multiplets $\Lambda^a_- = \lambda^a_- + \ldots$, and add the superpotential term

$$\int {\rm d}\theta^+ \Lambda^a_- J_a(\Phi) + {\rm c.c.}\,. \tag{3.1.25}$$

Here λ^a_-, the lowest component of the Fermi superfield Λ^a_-, is a left-moving chiral fermion. For such a superpotential to exist, the Fermi multiplets Λ^a_- must carry $U(1)_{\rm N}$ charge $J_{\rm N}[\Lambda^a_-] = \frac{1}{2}$. Furthermore we require Λ^a_- and $J_a(\Phi)$ to be uncharged under the spacetime gauge group. This requirement is motivated by our experience with explicit string theory models – see in particular Section 3.3.2 – and comes from the microscopic Green-Schwarz origin of the fermions λ^a_-. Indeed, EFT strings typically uplift to 'movable' brane configurations supporting corresponding Green-Schwarz fermions, which are neutral under the bulk gauge symmetries.[7]

A priori, the number of $U(1)_{\rm N}$ charged Fermi multiplets Λ^a_- and the number of unobstructed moduli of the NLSM might be uncorrelated. However, we propose that the NLSM should obey a certain minimality principle: The number of Fermi multiplets Λ^a_-, $n_{\rm N}$, should be given by the minimal number needed in order to account for the potential higher order obstructions in the target space of the NLSM. In other words, at the generic point of the moduli space the number of unobstructed directions should correspond to

$$n_{\rm C}^{\rm eff} := n_{\rm C} - n_{\rm N}\,. \tag{3.1.26}$$

In the sequel, we will assume this principle to hold at least for generators of the cone $\mathcal{C}^{\rm EFT}_{\rm S}$ of EFT strings. The intuition between this non-trivial assumption is that the $U(1)_{\rm N}$ charged Fermi multiplets Λ^a_- pair up with obstructed gauge-neutral chiral multiplets to form the analogue of a 'vector-like pair', i.e. an $\mathcal{N} = (2,2)$ chiral multiplet, leaving behind a collection of $n_{\rm C} - n_{\rm N}$ genuinely chiral $\mathcal{N} = (0,2)$ supermultiplets associated with the unobstructed directions in the moduli space of the NLSM. The minimality principle then states that there are no extra unpaired $U(1)_{\rm N}$ Fermi multiplets left.

In particular, the minimality principle implies that $n_{\rm C} - n_{\rm N} \geq 0$, at least for genuinely $(0,2)$ EFT strings.[8] On the other hand, in some cases the EFT string may support an enhanced UV $\mathcal{N} = (2,2)$ or higher non-chiral supersymmetry. For this to be possible, the Fermi multiplets Λ^a_- necessarily pair with *all* the $n_{\rm C} + 1$ chiral multiplets, including the universal one (3.1.23). As a result, a minimal non-chiral $\mathcal{N} = (2,2)$ symmetry requires that $n_{\rm C} - n_{\rm N} = -1$ and $n_{\rm F} = 0$. Hence, in general

$$n_{\rm C} - n_{\rm N} \geq -1\,, \tag{3.1.27}$$

[7] In the Green-Schwarz formulation, branes are described by their embedding in the bulk superspace, whose odd coordinates correspond to the brane fermions.

[8] Actually, we generically expect that $n_{\rm C} - n_{\rm N} \geq 1$, since generic EFT strings are associated with 'movable' internal configurations, whose deformations should be represented by unobstructed chiral multiplets. The extreme case $n_{\rm C} - n_{\rm N} = 0$ should instead be associated with the non-generic purely stringy rigid $w = 1$ EFT string flows mentioned above.

where the bound is saturated in presence of enhanced non-chiral world-sheet supersymmetry, while $n_{\rm C} - n_{\rm N} \geq 0$ for strictly chiral world-sheet supersymmetry.

The subset of the bosonic fields φ corresponding to the unobstructed moduli of the NLSM can enjoy an axionic shift symmetry which can be gauged by the bulk gauge fields. We will describe this effect in Section 3.2.2. In principle, some of the chiral multiplets might also carry a linear realization of the bulk gauge algebra.

Finally, the internal world-sheet sector can include also $n_{\rm F}$ chiral Fermi multiplets $\Psi_- = \psi_- + \dots$ which are neutral under $U(1)_{\rm N}$, taking values in some vector bundle defined on the NLSM target space [103–105]. This type of Fermi multiplets can be charged under the four-dimensional gauge symmetry and couple to the corresponding gauge vectors.

To sum up, we will henceforth assume that an EFT string supports a weakly-coupled $\mathcal{N} = (0, 2)$ NLSM, described by $n_{\rm C} + 1$ chiral multiplets, $n_{\rm N}$ $U(1)_{\rm N}$ charged Fermi multiplets as well as $n_{\rm F}$ $U(1)_{\rm N}$ neutral Fermi multiplets. The relevant possible charges of the corresponding world-sheet fermions are summarized in the following table:

Fermion	#	$U(1)_{\rm N}$ charge	$U(1)_A$ charge	G_I repr.	(0,2) multiplet
ρ_+	1	$\frac{1}{2}$	0	$\mathbf{1}$	chiral U
χ_+	$n_{\rm C}$	$-\frac{1}{2}$	$*$	$*$	chiral Φ
ψ_-	$n_{\rm F}$	0	q_A	$\mathbf{r}_I$	Fermi Ψ_-
λ_-	$n_{\rm N}$	$\frac{1}{2}$	0	$\mathbf{1}$	Fermi Λ_-

(3.1.28)

3.2 Anomaly matching and quantum gravity constraints

We are now in a position to derive in this section the main results of this chapter, the quantum gravity bounds (3.2.4) and (3.2.26). The strategy is to use that the 't Hooft anomaly associated with the world-sheet theory supported by the EFT string must match the expression (3.1.6), which was obtained by requiring the cancellation of the anomaly inflow contribution from the four-dimensional $\mathcal{N} = 1$ supersymmetric theory.

3.2.1 Gravitational/$U(1)_{\rm N}$ anomalies and curvature-squared bounds

Let us start with the gravitational and $U(1)_{\rm N}$ anomalies. From the fermionic charges (3.1.28) the corresponding anomaly polynomial is [106]

$$I^{\rm ws}_{4\,\mathbf{e}}|_{{\rm grav}+U(1)_{\rm N}} = -\frac{n_{\rm F} - n_{\rm C} + n_{\rm N} - 1}{192\pi^2}\,{\rm tr}(R_W \wedge R_W) + \frac{n_{\rm C} - n_{\rm N} + 1}{32\pi^2} F_{\rm N} \wedge F_{\rm N}\,. \tag{3.2.1}$$

Our conventions for the computation of the anomaly polynomial are summarised in Appendix A.1.

The 't Hooft anomaly (3.2.1) must match the third line of (3.1.17). We then arrive at the identifications

$$\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle = n_{\rm F} - n_{\rm C} + n_{\rm N} - 1\,, \tag{3.2.2a}$$

$$\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e} \rangle = 3(n_{\rm C} - n_{\rm N} + 1)\,. \tag{3.2.2b}$$

Note that, according to the above characterization of the world-sheet spectrum, an enhanced non-chiral world-sheet supersymmetry precisely corresponds to the separate vanishing of $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle$ and $\langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e} \rangle$.

Since by the assumed EFT string completeness the EFT string charges should generate the entire lattice of string charges, (3.2.2a) implies that

$$\boxed{\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle \in \mathbb{Z} \qquad \forall \mathbf{e} \in \mathcal{C}^{\rm EFT}_{\rm S}\,,} \tag{3.2.3}$$

which refines the EFT quantization condition (2.1.20). Furthermore, by combining the matching condition (3.2.2b) with (3.1.27) we deduce the bound

$$\boxed{\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e} \rangle \in 3\mathbb{Z}_{\geq 0} \qquad \forall \mathbf{e} \in \mathcal{C}^{\rm EFT}_{\rm S}\,,} \tag{3.2.4}$$

which should be saturated for enhanced non-chiral world-sheet supersymmetry. The equations (3.2.2) also imply that

$$4\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e} \rangle = 3n_{\rm F}\,, \tag{3.2.5}$$

from which, since $n_{\rm F} \geq 0$, we can then extract another similar bound:

$$\boxed{4\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e} \rangle \in 3\mathbb{Z}_{\geq 0} \qquad \forall \mathbf{e} \in \mathcal{C}^{\rm EFT}_{\rm S}\,.} \tag{3.2.6}$$

We observe that (3.2.6) follows from (3.2.4) if $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle \geq 0$, and vice versa if $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle \leq 0$.

If $\hat{\mathbf{C}}(\mathbf{e})$ is non-trivial, in principle (3.2.4) and (3.2.6) allow for negative $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle$, for some $\mathbf{e} \in \mathcal{C}^{\rm EFT}_{\rm S}$. On the other hand, in the explicit string theory models that we consider in this thesis, this does not occur. It is then tempting to promote this observation to a general property of quantum gravity models and, in this sense, we can regard (3.2.4) as the strongest bound. It would certainly be more satisfactory to derive this additional condition from a self-contained quantum gravity argument, but we leave this interesting question to the future.

3.2.2 Gauge anomalies and rank bound

We now turn to the world-sheet 't Hooft gauge anomalies, whose polynomial should match the second line of (3.1.17). The positive semi-definite coefficients $\langle \mathbf{C}^{AB}, \mathbf{e} \rangle$ and $\langle \mathbf{C}^{I}, \mathbf{e} \rangle$ appearing in (3.1.17) – see (3.1.18) – identify the gauge sector that 'interact' with the EFT string, in the sense that the corresponding (s)axionic couplings in (2.1.11) change along the EFT string flow

(2.2.9). In the sequel we will focus on the rank of this gauge sector, which is identified as

$$r(\mathbf{e}) \equiv \text{rank}\{\langle \mathbf{C}^{AB}, \mathbf{e}\rangle\} + \sum_{I|\langle \mathbf{C}^I,\mathbf{e}\rangle>0} \text{rk}(\mathfrak{g}_I)\,, \tag{3.2.7}$$

as one can easily realize by going to a basis of $U(1)$ gauge fields in which the symmetric matrix $\langle \mathbf{C}^{AB}, \mathbf{e}\rangle$ is diagonalized.[9] In particular, we aim at deriving an upper bound on $r(\mathbf{e})$ from the anomaly matching.

First of all, the Fermi multiplets in (3.1.28) yield the following contribution to the anomaly polynomial $I_{4\,\mathbf{e}}^{\text{ws}}|_{\text{gauge}}$:

$$-\frac{1}{8\pi^2}k_{\text{F}}^{AB}(\mathbf{e})F_A\wedge F_B - \frac{1}{16\pi^2}k_{\text{F}}^I(\mathbf{e})\text{tr}(F\wedge F)_I\,, \tag{3.2.8}$$

with

$$k_{\text{F}}^{AB}(\mathbf{e}) \equiv \sum_{\mathbf{q}\in\text{Fermi}} q^A q^B \quad , \quad k_{\text{F}}^I(\mathbf{e}) \equiv \sum_{\mathbf{r}^I\in\text{Fermi}} \ell(\mathbf{r}^I)\,. \tag{3.2.9}$$

The anomaly matching condition implies that these coefficients provide positive semi-definite contributions to the total anomaly coefficients $\langle \mathbf{C}^{AB}, \mathbf{e}\rangle$ and $\langle \mathbf{C}^{I}, \mathbf{e}\rangle$ appearing in (3.1.17). In particular

$$r_{\text{F}}(\mathbf{e}) \equiv \text{rank}\{k_{\text{F}}^{AB}(\mathbf{e})\} + \sum_{I|k_{\text{F}}^I(\mathbf{e})>0} \text{rk}(\mathfrak{g}_I) \tag{3.2.10}$$

can be regarded as the contribution of the Fermi multiplets to $r(\mathbf{e})$.

Since we are interested in the rank of the gauge algebra, we can actually focus on the Cartan sub-algebra $\mathfrak{h}_I \subset \mathfrak{g}_I$ of each simple gauge factor. On each $\mathfrak{h}_I$ we can pick a basis H^I_α, where $\alpha = 1,\dots,\text{rk}(\mathfrak{g}_I)$, normalised so that $\text{tr}(H^I_\alpha H^I_\beta) = 2\delta_{\alpha\beta}$. If we turn off all the non-Cartan field strength components in (3.2.8), the remaining purely $U(1)$ anomaly polynomial is

$$-\frac{1}{8\pi^2}\, k_{\text{F}}^{\mathcal{AB}}(\mathbf{e})\, F_{\mathcal{A}}\wedge F_{\mathcal{B}}\,, \tag{3.2.11}$$

where $F_{\mathcal{A}} = (F_A, F_{I\alpha})$ collectively represent the remaining $U(1)$ field strengths, and the symmetric matrix $k_{\text{F}}^{\mathcal{AB}}$ has block-diagonal entries given by k_{F}^{AB} and $k_{\text{F}}^I(\mathbf{e})\delta_{I\alpha,I\beta}$. The rank of the matrix $k_{\text{F}}^{\mathcal{AB}}$ can be identified with $r_{\text{F}}(\mathbf{e})$ as defined in (3.2.10). On the other hand we can still apply the first formula in (3.2.9) to this extended abelian sector and write

$$k_{\text{F}}^{\mathcal{AB}} = \sum_{\mathbf{q}\in\text{Fermi}} q^{\mathcal{A}} q^{\mathcal{B}}\,. \tag{3.2.12}$$

Hence, the matrix $k_{\text{F}}^{\mathcal{AB}}$ is the sum of n_{F} matrices of the form $q^{\mathcal{A}}q^{\mathcal{B}}$, which have either rank 0

[9] More explicitly, one can always diagonalize the matrix $\langle \mathbf{C}^{AB}, \mathbf{e}\rangle$ by means of an orthogonal matrix $O_A{}^B$. Then $\hat{A}_A \equiv O_A{}^B A_B$ have diagonal (s)axionic contributions to the kinetic terms couplings, with $\text{rank}\{\langle \mathbf{C}^{AB}, \mathbf{e}\rangle\}$ non-vanishing positive diagonal entries.

or 1, if either all the $q^{\mathcal{A}}$ charges are vanishing or not, respectively. We can now use the general property

$$\text{rank}(M_1 + M_2) \leq \text{rank}(M_1) + \text{rank}(M_2)\,, \tag{3.2.13}$$

which holds for any pair of matrices M_1, M_2. Repeatedly applied to (3.2.12) and combined with (3.2.5), it leads to

$$r_{\rm F}(\mathbf{e}) \leq n_{\rm F} = \frac{4}{3}\langle\tilde{\mathbf{C}}, \mathbf{e}\rangle + \frac{1}{3}\langle\hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle\,. \tag{3.2.14}$$

Consider next the anomaly contribution associated with the chiral multiplets Φ. If some of these multiplets carry a linear realisation of the gauge group, the charged chiral fermions χ_+ yield a negative definite contribution to the anomaly polynomial. One can convince oneself that this cannot increase the bound on the total rank of the gauge algebra. We therefore turn to the remaining possibility that some of the chiral fields φ corresponding to the unobstructed moduli enjoy an axionic shift symmetry which is gauged by the four-dimensional gauge symmetry. According to the minimality principle of the previous section, the number of unobstructed directions is $n_{\rm C}^{\rm eff} := n_{\rm C} - n_{\rm N}$. One can then introduce a set of 'axionic' $\mathcal{N} = (0,2)$ chiral multiplets $\tau_r \simeq \tau_r + 1$, with $r = 1, \ldots, n_{\rm A} \leq n_{\rm C}^{\rm eff}$, which transform as

$$\tau_r \to \tau_r + \frac{1}{2\pi} N_r{}^{\mathcal{A}} \lambda_{\mathcal{A}} \tag{3.2.15}$$

under the bulk $U(1)$ gauge transformations $A_{\mathcal{A}} \to A_{\mathcal{A}} + {\rm d}\lambda_{\mathcal{A}}$. Note that the fermions in the chiral multiplets τ_r do not transform under (3.2.15) and then do not contribute to the quantum anomaly. On the other hand, following [107, 108] one can write down the following Green-Schwarz-like contributions to world-sheet effective action,

$$-M^{\mathcal{A}r} \int_W {\rm Re}\,\tau_r\, F_{\mathcal{A}} - \frac{1}{8\pi} Q^{\mathcal{A}\mathcal{B}} \int_W A_{\mathcal{A}} \wedge A_{\mathcal{B}}\,, \tag{3.2.16}$$

with

$$Q^{\mathcal{A}\mathcal{B}} = -Q^{\mathcal{B}\mathcal{A}} \equiv (MN)^{\mathcal{A}\mathcal{B}} - (MN)^{\mathcal{B}\mathcal{A}}\,, \tag{3.2.17}$$

where $(MN)^{\mathcal{A}\mathcal{B}} \equiv M^{\mathcal{A}r} N_r{}^{\mathcal{B}}$. Note that the term (3.2.16) admits a supersymmetric extension in terms of $\mathcal{N} = (0,2)$ world-sheet vector multiplets [107, 108], while it cannot be added to an $\mathcal{N} = (2,2)$ theory [109].

From (3.2.15) we see that the variation of (3.2.16) under a gauge transformation $A_{\mathcal{A}} \to A_{\mathcal{A}} + {\rm d}\lambda_{\mathcal{A}}$ gives

$$-\frac{1}{4\pi} k_{\rm C}^{\mathcal{A}\mathcal{B}}(\mathbf{e}) \int_W \lambda_{\mathcal{A}}\, F_{\mathcal{B}}\,, \tag{3.2.18}$$

where we have introduced the symmetric matrix

$$k_{\rm C}^{\mathcal{A}\mathcal{B}}(\mathbf{e}) \equiv (MN)^{\mathcal{A}\mathcal{B}} + (MN)^{\mathcal{B}\mathcal{A}}\,. \tag{3.2.19}$$

This classical violation of the gauge symmetry has the form of an anomaly associated with the polynomial

$$-\frac{1}{8\pi^2}\,k_{\rm C}^{\mathcal{A}\mathcal{B}}(\mathbf{e})\,F_{\mathcal{A}}\wedge F_{\mathcal{B}}\,. \tag{3.2.20}$$

In particular, we can define a corresponding rank

$$r_{\rm C}(\mathbf{e})\equiv {\rm rank}\{k_{\rm C}^{\mathcal{A}\mathcal{B}}(\mathbf{e})\}\,. \tag{3.2.21}$$

Note that the rank of the matrix $(MN)^{\mathcal{A}\mathcal{B}}$, as well as its transposed, cannot exceed $n_{\rm A}\leq n_{\rm C}^{\rm eff}=n_{\rm C}-n_{\rm N}$. Hence applying the property (3.2.13) to (3.2.19) and the identity (3.2.2b), we obtain the upper bound

$$r_{\rm C}(\mathbf{e})\leq 2n_{\rm C}^{\rm eff}=\frac{2}{3}\langle\tilde{\mathbf{C}},\mathbf{e}\rangle+\frac{2}{3}\langle\hat{\mathbf{C}}(\mathbf{e}),\mathbf{e}\rangle-2\,. \tag{3.2.22}$$

By combining (3.2.11) and (3.2.20) we arrive at the maximally abelian anomaly polynomial

$$-\frac{1}{8\pi^2}\,k^{\mathcal{A}\mathcal{B}}(\mathbf{e})\,F_{\mathcal{A}}\wedge F_{\mathcal{B}} \tag{3.2.23}$$

with

$$k^{\mathcal{A}\mathcal{B}}(\mathbf{e})\equiv k_{\rm F}^{\mathcal{A}\mathcal{B}}(\mathbf{e})+k_{\rm C}^{\mathcal{A}\mathcal{B}}(\mathbf{e})\,. \tag{3.2.24}$$

By anomaly matching, this matrix must be positive semi-definite and its rank provides an upper bound for (3.2.7).[10] Hence, by recalling (3.2.14) and (3.2.22) and applying again (3.2.13), we can conclude that

$$r(\mathbf{e})\leq r_{\rm F}(\mathbf{e})+r_{\rm C}(\mathbf{e})\leq n_{\rm F}+2n_{\rm C}^{\rm eff}\,. \tag{3.2.25}$$

In view of (3.2.14) and (3.2.22), we can rewrite this in its final form

$$\boxed{r(\mathbf{e})\leq r(\mathbf{e})_{\rm max}\equiv 2\langle\tilde{\mathbf{C}},\mathbf{e}\rangle+\langle\hat{\mathbf{C}}(\mathbf{e}),\mathbf{e}\rangle-2\qquad\forall\mathbf{e}\in\mathcal{C}_{\rm S}^{\rm EFT}\,.} \tag{3.2.26}$$

The bounds (3.2.4), (3.2.6) and (3.2.26) are the main results of this chapter. In particular, (3.2.26) encodes a correlation between the ranks of the EFT gauge group and higher curvature corrections which is completely unexpected from a purely low-energy EFT viewpoint. We will illustrate the power of these bounds in Section 3.3.1. We stress that the bounds only apply to gauge sectors exhibiting a coupling of the form (2.1.11) and (2.1.19), from which they were derived. Furthermore, the bound (3.2.26) (but not (3.2.4) and (3.2.6)) relies on our assumption that the number of unobstructed moduli of the NLSM is given by (3.1.26).

We expect the bound (3.2.26) to be rather conservative because in many situations only a subset $n_{\rm A}$ of the $n_{\rm C}^{\rm eff}$ unobstructed chiral multiplets enjoy gauged axionic shift symmetries and

[10] Possible charged chiral multiplets would provide negative semi-definite contributions, which can only lower this upper bound.

hence participate in the anomaly matching. This will be elucidated further in concrete F-theoretic realisations in Section 3.3.2. Based on our explicit knowledge of the nature of the chiral multiplets in the NLSM of the EFT strings in F-theory, we will in fact propose the stronger bound (3.3.36) on the rank of the four-dimensional gauge group, which is expected to hold in F-theory compactifications on smooth geometric backgrounds not admitting extra isometries. In other cases, however, such as toroidal heterotic orbifolds commented on after (3.3.104), the bound (3.2.26) can in fact be saturated. Hence, even though in many instances the actual gauge group is of considerably smaller rank than the bound in (3.2.26), we have to content ourselves with this constraint in a general setting.

3.2.3 Some comments

So far we have been agnostic about the contribution $\hat{C}(\mathbf{e})$ to the anomaly inflow and the resulting quantum gravity bounds. In order to extract more precise information from (3.2.4), (3.2.6) and (3.2.26), we now analyse the possibilities identified in Section 3.1.2 in more detail, that is, either vanishing $\hat{C}(\mathbf{e})$ or a contribution of the form (3.1.22). Let us first assume that $\hat{C}(\mathbf{e}) \equiv 0$. In this case (3.2.4) and (3.2.6) reduce to

$$\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle \in 3\mathbb{Z}_{\geq 0} \qquad \forall\, \mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT} \,. \tag{3.2.27}$$

This shows that not only are the constants $\tilde{C}_i$ integral, rather than half-integral as in (2.1.20), but actually $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle$ should be a non-negative multiple of 3. Hence either $\tilde{\mathbf{C}}$ is vanishing or it lies in the cone $\mathcal{C}_{\rm I}$. Recalling the definition (2.1.3) of the saxionic cone, we therefore see that (3.2.27) implies $\langle \tilde{\mathbf{C}}, \mathbf{s} \rangle \geq 0$ (with equality only for $\tilde{\mathbf{C}} = 0$), which then follows from the quantum consistency of EFT strings, at least if $\hat{C}(\mathbf{e}) \equiv 0$. Note that the inequality (3.2.27) should be saturated precisely by the EFT strings with enhanced non-chiral supersymmetry, while for strict chiral world-sheet supersymmetry we should actually have $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle \geq 3$.[11]

Consider next an anomaly contribution of the form (3.1.22). This implies that

$$\langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e} \rangle = \hat{\mathbf{C}}(\mathbf{e}, \mathbf{e}, \mathbf{e}) \equiv \hat{C}_{ijk}\, e^i e^j e^k \,, \tag{3.2.28}$$

where $\hat{C}_{ijk} \in \mathbb{Z}$ are associated with some underlying five-dimensional $\mathcal{N} = 1$ structure which can be detected by the EFT string. Since the EFT string charges form the cone $\mathcal{C}_{\rm S}^{\rm EFT}$, if the condition (3.2.4) holds for a given string charge vector $\mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}$, then it should also hold for $\mathbf{e}' = N\mathbf{e}$, where $N > 0$ is any positive integer. If $\hat{\mathbf{C}}(\mathbf{e}, \mathbf{e}, \mathbf{e}) \neq 0$, by imposing the condition (3.2.4) for $\mathbf{e}' = N\mathbf{e}$ with $N \to \infty$, we obtain $N^3 \left[\hat{\mathbf{C}}(\mathbf{e}, \mathbf{e}, \mathbf{e}) + \mathcal{O}(1/N^2)\right] \geq 0$ and then

$$\hat{C}(\mathbf{e}, \mathbf{e}, \mathbf{e}) \geq 0 \,. \tag{3.2.29}$$

[11] If one considers models admitting a geometric higher dimensional UV-completion, by recalling (3.2.2b) and footnote 8 this bound could be further restricted to $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle \geq 6$.

We furthermore observe that the condition (3.2.29) follows also from the five-dimensional arguments of [39], and should then hold for our setting as well.

Note that (3.1.22) and (3.2.29) also imply that $\langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle \leq \langle \hat{\mathbf{C}}(N\mathbf{e}), N\mathbf{e}\rangle$ if $N \geq 1$. Since the rank of the gauge group interacting with an EFT string does not change if we rescale the string charge vector, that is $r(N\mathbf{e}) = r(\mathbf{e})$, we deduce that the strongest bounds (3.2.26) on the gauge group rank are obtained by using primitive EFT charge vectors. Of course this is also true if $\hat{\mathbf{C}}(\mathbf{e}) \equiv 0$. Furthermore, applying (3.2.13) to the matrix $\langle \mathbf{C}^{AB}, \mathbf{e}_1 + \mathbf{e}_2\rangle$, we obtain

$$
\begin{aligned}
r(\mathbf{e}_1+\mathbf{e}_2) \leq\ & r(\mathbf{e}_1) + r(\mathbf{e}_2) \\
& \leq 2\langle \tilde{\mathbf{C}}, \mathbf{e}_1+\mathbf{e}_2\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}_1), \mathbf{e}_1\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}_1), \mathbf{e}_2\rangle - 4 \\
& \leq 2\langle \tilde{\mathbf{C}}, \mathbf{e}_1+\mathbf{e}_2\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}_1), \mathbf{e}_1\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}_1), \mathbf{e}_2\rangle - 2\,.
\end{aligned} \tag{3.2.30}
$$

Now, a convexity property of the form

$$
\langle \hat{\mathbf{C}}(\mathbf{e}_1), \mathbf{e}_1\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}_2), \mathbf{e}_2\rangle \leq \langle \hat{\mathbf{C}}(\mathbf{e}_1+\mathbf{e}_2), \mathbf{e}_1+\mathbf{e}_2\rangle \tag{3.2.31}
$$

would guarantee that, if $\mathbf{e}_1$ and $\mathbf{e}_2$ satisfy (3.2.26) then $\mathbf{e}_1+\mathbf{e}_2$ satisfies (3.2.26) too. Hence, in this case the strongest constraints would be obtained by considering the generators of the cone $\mathcal{C}_{\rm S}^{\rm EFT}$ of EFT string charges, that is, the *elementary* EFT string charges. This property has a clear physical interpretation. One can think of any EFT string as a (possibly threshold) bound state formed by recombining elementary EFT strings. According to this property, if these elementary strings are separately consistent, then their bound state should be consistent too.

The condition (3.2.31) is trivially satisfied if $\hat{\mathbf{C}}(\mathbf{e}) \equiv 0$. If instead (3.1.22) holds, (3.2.31) is for instance guaranteed if

$$
\hat{C}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \geq 0 \qquad \forall \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathcal{C}_{\rm S}^{\rm EFT}\,. \tag{3.2.32}
$$

In [39] an analogous property has been proposed to hold for general $\mathcal{N}=1$ five-dimensional supergravities compatible with quantum gravity. This property would descend to (3.2.32) in our four-dimensional EFTs. We will see that (3.2.32), and hence (3.2.31), indeed holds in the explicit string models that we will consider.

Note also that these results trivialize in the case of enhanced non-chiral world-sheet supersymmetry, since in this case $\langle \mathbf{C}^{AB}, \mathbf{e}\rangle = \langle \mathbf{C}^{I}, \mathbf{e}\rangle = \langle \tilde{\mathbf{C}}, \mathbf{e}\rangle = \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle = 0$.

Finally, according to the general formulation with three-form potentials of [67], the two-form potentials $\mathcal{B}_{2,i}$ could be gauged under some two-form gauge transformations. This effect makes some axionic strings 'anomalous', forcing them to be the boundary of membranes. It would be interesting to extend the above arguments to these kinds of strings.

3.3 Test of the quantum gravity bounds

After having derived the bounds in the previous sections, we now turn to checking them in explicit models. Those will be effective $N = 1$ theories of gravity in 4 dimensions coming from string theory compactifications and include examples from type IIB/F-theory on smooth bases, heterotic models and M-theory on G_2 manifolds. In all cases we will find agreement with our most conservative bounds and in some of them also in their stronger form. We will motivate why this is to be expected by using geometric considerations in the F-theory models. In addition, we will underline links with known and conjectured geometric results. In the heterotic models, it will be stressed the interplay among different contributions to the effective action, whose presence is necessary for satisfying our proposed consistency conditions.

3.3.1 Simple examples

In this section we illustrate the implications of the quantum gravity constraints found in Section 4.2.2 for simple EFT models.

Let us start with the most basic possible example: The gauge group G consists of a simple gauge group factor and the perturbative EFT regime is associated with a single chiral field $t = a+\mathrm{i}s$, whose saxion s parametrizes the one-dimensional saxionic cone $\Delta = \mathbb{R}_{>0}$, plus possible additional chiral 'spectators' ϕ. According to the general definition (2.2.10), $\mathcal{C}_{\rm S}^{\rm EFT} = \mathbb{Z}_{\geq 0}$ then represents a single tower of EFT strings, labeled by the charge $e \in \mathbb{Z}_{\geq 0}$.

By supersymmetry, the relevant structure is specified by the saxionic couplings appearing in (2.1.7) and (2.1.16). Let us assume that these take the form

$$-\frac{1}{8\pi}\int (Cs + \ldots)\,\mathrm{tr}(F \wedge *F) - \frac{1}{192\pi}\int (\tilde{C}s + \ldots)\, E_{\rm GB} * 1\,, \tag{3.3.1}$$

where $E_{\rm GB}$ is the Gauss-Bonnet combination (2.1.17) and the ellipses represent possible additional constant and ϕ-dependent terms. As discussed in the previous sections, in general this is a non-trivial assumption about the EFT.

Furthermore we first suppose that the coupling (3.1.11) is vanishing. According to the discussion in Section 3.1.2, this means that we are considering models which do not have a hidden five-dimensional structure. At the low-energy EFT level the constants C and $\tilde{C}$ should just satisfy the quantization conditions (2.1.12) and (2.1.20) and could otherwise be chosen quite arbitrarily. On the other hand, the constraints derived in Section 4.2.2 imply that this is not true for EFTs admitting a quantum gravity completion. First of all, by (3.2.3) we must have $\tilde{C} \in \mathbb{Z}$, rather than just $2\tilde{C} \in \mathbb{Z}$. Furthermore, since (3.1.11) is vanishing, (3.2.4) and (3.2.6) reduce to (3.2.27) and then

$$\tilde{C} = 3k \geq 0 \quad \text{with} \quad k \in \mathbb{Z}_{\geq 0}\,. \tag{3.3.2}$$

We could actually be more precise: either $\tilde{C} = 0$, which means that the EFT strings should support an enhanced non-chiral supersymmetry, or otherwise $\tilde{C} = 3\tilde{k} + 3$ with $\tilde{k} \in \mathbb{Z}_{\geq 0}$.

Consider then the bound (3.2.26). In the present single saxion example, if $C > 0$ we can simply set $r(\mathbf{e}) = \text{rk}(\mathfrak{g})$ and (3.2.26) reduces to the bound

$$\text{rk}(\mathfrak{g}) \leq 2\tilde{C} - 2 = 6\tilde{k} + 4\,, \qquad \tilde{k} \in \mathbb{Z}_{\geq 0} \tag{3.3.3}$$

on the rank of the Lie algebra associated with the gauge group G.

Let us now allow for a non-trivial coupling (3.1.11) which, in view of (3.1.22), takes the form

$$-\frac{1}{24}\hat{C}e^2 \int_\Sigma h_1 \wedge A_{\rm N}\,, \tag{3.3.4}$$

where $\hat{C} \in \mathbb{Z}$ by (3.1.19). Now (3.2.4) and (3.2.6) require that

$$\tilde{C}e + \hat{C}e^3 \in 3\mathbb{Z}_{\geq 0}\,, \quad 4\tilde{C}e + \hat{C}e^3 \in 3\mathbb{Z}_{\geq 0} \qquad \forall e \in \mathbb{Z}_{\geq 0}\,. \tag{3.3.5}$$

The conditions can be solved by setting

$$\left\{\begin{array}{l} \tilde{C} = 3k - \hat{k} \\ \hat{C} = \hat{k}\,, \quad \hat{k} = 0, \dots, 3k \end{array}\right. \text{(case I)} \qquad \text{or} \qquad \left\{\begin{array}{l} \tilde{C} = -\tilde{k} \\ \hat{C} = 3k + 4\tilde{k} \end{array}\right. \text{(case II)}, \tag{3.3.6}$$

for all $k, \tilde{k} \in \mathbb{Z}_{\geq 0}$. The bound (3.2.26) reduces to $\text{rk}(\mathfrak{g}) \leq 2\tilde{C} + \hat{C} - 2$ and from (3.3.6) we then get

$$\text{rk}(\mathfrak{g}) \leq \left\{\begin{array}{ll} 6k - \hat{k} - 2 & \text{(case I)} \\ 3k + 2\tilde{k} - 2 & \text{(case II)} \end{array}\right. . \tag{3.3.7}$$

The above analysis can be immediately generalized to a non-simple gauge group, with saxionic couplings

$$-\frac{1}{4\pi}\int (C^{AB}s + \dots)\, F_A \wedge *F_B - \frac{1}{8\pi}\int (C^I s + \dots)\; \text{tr}(F \wedge *F)_I\,, \tag{3.3.8}$$

where by (2.1.13) we must impose that $C^I \in \mathbb{Z}_{\geq 0}$ and $C^{AB} \in \mathbb{Z}$ is a positive semi-definite matrix. Now $r(\mathbf{e})$ is the total rank of the linearly independent gauge fields which interact with the saxion s, and it is bounded as in the case of a simple gauge group discussed above.

Note that the mere presence of some gauge sector coupling to the (s)axion as in (3.3.8) implies that, necessarily, the higher derivative couplings appearing in (3.3.1) and (3.3.4) cannot be both trivial and the EFT string cannot exhibit enhanced $\mathcal{N} = (2,2)$ supersymmetry. Suppose for instance that, at the UV cut-off scale Λ, our EFT describes an $G = SU(5)$. Then in case I we must necessarily have either $(k, \hat{k}) = (1, 0)$ and then $(\tilde{C}, \hat{C}) = (3, 0)$, or $k \geq 2$, with any $\hat{k} = 0, \dots, 3k$. In case II we must instead impose $3k + 2\tilde{k} \geq 6$, which for instance implies $\hat{C} \geq 6$. If we consider instead a $G = SO(10)$ supersymmetric GUT model, then in case I we must necessarily have either $(\tilde{C}, \hat{C}) = (6 - \hat{k}, \hat{k})$ with $\hat{k} = 0, \dots, 5$ (and $k = 2$), or the other values corresponding to $k \geq 3$. In case II one must impose $3k+2\tilde{k} \geq 7$, which also implies that $\hat{C} \geq 7$. According to the proposal of Section 3.1.2, the non-vanishing of (3.3.4) should practically mean that the strict weak coupling limit of the gauge theory is an at least partial decompactification limit to five dimensions.

Consider next a model with two saxions s^1, s^2 with a general gauge group (2.1.5). One can choose a basis in which (s^1, s^2) parametrize the saxionic cone $\Delta = \mathbb{R}^2_{>0}$. In this basis, the EFT strings have charge vectors $\mathbf{e} = (e^1, e^2) \in \mathcal{C}^{\text{EFT}}_{\text{S}} = \mathbb{Z}^2_{\geq 0}$. By (3.1.18) the components of $\mathbf{C}^{AB} = (C^{AB}_1, C^{AB}_2)$ are positive semi-definite integral matrices, and $\mathbf{C}^I = (C^I_1, C^I_2) \in \mathbb{Z}^2_{\geq 0}$. Imposing for simplicity that the coupling (3.1.11) vanishes, (3.2.4) and (3.2.6) require that $\tilde{\mathbf{C}} = (\tilde{C}_1, \tilde{C}_2) = (3k_1, 3k_2)$, with $k_1, k_2 \in \mathbb{Z}_{\geq 0}$. Hence the EFT contains the Gauss-Bonnet term

$$-\frac{1}{64\pi}\int (k_1 s^1 + k_2 s^2 + \dots)\, E_{\text{GB}} * 1\,. \tag{3.3.9}$$

The values of k_1, k_2 constrain the ranks of the gauge groups coupling to the axions as in (2.1.11). The stricter bounds are obtained by setting $\mathbf{e} = \mathbf{e}_1 \equiv (1,0)$ and $\mathbf{e} = \mathbf{e}_2 \equiv (0,1)$ in (3.2.26), and read

$$r(\mathbf{e}_1) \leq 6k_1 - 2 \quad , \quad r(\mathbf{e}_2) \leq 6k_2 - 2\,. \tag{3.3.10}$$

Here $r(\mathbf{e}_1)$ and $r(\mathbf{e}_2)$ can be identified with the rank of the gauge sector which interact with the saxions s^1 and s^2, respectively, through couplings of the form (3.3.8). If for instance all gauge group factors interact with both saxions, we get a bound on the total rank $\text{rk}(\mathfrak{g}) \leq \min\{6k_1 - 2, 6k_2 - 2\}$. Otherwise, we can only say that $\text{rk}(\mathfrak{g}) \leq 6(k_1 + k_2) - 2$, which is what one gets by applying (3.2.26) to $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$. In any event, a non-trivial gauge group implies that $k_1 + k_2$ cannot vanish, and hence $\tilde{C} \neq 0$ (where we recall that we have assumed for simplicity that $\hat{C} = 0$).

We could systematically proceed by considering more complicated bottom-up EFTs. Instead, we turn to analyzing how our quantum gravity bounds are realised in large classes of $\mathcal{N} = 1$ compactifications of string or M-theory to four dimensions.

3.3.2 Microscopic checks in IIB/F-theory models

We begin our string theoretic tests of the quantum gravity bounds with IIB/F-theory models. Not only will this provide a microscopic confirmation of the general bounds derived so far, but we will also find a stronger version of these bounds which is valid for minimally supersymmetric F-theory compactifications on a *smooth* three-fold base.

F-theory models

Let us first analyse the large volume perturbative regime of F-theory compactifications to four dimensions. Our primary interest is in the part of the four-dimensional gauge theory sector that is supported by 7-branes, postponing a discussion of the D3-brane sector to Section 3.3.2.

The 7-brane gauge sector can be weakly coupled, for large volumes of the wrapped four-cycles, even if we relax the ten-dimensional weak string coupling condition. In this way one can also obtain more general gauge groups, including exceptional ones – see e.g. [86, 110] for reviews.

We then consider a generic F-theory compactification defined by an elliptically fibered Calabi-Yau four-fold Y with three-fold base X. For the time being we assume that the bulk and 7-brane gauge fluxes vanish and comment on the validity of this assumption in Section 3.3.2.

The type IIB axio-dilaton profile is geometrized into the non-trivial elliptic fibration $\pi : Y \to X$, described by the Weierstrass model

$$y^2 = x^3 + fxz^4 + gz^6 \,, \tag{3.3.11}$$

where $[x : y : z] \in \mathbb{P}^2_{(2,3,1)}$ and

$$f \in \Gamma(\overline{K}_X^4) \,, \qquad g \in \Gamma(\overline{K}_X^6) \,, \qquad \Delta \equiv 4f^3 + 27g^2 \in \Gamma(\overline{K}_X^{12}) \,. \tag{3.3.12}$$

In our notation a line bundle – e.g. the anti-canonical bundle $\overline{K}_X$ – and the corresponding divisor are denoted by the same symbol. The anti-canonical class is required to admit an effective representative in order for the Weierstrass model to exist. The non-abelian gauge theory sectors are supported on irreducible effective components $\mathcal{D}^I$ of the discriminant locus $\Delta \simeq 12\overline{K}_X$. Hence, we can write

$$\Delta = n_I \mathcal{D}^I + \mathcal{D}' \simeq 12\overline{K}_X \qquad \text{with} \quad n_I \equiv \text{ord}(\Delta)|_{\mathcal{D}^I} \,, \tag{3.3.13}$$

where $\mathcal{D}'$ is an *effective* divisor not supporting any gauge sector. According to Kodaira's classification [111–113] summarized in Table 3.3.1, the non-abelian gauge algebra G_I along each component $\mathcal{D}^I$ is determined by the vanishing orders ord(f, g, Δ) on $\mathcal{D}^I$.[12]

Abelian gauge algebra factors, on the other hand, are generated by rational sections $\mathbb{S}^A$ of the elliptic fibration independent of the zero-section $\mathbb{S}^0$. As reviewed e.g. in [110, 114], given a rational section $\mathbb{S}^A$, one defines the Shioda map

$$\sigma(\mathbb{S}^A) = \mathbb{S}^A - \mathbb{S}^0 - \pi^*(D(\mathbb{S}^A)) + \ell^A_{I_\alpha} E^{I_\alpha} \,, \tag{3.3.14}$$

where the divisor class $D(\mathbb{S}^A)$ on the base X is chosen such that $\sigma(\mathbb{S}^A)$ is orthogonal to the push-forward of all curve classes on X to the total space Y.[13] The $U(1)_A$ gauge potential is then obtained, in the dual M-theory, by expanding the M-theory three-form C_3 in terms of the

[12] The actual gauge algebra in four dimensions may be even smaller due to gauge flux and monodromies that would lead to non-simply laced algebras.

[13] Furthermore, the exceptional divisors E^{I_α} are the blowup divisors associated with the resolution of the singularities induced in the Weierstrass model of Y by the appearance of the non-abelian gauge algebra G_I. The coefficients $\ell^A_{I_\alpha} \in \mathbb{Q}$ are determined by requiring that $\sigma(\mathbb{S}^A)$ has vanishing intersection with the generic rational fiber of each E^{I_α}.

	$\mathrm{ord}_{\mathcal{D}}(f)$	$\mathrm{ord}_{\mathcal{D}}(g)$	$\mathrm{ord}_{\mathcal{D}}(\Delta)$	singularity
I_0	≥ 0	≥ 0	0	none
I_n, $n \geq 1$	0	0	n	A_{n-1}
II	1	1	≥ 2	none
III	1	≥ 2	3	A_1
IV	≥ 2	2	4	A_2
I_0^*	≥ 2	≥ 3	6	D_4
I_n^*, $n \geq 1$	2	3	$6+n$	D_{4+n}
IV^*	≥ 3	4	8	E_6
III^*	3	≥ 5	9	E_7
II^*	≥ 4	5	10	E_8

Table 3.3.1: Kodaira's classification of fiber degenerations.

two-forms dual to the classes $\sigma(\mathbb{S}^A)$.

The rank of the non-abelian subsector of the gauge group is constrained geometrically by the so-called *Kodaira bound* analysed in [84–87], which study its consequences for the effective theory that can be obtained from F-theory. From Table 3.3.1, the rank $r_I = \mathrm{rank}(G_I)$ of the non-abelian gauge group G_I on the divisor $\mathcal{D}^I$ satisfies the bound

$$r_I < n_I \equiv \mathrm{ord}(\Delta)|_{\mathcal{D}^I}\,. \tag{3.3.15}$$

This bound is an actual inequality which is never saturated. It can be translated into a bound on the total rank of the non-abelian part of the gauge group as follows: Consider a curve Σ on X with the property that $\Sigma \cdot D_{\mathrm{eff}} \geq 1$ for all effective divisors D_{eff}. Then the rank of the total non-abelian gauge sector is constrained by

$$\mathrm{rk}(G_{\mathrm{non-ab}}) \leq \sum_I \mathrm{rk}(G_I)(\Sigma \cdot \mathcal{D}^I) \leq \sum_I n_I(\Sigma \cdot \mathcal{D}^I) + \Sigma \cdot D' = \Sigma \cdot \Delta\,. \tag{3.3.16}$$

We will compare this geometrical Kodaira bound coming from the UV complete description of the model with the EFT string bounds. The rank of the abelian sector, on the other hand, is unconstrained by the Kodaira bound (3.3.16), while it also enters the EFT string constraints. The latter therefore contain valuable new information which cannot immediately be deduced from the Kodaira bound alone. This was already observed in F-theory compactifications to six dimensions in [37] and translated into a bound on the rank of the Mordell-Weil group on elliptic Calabi-Yau three-folds.

To complete the description of the EFT data, note that the relevant saxions s^a coupling to these gauge sectors can be defined as

$$s^a = \frac{1}{2}\int_{D^a} J \wedge J\,, \tag{3.3.17}$$

where J is the Einstein frame Kähler form on X in string units $\ell_{\mathrm{s}} = 1$ and D^a is a basis of divisors. In Type IIB language, the corresponding axions are then given by $a^a = -\int_{D^a} C_4^{\mathrm{RR}}$, where $C_4^{\rm RR}$ is the R-R four-form potential. Let us also introduce a dual basis of two-cycles Σ_a, such that $D^a \cdot \Sigma_b = \delta^a_b$. In the weak string coupling limit, the coupling to the non-abelian gauge sector appearing in (2.1.7) can be obtained by expanding the D7-brane CS term, leading to

$$C^I_a = \mathcal{D}^I \cdot \Sigma_a \,. \tag{3.3.18}$$

This result is usually extrapolated to configurations involving regions on X where the string coupling is non-perturbative.

Similarly the couplings C_a^{AB} between the abelian field strengths and the (s)axions can be identified geometrically as the intersection product

$$\mathbf{C}_a^{AB} = b^{AB} \cdot \Sigma_a \,, \tag{3.3.19}$$

where

$$b^{AB} = -\pi_*(\sigma(\mathbb{S}^A) \cdot \sigma(\mathbb{S}^B)) \tag{3.3.20}$$

is known as the height-pairing associated with the rational sections underlying the definition of the abelian gauge group factors $U(1)_A$ and $U(1)_B$, see (3.3.14). In particular, the (effective) divisors b^{AA} can be viewed as the analogue, for abelian gauge groups, of the divisor $\mathcal{D}^I$ wrapped by a stack of 7-branes supporting the non-abelian gauge group G_I.

It will be crucial for us that the anti-canonical divisor determines the couplings (2.1.19) [87]. By borrowing formula [94, Eq. (5.15)], we obtain

$$\tilde{C}_a = 6 \int_{\Sigma_a} c_1(B) = 6\, \Sigma_a \cdot \overline{K}_X \,. \tag{3.3.21}$$

The results of [87, 94] confirm that this formula (3.3.21) holds for general F-theory compactifications even if it was identified starting from the weak string coupling limit.

EFT strings

The BPS instanton cone $\mathcal{C}_{\rm I}$ characterizing the above perturbative regime can be identified with the cone of effective divisors $D_{\mathbf{m}} = m_a D^a \in {\rm Eff}^1(X)$. Hence, the associated saxionic cone Δ is defined by the conditions

$$\langle \boldsymbol{s}, \mathbf{m} \rangle = \frac{1}{2} \int_{D_{\mathbf{m}}} J \wedge J > 0 \,. \tag{3.3.22}$$

The physical interpretation of this condition is clear: Any instanton charge vector $\mathbf{m} \in \mathcal{C}_{\rm I}$ corresponds to a Euclidean D3-brane wrapping an effective divisor $D_{\mathbf{m}}$ and (3.3.22) just requires that its volume is positive.

As discussed in [34], the corresponding EFT string charges $\mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}$ can be identified with (effective) *movable* curves $\Sigma_{\mathbf{e}} = e^i \Sigma_i \in {\rm Mov}_1(X)$. Indeed, movable curves are characterized by a non-negative intersection with effective divisors [115] and then $\langle \mathbf{m}, \mathbf{e} \rangle = D_{\mathbf{m}} \cdot \Sigma_{\mathbf{e}} \geq 0$. The associated EFT strings are obtained from D3-branes wrapping these movable curves. A refined classification of the types of movable curves and their associated EFT string limits in F-theory has been obtained in [71].

By definition, a movable curve can sweep out the entire space X. The bosonic fields of the associated NLSM therefore have a clear geometric interpretation: they parametrize the moduli space $\mathcal{M}_{\mathbf{e}}$ of $\Sigma_{\mathbf{e}}$ inside X, including directions which are possibly obstructed at higher order. Its tangent space $T\mathcal{M}_{\mathbf{e}}$ can be identified with the cohomology group $H^0(N\Sigma_{\mathbf{e}})$, where $N\Sigma_{\mathbf{e}}$ is the $\Sigma_{\mathbf{e}}$ normal bundle. The metric on $T\mathcal{M}_{\mathbf{e}}$ is inherited from the components of the metric on X which are orthogonal to $\Sigma_{\mathbf{e}}$. These are precisely the directions which are stretched by the flow (2.2.9) generated by the EFT string, suggesting that the EFT string NLSM can be assumed to be weakly coupled.

Indeed, the volume of any effective divisor $\text{Vol}(D_{\mathbf{m}})$ changes as follows under (2.2.9):

$$\text{Vol}(D_{\mathbf{m}}) = \frac{1}{2}\int_{D_{\mathbf{m}}} J \wedge J = \langle \mathbf{m}, \boldsymbol{s}_0 \rangle + \langle \mathbf{m}, \mathbf{e} \rangle \sigma \,. \tag{3.3.23}$$

Hence we see that it increases precisely if $\langle \mathbf{m}, \mathbf{e} \rangle = D_{\mathbf{m}} \cdot \Sigma_{\mathbf{e}} > 0$, i.e. if the effective divisor $D_{\mathbf{m}}$ transversely intersects the generic curve $\Sigma_{\mathbf{e}}$. In this case $\text{Vol}(D_{\mathbf{m}})$ provides a measure of the directions transversal to $\Sigma_{\mathbf{e}}$, and then of the metric on the moduli space $\mathcal{M}_{\mathbf{e}}$. Hence these directions are stretched as we approach the string. From the RG viewpoint, this implies that the EFT string RG-flow scales up the NLSM metric as we move to the UV, in agreement with our general expectation of a weakly-coupled NLSM. In the non generic case in which $\langle \mathbf{m}, \mathbf{e} \rangle = D_{\mathbf{m}} \cdot \Sigma_{\mathbf{e}} = 0$, $\text{Vol}(D_{\mathbf{m}})$ remains constant but can anyway be assumed to be large, hence justifying again the weakly-coupled world-sheet description. A more precise justification of these claims is provided in Appendix A.2.

This expectation is further supported by the anomaly and central charge computations of [94]. Indeed, by using anomaly inflow arguments, it was found that the world-sheet anomaly and central charges precisely match what one gets from a massless spectrum obtained by geometrical arguments, combined with the topological duality twist of [116].[14] This is only possible if the world-sheet sector admits a weakly coupled NLSM description. Furthermore, the anomaly matching of [94] uses only bulk terms of the form (3.1.2), while it does not require any world-sheet term of the form (3.1.11). This further confirms that for these EFT strings we can set $\hat{C}_i(\mathbf{e}) = 0$, as suggested by the absence of an apparent five-dimensional bulk supersymmetric structure.

More precisely, by using the topological duality twist of [116], the spectrum of massless fields of the $\mathcal{N} = (0, 2)$ worldsheet theory was obtained by dimensional reduction of the $\mathcal{N} = 4$ Super-Yang-Mills theory of a single D3-brane wrapping the curve $\Sigma_{\mathbf{e}}$. Apart from one universal chiral

[14]The analysis of [94] assumes that $\Sigma \cdot \overline{K}_X \geq 0$, which is indeed satisfied by our EFT strings since $\overline{K}_X$ is effective.

multiplet U associated with the center-of-mass motion of the string in four dimensions, there are two types of extra chiral multiplets, $\Phi^{(1)}$ and $\Phi^{(2)}$, of respective multiplicity

$$\begin{aligned} & n_{\rm C}^{(1)} = h^0(\Sigma_{\mathbf{e}}, N_{\Sigma_{\mathbf{e}}/X})\,, \qquad n_{\rm C}^{(2)} = \overline{K}_X \cdot \Sigma_{\mathbf{e}} + g - 1\,, \\ & n_{\rm C} = n_{\rm C}^{(1)} + n_{\rm C}^{(2)}\,, \end{aligned} \tag{3.3.24}$$

where g denotes the genus of the curve $\Sigma_{\mathbf{e}}$. These are accompanied by two types of Fermi multiplets, $\Lambda_-^{(1)}$ and $\Lambda_-^{(2)}$, uncharged under the gauge group on the 7-branes, which are likewise obtained from reduction of the fermionic fields on the D3-brane world-volume and which come with multiplicities

$$\begin{aligned} & n_{\rm N}^{(1)} = h^1(\Sigma_{\mathbf{e}}, N_{\Sigma_{\mathbf{e}}/X}) = h^0(\Sigma_{\mathbf{e}}, N_{\Sigma_{\mathbf{e}}/X}) - \overline{K}_X \cdot \Sigma_{\mathbf{e}}\,, \qquad n_{\rm N}^{(2)} = g\,, \\ & n_{\rm N} = n_{\rm N}^{(1)} + n_{\rm N}^{(2)}\,. \end{aligned} \tag{3.3.25}$$

The uncharged (with respect to the 7-brane gauge group) Fermi multiplets Λ_- are to be contrasted with the

$$n_{\rm F} = 8\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X \tag{3.3.26}$$

charged Fermi multiplets Ψ_- localised at the intersection points of $\Sigma_{\mathbf{e}}$ with the divisors wrapped by the 7-branes. According to the logic of Section 3.2.2, only $n_{\rm C} - n_{\rm N}$ many chiral multiplets can potentially contribute to the gauge anomalies on the worldsheet via anomalous couplings of the form (3.2.16). From the above multiplicities one finds that

$$n_{\rm C}^{(1)} - n_{\rm N}^{(1)} = \overline{K}_X \cdot \Sigma_{\mathbf{e}}\,, \tag{3.3.27a}$$

$$n_{\rm C}^{(2)} - n_{\rm N}^{(2)} = \overline{K}_X \cdot \Sigma_{\mathbf{e}} - 1\,, \tag{3.3.27b}$$

and hence

$$n_{\rm C} - n_{\rm N} = 2\overline{K}_X \cdot \Sigma_{\mathbf{e}} - 1\,. \tag{3.3.28}$$

Note that $n_{\rm C}^{(1)} - n_{\rm N}^{(1)}$ agrees with the number of unobstructed complex geometric deformation moduli of the curve $\Sigma_{\mathbf{e}}$ inside the Kähler 3-fold X (see e.g. [117] for details on how to compute these). In particular, the subtraction of $n_{\rm N}^{(1)}$ accounts for the obstructions of some of the naive $h^0(\Sigma_{\mathbf{e}}, N_{\Sigma_{\mathbf{e}}/X})$ many geometric moduli. In this sector, therefore, the minimality principle invoked in Section 3.1.3 is manifestly realised.

The interpretation of the bosonic components of the chiral multiplets $\Phi^{(2)}$, on the other hand, is rather as a certain type of 'twisted' Wilson line moduli of the topologically twisted SYM theory reduced on $\Sigma_{\mathbf{e}}$. The nature of these fields becomes clearer by realising the strings in the dual M-theory compactified on the elliptic Calabi-Yau four-fold Y [94]. In this picture, the strings appear as MSW strings [118] obtained by wrapping an M5-brane on the surface $\hat{\Sigma}_{\mathbf{e}}$ which is given by the restriction of the elliptic fibration to $\Sigma_{\mathbf{e}}$. The expansion of the chiral two-form on the M5-brane in terms of a basis of primitive $(1,1)$ closed forms leads to left-moving chiral scalar fields on the string worldsheet. Of these, $n_{\mathrm{C}}^{(2)}$ many combine with the right-moving scalars obtained from the reduction of the chiral two-form in the $(0,2)$ and non-primitive $(1,1)$ closed forms on the M5-brane into the bosonic components of the chiral superfields $\Phi^{(2)}$, while the remaining chiral scalars can be dualized into the n_{F} Fermi multiplets charged under the bulk gauge group [94]. This suggests that it is the left-moving scalars of the bosonic components of the fields $\Phi^{(2)}$ which can enjoy an axionic shift symmetry that can be gauged by the part of the four-dimensional gauge group from the 7-brane sector in F-theory and which can participate in the anomaly cancellation. More precisely, taking into account the potential obstructions, the number n_{A} of such axionic moduli is bounded by

$$n_{\mathrm{A}} \leq n_{\mathrm{C}}^{(2)} - n_{\mathrm{N}}^{(2)} = \overline{K}_X \cdot \Sigma_{\mathbf{e}} - 1 \qquad (3.3.29)$$

rather than by $n_{\mathrm{C}} - n_{\mathrm{N}} = 2\overline{K}_X \cdot \Sigma_{\mathbf{e}} - 1$. Indeed, the unobstructed complex geometric deformation moduli counted by $n_{\mathrm{C}}^{(1)} - n_{\mathrm{N}}^{(1)}$ should not exhibit a gauged shift symmetry as long as we are considering F-theory over a smooth three-fold base X with vanishing isometries. By contrast, if the three-fold base X does admit isometries, these can manifest themselves in additional contributions to the gauge anomaly also from the fields of type $\Phi^{(1)}$. This, however, either requires that X is not smooth, for instance of toroidal orbifold type, or that the theory exhibits enhanced supersymmetry. In both scenarios, the additional gauge anomaly contributions from the $\Phi^{(1)}$ sector correspond to Kaluza-Klein gauge symmetry factors in F-theory rather than to the sector from 7-branes.

Microscopic realization of EFT predictions

We are now ready to test our EFT predictions.

Observe first that from (3.3.18) it obviously follows that

$$\langle \mathbf{C}^I, \mathbf{e} \rangle = \mathcal{D}^I \cdot \Sigma_{\mathbf{e}} \geq 0\,, \qquad (3.3.30)$$

for any $\mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}$, as in (3.1.18), because $\mathcal{D}^I$ is effective. The second constraint in (3.1.18) implies that

$$\{\langle \mathbf{C}^{AB}, \mathbf{e} \rangle\} = \{b^{AB} \cdot \Sigma_{\mathbf{e}}\} \geq 0\,, \qquad (3.3.31)$$

in the sense of being a positive semi-definite symmetric matrix. As discussed in [119], it follows from [120] that the diagonal components of the height-pairing b^{AB} can in general be written as the sum of effective divisors and hence the diagonal entries of the matrix in question are guaranteed to be positive. More generally, $\langle \mathbf{C}^{AB}, \boldsymbol{s} \rangle$ coincides with the kinetic matrix for the abelian gauge sector (see e.g.[110]) and must therefore indeed be positive definite, and hence

$\langle \mathbf{C}^{AB}, \mathbf{e}\rangle$ should be positive semi-definite.

The confirmation of the EFT string bounds on the curvature-squared couplings is immediate. Indeed from (3.3.21) we obtain

$$\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle = 6\, \Sigma_{\mathbf{e}} \cdot \overline{K}_X \geq 0 \qquad \forall \mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}\,. \tag{3.3.32}$$

This shows that (3.2.27), or equivalently (3.2.4) or (3.2.6) with $\hat{\mathbf{C}}(\mathbf{e}) = 0$, is manifestly realized in these F-theory models. Note also that, in fact, $\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle$ is a multiple of 6, which is in agreement with the refined bound which one would get from (3.2.2b) by imposing $n_{\rm C} - n_{\rm N} \geq 1$ rather than (3.1.27) – see footnote 8 – to strings with genuinely $\mathcal{N} = (0,2)$ supersymmetry. Indeed, the EFT strings with enhanced non-chiral spectrum correspond to D3-branes not intersecting the bulk 7-branes, that is, for which $\Sigma_{\mathbf{e}} \cdot \overline{K}_X = 0$.

Turning to the bound (3.2.26) on gauge group ranks, we expect it to be related to the Kodaira bound (3.3.15). To address this point, notice first that the rank of the part of the gauge group which is probed by the EFT string of charge vector $\mathbf{e}$ takes the form

$$r(\mathbf{e}) = \sum_{I|\mathcal{D}^I\cdot\Sigma_{\mathbf{e}}\neq 0} r_I + \text{rank}(b^{AB}\cdot \Sigma_{\mathbf{e}})\,. \tag{3.3.33}$$

On the other hand, from (3.3.32) and (3.3.13) we see that (3.2.26) becomes

$$r(\mathbf{e}) \leq r(\mathbf{e})_{\max} \equiv 2\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2 = 12\, \Sigma_{\mathbf{e}} \cdot \overline{K}_X - 2 = \Delta \cdot \Sigma_{\mathbf{e}} - 2\,. \tag{3.3.34}$$

Recall from the discussion in Section 3.2.2 that this bound includes the contribution to the anomaly polynomial both from the charged Fermi multiplets Ψ_- and from the $n_{\rm C} - n_{\rm N}$ unobstructed chiral multiplets, whose bosonic components may in principle transform via a shift under the four-dimensional gauge symmetry, i.e.

$$r(\mathbf{e})_{\max} = r_{\rm F}(\mathbf{e})_{\max} + r_{\rm C}(\mathbf{e})_{\max}\,, \tag{3.3.35a}$$

$$r_{\rm F}(\mathbf{e})_{\max} = \frac{4}{3}\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle = 8\, \Sigma_{\mathbf{e}} \cdot \overline{K}_X = \frac{2}{3}\Delta \cdot \Sigma_{\mathbf{e}}\,, \tag{3.3.35b}$$

$$r_{\rm C}(\mathbf{e})_{\max} = \frac{2}{3}\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2 = 4\, \Sigma_{\mathbf{e}} \cdot \overline{K}_X - 2 = \frac{1}{3}\Delta - 2\,. \tag{3.3.35c}$$

As an important consistency check, note that the value $r_{\rm F}(\mathbf{e})_{\max}$ is given by $n_{\rm F} = \frac{4}{3}\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle = 8\, \Sigma_{\mathbf{e}} \cdot \overline{K}_X$ identified by (3.2.5), which perfectly agrees with the microscopic counting (3.3.26) of left-moving Fermi multiplets charged under the gauge group supported by 7-branes, while $r_{\rm C}(\mathbf{e})_{\max}$ agrees with $2(n_{\rm C} - n_{\rm N})$ as computed in (3.3.28).

However, the discussion at the end of Section 3.3.2 suggests that the conservative bound (3.3.34) can in fact be sharpened in minimally supersymmetric F-theory models over smooth threefold bases. The point is that at best the unobstructed scalars of type $\Phi^{(2)}$ can contribute to the GS anomaly and hence to the bound on the rank. With this logic, we arrive at a stricter bound in F-theory given by

$$r(\mathbf{e})^{\text{strict}}_{\text{max}} = r_{\text{F}}(\mathbf{e})_{\text{max}} + r^{(2)}_{\text{C}}(\mathbf{e})_{\text{max}} = \frac{5}{3}\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2 = 10\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X - 2\,, \tag{3.3.36}$$

where

$$r^{(2)}_{\text{C}}(\mathbf{e})_{\text{max}} = 2(n^{(2)}_{\text{C}} - n^{(2)}_{\text{N}}) = \frac{1}{3}\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2 = 2\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X - 2 \tag{3.3.37}$$

denotes the contribution to the anomaly polynomial from the unobstructed scalars of type $\Phi^{(2)}$, see (3.3.27b).

Let us stress that the bound (3.3.36) refers to the gauge subsector which couples to the EFT string of charge $\mathbf{e}$. To bound the rank of the gauge group supported by all 7-branes, we need to take $\Sigma_{\mathbf{e}}$ to lie strictly in the interior of the movable cone $\text{Mov}_1(X)$ so that it has non-zero intersection with all non-abelian divisors $\mathcal{D}^I$ and height-pairing divisors. The strictest bound is then obtained as the minimal possible value $r(\mathbf{e})^{\text{strict}}_{\text{max}}$ with $\Sigma_{\mathbf{e}}$ in the interior of $\text{Mov}_1(X)$.

With this understanding, we propose (3.3.36), rather than (3.3.34), as a bound on the gauge group rank of minimally supersymmetric F-theory compactifications over smooth threefold base spaces, as these do not admit isometries acting on the internal geometric space. We now proceed to compare these quantum gravity bounds with the geometric bounds on the gauge group rank in F-theory.

Example 1: $X = \mathbb{P}^3$

We begin with the simple choice $X = \mathbb{P}^3$. The one-dimensional group of effective divisors $\text{Eff}^1(X)$ is spanned by the hyperplane class H, in terms of which $\overline{K}_X = 4H$. This implies that the sections (3.3.12) defining the Weierstrass model correspond to the classes

$$\Delta \simeq 48H \quad , \quad f \simeq 16H \quad , \quad g \simeq 24H\,. \tag{3.3.38}$$

In this case the saxionic cone is one-dimensional and $\mathcal{C}^{\text{EFT}}_{\text{S}} \simeq \text{Mov}_1(X)_{\mathbb{Z}} = \text{Eff}_1(X)_{\mathbb{Z}}$ is generated by the curve $H \cdot H \equiv H^2$. To obtain the strongest quantum gravity bounds from the EFT string, we can take $\Sigma_{\mathbf{e}} = H^2$. Since this curve class intersects all possible divisor classes $\mathcal{D}^I$, (3.3.34), and its proposed stronger version (3.3.36), constrains the total rank r_{tot} of abelian and non-abelian gauge algebra factors supported by all 7-branes, which can be detected by the EFT string associated with $\Sigma_{\mathbf{e}}$:

$$r_{\text{tot}} \leq \begin{cases} r(\mathbf{e})_{\text{max}} = 12\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X - 2 = 46 & \text{from (3.3.34)}\,, \\ r(\mathbf{e})^{\text{strict}}_{\text{max}} = 10\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X - 2 = 38 & \text{from (3.3.36)}\,. \end{cases} \tag{3.3.39}$$

We can use (4.3.18) to bound the maximal possible $SU(N)$ gauge algebra supported on any irreducible divisor $\mathcal{D}^{SU(N)}$ appearing in the decomposition (3.3.13) as

$$SU(N^{\rm EFT}_{\rm max}) = SU(r_{\rm max}+1) = \begin{cases} SU(47) & \text{from (3.3.34)}\,, \\ SU(39) & \text{from (3.3.36)}\,. \end{cases} \tag{3.3.40}$$

The first line is only slightly stronger than the naive Kodaira bound in its simple form (3.3.15), which together with Table 3.3.1 would give $SU(N^{\rm Kod}_{\rm max}) = SU(N^{\rm Kod}_{\rm max}) = 49$, as pointed out already in [85]. However, as stressed above, we in fact propose the stricter of the two bounds to hold in F-theory on a smooth base threefold X.

To compare this to actual realisations in F-theory, note first that the Kodaira upper bound cannot be saturated because this would require that all branes are mutually local. In fact, by an explicit analysis of the Weierstrass model, [85] showed that the maximal possible value which can be realised in F-theory on $\mathbb{P}^3$ corresponds to $N^{\rm F}_{\rm max} = 32$. Consistently, this is within the stricter of the two bounds in (3.3.40) based on (3.3.36).

The EFT string bounds are potentially very far-reaching when it comes to constraining the maximal rank of abelian gauge group factors [37]. From (4.3.18) we immediately constrain the maximal number of abelian gauge algebra factors as

$$n^{\rm EFT}_{\rm max}(U(1)) = \begin{cases} 46 & \text{from (3.3.34)}\,, \\ 38 & \text{from (3.3.36)}\,. \end{cases} \tag{3.3.41}$$

A comparable bound cannot be deduced in a straightforward manner from the Kodaira bound since the abelian gauge group factors are generated by extra rational sections of the elliptic fibration. While to each such section one can associate a divisor class on the base via the height-pairing, there does not exist any obvious constraint that forces this divisor class to be bounded by the class of the discriminant. Bounding the number of abelian gauge group factors is equivalent to finding an upper bound for the rank of the Mordell-Weil group of an elliptically fibered Calabi-Yau fourfold, which is an open problem in arithmetic geometry.[15]

For the special example of an $SU(N)$ gauge group, an interesting phenomenon occurs: The geometric bound $N^{\rm F}_{\rm max} = 32$ established in [85] is in fact only marginally stricter than the EFT string bound which one would deduce by taking into account, in the computation of the bound (3.3.35), only the contribution from the charged localised Fermi multiplets: This would give $r \leq r_{\rm F}(\mathbf{e})_{\rm max} = n_{\rm F} = 8\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X = 32$. However, it would be incorrect to conclude from this that $r_{\rm F}(\mathbf{e})_{\rm max}$, rather than the weaker bound $r(\mathbf{e})^{\rm strict}_{\rm max}$, represents the upper bound on the rank of the gauge group more generally.

[15] The upper bound (3.3.41) is the analogue, on fourfolds, of the bound $n(U(1)) \leq 32$ for F-theory compactifications to six dimensions on an elliptic three-fold over base $\mathbb{P}^2$ [37]. The highest Mordell-Weil rank obtained so far for elliptic threefolds is $r_{\rm max} = 10$ [121].

Indeed, it is certainly possible to violate the bound set by $r_{\rm F}(\mathbf{e})_{\rm max}$ in explicit Weierstrass models.[16] For example, consider the maximal possible gauge group of type $E_6^{n_1} \times E_7^{n_2}$ by engineering the various gauge group factors on separate divisors in class H. Compatibility with the (incorrect) bound $r_{\rm F}$ would give that $6n_1 + 7n_2 \leq 32$, with maximal solutions $(n_1, n_2) \in \{(0,4),(1,3),(2,2),(3,2),(4,1),(5,0)\}$. In actuality, a Weierstrass model allows for the construction of a gauge algebra $E_6^{n_1} \times E_7^{n_2}$ with maximal values given by $(n_1, n_2) \in \{(0,4),(1,4),(2,3),(3,2),(4,1),(5,0)\}$ (again on separate gauge divisors of class H). This follows from the vanishing orders of Table 3.3.1 that need to be engineered in the Weierstrass model. The two configurations $E_6 \times E_7^4$ and $E_6^2 \times E_7^3$ hence overshoot the bound from $r_{\rm F}(\mathbf{e})_{\rm max}$ by 2 and 1, respectively, while they respect the weaker bound from $r(\mathbf{e})^{\rm strict}_{\rm max}$, which we propose as the correct bound.

Two warnings concerning the validity of these examples are in place: First, even if all individual exceptional gauge groups are localised on 7-brane stacks wrapping separate divisors in class H, every pair of them intersects in a curve of class H^2. Along this curve, non-Kodaira type singularities appear because the vanishing orders of (f, g) are equal to or bigger than $(4, 6)$ and hence non-minimal. These non-minimalities in codimension-two arise at finite distance in the complex structure moduli space. Extrapolating from the analogous phenomenon in F-theory compactifications on elliptic Calabi-Yau threefolds [122], one expects a genuinely strongly coupled sector localised at the non-minimal loci. To reliably analyse the dynamics one must first remove the non-minimalities by blowing up the intersection loci, thereby separating the exceptional brane stacks. However, this will most likely not affect the original exceptional gauge group factors.

Another concern is pertinent, in fact, to all examples discussed in this section: Since the minimal gauge divisor of class H is non-spin, the Freed-Witten quantization condition [98] requires a half-integer quantized gauge flux in order for the model to be consistent. Depending on the nature of the available flux, the gauge background may reduce the rank of the four-dimensional gauge group or at least break the simple gauge group factors to a group involving abelian factors. Even if the rank were not reduced in this way, the $U(1)$ may become massive via a Stückelberg mechanism. If this is the case, the axion obtained by reducing the RR four-form field on $\mathbb{P}^3$ would likewise acquire a mass - a scenario which we excluded in our derivation of the quantum gravity bounds. To explicitly analyse the types of fluxes available and whether or not they necessarily induce a Stückelberg mass for the axion, one must resolve the fibral singularities of the Weierstrass model[17] and hence, in view of the non-minimalities in codimension two, first

[16] We thank Wati Taylor for pointing this out to us.

[17] In the dual M-theory description, the gauge fluxes are encapsulated in four-form fluxes on the resolved four-fold $\hat{Y}$ which take values in the vertical component of $H^{2,2}(\hat{Y})$ and must be quantized in such a way that $G_4 + \frac{1}{2}c_2(\hat{Y}) \in H^4(\hat{Y}, \mathbb{Z})$. For details we refer to [110] and references therein. Admissible fluxes in F-theory are subject to two transversality conditions and after implementing these it remains to be seen whether fluxes different from the so-called Cartan fluxes are available, which would break the simple gauge group factor. Fluxes which leave the gauge group intact are dual to the matter surfaces [123], but their explicit existence can only be determined after performing the resolution steps.

blow up the base. This involved surgery is beyond the scope of this thesis, but we stress that from the Weierstrass model alone it not yet clear which part of the gauge group survives in the four-dimensional effective theory after taking the flux background into account.

Let us take the optimistic point of view and assume that none of these technicalities undermines the validity of the models with a rank overshooting the stricter bound from $r_{\rm F}(\mathbf{e})_{\rm max}$. In this case, our analysis shows that some of the chiral multiplets, according to our proposal in fact the ones of type $\Phi^{(2)}$, indeed contribute to the anomaly matching. We will develop a better intuition behind this in the following section.

Example 2: $\mathbb{P}^1 \hookrightarrow X \to B$

Consider next an F-theory model in which the base X is a $\mathbb{P}^1$-fibration over a complex surface B. This example will be particularly illuminating for the interpretation of the scalar field contributions to the anomaly matching on the worldsheet and hence to the bound on the gauge group rank.

Quite generally, the twist of a rational fibration $p : X \to B$ is specified by a line bundle $\mathcal{T}$ on the base B,[18] in terms of which the anti-canonical class of X can be written as [124]

$$\overline{K}_X = 2S_- + p^* c_1(\mathcal{T}) + p^* c_1(B)\,. \tag{3.3.42}$$

Here S_- denotes the exceptional section of the rational fibration with the property

$$S_- \cdot S_- = -S_- \cdot p^* c_1(\mathcal{T})\,. \tag{3.3.43}$$

The cone $\mathcal{C}_{\rm I} \simeq \mathrm{Eff}^1(X)$ of effective divisors of X is generated by S_- together with the pullback of the generators of the effective cone of B.

Another section, S_+, of the fibration is related to S_- as $S_+ = S_- + p^* c_1(\mathcal{T})$. It satisfies the relations $S_+ \cdot S_+ = S_+ \cdot p^* c_1(\mathcal{T})$ and $S_- \cdot S_+ = 0$. This section S_+ generates the Kähler cone of X together with the pullback of the Kähler cone generators of B to the rational fibration.

Under F-theory/heterotic duality, this class of theories is dual to the heterotic string compactified on an elliptic fibration over the same base B. The gauge groups from 7-branes wrapped along the two sections S_- and S_+ map to perturbative heterotic gauge groups in either of the two E_8 factors, while gauge groups from 7-branes wrapping divisors pulled back to X from B are of non-perturbative nature in the heterotic frame.

For definiteness, let us specify now to $B = \mathbb{P}^2$ and adopt the notation of [34, p. 53],

$$D^1 = p^*(H)\,, \qquad D^2 = S_+\,, \tag{3.3.44}$$

[18] The threefold X is constructed as the projectivised bundle $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{T})$.

where H is the hyperplane class of $\mathbb{P}^2$. Parametrizing furthermore the twist of the fibration as $c_1(\mathcal{T}) = nH$, the general formula (3.3.42) becomes

$$\overline{K}_X = (3-n)D^1 + 2D^2\,. \tag{3.3.45}$$

According to the above discussion, D^1 and D^2 generate the Kähler cone of X. The dual curves

$$\Sigma_1 = S_- \cdot p^*(H) = (D^2 - nD^1)\cdot D^1\,, \quad \Sigma_2 = p^*(H)\cdot p^*(H) = D^1 \cdot D^1 \tag{3.3.46}$$

(such that $D^a \cdot \Sigma_b = \delta^a_b$) are effective and generate the Mori cone. In particular, Σ_1 can be identified with a $\mathbb{P}^1$ in the $\mathbb{P}^2$ base, while Σ_2 can be regarded as the $\mathbb{P}^1$ fibre of X.

The cone $\mathcal{C}_{\rm I} \simeq {\rm Eff}^1(X)$ of effective divisors is generated by $D^1 = p^*(H)$ and $S_- = D^2 - nD^1$. The dual cone $\mathcal{C}^{\rm EFT}_{\rm S} \simeq {\rm Mov}_1(X)$ of movable curves $\Sigma_{\mathbf{e}} = e^a\Sigma_a$ is given by

$$\mathcal{C}^{\rm EFT}_{\rm S} = \{\mathbf{e} \in (e^1, e^2) \in \mathbb{Z}_{\geq 0} | e^2 \geq n e^1\} \tag{3.3.47}$$

and is generated by the charges

$$\mathbf{e}_1 = (1, n)\,, \quad \mathbf{e}_2 = (0,1)\,. \tag{3.3.48}$$

Note that $\mathcal{C}^{\rm EFT}_{\rm S}$ is smaller than the cone of possible BPS strings, which can be identified with the Mori cone of effective curves. In particular, the elementary BPS charge $\mathbf{e} = (1,0)$ corresponding to a curve on the base alone does not give rise to an EFT string for $n \geq 1$, but it must be combined with n copies of the charge of an EFT string associated with the rational fiber.

To evaluate the bound (3.3.34), we compute the intersection number

$$\Sigma_{\mathbf{e}} \cdot \overline{K}_X = (3-n)e^1 + 2e^2. \tag{3.3.49}$$

The bound (3.3.34) and the proposed stronger form (3.3.36) then become

$$r(\mathbf{e}) \leq \begin{cases} r(\mathbf{e})_{\rm max} = 12(3-n)e^1 + 24e^2 - 2 & \text{from (3.3.34)}\,, \\ r(\mathbf{e})^{\rm strict}_{\rm max} = 10(3-n)e^1 + 20e^2 - 2 & \text{from (3.3.36)}\,, \end{cases} \tag{3.3.50}$$

which constrains the rank of the gauge group that can be detected by an EFT string with charge $\mathbf{e}$. The strongest constraints are obtained by applying this bound to $\mathbf{e}_1$ and $\mathbf{e}_2$. Specifying directly to the proposed stronger bound, one finds

$$r(\mathbf{e}_1) \leq 28 + 10n\,, \quad r(\mathbf{e}_2) \leq 18\,. \tag{3.3.51}$$

Suppose first that the semi-simple gauge sector is supported on effective divisors $\mathcal{D}^I = m^I_1 D^1 + m^I_2(D^2 - nD^1) \subset \{\Delta = 0\}$ with $m^I_1, m^I_2 > 0$. In this case both $r(\mathbf{e}_1)$ and $r(\mathbf{e}_2)$ correspond to the rank of the total semi-simple gauge algebra, since it is entirely detected by both elementary EFT strings. Since $n \geq 0$ we obtain the bound

$$\text{rk}(G_{\text{semi-simple}}) \leq 18\,. \tag{3.3.52}$$

We could instead assume that the discriminant locus splits in disconnected effective components

$$\Delta = n_1 \mathcal{D}^1 + n_2 \mathcal{D}^2 + \mathcal{D}'\,, \tag{3.3.53}$$

where $\mathcal{D}^1 = D^1$ and $\mathcal{D}^2 = D^2 - nD^1$ support different semi-simple gauge groups, G_1 and G_2 respectively. Then $r(\mathbf{e}_1)^{\text{strict}}_{\text{max}}$ bounds the rank of G_1 and $r(\mathbf{e}_2)^{\text{strict}}_{\text{max}}$ bounds the rank of G_2 as detected by the EFT strings. In this case we obtain

$$\text{rk}(G_1) \leq 28 + 10n\,, \quad \text{rk}(G_2) \leq 18\,. \tag{3.3.54}$$

We furthermore note that as in the case of $\mathbb{P}^3$, the contribution to the bound from the axionic sector is indeed needed - taking into account only the charged Fermi multiplets in (3.3.35) would lead to the (incorrect) bounds

$$\text{rk}(G_1) \leq 24 + 8n\,, \quad \text{rk}(G_2) \leq 16 \qquad \text{based on Fermi multiplets only,} \tag{3.3.55}$$

which are easily violated for instance in Weierstrass models with exceptional gauge group factors.

Indeed, for simplicity take $n = 0$, so that $X = \mathbb{P}^1 \times \mathbb{P}^2$. One can overshoot the first constraint in (3.3.55) for instance by realising a configuration with gauge group $G_1 = E_6^3 \times E_7$ and $\text{rk}(G_1) = 25$, or $G_1 = E_6^2 \times E_7^2$ and $\text{rk}(G_1) = 26$, where the simple gauge factors are supported on four separate (but intersecting) divisors in class D^1. These examples are analogous to the ones discussed in the previous section for the base $X = \mathbb{P}^3$, and the same remarks concerning the viability of these models apply.

More illuminating for us is the second bound in (3.3.55) compared to (3.3.54). It is violated (again for $n = 0$) for instance by a configuration of three simple gauge group factors along three separate divisors in class D^2 such that $G_2 = E_6^3$ or $G_2 = E_7^2 \times SO(8)$ or $G_2 = E_8 \times E_6 \times SO(8)$. The non-abelian gauge divisors can be separated along their common normal direction on the $\mathbb{P}^1$-fiber; nonetheless, there appear non-minimal Kodaira type fibers in codimension two or three, or both.[19] For example, the most general way to engineer the model with gauge group

[19] Also the second complication encountered already on $\mathbb{P}^3$ persists, namely that the gauge divisor D^2 is non-spin and gauge flux must be included to cancel the Freed-Witten anomaly; however we expect that this is possible without affecting the rank of the gauge group.

$G_2 = E_6^3$ is to set, in the Weierstrass equation,

$$f \equiv 0\,, \qquad g = p_1^4(u,v)\, q_1^4(u,v)\, r_1^4(u,v)\, s_{18}(u_1,u_2,u_3)\,, \tag{3.3.56}$$

where $[u_1 : u_2 : u_3]$ and $[u : v]$ denote homogeneous coordinates on $\mathbb{P}^2$ and $\mathbb{P}^1$, respectively, and p_1, q_1, r_1, s_{18} represent generic homogeneous polynomials of indicated degrees in the coordinates in brackets. Note that $\Delta = 4f^3 + 27g^2 = 27g^2$ because we were forced to set $f \equiv 0$ to accommodate the E_6^3 factor. As a result, over the divisor $s_{18} = 0$, the fiber enhances to Kodaira Type II with $\mathrm{ord}(f,g,\Delta) = (\infty, 1, 2)$. At the intersection locus of $s_{18} = 0$ with any of the three E_6 divisors $p_1 = 0$ or $q_1 = 0$ or $r_1 = 0$, the fiber enhances to Kodaira Type II*, but at the $s_{18} \cdot s_{18} = 18^2$ points on $\mathbb{P}^2$ given by the self-intersection of $s_{18} = 0$, there occurs a non-minimal fiber with vanishing orders $\mathrm{ord}(f,g,\Delta) = (\infty, 6, 12)$. To reliably analyse the model, this non-minimal singularity would first have to be removed via a blowup in the base.

Interestingly, however, the codimension-three non-minimalities can be avoided by replacing the base $\mathbb{P}^2$ by a rational elliptic surface dP_9, which is an elliptic fibration over $\mathbb{P}_1$ with generically 12 singular fibers. In this case, the section $s_{18}(u_1,u_2,u_3)$ in (3.3.56) is replaced by a general section $s \in \Gamma(6\bar{K}_{\mathrm{dP}_9})$. The anti-canonical class of dP_9 coincides with the class of the elliptic fiber and hence has vanishing self-intersection, $\bar{K}_{\mathrm{dP}_9} \cdot \bar{K}_{\mathrm{dP}_9} = 0$. As a result, there appear no codimension-three non-minimal fibers since $s \cdot s = 0$. For the sake of simplicity we will focus on this geometric example in the sequel. Clearly, the change of basis from $\mathbb{P}^2$ to dP_9 does not affect the second bound in (3.3.54) or (3.3.55), which only involves the EFT string with charge $\mathbf{e}_2$ associated with the $\mathbb{P}^1$-fiber of X (while the first bound in both equations changes).

Focusing on this second bound, to understand why (3.3.54), rather than (3.3.55), gives the correct bound, recall that the EFT string with charge $\mathbf{e}_2$ is the critical heterotic string of the dual compactification of the heterotic theory on an elliptic fibration over the base B of X. As discussed at the end of Section 3.3.2, there are two types of chiral scalar fields, $\Phi^{(1)}$ and $\Phi^{(2)}$, in the NLSM of this string. Since the genus of $\Sigma_{\mathbf{e}_2} = \mathbb{P}^1$ vanishes, $n_{\mathrm{N}}^{(1)} = 0 = n_{\mathrm{N}}^{(2)}$ and hence both types of scalars correspond to unobstructed moduli directions. The $n_{\mathrm{C}}^{(1)} - n_{\mathrm{N}}^{(1)} = n_{\mathrm{C}}^{(1)} = \bar{K}_X \cdot \Sigma_{\mathbf{e}_2} = 2$ complex moduli of the first type parametrize the motion of the heterotic string along the base of the dual elliptic fibration, while $n_{\mathrm{C}}^{(2)} - n_{\mathrm{N}}^{(2)} = n_{\mathrm{C}}^{(2)} = \bar{K}_X \cdot \Sigma_{\mathbf{e}_2} - 1 = 1$ counts the complex modulus of the dual heterotic string along the heterotic torus fiber. As long as the base of X admits no isometries - as is the case whenever X is smooth and the theory minimally supersymmetric - the scalars encoded in the chiral fields of type $\Phi^{(1)}$ cannot participate in the gauge anomaly cancellation in F-theory. Indeed, they are geometric moduli in a geometry without extra shift symmetries. By contrast, the two real scalars encoded in $\Phi^{(2)}$ can enjoy a gauged shift symmetry: They parametrize the motion of the heterotic string along the torus fiber of the dual heterotic geometry, which is responsible for the difference between the two bounds (3.3.54) compared to (3.3.55).

The rationale behind this claim is completely analogous to the simpler setup of F-theory compactified on an elliptic K3 surface to eight dimensions.[20] In such models, a D3-brane wrapped along the $\mathbb{P}^1$ base of the K3 surface is known to be dual to a heterotic F1 string in 8d. The zero mode spectrum of the $N = (0,8)$ supersymmetric worldsheet theory can be derived by dimensional reduction, as e.g. in [94]. It consists of one $\mathcal{N} = (0,8)$ hypermultiplet (8 real scalars in the $\mathbf{6}$ of $SO(6)_{\mathrm{T}}$ and two $SO(6)_{\mathrm{T}}$ singlets accompanied by 8 fermions in the $\mathbf{4}+\bar{\mathbf{4}}$) as well as 16 left-moving fermions transforming as $SO(6)_{\mathrm{T}}$ singlets and organising into Fermi multiplets. The scalars in the $\mathbf{6}$ of $SO(6)_{\mathrm{T}}$ parametrize the center-of-mass motion of the string in eight dimensions and must therefore be uncharged under the gauge group. The only charged fermions are then the $16 = 8\mathrm{deg}(\bar{K}_{\mathbb{P}^1})$ unpaired left-moving fermions. Taking only these into account would suggest a maximal rank for the gauge group of $r_{\mathrm{max}} = 16$, while in fact the maximal rank is known to be $r_{\mathrm{max}}^{\mathrm{K3}} = 18$. This of course matches the counting in the dual heterotic theory compactified on a torus T^2, where the gauge group comprises the ten-dimensional rank 16 gauge group together with 2 Kaluza-Klein $U(1)$ factors. The Fermi multiplets are charged under the part of the gauge group inherited from the ten-dimensional gauge group (in fact, the left-moving fermions generate the rank 16 current algebra present already in ten dimensions), but do not detect the KK $U(1)$s. To see these, one must allow for the two left-moving scalars in the $\mathcal{N} = (0,8)$ hypermultiplet to shift under the associated gauge transformation and in this way to contribute to the worldsheet anomalies. At special points in moduli space, the KK $U(1)$s are indistinguishable from the Cartan $U(1)$s inherited from ten dimensions and combine with part of them into higher rank non-abelian group factors. Examples include the rank 18 exceptional gauge group configurations of the form E_6^3, $E_7^2 \times SO(8)$ or $E_8 \times E_6 \times SO(8)$ studied in F-theory on K3 in [125]. More generally, the classification of maximal non-abelian gauge enhancements for the heterotic string on T^2 in [89] includes many configurations with rank 17 or 18 for the non-abelian sector.

Coming back to the bounds for the four-dimensional F-theory model on X, we see that the contribution to the anomaly from the chiral multiplets of type $\Phi^{(2)}$ is analogous to the way how the heterotic string detects the two KK $U(1)$s in eight dimensions. However, in order for such an interpretation to be possible, the dual heterotic compactification space, which is the target space of the NLSM, must be degenerate such as to admit isometries in two directions while at the same time preserving minimal supersymmetry. In general the dual heterotic string is compactified on an elliptic fibration over the base B. If this elliptic three-fold were smooth, there would be no room for an interpretation in terms of geometric KK $U(1)$s, not even from the elliptic fiber.[21] The way out is that whenever the bound on $r(\mathbf{e}_2) \leq 18$ is saturated, the dual elliptic fibration must degenerate to an orbifold such that the two KK $U(1)$s from the elliptic fiber can survive the projection. In the present example with base $B = \mathrm{dP}_9$, the dual

[20] Recall that D^2 is a copy of $\mathbb{P}^2$ and intersects the $\mathbb{P}^1$ fiber of X in a point; similarly, in F-theory on K3, the 7-branes are points on the base of the K3.

[21] While the generic elliptic fiber contains two 1-cycles, these do not survive as 1-cycles of the elliptic threefold due to monodromies, as required by the (strict) Calabi-Yau condition.

heterotic elliptic fibration is a Schoen manifold [126], which can be viewed as a fiber product $\mathrm{dP}_9 \times_{\mathbb{P}^1} \mathrm{dP}_9$. Schoen manifolds admit degenerations to toroidal orbifolds [127] (see also [128]). We propose such orbifolds as the heterotic duals of the F-theory models with $r(\mathbf{e}_2) = 18$. The orbifold must act in such a way that only two of the generally possible six KK $U(1)$s survive the projection.[22]

It would be desirable to explicitly construct the heterotic duals, also for the blowups of the F-theory models with base B different from dP_9, whenever the rank bound $r(\mathbf{e}_2) \leq 18$ is saturated. More ambitiously, we leave it for future work to determine the structure of the NSLM target space for non-heterotic EFT strings in models saturating the bounds (3.3.36) and to check it for isometries.

Perturbative IIB models with O3-planes

The EFT strings considered in Section 3.3.2 do not detect possible gauge sectors supported by D3-branes. The corresponding complexified gauge coupling can be identified with the type IIB axio-dilaton. Hence this gauge sector is fully perturbative in the ten-dimensional weak coupling limit, in which the compactification space is well described by a Calabi-Yau orientifold.

For simplicity, we will focus on the simpler compactifications with O3-planes only, but more general configurations with O3/O7 planes can be discussed along the same lines. The relevant chiral field associated with this regime is the type IIB axio-dilaton: $t = a + \mathrm{i}s \equiv \tau = C_0^{\mathrm{RR}} + \mathrm{i}e^{-\phi}$. The saxionic cone is one-dimensional and $\mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}} = \mathbb{Z}_{\geq 0}$. An EFT string of charge $e \in \mathbb{Z}_{\geq 0}$ corresponds to e D7-branes wrapping the internal space. Note that the associated EFT string flow drives the bulk sector to a strongly non-geometric regime in which the string frame volume of the compactification space naively goes to zero [34]. On the other hand, according to the estimate of [34] the scaling weight is $w = 1$ and then the Emergent String Conjecture suggests the existence of a weakly coupled dual description in terms of F1 strings.[23]

Without any additional assumption on the Kähler moduli, in this limit the only weakly coupled gauge sector is the one supported by D3-branes, and the total rank of the gauge group is simply given by the number n_{D3} of D3-branes. Taking the gauge group to be completely higgsed to $U(1)^{n_{\mathrm{D3}}}$, corresponding to non-coinciding D3-branes, one can expand the standard DBI action and identify the terms

$$-\frac{1}{4\pi} \sum_{A=1}^{n_{\mathrm{D3}}} \int s F_A \wedge * F_A \,. \tag{3.3.57}$$

[22]Heterotic orbifolds with $2+n$ KK $U(1)$s cannot be dual to elliptic fibrations over a smooth base X in F-theory. The additional n KK $U(1)$s must correspond to KK $U(1)$s, rather than 7-brane $U(1)$s, also on the F-theory side and be encoded in gauged shift symmetries of the scalars of type $\Phi^{(1)}$. Our assumption of a smooth base B of the rational fibration X excludes such situations in the minimally supersymmetric case. If for instance $X = T^4 \times \mathbb{P}^1$, the bulk theory preserves sixteen supercharges and the $n_{\mathrm{C}}^{(1)} = 2$ unobstructed $\Phi^{(1)}$ scalars describing the position of $\Sigma_{\mathbf{e}_2} = \mathbb{P}^1$ in the T^4 direction enjoy shift symmetries. The corresponding contribution to the anomaly raises the rank bound from $r(\mathbf{e}_2)^{\mathrm{strict}}_{\mathrm{max}} = 18$ to $r(\mathbf{e}_2)_{\mathrm{max}} = 22$, matching the bound found in [38].

[23]For example, D7 strings in the toroidal models – see for instance [75] for a discussion in a similar spirit – can be T-dualized to a D1-string and S-dualized to an F1-string.

By comparison with (2.1.11), we see that

$$C^{AB} = \delta^{AB} \tag{3.3.58}$$

and $C^I = 0$.

The higher curvature terms (2.1.19) come from both the D3-brane and O3-plane higher curvature couplings. The D3-brane contribution stems from the action [129–131]

$$-2\pi n_{\rm D3} \int_{\rm D3} C_0^{\rm RR} \sqrt{\hat{\mathcal{A}}\,(\mathcal{R})} = \frac{2\pi n_{\rm D3}}{48} \int C_0^{\rm RR}\, p_1(M) + \dots\,, \tag{3.3.59}$$

where $\hat{\mathcal{A}}$ is the A-roof genus and we are simplifying the formulas by setting $\ell_{\rm s} \equiv 2\pi\sqrt{\alpha'} = 1$.[24] The O3-planes instead contribute the terms [132]

$$\frac{2\pi n_{\rm O3}}{4} \int C_0^{\rm RR} \sqrt{L\left(\frac{1}{4}\mathcal{R}\right)} = \frac{2\pi n_{\rm O3}}{4\cdot 96} \int C_0^{\rm RR}\, p_1(M)\,, \tag{3.3.60}$$

where L is the Hirzebruch L-polynomial. By using the identification $C_0^{\rm RR} = a$ and the D3 tadpole cancellation condition

$$n_{\rm O3} = 4 n_{\rm D3}\,, \tag{3.3.61}$$

we then obtain the total contribution

$$\frac{2\pi(8n_{\rm D3} + n_{\rm O3})}{4\cdot 96} \int a\, p_1(M) = -\frac{3}{16}\frac{n_{\rm O3}}{96\pi} \int a\,{\rm tr}(R\wedge R)\,. \tag{3.3.62}$$

By matching with (2.1.19) we conclude that

$$\tilde{C} = \frac{3}{16} n_{\rm O3}\,. \tag{3.3.63}$$

Note that these models do not encode any five-dimensional structure and indeed the D7-brane string does not give rise to effective couplings of the form (3.1.11). Hence we can set $\hat{C} = 0$.

We can now compare these results with our quantum gravity constraints. Note that the positivity constraints (3.1.18) are obviously satisfied. On the other hand, (3.2.3) requires that $\tilde{C} \in \mathbb{Z}$. This is possible only if

$$n_{\rm O3} \in 16\mathbb{N}\,. \tag{3.3.64}$$

This non-trivial prediction in fact agrees with Theorem 1.5 of [133], which implies that $n_{\rm O3} = 16$ for all (strict) Calabi-Yau covering spaces. This mathematical result can also be explicitly

[24]Note that the sign is fixed by requiring that the internal cycles wrapped by mutually supersymmetric D-branes are calibrated by $-e^{{\rm i}J}$ so that, for instance, D7-branes would be calibrated by $\frac{1}{2}J\wedge J$ and then would wrap internal holomorphic cycles. With this choice supersymmetric space-filling D3-branes must be calibrated by -1 and may be considered as *anti* D3-branes.

checked in all the models of the database of [134].[25]

The bound (3.2.26) on the rank of the gauge group detected by the EFT strings becomes

$$r(\mathbf{e}) \leq \frac{3}{8} n_{\rm O3} - 2\,, \tag{3.3.65}$$

where we used (3.3.63) and $\hat{C} = 0$ and the r.h.s. is integral by (3.3.64).

From the explicit string theory model, however, we know that the gauge group probed by the EFT strings in question comes from the perturbative D3-brane sector and its rank is therefore bounded by $n_{\rm D3} = \frac{1}{4} n_{\rm O3}$. Interestingly, this actual bound coincides with the stronger bound obtained by taking into account only the contribution (3.2.14) from the charged localised Fermi multiplets in (3.2.25), i.e.

$$r_{\rm F}(\mathbf{e}) \leq \frac{4}{3}\tilde{C} = \frac{1}{4} n_{\rm O3}\,. \tag{3.3.66}$$

This match seems to be characteristic of the perturbative (with respect to the $SL(2,\mathbb{Z})$ duality group of Type IIB string theory) nature of the D3-brane sector.

In these models the stricter bound set by (3.2.25) is always saturated, see (3.3.61). However, it is microscopically clear that it remains valid even in presence of internal supersymmetric three-form fluxes, which contribute positively to the D3 tadpole condition and then reduce the number of D3-branes. From the four-dimensional viewpoint, the introduction of these fluxes makes the EFT string 'anomalous': The string is forced to be the boundary of a membrane, which microscopically corresponds to a D5-brane ending on the D7 string. We leave for future work the extension of our four-dimensional quantum gravity arguments to these kinds of anomalous strings.

Complex structure EFT strings

The third qualitatively different gauge sector in F-theory is associated with the R-R four-form $C_4^{\rm RR}$ expanded in terms of non-trivial elements of $H^3(X,\mathbb{Z})$. The gauge kinetic function and hence also the characteristic couplings (2.1.7) in this sector depend on the complex structure moduli of the four-fold Y [135]. Correspondingly, the weak coupling limits are attained in large complex structure limits on Y. An explicit description of the EFT strings associated with this sector is an interesting challenge for future work.

Suffice it here to mention that under heterotic/F-theory duality the R-R gauge sector is expected to map to the abelian gauge sector from heterotic 5-branes wrapped along curve classes of genus $g \geq 1$ [95] in the base of the dual heterotic elliptic fibration [136, 137]. This implies a duality between the EFT strings associated with this heterotic 5-brane sector and the complex structure EFT strings in F-theory.

[25]We thank X. Gao and in particular F. Carta and R. Valandro for discussions on the available examples of Calabi-Yau orientifolds with just O3 planes. We also thank F. Carta for helping us in checking (3.3.64) by scanning the database [134], and for pointing out to us reference [133].

3.3.3 Microscopic checks in $E_8 \times E_8$ heterotic models

We now turn to compactifications of the $E_8 \times E_8$ heterotic theory on Calabi-Yau three-folds X, and their M-theory uplift. We will discuss the $SO(32)$ case instead in Section 3.3.4, but the following initial considerations apply to both.

In order to understand how our quantum gravity bounds on the gauge sector and the curvature-squared terms are realized, we need to revisit and complete the discussion presented in [34], which neglects corrections coming from higher derivative terms in ten/eleven dimensions. Indeed, we will see that these corrections modify the structure of the saxionic cone and of the corresponding EFT string spectrum in a non-trivial manner. We will also allow for possible background NS5/M5-branes, not considered in [34], which significantly broaden this class of vacua and enrich the structure of their EFT in an interesting way.

In the heterotic picture, we focus on the chiral fields $(\hat{t}, t^a)$ where $t^a = a^a + \mathrm{i}s^a$ appear in the expansion

$$t^a D_a = a^a D_a + \mathrm{i}s^a D_a \equiv B_2 + \mathrm{i}J \tag{3.3.67}$$

and

$$\hat{t} = \hat{a} + \mathrm{i}\hat{s} \equiv \int_X B_6 + \mathrm{i}e^{-2\phi} V_X \,. \tag{3.3.68}$$

Here D_a is Poincaré dual to a basis of $H_4(X, \mathbb{Z})$, B_6 is the electromagnetic dual of the NS-NS B_2 and

$$V_X = \frac{1}{3!} \int_X J \wedge J \wedge J = \frac{1}{3!} \kappa_{abc} s^a s^b s^c \quad , \quad \kappa_{abc} \equiv D_a \cdot D_b \cdot D_c \tag{3.3.69}$$

is the string frame volume of X measured in string units $\ell_s = 2\pi\sqrt{\alpha'} = 1$.

If we neglect higher derivative corrections as in [34], the saxionic cone described by $(\hat{s}, s^a)$ can simply be identified with $\mathbb{R}_{>0} \oplus \mathcal{K}(X)$, where $\mathcal{K}(X)$ is the Kähler cone of X. It will be important for us to understand how the higher derivative corrections affect this naive result. To this end we will collect the relevant threshold corrections to the gauge couplings of the four-dimensional EFT, and then argue how these corrections are compatible with a corresponding modification of the relevant sets of BPS instanton and EFT string charges. As we will see, the saxionic cone will also be enlarged into directions corresponding to the moduli of the possible background NS5/M5-branes. We will then check that the curvature-squared terms obey the expected quantum gravity constraints and microscopically verify the corresponding upper bounds on the ranks of the gauge group.

The EFT terms

As a first step, we need to identify the couplings (2.1.11), including possible contributions coming from heterotic higher derivative corrections. All the relevant complexified gauge couplings may be extracted directly from [138], and from various previous partial results cited therein such as [139–141]. However, in order to render the discussion more self-contained and to carefully pin down the possible convention ambiguities, in Appendix A.3 we will briefly go through the main steps of the derivation. Furthermore, we will similarly derive also the curvature-squared terms, which were not explicitly considered in [138].

The two heterotic E_8 sectors can have a non-trivial bundle structure along the internal space. Let us denote by $\hat{F}_1$ and $\hat{F}_2$ the corresponding fields strength. For simplicity, we assume that $\hat{F}_{1,2}$ take values in semi-simple sub-algebras of $\mathfrak{e}_8$. This in particular allows us to avoid possible Stückelberg masses and non-trivial kinetic mixing between the $U(1)$ factors.[26] We also allow for possible NS5-branes wrapping irreducible holomorphic curves $\mathcal{C}^k$, $k = 0, \ldots, N_{\text{NS5}}$. The internal gauge bundles and the curves $\mathcal{C}^k$ enter the cohomological tadpole condition

$$\lambda(E_1) + \lambda(E_2) + [\mathcal{C}] = c_2(X)\,, \tag{3.3.70}$$

where $[\mathcal{C}]$ is the Poincaré dual of the two-cycle $\mathcal{C} \equiv \sum_k \mathcal{C}^k$ and

$$\lambda(E) \equiv -\frac{1}{16\pi^2}\,\text{tr}(\hat{F} \wedge \hat{F})\,, \tag{3.3.71}$$

with the trace defined as in Section 2.1.1.

The background NS5-branes support additional (s)axionic fields. These can be more easily identified by considering the Hořava-Witten (HW) M-theory uplift of the heterotic theory [142, 143], compactified over $I \times X$, where $I = S^1/\mathbb{Z}_2$ is the HW interval. In the upstairs picture, we parametrize the M-theory circle by the coordinate $y \simeq y + 2$, on which we impose the $\mathbb{Z}_2$ identification $y \simeq -y$. The E_8 sectors are then supported on the walls $\{y = 0\}$ and $\{y = 1 \simeq -1\}$, and we can parametrize $I = S^1/\mathbb{Z}_2$ by $y \in [0, 1]$. The background NS5-branes uplift to M5-branes sitting at points $\hat{y}^k$ along the HW interval I. Their worldvolume supports a self-dual two-form $\tilde{\mathcal{B}}_2^k$. This sector hence gives rise to N_{NS5} additional axions

$$\tilde{a}^k \equiv \int_{\mathcal{C}^k} \tilde{\mathcal{B}}_2^k\,, \tag{3.3.72}$$

By supersymmetry the axions pair up with the saxions [138]

$$\tilde{s}^k = \left(\hat{y}^k - \frac{1}{2}\right) \int_{\mathcal{C}^k} J = \left(\hat{y}^k - \frac{1}{2}\right) m_a^k s^a\,, \qquad \text{(no sum over } k)\,, \tag{3.3.73}$$

where

$$m_a^k \equiv D_a \cdot \mathcal{C}^k\,. \tag{3.3.74}$$

By dimensionally reducing the bulk action, including the Green-Schwarz anomaly cancelling term as well as certain terms supported on the M5-branes, one obtains the following contribution

[26] A (partial) verification of our quantum gravity constraints in presence of kinetic mixing of $U(1)$ factors can be obtained by duality to other corners of the string landscape considered in this thesis.

to the four-dimensional axionic couplings:

$$
\begin{aligned}
&-\frac{1}{8\pi}\int\left(\hat{a}+\frac{1}{2}p_a a^a-\frac{3}{8}q_a a^a-\frac{1}{2}\sum_k \tilde{a}^k\right)\mathrm{tr}(F_1\wedge F_1)\\
&-\frac{1}{8\pi}\int\left(\hat{a}-\frac{1}{2}p_a a^a+\frac{1}{8}q_a a^a+\frac{1}{2}\sum_k \tilde{a}^k\right)\mathrm{tr}(F_2\wedge F_2)\,.
\end{aligned}
\tag{3.3.75}
$$

Here $F_{1,2}$ denote the field strengths of the four-dimensional gauge sectors coming from the two heterotic E_8 sectors, respectively, and

$$
p_a\equiv-\int_{D_a}\left[\lambda(E_2)-\frac{1}{2}c_2(X)\right]\quad,\quad q_a\equiv D_a\cdot\mathcal{C}=\sum_k m_a^k\,. \tag{3.3.76}
$$

We furthermore recall the definition of the axions in (3.3.68), (3.3.67) and (3.3.72). The corresponding saxionic couplings are fixed by supersymmetry.

As discussed in detail in Appendix A.3, one can similarly compute the relevant curvature-squared terms. By focusing again on the axionic couplings, the final result is

$$
-\frac{1}{96\pi}\int\left(12\hat{a}+n_a a^a-\frac{3}{2}q_a a^a\right)\mathrm{tr}(R\wedge R)\,, \tag{3.3.77}
$$

where

$$
n_a\equiv\frac{1}{2}\int_{D_a}c_2(X)\,. \tag{3.3.78}
$$

By applying the Hirzebruch-Riemann-Roch theorem to a line bundle $\mathcal{O}_X(D_a)$ one can easily conclude that $n_a\in\mathbb{Z}$. Since $\int_{D_a}\lambda(E_2)$ is an instanton number, $p_a\in\mathbb{Z}$ as well.

Note that the constants appearing in both (3.3.75) and (3.3.77) do not satisfy the naively expected quantization conditions (2.1.12) and (2.1.20), respectively. This signals the presence of a rational mixing of the axionic periodicities induced by the higher derivative corrections. In other words, the axions $(\hat{a},a^a,\tilde{a}^k)$ defined in (3.3.68), (3.3.67) and (3.3.72) do not satisfy the integral periodicity $a^i\simeq a^i+1$ assumed here. This issue can be solved by passing to a better axionic basis, obtained from a rational linear redefinition of the axions. For instance, we can replace the axions $\hat{a}$ and $\tilde{a}^k$ with the linear combinations

$$
a^0=\hat{a}+\frac{1}{2}p_a a^a-\frac{3}{8}q_a a^a-\frac{1}{2}\sum_k\tilde{a}^k\,,\quad a^k=\tilde{a}^k+\frac{1}{2}m_a^k a^a\,. \tag{3.3.79}
$$

In this way the sum of (3.3.75) and (3.3.77) can be rewritten as

$$
\begin{aligned}
&-\frac{1}{8\pi}\int a^0\,\mathrm{tr}\,(F_1\wedge F_1)-\frac{1}{8\pi}\int\left(a^0-p_a a^a+\sum_k a^k\right)\mathrm{tr}\,(F_2\wedge F_2)\\
&-\frac{1}{96\pi}\int\left(12a^0-6p_a a^a+n_a a^a+6\sum_k a^k\right)\mathrm{tr}(R\wedge R)\,.
\end{aligned}
\tag{3.3.80}
$$

In the axionic basis $a^i = (a^0, a^a, a^k)$, these terms indeed have the form of the axionic terms appearing in (2.1.11) and (2.1.19) and then determine the corresponding constants

$$\begin{aligned} \mathbf{C}^1 &= (C^1_0, C^1_a, C^1_k) = (1, \vec{0}, \vec{0})\,, \qquad \mathbf{C}^2 = (C^2_0, C^2_a, C^2_k) = (1, -p_a, \vec{1})\,, \\ \tilde{\mathbf{C}} &= (\tilde{C}_0, \tilde{C}_a, \tilde{C}_k) = (12, -6p_a + n_a, \vec{6})\,. \end{aligned} \tag{3.3.81}$$

The saxionic couplings appearing in (2.1.11) and (2.1.19) are completely fixed by supersymmetry and involve the saxions $s^i = (s^0, s^a, s^k)$, with

$$s^0 \equiv \hat{s} + \frac{1}{2} p_a s^a - \frac{3}{8} q_a s^a - \frac{1}{2} \sum_k s^k\,, \tag{3.3.82a}$$

$$s^k \equiv \tilde{s}^k + \frac{1}{2} m^k_a s^a = \hat{y}^k m^k_a s^a \quad (\text{no sum over } k)\,. \tag{3.3.82b}$$

Note that $s^k = 0$ if the k-th M5-brane sits on the HW wall $y = 0$ and $s^k = m^k_a$ if the k-th M5-brane sits on the HW wall $y = 1$. We also observe that the microscopic symmetry under the exchange of the two HW walls, together with the coordinate change $y \leftrightarrow 1 - y$, descends to the invariance of the four-dimensional action under exchange of the two gauge sectors $F_1 \leftrightarrow F_2$, together with

$$p_a \leftrightarrow q_a - p_a\,, \qquad s^0 \leftrightarrow s^0 - p_a s^a + \sum_k s^k\,, \qquad s^k \leftrightarrow m^k_a s^a - s^k\,, \tag{3.3.83}$$

plus similar transformations for the axions. This may be regarded as a discrete $\mathbb{Z}_2$ gauge symmetry of the theory, since it describes the same microscopic configuration.

From (3.3.81) we note that the saxions s^k enter the EFT terms (2.1.11) and (2.1.19) in the combination

$$p_a s^a - \sum_k s^k\,. \tag{3.3.84}$$

Interestingly, this combination is continuous under 'small instanton transitions' [144] in which some M5-brane is absorbed or emitted by the HW walls. As an example, take the limit $\hat{y}^k \to 1$ in which all M5-branes are moved on top of the second HW wall, and are then absorbed by a small instanton transition in which $\lambda(E_2) \to \lambda(E'_2) = \lambda(E_2) + [\mathcal{C}]$ and then $p_a \to p'_a = p_a - \sum_k m^k_a = p_a - q_a$. Along this process, $p_a s^a - \sum_k s^k$ first becomes $p_a s^a - \sum_k m^k_a s^a$, which is indeed equal to $p'_a s^a$. The combination (3.3.84) is also continuous under a small instanton transition in which all M5-branes are absorbed by the first HW wall, which is simply described by the limit $s^k \to 0$. Clearly (3.3.84) is also continuous under more general small instanton transitions.

Note that the M5-branes can provide an additional gauge sector, coming from the expansion of the M5 self-dual two-form potentials in harmonic one-forms of $\mathcal{C}^k$, as in [145]. However, the corresponding gauge couplings are controlled by the complex structure of the curves $\mathcal{C}^k$, rather than by their volume. Hence, this sector is generically strongly coupled in the perturbative regime considered here, and one should consider some large complex structure limit in order to identify corresponding axionic strings. The F-theory dual sector was briefly mentioned in Section 3.3.2.

Saxionic cone

The saxionic cone is determined by the set of possible BPS instantons, as in (2.1.3). In order to clarify its global structure, it is convenient to work in the M-theory frame. There are three types of BPS instantons to consider for us: Heterotic worldsheet instantons, open membrane instantons ending with one or both ends on an M5-brane, or M5-brane instantons.

Consider first the heterotic world-sheet instantons, which in M-theory are represented by Euclidean open M2-brane wrapping some effective curve $\Sigma \subset X$ and stretching between the two HW walls. Their action is given by $2\pi m_a(\Sigma) s^a$, with $m_a(\Sigma) = D_a \cdot \Sigma$. The condition $m_a(\Sigma)s^a > 0$ defines the standard Kähler cone $\mathcal{K}(X)$.

If in particular we choose $\Sigma = \mathcal{C}^k$, we obtain the condition $m^k_a s^a > 0$. Hence, by using the restriction $y \in (0,1)$ in (3.3.82b) we deduce that s^k must satisfy the condition

$$0 < s^k < m^k_a s^a \,. \tag{3.3.85}$$

This condition guarantees the exponential suppression of instanton contributions coming from the second type of instantons, Euclidean open M2-branes stretched between the background M5-branes and the HW walls [146–148]. For instance, consider an M2-brane along the curve $\mathcal{C}^k$ wrapped by the k-th M5-brane and connecting the HW wall at $y = 0$ with the k-th M5-brane. Its action is given precisely by $2\pi s^k$, which is indeed positive by (3.3.85). If instead the M2-brane connects the k-th M5-brane to the HW wall at $y = 1$, then its action is given by $2\pi(1 - \hat{y}^k) m^k_a s^a = 2\pi(m^k_a s^a - s^k)$, which is again positive by (3.3.85). One can similarly consider an open Euclidean M2-brane stretching between two background M5-branes.

It remains to discuss the more subtle BPS instantons corresponding to Euclidean M5-branes wrapping the entire Calabi-Yau X. A crucial role will be played by the internal M-theory G_4 flux, which is generically non-vanishing. In the downstairs picture of the HW orbifold, the internal G_4 flux must satisfy specific boundary conditions [142, 143]. In our setting, these read

$$\lim_{y\to 0^+} G_4|_{\{y\}\times X} = \ell^3_{\rm M} \left[\lambda(E_1) - \frac{1}{2}c_2(X)\right] \,, \tag{3.3.86a}$$

$$\lim_{y\to 1^-} G_4|_{\{y\}\times X} = -\ell^3_{\rm M} \left[\lambda(E_2) - \frac{1}{2}c_2(X)\right] \,, \tag{3.3.86b}$$

where $\ell_{\rm M}$ is the M-theory Planck length, which we choose to coincide with $\ell_{\rm s}$ under dimensional reduction.

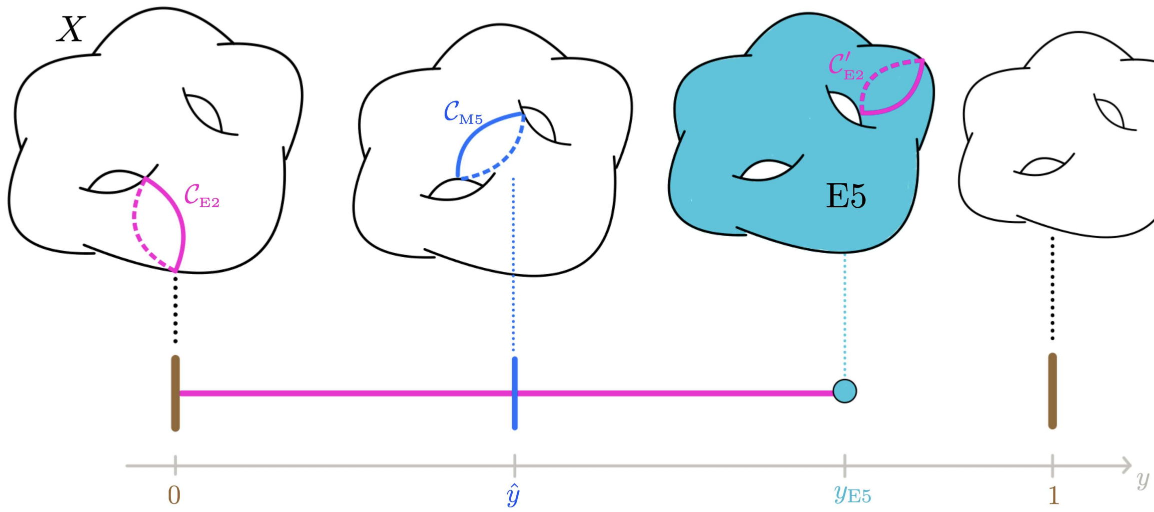


Figure 3.1: **BPS instantons in $E_8 \times E_8$ heterotic models**. The internal seven-dimensional space is a fibration of the Calabi-Yau X over the horizontal M-theory interval $I = \{0 \leq y \leq 1\}$. The vertical brown and blue lines denote the HW walls and a background M5-brane wrapping the internal curve $\mathcal{C}_{\rm M5}$, respectively. The picture includes a Euclidean M5/M2-instanton: The light blue Euclidean M5-brane (E5) sits at a given position $y_{\rm E5}$ along I and wraps the entire internal Calabi-Yau space. It is connected to the left HW wall by two purple Euclidean M2-branes (E2) wrapping $\mathcal{C}_{\rm E2}$ and $\mathcal{C}'_{\rm E2}$ respectively, such that $\mathcal{C}'_{\rm E2} \simeq \mathcal{C}_{\rm E2} + \mathcal{C}_{\rm M5}$, which reconnect on the bulk M5-brane. See also Appendix A.4 for further explanations.

Inside the HW interval G_4 must satisfy the Bianchi identity

$$\mathrm{d}G_4 = \ell_{\rm M}^3 \sum_k \delta(y - \hat{y}^k)\mathrm{d}y \wedge \delta_X^4(\mathcal{C}^k)\,. \tag{3.3.87}$$

This implies that the cohomology of G_4 jumps by $\ell_{\rm M}^3[\mathcal{C}^k]$ as one crosses the k-th M5-brane. The combination of (3.3.86) and (3.3.87) indeed implies the consistency condition (3.3.70).

As first discussed in [149], this non-trivial flux and the HW walls induce a non-trivial deformation of the internal geometry. Furthermore, the presence of a non-trivial G_4 implies that a Euclidean M5 wrapping the entire Calabi-Yau X and sitting at an intermediate position $0 < y_{\rm E5} < 1$ is not consistent by itself, because of the world-volume tadpole condition. Rather, one must add Euclidean open M2-branes ending on the M5-brane and must then consider a composite M5/M2-instanton – see Figure 3.1. All this complicates the direct computation of the instanton Euclidean action and of the corresponding saxionic conditions. However, we can deduce this information by indirect arguments, exploiting the holomorphy of the BPS instanton corrections.

We devote Appendix A.4 to a detailed discussion of these arguments and here present only the final result. Namely, the positivity of the action of any possible BPS M5 instanton (including possible open M2-brane insertions) is guaranteed if we impose the conditions $s^0 > 0$ and

$s^0 - p_a s^a + \sum_k s^k > 0$. This allows us to complete our identification of the saxionic cone:

$$\Delta = \left\{ \boldsymbol{s} = (s^0, s^a, s^k) \mid s^a D_a \in \mathcal{K}(X)\,, s^0 > 0\,, s^0 - p_a s^a + \sum_k s^k > 0\,, 0 < s^k < m^k_a s^a \right\}. \quad (3.3.88)$$

Note that this saxionic cone is indeed invariant under (3.3.83).

Recalling (3.3.81), one immediately checks that the combinations $\langle \mathbf{C}^1, \boldsymbol{s} \rangle$ and $\langle \mathbf{C}^2, \boldsymbol{s} \rangle$, which define the gauge couplings, are positive within the saxionic cone, as expected. Furthermore, a theorem [150] guarantees that

$$n_a s^a = \frac{1}{2} \int_X J \wedge c_2(X) > 0 \quad (3.3.89)$$

if $J = s^a D_a$ belongs to the Kähler cone. Recalling (3.3.81), it follows that $\langle \tilde{\mathbf{C}}, \boldsymbol{s} \rangle > 0$ within the saxionic cone.

EFT strings

We now come to specifying the cone of EFT strings. There are three types of BPS strings dual to the three types of BPS instantons discussed in the previous section: The critical heterotic string, i.e. an F1 string, corresponding to an open M2-brane stretched between the two HW walls, furthermore the strings obtained by wrapping an M5-brane along an effective divisor on X, and finally open M2-branes stretched between two M5-branes or between an M5-brane and an HW wall. The cone of EFT string charges, (2.2.10), can be identified with the help of the saxionic cone (3.3.88) as follows:

$$\begin{aligned} \mathcal{C}^{\text{EFT}}_{\text{S}} = \Big\{ \mathbf{e} = (e^0, e^a, e^k) \mid D_{\mathbf{e}} \equiv e^a D_a \in \text{Nef}^1(X)\,, \\ e^0 \geq 0\,,\ e^0 - p_a e^a + \sum_k e^k \geq 0\,,\ 0 \leq e^k \leq m^k_a e^a \Big\}\,. \end{aligned} \quad (3.3.90)$$

Recalling (3.3.74) and (3.3.76), we note that $m^k_a e^a = \mathcal{C}^k \cdot D_{\mathbf{e}} \geq 0$ if $D_{\mathbf{e}}$ is nef, and then $q_a e^a = \sum_k \mathcal{C}^k \cdot D_{\mathbf{e}} \geq 0$ as well. $\mathcal{C}^{\text{EFT}}_{\text{S}}$ has a richer structure than the cone identified in [34], not only because of the additional charges e^k associated with the presence of background M5-branes, but also because of the additional condition involving the background constants p_a, which come from 10d/11d higher derivative corrections. As a result, the analysis of section 6.1 of [34] must be updated.

A charge vector $\mathbf{e} = (e^0, \vec{0}, \vec{0})$ corresponds to e^0 F1 strings and, as in [34], is associated with an EFT string flow along which the ten-dimensional string coupling vanishes, $e^\phi \to 0$, while the string frame Kähler moduli and the M5 positions $\hat{y}^k$ remain fixed. As noted already, from the M-theory viewpoint, these EFT strings correspond to M2-branes stretching between the two HW walls.

A choice $\mathbf{e} = (0, \vec{0}, e^k)$ corresponds to e^k M2-branes filling two external directions and stretching between the k-th background M5 and the second HW wall (at $y = 1$). From the ten-dimensional viewpoint, they appear as 'fractional' F1 strings bound to the NS5. However the condition $0 \le e^k \le m^k_a e^a$ appearing in (3.3.90) excludes a charge vector of the form $\mathbf{e} = (0, \vec{0}, e^k)$ from $\mathcal{C}^{\rm EFT}_{\rm S}$, and hence these strings are *not* EFT strings. Note that indeed such strings cannot explore the entire internal space, as would be characteristic for an EFT string. Rather, such BPS strings can become classically tensionless [151] at finite distance in the moduli space, where the theory develops a strongly coupled sector in which the open M2-brane instantons discussed in the previous section become unsuppressed.

In order to interpret the condition $0 \le e^k \le m^k_a e^a$, we must then turn on the charges e^a, associated with a string obtained from an M5-brane along a nef divisor $e^a D_a$. Assume first that $e^a(p_a - q_a) \ge 0$ (and thus $e^a p_a \ge 0$ as well) and suppose that the M5-string wrapping $D_{\mathbf{e}}$ sits on top of the second HW wall at $y = 1$. From (3.3.76) and (3.3.86b) we know that $p_a e^a = \frac{1}{\ell^3_{\rm M}} \lim_{y_{\mathbf{e}} \to 1} \int_{\{y_{\mathbf{e}}\} \times D_{\mathbf{e}}} G_4$. In order to move the M5 string away from the HW wall, there must therefore be $p_a e^a$ M2-branes ending on it from the left (to solve the world-volume tadpole condition along the M5-brane). If $e^k = 0$ for any k, all these M2-branes must originate on the first HW wall, and hence initially there must be $e^0 \ge p_a e^a$ M2-branes connecting the two HW walls, which is precisely the content of (3.3.90). Now start moving the M5 string to the left, along the y direction. When it crosses the k-th bulk M5, the G_4-flux across $D_{\mathbf{e}}$ jumps by $-m^k_a e^a$. Hence, at a more general point $y_{\mathbf{e}}$ the number of M2 strings ending on the M5 string from the left is given by

$$\frac{1}{\ell^3_{\rm M}} \int_{\{y_{\mathbf{e}}\} \times D_{\mathbf{e}}} G_4 = p_a e^a - \sum_{k | \hat{y}^k > y_{\mathbf{e}}} m^k_a e^a \,, \tag{3.3.91}$$

while the remaining $m^k_a e^a$ M2 strings stretch between the first HW wall and the k-th background M5-brane with $\hat{y}^k > y_{\mathbf{e}}$ – see Figure 3.2 for an example with one background M5-brane.

More generically, if $e^k \neq 0$, then initially, when the M5 string is at $y_{\mathbf{e}} = 1$, there are e^k open M2-branes connecting the second HW wall to the k-th background M5-brane. This implies that in order to allow the M5 string to move away from $y_{\mathbf{e}} = 1$ it is sufficient to take $e^0 \ge p_a e^a - \sum_k e^k$, which is indeed one of the conditions appearing in (3.3.90). Moreover, the last condition $0 \le e^k \le m^k_a e^a$ appearing in (3.3.90) implies that, after the M5 string crosses a background M5-brane, there remain no open M2-brane ending on it from the right. In the extreme case in which $e^k = m^k_a e^a$ and $e^0 = (p_a - q_a)e^a$, when the M5 string arrives at $y_{\mathbf{e}} = 0$, no M2 strings are left in the bulk.

The case $p_a e^a \le 0$ can be treated in complete analogy, being related to the case $(p_a - q_a)e^a \ge 0$ by the $\mathbb{Z}_2$ symmetry (3.3.83) which swaps the role of two HW walls. One may similarly discuss some intermediate case with $p_a e^a \ge 0$ and $(q_a - p_a)e^a \ge 0$.

As reviewed in Section 2.2, the EFT strings are associated to an infinite distance limit. In the

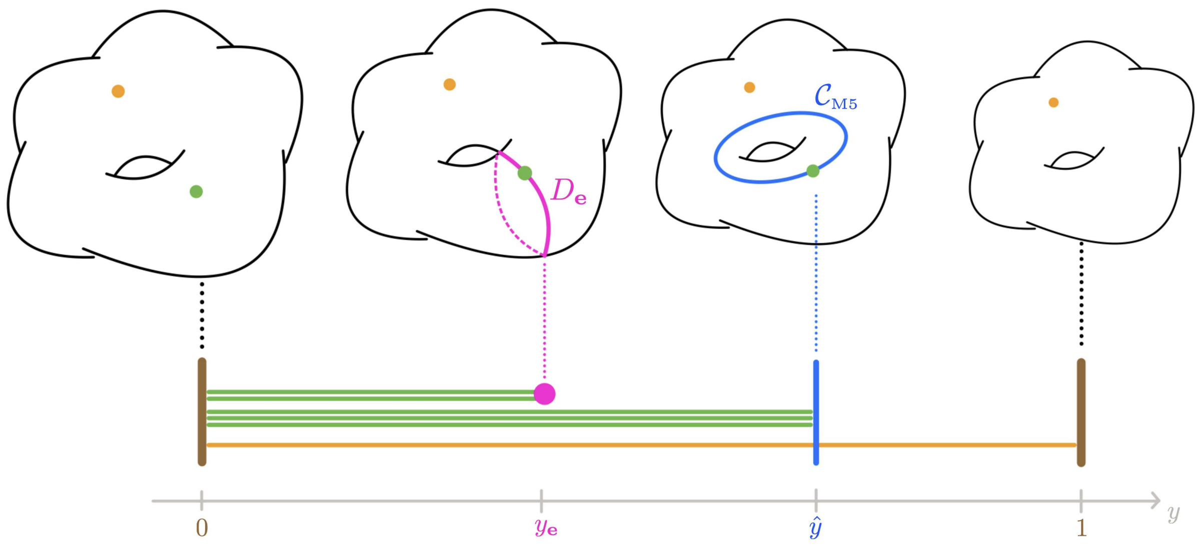


Figure 3.2: **EFT strings in $E_8\times E_8$ heterotic models**. The bulk sector is as in Fig. 3.1, but now the picture includes two EFT strings: an orange M2-brane stretching between the HW walls, which descends to the critical heterotic string in ten dimensions; a bound state of $(p_a - m_a)e^a = 2$ green open M2-branes and a purple M5-brane wrapping the nef divisor $D_{\mathbf{e}} \subset X$ and sitting at $y = y_{\mathbf{e}}$. There also appear $m_a e^a = \mathcal{C} \cdot D_{\mathbf{e}} = 3$ green open M2-branes, connecting the left HW wall with the background M5-brane, which tie to the purple M5-brane if we move it to $y_{\mathbf{e}} > \hat{y}$.

present model, the physical properties can be analysed by slightly adapting the discussion of [34], taking into account two important differences. First, as highlighted by the above discussion, the flows corresponding to NS5/M5 strings can generically involve also the dilaton s^0 and the 5-brane moduli s^k, in addition to the Kähler moduli s^a. Furthermore, the identifications (3.3.82) complicate the microscopic interpretation of the various EFT string flows and in general affect also the value of the corresponding scaling weight. However, a conclusion of [34] still holds: the EFT string flows such that

$$\kappa(\mathbf{e},\mathbf{e},\mathbf{e}) \equiv \kappa_{abc}e^a e^b e^c = D_{\mathbf{e}}^3 > 0 \tag{3.3.92}$$

lead to a rapid growth of the HW interval, so that we have a dynamically generated sharp hierarchy between its length and the length scale of the internal Calabi-Yau. Hence, close to the string, one internal direction opens up, and the 4d string uplifts to a string/membrane bound state in an HW-like 5d theory on $M_4 \times I$, where $I = S^1/\mathbb{Z}_2$. In particular, the string probes a local $\mathcal{N} = 1$ 5d supergravity, whose 8 supercharges are spontaneously broken to 4 by the presence of the HW walls. These types of EFT have been studied for instance in [152] but the relevant term can be quite easily obtained by reducing on the Calabi-Yau X the M-theory CS term (in upstairs picture). By using the decomposition

$$C_3 = \frac{\ell_{\rm M}^3}{2\pi} A^a \wedge D_a \tag{3.3.93}$$

one obtains the five-dimensional CS term

$$S_{\rm CS}^{\rm 5d} = \frac{\pi}{6(2\pi)^3}\kappa_{abc}\int_{M_4\times S^1} A^a\wedge F^b\wedge F^c \qquad \text{with} \quad \kappa_{abc}\equiv D_a\cdot D_b\cdot D_c\,, \tag{3.3.94}$$

in the upstairs picture, in which $y\in[-1,1]$. Note that C_3 is odd under parity. Hence the five-dimensional $U(1)$ gauge fields A^a are odd under the $\mathbb{Z}_2$ parity $\iota: y\mapsto -y$. The $S^1/\mathbb{Z}_2$ orbifold projection then imposes that $\iota^* A^a = -A^a$. If we restrict to zero modes along S^1, we are then forced to set $A^a = 2\pi a^a \mathrm{d}y$, where a^a are our 4d axions. This in particular implies that, from the 4d viewpoint, the 5d gauge fields A^a contain a finite number of massless axions plus an infinite tower of massive vectors and pseudoscalars. Furthermore, the restriction of (3.3.94) to the zero modes identically vanishes.

On the other hand, as in Section 3.1.2 this additional CS term produces an extra world-sheet term of the from (3.1.11), with

$$\hat{C}_i(\mathbf{e}) = \delta_i^a\kappa_{abc}e^b e^c\,. \tag{3.3.95}$$

This provides an explicit microscopic realization of effect described in Section 3.1.2, with

$$\hat{C}_{ijk} = \begin{cases} \kappa_{abd} & \text{if } (i,j,k)=(a,b,c) \\ 0 & \text{otherwise} \end{cases}\,. \tag{3.3.96}$$

Microscopic check of quantum gravity bounds

We are now ready to test our EFT quantum gravity constraints. From (3.3.81) we obtain the relations

$$\begin{aligned} &\langle \mathbf{C}^1,\mathbf{e}\rangle = e^0 \quad , \quad \langle \mathbf{C}^2,\mathbf{e}\rangle = e^0 - p_a e^a + \sum_k e^k\,, \\ &\langle \tilde{\mathbf{C}},\mathbf{e}\rangle = 6\langle \mathbf{C}^1,\mathbf{e}\rangle + 6\langle \mathbf{C}^2,\mathbf{e}\rangle + n_a e^a\,. \end{aligned} \tag{3.3.97}$$

First of all, the positivity bounds (3.1.18) are satisfied by definition of (3.3.90). It follows that not only is (3.2.3) obeyed, but actually $\langle \tilde{\mathbf{C}},\mathbf{e}\rangle\in\mathbb{Z}_{\geq 0}$, since (3.3.89) implies that

$$n_a e^a = \frac{1}{2}\int_{D_{\mathbf{e}}} c_2(X)\geq 0 \tag{3.3.98}$$

for any nef divisor $D_{\mathbf{e}}$. In turn, this guarantees that the combinations appearing in (3.2.4) and (3.2.6) are non-negative, since

$$\langle \hat{C}(\mathbf{e}),\mathbf{e}\rangle = \hat{C}(\mathbf{e},\mathbf{e},\mathbf{e}) = \kappa(\mathbf{e},\mathbf{e},\mathbf{e}) = D_{\mathbf{e}}^3 \tag{3.3.99}$$

and the triple self-intersection of a nef divisor is non-negative.

On the other hand (3.2.4) makes the stronger prediction that $\langle \tilde{\mathbf{C}},\mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}),\mathbf{e}\rangle$ should actually be a (positive) integral multiple of 3. The contributions to $\langle \tilde{\mathbf{C}},\mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}),\mathbf{e}\rangle$ coming from the first two terms appearing in the second line of (3.3.97) clearly satisfy this property, being positive multiples of 6. It then remains to check that

$$n_a e^a + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e} \rangle = \frac{1}{2} \int_{D_{\mathbf{e}}} c_2(X) + D_{\mathbf{e}}^3 \in 3\mathbb{Z}_{\geq 0} \,. \tag{3.3.100}$$

In order to prove it, it is sufficient to prove that $D_{\mathbf{e}}^3 - \int_{D_{\mathbf{e}}} c_2(X)$ is a multiple of 3. The latter statement follows from the index theorem applied to the signature complex twisted by the line bundle $\mathcal{O}_X(D_{\mathbf{e}})$ – see for instance [153] – which implies that[27]

$$\int_X L(X) \wedge \widetilde{\mathrm{ch}}(\mathcal{O}_X(D_{\mathbf{e}})) = \frac{2}{3} \int_{D_{\mathbf{e}}} p_1(X) + \frac{4}{3} D_{\mathbf{e}}^3 = \frac{4}{3} \left(D_{\mathbf{e}}^3 - \int_{D_{\mathbf{e}}} c_2(X) \right) \in \mathbb{Z} \,. \tag{3.3.101}$$

Since $D_{\mathbf{e}}^3 - \int_{D_{\mathbf{e}}} c_2(X)$ is guaranteed to be integral, this result confirms the quantization condition (3.2.4) (and then (3.2.6)).

Let us now test (3.2.26), which bounds the rank of the gauge group detected by an EFT string of charge $\mathbf{e}$ and presently takes the form

$$r(\mathbf{e}) \leq 2\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle + D_{\mathbf{e}}^3 - 2 \,. \tag{3.3.102}$$

As explained in Section 4.2.2, the strongest bounds are obtained by picking the generators of $\mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}$. First of all, we can consider

$$\mathbf{e} = (1, \vec{0}, \vec{0}) \,, \tag{3.3.103}$$

which corresponds to the saxionic direction s^0 and 'detects' all perturbative gauge groups. With such a choice $D_{\mathbf{e}}^3 = 0$ and $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle = 12$ and then the bound (3.3.102) gives

$$r(\mathbf{e}) \leq 22 \,. \tag{3.3.104}$$

Here $r(\mathbf{e})$ includes the rank of the perturbative $E_8 \times E_8$ gauge group present already in ten dimensions plus a maximal extra contribution of six to the total rank as encoded in the chiral superfields. In the following we will refer to this gauge sector as the 'perturbative' one. As we will see more explicitly at the end of Section 3.3.3, compactifications on singular Calabi-Yau three-folds can host also 'non-perturbative' gauge sectors, which are not accounted for by (3.3.104). The bound (3.3.104) agrees with expectations from heterotic compactifications for instance on toroidal orbifolds, which can admit a maximum of six additional $U(1)$ group factors associated with the six KK $U(1)$ gauge fields [154].[28]

On the other hand, the bound (3.3.104) does not include possible $U(1)$s coming from the M5-branes, whose gauge coupling is of order one for generic complex structure, or more generally

[27] Here $L(X)$ is the Hirzebruch L-polynomial and $\widetilde{\mathrm{ch}}(V)$ is the Chern character in which one replaces the curvature F of V by $2F$ in all expressions.

[28] In such situations, all of the chiral superfields can contribute in the anomaly cancellation, showing that the bound (3.2.26) can indeed be saturated.

any other gauge factors which avoid a coupling of the form (2.1.11) to the axion associated with the fundamental heterotic string.

Assume next that there exists a nef divisor $D_{\mathbf{e}} = e^a D_a$ such that $p_a e^a = 0$ and $q_a e^a = 0$, so that $\mathbf{e} = (0, e^a, 0)$ belongs to the cone $\mathcal{C}_{\rm S}^{\rm EFT}$. In this case $\langle \mathbf{C}^1, \mathbf{e}\rangle = \langle \mathbf{C}^2, \mathbf{e}\rangle = 0$.[29] The corresponding EFT string does not interact with the heterotic gauge bundles and does not provide any bound on their rank.[30]

More generally, let us assume that $(p_a - q_a)e^a > 0$. Then we can pick

$$\mathbf{e} = ((p_a - q_a)e^a, e^a, m^k_a e^a)\,. \tag{3.3.105}$$

In this case $\langle \mathbf{C}^1, \mathbf{e}\rangle = (p_a - q_a)e^a$ and $\langle \mathbf{C}^2, \mathbf{e}\rangle = 0$, where we recall (3.3.97) and (3.3.76). Hence, as far as the perturbative gauge group is concerned, $r(\mathbf{e})$ is sensitive to the rank of the gauge sector coming only from the first HW wall. Since $\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle = 6(p_a - q_a)e^a + n_a e^a$, with $n_a e^a = \frac{1}{2}\int_{D_{\mathbf{e}}} c_2(X)$, (3.3.102) yields the bound

$$r(\mathbf{e}) \leq 12(p_a - q_a)e^a + \int_{D_{\mathbf{e}}} c_2(X) + D^3_{\mathbf{e}} - 2\,. \tag{3.3.106}$$

This bound is clearly satisfied by the pertubative gauge sector supported on the first HW wall, since we know from the microscopic uplift that its rank is at most 8, and can provide a non-trivial bound on the non-perturbative gauge sector. By applying the $\mathbb{Z}_2$ symmetry (3.3.83), we obtain an analogous bound for the second gauge sector if $p_a e^a < 0$.

If we specialise to models with standard embedding $\lambda(E_1) = \lambda(X)$, from (3.3.76) and (3.3.78) we read off that $p_a = n_a$ and $q_a = 0$ (i.e. there are no background M5s).[31] Noting that in this case $p_a e^a \geq 0$, we could pick $\mathbf{e}$ as in (3.3.105) and the bound (3.3.106) becomes

$$r(\mathbf{e}) \leq 7\int_{D_{\mathbf{e}}} c_2(X) + D^3_{\mathbf{e}} - 2\,. \tag{3.3.107}$$

We now turn to discuss some concrete models. As we will see, at least in these models, the bounds on the perturbative gauge sector coming from M5 strings are always weaker than (3.3.104).

Example 1: The quintic

The quintic three-fold X is defined as the vanishing locus of a section of $\mathcal{O}_{\mathbb{P}^4}(5)$ inside $\mathbb{P}^4$. In this case the effective divisor $D_{\mathbf{e}}$ generating the (1-dimensional) nef cone is obtained by

[29] These strings are related to the $(0,4)$ supergravity strings in five dimensions discussed in [39] by dimensional reduction on the HW interval. If in addition $D^3_{\mathbf{e}} = 0$ and $n_a e^a = 0$, then they support an enhanced $(4,4)$ or $(8,8)$ non-chiral supersymmetric spectrum.

[30] By contrast, based on duality with F-theory, we will argue at the end of this section that EFT strings with $e^a \neq 0$ detect a certain non-perturbative gauge sector of the heterotic compactification.

[31] The case $\lambda(E_2) = \lambda(X)$ can again be obtained by applying (3.3.83).

restricting the hyperplane $H \subset \mathbb{P}^4$ to the hypersurface. Hence $D_{\mathbf{e}}^3 = 5H^4 = 5$, and by using the adjunction formula one obtains $c_2(X) = 10[H^2]|_X$ and then $\int_D c_2(X) = 50$. Hence, if we consider a standard embedding, the bound (3.3.106) becomes $r(\mathbf{e}) \leq 353$ which, if applied to the perturbative gauge sector, is clearly much weaker than (3.3.104).

Example 2: Elliptically fibered CYs and duality with F-theory

Consider a smooth CY three-fold X which is given by an elliptic fibration $\pi : X \to B$ over some weak-Fano two-fold B and which is described by a smooth Weierstrass model. The weak Fano condition implies that $c_1(B) = c_1(\overline{K}_B)$ is Poincaré dual to an effective nef divisor.

One can compute the second Chern class of X as

$$c_2(X) = \pi^* c_2(B) + 12S \cdot \pi^* \overline{K}_B + 11\pi^* \overline{K}_B^2 \,, \tag{3.3.108}$$

where Poincaré duality is implicit, $\overline{K}_B$ is identified with its divisor and S denotes the divisor associated with the global section of the Weierstrass model. The effective curves are generated by the elliptic T^2 fibre and the push forward $\sigma_*(c)$ of the base effective curves $c \subset B$. The nef cone is generated by the vertical divisors $V = \pi^* j$ which project to a nef divisor j of the base and the 'horizontal' divisor $H = S + \pi^* \overline{K}_B$.

Notice that, if we pick an EFT charge vector $\mathbf{e} = (e^0, e^a, e^k)$ with e^a such that $D_{\mathbf{e}} = e^a D_a = V$, then $\hat{\mathbf{C}}(\mathbf{e}) = D_{\mathbf{e}}^3 = 0$ and – see (3.3.97) –

$$\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle = 6\langle \mathbf{C}^1, \mathbf{e} \rangle + 6\langle \mathbf{C}^2, \mathbf{e} \rangle + 6\, j \cdot \overline{K}_B \,. \tag{3.3.109}$$

Heterotic compactifications on such threefolds are dual to F-theory compactified on an elliptic four-fold whose base is a $\mathbb{P}^1$-fibration over the same weak-Fano two-fold B, blown up in the fiber over curves on B wrapped by heterotic 5-branes. To avoid confusion we call this F-theory threefold base X_F in this section. In the notation of Section 3.3.2, the anti-canonical class of X_F is[32]

$$\overline{K}_{X_F} = 2S_- + p^* c_1(\mathcal{T}) + p^* c_1(B) - \sum_k E_k \,. \tag{3.3.110}$$

Here the twist bundle $\mathcal{T}$ characterising the rational fibration is related to the heterotic invariants p_a as

$$p_a = \int_{d_a} c_1(\mathcal{T}) \,, \tag{3.3.111}$$

with d_a a basis of divisors on B. In particular, for a positive bundle, i.e. $c_1(\mathcal{T})$ effective, the gauge sector on the second HW wall maps to the gauge theory on a stack of 7-branes wrapping

[32]For simplicity we are considering single blowups over separate curves.

the exceptional section S_- in the F-theory base X_F.[33] Furthermore E_k denotes a blowup divisor on X_F over the curve $\mathcal{C}^k$ wrapped by the k-th M5-brane. Under the duality, an EFT string with charges $\mathbf{e} = (e^0, e^a, e^k)$ in the heterotic theory maps to an EFT string obtained by wrapping a D3-brane along the curve

$$\Sigma_{\mathbf{e}} = e^0 F_0 + e^a S_- \cdot p^*(d_a) + e^k F_k \,, \tag{3.3.112}$$

where F_0 denotes the generic rational fiber of X_F, d_a continues to denote a basis of divisors on B and F_k is the rational fiber of the exceptional divisor E_k with $E_i \cdot F_j = -\delta_{ij}$. The inequalities (3.3.90) translate into the condition for the curve $\Sigma_{\mathbf{e}}$ to lie inside the movable cone. For instance, in the simple example discussed in Section 3.3.2 (with all $e^k = 0$), the EFT string condition (3.3.47) maps to the condition (3.3.90) once we identify p_a with the positive integer n defining the twist. With these identifications, one can convince oneself that (3.3.109) agrees with the dual expression (3.3.21), more precisely with $6\bar{K}_{X_F} \cdot \Sigma_{\mathbf{e}}$, in F-theory.

Let us now turn to the bounds: By assuming for instance a standard embedding, the bound (3.3.107) resulting from a charge (3.3.105) with $q_a = 0$ and $p_a = n_a$ and with $D_{\mathbf{e}} = V = \pi^*(j)$ becomes

$$r(\mathbf{e}) \leq 7c_2(X) \cdot D_{\mathbf{e}} - 2 = 84j \cdot \overline{K}_B - 2 \,. \tag{3.3.113}$$

This bound does not contain any new information. If $j \cdot \overline{K}_B = 0$ then $\langle \mathbf{C}^1, \mathbf{e} \rangle = \langle \mathbf{C}^2, \mathbf{e} \rangle = 0$, because in the charge vector (3.3.105) the F1 component vanishes for the standard embedding: $p_a e^a = n_a e^a = \frac{1}{2} \int_{\pi^*(j)} c_2(X) = 0$; in this case we already know that $r(\mathbf{e}) = 0$. If $j \cdot \overline{K}_B \geq 1$ and we focus just on the perturbative gauge sector, we arrive at a bound much weaker than (3.3.104). As another example, pick a charge of the form (3.3.105) but with $D_{\mathbf{e}} = H = S + \pi^*(\bar{K}_B)$, still assuming a standard embedding. Then the bound (3.3.106) becomes

$$r(\mathbf{e}) \leq 7\chi(B) + 78\overline{K}_B \cdot \overline{K}_B - 2 \,. \tag{3.3.114}$$

For B a weak-Fano space, $\overline{K}_B \cdot \overline{K}_B \geq 1$, and then again this bound does not improve the bound (3.3.104) for the perturbative gauge sector.

We have emphasized that these bounds test the perturbative part of the heterotic gauge group probed by the EFT string with charge $\mathbf{e}$ as given. In F-theory this gauge sector corresponds to gauge symmetry from 7-branes on the two sections S_- and S_+ of X_F. On the other hand, on the F-theory side there can also be non-abelian gauge groups supported on a vertical divisor $D_V = p^* d$ with d an effective divisor on the base B of X_F. Note that such gauge sectors are independent of the existence of blowup divisors in F-theory, which map to the background M5-branes in heterotic M-theory whose gauge coupling is controlled by the complex structure of the heterotic theory. An example of such a vertical gauge sector was discussed in Section 3.3.2 by

[33] Consistently, for bigger and bigger effective twist class $c_1(\mathcal{T})$, or larger positive values of p_a, the gauge flux on the second HW wall becomes smaller and smaller, as follows from the definition (3.3.76). This eventually results in a non-Higgsable remnant gauge group, which in F-theory must be localised on the rigid section S_-, with negative self-intersection (3.3.43), rather than on S_+.

taking the 7-branes in class D^1. The vertical gauge sector in F-theory must be dual to a non-perturbative gauge sector in the heterotic theory which arises when the elliptic fibration becomes singular over the divisor $d \subset B$. The EFT strings detecting the vertical part of the gauge group in F-theory map to heterotic EFT strings with non-vanishing charges e^a. By duality, therefore, such heterotic EFT strings must contain information about the non-perturbative sectors in question. It would be very interesting to study this effect further from the heterotic point of view.

3.3.4 $SO(32)$ heterotic/Type I models

The $SO(32)$ heterotic models can be discussed in a similar manner, but are somewhat simpler and so we will be brief. We first assume that there are no background NS5-branes. The relevant EFT terms can be derived as in the $E_8 \times E_8$ case. The details are provided in Appendix A.3.2. Assume an internal gauge bundle and suppose for simplicity that it takes values in a semi-simple sub-algebra $\mathfrak{g} \subset \mathfrak{so}(32)$ with vanishing forth order Casimir. Then the four-dimensional EFT contains the terms

$$-\frac{1}{8\pi}\int s^0\, \text{tr}(F \wedge *F) - \frac{1}{96\pi}\int (12 s^0 + 3 n_a s^a)\, \text{tr}(R \wedge *R)\,, \tag{3.3.115}$$

where F takes values in the commutant of $\mathfrak{g} \subset \mathfrak{so}(32)$, n_a is defined as in (3.3.78), and

$$s^0 = \hat{s} - \frac{1}{6} n_a s^a\,, \tag{3.3.116}$$

with $\hat{s}$ as in (3.3.68). Hence

$$\mathbf{C} = (C^0, C^a) = (1, 0) \quad , \quad \tilde{\mathbf{C}} = (\tilde{C}^0, \tilde{C}^a) = (12, 3n_a)\,. \tag{3.3.117}$$

One can also discuss the instanton corrections and the saxionic cone as in Section 3.3.3. In this case the ten-dimensional curvature corrections do not affect the form of the saxionic cone, but only the microscopic definition (3.3.116) of s^0. The saxionic cone is simply given by $\mathbb{R}_{>0} \oplus \mathcal{K}(X)$, where $\mathbb{R}_{>0}$ is parametrized by s^0. Correspondingly the EFT string charges are given by $\mathcal{C}_{\rm S}^{\rm EFT} = \{\mathbf{e} = (e^0, e^a) | e^0 \geq 0\,, D_{\mathbf{e}} \equiv e^a [D_a] \in \text{Nef}^1(X)\}$. It is then easy to see that (3.1.18) and our quantum gravity constraints (3.2.4) and (3.2.6) are satisfied.

In particular, by picking $\mathbf{e}_{\rm F1} = (1, \vec{0}) \in \mathcal{C}_{\rm S}^{\rm EFT}$ in (3.2.26) we obtain $r(\mathbf{e}_{\rm F1}) \leq 22$, correctly reproducing the rank of the 'perturbative' gauge sector detected by the perturbative heterotic string: the $\mathfrak{so}(32)$ sector explicitly appearing in (3.3.115), plus the up to six possible additional KK $U(1)$s, depending on the type of background. (As before, potential gauge group factors whose axionic couplings are not of the form (2.1.11) cannot be detected in this manner.)

One may equivalently start from the S-dual Type I description of these backgrounds. In particular, the EFT string charge $\mathbf{e}_{\rm F1} = (1, \vec{0})$ corresponds to a D1-brane. However, as discussed in [34], the D1-brane string flow drives the Type I dilaton to $+\infty$, and then the heterotic formulation is better suited for describing the UV completion of the corresponding perturbative regime.

We next consider EFT strings of charges $\mathbf{e}_{\rm NS5} = (0, e^a)$, which correspond to NS5-branes wrapping internal nef divisors $D_{\mathbf{e}} \equiv e^a[D_a] \in \text{Nef}^1(X)$. These EFT strings can detect the 'non-perturbative' gauge sector supported by bulk NS5-branes, which is not included in (3.3.115). These NS5-branes can appear through small instanton transitions [155] and correspond to D5-branes in the dual Type I description. In fact, as in the $E_8 \times E_8$ case, the heterotic dilaton $e^{2\phi_{\rm het}} = 6(\kappa_{abc}s^a s^b s^c)/\hat{s}$ generically diverges along the flows of these EFT strings. Hence the type I description is better suited.[34]

The bound (3.2.26) now takes the form

$$r(\mathbf{e}_{\rm NS5}) \leq 3\int_{D_{\mathbf{e}}} c_2(X) - 2 \tag{3.3.118}$$

In order to show that this bound is indeed satisfied, recall the $SO(32)$ counterpart of the tadpole condition (3.3.70):

$$\lambda(E) + [\mathcal{C}] = c_2(X)\,. \tag{3.3.119}$$

Here $\lambda(E)$ is defined as in (3.3.71) and

$$\mathcal{C} = N_A \mathcal{C}^A\,, \tag{3.3.120}$$

where $N_A > 0$ counts the NS5-branes wrapping the irreducible curve $\mathcal{C}^A$. The supersymmetry condition on the internal bundle implies that $\int_{D_{\mathbf{e}}} \lambda(E) \geq 0$. Hence from (3.3.119) we get

$$\mathcal{C} \cdot D_{\mathbf{e}} \leq \int_{D_{\mathbf{e}}} c_2(X)\,. \tag{3.3.121}$$

We can now use the dual type I description to compute the rank of $r(\mathbf{e}_{\rm NS5})$, which just counts the bulk D5-branes which intersect the nef divisor $D_{\mathbf{e}}$:

$$r(\mathbf{e}_{\rm NS5}) = \sum_{A|\mathcal{C}^A\cdot D_{\mathbf{e}}\neq 0} N_A\,. \tag{3.3.122}$$

Since $D_{\mathbf{e}}$ is nef, we know that $\mathcal{C}^A \cdot D_{\mathbf{e}} \geq 0$ and then

$$r(\mathbf{e}_{\rm NS5}) \leq \sum_A N_A(\mathcal{C}^A \cdot D_{\mathbf{e}}) = \mathcal{C} \cdot D_{\mathbf{e}} \leq \int_{D_{\mathbf{e}}} c_2(X)\,, \tag{3.3.123}$$

where in the last step we have used (3.3.121). The microscopic bound (3.3.123) implies our

[34]The microscopic description of these infinite distance limits can be done as in [34], but should be revised by taking into account the curvature correction entering (3.3.116).

quantum gravity bound (3.3.118) (in the non-trivial case $\langle \tilde{\mathbf{C}}, \mathbf{e}_{\text{NS5}} \rangle = \frac{3}{2}\int_{D_{\mathbf{e}}} c_2(X) > 0$), which is then always satisfied.

3.3.5 Microscopic checks in G_2 M-theory models

As a last class of models, we consider M-theory compactified on a G_2-manifold X. In this case, the 4d $U(1)$ gauge sector comes from the expansion of the M-theory three-form C_3 into a basis of integral harmonic two-forms $[\Gamma^A]_{\text{harm}} \in H^2(X,\mathbb{Z})$, where $\Gamma^A \in H_5(X,\mathbb{Z})$ denotes the Poincaré dual basis:

$$C_3 = \frac{\ell_{\text{M}}^3}{2\pi} A_A \wedge [\Gamma^A]_{\text{harm}} \,. \tag{3.3.124}$$

One can also have non-abelian gauge sectors localised at singularities, whose $U(1)$ Cartan sector can be identified by resolving the singularity. Our EFT constraints should also hold in the singular case, but in order to check them we will assume that all these singularities have been resolved.

The (s)axions are obtained from the expansion

$$C_3 + \mathrm{i}\Phi = \ell_{\text{M}}^3 (a^i + \mathrm{i}s^i)[\Pi_i]_{\text{harm}} \,, \tag{3.3.125}$$

where $\Pi_i \in H_4(X,\mathbb{Z})$ is a basis of 4-cycles, $[\Pi_i]_{\text{harm}} \in H^3(X,\mathbb{Z})$ is the Poincaré dual basis of harmonic representatives, and Φ is the associative three-form. As a key property of G_2 manifolds, any harmonic two-form ω satisfies the relation [156]

$$*_X \omega = -\Phi \wedge \omega \,, \tag{3.3.126}$$

where $*_X$ is the Hodge star associated with the internal G_2 metric. By expanding the eleven-dimensional terms

$$-\frac{\pi}{\ell_{\text{M}}^9} \int G_4 \wedge * G_4 + \frac{2\pi}{\ell_{\text{M}}^9} \int C_3 \wedge G_4 \wedge G_4 \tag{3.3.127}$$

one obtains the four-dimensional terms of the form (2.1.7), with $C_i^{AB} = -\Pi_i \cdot \Gamma^A \cdot \Gamma^B$ and $C_i^I = 0$. Note that (3.3.126) implies that the matrix

$$\langle \mathbf{C}^{AB}, \boldsymbol{s} \rangle = -\int_X \Phi \wedge [\Gamma^A] \wedge [\Gamma^B] \tag{3.3.128}$$

is positive definite.

Consider now the eleven-dimensional term

$$\frac{1}{192(2\pi)^3 \ell_{\text{M}}^3} \int C_3 \wedge \left[\operatorname{tr} \mathbf{R}^4 - \frac{1}{4} (\operatorname{tr} \mathbf{R}^2)^2 \right] \,, \tag{3.3.129}$$

where $\mathbf{R}$ is the eleven-dimensional curvature two-form. By splitting $\mathbf{R} = R + \hat{R}$ according to the $4+7$ split, one finds an axionic coupling of the form appearing in (2.1.19), with

$$\tilde{C}_i = -\frac{1}{4}\int_{\Pi_i} p_1(X)\,. \tag{3.3.130}$$

A four-dimensional BPS axionic string of charges e^i is obtained by wrapping an M5-brane along a coassociative 4-cycle $\Pi = e^i\Pi_i$. Imposing that this string is EFT, $\mathbf{e}\in\mathcal{C}_{\rm S}^{\rm EFT}$, corresponds to requiring that the Poincaré dual three-cocycle $[\Pi]$ can be represented by an associative three-form $\Phi_{\mathbf{e}}$, or a limit thereof reached by approaching the boundary of the corresponding saxionic cone [34]. From the positive-definiteness of (3.3.128), we conclude that $\langle \mathbf{C}^{AB},\mathbf{e}\rangle$ is positive semi-definite, hence realizing (3.1.18). Furthermore an argument of footnote 2 of [157] and Lemma 1.1.2 of [158] implies that

$$\langle\tilde{\mathbf{C}},\mathbf{e}\rangle = -\frac{1}{4}\int_X \Phi_{\mathbf{e}}\wedge p_1(X)\in\mathbb{Z}_{\geq 0}\,. \tag{3.3.131}$$

By splitting $TX|_\Pi = T\Pi\oplus N\Pi$ and using the identifications $N\Pi\simeq\Lambda^2_+\Pi$ (the bundle of self-dual two-forms) and $p_1(\Lambda^2_+\Pi) = p_1(\Pi)+2e(\Pi)$ [159][35], we can rewrite $\langle\tilde{\mathbf{C}},\mathbf{e}\rangle = -\frac{1}{4}\int_\Pi p_1(X)$ in the form

$$\langle\tilde{\mathbf{C}},\mathbf{e}\rangle = b_2^-(\Pi) - 2b_2^+(\Pi) + b_1(\Pi) - 1\,. \tag{3.3.132}$$

By the anomaly matching argument, this formula should agree with (3.2.2a). We can indeed check this result microscopically, following [160]. More precisely, the numbers of massless fields on the string are counted by

$$n_{\rm C} = b_2^+(\Pi)\,,\quad n_{\rm F} = b_2^-(\Pi) - b_2^+(\Pi)\,,\quad n_{\rm N} = b_1(\Pi)\,. \tag{3.3.133}$$

To see this, note first that the world-sheet theory contains $b_2^+(\Pi)$ massless real scalars, which parametrize the geometric deformations of Π [159]. Furthermore, by dimensionally reducing the self-dual M5 two-form on Π one obtains $b_2^+(\Pi)$ right-moving plus $b_2^-(\Pi)$ left-moving real chiral scalars. Supersymmetry then fixes the orientation of Π so that $b_2^-(\Pi) - b_2^+(\Pi)\geq 0$ and we can combine the above modes into the $b_2^+(\Pi)$ complex scalars. These form the scalar components of $n_{\rm C} = b_2^+(\Pi)$ chiral multiplets, whose $b_2^+(\Pi)$ right-moving fermions should come from the reduction of the M5 fermions. The remaining $b_2^-(\Pi)-b_2^+(\Pi)\geq 0$ left moving real chiral bosons can be fermionized and, completed by $b_2^-(\Pi)-b_2^+(\Pi)$ left-moving fermions coming from the M5-brane fermions, form $n_{\rm F} = b_2^-(\Pi)-b_2^+(\Pi)\geq 0$ Fermi multiplets. Furthermore, reducing the M5 self-dual two-form on the harmonic one-forms yields $b_1(\Pi)$ $U(1)$ vectors, which are completed into vector multiplets by a corresponding number of left-moving fermions λ_- with charges as in Table (3.1.28). The corresponding $n_{\rm N} = b_1(\Pi)$ Fermi superfields Λ_- can be identified with the super field strengths of the vector multiplets, which contribute to our anomaly matching in the

[35] Our self-dual forms corresponds to the anti-self-dual ones of [159], and viceversa.

same way as fundamental Fermi multiplets.[36] Altogether, we have therefore derived the values of $n_{\rm C}$, $n_{\rm F}$ and $n_{\rm N}$ in (3.3.133), which together with (3.2.2a) reproduce the geometric prediction (3.3.132).

From (3.2.25) and with the help of (3.3.133), the bound on the rank of the gauge group detected by the EFT string is given by[37]

$$r(\mathbf{e}) \leq n_{\rm F} + 2(n_{\rm C} - n_{\rm N}) = b_2(\Pi) - 2b_1(\Pi). \tag{3.3.134}$$

On the other hand, (3.2.26) and (3.3.132) give

$$r(\mathbf{e}) \leq 2\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle - 2 = 2b_2(\Pi) - 6b_2^+(\Pi) + 2b_1(\Pi) - 4 + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle\,. \tag{3.3.135}$$

These two bounds must coincide and their difference can therefore be read as an equation for $\langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle$.

As an example, we may assume that Π has trivial normal bundle $N\Pi \simeq \Lambda^2_+\Pi$ as in [162]. In this case $p_1(N\Pi) = 0$ and then Π is spin [162]. Hence by Rochlin's theorem $\int_\Pi p_1(\Pi) \in 48\mathbb{Z}$ and $\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle = -\frac{1}{4}\int_\Pi p_1(X) \in 12\mathbb{Z}$. For instance, one can consider a so called 'Twisted Connected Sum' G_2 manifold [163], which can be viewed as a coassociative K3-fibration over an S^3. By fiber-wise duality it should be dual to a heterotic compactification on a Calabi-Yau three-fold – see for instance [164]. More precisely, the M5-brane wrapping the K3 fiber is the dual heterotic fundamental string and our quantum gravity bounds derived from this string must therefore coincide with the bounds obtained in the heterotic case. Indeed, $\int_\Pi p_1(X) = -48$ and then $\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle = 12$, as found for the heterotic fundamental string in Section 3.3.3. Furthermore, by comparing the bounds (3.3.134) and (3.3.135) and using the K3 Betti numbers $b_2^+(\Pi) = 3$, $b_2^-(\Pi) = 19$ and $b_1(\Pi) = 0$, one gets $\langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle = 0$ as expected. The bound (3.3.135) – or equivalently (3.3.134) – then gives

$$r(\mathbf{e}) \leq 2\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2 = 22, \tag{3.3.136}$$

which reproduces the general bound (3.3.104) on the rank of a heterotic perturbative gauge group in four dimensions.

In the large volume heterotic regime, we have encountered EFT strings corresponding to NS5-branes wrapping nef divisors. These should also correspond to M5 EFT strings in the M-theory picture. This in particular implies that there should exist EFT strings with non-vanishing

[36] In two dimensions a $U(1)$ gauge field does not carry propagating degrees of freedom and, in absence of charged matter, its field strength can be traded for the auxiliary field of the Fermi multiplet. This is the two-dimensional counterpart of the relation between three-form multiplets and chiral multiplets in four-dimensions discussed in [67, 161].

[37] Recall that the subtraction of the $n_{\rm N}$ Fermi multiplets accounts for potential obstructions of the scalar moduli. In the present situation, these would have to be due to non-perturbative effects from M2-brane instantons ending on the M5-brane, since the b_2^+ massless modes describing the geometric deformations of a coassociative cycle are classically unobstructed [159].

$\hat{C}_i(\mathbf{e}) = \hat{C}_{ijk}e^j e^k$. At a first sight, this may appear in contradiction with our general expectation that such a $\hat{C}_i(\mathbf{e})$ should be associated with some hidden five-dimensional structure, since apparently no such structure is present. However, the strings with non-trivial $\hat{C}_i(\mathbf{e})$ may dynamically generate a preferred fifth direction along their flow.

In order to support this proposal, let us consider an M-theory compactification admitting a weakly-coupled type IIA limit, in which the G_2 manifold X becomes the orbifold [165]

$$X_\circ = (Y \times S^1)/(\sigma, -1)\,. \tag{3.3.137}$$

Here Y represents a Calabi-Yau three-fold admitting an O6-plane involution $\sigma : Y \to Y$. In this limit, each irreducible component of the fixed locus of σ is occupied by one O6-plane and four D6-planes. Furthermore the associative three-form becomes $\Phi = \mathrm{Re}\,\Omega + \mathrm{d}y \wedge J$, where J and Ω are respectively the Kähler form and (appropriately normalized) holomorphic $(3,0)$ form of Y. Then the coassociative four-cycles are calibrated by $*_{X_\circ}\Phi = \frac{1}{2}J \wedge J + \mathrm{d}y \wedge \mathrm{Im}\,\Omega$. Note that (J, Ω) must satisfy the orientifold projection $\sigma^* J = -J$ and $\sigma^*\mathrm{Re}\,\Omega = \mathrm{Re}\,\Omega$.

Let us then focus on those coassociative four-cycles Π in $X_\circ$ which are calibrated by $\frac{1}{2}J \wedge J$. These can be regarded as orientifold-even effective divisors D in Y. In this case, an M5-brane wrapping Π reduces to an NS5-brane wrapping D and represents an EFT string if D is a nef divisor. The analysis of these EFT strings is completely analogous, *mutatis mutandis*, to the analysis carried for the $E_8 \times E_8$ models in Section 3.3.3 (and in [34]), without the complications due to the higher derivative corrections discussed therein. In particular, the EFT strings should support a term of the form (3.1.11) with $\hat{C}_i(\mathbf{e})$ as in (3.3.95), where κ_{abc} now denotes the triple intersection number defined on $H^2_-(Y, \mathbb{Z})$.

Now the key point is that, along the string flows associated with the EFT strings with non-vanishing $\hat{C}_i(\mathbf{e})$, the M-theory circle decompactifies much faster than the Calabi-Yau Y. So, even if one starts from a more generic field space point, in which the factorized geometry of the form (3.3.137) is not manifest, a preferred fifth direction would dynamically emerge.

In type IIA language, up to $U(1)$ mixing effects [166], these EFT strings more naturally detect the R-R $U(1)$ gauge sectors, while the EFT strings corresponding to D4-branes on appropriate special Langrangian three-cycles detect the D6 gauge sectors. The M-theory analysis nicely unifies these sectors. The corresponding EFT constraints may be tested as already done above in other models but, since $X_\circ$ is not complex, in this case it is harder to both extract further general results or perform a case by case analysis. We leave this interesting task for future explorations.

Chapter 4

Wormholes in the axiverse and the Species Scale

Wormholes are a deeply studied subject in scientific literature, nevertheless so far no clear consensus has been reached on their physical relevance as quantum gravity effects. Arguments both for or against the necessity of their non-perturbative contributions have been formulated, with recently the so-called factorization problem in holography and the conjecture about Baby Universes sparking new discussions around the topic [13, 167]. Here, we will conservatively assume that such objects can contribute to the Euclidean path integral and, when charged, can account for a violation of global symmetries. Subsequently we will discuss the consequences of this assumption.

The global symmetries we are interested in are the shift symmetries associated to axions, and the explicit breaking effect expected to be induced by wormholes is materialized at low energies in an axion potential. Quantifying these effects is especially relevant in the context of models of inflation and the QCD axion solution to the strong CP problem, where even small symmetry breaking effects can have important phenomenological consequences.

It is worth briefly reviewing how this phenomenon might manifest itself in this context. This is best understood via an effective field theory defined at distances much larger than the wormhole throat L. In that regime, and working for now in a dilute instanton gas approximation, wormholes are described by bi-local corrections to the action [168, 169]

$$S_{\text{vert.}} = \sum_{IJ} \int d^4x \int d^4y \; C_{IJ} \mathcal{O}_I^\dagger(x) \mathcal{O}_J(y), \tag{4.0.1}$$

with $C_{IJ} = C_{JI}^*$ constant parameters proportional to the amplitude of wormhole production and $\mathcal{O}_I$ local gauge-invariant operators. Consider for concreteness the effect of wormholes carrying charge under a global $U(1)$ realized non-linearly via a compact axion θ in the EFT. By assumption wormholes subtract charge from x and posit it in y. Accordingly, the bi-local action of eq. (4.0.1) must be written in terms of operators $\mathcal{O}_I$, $\mathcal{O}_J$ with equal and opposite charges ($q_I = -q_J$). An observer around x would thus experience a *local* violation of charge conservation. This is signaled by an anomaly in the Noëther current:

$$\partial_\mu J^\mu = i \sum_{IJ} q_J C_{IJ} \left[\left(\int \mathcal{O}_I^\dagger \right) \mathcal{O}_J - \mathcal{O}_I^\dagger \left(\int \mathcal{O}_J \right) \right] . \tag{4.0.2}$$

Globally, though, charge is conserved. Because wormholes act as conduits of charge, there is vanishing charge flowing from or outside any region that encompasses both the origin and the end of the wormhole. The anomaly in eq. (4.0.2), when integrated over the whole space-time, vanishes identically. This conclusion continues to hold when corrections to the dilute instanton gas approximation, in the form of wormhole and instanton interactions as described in [169], are included. If wormholes are mere charge carriers, at distances $\gg L$ these corrections are parametrized by multi-local vertices involving operators $\mathcal{O}_I$ with constant coefficients and, crucially, vanishing overall charge. None of these interactions can break charge globally by assumption, nor generate an effective potential for the axion. In order to induce such effect, the anomaly should have an overlap with the vacuum and a state of n axions at rest, and this cannot certainly happen here because the right-hand side of (4.0.2) is always completely neutral under the $U(1)$, as so does $S_{\text{vert.}}$ or its multi-local generalization. How can wormholes generate an axion potential, then?

It seems that the only logical possibility is that for some reason some of the charged wormholes do not bring charge back to our Universe. Wormholes should swallow charge and seclude it into another world. Effectively, one end of the wormhole should disappear, so that from our perspective charge would be badly violated both locally and globally.

One may gain some insight as to how this possibility might be realized by reformulating this story in the language of the famous α-parameters of Coleman [170]. The bi-local vertex (4.0.1) can be equivalently written as a sum of local interactions $-\sum_{IJ} \alpha_I^* C_{IJ}^{-1} \alpha_J + \sum_I \int d^4x \; \alpha_I^* \mathcal{O}_I(x) +$ hc provided we integrate also over constant α-parameters. The main effect of wormhole and instanton interactions is to turn the Gaussian distribution of the α_I's into a more involved one [169]. Within the α-parameters language, wormholes are thus conjectured to renormalize the couplings of the theory via α-dependent terms. In addition, by opening up gateways to otherwise disconnected geometries, they introduce new (α-dependent) contributions to the path integral [171]. After having summed over all allowed geometries, the partition function describing our Universe at length scales $\gg L$ (where physics in the absence of wormholes would be captured by fields Φ with action $S[\Phi]$) is determined by the effective action $S[\Phi] - \sum_I \int d^4x \; \alpha_I^* \mathcal{O}_I(x) + \text{hc}$ weighted with a complicated distribution $\mathcal{P}(\alpha, \alpha^*)$ for the α-parameters — one that takes into account both the existence of a priori disconnected Universes communicating to ours only via wormholes as well as corrections to the dilute instanton gas approximation [168, 169]:

$$Z = \int d\alpha \int d\alpha^* \; \mathcal{P}(\alpha, \alpha^*) \int \mathcal{D}\Phi \; e^{-S[\Phi] - \left[\sum_I \int d^4x \; \alpha_I^* \mathcal{O}_I(x) + \text{hc}\right]}. \tag{4.0.3}$$

While locality seems to be restored by the introduction of α parameters, the cluster decomposition principle is in general badly violated by the presence of an integral over α, α^* [14]. A possible way out is that for some reason $\mathcal{P}$ is sharply peaked at specific values $\alpha_I = \overline{\alpha}_I$ [171] (see also [172, 173] for more recent considerations). A (not necessarily sharp) peak at some non-trivial value also seems to be necessary for the production of an axion potential. In fact, with fixed $\alpha_I = \overline{\alpha}_I$ our effective field theory includes new symmetry-breaking terms $\sum_I \int d^4x \; \overline{\alpha}_I^* \mathcal{O}_I(x) + \text{hc}$, and a potential for θ is generated provided $\mathcal{O}_I$ has a non-trivial charge and a non-trivial expectation value:

$$V_{\text{wh}}(\theta) = \sum_I \overline{\alpha}_I^* \langle \mathcal{O}_I \rangle e^{iq_I\theta} + \text{hc} = 2 \sum_I |\overline{\alpha}_I^* \langle \mathcal{O}_I \rangle| \cos\left(q_I\theta + \text{Arg}[\overline{\alpha}_I^* \langle \mathcal{O}_I \rangle]\right). \tag{4.0.4}$$

The physical implications of wormholes therefore crucially depend on the unknown distribution $\mathcal{P}(\alpha, \alpha^*)$. The naive symmetry around $\alpha_I = 0$, or between wormholes swallowing charge and spitting it back, must be broken in order to generate measurable $U(1)$-violating effects. Throughout the discussion we will assume that such asymmetry in fact is present and in particular that wormholes can generate an axion potential.

It is useful for what follows to have at least a rough idea of the size of this potential. On purely dimensional grounds $\overline{\alpha}_I^* \langle \mathcal{O}_I \rangle$ is expected to be proportional to some dimensionless number times $1/L^4$, namely four powers of the inverse wormhole length. The estimate of the dimensionless number is a bit subtle. Because by $\mathcal{O}_I$ we denote any allowed operator with the appropriate charge, it is natural to conservatively take $\langle \mathcal{O}_I \rangle \sim 1/L^{d_I}$, with d_I its engineering dimension. To make an educated guess of the size of the dimensionless number $\overline{\alpha}_I L^{d_I - 4}$ we go back to eqs.(4.0.1). In a semi-classical description of wormholes we have $C_{IJ} \propto e^{-S_{I,\text{wh}}}$, with $S_{I,\text{wh}}$ the Euclidean classical action of a wormhole of charge q_I. By defining $\alpha_I \equiv \alpha'_I e^{-S_{I,\text{wh}}/2}$ the α'_I distribution is controlling by order one numbers in natural units. Hence one anticipates that $\overline{\alpha}'_I \sim 1$ and finally

$$\overline{\alpha}_I \propto e^{-S_{I,\text{wh}}/2}. \tag{4.0.5}$$

The exponential of half-wormhole is therefore a measure of the strength of the effective wormhole interactions appearing in (4.0.3), and thus of the axion potential shown in eq. (4.0.4).

In this Chapter, based upon the upcoming paper *"Wormholes in the axiverse and the species scale"*, we will provide a detailed study of a particular class of homogeneous multi-saxionic wormholes, some of their limit cases and generalizations, and their contributions to low energy effective theories. The main aspects of the theoretical setting, $N = 1$ supergravity in four dimension, are collected in chapter 2, but we will anyway begin with some essential remarks and recall for the reader about models with many saxions in Section 4.1. In doing so, we will stress the role of the Gauss-Bonnet correction to the 2-derivative action as the relevant one for our purposes.

Before diving into the wormholes themselves, in Section 4.2 we will first address the issue of the perturbative control and of the important energy scales involved in the problem. We will be able to identify a specific perturbative domain in saxionic space where the bounds derived in Chapter 3 lead to a suppression of the higher derivative corrections. In addition we will explore the interplay of the notion of energy scale of strong coupling for gravity with those of EFT string limits and their role as natural cut-offs for the wormhole solutions. There we will argue on general grounds for the strongly coupled gravity scale to be always controlled from above by the square root of the EFT string tensions in asymptotic limits.

Those statements will be made more manifest with explicit examples from String Theory in Section 4.3. There we will briefly review some F-theory constructions and apply, to both them and heterotic models, the previously discussed arguments about the relevant energy scales, finding agreement with our expectations.

In Section 4.4 we will show the possibility of sub-extremal and extremal Euclidean wormhole solutions in our setting. We will provide explicitly the equations of motion, solve them through a $SO(4)$-symmetric ansatz and compute the on-shell action associated to these configurations. In the extremal case we will delve into the possible connection to fundamental instantons with analogous instanton charges and present a peculiar BPS bound on the on-shell action of sub-extremal wormholes.

In Section 4.5 we will introduce a class of homogeneous multi-saxionic wormholes, made possible by the homogeneity of the intersection polynomial controlling the kinetic terms of the saxions in our concrete examples, and provide arguments for the interplay between their instanton charges and the set of charges which characterize BPS and EFT strings. In particular we will be able to associate the subset of BPS instanton charges, which we will call EFT instanton charges, characterizing controlled configurations with those that have a non-negative pairing with BPS string charges, in an analogous way as EFT strings are defined within the set of BPS strings. We will consider the non-homogeneous generalizations and discuss some of their expected features. For such scope, a perturbative analysis in the space of solutions around the analytically controlled ones will be necessary and fruitful. A marginally degenerate case, that with generating polynomial of degree 3, will appear and spawn an in-depth discussion all on its own, where the connection to BPS fundamental instantons will be highlighted. Finally, in absence of analytical control, a numerical scan will give us further evidence of the existence and confirmation of the features of the space of non-homogeneous solutions close to the universal ones.

The main conclusions and a set of open questions are presented in chapter 5.

4.1 Saxions and semitopological higher derivative terms

The basic assumption underlying our work is the existence of an effective four-dimensional field theory (EFT) with a large number N of light axions. We will focus on *fundamental* axions, namely periodic axions that originate from string theory, as opposed to quantum field theory. Because in quantum gravity (QG) global symmetries are expected to be at most approximate, the global shift symmetries that prevent our axions to acquire large masses must ultimately be broken. It is then crucial to identify a concrete and realistic framework in which the axionic shift symmetries can be considered exact up to small corrections dictated by QG. Such a framework is provided by the $\mathcal{N}=1$ setup outlined in [51, 52, 64]. The associated EFTs emerge from large classes of string theory models and naturally take into account the relevant QG constraints.

Our basic setup, analogous to the one presented in [51, 52, 64], is essentially the same as the one setting the stage for the previous chapter and we recommend the reader for Section 2 for a quick review. Here we will just recall some very basic features of such setting, mostly to set the conventions. We focus on the leading 2-derivative approximation but, for reasons that will become clearer later, also keep an eye on the (semi-)topological couplings to gravity, and in particular the Gauss-Bonnet interaction, for which we will provide a deeper introduction. In Section 4.2.1 we provide an unambiguous definition of the domain of validity of our EFT and quantify the size of the QG effects that violate the axionic shift-symmetry. Remarkably, within this perturbative domain the coefficient of the Gauss-Bonnet interaction will be shown to be subject to an interesting lower bound (see Section 4.2.2). We point to the reader Section 2.2.1 for the basics in the reformulation of the EFT in a dual 2-form language in preparation of the subsequent sections. An explicit connection between our four-dimensional EFT and several string theory models is given in Section 4.3.

4.1.1 The two-derivative theory

The exact shift symmetry combined with supersymmetry constrain significantly the EFT. The only manifestly supersymmetric non-derivative couplings of the axions can be either topological or functions of $e^{2\pi i t^i q_i}$ with $q_i \in \mathbb{Z}$, and so exponentially suppressed by the saxions. We will discuss the topological couplings shortly and postpone an analysis of the exponentially suppressed instanton-like corrections to a following subsection, where we also provide a quantitative definition of the perturbative regime in which such corrections can be considered small.

Up to topological couplings and instanton-like effects, the EFT is accidentally invariant under arbitrary constant shifts of the axions. At the 2-derivative level the most general shift-symmetric Minkowskian action involving gravity and t^i is

$$S^{(\mathrm{M})} = \frac{1}{2}M_{\mathrm{P}}^2 \int R * 1 - \frac{1}{2}M_{\mathrm{P}}^2 \int \mathcal{G}_{ij}(s) \left(\mathrm{d}s^i \wedge *\mathrm{d}s^j + \mathrm{d}a^i \wedge *\mathrm{d}a^j\right) + \mathrm{BT} \tag{4.1.1}$$

where $\mathcal{G}_{ij}(s)$ is a symmetric positive matrix function of the saxion fields and BT denotes the appropriate Hawking-Gibbons boundary term. By supersymmetry, the kinetic terms of the scalars are specified by a Kähler potential K. Within our perturbative regime that depends on the complex fields t^i only through their saxionic component $s^i = -\frac{\mathrm{i}}{2}(t^i - \bar{t}^i)$, i.e. $K = K(s)$ (a possible dependence on spectator chiral multiplets is ignored), via the relation

$$\mathcal{G}_{ij} \equiv \frac{1}{2}\frac{\partial^2 K}{\partial s^i \partial s^j}. \tag{4.1.2}$$

We can actually say more about our EFT if we take into account additional non-trivial inputs from the UV completion. Indeed, within the perturbative regime we are considering (to be rigorously defined later), for a large class of string theory models the Kähler potential reads

$$K(s) = -\log P(s)\,, \tag{4.1.3}$$

where $P(s)$ is a positive homogeneous function, $P(\lambda s) = \lambda^n P(s)$ and n is an integer ranging from 1 to 7. As discussed in [51, 64] the perturbative structure (4.1.3) conforms with various formulations of the weak gravity conjecture [69] and the distance conjecture [22] in the present setting. The homogeneity of P and the relation (4.1.3) will play a crucial role in some of the subsequent sections.

Note that by the homogeneity of $P(s)$ we have $\ell_i s^i = \frac{n}{2}$ and then the dual saxion kinetic function (2.2.5) takes the form

$$\mathcal{F}(\ell) = n + \log \tilde{P}(\ell)\,, \tag{4.1.4}$$

where n always turns out to be an irrelevant constant factor whereas

$$\tilde{P}(\ell) \equiv \frac{1}{P(s(\ell))} \tag{4.1.5}$$

is a homogeneous function of degree n:

$$\tilde{P}(\lambda\ell) = \lambda^n \tilde{P}(\ell)\,. \tag{4.1.6}$$

Similarly, the (semi-)topological couplings to gravity can be written as in (4.1.10) provided we interpret s^i as a function of the dual saxions, as dictated by (2.2.8).

4.1.2 (Semi-)Topological couplings to gravity

Eq. (4.1.1) just represents the leading 2-derivative term in our EFT. In general one should also allow the presence of higher dimensional interactions suppressed by the UV cutoff Λ. Restricting our EFT considerations to energy scales satisfying $E \ll \Lambda$ one may naively presume that the effect of higher dimensional operators can be safely neglected. But how can we know what Λ is from the EFT point of view? We will come back to this important question in Section 4.2.3. For the moment we observe that a very special class of higher-dimensional operators are not suppressed at low scales. These are the *topological operators*. Indeed, while topological operators do not affect the equations of motion nor induce particle vertices, they can contribute to the on-shell action and therefore impact semiclassical calculations. It is therefore important to investigate what operators belong to this class in our EFT. Even if we have introduced those higher derivative terms already in chapter 2, let us delve a bit deeper into them here, since they will become important for the successive discussion.

In a gravitational theory there are two potentially topological terms, associated to the Gauss-Bonnet operator

$$E_{\rm GB} \equiv \frac{1}{32\pi^2}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right) \tag{4.1.7}$$

and the Pontryagin operator $E_{\rm P} \equiv \epsilon^{cdef} R^a{}_{bef} R^b{}_{acd}$. In an $\mathcal{N}=1$ SUSY framework these are described by

$$\begin{aligned} S^{(\rm M)} &\supset \int \frac{1}{96\pi}{\rm Re}[F(t)]\,{\rm tr}(\mathcal{R}\wedge\mathcal{R}) + \int \frac{1}{96\pi}{\rm Im}[F(t)]\left[{\rm tr}(\mathcal{R}\wedge *\mathcal{R}) + \mathcal{O}(R^2_{ab}, R^2)\right] + {\rm BT} \\ &= \int {\rm d}^4x\,\sqrt{|g|}\,\frac{1}{384\pi}({\rm Re}[F(0)] + \tilde{C}_i a^i)\,E_{\rm P} \\ &+ \int {\rm d}^4x\,\sqrt{|g|}\,\frac{\pi}{6}({\rm Im}[F(0)] + \tilde{C}_i s^i)\left[E_{\rm GB} + \mathcal{O}(R^2_{ab}, R^2)\right] + {\rm BT} \end{aligned} \tag{4.1.8}$$

and originate from the superspace F-term $\int {\rm d}^4x\,{\rm d}^2\theta\,E\,F(t)\,\mathcal{W}^{\alpha\beta\gamma}\mathcal{W}_{\alpha\beta\gamma} + {\rm c.c.}$, where $F(t)$ is a holomorphic function of t^i, see [54–57]. The topological charge $\int {\rm tr}(\mathcal{R}\wedge\mathcal{R})$ is quantized in multiples of $192\pi^2$. Therefore, the basic assumption (2.1.1) forces $F(t)$ to be of the form $F(t) = F(0) + \tilde{C}_i t^i$ with $F(0)$ an arbitrary constant and $\tilde{C}_i \in \mathbb{Z}$, modulo exponentially suppressed instanton-like corrections. Perturbatively, the quantized linear coupling is thus exact, whereas the contribution proportional to $F(0)$ may receive at most 1-loop corrections $\propto N\ln\Lambda_{\rm UV}$ in a manifestly supersymmetric framework. [1] The first line in eq. (4.1.10) includes an axion coupling to the Pontryagin operator ${\rm tr}(\mathcal{R}\wedge\mathcal{R}) = \frac{1}{4}{\rm d}^4x\,\sqrt{|g|}\,E_{\rm P}$ and a coupling of the saxions to the Weyl density, which we write as ${\rm tr}(\mathcal{R}\wedge *\mathcal{R}) + \mathcal{O}(R^2_{ab}, R^2) = \frac{1}{2}{\rm d}^4x\,\sqrt{|g|}\,R_{abcd}R^{abcd} + \mathcal{O}(R^2_{ab}, R^2) = {\rm d}^4x\,\sqrt{|g|}\left[16\pi^2 E_{\rm GB} + \mathcal{O}(R^2_{ab}, R^2)\right]$. Neglecting $\mathcal{O}(R^2_{ab}, R^2)$ terms, therefore, we find that the coefficient of the Gauss-Bonnet operator is $\frac{\pi}{6}({\rm Im}[F(0)] + \tilde{C}_i s^i)$, up to non-perturbative effects. We have been intentionally sloppy about the coefficients of the non-derivative saxions coupling to $R_{ab}R^{ab}$ and R^2 because these can also receive contributions from superspace D-terms. Hence, in general those interactions can contain more complicated functions of t^i, and moreover be

[1] In complete analogy, anomalous couplings of axions to vector fields are described by

$$S^{(\rm M)}_{\rm top-gauge} = \int \frac{1}{8\pi} C_i a^i\,{\rm tr}(\mathcal{F}\wedge\mathcal{F}) + \int \frac{1}{8\pi} C_i s^i\,{\rm tr}(\mathcal{F}\wedge *\mathcal{F}), \tag{4.1.9}$$

and originate from a supersymmetric F-term of the form $\int {\rm d}^4x\,{\rm d}^2\theta\,f(t)\,\mathcal{W}^\alpha\mathcal{W}_\alpha + {\rm c.c.}$, where $f(t) = f(0) + C_i t^i$ with $f(0)$ an arbitrary constant and $C_i \in \mathbb{Z}$, modulo exponentially suppressed instanton-like corrections. We will study only solutions with trivial gauge configurations, and so the above couplings are not of primary interest here. See section 2 of [65] for a more detailed discussion of the topological terms in the present setting.

affected by radiative corrections. Supersymmetry is not enough to provide robust information about them. Fortunately, the latter terms are also basis-dependent, in the sense that re-defining the metric one can always trade them for operators involving derivatives of a^i and s^i. Hence, without loss of generality, we can choose a field basis in which the non-derivative saxions couplings to curvature squared operators reduce to the Gauss-Bonnet term with the coefficient given in eq. (4.1.10).

We have been a bit cavalier in writing (4.1.10), though, because strictly speaking the manifestly supersymmetric formulation adopted there (see [50]) is not automatically in the Einstein frame, which instead we used in (4.1.1). In order to pass to the Einstein frame a Weyl rescaling $\Phi \to e^{\beta K}\Phi$ of all fields, with β some number, is necessary. Such transformation is anomalous and brings a non-manifestly supersymmetric correction to the coefficient of the Gauss-Bonnet term of the parametric form $\sim N \ln e^{\beta K} = \beta N K$. [2] Including the 1-loop renormalization discussed earlier, of order $F(0) \sim N \ln \Lambda$, we conclude that in our perturbative regime, and working in the Einstein frame, the potential topological couplings of t^i to gravity are described by

$$S^{(\mathrm{M})}_{\text{top}-\text{gravity}} \equiv \int \mathrm{d}^4x\,\sqrt{|g|}\,[\delta_{\text{tot}}(a)\,E_{\mathrm{P}} + \gamma_{\text{tot}}(s)\,E_{\mathrm{GB}}] + \mathrm{BT} \tag{4.1.10}$$

with

$$\gamma_{\text{tot}}(s) = c_0 + c_1 N K(s) + c_2 N \ln \Lambda + \frac{\pi}{6}\tilde{C}_i s^i \tag{4.1.11}$$

$$\delta_{\text{tot}}(a) = c_0' + c_2' N \ln \Lambda + \frac{1}{384\pi}\tilde{C}_i a^i. \tag{4.1.12}$$

Here $c_{1,2}, c_2'$ are order one coefficients calculable within the EFT. Despite their $\propto N$ nature, we will see in Section 4.2.2 that the terms proportional to $c_{1,2}$ are subleading compared to

$$\gamma(s) \equiv \frac{\pi}{6}\tilde{C}_i s^i \tag{4.1.13}$$

in any tractable framework. The constants c_0, c_0', however, are UV sensitive and incalculable within our $\mathcal{N} = 1$ description. Yet, there is no reason for them to be sensitive to the number of UV degrees of freedom, especially in view of the fact that the UV completion of our EFT features an extended supersymmetry that is expected to protect $\gamma_{\text{tot}}, \delta_{\text{tot}}$ from radiative corrections. In the following we will thus assume that $c_0 = c_0' = 0$, consistently with the explicit $\mathcal{N} = 2$ models discussed in [174].

In summary, the Pontryagin and Gauss-Bonnet combinations are singled out from the infinite

[2] Similarly, the re-definition of the metric necessary to remove the saxion couplings to $R_{ab}R^{ab}$ and R^2 may induce a Weyl anomaly, but that does not carry an $\sim N$ enhancement and is hence parametrically smaller.

set of higher-derivative interactions for two important reasons. The first one is their quasi-topological nature (note that the boundary Hawking-Gibbons terms are essential for this property to hold). After stabilization of the saxions, for example, the GB operator in (4.1.10) becomes a purely topological term that does not alter the axions' equations of motion but nevertheless contributes to the on-shell action of topologically non-trivial space-times. In particular it plays an important role in wormhole physics, as we will see. At the perturbative level, the topological nature of these terms is connected to the absence of ghosts, which is why string theory effective actions of any dimension seem to favor, say, the Gauss-Bonnet combination over other higher curvature terms [59]. The second reason that make the Pontryagin and Gauss-Bonnet terms special is that their coefficients are significantly constrained by supersymmetry and more generally by the string theory models they emerge from, as we have just shown. This information will provide very useful in the sections that follow.

4.2 Regime of validity of the EFT

4.2.1 Perturbative domain and saxionic cones

The EFT described in our discussion assumes the existence of a perturbative saxion domain in which the axionic shift symmetries are broken only by exponentially suppressed non-perturbative corrections. We already introduced the topic of the perturbative saxionic domain in Chapter 2, anyway in this section we would like to better refine such definition by identifying it with what we call *α-saxionic convex hull* $\hat{\Delta}_\alpha$. The reasons for that will be manifest in the successive discussion about consistency bounds on the coefficient of the Gauss-Bonnet term.

We begin making a few basic considerations about power-counting. As usual, the 2-derivative action (4.1.1) remains perturbative as long as the UV cutoff Λ of the EFT is not too large

$$\Lambda \leq \Lambda_{\text{strong,EFT}} \equiv \frac{2\pi M_{\text{P}}}{\sqrt{N}} \,. \tag{4.2.1}$$

The reason is that the effective coupling of t^i to gravity, namely $g^2_{\text{EFT}} \equiv p^2/M^2_{\text{P}}$ with $p^2 < \Lambda^2$ the typical momentum transfer, must satisfy $g^2_{\text{EFT}} N/(2\pi)^2 \ll 1$ in order to retain perturbative control. This consistency requirement translates into (4.2.1). We may call $\Lambda_{\text{strong,EFT}}$ the "strong EFT scale", as it only depends on the degrees of freedom appearing in the EFT, as in [175]. This is to be distinguished from the "strong-coupling scale" that contains information about the tower of states beyond the 4-dimensional effective field theory, which will be discussed in Section 4.2.3. The constraint (4.2.1) also ensures that the radiative corrections to the non-linear sigma model describing the leading saxions dynamics be small.

The quantity $p^2 N/(2\pi M_{\text{P}})^2$ is not the only small parameter characterizing the domain of validity of our EFT, though. To appreciate this important point it is sufficient to make a simple consideration based on dimensional analysis. If we momentarily restore the powers of $\hbar$ while keeping the speed of light equal to one, and insist that $\tilde{C}_i$ in (4.1.10) (see also C_i in (4.1.9)) be truly dimensionless integers, one infers that a^i has units $[a^i] = \hbar$ and similarly $[s^i] = \hbar$. Because quantum corrections come with powers of $\hbar$, one realizes that $2\pi\hbar/s^i$ must represent some sort of "hidden" dimensionless loop counting parameter. [3] Consistently, besides (4.2.1), the other necessary condition for our EFT to make sense is given by a relation of the form

$$1/(2\pi s^i) \ll 1 \qquad (\text{or} \;\; \ell_i/(2\pi) \ll 1). \tag{4.2.2}$$

That a relation like (4.2.2) is a necessary requirement in any perturbative low energy description of string theory is hardly news. For example, eq. (4.1.9) identifies the gauge coupling g_{gauge} with $1/g^2_{\text{gauge}} = C_i s^i/4\pi$ and so, up to numerical factors, the perturbative regime for the gauge theory ($g^2_{\text{gauge}} \ll (2\pi)^2$) is set by $2\pi C_i s^i \gg 1$. What the effective field theorist cannot know, however, is that the large saxions limit is in fact instrumental in reproducing the EFT from a string theory model, where (4.2.2) is often disguised as a large volume expansion. In particular, of crucial importance for the present discussion is that the large-saxions regime is essential to derive eq. (4.1.1) (see Section 4.3). Sub-leading corrections to that expression consistently appear as terms in $P(s)$ with a smaller order of homogeneity. Said differently, the EFT per se does not appear to have any pathology when violating (4.2.2); it is the UV completion from which it is derived that becomes intractable in that limit, and the very notion of EFT comes into question.

In view of our dimensional analysis argument it should not come as a surprise that the non-perturbative effects that break the axionic shift symmetries also turn out to be suppressed in the semi-classical regime by a requirement of the form (4.2.2). Non-perturbative effects may be due to fundamental instantons beyond the EFT or by physics within the EFT, e.g. wormholes. The latter will be analyzed in the following, here our main concern is understanding when the former can be considered small. The contribution of point-like fundamental instantons are ubiquitous in string theory compactifications, in which they are associated to Euclidean branes wrapping internal cycles – see for instance [176] for a review. Their effects show up in the four-dimensional EFT defined at its highest possible UV cutoff as shift-symmetry breaking local operators. Imposing that these are sufficiently small is what defines our perturbative regime. Among all fundamental instantons, the BPS ones — preserving $\frac{1}{2}$ of the bulk supersymmetry — are expected to be the most relevant. [4] These carry a set of quantized axionic charges $q_i \in \mathbb{Z}$

[3]The factor of 2π arises because of our normalization (2.1.1). Also, the analogous combination $2\pi\hbar/a^i$ simply cannot appear because of the approximate shift symmetry.

[4]Experience with supersymmetric instantons suggests that the action S_{inst} of a possible non-BPS fundamental instanton carrying charges q_i obeys a BPS bound $\text{Re}S_{\text{inst}} > 2\pi\langle \mathbf{q}, \boldsymbol{s}\rangle$. On the other hand, the axion form of the weak gravity conjecture [69] suggests the possible existence of non-BPS instantons violating such bound – see for instance [177, 178] and appendix B of [45] for for related discussions in string theory contexts. We nevertheless expect the violation of the BPS bound not to be large and then not to substantially affect our characterization of the perturbative regime in terms of BPS instantons.

and their contribution to the path-integral is forced by (2.1.1) to be of the form

$$e^{-2\pi\langle \mathbf{q},\boldsymbol{s}\rangle} \tag{4.2.3}$$

where we have introduced the definition

$$\langle \mathbf{q},\boldsymbol{s}\rangle \equiv q_i s^i. \tag{4.2.4}$$

In our notation $\langle\, .\, ,\, .\,\rangle$ is the canonical pairing between the elements of dual vector spaces $V_\mathbb{R}$ and $V^*_\mathbb{R}$, and corresponding dual lattices $V_\mathbb{Z}\subset V_\mathbb{R}$ and $V^*_\mathbb{Z}\subset V^*_\mathbb{R}$. One can introduce an integral basis $\{\mathbf{v}_i\}_{i=1}^N$ of generators of $V_\mathbb{Z}$ and the dual basis $\{\mathbf{w}^i\}_{i=1}^N$ of generators $V^*_\mathbb{Z}$, such that $\langle \mathbf{w}^i,\mathbf{v}_j\rangle = \delta^i_j$. The saxions s^i and the charges q_i are the components of the vectors $\boldsymbol{s} = s^i\mathbf{v}_i\in V_\mathbb{R}$ and $\mathbf{q} = q_i\mathbf{w}^i\in V^*_\mathbb{R}$ respectively. The set of all BPS instanton charges is denoted by

$$\mathcal{C}_\text{I} = \{\text{set of BPS instanton charges } \mathbf{q}\}\subset V^*_\mathbb{Z}\,. \tag{4.2.5}$$

Given two BPS instantons of charge vectors $\mathbf{q}_1$ and $\mathbf{q}_2$, being mutually BPS, they can be superimposed to form a BPS instanton of charge vector $\mathbf{q}_1+\mathbf{q}_2$. Hence $\mathcal{C}_\text{I}$ can be regarded as discrete convex cone, generated by a set of 'elementary' BPS instanton charges.[5]

The combination $2\pi\langle \mathbf{q},\boldsymbol{s}\rangle$ appearing in (4.2.3) represents the real part of the BPS instanton Euclidean action, and must be positive. Hence the saxions necessarily take values in the *saxionic cone*:[6]

$$\Delta \equiv \{\boldsymbol{s}\in V_\mathbb{R} | \langle \mathbf{q},\boldsymbol{s}\rangle \geq 0,\ \forall \mathbf{q}\in\mathcal{C}_\text{I}\}\,. \tag{4.2.6}$$

This is a convex cone, which we will assume to be polyhedral, that is, to be generated by a finite number of vectors – see [64] for a more detailed discussion on this assumption. The prototypical example of such saxionic cone is provided by the Kähler cone in heterotic or type IIA string compactifications on Calabi-Yau spaces, where $\mathcal{C}_\text{I}$ represents the cone of effective curves which can be wrapped by world-sheet instantons.

The realization that saxions must necessarily belong to Δ is bringing us closer to a rigorous definition of the domain of validity of our EFT. Unfortunately, $\boldsymbol{s}\in\Delta$ is not sufficient to suppress instantons nor to ensure the perturbativity requirement suggested in eq. (4.2.2) holds. Actually, it is not even clear how to assign a rigorous meaning to (4.2.2) since, as written, it represents a basis-dependent statement. To make some progress we identify a preferred saxion basis in which our perturbative domain can be unambiguously defined. A natural choice is provided by the magnetic axionic charges $e^i\in\mathbb{Z}$ of the strings, as they specify an element $\mathbf{e} = e^i\mathbf{v}_i$ of $V_\mathbb{Z}$.

[5]For each BPS instanton of charge vector $\mathbf{q}$ there exists an *anti*-instanton of charge vector $-\mathbf{q}$ preserving the opposite $\frac{1}{2}$ supersymmetry and contributing to the effective action by the anti-holomorphic combination $\overline{(e^{2\pi \mathrm{i} q_i t^i}})$, and then by the same real exponential factor (4.2.3).

[6]The definition of saxionic cone of [51, 64] slightly differs from (4.2.6), in that it does not include the boundary faces at which $\langle \mathbf{q},\boldsymbol{s}\rangle = 0$ for some $\mathbf{q}$, and then some instanton action degenerates. We include these faces since we will anyway restrict ourselves to more interior regions – see subsection 4.2.1.

As it was presented in Chapter 2, EFT strings are those axionic strings associated to charges $\mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}$, where the latter domain is obtained restricting $V_{\mathbb{Z}}$ to the saxionic cone [51, 52, 64]

$$\mathcal{C}_{\rm S}^{\rm EFT} = V_{\mathbb{Z}} \cap \Delta \,. \tag{4.2.7}$$

According to this definition, one may regard the saxionic cone Δ as being generated by the EFT string charges. We thus propose to define the perturbative saxions domain in the basis defined by the charges e^i of the EFT strings.

More precisely, consider the set of all the elementary generators[7] $\{\mathbf{e}_A\}_{A\in\mathcal{J}}$ of $\mathcal{C}_{\rm S}^{\rm EFT}$, where $\mathcal{J}$ denotes the corresponding set of indices. Take any subset $\mathcal{J}_\sigma \subset \mathcal{J}$ of N elements, such that the corresponding elementary charges $\{\mathbf{e}_A\}_{A\in\mathcal{J}_\sigma}$ are linearly independent. Each of these subsets is associated to a regular simplicial cone

$$\sigma = \{\sum_{A\in\mathcal{J}_\sigma} \lambda^A \mathbf{e}_A | \lambda^A \geq 0\}. \tag{4.2.8}$$

For each of these cones we construct a corresponding "α-stretched" cone

$$\tilde{\sigma}_\alpha = \{\sum_{A\in\mathcal{J}_\sigma} \lambda^A \mathbf{e}_A | \lambda^A \geq \frac{1}{\alpha}\} \tag{4.2.9}$$

with $0 < \alpha \ll 1$ some arbitrary number which plays a role analogous to $\alpha = g^2_{\rm gauge}/4\pi$ in gauge theories. The *α-saxionic convex hull* $\hat{\Delta}_\alpha$ anticipated at the beginning of this subsection is defined as the convex hull of all the stretched sub-cones $\tilde{\sigma}_\alpha$, that is

$$\hat{\Delta}_\alpha = \Big\{\boldsymbol{s} = \sum_\sigma \lambda^\sigma \boldsymbol{s}_\sigma \in \Delta \Big| \; \lambda^\sigma \geq 0,\; \sum_\sigma \lambda^\sigma = 1,\; \boldsymbol{s}_\sigma \in \tilde{\sigma}_\alpha \Big\}. \tag{4.2.10}$$

The perturbative domain for the saxions is now rigorously identified by the requirement $\boldsymbol{s} \in \hat{\Delta}_\alpha$. This definition simultaneously formalizes the perturbativity requirement (4.2.2) as well as guarantees the suppression of non-perturbative corrections. More explicitly, in the domain $\hat{\Delta}_\alpha$ the saxion components measured in units of the EFT strings charges satisfy $s^A \geq 1/\alpha$. The condition (4.2.2) is thus reduced to an upper bound on α, and perturbative corrections to our EFT (say to the relation (4.1.3)) are suppressed by powers of α. Furthermore, for any $\boldsymbol{s} \in \hat{\Delta}_\alpha$ the BPS action must be larger than $2\pi\alpha$. Indeed, by definition $\hat{\Delta}_\alpha \subset \Delta$ and so the positivity of the BPS instantons action is obviously ensured; but because our $\tilde{\sigma}_\alpha$ are subsets of the saxionic analogue of the stretched Kähler cones introduced in [44], namely the *α-stretched saxionic cone*[8]

$$\tilde{\sigma}_\alpha \subset \tilde{\Delta}_\alpha \equiv \{\boldsymbol{s} \in \Delta | \langle \mathbf{q}, \boldsymbol{s}\rangle \geq \frac{1}{\alpha},\; \forall \mathbf{q} \in \mathcal{C}_{\rm I}\} \tag{4.2.11}$$

[7] A charge vector $\mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}$ is 'elementary' if it cannot be written as the sum of other elements of $\mathcal{C}_{\rm S}^{\rm EFT}$.

[8] We are adapting the terminology of [44] which may be misleading, since $\tilde{\Delta}_\alpha$ is generically not a cone, but rather a convex polyhedron.

by convexity the same inclusion extends to the entire saxionic convex hull, so that:

$$\hat{\Delta}_\alpha \subset \tilde{\Delta}_\alpha \subset \Delta\,. \tag{4.2.12}$$

Hence the condition $\boldsymbol{s} \in \hat{\Delta}_\alpha$ is stronger, though qualitatively similar, to the condition $\boldsymbol{s} \in \tilde{\Delta}_\alpha$. (Clearly $\hat{\Delta}_\alpha = \tilde{\Delta}_\alpha$ if Δ is a simplicial cone.) As a result, the non-perturbative corrections to our EFT are at least suppressed by $e^{-2\pi/\alpha}$ when $\boldsymbol{s} \in \hat{\Delta}_\alpha$.[9]

The perturbative saxionic regions $\tilde{\Delta}_\alpha, \hat{\Delta}_\alpha \subset \Delta$ are associated to corresponding dual saxionic regions $\tilde{\mathcal{P}}_\alpha, \hat{\mathcal{P}}_\alpha \subset \mathcal{P}$ through the map (2.2.4). In particular, we can consider the saxionic domain $\mathcal{P}$ dual to Δ:

$$\mathcal{P} = \left\{ \boldsymbol{\ell} = \ell_i \mathbf{w}^i \in V^*_{\mathbb{R}} \middle| \ell_i = -\frac{1}{2}\frac{\partial K}{\partial s^i}\bigg|_{\boldsymbol{s}\in\Delta} \right\}. \tag{4.2.13}$$

While the general structure of $\mathcal{P}$ can be a priori complicated, if K takes the form (4.1.3) then $\mathcal{P}$ becomes conical. Indeed, if $\boldsymbol{\ell} \in \mathcal{P}$ is the image of $\boldsymbol{s} \in \Delta$, $\lambda\boldsymbol{\ell}$ is the image of $\lambda^{-1}\boldsymbol{s}$ then for any $\lambda > 0$. Since also $\lambda^{-1}\boldsymbol{s}$ belongs to Δ, then $\lambda\boldsymbol{\ell}$ belongs to $\mathcal{P}$.[10] In the case of Kähler potential of the form (4.1.3) the regions dual to $\tilde{\Delta}_\alpha, \hat{\Delta}_\alpha$ are neighborhoods of the dual saxion origin $\boldsymbol{\ell} = 0$.

To summarize, by assuming $\boldsymbol{s} \in \hat{\Delta}_\alpha$ (or $\boldsymbol{\ell} \in \hat{\mathcal{P}}_\alpha$) we are certain that our EFT (4.1.1) with (4.1.3) provides a reliable low-energy description of a large class of string theory models up to controllable powers of $\alpha \ll 1$ and $e^{-2\pi/\alpha}$. In the following we will therefore assume that $\boldsymbol{s} \in \hat{\Delta}_\alpha$ and for concreteness have in mind $\alpha \sim 0.1$ as benchmark value. Indeed, $\alpha \sim 1$ might already be enough to sufficiently suppress non-perturbative effects. However, such values of α do not necessarily ensure the reliability of the leading order expression (4.1.3). Concretely, in the case of the Kähler cone of heterotic compactifications, setting $\alpha \sim 1$ allows for string-size internal cycles, which cast doubts on the geometric formula corresponding to (4.1.3) – see Section 4.3 for more details. From these consideration $\alpha \sim 0.1$ seems a more appropriate choice.

4.2.2 Quantum gravity bounds

In Chapter 3 it was shown how quantum consistency in the presence of EFT strings imposes strong constraints on the structure of the bulk theory. These constraints crucially involve the constants $\tilde{C}_i$ appearing in (4.1.13). In particular $\tilde{C}_i$ must satisfy the quantization condition $\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle \in \mathbb{Z}$, for any string charge vector $\mathbf{e} \in \mathbb{Z}^N$. More importantly for us, in [65] it was argued that $\langle \tilde{\mathbf{C}}, \boldsymbol{s} \rangle$ enter some positivity bounds which, in their weakest form, imply that

$$\langle \tilde{\mathbf{C}}, \mathbf{e} \rangle \geq 0 \qquad\qquad \forall \mathbf{e} \in \mathcal{C}^{\text{EFT}}_{\text{S}} \tag{4.2.14}$$

[9] Note that, given a saxionic cone Δ, its boundary can be regarded as the union of conical faces $\Delta' \subset \partial\Delta$ of various codimensions. These may be associated with corresponding perturbative domains $\tilde{\Delta}'_\alpha$ or $\hat{\Delta}'_\alpha$ (which are *not* subsets of $\tilde{\Delta}_\alpha$ or $\hat{\Delta}_\alpha$).

[10] On the other hand, $\mathcal{P}$ is not necessarily convex.

Recalling (4.1.13) and the fact that the EFT string charges generate the saxionic cone, (4.2.14) has as a consequence that $\gamma(s) > 0$ for any $\boldsymbol{s} \in \Delta$.

In order to get a stronger lower bound for $\gamma(s)$ we observe that $\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle$ is proportional to the two-dimensional gravitational anomaly of the EFT string of charge vector $\mathbf{e}$. Since such strings break half of the bulk supersymmetry and support a chiral $(0,2)$ world-sheet, they generically have a chiral spectrum with non-vanishing gravitational anomaly. This means that (4.2.14) generically translates into the stricter bound

$$\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle \in \mathbb{Z}_{\geq 1} \qquad \forall \text{ generic } \mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}\,. \tag{4.2.15}$$

Note that in many models (4.2.15) can be strengthened to $\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle \in 3\mathbb{Z}_{\geq 1}$. This stronger bound holds if the world-sheet normal bundle $U(1)_{\rm N}$ symmetry is classically preserved (though generically anomalous at the quantum world-sheet level). This is a conceivable expectation, which is indeed realized in large classes of string theory models, such as the F-theory/type IIB ones of subsection 4.3.1. On the other hand, in [65] it was pointed out that in addition to the standard quantum $U(1)_{\rm N}$ anomaly there could be classical Green-Schwarz-like terms on the world-sheet, which signal the existence of an intermediate microscopic description in terms of a five dimensional $\mathcal{N} = 1$ supergravity (in presence of possible supersymmetry breaking defects). The five-dimensional arguments of [39] hence lead to (4.2.14), and then also (4.2.15). For instance, this weaker bound can hold in the $E_8 \times E_8$ heterotic models of subsection 4.3.2, when $D_{\mathbf{e}}$ has non-vanishing triple intersection number.

In our models there are $N \gg 1$ (s)axions and an even larger number of elementary EFT string charge vectors $\{\mathbf{e}_A\}_{A\in\mathcal{J}}$ (since these generate Δ). We can then assume (4.2.15) to be satisfied for *all* $\mathbf{e}_A$, since for $N \gg 1$ any non-generic violation of this assumption would affect our conclusions by negligible corrections. Hence we will assume that

$$\langle \tilde{\mathbf{C}}, \mathbf{e}_A\rangle \in \mathbb{Z}_{\geq 1} \quad , \tag{4.2.16}$$

for any elementary $\mathbf{e}_A \in \mathcal{C}_{\rm S}^{\rm EFT}$. Take now any regular simplicial sub-cone (4.2.8) and an element $\boldsymbol{s}_\sigma$ of the corresponding α-stretched cone: $\boldsymbol{s}_\sigma \in \tilde{\sigma}_\alpha$. We can write $\boldsymbol{s}_\sigma = \frac{1}{\alpha}\sum_{A\in\mathcal{J}_\sigma} \mathbf{e}_A + \boldsymbol{v}_\sigma$, where the first contribution represents the tip of $\boldsymbol{s}_\sigma$ and $\boldsymbol{v}_\sigma$ is an element of σ. As a consequence, the lower bounds (4.2.16) imply that

$$\langle \tilde{\mathbf{C}}, \boldsymbol{s}_\sigma\rangle = \frac{1}{\alpha}\sum_{A\in\mathcal{J}_\sigma} \langle \tilde{\mathbf{C}}, \mathbf{e}_A\rangle + \langle \tilde{\mathbf{C}}, \boldsymbol{v}_\sigma\rangle \geq \frac{1}{\alpha}\sum_{A\in\mathcal{J}_\sigma} \langle \tilde{\mathbf{C}}, \mathbf{e}_A\rangle \geq \frac{N}{\alpha}\,. \tag{4.2.17}$$

Consider next a more general point $\boldsymbol{s}$ of the saxionic convex hull (4.2.10). By definition, we can write it as $\boldsymbol{s} = \sum \lambda^\sigma \boldsymbol{s}_\sigma$, with $\sum_\sigma \lambda^\sigma = 1$ and $\lambda^\sigma \geq 0$. According to (4.2.17) we thus have $\langle \tilde{\mathbf{C}}, \boldsymbol{s}\rangle = \sum_\sigma \lambda^\sigma \langle \tilde{\mathbf{C}}, \boldsymbol{s}_\sigma\rangle \geq \frac{N}{\alpha}\sum_\sigma \lambda^\sigma = \frac{N}{\alpha}$. Hence we conclude that for $\boldsymbol{s} \in \hat{\Delta}_\alpha$ the part of the coefficient of the Gauss-Bonnet term defined in eq. (4.1.13) satisfies the lower bound

$$\gamma(s)|_{\hat{\Delta}_\alpha} \geq \frac{N\pi}{6\alpha} \,. \tag{4.2.18}$$

This bound may receive $1/N$ corrections due to non-generic violations of (4.2.16), but in the large N regime these can be safely neglected. We also observe that, as stressed above, in many models we could alternatively adopt $\langle \tilde{\mathbf{C}}, \mathbf{e}_A \rangle \in 3\mathbb{Z}_{\geq 1}$ instead of (4.2.16), obtaining a slightly stronger lower bound $\gamma(s)|_{\hat{\Delta}_\alpha} \geq \frac{N\pi}{2\alpha}$.

In eq. (4.1.11) we collected all contributions to the coefficient $\gamma_{\rm tot}$ of the Gauss-Bonnet operator. Within the perturbative regime $\alpha \ll 1$ the radiative effects controlled by $c_{1,2}$ are parametrically smaller than the one in (4.2.18). We can thus conclude with confidence that in the class of theories under consideration the calculable part of the Gauss-Bonnet coefficient $\gamma_{\rm tot} \approx \gamma(s)$ satisfies a universal lower bound of order $\frac{N\pi}{\alpha}$. Note that while this result was derived by restricting $\boldsymbol{s}$ to the α-stretched saxionic convex hull, we expect it to qualitatively hold (up to a possible overall constant) also if we consider the stretched saxionic cone. We will provide some numerical evidence of this claim in Section 4.3.2, where we will show that (4.2.18) is actually very conservative in a set of concrete UV completions.

4.2.3 The UV cutoff

Any EFT is associated with a cutoff energy Λ, and an important question regards the identification and interpretation of the possible UV energy scales at which the EFT description breaks down. In our quantum gravity context, we can identify at least two important universal energy scales: the *tower* scale m_* and the *species* scale $\Lambda_{\rm QG}$ [47, 179]. The tower scale m_* represents the mass of the lightest particle appearing in the towers of massive single particle states that populate the UV completion of the EFT. When $\Lambda \geq m_*$ the four-dimensional EFT does not make sense anymore, but may admit another weakly coupled higher dimensional EFT description. On the other hand, at $\Lambda \geq \Lambda_{\rm QG}$ the (semi-)classical geometrical description of the gravitational interactions breaks down.

In the perturbative four-dimensional framework outlined in the previous sections the zero-coupling limit $\alpha \to 0$ corresponds to an infinite distance limit in field space. The presence of UV towers of massive states is then predicted by the Swampland Distance Conjecture (SDC) [22]. Even if in general m_* can be precisely determined only by knowing the EFT UV completion, in [52, 64] it was pointed out that in a large class of string theory models there is a direct relation between m_* and the tension associated with EFT strings. Namely, each EFT string charge $\mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}$ identifies an infinite distance saxionic flow $\boldsymbol{s} = \boldsymbol{s}_0 + \mathbf{e}\sigma$, with $\sigma \to \infty$, and is associated with an integral *scaling weight* $w_{\mathbf{e}} \in \{1, 2, 3\}$. Along this EFT string flow $\mathcal{T}_{\mathbf{e}} \sim M_{\rm P}^2/\sigma$ and the scaling weight relates this asymptotically vanishing tension to m_* as follows:

$$m_*^2 \sim \left(\frac{\mathcal{T}_{\mathbf{e}}}{M_{\rm P}^2}\right)^{w_{\mathbf{e}}} M_{\rm P}^2\,. \tag{4.2.19}$$

The Integral Weight Conjecture (IWC) [52, 64] proposes that this relation holds in any quantum gravity model of the type considered in the present thesis.

The combination of the IWC with the Emergent String Conjecture (ESC) [70] gives further important information on the UV nature of the EFT strings and of the corresponding infinite distance limits. First of all, any $w_{\mathbf{e}} = 1$ EFT string must be dual to a critical heterotic string (or a type II string in higher supersymmetric settings). This in turn implies that any $w_{\mathbf{e}} = 1$ EFT string flow[11] is dual to a ten-dimensional weak string coupling limit $g_{\rm s} \to 0$, along which the tower scale m_* can be identified with the mass of the first excited string mode and (4.2.19) can be actually promoted to the identity $m_*^2|_{w_{\mathbf{e}}=1} = 2\pi\mathcal{T}_{\mathbf{e}}$. On the other hand, EFT strings with $w_{\mathbf{e}} \geq 2$ *cannot* be identified with critical strings, and the corresponding infinite distance limits must correspond to decompactification limits along which m_* corresponds to a Kaluza-Klein (KK) mass scale.

Regarding the species scale, perturbative as well as non-perturbative arguments [47, 179] lead to the formula $\Lambda_{\rm QG}^{(d)} \simeq M_{\rm P}^{(d)}/N_{\rm tot}^{\frac{1}{d-2}}$ valid in a generic d-dimensional theory, where $N_{\rm tot}$ represents the total number of species with mass lower than $\Lambda_{\rm QG}^{(d)}$. A priory this formula holds only at the parametric level and neglects possible numerical factors. Hence, in our four-dimensional axiverse setting we can roughly split $N_{\rm tot} = N + N_{\rm UV}$ and write

$$\Lambda_{\rm QG}^2 \simeq \frac{M_{\rm P}^2}{N_{\rm tot}} = \frac{M_{\rm P}^2}{N + N_{\rm UV}}\,, \tag{4.2.20}$$

where N counts the number of axionic species and $N_{\rm UV}$ the total number of UV species of mass smaller than $\Lambda_{\rm QG}$. We clearly have $\Lambda_{\rm QG}^2 \leq \Lambda_{\rm sp}^{\rm EFT}$, where $\Lambda_{\rm sp}^{\rm EFT}$ was introduced in (4.2.1).

Assuming the ESC, the species scale should be determined by KK and/or string modes. In the cases in which the dominant contribution to $N_{\rm tot}$ comes a KK tower, then $\Lambda_{\rm QG}^2$ can be identified with the higher dimensional Planck scale. On the other hand, in an emergent string theory limit $\Lambda_{\rm QG}^2$ should dominated by the excitation modes critical string itself, which should hence correspond to some EFT charge $\mathbf{e}_*$ with $w_{\mathbf{e}_*} = 1$. Note that at energy-squared larger than $\mathcal{T}_* \equiv M_{\rm P}^2\langle \boldsymbol{\ell}, \mathbf{e}_*\rangle$, the EFT description breaks-down and should be replaced by a superstring perturbation theory. The identification of what is the relevant species scale is then less obvious. By applying black-hole arguments [180, 181] one gets the estimate $\Lambda_{\rm QG}^2 \simeq \mathcal{T}_*$ (see also [182]), while from the QFT-based formula (4.2.20) one gets logarithmic corrections thereof as in [183, 184], – see also related discussions in [185, 186]. For most of our purposes we could remain agnostic about the precise form of the stringy species scale. So, for concreteness we will mostly

[11] If the $w_{\mathbf{e}} = 1$ string is dual to an $E_8 \times E_8$ string, then this limit corresponds to $s^0 \sim \sigma \to \infty$, with fixed s^a and s^0 as in (4.3.28).

adopt the first one, but we will also discuss how to adapt our discussion in order to make it compatible with the second choice.

As the tower scale, also the species scale $\Lambda_{\rm QG}$ is generically expected to have a non-trivial field dependence, which has been intensively studied in the last year [174, 187–192]. In particular, in [174] it is proposed that coefficients of higher-curvature terms of the form (curvature)^n protected by supersymmetry should provide upper bounds on $\Lambda_{\rm QG}$ – see also [186, 189, 190, 193] for further discussions. Applying this proposal to our GB term (4.4.2), one then gets the following upper bound on the species scale set by the GB term (4.4.2):

$$\Lambda^2_{\rm QG} \lesssim \frac{2\pi M^2_{\rm P}}{\gamma(s)}\,, \tag{4.2.21}$$

where we have fixed the somewhat arbitrary 2π-factor in order to better realize some of the following properties. This also implies the bound $2\pi N_{\rm tot} > \gamma(s)$ from the combination of (4.2.20) and (4.2.21).

As a simple check for this bound, we can combine (4.2.21) with (4.2.18), and just require that $\alpha \lesssim 0.1$, to get an upper bound on $\Lambda_{\rm QG}$:

$$\Lambda^2_{\rm QG} \lesssim \frac{12\alpha M^2_{\rm P}}{N} \lesssim \left(\Lambda^{\rm EFT}_{\rm sp}\right)^2 \,. \tag{4.2.22}$$

The upper bound (4.2.22) is fully consistent with the identification (4.2.20) of $\Lambda_{\rm QG}$. The bound (4.2.22) is consistent with the expectation that $\Lambda_{\rm QG}$ should be smaller than $\Lambda^{\rm EFT}_{\rm sp}$, and actually much smaller if $N_{\rm UV} \gg 1$.

The bound (4.2.21) provides useful information which can be extracted from the EFT. On the other hand, as emphasized in [189], nothing prevents the right-hand side of (4.2.21) to be much larger than $\Lambda_{\rm QG}$. In such cases, other protected higher-curvature couplings would be needed to get a better estimate of $\Lambda_{\rm QG}$. It can then be useful to have alternative/complementary constraints on $\Lambda_{\rm QG}$. Fortunately, in our framework additional information on $\Lambda_{\rm QG}$ can be identified by recalling the implications of the IWC and ESC discussed above.

Take the set of EFT string charges (4.2.7). As emphasized above, if an EFT charge $\mathbf{e}$ has scaling weight $w_{\mathbf{e}} \geq 2$ then the corresponding string is not a fundamental critical superstring. In other words, one should not be allowed to quantize it. By investigating EFT strings in F-theory models, the authors of [71] observed that this is indeed the case because for such strings the EFT string tension $\mathcal{T}_{\mathbf{e}}$ is larger than $\Lambda^2_{\rm QG}$. We promote this observation to a general principle, requiring that $\Lambda^2_{\rm QG} \leq \mathcal{T}_{\mathbf{e}}$, for any EFT string charge with $w_{\mathbf{e}} \geq 2$. Furthermore, according to the above discussion in an emergent string limit the species scale should be the stringy one, $\Lambda^2_{\rm QG} \simeq \mathcal{T}_*$, while away from it we expect $\Lambda^2_{\rm QG} \leq \mathcal{T}_{\mathbf{e}}$ for any charge with scaling weight $w_{\mathbf{e}} = 1$ as well. In conclusion, everywhere in our perturbative regime the species scale should satisfy the upper bound:

$$\Lambda^2_{\rm QG}(\ell) \le \Lambda^2_{\rm max}(\ell) \equiv \min\left\{\mathcal{T}_{\bf e}(\ell)\,|\,{\bf e}\in\mathcal{C}_{\rm S}^{\rm EFT}\right\}, \tag{4.2.23}$$

where $\mathcal{T}_{\bf e}(\ell)$ is as in (2.2.3) and we have emphasized the dependence of $\Lambda_{\rm QG}$ and $\Lambda^2_{\rm max}$ on the dual saxions ℓ_i. Note that in order to compute the upper bound (4.2.23) it is sufficient to restrict to the generators of $\mathcal{C}_{\rm S}^{\rm EFT}$, compute the corresponding tensions and identify the lowest one. Of course, the charge corresponding to the lowest tension generically changes as we move in the saxionic domain. Hence $\Lambda_{\rm max}(\ell)$ will be a continuous but possibly non-smooth function of the dual saxions. Furthermore, in an emergent string limit the bound should be saturated by $\Lambda^2_{\rm max} = \mathcal{T}_*$.

The bound (4.2.23) scales as (4.2.21) under an overall constant rescaling of the saxions. On the other hand, it is determined just by data characterizing the two-derivative EFT, and moreover with an unambiguous overall coefficient. Once one knows the Kähler potential and the saxionic cone, and then the dual saxions ℓ_i and the set of EFT string charges (4.2.7), at each point of the perturbative region one can in principle compute (4.2.23).

Let us finally discuss how possible logarithmic corrections to the species scale may require a modification of the bound (4.2.23). In order to have some control over this possibility, one should be able to isolate the elementary EFT string charge ${\bf e}_*$ with $w_{{\bf e}_*} = 1$ by using just EFT data. Luckily, this is possible by applying the following criterion [194]: $w_{\bf e} = 1$ if and only if the Kähler potential along the associated flows $\boldsymbol{s} = \boldsymbol{s}_0 + {\bf e}\,\sigma$ behaves asymptotically as $K \simeq -\log\sigma$ for $\sigma \to \infty$. Hence ${\bf e}_*$ can be identified with the smallest EFT string charge satisfying this condition. As we will review in subsection (4.3.2), the corresponding flow $\boldsymbol{s} = \boldsymbol{s}_0 + {\bf e}_*\sigma$, with $\sigma \to \infty$, corresponds to an emergent string limit, along which $g_{\rm s} = e^{\phi} \to 0$ and $\mathcal{T}_* \to 0$, while the other tensions $\mathcal{T}_{\bf e}|_{w_{\bf e}\ge 2}$ remain finite [64]. In this regime, at energies of order $\sqrt{\mathcal{T}_*}$ the dynamics should be described by the full perturbative string theory. Rather then starting from the QFT-motivated formula (4.2.20) as in [183, 184], one can identify the potential logarithmic correction from the following purely string theory argument – see also the recent discussion in [186].

As discussed in [195], at weak string coupling $g_{\rm s} \ll 1$ the high-energy and fixed-angle string scattering amplitudes [196, 197] enter a fully quantum phase at an energy-squared of order $\hat\Lambda^2_{\rm QG} = M_{\rm s}^2\log(1/g_{\rm s}^2)$, where $M_{\rm s}$ is the ten-dimensional string scale. At this scale, higher loop amplitudes dominate over the tree-level ones and can be Borel resummed. Observe now that, from the point of view of a four-dimensional observer, $M_{\rm s}^2$ can be identified with $\mathcal{T}_*$. Hence $\hat\Lambda_{\rm QG}$ is logarithmically larger than $\mathcal{T}_*$ for $g_{\rm s} \ll 1$ (for fixed string frame internal metric), and then violates (4.2.23). On the other hand, the discussion of the following subsection 4.3.2 shows that $\mathcal{T}_* \lesssim g_{\rm s}^2 M_{\rm P}^2$, which implies that

$$\hat\Lambda^2_{\rm QG} \lesssim \mathcal{T}_*\log\frac{M_{\rm P}^2}{\mathcal{T}_*} \qquad (\text{if } g_{\rm s} \ll 1)\,. \tag{4.2.24}$$

Since in the emergent string limit we must hence have $\mathcal{T}_* \ll \mathcal{T}_{\mathbf{e}}|_{w_{\mathbf{e}}\geq 2}$, one may modify (4.2.23) into the following upper bound

$$\hat{\Lambda}^2_{\rm QG}(\ell) \leq \hat{\Lambda}^2_{\max}(\ell) \equiv \min\left\{\mathcal{T}_{\mathbf{e}}(\ell)\,|\,\mathbf{e}\in\mathcal{C}^{\rm EFT}_{\rm S}|_{w_{\mathbf{e}}\geq 2}\right\}, \tag{4.2.25}$$

which should hold for either definition of stringy species one decides to adopt. The bound (4.2.25) is clearly weaker than (4.2.23) since by definition $\hat{\Lambda}^2_{\max} \geq \Lambda^2_{\max}$.

4.3 String theory models

In order to make the general discussion of section 4.1 less abstract, we now describe two large classes of string theory models, namely the heterotic and F-theory models in the large volume regime. These have the advantage that can be described quite explicitly in our general framework, and will allow us to provide a few concrete examples thereof to better illustrate and check our main points. As in [64, 65], our general claims also apply to other string theory models or perturbative regimes – e.g. type I, type IIA and M-theory – which are however either very similar/dual to the heterotic and F-theory cases, or admit a less explicit EFT description. Hence, for concreteness and clarity, in this section we focus on the F-theory and heterotic ones, while we will encounter again the IIA and M-theory models in section 4.5.1.

4.3.1 F-theory/type IIB models

An important large class of examples is provided by the F-theory compactifications – see e.g. [110, 198] for reviews. An F-theory model corresponds to a type IIB compactification on a Kähler space X in presence of 7-branes. The space X can be regarded as the base of an elliptically fibered Calabi-Yau four-fold, whose fiber's complex structure can be identified with the type IIB axio-dilaton. In particular, this requires the base X to have an effective anti-canonical divisor $\overline{K}_X$.[12] In the following we will for simplicity assume that the Calabi-Yau four-fold elliptically fibred over X has vanishing third Betti number, $b_3 = 0$, so to avoid technical complications associated with moduli of the M-theory gauge three-form.

These models admit a natural perturbative regime corresponding to the large volume limit. Let us pick a basis of divisors $D^a \in H_4(X,\mathbb{Z})$ of X and a dual basis of two-cycles $\Sigma_a \in H_2(X,\mathbb{Z})$, such that $D^a \cdot \Sigma_b = \delta^a_b$, $a = 1,\dots,b_2(X)$. The Kähler moduli v_a are obtained by expanding the (Einstein frame) Kähler form J of X in the Poincaré dual basis $[D^a] \in H^2(X,\mathbb{Z})$:

$$J = v_a[D^a] \qquad \Leftrightarrow \qquad v_a = \int_{\Sigma_a} J\,. \tag{4.3.1}$$

[12]We adopt the quite common usage of denoting holomorphic line bundles and corresponding divisors by the same symbol.

The corresponding saxions s^a are then defined as follows:

$$s^a = \frac{1}{2}\kappa^{abc} v_b v_c \,, \tag{4.3.2}$$

where we have introduced the triple intersection numbers $\kappa^{abc} = D^a \cdot D^b \cdot D^c$. Hence in this case $N = b_2(X)$.

As discussed in [64], one can identify the saxionic cone with the cone $\text{Mov}_1(X)$ generated by *movable* curves.[13] We can then write

$$\boldsymbol{s} = s^a \Sigma_a \in \Delta_{\text{K}} \simeq \text{Mov}_1(X)\,. \tag{4.3.3}$$

Note that the string charge vectors $\mathbf{e}$ can be identified with curves $\Sigma_{\mathbf{e}} = e^a \Sigma_a$ and, in particular, the EFT string charges correspond to movable curves

$$\mathcal{C}_{\text{S}}^{\text{EFT}} \simeq \text{Mov}_1(X)_{\mathbb{Z}}\,. \tag{4.3.4}$$

Physically, EFT strings are realized by D3-branes wrapping movable curves.

The constants $\tilde{C}_a$ appearing in the Gauss-Bonnet term admit a nice geometrical interpretation [65]:

$$\tilde{C}_a = 6\overline{K}_X \cdot \Sigma_a\,. \tag{4.3.5}$$

In particular, the pairing appearing in (4.2.14) corresponds to the intersection number

$$\langle \mathbf{C}, \mathbf{e} \rangle = 6\overline{K}_X \cdot \Sigma_{\mathbf{e}} \tag{4.3.6}$$

Recalling that $\overline{K}_X$ is an effective divisor, the bound is (4.2.14) always satisfied, since movable curves can be precisely characterized as those curves that have non-negative intersection with all effective divisors [115]. In order to test the stronger bound (4.2.18), which is expected to hold up to subleading corrections in $1/N \ll 1$, let us focus on the large class of models with toric X – for more details see [199]. Then the anti-canonical divisor is given by

$$\overline{K}_X = \sum_{I \in \text{toric div.}} \mathcal{D}_I \tag{4.3.7}$$

where the sum is over the set of all toric divisors $\mathcal{D}_I$, $I = 1, \ldots, N+3$. Each toric divisor is effective, and in fact generate the whole cone of effective divisors. So any movable curve $\Sigma_{\mathbf{e}}$ has strictly positive intersection number with at least one toric divisor D_I, and then

$$\overline{K}_X \cdot \Sigma_{\mathbf{e}} \geq 1\,. \tag{4.3.8}$$

[13] Movable curves and get their names from the fact that they can freely explore the entire internal space. Below we will also encounter movable divisor, which enjoy the same property.

Combined with (4.3.7), this implies that

$$\langle \mathbf{C}, \mathbf{e} \rangle \geq 6\,, \qquad\qquad \forall \mathbf{e} \in \mathcal{C}_{\rm S}^{\rm EFT}\,. \tag{4.3.9}$$

We then see that, in this large class of models, in the saxionic convex hull the bound (4.2.18) is realized in a quite stronger form:

$$\gamma(s) \geq \frac{N\pi}{\alpha}\,. \tag{4.3.10}$$

In order to describe the corresponding contribution EFT Kähler potential, it is convenient to use the dual saxionic formulation

$$\ell_a = \frac{3v_a}{\kappa(J,J,J)}\,, \tag{4.3.11}$$

where $\kappa(J,J,J) \equiv J \cdot J \cdot J = \kappa^{abc} v_a v_b v_c$. The dual saxionic cone $\mathcal{P}_{\rm K}$ can then be identified with an 'extended' Kähler cone $\mathcal{K}(X)_{\rm ext}$ obtained by gluing different spaces connected by flop transitions, in which curves collapse or blow-up:

$$\mathcal{P}_{\rm K} = \mathcal{K}_{\rm ext}(X) = \bigcup_{X' \sim X} \overline{\mathcal{K}(X')}\,. \tag{4.3.12}$$

Here $X' \sim X$ means that X' can be obtained from X by a chain of flops (which may be also trivial, corresponding to $X' = X$). Hence $\boldsymbol{\ell} \in \mathcal{P}_{\rm K}$ if there exists one chamber of $\mathcal{K}_{\rm ext}(X)$, associated with a compactification space $X' \sim X$, in which $\boldsymbol{\ell} = \ell_a\, D^a$ is a *nef* $\mathbb{R}$-divisor, that is $\boldsymbol{\ell} \in \overline{\mathcal{K}(X')}$.[14]

At large volume, the kinetic potential $\mathcal{F}(\ell)$ takes the form (4.1.4):

$$\mathcal{F}_{\rm K}(\ell) = \log \kappa(\boldsymbol{\ell}, \boldsymbol{\ell}, \boldsymbol{\ell})\,. \tag{4.3.13}$$

Hence $\tilde{P}(\ell) = \kappa(\boldsymbol{\ell}, \boldsymbol{\ell}, \boldsymbol{\ell})$, which is clearly homogeneous as in (4.1.6), with $n = 3$.

If one can take Sen's orientifold limit, the space X can be regarded as the $\mathbb{Z}_2$-orientifold quotient of a Calabi-Yau three-fold $\hat{X}$. A new saxion $\hat{s} \equiv e^{-\phi}$ appears, detected by $D(-1)$-instantons, where ϕ is the standard type IIB dilaton, so that we now have $N = b_2(X)+1$. The corresponding dual saxion is

$$\hat{\ell} = \frac{1}{2} e^{\phi}\,. \tag{4.3.14}$$

In the perturbative regime described by the dual saxions $\ell_i = (\hat{\ell}, \ell_a)$, the leading contribution to the Kähler potential is given by

$$\mathcal{F} = \log \hat{\ell} + \mathcal{F}_{\rm K}(\ell) = \log \hat{\ell} + \log \kappa(\boldsymbol{\ell}, \boldsymbol{\ell}, \boldsymbol{\ell}) \tag{4.3.15}$$

[14] In the following we will often focus spaces X which are toric or orientifold quotients of Calabi-Yau three-folds. In this case $\mathcal{K}(X)_{\rm ext}$ can be identified with the space of the so-called movable divisors. Hence we can write $\boldsymbol{\ell} = \ell_a\, D^a \in \mathcal{P}_{\rm K} \simeq {\rm Mov}^1(X)$. This identification can actually hold more generically – see [64] for more details.

and can then be written as in (4.1.4) with $\tilde{P} = \hat{\ell}\,\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell})$, which has homogeneity $n = 4$.[15]

Model 1: $\mathbb{P}^3$

For illustrative purposes, it is useful to describe a couple of simple explicit models (though with a small number of (s)axions) and their relevant energy scales. The first and easiest example is when $X = \mathbb{P}^3$.

In this case, the set of effective divisors $\text{Eff}^1(X)$ is spanned by a single element, the hyperplane class H. The saxionic cone is also one-dimensional and spanned by the curve $\Sigma = H^2$. The only saxion of this model encodes the volume of the hyperplane divisor and has a Kähler potential given by

$$K = -3\log s\,, \tag{4.3.16}$$

and a corresponding dual saxion $\ell = \frac{3}{2s}$. The condition for the inside the saxionic cone is extremely simple, being $s > 0 \iff \ell > 0$.

The single relevant curve leads to a single elementary EFT string and a set of string charges $\mathcal{C}_{\rm S}^{\rm EFT} = \{\mathbf{e} = e\Sigma = eH^2 | e \in \mathbb{Z}_{\geq 0}\}$. The tension of this elementary string, that represents a $w = 2$ EFT string limit, is given by $\mathcal{T} = M_{\rm P}^2\,\ell$.

The anti-canonical divisor in this setting is just $\overline{K}_X = 4H$ and, by using (4.4.3) and (4.3.5), it yields

$$\gamma(s) = 4\pi s\,. \tag{4.3.17}$$

This is manifestly positive in the saxionic cone and we have also $\gamma(s)|_{\hat{\Delta}_\alpha} \geq \frac{4\pi}{\alpha}$, stricter than (4.2.18) with $N = 1$.

The bound (4.2.23) then reduces to

$$\Lambda_{\rm QG}^2(\ell) \leq M_{\rm P}^2\ell = \frac{3M_{\rm P}^2}{2s}\,. \tag{4.3.18}$$

For generic values of the complex structure and 7-brane moduli, the species scale should correspond to the ten-dimensional Planck scale converted to the four-dimensional Einstein-frame. This is given the general formula

$$\Lambda_{\rm QG}^2 = \frac{M_{\rm P}^2}{2V(X)} = M_{\rm P}^2\sqrt{\frac{\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell})}{3}}\,, \tag{4.3.19}$$

where $V(X) = \frac{1}{6}\kappa(J,J,J)$ is the internal volume measured in Planck units units and we have

[15] In fact, the saxionic and dual saxionic cones are expected to receive corrections coming from higher derivative terms. This type of effect has been discussed in some detail for heterotic models in [65] and we will encounter it in subsection 4.3.2 – see e.g. (4.3.28). In particular, if we choose ℓ_0 so that $\ell_0 M_{\rm P}^2$ gives the tension of lightest D7-string, we generically have $\ell_0 = \hat{\ell} + c^a\ell_a$, where $c^a \in \mathbb{Q}$ accounts for possible world-volume curvature/bundle corrections. This means that in (4.3.15) we should set $\hat{\ell} = \ell_0 - c^a\ell_a$, which induces also a shift $s^a \to \hat{s}^a = s^a + c^a s^0$ in the Kähler potential.

used (4.3.11).[16] In this simple case we have $\boldsymbol{\ell} = \ell H$ and this general formula reduces to $\Lambda^2_{\rm QG} = M^2_{\rm P}\ell^{\frac{3}{2}}/\sqrt{3}$. We then see that (4.3.18) is satisfied if $\ell \leq 3$ or equivalently $s \geq \frac{1}{2}$. It is also interesting to observe that the bound (4.2.21) becomes $\Lambda^2_{\rm QG} \lesssim \frac{M^2_{\rm P}}{2s} = M^2_{\rm P}\ell/3$, which is basically equivalent to (4.3.18).

Model 2: $\mathbb{P}^1$ fibration over $\mathbb{P}^2$

This model has already been discussed in [64], which we can then follow. The internal space X is a $\mathbb{P}^1$ fibration over $\mathbb{P}^2$, and the fibration is specified by the integer $n \geq 0$.

The cone of effective divisors, which can be identified with the cone of BPS instanton charges $\mathcal{C}_{\rm I}$,[17] is simplicial and is generated by two effective divisors E^1, E^2: E^1 is the divisor obtained by restricting the $\mathbb{P}^1$ fibration over $\mathbb{P}^1 \subset \mathbb{P}^2$, while E^2 corresponds to a global section of the $\mathbb{P}^1$ fibration. One can then identify a basis of nef divisors $D^1 = E^1$ and $D^2 = E^2 + nE^1$, which generate the Kähler cone: $J = v_1 D^1 + v_2 D^2$, with $v_{1,2} > 0$. The triple intersection numbers are given by the coefficients of the formal object $\mathcal{I}(X) = (D^1)^2 D^2 + nD^1(D^2)^2 + n^2(D^2)^3$. Hence, by using the expansion $\boldsymbol{\ell} = \ell_a D^a$ the kinetic potential (4.3.13) becomes

$$\mathcal{F}_{\rm K} = \log\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell}) = \log\left(3\ell_1^2\ell_2 + 3n\ell_1\ell_2^2 + n^2\ell_2^3\right) \tag{4.3.20}$$

In this model the dual saxionic cone coincides with the Kähler cone: $\mathcal{P}_{\rm K} = \{\boldsymbol{\ell} = \ell_1 D^1 + \ell_2 D^2 | \ell_1 > 0, \ell_2 > 0\}$. The corresponding saxions are

$$s^1 = \frac{6\ell_1 + 3n\ell_2}{6\ell_1^2 + 6n\ell_1\ell_2 + 2n^2\ell_2^2} \quad , \quad s^2 = \frac{3(\ell_1 + n\ell_2)^2}{6\ell_1^2\ell_2 + 6n\ell_1\ell_2^2 + 2n^2\ell_2^3} \, . \tag{4.3.21}$$

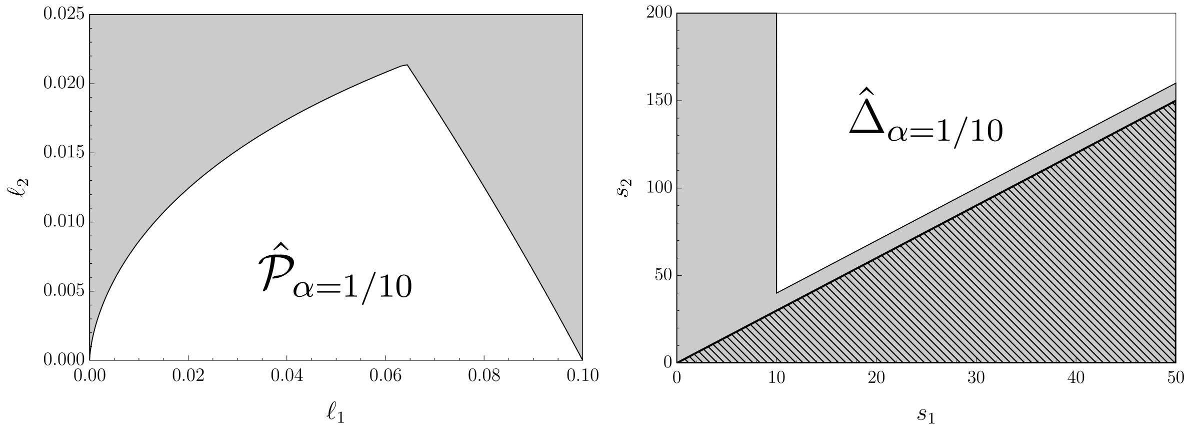
The (Mori) cone of effective curves is generated by $\Sigma_1 = E^1 \cdot E^2$ and $\Sigma_2 = (E^1)^2$, which are dual to the nef divisors D^a: $D^a \cdot \Sigma_b = \delta^a_b$. The cone of movable curves is instead generated by $\hat\Sigma_1 = D^1 \cdot D^2 = \Sigma_1 + n\Sigma_2$ and $\hat\Sigma_2 = (D^1)^2 = \Sigma_2$, which are dual to the effective divisors E^a: $E^a \cdot \hat\Sigma_b = \delta^a_b$. If we use the expansion $\boldsymbol{s} = s^1\Sigma_1 + s^2\Sigma_2 = s^1\hat\Sigma_1 + (s^2 - ns^1)\hat\Sigma_2$. Hence the saxionic cone is $\Delta = \{\boldsymbol{s} = s^a\Sigma_a | s^1 \geq 0\,,\ s^2 \geq ns^1\}$. One can also invert the relation between saxions and dual saxions:

$$\ell_1 = \frac{3n\sqrt{s^2 - ns^1}}{2\left[(s^2)^{\frac{3}{2}} - (s^2 - ns^1)^{\frac{3}{2}}\right]} \quad , \quad \ell_2 = \frac{3\left(\sqrt{s^2} - \sqrt{s^2 - ns^1}\right)}{2\left[(s^2)^{\frac{3}{2}} - (s^2 - ns^1)^{\frac{3}{2}}\right]} \tag{4.3.22}$$

[16] More precisely, we use conventions in which the d-dimensional Planck length $l_{(d)}$ appears in the Einstein-frame d-dimensional Einstein-Hilbert term through the combination $2\pi l_{(d)}^{2-d}\int {\rm d}^d x\sqrt{-g}R$. Then $V(X)l^6_{(10)}$ is the internal compactification volume and (4.3.19) is the conversion to four dimensions of the ten-dimensional energy-squared $2\pi/l^2_{(10)}$. Note that in four dimensions we can also introduce the alternative Planck length $l_{\rm P} = M_{\rm P}^{-1}$, so that $l^2_{(4)} = 4\pi l^2_{\rm P}$. In the checks of the asymptotic behaviour of the species scale we will often drop the 2π factors which are irrelevant for our purposes.

[17] The fundamental instantons correspond to Euclidean D3-branes wrapping effective divisors.

As in our general discussion, we can characterize the boundaries of $\mathcal{P}_{\rm K}$ in terms of tensionless strings. The set of EFT string charges is $\mathcal{C}_{\rm S}^{\rm EFT} = \{\mathbf{e} = e^1\Sigma_1 + e^2\Sigma_2 | (e^1, e^2) \in \mathbb{Z}^2\,,\; e^1 \geq 0\,,\; e^2 \geq ne^2\}$ is generated by $\hat{\Sigma}_1 = \Sigma_1 + n\Sigma_2$ and $\hat{\Sigma}_2 = \Sigma_2$, which have tensions $\mathcal{T}_{\hat{\Sigma}_1} = M_{\rm P}^2(\ell_1 + n\ell_2)$ and $\mathcal{T}_{\hat{\Sigma}_2} = M_{\rm P}^2\ell_2$. We notice that $\mathcal{T}_{\hat{\Sigma}_2}$ vanish at $\ell_2 = 0$, while $\mathcal{T}_{\hat{\Sigma}_1}$ vanish at the tip $\ell_1 = \ell_2 = 0$. These are infinite distance boundary components of $\mathcal{P}_{\rm K}$. On the other hand, on the boundary component $\ell_1 = 0$ no EFT string tension vanishes. This is instead characterized by the vanishing of the tension $\mathcal{T}_{\Sigma_1} = M_{\rm P}^2\ell_1$ associated with the non-EFT string charge Σ_1, which together with Σ_2 generates the set of BPS charges $\mathcal{C}_{\rm S}$. This implies that, even if the saxionic convex hull is simply given by $\hat{\Delta}_\alpha = \{s^1 \geq \frac{1}{\alpha}, s^2 - ns^1 \geq \frac{1}{\alpha}\}$, the corresponding dual saxionic convex hull $\hat{\mathcal{P}}_\alpha$ is more complicated – see figure 4.1.



(a) Saxionic convex hull $\hat{\Delta}_\alpha$.

(b) Dual saxionic convex hull $\hat{\mathcal{P}}_\alpha$.

Figure 4.1: Saxionic convex hull $\hat{\Delta}_\alpha$ and dual saxionic convex hull $\hat{\mathcal{P}}_\alpha$ for the F-theory model 2. The plot has been drawn with the reference values $\alpha = 1/10$ and $n = 3$. The hatched area in figure 4.1a is outside the saxionic cone Δ.

Remember that the GB term is identified by the anti-canonical divisor – see (4.3.5). In the present examples, it is given by

$$\overline{K}_X = (3-n)D^1 + 2D^2 = (3+n)E^1 + 2E^2\,. \tag{4.3.23}$$

From (4.4.3) and (4.3.5) we get

$$\gamma(s) = \pi\left[(3-n)s^1 + 2s^2\right]\,, \tag{4.3.24}$$

which is positive since $s^2 \geq ns^1$. We then see that $\gamma(s)|_{\hat{\Delta}_\alpha} \geq \frac{(5+n)\pi}{\alpha}$, which is stronger than (4.2.18) with $N = 2$. It is for instance sufficient to take $\alpha \leq \frac{1}{10}$ and $n \geq 1$ to get $\gamma(s)|_{\hat{\Delta}_\alpha} > 188$.

Now let us turn our attention to the relevant energy scales at play. It is easy to check that $K \simeq -\log\sigma$ asymptotically along the EFT string flow associated with $\mathbf{e}_{(2)} = \hat{\Sigma}_2$, while $K \simeq -3\log\sigma$ along the EFT string flow associated with $\mathbf{e}_{(1)} = \hat{\Sigma}_1$. Hence, only $\mathbf{e}_{(2)}$ should have $w = 1$, as one can explicitly check [64], and the corresponding string obtained by wrapping a D3 on $\hat{\Sigma}_2$ is indeed dual to a fundamental heterotic superstring via F-theory/heterotic duality. We can then distinguish two regimes set by the elementary EFT string tensions $\mathcal{T}_{(1)} = M_{\rm P}^2(\ell_1 + n\ell_2)$ and $\mathcal{T}_* = \mathcal{T}_{(2)} = M_{\rm P}^2\ell_2$. Note that $\mathcal{T}_* \leq \mathcal{T}_{(1)}$ for any value of the saxions assuming $n > 0$. Hence the upper bound (4.2.23) is given by

$$\Lambda^2_{\rm max} = \mathcal{T}_* = M_{\rm P}^2\ell_2\,. \tag{4.3.25}$$

If $\ell_2 \ll \ell_1$ and hence $\mathcal{T}_* \ll \mathcal{T}_{(1)}$ should indeed correspond to the tension of a weakly coupled critical string, which then determines the species scale, so that (4.2.23) is satisfied and actually saturated.

If instead $\mathcal{T}_* \simeq \mathcal{T}_{(1)}$, there should not exist a controlled dual weakly couple string theory description. Since $\mathcal{T}_{(1)}/\mathcal{T}_* = n + \ell_1/\ell_2 > n$, this regime can be reached only if $n \sim \mathcal{O}(1)$ and $\ell_1/\ell_2 \lesssim 1$. Furthermore, the weakly-coupled EFT description requires that $\mathcal{T}_{(1)} \lesssim M_{\rm P}^2$, which implies that $\ell_2 \lesssim \frac{1}{n}$. In this regime the species scale could be identified with the ten-dimensional Planck scale $1/l_{(10)}$ as measured by the four-dimensional observer. By combining (4.3.19) and (4.3.20) we then get

$$\Lambda^2_{\rm QG} = M_{\rm P}^2\sqrt{\ell_1^2\ell_2 + n\ell_1\ell_2^2 + \frac{1}{3}n^2\ell_2^3} \lesssim M_{\rm P}^2\ell_2\,, \tag{4.3.26}$$

where in the second step we have used $\ell_1 \lesssim \ell_2 \lesssim \frac{1}{n}$ and $n \sim \mathcal{O}(1)$, and we have neglected an $\mathcal{O}(1)$ overall constant. The result is compatible with the bound (4.2.23). Other regimes can be better studied through the dual heterotic M-theory description, which will be discussed in subsection (4.3.2) and will confirm that $\Lambda_{\rm QG}$ is still bounded by (B.3.13).

Let us also discuss the bound (4.2.21) for this model. By recalling (4.3.24) and (4.3.21) we get

$$\frac{2\pi M_{\rm P}^2}{\gamma(s)} = \frac{2M_{\rm P}^2}{(3-n)s^1 + 2s^2} = \frac{4\ell_2(3\ell_1^2 + 3n\ell_1\ell_2 + n^2\ell_2^2)}{6\ell_1^2 + (3+n)(6\ell_1\ell_2 + 3n\ell_2^2)}M_{\rm P}^2 \tag{4.3.27}$$

If for instance $\ell_2 \ll \ell_1$, then this upper bound reduces to $2M_{\rm P}^2\ell_2$, which is then close to (B.3.13). If instead one considers a limit $\ell_1 \ll \ell_2$ then (4.3.27) is well approximated by $\frac{4\ell_2}{3+9/n}M_{\rm P}^2$, which is again of the same order of (B.3.13). A similar discussion can be carried out for the model where X is $\mathbb{P}^1$ fibration over the Hirzebruch surface $\mathbb{F}_n$ and is presented in appendix B.3.1.

4.3.2 Heterotic models

Our second class of models is given by $E_8 \times E_8$ heterotic compactifications on Calabi-Yau spaces, and their M-theory counterpart, at large volume. (The $SO(32)$ case is completely analogous.) As discussed in [65], the relevant saxionic cone is affected by higher derivative terms. Here we summarize only the necessary information.

The saxions are $s^i = (s^0, s^a)$, which include the Kähler moduli s^a of the Calabi-Yau compactification space X. Hence $N = b_2(X) + 1$. These are obtained by expanding the (string frame) Kähler form $J = s^a[D_a]$ in a basis Poincaré dual to a basis of divisors D_a, $a = 1, \ldots, b_2(X)$. The remaining saxion s^0 combines the dilaton and the Kähler moduli:

$$s^0 = \frac{1}{6} e^{-2\phi} \kappa_{abc} s^a s^b s^c + \frac{1}{2} p_a s^a \,. \tag{4.3.28}$$

where $\kappa_{abc} \equiv D_a \cdot D_b \cdot D_c$ and

$$p_a \equiv - \int_{D_a} \left[\lambda(E_2) - \frac{1}{2} c_2(X) \right] \in \mathbb{Z} \,, \tag{4.3.29}$$

where E_1 and E_2 denote the two E_8 internal bundles, and

$$\lambda(E) = -\frac{1}{16\pi^2} \, \mathrm{tr}(F \wedge F) \,. \tag{4.3.30}$$

Note that the tadpole cancellation condition imposes the topological constraint $\lambda(E_1) + \lambda(E_2) = c_2(X)$. One could also include NS5/M5-branes wrapping internal curves (see [65]), but here for simplicity we will not do that. The saxionic cone is given by

$$\Delta = \left\{ \boldsymbol{s} = (s^0, s^a) \;\middle|\; s^a D_a \in \overline{\mathcal{K}(X)} \,,\; s^0 \geq 0 \,,\; s^0 \geq p_a s^a \right\} \,, \tag{4.3.31}$$

and the Gauss-Bonnet coupling (4.1.13) takes the form

$$\gamma(s) = \pi \left(2s^0 - p_a s^a + \frac{1}{6} n_a s^a \right) \,, \tag{4.3.32}$$

where

$$n_a \equiv \frac{1}{2} \int_{D_a} c_2(X) \,. \tag{4.3.33}$$

Let us also recall that in the M-theory realization [142, 143], the Calabi-Yau X three-fold is fibered over an interval, representing the 11-th M-theory direction. Then s^0 and $s^0 - p_a s^a$ can be interpreted as the volume of X, as measured by a Euclidean M5-brane, at the two endpoints of this interval [65].

For simplicity, we will henceforth assume that $p_a s^a \geq 0$. In this case the saxionic cone reduces to

$$\Delta = \left\{ \boldsymbol{s} = (s^0, s^a) \;\middle|\; s^a D_a \in \overline{\mathcal{K}(X)} \,,\; s^0 \geq p_a s^a \right\} \,, \tag{4.3.34}$$

and the Gauss-Bonnet coupling (4.3.35) becomes

$$\gamma(s) = \pi \left[2(s^0 - p_a s^a) + (p_a + \frac{1}{6} n_a) s^a \right] \,. \tag{4.3.35}$$

The Kähler potential can be obtained by dimensionally reducing the heterotic M-theory [143], taking into account the deformations discussed in [149] – see also [138] – and using the above parametrization:

$$K = -\log(s^0 - \frac{1}{2}p_a s^a) - \log \kappa(\boldsymbol{s}, \boldsymbol{s}, \boldsymbol{s})\,, \tag{4.3.36}$$

where $\kappa(\boldsymbol{s}, \boldsymbol{s}, \boldsymbol{s}) \equiv \kappa_{abc}s^a s^b s^c$. Hence

$$\ell_0 = \frac{1}{2(s^0 - \frac{1}{2}p_b s^b)} \quad , \quad \ell_a = \frac{3\kappa_{abc}s^b s^c}{2\kappa(\boldsymbol{s}, \boldsymbol{s}, \boldsymbol{s})} - \frac{p_a}{4(s^0 - \frac{1}{2}p_b s^b)}\,. \tag{4.3.37}$$

The dual saxionic cone is then given

$$\mathcal{P} = \{(\ell_0, \ell_a\Sigma^a) \in \mathbb{R}_{\geq 0} \times H_2(X, \mathbb{R}) | \big(\ell_a + \frac{1}{2}\ell_0 p_a\big)\Sigma^a \in \mathcal{P}^{\rm het}_{\rm K}\}\,, \tag{4.3.38}$$

where Σ^a is the basis of curves dual to D_a ($D_a \cdot \Sigma^b = \delta^b_a$), and $\mathcal{P}^{\rm het}_{\rm K} \subset H_2(X, \mathbb{R})$ is the Poincaré dual of the closure of the image of $\mathcal{K}(X)$ under the map $J \to \frac{1}{2}J \wedge J$. Note that $\mathcal{P}^{\rm het}_{\rm K}$ is a sub-cone of the cone $\text{Mov}_1(X)$ introduced in (4.3.3).

The saxionic convex hull is now given by

$$\hat{\Delta}_\alpha = \left\{ \boldsymbol{s} = (s^0, s^a D_a) \;\middle|\; s^a D_a \in \hat{\mathcal{K}}_\alpha(X)\,,\; s^0 \geq \left(\frac{1}{\alpha} + p_a s^a\right) \right\}\,. \tag{4.3.39}$$

where $\hat{\mathcal{K}}_\alpha(X)$ is the Kähler convex hull defined as in the generic saxionic case, which is contained in the stretched Kähler cone introduced in [44]. We then see that

$$\gamma(s)|_{\hat{\Delta}_\alpha} \geq \pi \left[\frac{2}{\alpha} + \big(p_a + \frac{1}{6}n_a\big)s^a|_{\hat{\mathcal{K}}_\alpha(X)}\right]\,. \tag{4.3.40}$$

For concreteness, consider for instance the models with $\lambda(E_2) = 0$. In this case $p_a = n_a$, which satisfies the above condition $p_a s^a \geq 0$ since $n_a s^a = \frac{1}{2}\int_X c_2(X) \wedge J \geq 0$ [150]. By applying the same arguments that led us to (4.2.18), we expect $\big(n_a s^a\big)|_{\hat{\mathcal{K}}_\alpha(X)} \geq b_2(X)/\alpha$, where $b_2(X) = N - 1$. Hence we should have

$$\gamma(s)|_{\hat{\Delta}_\alpha} = \pi \left(\frac{2}{\alpha} + \frac{7}{6}n_a s^a|_{\hat{\mathcal{K}}_\alpha(X)}\right) \geq \frac{(5 + 7N)\pi}{6\alpha}\,. \tag{4.3.41}$$

Note that is lower bound is stronger than (4.2.18). We numerically tested this bound against a set of explicit Calabi-Yau compactifications with `CYtools` [48]. The result is reported in figure 4.2, in which we plot the value of $\alpha\gamma(s)$ evaluated at the tip of stretched Kähler cone against $b_2(X) = N - 1$. Such values clearly satisfy the lower bound in (4.3.41). Since the Kähler sector of the saxionic convex hull $\hat{\Delta}_\alpha$ is contained in the stretched Kähler cone, also $\gamma(s)|_{\hat{\Delta}_\alpha}$ satisfies this bound. Hence the plot confirms how (4.3.41) provides a good, if not conservative lower bound of the scaling with respect to N.

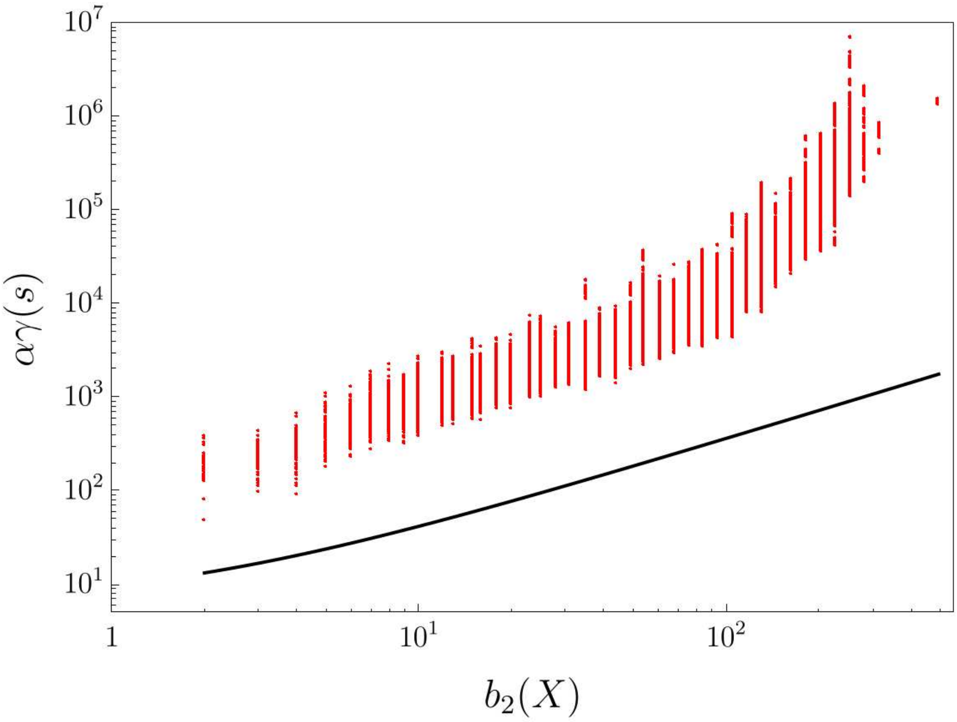


Figure 4.2: Value of $\alpha\gamma(s)$ at the tip of the stretched Kähler cone as a function of $b_2(X) = N-1$ in an explicit set of Calabi-Yau compactifications analyzed with `CYtools` [48]. The set consists of 71908 Calabi-Yau manifolds with $b_2(X)$ ranging from 2 to 491, with up to 100 polytopes per fixed $b_2(X)$ and with 25 triangulations per polytope obtained with the `random_triangulation_fast` method. The quantity $n_a s^a$ of equation (4.3.35) has been obtained with the `second_chern_class` method and evaluated at the tip of the stretched Kähler cone. The black line refers to the lower bound of equation (4.3.41).

Energy scales in heterotic models

Finally, let us consider the heterotic models discussed in section 4.3.2. By discretizing (4.3.34) one gets $\mathcal{C}_{\rm S}^{\rm EFT}$, which is generated by $\mathbf{e}_* = (1,0)$ and vectors of the form $(e^a p_a, D_{\mathbf{e}})$, where $D_{\mathbf{e}} \equiv e^a D_a$ are generators of the cone of nef divisors. By looking at the behavior of (4.3.36) under the corresponding EFT string flows we can check that $K \sim -\log\sigma$ under the $\mathbf{e}_*$ flow, consistently with the fact that $\mathbf{e}_*$ indeed represents a critical heterotic string. On the other hand, under the flow of the EFT string charges $(e^a p_a, D_{\mathbf{e}})$ we have $K \sim -n\log\sigma$, with integer $n = 2,3,4$ determined by the self-intersections of $D_{\mathbf{e}}$. Indeed these strings have scaling weights $w = 2$ or $w = 3$ [65]. These EFT strings have tensions

$$
\begin{aligned}
\mathcal{T}_* &= M_{\rm P}^2 \ell_0 = \frac{M_{\rm P}^2}{2(s^0 - \frac{1}{2}p_b s^b)} = \frac{3M_{\rm P}^2 e^{2\phi}}{\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})} \qquad (w=1)\,,\\
\mathcal{T}_{D_{\mathbf{e}}} &= M_{\rm P}^2(e^a\ell_a + e^a p_a \ell_0) = \left(\frac{3\kappa(\mathbf{e},\boldsymbol{s},\boldsymbol{s})}{2\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})} + \frac{e^a p_a}{2(s^0 - \frac{1}{2}p_b s^b)}\right) M_{\rm P}^2 \qquad (w=2,3)\,,
\end{aligned}
\tag{4.3.42}
$$

where we have used (4.3.37). By recalling (4.3.28) one can immediately see that in the limit $s^0 \gg 1$, with s^a fixed within (4.3.39), corresponds to the weak string coupling limit $g_s = e^\phi \ll 1$, and that in this limit $\mathcal{T}_* \ll \mathcal{T}_{D_\mathbf{e}}$, for any nef $D_\mathbf{e}$. Hence in this regime $\Lambda^2_{\rm max} = \mathcal{T}_*$ and the the bound (4.2.23) is saturated, since $\mathcal{T}_*$ corresponds to the critical string tension.

On the other hand, in the strong string coupling regime $e^{2\phi} \gg 1$ the determination of $\Lambda^2_{\rm max}$ depends on the constants p_a. For simplicity, suppose that $e^a p_a > 0$ for any e^a identifying a nef divisor $e^a D_a$. This is for instance the case if $\int_D c_2(X)$ is strictly positive for any nef divisor D – see e.g. [200] – and we assume that $\lambda(E_2) = 0$, so that $e^a p_a = e^a n_a = \frac{1}{2}\int_{D_\mathbf{e}} c_2(X) > 0$. In this case tension $\mathcal{T}_*$ is just slightly smaller than any $\mathcal{T}_{D_\mathbf{e}}$, hence we again have $\Lambda^2_{\rm max} = \mathcal{T}_*$. One can then check (4.2.23) in different regimes. Consider first the M-theory supergravity regime. Since the internal six-dimensional M-theory and string frame IIA metrics are related by $\mathrm{d}\hat{s}^2_{\rm M}(X) = e^{-\frac{2\phi}{3}}\mathrm{d}s^2_{\rm IIA\text{-}st}$, this requires $\hat{\boldsymbol{s}} \equiv e^{-\frac{2\phi}{3}}\boldsymbol{s}$ to be large enough – for instance we may require that $\hat{\boldsymbol{s}} \in \hat{\mathcal{K}}_\alpha$. In this case the species scale $\Lambda_{\rm QG}$ should coincide with the M-theory Planck scale $1/l_{(11)}$ (see footnote 16) as measured by the four-dimensional Einstein observer, that is:

$$\Lambda^2_{\rm QG} = \frac{6^{\frac{1}{3}} M^2_{\rm P}}{4\pi(s^0 - \frac{1}{2}p_a s^a)^{\frac{2}{3}}\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})^{\frac{1}{3}}}\,. \tag{4.3.43}$$

Hence

$$\frac{\Lambda^2_{\rm max}}{\Lambda^2_{\rm QG}} = 2\pi\left(\frac{\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})}{6(s^0 - \frac{1}{2}p_a s^a)}\right)^{\frac{1}{3}} = 2\pi e^{\frac{2\phi}{3}}\,. \tag{4.3.44}$$

This formula can be more directly derived by identifying $\Lambda^2_{\rm max} = \mathcal{T}_*$ with the tension of an open M2-branes stretching along the M-theory interval, which has length $l_{(11)}e^{2\phi/3}$. It is then clear that the bound (4.2.23) is satisfied if $e^\phi \gtrsim 1$. It is also interesting to check (4.2.23) in other limits. These are discussed in appendix B.3.3, together with examples with $p_a e^a = 0$, in both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry.

4.4 $SO(4)$-symmetric configurations

In this section we show that the broad class of models described by the 2-derivative action (4.1.1), or equivalently (2.2.6), admits sub-extremal and extremal wormhole configurations with $SO(4)$ symmetry, after continuation to Euclidean space. These solutions can be considered as generalizations of the ones provided in the seminal paper [201] and encompass several other generalizations already appeared in the literature. In Section 4.4.2 we derive the equations of motion associated to our $SO(4)$-symmetric ansatz and of the on-shell action for sub-extremal wormholes. This part of our work is not new but serves to set the stage for the original results presented in the subsequent sections. Extremal solutions are discussed in Section 4.4.3 and their relation with fundamental instantons is clarified. A curious BPS bound on the on-shell action of sub-extremal wormholes, together with a possible solution of the associated puzzle, is briefly commented upon in Section 4.4.4.

4.4.1 Euclidean action

When discussing non-perturbative effects such as wormholes we have to pass to the Euclidean formulation. The way in which one identifies charged saddles in the axions-saxions Euclidean action has been clarified in [6] (see also [14]) and involves a continuation to imaginary axion fields as well as the introduction of boundary terms. No such subtleties appear if one works with the dual formulation, which is what we do. One just has to Wick rotate (2.2.6), giving

$$\begin{aligned} S = &-\frac{1}{2}M_{\rm P}^2\int_{\mathcal{M}}\sqrt{g}\,R - \frac{1}{2}M_{\rm P}^2\int_{\partial\mathcal{M}}\sqrt{h}(K-K_0) \\ &+\frac{1}{2}M_{\rm P}^2\int_{\mathcal{M}}\mathcal{G}^{ij}{\rm d}\ell_i\wedge *{\rm d}\ell_j + \frac{1}{2M_{\rm P}^2}\int_{\mathcal{M}}\mathcal{G}^{ij}\mathcal{H}_{3,i}\wedge *\mathcal{H}_{3,j}\,, \end{aligned} \tag{4.4.1}$$

where the Hawking-Gibbons term has been written explicitly. As already anticipated in Section 4.1, our analysis also aims at investigating the impact of the semi-topological operators in (4.4.1). The Pontryagin operator turns out to be trivial on the $SO(4)$-invariant solutions that we will be studying.[18] We therefore focus on the Gauss-Bonnet term. Rotating to Euclidean signature, this reads

$$S_{\rm GB} \equiv -\int_{\mathcal{M}}\sqrt{g}\,\gamma(s)\,E_{\rm GB} - \int_{\partial\mathcal{M}}\sqrt{h}\,\gamma(s)\,(Q-Q_0)\,, \tag{4.4.2}$$

where we included the appropriate boundary term and we approximated $\gamma_{\rm tot}$ (see (4.1.11)) as

$$\gamma(s) = \frac{\pi}{6}\tilde{C}_i s^i \equiv \frac{\pi}{6}\langle\tilde{\mathbf{C}},\boldsymbol{s}\rangle \tag{4.4.3}$$

consistently with the bound (4.2.18). In (4.4.2) we regard s^i as functions of the dual saxions ℓ_i as defined by (2.2.8). In this dual formulation the presence of accidental axionic shift symmetries is encoded in the Bianchi identity ${\rm d}\mathcal{H}_3 = 0$. This is in general broken by the Pontryagin operator (4.1.9), which we just argued that will be ignored, or (4.1.10) (see for instance section 3.2 of [65]) as well as non-perturbative effects. [19] Because in our work gauge fields are assumed to have trivial backgrounds, the interaction in (4.1.9) can be safely ignored as well.

4.4.2 Effective one-dimensional action and equations of motion

In deriving the relevant equations of motion we will follow the approach of [14], but will work with the formulation in terms of gauge two-forms $\mathcal{B}_{2\,i}$ and dual linear multiplets.

[18] Because of the axion-dependent coefficient in (4.1.10), it is not fully topological and actually affects the equations of motion. However here we will treat the higher-dimensional operators as perturbations and estimate their effect by evaluating them on the leading order solution. It is in this sense that the Pontryagin operator can be neglected.

[19] The Bianchi identity ${\rm d}\mathcal{H}_3 = 0$ can also be broken by field-strengths of gauge three-forms. As discussed in [67], in $\mathcal{N}=1$ supersymmetry this effect is dual to the presence of special multi-branched superpotentials which are often realized in flux compactifications [202]. In this section we assume that such superpotential terms are not present.

We are looking for Euclidean wormholes preserving $SO(4)$ rotational symmetry. This means that we can restrict the metric to take the form

$$\mathrm{d}s^2 = \frac{1}{M_{\mathrm{P}}^2}\left[e^{2A(\rho)}\mathrm{d}\rho^2 + e^{2B(\rho)}\mathrm{d}\Omega^2\right] . \tag{4.4.4}$$

Here $\rho \in I$ is an arbitrary dimensionless radial coordinate taking values in some interval $I \subset \mathbb{R}$, to be defined below, and $\mathrm{d}\Omega^2$ is the line element of a three-sphere of unit radius, which has volume $2\pi^2$. One can of course remove the arbitrariness of ρ by gauge-fixing $e^{A(\rho)}$ or $e^{B(\rho)}$. A particularly convenient choice is given by

$$e^B = \rho \equiv M_{\mathrm{P}} r \tag{4.4.5}$$

where r represents the radius of the three-sphere. This gauge fixing corresponds to the following line element

$$\mathrm{d}s^2 = e^{2A(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{4.4.6}$$

Note that, however, in this case r can smoothly parametrize only half wormhole.[20] Another useful parametrization is discussed below, but for the moment we keep ρ arbitrary.

In addition to the metric, we will allow the fields $\ell_i, \mathcal{H}_{3,i}$ appearing in (4.4.1) to have non-trivial profiles. By $SO(4)$ symmetry the (dual) saxions can only depend on the radial coordinate: $\ell_i = \ell_i(\rho)$, or equivalently $s^i = s^i(\rho)$. Instead the field-strengths $\mathcal{H}_{3\,i}$ must necessarily take the form

$$\mathcal{H}_{3\,i} = \frac{1}{\pi} q_i \, \mathrm{vol}_{S^3} , \tag{4.4.7}$$

with q_i constants. Observing that $\mathcal{J}_{1\,i} = M_{\mathrm{P}}^2 \mathcal{G}_{ij}\mathrm{d}a^i = *\mathcal{H}_{3\,i}$ corresponds to the 1-form current associated to the axion shift symmetry, the quantities q_i are to be interpreted as the wormhole charges

$$\frac{1}{2\pi}\int_{S^3} \mathcal{H}_{3\,i} = q_i \in \mathbb{Z} . \tag{4.4.8}$$

Charge quantization can be either seen as a consequence of the axion periodicity (2.1.1) (i.e. the momentum conjugate to a periodic variable is quantized) or, in the dual language, to the quantization of the $\mathcal{H}_{3\,i}$ field-strengths.

The $SO(4)$ symmetry ensures that our ansatz can be regarded as a consistent truncation of the

[20] This issue is avoided if instead we use the 'geodesic' radial coordinate χ defined by $M_{\mathrm{P}}\mathrm{d}\chi = e^A \mathrm{d}\rho$. With this coordinate the metric takes the form $\mathrm{d}s^2 = \mathrm{d}\chi^2 + M_{\mathrm{P}}^{-2} e^{2B(\chi)}\mathrm{d}\Omega^2$, corresponding to the gauge-fixing $e^A = 1$ and the identification $\rho = \chi M_{\mathrm{P}}$.

full theory. Inserting it in eq. (4.4.1), the action reduces to

$$
\begin{aligned}
S|_{\text{ansatz}} =& -6\pi^2 \int_I \mathrm{d}\rho \left[e^{A+B} + \left(\frac{\mathrm{d}B}{\mathrm{d}\rho}\right)^2 e^{3B-A} \right] + 6\pi^2 \left[e^{2B}\right]_{\partial I} \\
&+ \int_I \mathrm{d}\rho \left[e^{3B-A} \pi^2 \mathcal{G}^{ij} \frac{\mathrm{d}\ell_i}{\mathrm{d}\rho}\frac{\mathrm{d}\ell_j}{\mathrm{d}\rho} + e^{A-3B} \mathcal{G}^{ij}\, q_i q_j \right] .
\end{aligned}
\tag{4.4.9}
$$

Extremizing (4.4.9) with respect to A we have the constraint

$$
e^{6B-2A}\pi^2 \mathcal{G}^{ij} \frac{\mathrm{d}\ell_i}{\mathrm{d}\rho}\frac{\mathrm{d}\ell_j}{\mathrm{d}\rho} - \mathcal{G}^{ij}\, q_i q_j = 6\pi^2 \left(\frac{\mathrm{d}B}{\mathrm{d}\rho}\right)^2 e^{6B-2A} - 6\pi^2\, e^{4B} . \tag{4.4.10}
$$

The equations of motion for ℓ_i are derived from the second line of (4.4.9) and will be shown shortly. The variation of (4.4.9) with respect to B is on the other hand redundant once the former two conditions are satisfied, and hence will not be discussed.

To write the field equations for ℓ_i it is convenient to introduce [14] a 'proper' radial coordinate τ such that

$$
\mathrm{d}\tau = \pm \frac{1}{\pi} e^{A-3B} \mathrm{d}\rho , \tag{4.4.11}
$$

where the two signs correspond to the two possible relative orientations between τ and ρ. With this notation the second line of (4.4.9) resembles the action for a particle with non-canonical kinetic term moving in a non-trivial potential:

$$
2\pi \int \mathrm{d}\tau \left[\frac{1}{2} \mathcal{G}^{ij}(\ell)\, \dot{\ell}_i \dot{\ell}_j - V_{\mathbf{q}}(\ell) \right] \tag{4.4.12}
$$

with $\dot{\ell}_i \equiv \frac{\mathrm{d}\ell_i}{\mathrm{d}\tau}$. The potential is

$$
V_{\mathbf{q}}(\ell) \equiv -\frac{1}{2} \mathcal{G}^{ij}(\ell)\, q_i q_j \equiv -\frac{1}{2} \|\mathbf{q}\|^2 \tag{4.4.13}
$$

where we have introduced the norm defined by the metric (2.2.7):

$$
\|\mathbf{q}\|^2 \equiv \mathcal{G}^{ij}(\ell)\, q_i q_j . \tag{4.4.14}
$$

The (dimensionless) particle energy

$$
E \equiv \frac{1}{2} \mathcal{G}^{ij}(\ell)\, \dot{\ell}_i \dot{\ell}_j + V_{\mathbf{q}}(\ell) \tag{4.4.15}
$$

is thus manifestly conserved. From (4.4.12) – or equivalently (4.4.9) – we may now obtain the equations of motion for ℓ_i

$$
\mathcal{G}^{ij} \frac{D\dot{\ell}_j}{\mathrm{d}\tau} = -\frac{\partial V_{\mathbf{q}}}{\partial \ell_i} \quad \Leftrightarrow \quad \mathcal{F}^{ij} \ddot{\ell}_j = \frac{1}{2} \mathcal{F}^{ijk} \left(q_j q_k - \dot{\ell}_j \dot{\ell}_k \right) , \tag{4.4.16}
$$

where $D\dot{\ell}_j/\mathrm{d}\tau$ is the Levi-Civita covariant derivative associated with the metric $\mathcal{G}^{ij}$; we have used (2.2.7) and the shorthand notation $\mathcal{F}^{ij} \equiv \partial^2\mathcal{F}/\partial\ell_i\partial\ell_j$, $\mathcal{F}^{ijk} \equiv \partial^3\mathcal{F}/\partial\ell_i\partial\ell_j\partial\ell_k$. In addition the dual saxions ℓ_i satisfy appropriate boundary conditions, which we assume to be Dirichlet.[21]

We are interested in asymptotically flat wormhole configurations manifesting a Euclidean "time-reversal" symmetry.[22] Namely, the dual saxions flow from some asymptotic value $\boldsymbol{\ell}_\infty$ at infinity ($\rho = -\infty$) to some value $\boldsymbol{\ell}_*$ at the neck of the wormhole ($\rho = 0$), and then go back to the asymptotic value $\boldsymbol{\ell}_\infty$ ($\rho = +\infty$). The analogy with the point particle makes it clear that the saxionic flow can be interpreted as a scattering process, in which the particle climbs the potential (4.4.13) until it reaches the turning point $\boldsymbol{\ell}_*$, at which it stops and then rolls down back to its original position. Since $V_{\mathbf{q}} < 0$ the saxions can bounce only if $E = -|E| < 0$. We will however be a bit more general and consider

$$E = -|E| \leq 0\,. \tag{4.4.17}$$

The cases $E = 0$ and $E < 0$ are usually referred to as extremal and sub-extremal (non-supersymmetric) solutions – see for instance the review papers [203–205]. In this section we will concentrate on these cases, and not consider super-extremal wormholes, which have $E > 0$ and a singular geometry.

Inserting (4.4.15) in the constraint (4.4.10), and writing the result in the gauge (4.4.5) we get

$$e^{2A} = \frac{1}{1 - \frac{L^4}{r^4}} \tag{4.4.18}$$

where

$$L^4 \equiv \frac{|E|}{3\pi^2 M_{\rm P}^4} = \frac{\|\mathbf{q}\|_*^2}{6\pi^2 M_{\rm P}^4}\,, \tag{4.4.19}$$

with $\|\mathbf{q}\|_*^2 \equiv \|\mathbf{q}\|^2(\ell_*)$. Hence the metric takes the form

$$\mathrm{d}s^2 = \frac{1}{1 - \frac{L^4}{r^4}}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2, \tag{4.4.20}$$

with $r \in [L, \infty)$ and L can be identified with the minimal S^3 radius $r_* = L$. In the second equality in (4.4.19) we used the fact that at the turning point, the wormhole throat, the particle stops $\dot{\ell}_i = 0$ and E reduces to $V_{\mathbf{q}}(\ell_*)$. We see that the wormhole minimal S^3 radius is controlled by the charge vector norm $\|\mathbf{q}\|$ at such radius.

By using (4.4.18) in (4.4.11) (in the gauge (4.4.5)) one gets a differential relation between r and τ, which can be easily integrated. For sub-extremal ($E < 0$) wormholes we will impose that

[21] In presence of low-energy supersymmetry breaking, the asymptotic values $\boldsymbol{\ell}_\infty$ may ultimately be determined by an hypothetical stabilizing potential for the saxions. We will come back to this in the conclusions.

[22] The Euclidean time-reversal symmetry ensures that by cutting these solutions at the minimal radius one gets half-wormholes which, once analytically continued back to a Lorentzian spacetime, describe the nucleation or absorption of a baby universe, with real boundary values of the fields and of their time derivatives [201].

$\tau_* = 0$ at the turning point ℓ_*, that is

$$\tau_* = 0 \qquad \Leftrightarrow \qquad r_* = L\,. \tag{4.4.21}$$

Then (4.4.11), with $+/-$ sign for positive/negative τ, integrates to

$$\tau = \tau = \pm\frac{1}{2\pi M_{\rm P}^2 L^2}\left[\frac{\pi}{2} - \arcsin\left(\frac{L^2}{r^2}\right)\right] \tag{4.4.22}$$

or similarly $L^2/r^2 = \cos\left(2\pi M_{\rm P}^2 L^2 \tau\right)$. This in particular implies that e^{-2A} as a function of τ is given by $e^{-2A} = \sin^2\left(2\pi M_{\rm P}^2 L^2 \tau\right)$. The maximal extension of the τ interval is

$$|\tau| < \tau_\infty \equiv \frac{1}{4M_{\rm P}^2 L^2} \qquad \Leftrightarrow \qquad r_\infty = \infty\,, \tag{4.4.23}$$

where $\cos\left(2\pi M_{\rm P}^2 L^2 \tau_\infty\right) = 0$. In particular, the region $0 \leq \tau < \tau_\infty$ parametrizes the first half-wormhole, while the interval $-\tau_\infty < \tau \leq 0$ parametrizes the second half-wormhole. In other words, the particle takes a proper time τ_∞ to make half of its journey.

The particle interpretation also suggests a simple characterization of regular wormholes. The point particle is at rest at $\tau = 0$ and starts rolling down the potential $V_{\mathbf{q}}$. If it reaches τ_∞ remaining inside the perturbative dual saxionic domain, then the corresponding trajectory describes an everywhere regular on-shell wormhole. Whenever along its path the particle exits the allowed domain, the solution develops a singularity and should either be somehow regularized or discarded.

Finally, the on-shell wormhole action can be obtained by plugging (4.4.10) into (4.4.9), using the gauge (4.4.5) and the explicit solution (4.4.18). Alternatively, we can use the traced Einstein equation

$$R * 1 = \mathcal{G}^{ij}\left(\mathrm{d}\ell_i \wedge *\mathrm{d}\ell_j - \frac{1}{M_{\rm P}^2}\mathcal{H}_{3\,i} \wedge *\mathcal{H}_{3\,j}\right) \tag{4.4.24}$$

in (4.4.1), observing that the Gibbons-Hawking boundary term vanishes for the metric (B.1.1). In either case, we obtain the wormhole action

$$S|_{\rm w} = 2\pi \int \mathrm{d}\tau\, \|\mathbf{q}\|^2 = \frac{1}{M_{\rm P}^2}\int \mathcal{G}^{ij}\mathcal{H}_{3,i} \wedge *\mathcal{H}_{3,j}\,. \tag{4.4.25}$$

Considering only half of the domain, we finally arrive at the on-shell action for half-wormhole:

$$S|_{\rm hw} = 2\pi \int_{\tau_*}^{\tau_\infty} \mathrm{d}\tau\, \|\mathbf{q}\|^2\,. \tag{4.4.26}$$

The latter can also be written using (4.4.13), (4.4.19), and (4.4.23) in an alternative form:

$$\begin{aligned} S|_{\rm hw} &= -4\pi \int_{\tau_*}^{\tau_\infty} {\rm d}\tau\, V_{\mathbf{q}} \\ &= 3\pi^3 M_{\rm P}^2 L^2 + 4\pi \int_{\tau_*}^{\tau_\infty} {\rm d}\tau\, (E - V_{\mathbf{q}})\,. \end{aligned} \tag{4.4.27}$$

The first term coincides with what one would find for a purely axionic theory, where the dual saxions are essentially 'frozen' at $\boldsymbol{\ell}_* = \boldsymbol{\ell}_\infty$.[23] The second contribution can be identified with the integrated 'kinetic' energy of the dual saxions, see (4.4.15), and is hence always positive.

We remind the reader that the results obtained in this section relies on the assumption that a truncation of the action at the 2-derivative level is justified. This requires $L > 1/\Lambda$, where Λ is an appropriate UV cutoff. While the agnostic EFT physicist would just impose on Λ the upper bound in (4.2.1), the discussion of Section 4.2.3 implies that Λ should in fact satisfy much stronger upper bounds. The most conservative bound would be $\Lambda < m_*$, since the tower scale m_* represents the highest possible energy at which our four-dimensional EFT makes sense. However, in principle a wormhole with $L \lesssim m_*^{-1}$ could still make sense in a weakly-coupled higher-dimensional EFT. This is certainly not possible if L^{-1} exceeds the scale $\Lambda_{\rm QG}$. So, we will conservatively impose the bound

$$L^2 > \Lambda_{\rm QG}^{-2} \tag{4.4.28}$$

as the most fundamental consistency condition. We will come back to this point later on. Note that, in any case, once a wormhole solution exists these bounds can be always satisfied by picking a large enough charge $||\mathbf{q}||$.

4.4.3 Extremal wormholes and fundamental BPS instantons

So far in this section we have not really exploited supersymmetry, and what we have found would hold even if the kinetic matrices of ℓ_i and $\mathcal{H}_{3,i}$ were different from each other (see for instance the recent work [208]). On the other hand supersymmetry forces such matrices to be identical, see (4.4.1), and this allows for a particularly simple class of extremal $E = 0$ configurations.

Indeed, given the form of the potential (4.4.13), we see that taking $\dot{\ell}_i = q_i$ obviously satisfies (4.4.15) with $E = 0$ (another possible choice is $\dot{\ell}_i = -q_i$ and can be viewed as being associated to conjugate charges). The equations of motion (4.4.16) are manifestly solved as well. The condition can be easily integrated to $\boldsymbol{\ell}(\tau) = \boldsymbol{\ell}(\tau_\infty) + \mathbf{q}(\tau_\infty - \tau)$. The throat radius (4.4.19)

[23] Up to a different convention for the Planck scale, our semi-wormhole action agrees with the one first derived in [201]. We also agree with [7] provided no Gibbons-Hawking term is added at the wormhole throat (recall we are interested in 1/2 of the wormhole action and in such a calculation the Gibbons-Hawking term does not contribute to the on-shell action because the geometry of a full wormhole is asymptotically flat). We also agree with the result shown in eq.(12) of [203], up to a typo in the last equality. The expression for half a wormhole presented in [206, 207], however, includes a 1/2 in front and restricts the integration over half wormhole, and thus effectively corresponds to the action of a fourth of a wormhole.

tends to zero as $E \to 0$ and so the wormhole metric (4.4.6) becomes flat in the extremal limit, i.e. in the gauge (4.4.5) we have $e^{2A} = 1$.

In the present case, it is more convenient introduce a new proper radial coordinate

$$\hat{\tau} \equiv \tau_\infty - \tau\,, \tag{4.4.29}$$

which has opposite orientation with respect to τ and is such that $r = \infty$ corresponds to $\hat{\tau}_\infty = 0$, while $r = 0$ corresponds to $\hat{\tau}_* = \tau_\infty$. This is motivated by the fact that in the extremal solution $r = 0$ corresponds to a singular point, which is then moved to infinite $\hat{\tau}$-distance, and this simplifies our presentation. By solving (4.4.11) one then obtains

$$\hat{\tau} = \frac{1}{2\pi M_{\rm P}^2 r^2} \tag{4.4.30}$$

and

$$\boldsymbol{\ell}(r) = \boldsymbol{\ell}_\infty + \mathbf{q}\hat{\tau} = \boldsymbol{\ell}_\infty + \frac{\mathbf{q}}{2\pi M_{\rm P}^2 r^2}\,. \tag{4.4.31}$$

The nature of the singularity will be explored shortly. For the moment let us point out a suggestive coincidence. By recalling (2.2.4), one can easily derive the identity $\mathcal{G}^{ij} q_i q_j = \mathcal{G}^{ij} q_i \dot{\ell}_j = -q_i \dot{s}^i$ – in this section $\dot{\ell}_i \equiv \frac{{\rm d}\ell_i}{{\rm d}\hat{\tau}}$. Thus, paying attention to the relation (4.4.29), the on-shell action (4.4.26) of our extremal solution reads

$$S|_{\text{hw-extr.}} = 2\pi \int_{\hat{\tau}_\infty}^{\hat{\tau}_*} {\rm d}\hat{\tau}\, \mathcal{G}^{ij} q_i q_j = -2\pi \int_{\hat{\tau}_\infty}^{\hat{\tau}_*} {\rm d}\tau\, q_i \dot{s}^i = 2\pi q_i s^i_\infty - 2\pi q_i s^i_*\,. \tag{4.4.32}$$

This action reminds us of that of a BPS instanton but differs from it because of the term $-2\pi q_i s^i_*$. Yet, inspecting (4.4.31) we see that the dual saxions diverge as $r \to 0$. If the Kähler potential is as in (4.1.3), this implies that $\boldsymbol{s}_* \to 0$ in this limit. So, apparently, the two actions do in fact coincide. This coincidence suggests that perhaps one should view extremal wormholes as low energy manifestations of fundamental BPS instantons. But this interpretation, as it stands, is a bit naive. We will now support it with a more precise argument.

The singularity at $r = 0$ indicates that the expression (4.4.31) is not fully reliable. From a genuinely EFT perspective, that solution should be interpreted as viable only in a region $r \geq 1/\Lambda$ outside the origin, and the singularity should be regularized by some local counterterm placed around $r = 0$. So the integral in (4.4.32) should be limited to the region $r \geq 1/\Lambda$, hence giving the on-shell contribution

$$S^\Lambda_{\rm bulk} = 2\pi q_i (s^i_\infty - s^i_\Lambda), \tag{4.4.33}$$

with $s^i_\Lambda \equiv s^i(r = \Lambda^{-1})$, and should be supplemented by a cutoff dependent localized term $S^\Lambda_{\rm loc}$ contributing to the total action. Such a localized term cannot be determined by sole considerations of the low energy observer. [24] Fortunately we have crucial information about the UV. The analysis presented in appendix E of [64] shows that fundamental BPS instantons appear precisely in this way in the EFT: the introduction of a fundamental BPS instanton at $r \sim 1/\Lambda$ is captured by the addition of a localized term

$$S^{\Lambda}_{\text{loc}} = 2\pi q_i s^i_{\Lambda} = 2\pi\langle \mathbf{q}, \boldsymbol{s}_{\Lambda}\rangle\,. \tag{4.4.34}$$

This acts as a magnetic source for the potentials $\mathcal{B}_{2,i}$ and leads to the solution (4.4.31). Hence the complete on-shell action, including the localized term, exactly reproduces the expression

$$S_{\text{BPS}} = S^{\Lambda}_{\text{bulk}} + S^{\Lambda}_{\text{loc}} = 2\pi q_i s^i_{\infty} \equiv 2\pi\langle \mathbf{q}, \boldsymbol{s}_{\infty}\rangle \tag{4.4.35}$$

familiar from Section 4.2.1. Note that for fixed s^i_∞ the dependence of s^i_Λ and then of S^{Λ}_{loc} on Λ is dictated by the dual saxionic flow (4.4.31). This can be interpreted as an RG-flow of S^{Λ}_{loc}, as discussed in the same context but for strings and membranes in [51, 64]. This flow has an IR-fixed point which is given by $2\pi q_i s^i_\infty$, and of course coincides with (4.4.35), which is RG-flow invariant.

This observation strongly supports our conclusion that a subclass of fundamental BPS instantons can admit a description within the EFT in terms of extremal wormholes. As in [64], we will refer to such instantons as *EFT instantons*. Note also that all BPS instantons are mutually BPS and the derivation described in [64] makes it clear that multiple non-coincident EFT instantons admit an extremal multi-center wormhole description. In fact, there exists a more concrete criterion to distinguish EFT instanton from BPS but non-EFT ones [64]. We now revisit this criterion and present a refinement of its interpretation based on the discussion of Sections 2.2.1 and 4.2.3.

According to the discussion of Section 4.2.1, a BPS instanton must have charge $\mathbf{q} \in \mathcal{C}_{\text{I}}$. But EFT instantons must satisfy an additional requirement: the corresponding extremal wormhole profile (4.4.31) must lie within the perturbative regime along the radial trajectory all the way down to $r = \Lambda^{-1}$. As observed in Section 4.2.1, in the perturbative models in which the Kähler potential can be approximated by the form (4.1.3), $\mathcal{P}$ has conical shape. Since (4.4.31) describes a straight dual saxionic line generated by $\mathbf{q}$, we conclude that any $\mathbf{q} \in \mathcal{P}$ potentially identifies an EFT instanton. As in [64], we therefore define the set of charges of the EFT instantons as

$$\mathcal{C}^{\text{EFT}}_{\text{I}} \equiv \mathcal{P} \cap \mathcal{C}_{\text{I}}\,. \tag{4.4.36}$$

In general $\mathcal{P}$ may be non-convex, and hence the trajectory (4.4.31) may not be completely contained in $\mathcal{P}$ for any $\boldsymbol{\ell}_\infty \in \mathcal{P}$. However, if $\mathbf{q}$ is in the interior of $\mathcal{P}$ the solution certainly exists if $\boldsymbol{\ell}_\infty$ is chosen to be proportional to $\mathbf{q}$, and hence by continuity also for nearby choices of $\boldsymbol{\ell}_\infty$.

[24] Since $\Lambda/M_{\text{P}} \ll 1$ the dual saxion displacement is quite small and the requirement that the dual saxions are contained in our perturbative domain is easily ensured.

For the same reason, for any $\mathbf{q} \in \mathcal{C}_{\mathrm{I}}^{\mathrm{EFT}}$, one can judiciously adjust $\boldsymbol{\ell}_\infty$ in order to get a sensible extremal solution. On the other hand, as will be shortly discussed in detail, if a BPS instanton charge $\mathbf{q}$ is *not* EFT, specifically if $\mathbf{q} \in \mathcal{C}_{\mathrm{I}} - \mathcal{C}_{\mathrm{I}}^{\mathrm{EFT}}$, then the solution (4.4.17) breaks down at finite radial distance.

Consider the evolution of the tension (2.2.3) along the extremal wormhole flow (4.4.31):

$$\mathcal{T}_{\mathbf{e}}(r) = \mathcal{T}_{\mathbf{e}}^{\infty} + \frac{\langle \mathbf{q}, \mathbf{e} \rangle}{2\pi r^2}\,, \tag{4.4.37}$$

where $\mathcal{T}_{\mathbf{e}}^{\infty} \equiv \mathcal{T}_{\mathbf{e}}(\ell_\infty)$. Eq. (4.4.37) gives some important information about the consistency of the solution. We first note that the semiclassical description requires the (fixed) EFT cutoff Λ to be smaller than the (field dependent) scale Λ_{QG}, i.e. $\Lambda < \Lambda_{\mathrm{QG}}$. Now, while the EFT does not have direct information on Λ_{QG}, the upper bound (4.2.23) tells us that Λ should at least satisfy the non-trivial condition $\Lambda < \Lambda_{\max}$. In particular, it should certainly hold in the unperturbed vacuum, that is, one must impose that $\Lambda < \Lambda_{\max}^{\infty}$. Consider now (4.4.37), where $\mathbf{q} \in \mathcal{C}_{\mathrm{I}}$ while $\mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}$. This implies that $\langle \mathbf{q}, \mathbf{e} \rangle \geq 0$ and thus the tension (4.4.37) of EFT strings can never decrease along the extremal wormhole flow. We then arrive at the following important conclusion: if the asymptotic vacuum satisfies $\Lambda < \Lambda_{\max}^{\infty}$ then the condition $\Lambda < \Lambda_{\max}$ automatically holds along the entire flow, regardless of whether $\mathbf{q}$ is EFT or not. Furthermore, while $\Lambda_{\max}$ cannot generically be identified with Λ_{QG}, we expect these scales to have a similar behavior, in the sense that an increasing $\Lambda_{\max}$ should correspond to an increasing Λ_{QG} (though possibly with a different rate). This reasoning then leads to the conclusion that the stronger consistency condition $\Lambda < \Lambda_{\mathrm{QG}}$ holds everywhere as well. Let us now separately discuss the two cases corresponding to EFT and non-EFT BPS charges.

Suppose first that $\mathbf{q} \in \mathcal{C}_{\mathrm{I}}^{\mathrm{EFT}}$. As already argued above, this implies that (by possibly adjusting $\boldsymbol{\ell}_\infty$) the flow is entirely contained in $\mathcal{P}$. On the other hand, since $\langle \mathbf{q}, \mathbf{e} \rangle \geq 0$ for any BPS (non-necessarily EFT) string charge $\mathbf{e} \in \mathcal{C}_{\mathrm{S}}$, the tension (4.4.37) generically grows and formally diverges as $r \to 0$. Recalling that weakly and strongly coupled regions are detected by small and large elementary EFT string tensions (in Planck units), this suggests that the solution could exit the perturbative domain as one approaches the instanton. Fortunately, this is not necessarily the case, since the flow must stop at $r_\Lambda \equiv \Lambda^{-1}$ and as argued above we necessarily have $\Lambda < \Lambda_{\mathrm{QG}}^{\infty}$. We conclude that

$$\mathcal{T}_{\mathbf{e}} \leq \mathcal{T}_{\mathbf{e}}^{\infty} + \frac{\langle \mathbf{q}, \mathbf{e} \rangle}{2\pi r_\Lambda^2} \leq \mathcal{T}_{\mathbf{e}}^{\infty} + \frac{\langle \mathbf{q}, \mathbf{e} \rangle}{2\pi} (\Lambda_{\mathrm{QG}}^{\infty})^2\,. \tag{4.4.38}$$

Since we are assuming that the asymptotic vacuum is in the perturbative regime, we know that $\mathcal{T}_{\mathbf{e}}^{\infty} \ll M_{\mathrm{P}}^2$ and $\Lambda_{\mathrm{QG}}^{\infty} \leq \Lambda_{\max}^{\infty} \ll M_{\mathrm{P}}$. We then conclude that if $\langle \mathbf{q}, \mathbf{e} \rangle$ is not too large, that is if

$$\langle \mathbf{q}, \mathbf{e} \rangle \ll \frac{2\pi M_{\mathrm{P}}^2}{(\Lambda_{\mathrm{QG}}^{\infty})^2}\,, \tag{4.4.39}$$

then $\mathcal{T}_{\mathbf{e}} \ll M_{\rm P}^2$ and the EFT remains weakly coupled along the entire flow (up to r_Λ). Since $\Lambda_{\rm QG}^\infty \ll M_{\rm P}$ and we are assuming that the string charge is elementary, we conclude that (4.4.39) can indeed be satisfied for $\mathbf{q}$ small enough. Note also that we can always decompose a large charge instanton into smaller ones and separate them into a controlled extremal multi-wormhole solution.

Let us now consider an elementary BPS but non-EFT instanton charge $\mathbf{q} \in \mathcal{C}_{\rm I} - \mathcal{C}_{\rm I}^{\rm EFT}$. As already noticed in [64], in this case there must exists a BPS but non-EFT string charge $\mathbf{e} \in \mathcal{C}_{\rm S} - \mathcal{C}_{\rm S}^{\rm EFT}$ such that $\langle \mathbf{q}, \mathbf{e} \rangle < 0$. This means the corresponding tension (4.4.37) vanishes at $r_{\rm crit}^2 = \frac{|\langle \mathbf{q},\mathbf{e}\rangle|}{2\pi \mathcal{T}_{\mathbf{e}}^\infty}$, and becomes negative for $r < r_{\rm crit}$. Now, as recalled in Section 2.2.1, the vanishing of a non-EFT string tension signals the presence of strongly-coupled finite distance boundary. The key point is that elementary non-EFT strings are not directly linked to the quantum gravity aspects of the theory. In particular, their tension can be arbitrarily small compared to the Planck mass or the species scale, hence implying that along the extremal non-EFT wormhole one enters a strongly coupled region, and hence loses control, well before reaching the minimal allowed length scale $\Lambda_{\rm QG}^{-1}$.[25] This fits well with the idea that non-EFT BPS instantons are not gravitational in nature, but are rather associated with some strongly coupled non-gravitational sector with possible tensionless strings as in [97, 151].

We will see how, on the other hand, not only are EFT instanton charges $\mathbf{q} \in \mathcal{C}_{\rm I}^{\rm EFT}$ associated with controlled extremal wormholes, but also with smooth sub-extremal deformations thereof.

4.4.4 A non-standard BPS bound

Given the underlying supersymmetric structure of our EFT, one expects some kind of BPS bound relating the action of the (non-BPS) wormhole on-shell action and the BPS instanton action (4.4.35). Take any EFT instanton of charges $\mathbf{q} \in \mathcal{C}_{\rm I}^{\rm EFT}$ and consider a sub-extremal wormhole carrying the same charges. Recalling (4.4.15), we observe that the extremality bound (4.4.17) is equivalent to $\|\mathbf{q}\|^2 \geq \|\dot{\boldsymbol{\ell}}\|^2$. Employing (2.2.7) and (2.2.8) we then have

$$\|\mathbf{q}\|^2 \geq \|\mathbf{q}\| \|\dot{\boldsymbol{\ell}}\| \geq |q_i \mathcal{G}^{ij} \dot{\ell}_j| = |q_i \dot{s}^i| \,, \tag{4.4.40}$$

which is saturated only in the extremal BPS instanton case: $\dot{\boldsymbol{\ell}} = \mathbf{q}$. The action (4.4.26) is therefore subject to the following bound:

$$S|_{\rm hw} \geq 2\pi \int_0^{\tau_\infty} {\rm d}\tau \, \left| q_i \dot{s}^i \right| \geq 2\pi \left| q_i \int_0^{\tau_\infty} {\rm d}\tau \, \dot{s}^i \right| = \left| S_{\rm BPS} - 2\pi \langle \mathbf{q}, \boldsymbol{s}_* \rangle \right| , \tag{4.4.41}$$

where we have again adopted the convention (4.4.21). The bound involves in a crucial way the neck contribution $\langle \mathbf{q}, \boldsymbol{s}_* \rangle$. In the EFT instanton case, as we have seen in the previous

[25] A probably more precise way of identifying the exit from a weakly-coupled phase is provided by the violation of the EFT condition $\mathcal{T}_{\mathbf{e}} \geq \Lambda^2$, which then leads to the critical radius $r_{\rm crit}^2 = \frac{|\langle \mathbf{q},\mathbf{e}\rangle|}{2\pi(\mathcal{T}_{\mathbf{e}}^\infty - \Lambda^2)}$. We then see that for by taking Λ^2 closer to $\mathcal{T}_{\mathbf{e}}^\infty$ we have an arbitrarily large $r_{\rm crit}$.

section, that contribution is actually absent once one takes into account the counterterm induced by the insertion of the fundamental instanton at the singularity. On the other hand, in the sub-extremal wormhole case there is no singularity and no need for a local counterterm. A large enough $\langle \mathbf{q}, \boldsymbol{s}_* \rangle$ in that case may indicate that $S|_{\rm hw} < S_{\rm BPS}$, which would contradict the expectation that supersymmetric solutions saturate inequalities like (4.4.41). This puzzling relation has indeed been observed in some explicit wormhole solutions (see e.g. [14]) and will be encountered in the following section.

Perhaps the claim that sub-extremal wormholes can have actions smaller than the BPS solutions is a bit rushed, though. Other contributions, beyond the 2-derivative approximation, may not always be negligible. For example the Gauss-Bonnet term (4.4.2) contributes to the wormhole action. It does not however affect the (flat) BPS solution. Hence, once Gauss-Bonnet is taken into account one may recover a more familiar inequality of the type $S_{\rm BPS} < (S + S_{\rm GB})|_{\rm hw}$ despite of $S|_{\rm hw} < S_{\rm BPS}$. This is particularly relevant for EFTs with many axions, where we have seen that $S_{\rm GB}$ has a large coefficient. We may provide a suggestive perturbative argument in favor of the plausibility of our resolution of the puzzle. If we are allowed to treat the Gauss-Bonnet as a perturbation, at leading non-trivial order its effect on the equations of motion can be neglected when evaluating the on-shell action. Within this approximation $(S + S_{\rm GB})|_{\rm hw}$ is simply obtained by plugging in the leading order solution. The Gauss-Bonnet density of a (leading-order) wormhole is negative definite, $E_{\rm GB}|_{\rm wh} = -3L^8/(2\pi^2 r^{12})$, whereas the boundary term in (4.4.2) identically vanishes because of the flat asymptotic condition. Denoting by $\gamma_{\rm min}$ the minimum value attained by $\gamma(s)$ along the saxions flow, we find that the Gauss-Bonnet action of a semi-wormhole satisfies a lower bound

$$S_{\rm GB}|_{\rm hw} = -\int_{\mathcal{M}} \sqrt{g}\, \gamma(s)\, E_{\rm GB}|_{\rm wh} \geq -\gamma_{\rm min} \int_{\mathcal{M}} \sqrt{g}\, E_{\rm GB} = \gamma_{\rm min}\,. \tag{4.4.42}$$

Restricting our considerations to the controllable domain $\hat{\Delta}_\alpha$ we thus identify a sizable contribution $S_{\rm GB}|_{\rm hw} \geq N\pi/(6\alpha)$ that can conceivably result in $S_{\rm BPS} < (S + S_{\rm GB})|_{\rm hw}$. We will see that this in fact happens for some of the explicit wormhole solutions we find in Section 4.5.1.

4.5 Wormholes in the $\mathcal{N} = 1$ axiverse

Having introduced all the necessary machinery, we are finally ready to analyze explicit sub-extremal (non-supersymmetric) wormhole configurations. In Section 4.5.1 we introduce novel solutions involving an arbitrary number of axions and saxions. Crucial to their existence is the homogeneity of the function $\tilde{P}$ in eq. (4.1.4). In Section 4.5.3 we will discuss a particularly interesting subclass of these solutions. We will then draw some general lessons in Section 4.5.2.

4.5.1 The homogeneous solution

Let us now restrict our focus on the large class of models characterized by a Kähler potential determined by the dual saxion kinetic function (4.1.4) (or equivalently (4.1.3)). We would like to show that, under rather general conditions, the set of EFT instanton charges (4.4.36) admits a corresponding set of wormhole configurations, no matter how complicated $P(\boldsymbol{s})$ or $\tilde{P}(\boldsymbol{\ell})$ are, as long as they are homogeneous functions. In a certain sense, our result can be considered as a four-dimensional generalization of the string theory solutions found in [209], which also applies to many more string models.

The sharpest statements can be made for wormhole charges $\mathbf{q}$ belonging to the *interior* of the set (4.4.36), that is to

$$\mathcal{C}_{\rm WH} \equiv \mathcal{C}_{\rm I} \cap {\rm Int}\mathcal{P} \subset \mathcal{C}_{\rm I}^{\rm EFT} \,. \tag{4.5.1}$$

Since $\mathcal{P}$ is conical, the whole 'ray' generated by $\mathbf{q}$ is contained in $\mathcal{P}$. It is then natural to consider *homogeneous* wormhole configurations with

$$\boldsymbol{\ell}(\tau) = \tilde{\ell}(\tau)\,\mathbf{q} \,. \tag{4.5.2}$$

Note that the condition $\boldsymbol{\ell} \in \mathcal{P}$ simply reads $\tilde{\ell} > 0$. Inserting the ansatz (4.5.2) into (4.4.12) leads to

$$2\pi \int {\rm d}\tau \left[\frac{n}{4\tilde{\ell}^2}\left(\frac{{\rm d}\tilde{\ell}}{{\rm d}\tau}\right)^2 + \frac{n}{4\tilde{\ell}^2} \right] , \tag{4.5.3}$$

where we used the identity $\mathcal{G}^{ij}(\mathbf{q})q_i q_j = \frac{n}{2}$, which follows from degree-n homogeneity of $\tilde{P}$. Furthermore, one can verify that the homogeneity of $\tilde{P}$ guarantees that the equations of motion (4.4.16) for ℓ_i coincide with those derived from the effective action in eq. (4.5.3). What we have found is therefore a consistent multi-saxion wormhole with axionic charges $\mathbf{q}$.

We have basically reduced the problem to a simple axion-dilaton model with kinetic function

$$\tilde{\mathcal{F}} = n \log \tilde{\ell} \tag{4.5.4}$$

and (effective) unit charge, as the ones studied in refs. [201, 209]. Similar results also follow. In particular, since the reduced effective potential is

$$\tilde{V}(\tilde{\ell}) = -\frac{n}{4\tilde{\ell}^2} \,, \tag{4.5.5}$$

the wormhole radius (4.4.19) and the maximal proper time (4.4.23) are given respectively by

$$L^4 = \frac{n}{12\pi^2 \tilde{\ell}_*^2 M_{\rm P}^4} \qquad \Rightarrow \qquad \tau_\infty = \frac{\pi}{2}\sqrt{\frac{3}{n}}\,\tilde{\ell}_* \,. \tag{4.5.6}$$

We note in particular that L explicitly depends on $\tilde{\ell}_*$, but not on $\mathbf{q}$. The explicit profile $\tilde{\ell}(\tau)$ can be obtained by integrating directly (4.4.15) in our reduced one-saxion model. Imposing the boundary conditions $\tilde{\ell}(\tau_*) = \tilde{\ell}_*$ and $\mathrm{d}\tilde{\ell}/\mathrm{d}\tau(\tau_*) = 0$ fixes completely the solution:

$$\tilde{\ell}(\tau) = \tilde{\ell}_* \cos\left(\frac{\tau}{\tilde{\ell}_*}\right) = \tilde{\ell}_* \cos\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\,\frac{\tau}{\tau_\infty}\right) , \tag{4.5.7}$$

where the asymptotic value is implicitly determined by

$$\tilde{\ell}_\infty = \tilde{\ell}_* \cos\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right) . \tag{4.5.8}$$

Since by definition $\tilde{\ell}$ must be positive, eq. (4.5.7) describes a completely smooth wormhole if and only if

$$n > 3 . \tag{4.5.9}$$

This represents a lower bound on the coefficient of the scalar kinetic term first identified in [201, 209] for the case of simple dilatonic models, and then generalized in [14]. In all string theory models we are aware of, n is an integer taking values from 1 to 7 – see subsection 4.5.5. We hence have a wormhole regular everywhere only in models with $n = 4, 5, 6, 7$. Note that the wormhole corresponding to $n < 3$ become singular at finite distance from the wormhole throat, while the limit case $n = 3$ is singular only at spatial infinity. We will further investigate this case below in Section 4.5.3.

Our homogeneous solution can equivalently be expressed in terms of saxionic fields. These move along a straight line as well. Indeed from (2.2.8) and (4.1.6) we get

$$s^i(\tau) = \frac{\mathcal{F}^i(\mathbf{q})}{2\tilde{\ell}(\tau)} , \tag{4.5.10}$$

where $\mathcal{F}^i(\mathbf{q}) \equiv \partial\mathcal{F}/\partial\ell_i(\mathbf{q})$ is a charge dependent constant. Homogeneity also implies that

$$\langle \mathbf{q}, \boldsymbol{s}\rangle = \frac{n}{2\tilde{\ell}} . \tag{4.5.11}$$

Because s^i increases as we move away from the wormhole throat, the most constraining perturbative requirement on our solution comes from the internal region. In particular, by repeating the same argument leading to (4.2.18) with $\mathbf{q}$ now playing the role of $\tilde{\mathbf{C}}$, we get the lower bound $\langle \mathbf{q}, \boldsymbol{s}_*\rangle \geq c_{\mathbf{q}} N/\alpha$, where $c_{\mathbf{q}} \geq 1$ is some $\mathbf{q}$-dependent constant scaling as $c_{k\mathbf{q}} = k c_{\mathbf{q}}$. In terms of $\tilde{\ell}$, this reads

$$\tilde{\ell}(\tau) \leq \tilde{\ell}_* \leq \frac{n\alpha}{2c_{\mathbf{q}}N} . \tag{4.5.12}$$

By applying this upper bound to (4.5.6), we get

$$L^2 \geq \frac{c_{\mathbf{q}}}{\alpha n\pi\sqrt{3}}\frac{N}{M_{\mathrm{P}}^2} = \frac{c_{\mathbf{q}}}{\alpha n\pi\sqrt{3}}\frac{1}{(\Lambda_{\mathrm{sp,EFT}})^2} . \tag{4.5.13}$$

Hence for moderately small α and/or moderately large $\mathbf{q}$, L^{-1} is automatically smaller than low-energy length scale $\Lambda_{\rm sp,EFT}$ introduced in (4.2.1). While this is already reassuring, from the discussion of Section 4.2.3 it is clear that L^{-1} should actually be smaller than microscopic species scale $\Lambda_{\rm QG} \leq \Lambda_{\rm sp,EFT}$.

We can get more information about this point from the tension of EFT strings. By applying the general formula (2.2.3) to the present case, we get

$$\mathcal{T}_{\mathbf{e}}(\tau) = M_{\rm P}^2\, \tilde{\ell}(\tau)\langle \mathbf{q}, \mathbf{e}\rangle\,. \tag{4.5.14}$$

Notice that $\langle \mathbf{q}, \mathbf{e}\rangle > 0$ for any EFT (or even non-EFT) BPS string charge $\mathbf{e}$, since $\mathbf{q} \in \mathcal{C}_{\rm WH}$ – see (4.5.1). Recalling (4.2.23), we then get the following τ-dependent upper bound on the species scale

$$\Lambda^2_{\rm max}(\tau) = M_{\rm P}^2\, \tilde{\ell}(\tau) \min_{\mathbf{e}\in\mathcal{C}_{\rm S}^{\rm EFT}} \langle \mathbf{q}, \mathbf{e}\rangle\,. \tag{4.5.15}$$

This increases monotonically as one approaches the wormhole throat, $\Lambda^\infty_{\rm max} \leq \Lambda_{\rm max}(\tau) \leq \Lambda^*_{\rm max}$ with

$$\Lambda^\infty_{\rm max} = \Lambda^*_{\rm max} \cos\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right)\,, \tag{4.5.16}$$

which signals a similar behavior of the species scale $\Lambda_{\rm QG}(\tau)$. Notice also that $\cos\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right) \gtrsim 0.2$ for $n \geq 4$. Hence $\Lambda^*_{\rm max}$ is of the same order of $\Lambda^\infty_{\rm max}$, exceeding it at most by a factor of 5.

As already mentioned, a semiclassical wormhole configuration can make sense only if its throat's radius is larger than $\Lambda^{-1}_{\rm QG}$, and hence than $\Lambda^{-1}_{\rm max}$. On the other hand, the actual four-dimensional EFT cutoff Λ must be in principle smaller than the tower scale m_*. If m_* is associated with a KK tower, it can in turn be much smaller than $\Lambda_{\rm QG}$, and precisely in this case we may also have a large hierarchy between $\Lambda_{\rm QG}$ and $\Lambda_{\rm max}$.[26] Hence the wormhole solution that we have described, and in particular the formula (4.5.6), is actually reliable only if $L > m_*^{-1}$. Notice that, in fact, it is always possible to get a wormhole satisfying this condition by simply rescaling $\mathbf{q} \to k\mathbf{q}$ and $\tilde{\ell} \to k^{-1}\tilde{\ell}$ for large enough k, since $L^2 \to kL^2$ while the saxionic profile and hence all the microscopic energy scales do note change. However, if m_* is a KK tower scale, it may be interesting to consider also wormholes of smaller charges. These may be better described by a higher dimensional configuration with a corrected throat radius $L' = L + \Delta L$, which would still make sense at the semiclassical level.[27] In such a case, on purely dimensional grounds, we would expect the correction ΔL to be set by m_*^{-1} and to make L' *larger* than L. This reasoning leads to the conclusion that a condition of the form

$$L\Lambda^*_{\rm max} > 1 \tag{4.5.17}$$

[26] Assuming the ESC [70], the only other possibility is that m_* corresponds to the microscopic critical string scale, and then coincides with $\Lambda_{\rm QG} = \Lambda_{\rm max}$.

[27] Of course, here we are assuming that the wormhole survives the higher dimensional UV completion.

already provides a quite non-trivial criterion for the validity of a semiclassical wormhole configuration. In particular, in case of a hierarchy between $\Lambda_{\rm QG}$ and $\Lambda_{\rm max}$ we expect a corresponding significant correction ΔL which can help in realizing the actual semiclassical condition $L'\Lambda^*_{\rm QG} > 1$. Hence in the following we will just focus on (4.5.17).

Now, from (4.5.6) and (4.5.15) we get the relation

$$(L\Lambda^*_{\rm max})^2 = \frac{1}{2\pi}\sqrt{\frac{n}{3}}\min_{\mathbf{e}\in\mathcal{C}^{\rm EFT}_{\rm S}}\langle\mathbf{q},\mathbf{e}\rangle\,. \tag{4.5.18}$$

Notice that any explicit dependence on $\tilde{\ell}$ has disappeared from $L\Lambda^*_{\rm max}$, which instead depends linearly only on $\mathbf{q}$. The condition $L\Lambda^*_{\rm max} > 1$ then translates into the condition

$$\langle\mathbf{q},\mathbf{e}\rangle \geq 2\pi \quad \forall\mathbf{e}\in\mathcal{C}^{\rm EFT}_{\rm S}\,, \tag{4.5.19}$$

which is basically always satisfied. This conclusion provides an important and non-trivial consistency check on the universal reliability of our homogeneous wormhole solutions, even for small charges $\mathbf{q}$.

Having assessed the validity of our solution, we can finally discuss the on-shell action. Observing that $\|\mathbf{q}\|^2 = \frac{n}{2\tilde{\ell}^2}$, eq. (4.4.26) is easily computed:

$$\begin{aligned} S|_{\rm hw} &= \frac{n\pi}{\tilde{\ell}_*}\tan\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right) = 6\pi^2 M_{\rm P}^2 L^2\sqrt{\frac{n}{3}}\tan\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right) \\ &= 2\pi\sin\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right)\langle\mathbf{q},\boldsymbol{s}_\infty\rangle\,. \end{aligned} \tag{4.5.20}$$

The equivalent expression shown at the end of the first line serves as a consistency check of our result. In the regime $n\to\infty$ and $\tilde{\ell}_*\sim\sqrt{n}$ the saxion profile flattens while L stays finite, and consistently the on-shell action becomes the same as the purely-axionic scenario, as seen in the second line of (4.4.27). Yet, we should assign no physical meaning to this mathematical check because in such a limit the corresponding solution has $\ell_i \gg 1$ and is therefore well outside the regime of validity of our perturbative analysis. Indeed, so far string theory realizations have only allowed for $n\leq 7$. Scenarios with $n\leq 3$ have divergent actions, as expected by the considerations above. We will clarify the physics of the limiting case $n=3$ below in Sections 4.5.3 and 4.5.4.

Sticking for now to the controllable solutions with $n>3$, we observe that the last line of (4.5.20), where the relation (4.5.11) was used, implies $S|_{\rm hw} < S_{\rm BPS}$, explicitly realizing the puzzling possibility mentioned in subsection 4.4.4. The impact of the Gauss-Bonnet term on this inequality is quantified in Appendix B.4. It is found that – in the approximation discussed in Section 4.4.4 – the more conventional inequality $(S+S_{\rm GB})|_{\rm hw} > S_{\rm BPS}$ is recovered for $n=4,5,6$.

What we infer from this simple exercise is that small actions for non-supersymmetric solutions like the one of this subsection do not necessarily indicate that such configurations are more important than the supersymmetric ones, as one may have feared. In fact, corrections to those solutions from higher order terms cannot always be neglected.

4.5.2 Non-homogeneous generalization

Coming back to these wormholes solutions, we observe that $\boldsymbol{s}_*$, or equivalently $\boldsymbol{s}_\infty$, can be freely chosen to be any point of the ray in $\mathcal{P}$ generated by $\mathbf{q}$. This means that we can always choose $\boldsymbol{s}_*$ such that the entire trajectory lies in the perturbative region $\hat{\Delta}_\alpha$ (or $\tilde{\Delta}_\alpha$). This property actually comes from the more general scaling symmetry of the wormhole equations. Namely, if $\boldsymbol{\ell}(\tau)$ is a solution of the equations of motion then $\boldsymbol{\ell}'(\tau) = \lambda\boldsymbol{\ell}(\frac{\tau}{\lambda})$ $(\lambda > 0)$ is a solution too, which starts from $\boldsymbol{\ell}'_* = \lambda\boldsymbol{\ell}_*$ (at $\tau = 0$) and arrives in $\boldsymbol{\ell}'_\infty = \lambda\boldsymbol{\ell}_\infty$ (at $\tau'_\infty = \lambda\tau_\infty$). The corresponding saxionic flow is $\boldsymbol{s}'(\tau) = \frac{1}{\lambda}\boldsymbol{s}(\frac{\tau}{\lambda})$, which shows that we can always make such a rescaling in order to ensure that the flow is inside $\hat{\Delta}_\alpha$ (or $\tilde{\Delta}_\alpha$). We would then like to use this property to identify more general wormhole solutions corresponding to an EFT instanton charge $\mathbf{q} \in \mathcal{C}_{\rm WH}$. In order to understand this possibility, it is useful to discuss some properties of the metric $\mathcal{G}_{ij}$ and the potential $V_{\mathbf{q}}$.

First of all, the saxionic metric admits as particular geodesics the radial direction. This 'isotropy' is broken by the presence of the potential $V_{\mathbf{q}}(\ell)$, which in fact identifies $\mathbf{q}$ as preferred direction. Indeed, if we restrict along the $\mathbf{q}$ direction as in (4.5.2), we get $\mathcal{G}_{ij}\frac{\partial V_{\mathbf{q}}}{\partial \ell_j} = \frac{q_i}{\ell}$. Hence we see that along the radial direction identified by $\mathbf{q}$ the 'force' associated with $V_{\mathbf{q}}$ is directed along $-\mathbf{q}$. The sign can be understood by noticing that $V_{\mathbf{q}}(\lambda\ell) = \frac{1}{\lambda^2}V_{\mathbf{q}}(\ell)$ and then, since $V_{\mathbf{q}}$ is negative definite, it decreases as one moves radially towards the tip of $\mathcal{P}$. Correspondingly, $V_{\mathbf{q}}(\lambda s) = \lambda^2 V_{\mathbf{q}}(s)$ and then $V_{\mathbf{q}}(s)$ decreases as we move radially away from the tip of the saxionic cone Δ.

We can say something more about the shape of the potential $V_{\mathbf{q}}$. Take a particular point $\hat{\boldsymbol{\ell}}$ along the ray in $\mathcal{P}$ generated by $\mathbf{q}$ and the corresponding point $\hat{\boldsymbol{s}}$ along the ray generated by $\boldsymbol{s}|_{\boldsymbol{\ell}=\mathbf{q}} = \frac{1}{2}\frac{\partial\mathcal{F}}{\partial\boldsymbol{\ell}}(\mathbf{q})$,[28] and consider the saxionic plane $\mathcal{S}_{\hat{\boldsymbol{s}}}$ passing through it and 'orthogonal' to $\mathbf{q}$:

$$\mathcal{S}_{\hat{\boldsymbol{s}}} = \left\{ \boldsymbol{s} \in \Delta \middle| \langle \mathbf{q}, \boldsymbol{s}\rangle = \langle \mathbf{q}, \hat{\boldsymbol{s}}\rangle \right\} . \tag{4.5.21}$$

By using again the homogeneity, one can easily show that $V_{\mathbf{q}}(\hat{s}) = -\frac{1}{n}\left|\langle \mathbf{q}, \hat{\boldsymbol{s}}\rangle\right|^2 = -\frac{1}{n}\left|\langle \mathbf{q}, \boldsymbol{s}\rangle\right|^2|_{\mathcal{S}_{\hat{\boldsymbol{s}}}}$. On the other hand $\left|\langle \mathbf{q}, \boldsymbol{s}\rangle\right|^2 \leq \|\mathbf{q}\|^2\|\boldsymbol{\ell}\|^2 = -nV_{\mathbf{q}}(s)$. We deduce that

$$V_{\mathbf{q}}(\hat{s}) \geq V_{\mathbf{q}}(s)|_{\mathcal{S}_{\hat{\boldsymbol{s}}}} \tag{4.5.22}$$

[28] By $\frac{\partial\mathcal{F}}{\partial\boldsymbol{\ell}}(\mathbf{q})$ we mean the saxionic vector with components $s^i = \frac{\partial\mathcal{F}}{\partial\ell_i}(\mathbf{q})$. Since $\mathbf{q}$ belongs to $\mathcal{P}$, then by definition $\frac{\partial\mathcal{F}}{\partial\boldsymbol{\ell}}(\mathbf{q})$ belongs to Δ.

and then the potential restricted to $\mathcal{S}_{\hat{s}}$ has an absolute maximum at $\hat{\boldsymbol{\ell}}$. Hence $V_{\mathbf{q}}(s)$ is negative definite and has a hill shape with crest decreasing along $\boldsymbol{s}|_{\boldsymbol{\ell}=\mathbf{q}}$ (which belongs to Δ) or, correspondingly, $V_{\mathbf{q}}(\ell)$ has hill shape with crest decreasing $-\mathbf{q}$.

It is now clear why, if we start from a point $\boldsymbol{s}_*$ at rest along the ray identified by $\boldsymbol{s}|_{\boldsymbol{\ell}=\mathbf{q}}$, the point is radially driven away from the tip of the cone $\mathcal{P}$ along the hill's crest down to the point $\boldsymbol{s}_\infty$. This is precisely the saxionic counterpart of the radial solution described above and, as already discussed, we can rescale it in order to make it lie inside $\hat{\Delta}_\alpha$.

Suppose now to slightly move $\boldsymbol{s}_*$ away from the the ray generated by $\boldsymbol{s}|_{\boldsymbol{\ell}=\mathbf{q}}$. The potential will drive it further away the radial direction, but if the $\boldsymbol{s}_*$ is close enough to the initial position one should still get a sensible wormhole solution, ending at some $\boldsymbol{s}_\infty$ inside $\hat{\Delta}_\alpha$. The perturbative results of Appendix B.2 confirm this qualitative idea. Combining this observation with the above scaling symmetry, we then expect to be able to fine-tune the initial value $\boldsymbol{s}_*$ to reach any final point $\boldsymbol{s}_\infty$ inside $\hat{\Delta}_\alpha$. This is certainly true in the case of a diagonal metric (which corresponds to a completely factorized of the form $P(s) = (s^1)^{n_1}(s^2)^{n_2}(s^3)^{n_3}\ldots$, with $n_1 + n_2 + \ldots > 3$), which is a case already discussed in the literature – see for instance [14]. More generically, if the metric factorizes into decoupled saxionic sub-sectors (corresponding to a factorization of $P(s)$), then one can clearly construct non-radial solutions by freely combining radial solutions in each subsector – see appendix B.5 for more details. The above argument strongly suggest that this property holds also for more general models, with non-factorizable $P(s)$. We will provide further support for this claim in subsection 4.5.5.

This provides further support to the main claim of this section: if the perturbative regime is described by a Kähler potential (4.1.3) (or a kinetic potential (4.1.4)) associated with a homogeneous function $P(s)$ (or $\tilde{P}(\ell)$) of degree $n > 3$, then for each $\mathbf{q} \in \mathcal{C}_{\rm WH}$ there exists a corresponding smooth wormhole solution. Some comments are in order.

- The existence of the wormholes does not depend on the overall sign of $\mathbf{q}$. Hence a wormhole exists also for $-\mathbf{q} \in \mathcal{C}_{\rm WH}$. We will refer to it as *anti*-wormhole.
- The condition (4.5.1) excludes the cases in which $\mathbf{q} \in \mathcal{C}^{\rm EFT}_{\mathbf{I}}$ belongs to the boundary of $\mathcal{P}$. However, some boundary regions $\mathcal{P}' \subset \partial\mathcal{P}|_{\rm fin.\,dist.}$ are at *finite* distance in field space and can correspond to a restricted perturbative regime. In these cases, we can apply the same construction to this restricted perturbative regime.
- If instead a $\mathbf{q} \in \mathcal{C}^{\rm EFT}_{\rm I}$ belongs to an *infinite* distance boundary $\partial\mathcal{P}|_{\rm inf.\,dist.}$, then the homogeneous ansatz (4.5.2) does not make sense. However, the arguments of this section suggest that if we move $\boldsymbol{\ell}_*$ slightly away from the unphysical homogeneous direction, inside $\mathcal{P}$, than a wormhole may still exists. In particular, this is certainly true if $\mathcal{P}$ can be factorized into two subcones, e.g. $\mathcal{P} = \mathcal{P}_{(1)} \times \mathcal{P}_{(2)}$, with block-diagonal metric with corresponding factorized homogeneous polynomials $\tilde{P} = \tilde{P}_1\tilde{P}_2$, with at least one of degree ≥ 4 and $\mathbf{q}$ contained in it.

We can summarize the above conclusions on the more general wormholes as follows. First, it is natural to extend of the set (4.5.1) of wormhole charges as follows:

$$\mathcal{C}^{\rm ext}_{\rm WH} \equiv \mathcal{C}_I \cap ({\rm Int}\mathcal{P} \cup \partial\mathcal{P}|_{\rm fin.\,dist.}) = \mathcal{C}_{\rm WH} \cup (\mathcal{C}_I \cap \partial\mathcal{P}|_{\rm fin.\,dist.}) \tag{4.5.23}$$

Second, if $\mathbf{q} \in \mathcal{C}^{\rm ext}_{\rm WH}$ and $\tilde{P}(\ell)$ restricted to the ray (4.5.2) is homogeneous of degree $n_{\mathbf{q}} \geq 4$, then there should still exists wormhole solutions approaching any $\mathbf{s}_\infty \in \mathcal{P}$. Moreover, there could exist regular wormholes also for $\mathbf{q} \in \partial\mathcal{P}|_{\rm inf.\,dist.}$, under some additional conditions. Finally, the critical $n_{\mathbf{q}} = 3$ case is in fact quite interesting too, as we will be discuss in the following subsection.

4.5.3 Marginally degenerate case

It is instructing to more quantitatively discuss how the homogeneous wormholes with $n \leq 3$ fail to be globally well defined. This will highlight how the 'marginally' degenerate case $n = 3$ can, and in fact is, physically relevant.

We start by rewriting the universal solution (4.5.7) in terms of the three-sphere radius r. This can be more easily done by passing through the alternative proper parameter $\hat{\tau}$ introduced in (4.4.29). Indeed, from (4.4.22) and (4.4.23) we get the relation

$$\hat{\tau} = \frac{1}{2\pi M_{\rm P}^2 L^2} \arcsin\left(\frac{L^2}{r^2}\right) = \frac{2\tau_\infty}{\pi} \arcsin\left(\frac{L^2}{r^2}\right) \tag{4.5.24}$$

By using it in (4.5.7) we get

$$\tilde{\ell}(r) = \tilde{\ell}_* \cos\left[\sqrt{\frac{3}{n}}\left(\frac{\pi}{2} - \arcsin\left(\frac{L^2}{r^2}\right)\right)\right] . \tag{4.5.25}$$

This more manifestly shows that if $n \leq 3$ then $\tilde{\ell}(r)$ vanishes, and then the solution degenerates, at the radius

$$r_{\rm deg} = \frac{L}{\sqrt{\sin\left[\frac{\pi}{2}\left(1 - \sqrt{\frac{n}{3}}\right)\right]}} . \tag{4.5.26}$$

We will assume that in any physically sensible theory n is an integer, as clearly indicated by all string theory models n. Hence we have basically three possibilities: $n = 1, 2, 3$. From (4.5.26) one gets $r_{\rm deg}|_{n=1} \simeq 1.27L$ and $r_{\rm deg}|_{n=2} \simeq 1.88L$.

Physically, at these degeneration points the theory reaches the infinite distant tip $\boldsymbol{\ell} = 0$ of the dual saxionic cone. This means that all BPS string tensions (2.2.3) and $\Lambda_{\rm max}$ defined in (4.2.23) vanish. Note that all the other relevant four-dimensional energy scales $\Lambda \leq m_* \leq \Lambda_{\rm QG}$ vanish as well. Hence, for $n = 1, 2$ already at $r_{\rm deg} \simeq L$ the solution ceases to make any sense. This clearly indicates that the charges $\mathbf{q} \in \mathcal{C}^{\rm ext}_{\rm WH}$ with $n_{\mathbf{q}} = 1, 2$ can only be regarded as charges of fundamental BPS instantons of the type discussed in Section 4.4.3.

The story is very different for the $n = 3$ case. First of all, in this case $r_{\rm deg}|_{n=3} \simeq \infty$. So, in fact, the solution is everywhere well defined and degenerates only asymptotically at spatial infinity. Moreover, for $n = 3$ the profile (4.5.25) assumes a particularly simple form. Indeed (4.5.6) implies that

$$L^2 = \frac{1}{2\pi\tilde{\ell}_* M_{\rm P}^2} \tag{4.5.27}$$

and (4.5.25) reduces to

$$\tilde{\ell}(r) = \frac{1}{2\pi M_{\rm P}^2 r^2} \quad \Rightarrow \quad \boldsymbol{\ell}(r) = \frac{\mathbf{q}}{2\pi M_{\rm P}^2 r^2}\,. \tag{4.5.28}$$

We then see that, as a function of r, the dual saxionic profile is *identical* to extremal BPS profile (4.4.31), with vanishing (and hence degenerate) asymptotic saxionic value $\boldsymbol{\ell}_\infty = 0$. The only difference with the extremal case is in the metric, which has non vanishing warping (4.4.18), and in the radial range $r \in [L, \infty)$ (if we restrict to only half wormhole). This means that, even if the two solutions are not exactly identical, at a distance moderately larger then L they do not look much different. Already at a radius $r \geq 10L$, the two solutions differ by a relative error of order $\mathcal{O}(10^{-4})$! The degeneration $\boldsymbol{\ell} \to 0$ is of course again associated to an infinite field distance limit, but now it corresponds to infinite radius limit $r \to \infty$. In particular, all BPS string tensions (2.2.3) and $\Lambda^2_{\rm max}$ decrease as r^{-2}. This is in fact what one would have guessed just from dimensional analysis, if we regard r^{-1} as and energy scale.[29]

These considerations become particularly important if the wormhole solution is relevant only up to some infra-red length scale $r_{\rm IR} \gg L$. For instance, as one moves to large distances some low supersymmetry-breaking scale $\Lambda_{\rm SB}$ might become relevant, and stabilize the saxions to some finite value at a distance $r_{\rm IR} \simeq 1/\Lambda_{\rm SB}$. Or one may be interested in studying the wormhole contributions to the EFT with low sliding cutoff $\Lambda \ll 1/L$, as we will do in in Section 4.5.4. In such a case, one 'integrates-out the wormhole' only up to a distance $r_{\rm IR} = r_\Lambda \equiv 1/\Lambda$, and so the behavior of the wormhole solution at larger distances does not matter [5, 6, 209]. In the following, we will focus on the latter Wilsonian interpretation of the IR cutoff.

As an example, in the low energy effective action defined at $\Lambda \ll L^{-1}$ the contribution of the half-wormhole is represented by the insertion of the localized operator obtained by cutting-off the on-shell action (4.4.26) at the proper time $\tau_\Lambda = \tau_\infty - \hat{\tau}_\Lambda$, with $\hat{\tau}_\Lambda$ corresponding to r_Λ through (4.5.24). This gives

$$S|_{\rm hw} = \frac{3\pi}{\tilde{\ell}_\Lambda} \sin\left(\frac{\pi}{2}\frac{\tau_\Lambda}{\tau_\infty}\right) = 2\pi\langle \mathbf{q}, \boldsymbol{s}_\Lambda\rangle \sqrt{1 - L^4\Lambda^4} = 2\pi\langle \mathbf{q}, \boldsymbol{s}_\Lambda\rangle[1 + \mathcal{O}(\epsilon^2)]\,.\,, \tag{4.5.29}$$

where in the second step we have used (4.5.11) (with $n = 3$) and (4.5.24), and we have introduced

[29]From the saxionic viewpoint, (4.5.10) tells us that the flow may be regarded as a sort of EFT string flow. So (4.2.19) suggests that m_*^2 should generically decrease faster than $\Lambda^2_{\rm max}$ as r. However we again regard the species scale, rather than the tower scale, as the relevant one in the present setting. This is supported by the 'fake' BPS-ness of the $n = 3$ wormholes, which suggests a protection mechanism.

the expansion parameter

$$\epsilon_\Lambda \equiv L^2\Lambda^2 \ll 1 \qquad \Rightarrow \qquad \ell_\Lambda = \epsilon_\Lambda \tilde{\ell}_* \,. \tag{4.5.30}$$

For instance, we can take $r_\Lambda = 10L$, and hence $\epsilon_\Lambda = 10^{-2}$. We then see that, up to $\mathcal{O}(\epsilon_\Lambda^2)$ corrections, the cut-off on-shell action (4.5.37) is precisely the localized operator (4.4.34) accounting for the insertion of a BPS fundamental instanton! This confirms the above observation that, from far away, the $n = 3$ wormhole looks like a fundamental BPS instanton.

We can also study the effect of slightly moving $\boldsymbol{\ell}_*$ away from the choice (4.5.2). The general non-perturbative arguments of Section 4.5.2 and the perturbative results of Appendix B.2 clearly indicate that a tiny displacement of $\boldsymbol{\ell}_*$ transversal to $\mathbf{q}$ evolves to a large relative deviation along as we go to infinite radial distance.

As in Appendix B.2, let us consider a small deformation $\ell_i(r) \to \hat{\ell}_i(\tau) = \ell_i(r) + \delta\ell_i(r)$ of the dual saxionic flow. We can set

$$\delta\ell_i(r) = \tilde{\ell}(r) f_i(r) \quad , \quad f_i(r) \ll 1 \,, \tag{4.5.31}$$

with $f_i(r)$ representing the relative small deformations. Only transversal deformations are relevant, and one can impose the transversality condition (B.2.6) on the initial deformations f_{*i}, so that the wormhole radius L receives only second order corrections in f_{*i} – see (B.2.7).

By using (4.4.29) and (4.5.24) in (B.2.8) (with $n = 3$), one gets the relative deviation:

$$f_i(r) = g(r) f_{*i} \quad , \qquad g(r) \equiv 1 + \sqrt{\frac{r^4}{L^4} - 1}\left[\frac{\pi}{2} - \arcsin\left(\frac{L^2}{r^2}\right)\right] \,, \tag{4.5.32}$$

which clearly diverges for $r \to \infty$. However, for fixed large radius $r \geq r_\Lambda \gg L$ we have

$$g(r) = \frac{\pi r^2}{2L^2}\left[1 + \mathcal{O}\left(\epsilon_\Lambda^2\right)\right] \,. \tag{4.5.33}$$

and then the perturbative solution can still make sense up to this radius if f_{*i} is slightly smaller than L^2/r^2. In particular, since $r \geq r_\Lambda$ we necessarily have $f_{*i} < \epsilon_\Lambda$ and thus L^2 can be identified with (4.5.27) up to $\mathcal{O}(\epsilon_\Lambda^2)$ corrections.

By plugging (4.5.28) and (4.5.32) in (B.2.2), we get the absolute deviation

$$\begin{aligned} \delta\ell_i(r) &= \tilde{\ell}_* f_{*i} \, \frac{L^2 g(r)}{r^2} \\ &\simeq \frac{f_{*i}}{4M_{\rm P}^2 L^2}\left[1 + \mathcal{O}\left(\epsilon_\Lambda^2\right)\right] \qquad \text{for } r \geq r_\Lambda \,. \end{aligned} \tag{4.5.34}$$

Hence $\delta\ell_i(r)$ is asymptotically constant and then, for $r \geq r_\Lambda$, the complete perturbed solution can be written as

$$\begin{aligned}\hat{\boldsymbol{\ell}}(r) &\equiv \boldsymbol{\ell}(r) + \delta\boldsymbol{\ell}(\tau) = \tilde{\ell}\,(\mathbf{q} + g(\tau)\boldsymbol{f}_*) \\ &= \frac{\mathbf{q}}{2\pi M_{\rm P}^2 r^2} + \delta\boldsymbol{\ell}_\Lambda\left[1 + \mathcal{O}(\epsilon_\Lambda^2)\right] \qquad \text{with} \quad \delta\boldsymbol{\ell}_\Lambda \equiv \frac{\boldsymbol{f}_*}{4M_{\rm P}^2 L^2}\,.\end{aligned} \tag{4.5.35}$$

This clearly shows that, up to $\mathcal{O}(\epsilon_\Lambda^2)$ corrections, the perturbed solution still has the BPS form (4.4.31), but with a non-vanishing constant term $\delta\boldsymbol{\ell}_\Lambda$, which must be sufficiently small but otherwise completely arbitrary.

In the assumed approximation scheme, the cut-off on-shell action (4.4.26) is still given by (4.5.37). Indeed, by using the second expansion in (4.5.35) and the transversality condition (B.2.6), one can easily check that

$$\|\mathbf{q}\|^2(\hat{\boldsymbol{\ell}}) = \|\mathbf{q}\|^2(\boldsymbol{\ell})\left[1 + \mathcal{O}(\epsilon_\Lambda^2)\right]\,. \tag{4.5.36}$$

Correspondingly, $\langle \mathbf{q}, \hat{\boldsymbol{s}}\rangle = \langle \mathbf{q}, \boldsymbol{s}\rangle[1 + \mathcal{O}(\epsilon_\Lambda^2)]$, where $\hat{s}^i$ are the saxions corresponding to $\hat{\ell}_i$, as can be verified by using the saxionic expansion $\hat{s}^i = s^i - g(r)\mathcal{G}^{ij}(\boldsymbol{\ell})f_{*j}$ associated to (4.5.35) and (B.2.6). We conclude that the perturbed solution still has BPS-like cut-off action

$$S|_{\rm hw} = 2\pi\langle \mathbf{q}, \hat{\boldsymbol{s}}_\Lambda\rangle[1 + \mathcal{O}(\epsilon_\Lambda^2)]\,. \tag{4.5.37}$$

In the following Subsection we will provide numerical non-perturbative evidence that this conclusion holds also for larger deformations of the homogeneous solution.

It is also interesting to evaluate the contribution of the GB term to the on-shell action. Consider first the homogeneous case. The matching procedure amounts to computing (see Appendix B.4)

$$\begin{aligned}S|_{\rm GB,\,hw} &= \frac{\pi \tilde{C}_i \mathcal{F}^i(\mathbf{q})}{8\tilde{\ell}_*}\int_0^{x_\Lambda} {\rm d}x\,\cos^2 x = \frac{\pi^2 \tilde{C}_i \mathcal{F}^i(\mathbf{q})}{32\tilde{\ell}_*}\left[1 + \mathcal{O}(\epsilon_\Lambda^3)\right] \\ &= \frac{\pi^2\epsilon_\Lambda}{16}\langle \tilde{\mathbf{C}}, \boldsymbol{s}_\Lambda\rangle\left[1 + \mathcal{O}(\epsilon_\Lambda^3)\right]\,,\end{aligned} \tag{4.5.38}$$

where $x_\Lambda = \frac{\pi}{2} - \arcsin\epsilon_\Lambda$. We see that compared to the leading order action (4.5.37) the GB contribution is $\mathcal{O}(\epsilon_\Lambda)$ subleading. If we consider the perturbed solution (4.5.35), we must impose $f_{*i} < \epsilon_\Lambda$ and hence their correction to the GB contribution is always negligible. So, the corresponding GB correction is simply given by (4.5.38) with $\hat{\boldsymbol{s}}$ instead of $\boldsymbol{s}$.

Finally, being (4.5.38) of order $\mathcal{O}(\epsilon_\Lambda)$, for sufficiently small ϵ_Λ it always overwhelms a possible negative $\mathcal{O}(\epsilon_\Lambda^2)$ correction in (4.5.37), giving an overall positive correction to the BPS on-shell action $2\pi\langle \mathbf{q}, \hat{\boldsymbol{s}}_\Lambda\rangle$.

4.5.4 Marginally degenerate wormholes and BPS instantons

We would like now to explore in more depth the singular case $n = 3$. As a preliminary step, though, we note that, quite suggestively, sending $n \to 3^+$ and $\tilde{\ell}_* \propto 1/(n-3) \to \infty$ with $\boldsymbol{\ell}_\infty$ (or equivalently $\boldsymbol{s}_\infty$) held fixed one recovers from (4.5.7) the same solution as in (4.4.31). In addition, the on-shell action becomes the BPS one, as seen in the second line of (4.5.20). [30] This intriguing result suggests a correlation between homogeneous wormholes with $n = 3$ and BPS instantons with $\boldsymbol{\ell}_\infty \propto \mathbf{q}$. However, since the curvature length tends to zero (or equivalently $\mathbf{s}_* \to 0$) this limit is not fully under control physically. We thus need to look at this limit a bit more carefully.

With the boundary condition $\tilde{\ell}(\tau_\infty) = \tilde{\ell}_\infty$ kept fixed, the singular homogeneous solution (4.5.7) with $n = 3$ becomes progressively larger and larger as we approach the wormhole throat, and eventually we reach a scale at which the EFT cannot be trusted anymore. We should confine our considerations to distances $r > 1/\Lambda > \sqrt{N}/(4\pi M_{\rm P})$ away from the throat. For any $r \gg L > 1/\Lambda$ the proper time in the + branch of (4.4.22) approximates to

$$\tau = \frac{1}{4M_{\rm P}^2 L^2} - \frac{1}{2\pi M_{\rm P}^2 r^2}\left(1 + O(L^2/r^2)\right). \tag{4.5.39}$$

Hence, at a time $\tau = \tau_\infty - \Delta\tau$ our solution (4.5.7) can be written as

$$\begin{aligned}\tilde{\ell}(\tau) &= \tilde{\ell}_* \left[\cos\left(\frac{\tau_\infty}{\tilde{\ell}_*}\right)\cos\left(\frac{\Delta\tau}{\tilde{\ell}_*}\right) + \sin\left(\frac{\tau_\infty}{\tilde{\ell}_*}\right)\sin\left(\frac{\Delta\tau}{\tilde{\ell}_*}\right)\right] \\ &= \left[\tilde{\ell}_\infty + \Delta\tau\right]\left[1 + O(\Delta\tau/\tilde{\ell}_*)^2\right]\end{aligned} \tag{4.5.40}$$

where the correction can be ignored since

$$\frac{\Delta\tau}{\tilde{\ell}_*} \equiv \frac{\tau_\infty - \tau}{\tilde{\ell}_*} = \frac{L^2}{r^2}\left(1 + O(L^2/r^2)\right) \ll 1. \tag{4.5.41}$$

Eq. (4.5.40) exactly reproduces the form of a BPS instantons with $\boldsymbol{\ell}_\infty \propto \mathbf{q}$, see eq. (4.4.31), up to small corrections of order L^4/r^4. In other words, within the EFT the latter configuration is in fact indistinguishable from a homogeneous wormhole with $n = 3$. The difference, if any, must appear at scales as close as $r < 1/\Lambda$ to the throat, and is not visible at large distances.

The apparent loss of calculability in the far UV is just a red-herring. There is an alternative way to confirm that homogeneous $n = 3$ wormholes are actually BPS instantons and that avoids curvature singularities in the UV. Rather than fixing the asymptotic value of the field and letting $\tilde{\ell}_*$ grow as we approach the throat, as done so far, we now fix L to some reliable value and instead introduce an arbitrary IR cutoff $1/r_{\rm IR} \ll 1/L$. Restricting the integration to the domain $L < r < r_{\rm IR}$, and keeping n arbitrary for the moment, the on-shell action for half

[30] If we instead kept $\boldsymbol{\ell}_*$ (hence L) finite we would have obtained a divergent $\mathbf{s}_\infty$ as well as a divergent action. We will come back to this alternative perspective below.

a wormhole becomes

$$S|_{\text{hw}} = \frac{n\pi}{\tilde{\ell}_*}\tan\frac{\tau(r_{\text{IR}})}{\tilde{\ell}_*} = \frac{n\pi}{\tilde{\ell}_*}\tan\left(\sqrt{\frac{3}{n}}\arctan\sqrt{\frac{r_{\text{IR}}^2}{L^2}-1}\right). \tag{4.5.42}$$

For any finite curvature length L (and hence $\tilde{\ell}_*$), we see that this expression diverges as $r_{\text{IR}}/L \to \infty$ if $n = 3$, and only in that case [31]. But the solution at least does not develop any curvature singularity. One may superficially think that the IR divergence indicates that the associated wormhole configuration has no physical impact semi-classically, but this is not true [5]: at scales larger than a matching scale $r_\Lambda \gg L$ ($r_\Lambda \ll r_{\text{IR}}$) the effect of the wormhole is captured by a local operator of the form $Ce^{2\pi i\langle \mathbf{q},\mathbf{a}_\Lambda\rangle}$ [6], with $\mathbf{a}_\Lambda \equiv \mathbf{a}(\tau_\Lambda)$ and a *finite* coefficient C. One can compute the latter coefficient by matching the IR theory valid at $r > r_\Lambda$, where the geometry is essentially flat, with our earlier description of the wormhole. The result is approximately $C \propto e^{-(S|_{\text{hw}}-S_\Lambda)}$ [5], where S_Λ is the action evaluated with the IR description at lengths $r > r_\Lambda$ for configurations with axionic charges q_i equal to those of the wormhole. In both cases one observes the same IR divergence regulated by our IR cutoff r_{IR}, because IR divergences are insensitive to the UV. As a result $S|_{\text{hw}} - S_\Lambda$ is finite and approximately equal to our earlier result (4.5.42) with r_{IR} replaced by the matching scale r_Λ [5]. For $n = 3$ one obtains:

$$C \propto e^{-\frac{3\pi}{\tilde{\ell}_\Lambda}\sin\frac{\tau(r_\Lambda)}{\tilde{\ell}_*}} = e^{-2\pi\langle \mathbf{q},\mathbf{s}_\Lambda\rangle\sin\frac{\tau(r_\Lambda)}{\tilde{\ell}_*}} = e^{-2\pi\langle \mathbf{q},\mathbf{s}_\Lambda\rangle\left[1+O(L^4/r_\Lambda^4)\right]}, \tag{4.5.43}$$

where we used (4.5.7) to write $\tilde{\ell}_\Lambda \equiv \tilde{\ell}(\tau_\Lambda) = \tilde{\ell}_*\cos(\tau_\Lambda/\tilde{\ell}_*)$ as well as the relation $n/\tilde{\ell} = 2\langle \mathbf{q},\mathbf{s}\rangle$. The conclusion of this exercise is that the effect of a $n = 3$ wormhole is parametrized at energies below $1/r_\Lambda$ by adding to the action of the IR EFT a term $2\pi i\langle \mathbf{q},\mathbf{t}_\Lambda\rangle$ localized at the matching scale. Because we have argued earlier in Section 4.4.3 that BPS instantons are described exactly in the same way, we are lead to conjecture that *singular homogeneous wormholes with $n = 3$ are the low energy manifestation of a class of BPS instantons*.

4.5.5 Examples in string theory models

Let us now discuss some possible realizations of the above wormholes in string theory models.

Consider the F-theory models discussed in subsection 4.3.1. In this case case, we recall that the set $\mathcal{C}_{\text{I}}$ of BPS instanton charges can be identified with the cone of effective divisors. In order to identify the possible wormhole configurations, we must identify the set (4.4.36) of possible EFT instanton charges. Recalling the discussion around (4.3.12), we immediately conclude that $\mathbf{q} = q_a D^a \in \mathcal{C}_{\text{I}}^{\text{EFT}}$ is a nef divisor in some space $X' \sim X$. In the following we will for simplicity take $\mathbf{q} = q_a D^a$ to be nef already in X. Note also that we will mostly focus on toric models, or orientifold quotients of Calabi-Yau three-folds, in which the comments of footnote (14) hold and then $\mathbf{q} = q_a D^a \in \mathcal{C}_{\text{I}}^{\text{EFT}}$ can be regarded as movable divisors.

[31] When $n < 3$ the solution develops a singularity at a small distance from the throat. We will not consider these cases in the following.

Since the ℓ_a-sector alone has $n = 3$, in order to get everywhere regular wormhole solutions we are led to focus on Sen's weak coupling limit, in which we have the additional dual saxion $\hat{\ell}$, so that we can pass to $n = 4$ by turning on a corresponding charge $\hat{q} > 0$.[32] In this case we know that there certainly exists the family of homogeneous wormholes for any wormhole charge vector $(\hat{q}, \mathbf{q}) \in \mathcal{C}_{\rm WH}$ – see 4.5.1 – which means that $\hat{q} > 0$ and that $\mathbf{q} = q_a D^a$ represents an ample divisor in X. In these wormholes the dual saxions have a profile of the form $\hat{\ell}(\tau) = \hat{q}\,\tilde{\ell}(\tau)$ and $\boldsymbol{\ell}(\tau) = \ell_a(\tau) D^a = \mathbf{q}\,\tilde{\ell}(\tau)$, where $\tilde{\ell}(\tau)$ is as in (4.5.7) with $n = 4$.

According to the general argument, we also expect that there exist other wormhole solutions whose initial position $(\hat{\ell}_*, \boldsymbol{\ell}_*)$ is non-aligned but sufficiently closed to the radial direction identified by the charge vector $(\hat{q}, \mathbf{q})$. Since the scaling discussed below (4.5.20) identifies equivalent classes of solutions, it is convenient to rephrase our statement in terms of the vector

$$\boldsymbol{x}(\tau) \equiv x_a(\tau) D^a \equiv \frac{\hat{q}\,\boldsymbol{\ell}(\tau)}{\hat{\ell}(\tau)}\,, \tag{4.5.44}$$

which characterizes wormholes that are not equivalent under scaling symmetry. Note that the universal solution corresponds to the constant profile $\boldsymbol{x}(\tau) \equiv \mathbf{q}$ (hence with $\boldsymbol{x}_* = \boldsymbol{x}_\infty = \mathbf{q}$), while other equivalence classes of solutions correspond to any other profile $\boldsymbol{x}(\tau)$ completely contained in $\mathcal{P}_{\rm K}$ and with $\boldsymbol{x}_* \neq \mathbf{q}$. Not only do we expect to find admissible solutions for any $\boldsymbol{x}_*$ in a sufficiently small neighborhood of $\mathbf{q}$, but we also expect that we can tune $\boldsymbol{x}_*$ in this neighborhood to get any $\boldsymbol{x}_\infty \in \mathcal{P}_{\rm K}$. Note also that by the scaling symmetry, for any admissible solution $\boldsymbol{x}(\tau)$, we can always find a non-empty subset of corresponding flows $(\hat{\ell}(\tau), \boldsymbol{\ell}(\tau))$ that lie inside the α-saxionic convex hull.

We can more explicitly test our expectation by considering the simple models described in subsections 4.3.1 and in the appendices B.3.1, in which $\boldsymbol{x}(\tau)$ should belong to $\mathbb{R}^2_{>0}$ and $\mathbb{R}^3_{>0}$ respectively. Consider first the model of subsection 4.3.1. Ignoring for the moment the sector associated with the dilaton (4.3.14), the cone of BPS instanton charges is generated by the effective divisors E^1 and E^2. Hence, in the basis of nef divisors D^1, D^2 the components of a BPS instanton charge $\mathbf{q} = q_1 D^1 + q_2 D^2 \in \mathcal{C}_{\rm I}$ must satisfy $q_1 + n q_2 \geq 0$ and $q_2 \geq 0$. On the other hand, $\mathbf{q}$ belongs to the subset $\mathcal{C}^{\rm EFT}_{\rm I}$ of EFT instanton charges only if $q_1 \geq 0$ and $q_2 \geq 0$. For instance the charge vector $\mathbf{q} = E^2$, which corresponds to $(q_1, q_2) = (-n, 1)$, is BPS but not EFT. Including back the dilaton, according to our general claim there should exist a large family of wormhole solutions for charges in the set (4.5.1), that is $\hat{q}, q^1, q^2 > 0$, in addition to the corresponding homogeneous wormholes which certainly exist. One can numerically integrate the equations (4.4.16) for different choices of the twisting constant $n > 0$ (the case $n = 0$ reducing to a trivial factorized one), charges $\hat{q}, q^1, q^2 > 0$ and initial values (x_{*1}, x_{*2}). The results confirm our expectations, as for instance exemplified in figures 4.3 and 4.4. In particular, the plots show how, even if the allowed throat values (x_{*1}, x_{*2}) are confined to a sharp specific sub-region, the

[32] This is just the simplest choice but, for instance, we may also turn on charges corresponding to the (s)axionic sector appearing in some asymptotic region of the complex structure moduli space.

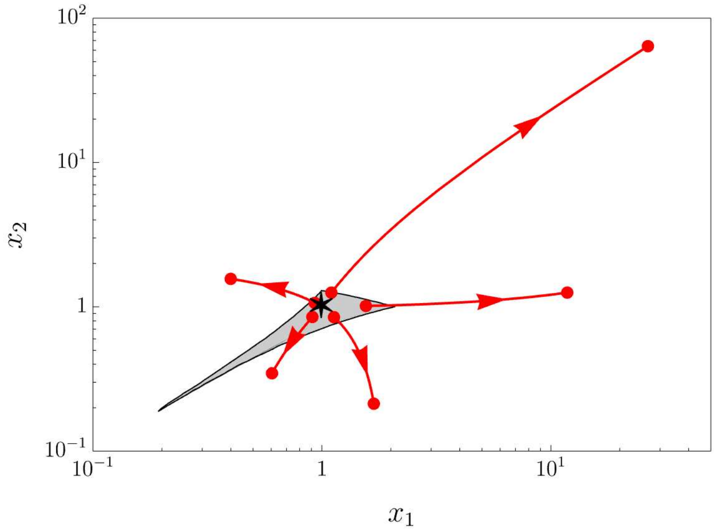
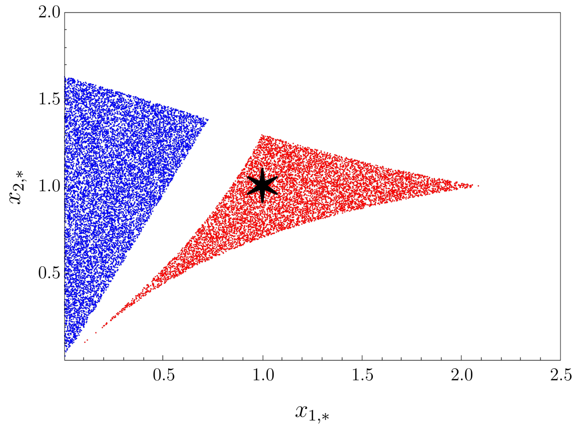


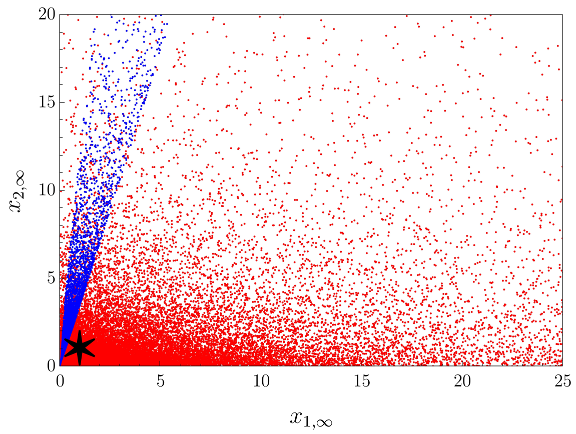
Figure 4.3: Sample $\boldsymbol{x}(\tau)$ trajectories with initial condition $\boldsymbol{x}_*$ displaced from the homogeneous solution, indicated by a star in the plot. The gray region is an interpolation of the red $\boldsymbol{x}_*$ acceptable domain of figure 4.4a. Small displacements around the star lead to completely different trajectories in a manner resembling the behavior of an unstable system.

corresponding asymptotic values $(x_{\infty 1}, x_{\infty 2})$ spread all over $\mathbb{R}^2_{>0}$, as predicted by our general arguments. Recalling that the boundary $\ell_1 = 0$ is at finite distance, we also expect to find wormhole solutions for charges $\hat{q}, q^2 > 0$ and $q^1 = 0$ around the homogeneous solution. This is indeed what we find, as shown in the plots of Fig. 4.5. Almost surprisingly, we also find solutions with $\hat{q}, q^1 > 0$ and $q^2 = 0$ where no homogeneous solution is in principle possible. A closer inspection of the plots reveals however how these solutions do not form a dense open set around the would-be homogeneous solution, but rather belong to the class of "accidental" solutions as for the ones of Fig. 4.4.

It is also interesting to explicitly check the correspondence between fundamental instantons and $n = 3$ wormholes of section 4.5.3 beyond the homogeneous and perturbative regime. We can do this by keeping only the ℓ_a sector, which is described by a $n = 3$ Kähler potential, and performing the usual numerical scan. Following the cut-off procedure outlined in section 4.5.3, we can test the equivalence by comparing the complete numerical action of the solutions to the associated BPS value (4.4.34). As on general grounds we expect all the solutions to degenerate at most at $r = \infty$, identifying valid solutions is somehow ambiguous. We therefore employ the empirical criterion that the dual saxions must degenerate at distances at least greater than r_Λ, where we assume that they will then be stabilized by some mechanism within the low-energy EFT. The result of the scan is reported in the plots of Fig. 4.6, where we both show the usual spread in the asymptotic values associated to deformations around the homogeneous solutions and the comparison between the complete and the BPS action. In the neighbourhood of the homogeneous solutions the actions coincide up to $\mathcal{O}(\epsilon_\Lambda^2)$, exactly as predicted by (4.5.37). Interestingly, a new region around $\ell_1 = 0$ also appears. This is not surprising as it closely



(a) Scanned throat values $\boldsymbol{x}_* \equiv \boldsymbol{x}|_{r=L}$.

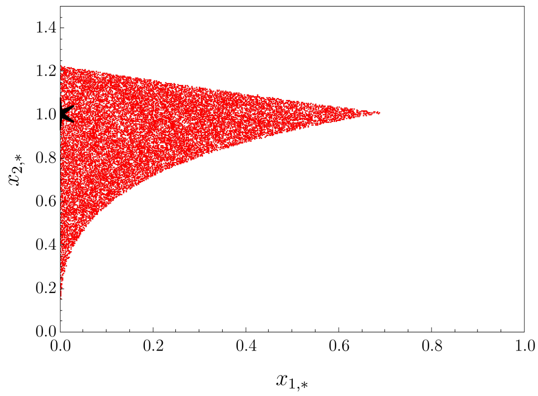


(b) Corresponding asymptotic values $\boldsymbol{x}_\infty \equiv \boldsymbol{x}|_{r=\infty}$.

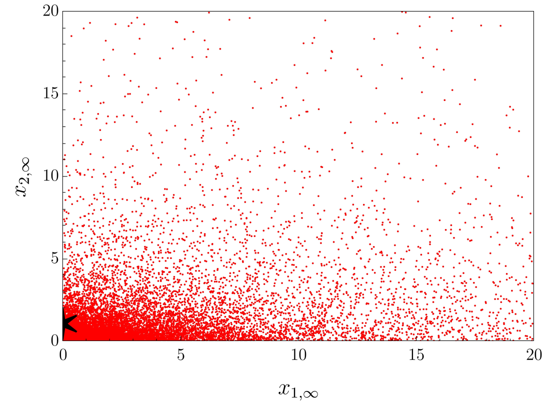
Figure 4.4: Numerical scan obtained by integrating the equations of motion with charge vector $\mathbf{q} = (1, 1, 1)$ in the F-theory model 2 with random initial conditions for the dual saxions. The first plot contains the allowed throat points $\boldsymbol{x}_*$ identified by the scan: all the points in the box has been randomly scanned, but we just show the ones that lead to acceptable solutions, i.e. with $x_1, x_2 > 0$ all along their trajectory. The shape of the domains of acceptable solutions are clearly distinguished and visible and have been indicated with different colors. The black star denotes the point corresponding to the radial solution and the red dots in its neighborhood can be identified with the solutions predicted by our general arguments, while the blue set on the left represents "accidental" solutions, not predicted by our argument. The second plot shows the asymptotic values $\boldsymbol{x}_\infty$ associated to the acceptable solutions. We can again clearly distinguish two sets of points, corresponding to the two disjoint sets of the first plot: the set corresponding to the red neighborhood of the black star in the first plot, which is spread out all over $\mathbb{R}^2_{>0}$, as predicted by our general argument; the set corresponding to the accidental solutions, which identifies the blue thin denser cone which is visible in the second plot.

resemble the situation of Figs. 4.4, 4.5. This time the agreement between the complete and BPS action is roughly of $\mathcal{O}(\epsilon_\Lambda)$.

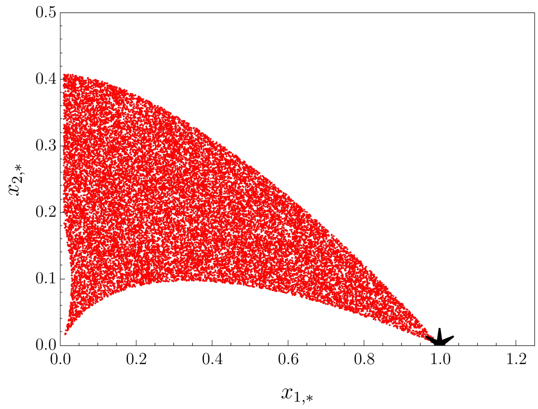
The model described in the appendix B.3.1 can be analyzed in the same way. In this case, in addition to the dilaton (4.3.14), we have a three-dimensional vector $\boldsymbol{\ell} = \ell_1 D^1 + \ell_2 D^2 + \ell_3 D^3$ that takes values in the Kähler cone. So, with focus on this sector, any $\mathbf{q} \in \mathcal{C}_{\rm I}^{\rm EFT}$ can be identified with the generic nef divisor $\mathbf{q} = q_1 D^1 + q_2 D^2 + q_3 D^3$, with $q_1, q_2, q_3 \geq 0$. Again, the set $\mathcal{C}_{\rm I}$ of BPS instanton charges is larger, and is generated by the effective divisors. In particular, if $n, h > 0$, the generators E^1 and E^3 identify BPS instanton charges $(q_1, q_2, q_3) = (1, -n, 0)$ and $(q_1, q_2, q_3) = (-h, 0, 1)$, respectively, which do *not* belong to $\mathcal{C}_{\rm I}^{\rm EFT}$. As in the previous model, including the dilaton, we expect that for any choice of charges in $\mathcal{C}_{\rm WH}$, that is $\hat{q}, q^1, q^2, q^3 > 0$, there should exist a large family of wormhole solutions reaching all possible asymptotic values $(x_{\infty 1}, x_{\infty 2}, x_{\infty 3}) \in \mathbb{R}^3_{>0}$. By numerically integrating (4.4.16) with initial conditions $(x_{\star 1}, x_{\star 2}, x_{\star 3}) \in \mathbb{R}^3_{>0}$, one gets another clear confirmation of our expectation as evident from the plots in figure 4.7.



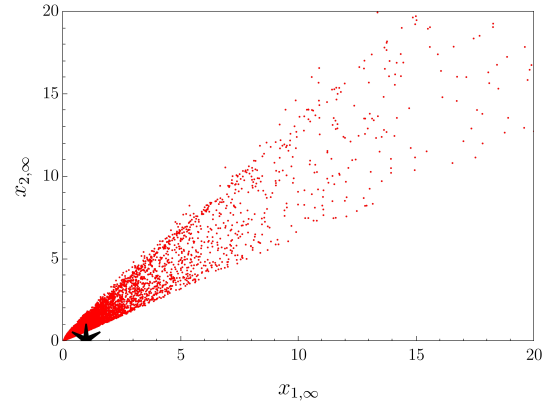
(a) Throat values $\boldsymbol{x}_* \equiv \boldsymbol{x}|_{r=L}$ of solutions with charge vector $\mathbf{q} = (1, 0, 1)$.



(b) Asymptotic values $\boldsymbol{x}_\infty \equiv \boldsymbol{x}|_{r=\infty}$ of solutions with charge vector $\mathbf{q} = (1, 0, 1)$.

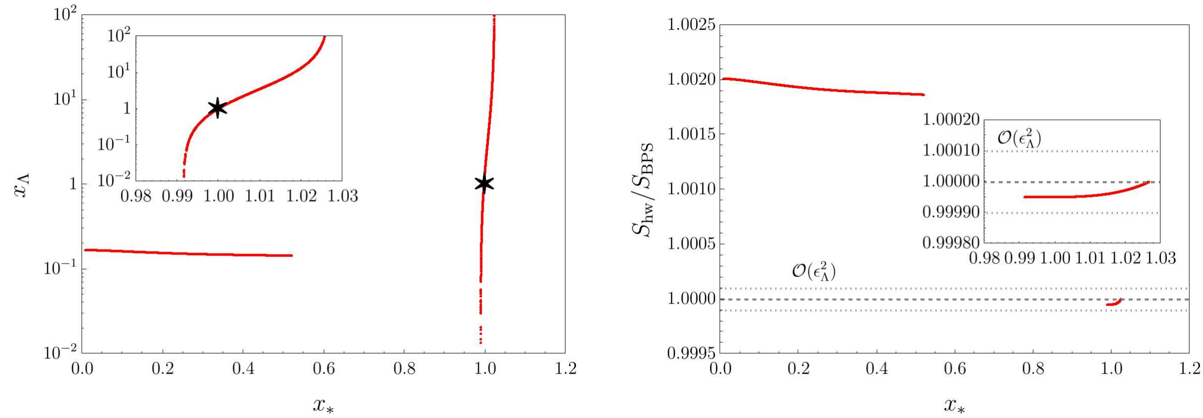


(c) Throat values $\boldsymbol{x}_* \equiv \boldsymbol{x}|_{r=L}$ of solutions with charge vector $\mathbf{q} = (1, 1, 0)$.



(d) Asymptotic values $\boldsymbol{x}_\infty \equiv \boldsymbol{x}|_{r=\infty}$ of solutions with charge vector $\mathbf{q} = (1, 1, 0)$.

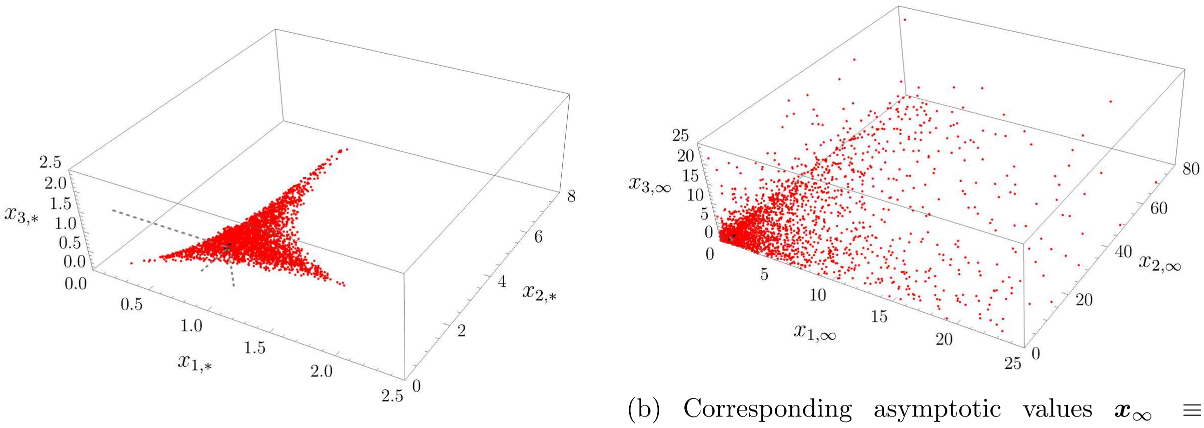
Figure 4.5: Same scan as in Figure 4.4 but with charges $\mathbf{q} = (1, 0, 1)$ and $\mathbf{q} = (1, 1, 0)$, as indicated in the captions of the plots. While the solution with vanishing ℓ_1 charge is expected, the one with vanishing ℓ_2 charge is apparently surprising as the point $\ell_2 = 0$ is at infinite distance. Indeed, a closer look at 4.5c reveals how getting closer to the point associated to the homogeneous solution (indicated as star in the plot) the domain shrinks so that the the homogeneous solution is never actually reached, confirming that $\ell_{2,*} = 0$ never delivers a valid configuration. This is a completely different behavior with respect to figure 4.5a, where the homogeneous solution has a "dense" neighbourhood of valid solutions.



(a) Scanned throat and asymptotic values of $x = \ell_1/\ell_2$.

(b) Ratio between the complete and the BPS action as a function of x_*.

Figure 4.6: Numerical scan obtained by integrating the equations of motion with charge vector $\mathbf{q} = (1, 1)$ in the F-theory model 2 without the dilaton and with random initial conditions for the dual saxions. The left panel compares the allowed throat values of $x_* = \ell_{1,*}/\ell_{2,*}$ to the asymtpotic values $x_\Lambda = \ell_{1,\Lambda}/\ell_{2,\Lambda}$ found by the scan, where acceptable solutions have been identified as the ones in which the dual saxions degenerate at a distance greater than $r_\Lambda = 10L$ from the throat. The black star denotes the point corresponding to the radial solution and the red dots in its neighborhood can be identified with the solutions predicted by the general perturbative argument, while the set on the left represents the usual accidental solutions. In this case, the set of points around the homogeneous solution is significantly more restricted compared to the accidental solutions one. The second plot shows the ratio $S_{\rm hw}/S_{\rm BPS}$ associated to the acceptable solutions, where $S_{\rm hw}$ is obtained by cutting off the integration up to r_Λ as outlined in the main text and $S_{\rm BPS}$ is given by (4.4.34) . Around the homogeneous solution the two actions are compatible up to $\mathcal{O}(\epsilon_\Lambda^2)$ terms as predicted by (4.5.37), while in the accidental region the deviation is slightly bigger (but still below $\mathcal{O}(\epsilon_\Lambda)$).



(a) Scanned throat values $\boldsymbol{x}_* \equiv \boldsymbol{x}|_{r=L}$.

(b) Corresponding asymptotic values $\boldsymbol{x}_\infty \equiv \boldsymbol{x}|_{r=\infty}$.

Figure 4.7: Numerical scan obtained by integrating the equations of motion with charge vector $\mathbf{q} = (1, 1, 1, 1)$ in the $\mathbb{P}^1$ over $\mathbb{F}_n$ F-theory model with random initial conditions for the dual saxions. As in figure 4.4, the first plot contains the acceptable throat points $\boldsymbol{x}_*$ identified by the scan. The shape of the domain of acceptable solutions is clearly visible and generates a dense set around the homogeneous solution, indicated as a black ball connected by dashed gray lines for clarity. The second plot shows the asymptotic values $\boldsymbol{x}_\infty$ associated to the acceptable solutions, that as predicted by our general argument spreads all over $\mathbb{R}^3_{>0}$.

Chapter 5

Conclusions

In this thesis we have presented the results of two main lines of work linked by the use of certain axionic, or EFT strings, previously introduced in [34, 51], as the main tools.

In Chapter 3 we provided a number of quantum gravity constraints on the effective action of an $\mathcal{N} = 1$ supergravity theory in four dimensions. The constraints arise, modulo certain assumptions, by demanding the consistent cancellation of the gauge and gravitational anomalies induced by inflow from the bulk on the worldsheet of the EFT strings. This logic is generally in the spirit of the analysis of [36, 37, 39, 49, 77–79] treating effective theories with more supersymmetry or in a larger number of dimensions.

Our derivation of the anomaly inflow and hence the quantum gravity bounds is valid for theories in which the gauge and gravity sector couple in a standard way to the axionic sector of the supergravity as in (2.1.11) and (2.1.19). For theories with this property, we have derived the positivity bounds and quantization conditions (3.2.3), (3.2.4) and (3.2.6) on the axionic couplings and furthermore argued for the bound (3.2.26) on the rank of the gauge sector coupling to the EFT strings. These bounds indeed rule out many supergravity theories as quantum gravity theories which would otherwise seem perfectly consistent, as we have demonstrated in simple settings in Section 3.3.1.

At a technical level the derivation of the bounds rests on the fact that the worldsheet theory of the EFT strings can be assumed to be weakly coupled. This is a consequence of the backreaction of the strings on the supergravity background in four dimensions [34, 51] and distinguishes EFT strings from intrinsically strongly coupled strings. At the same time, the class of EFT strings cannot be decoupled from the gravitational sector, and in this sense are similar to the supergravity strings in higher dimensions [36, 39]. This makes them ideal candidates to constrain the quantum gravity effective action.

Our derivations furthermore rest on the assumption, specified around (3.1.26), that the spectrum of the weakly coupled NLSM obeys a certain minimality principle: The $n_{\rm N}$ Fermi multiplets charged under the group of transverse rotations, $U(1)_{\rm N}$, are all assumed to participate in the lifting of a subset of the $n_{\rm C}$ chiral multiplets in such a way that at best the difference $n_{\rm C} - n_{\rm N}$ of remaining unobstructed chiral multiplets can contribute to the worldsheet anomalies via a Green-Schwarz type coupling. This assumption is natural in concrete string theory realizations of the effective theory. As one of the classes studied in more detail in this paper, we have analyzed EFT strings in F-theory compactifications by wrapping D3-branes on movable curves on the base of the elliptic Calabi-Yau four-fold. In this context, we have seen that the difference $n_{\rm C} - n_{\rm N}$ is a topological index; in the geometric subsector it corresponds precisely to the number of unobstructed geometric moduli which enter the NLSM. In fact, the minimality principle is manifestly satisfied for EFT strings obtained from D3-branes wrapping movable curves of genus $g = 0$, as in this case $n_{\rm N} = 0$. If the cone of movable curves is generated by rational curves, our assumptions are thus indeed realised. Incidentally, for Fano 3-folds this is the case (see e.g. [210] and references therein), and it is tempting to speculate if this pattern generalises to non-Fano base spaces of Calabi-Yau fourfolds with minimal holonomy. More generally, it would be important to better understand the validity of our working assumption (3.1.26) independently of concrete realizations in string theory.

In full generality, the quantum gravity bounds depend not only on the axionic couplings C and $\tilde{C}$ appearing in the four-dimensional effective field theory, (2.1.11) and (2.1.19), but also on a contribution to the worldsheet anomalies encoded in the quantity $\hat{C}$ as defined in (3.1.11). We propose that such contributions should be present only for EFT strings whose backreaction probes a five-dimensional substructure of the theory. Examples of such EFT strings can arise in the heterotic theory, for strings obtained by dimensional reduction of heterotic 5-branes along nef divisors with non-vanishing triple self-intersection on the Calabi-Yau threefold. In a general setting, the appearance of the coupling $\hat{C}$ could be used to deduce an underlying five-dimensional structure of the four-dimensional effective theory whenever a comparison of our bounds with the four-dimensional effective action implies that $\hat{C} \neq 0$. We believe that this interesting effect deserves further investigation.

The bounds on the rank of the gauge algebra in their general form (3.2.26) are oftentimes rather conservative. In the context of minimally supersymmetric F-theory on elliptic fibrations with a *smooth* base, we have proposed a sharper bound, (3.3.36), based on a detailed understanding of the NLSM massless fields in this case. On the other hand, the more general bound (3.2.26) can in fact be saturated for instance in heterotic orbifolds (and therefore also their F-theory duals, which necessarily involve a non-smooth base) and is hence to be regarded as the more general bound. Irrespective of this, it would be very important to verify if the sharper bound (3.3.36) indeed withstands scrutiny in F-theory on smooth three-fold bases, as our preliminary analysis in a handful of examples is currently suggesting. We believe that the analysis of EFT strings adds a fruitful new line of investigation to the active program [84–93] of constraining the possible effective field theories which can be obtained in F-theory. An important achievement would be to establish a universal bound on the rank of the gauge group in this class of minimally supersymmetric compactifications, as was possible for the abelian subsector in six dimensions in [37]. In view of (3.3.36), this challenge can be re-phrased as the task of finding a universal bound for the intersection product between a curve in the interior of the movable cone $\text{Mov}_1(X)$ and the anti-canonical class $\bar{K}_X$ on all possible three-fold base spaces X over which an elliptic Calabi-Yau can be constructed.

Finally our focus in concrete string realizations has been on strings probing the Kähler sector e.g. of F-theory or heterotic compactifications. By contrast, the sector of EFT strings probing the complex structure sector of these theories is far less understood. We leave an investigation of this sector of EFT strings as a challenge for future work.

In Chapter 4 we presented a detailed study of a class of homogeneous multi-saxionic wormhole solutions in $N = 1$ effective theories of gravity coming from String Theory compactifications, their non-homogeneous generalizations and begin to discuss their physical consequences on the low energy action. In doing so, we also delved into the topic of the energy scale of validity of such solutions and hence into the notion of species scale in our context.

The concept of EFT strings guided us in our inquiry, providing both a consistency bound on the coefficient of the relevant higher derivative correction to the on-shell action of the wormholes, the Gauss-Bonnet term, derived in Section 4.2.2 on the results of Chapter 3, an upper bound on the species scale through their tension and an organising principle for such bound in the neighbourhood of various infinite distance limits, as shown in Section 4.2.3. The issue of the energy cut-off is particularly important when dealing with geometric configurations controlled by a length scale, in this case the size of the wormhole throat, to assess their validity in the EFT regime and for this we dedicated resources to it in our discussion. We proposed the square of the corresponding EFT string tension as an upper bound on the scale of strongly coupled gravitational effects and motivated it both in EFT string infinite distance limits and the asymptotic region around them. This proposal was at the core motivated by the Emergent String Conjecture [70] together with the Scaling Weight Conjecture of [34], which imply that such a scale can be either the mass of the excitations of a fundamental string (for which a perturbative quantum description can be achieved and the bound is expected to be saturated) or of an extended object (for which a perturbative description cannot be achieved anyway). Several checks in explicit F-theory and heterotic compactifications confirmed our proposal.

We have described and analysed a wide class of Euclidean wormholes which are characterized by homogeneous dual saxion profiles and delved into their extensions thanks to both perturbative and numerical tools. The dual framework is specifically suited for simplifying the exploration of their features. We have proved the existence of such controlled solutions whenever the configuration carries an instanton charge in a peculiar subset of the dual of the set of possible EFT string charges. This both confirms and widens their role as the charges associated to perturbatively controlled regimes. It has already been proposed that EFT string charges are in biunivocal correspondence with weakly coupled asymptotic regimes in [34] and [35], but this result might point to them and BPS string charges in general being also crucial for the understanding of geometric configurations which can be described consistently within the EFT approximation. Such general statement is clearly outside the scope of our discussion, but it is nevertheless an intriguing possibility.

The existence of the above solutions is granted by the specific form of the kinetic term of the dual saxions, controlled by a Käler potential generated by a homogeneous intersection polynomial of degree 3 or greater. This condition is automatically satisfied in F-theory models when the axio-dilaton is taken into account. The marginal case of degree 3 is not without meaning: thanks to it leading to solutions which are well-defined everywhere outside infinity we were able to highlight an interesting connection to extremal BPS instantons. The study of the physical consequences of such wormholes in the marginal cases allowed us to entertain this possibility and propose that such wormholes might be seen as the low energy effective realizations of extremal instantons of the UV complete theory. It would be interesting to gather additional evidence for such claims in future work.

Finally we should stress that all of our discussion is based on the assumptions that we are in a minimal supersymmetric setting and that the saxions involved get a potential only through the effective one induced by the non-vanishing flux of the 2-form field on the compact patches of the wormhole. A possible future line of research, beyond the scopes of this thesis, is represented by the addition to the framework of the effects coming from supersymmetry breaking and an explicit potential, and check whether either of those are enough to change substantially our conclusions and proposals.

Appendix A

A.1 Conventions for 2d anomalies

Here we summarise our conventions for the computation of the two-dimensional gauge and gravitational anomalies – see for instance [211] for a review and references to the original literature.

The notion of positive and negative chirality fermions along the two-dimensional worldsheet of the string is induced from the four-dimensional bulk theory by identifying the two-dimensional chirality matrix γ_* with $-\Gamma_\alpha{}^\beta = -(\sigma_3)_\alpha^{\ \beta}$, using the notation introduced in Section 3.1.3. In matrix notation,

$$\gamma_* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.1.1}$$

and a two-dimensional Dirac spinor decomposes as

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} . \tag{A.1.2}$$

In order to be compatible with the terminology adopted in several related works, we will refer to the negative (positive) chirality spinors ψ_- (ψ_+) as left-moving (right-moving). (Admittedly, this may be confusing, since on-shell $\psi_\mp$ depends only on $y^{\mp\mp} \equiv y^0 \mp y^1$.)

The variation of the quantum effective action can be written as

$$\delta\Gamma \equiv (\delta_{\text{gauge}} + \delta_{\text{Lorentz}})\Gamma = 2\pi \int I_2^{(1)} , \tag{A.1.3}$$

where the polynomial $I_2^{(1)}$ obeys the descent relations

$$I_4 = \mathrm{d}I_3^{(0)} , \qquad \delta I_3^{(0)} = \mathrm{d}I_2^{(1)} . \tag{A.1.4}$$

In $2n$ dimensions, the anomaly polynomial I_{2n+2} associated with a complex positive-chirality Weyl fermion transforming in a representation $\mathbf{r}$ of the gauge group is given by the following general formula:

$$I_{2n+2} = [\mathcal{A}(M)\,\mathrm{ch}_{\mathbf{r}}(-F)]_{2n+2}\,, \tag{A.1.5}$$

where $\mathcal{A}(M)$ is the Dirac genus of the manifold M and $\mathrm{ch}(-F)$ the Chern character:

$$\begin{aligned}\mathcal{A}(M) &= 1 + \frac{1}{12(4\pi)^2}\,\mathrm{tr}\,R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{360}\,\mathrm{tr}\,R^4 + \frac{1}{288}(\mathrm{tr}\,R^2)^2\right] + \ldots \\ \mathrm{ch}_{\mathbf{r}}(-F) &= \mathrm{tr}_{\mathbf{r}}\exp\left(-\frac{1}{2\pi}F\right) = \dim\mathbf{r} - \frac{1}{2\pi}\,\mathrm{tr}_{\mathbf{r}}\,F + \frac{1}{2(2\pi)^2}\,\mathrm{tr}_{\mathbf{r}}\,F^2 + \ldots\,.\end{aligned} \tag{A.1.6}$$

In two dimensions, the gauge and gravitational anomaly generated by a single complex Weyl fermion with positive chirality (i.e. a right-moving fermion) is therefore encoded in

$$I_4 = \frac{1}{8\pi^2}\,\mathrm{tr}_{\mathbf{r}}\,F^2 + \frac{\dim\mathbf{r}}{192\pi^2}\,\mathrm{tr}\,R^2\,. \tag{A.1.7}$$

Notice that in this form the possibility of mixed abelian anomalies is automatically taken into consideration. In fact, if the gauge group G can be written as a direct product as in (2.1.5), the field strength has the form $F = \sum_A F_A + \sum_I F_I$ and we obtain

$$\begin{aligned}I_4 &= \frac{1}{8\pi^2}\,\mathrm{tr}_{\mathbf{r}}\,F^2 + \frac{\dim\mathbf{r}}{192\pi^2}\,\mathrm{tr}\,R^2 \\ &= \frac{1}{8\pi^2}\,\mathrm{tr}_{\mathbf{r}}(\sum_A F_A + \sum_I F_I)(\sum_B F_B + \sum_J F_J) + \frac{\dim\mathbf{r}}{192\pi^2}\,\mathrm{tr}\,R^2 \\ &= \frac{1}{8\pi^2}\sum_i q_i^A q_i^B F_A\wedge F_B + \sum_I\sum_k \frac{\ell(\mathbf{r}_k^I)}{16\pi^2}\,\mathrm{tr}\,F_I^2 + \frac{\dim\mathbf{r}}{192\pi^2}\,\mathrm{tr}\,R^2\,,\end{aligned} \tag{A.1.8}$$

where q_i^A denotes the $U(1)_A$ charge of the i-th component of the representation $\mathbf{r}$ and we have decomposed $\mathbf{r} = \bigoplus_k \mathbf{r}_k^I$ of $\mathbf{r}$ into G_I representations $\mathbf{r}_k^I$. In the second line we have used the trace $\mathrm{tr} \equiv \frac{2}{\ell(\mathbf{r})}\,\mathrm{tr}_{\mathbf{r}}$ introduced in section 2.1.1. A negative chirality (i.e. left-moving) complex fermion contributes with the opposite sign. Taking this into account and summing over all the possible contributions of (0,2) scalar and Fermi chiral multiplets, one gets (3.2.1) and (3.2.8).

A.2 Weakly coupled NLSMs on EFT strings

In this appendix we provide some non-trivial evidence that the NLSM supported on EFT strings can be treated as weakly coupled.

A.2.1 EFT strings in F-theory models

Consider an F-theory model of the type described in Section 3.3.2 and focus on the EFT strings corresponding to D3-branes wrapping movable curves. Note that by the EFT Completeness Conjecture the movable curves should admit an effective representative, which is furthermore expected to be generically smooth. The geometric moduli space $\mathcal{M}_{\rm geom}$ of such a movable curve $\Sigma \subset X$ should provide part of the target space $\mathcal{M}_{\rm NLSM}$ of the NLSM model supported by the EFT string. Other possible directions along $\mathcal{M}_{\rm NLSM}$ are represented by $SL(2,\mathbb{Z})$-twisted 'Wilson lines', which are part of the chiral fields $\Phi^{(2)}$ in the language of Section 3.3.2. By appropriately tuning the initial value of the moduli, the NLSM should remain weakly coupled along the EFT string flow. In this appendix we more explicitly check these expectations.

If we introduce some complex coordinates Z^i on X, φ^I on $\mathcal{M}_{\rm geom}$ and ζ along Σ, then the moduli space corresponds to a family of local embeddings $\zeta \mapsto Z^i(\zeta;\varphi)$. A first-order infinitesimal deformation of Σ corresponds to an element of $T\mathcal{M}_{\rm geom}|_\Sigma \simeq H^0(N\Sigma)$, where $N\Sigma = TX|_\Sigma/T\Sigma$. By introducing a basis ω_I of $H^0(N\Sigma)$ we can regard these elements as vectors $\omega_I = \omega_I^i\partial_i$ describing the deformations of the local embedding:

$$\delta Z^i = \omega_I^i(\zeta;\varphi)\delta\varphi^I\,. \tag{A.2.1}$$

The IIB Einstein frame spacetime metric takes the form

$$\mathrm{d}s^2 = e^{2A}\mathrm{d}s_4^2 + \ell_{\rm s}^2\mathrm{d}\hat{s}_X^2\,, \tag{A.2.2}$$

with $e^{2A} = \frac{M_{\rm P}^2\ell_{\rm s}^2}{4\pi V_X}$, where the dimensionless quantity V_X is the volume of X in string units (i.e. measured by $\mathrm{d}\hat{s}_X^2$, which we also take to be dimensionless).

Along the D3-brane world-volume $W\times\Sigma$, where W is a two-dimensional world-sheet in the external directions, we can use coordinates $\xi^A = (\sigma^\alpha, u^a)$, where $u^a = u^a(\zeta,\bar\zeta)$ are real coordinates along Σ. The complete embedding is then defined by $X^\mu(\sigma), Z^i(\zeta;\varphi(\sigma))$, and we can consider $\varphi^I(\sigma)$ as NLSM fields. The metric $\tilde h_{AB}$ induced on $W\times\Sigma$ splits as follows:

$$\begin{aligned}\mathrm{d}s^2|_{W\times\Sigma} &= \tilde h_{AB}\mathrm{d}\xi^A\mathrm{d}\xi^B = \tilde h_{\alpha\beta}\mathrm{d}\sigma^\alpha\mathrm{d}\sigma^\beta + 2\tilde h_{\alpha a}\mathrm{d}\sigma^\alpha\mathrm{d}u^b + \ell_{\rm s}^2\hat h_{ab}\mathrm{d}u^a\mathrm{d}u^b\\ &= \left[e^{2A}h_{\alpha\beta} + 2\ell_{\rm s}^2\hat g_{i\bar\jmath}(Z,\bar Z)\omega_I^i\overline{\omega}_{\bar J}^{\bar\jmath}\partial_\alpha\varphi^I\partial_\beta\bar\varphi^{\bar J}\right]\mathrm{d}\sigma^\alpha\mathrm{d}\sigma^\beta\\ &\quad + 2\ell_{\rm s}^2\left[\hat g_{i\bar\jmath}(Z,\bar Z)\omega_I^i\partial_\alpha\varphi^I\partial_b\bar Z^{\bar\jmath} + \text{c.c.}\right]\mathrm{d}\sigma^\alpha\mathrm{d}u^b + \ell_{\rm s}^2\,\hat h_{ab}\mathrm{d}u^a\mathrm{d}u^b\,,\end{aligned} \tag{A.2.3}$$

where $h_{\alpha\beta} \equiv g_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu$ is the pull-back to the EFT string world-sheet of the four-dimensional Einstein frame metric, and $\hat h_{ab} \equiv \hat g_{i\bar\jmath}(Z,\bar Z)\partial_a Z^i\partial_b\bar Z^{\bar\jmath} + \text{c.c.}$. Then

$$\det(\tilde h_{AB}) = \ell_{\rm s}^4\det\left[e^{2A}h_{\alpha\beta} + \ell_{\rm s}^2\left(\hat g^\perp_{i\bar\jmath}(Z,\bar Z)\,\omega_I^i\overline{\omega}_{\bar J}^{\bar\jmath}\partial_\alpha\varphi^I\partial_\beta\bar\varphi^{\bar J} + \text{c.c.}\right)\right]\det(\hat h_{ab})\,, \tag{A.2.4}$$

where

$$\hat g^\perp_{i\bar\jmath} \equiv \hat g_{i\bar\jmath} - \hat g_{i\bar k}\,\hat g_{l\bar\jmath}\,\hat h^{\bar\zeta\zeta}\bar\partial_{\bar\zeta}\bar Z^{\bar k}\partial_\zeta Z^l \tag{A.2.5}$$

is the projection of the metric in the orthogonal directions, in the sense that $\hat g^\perp_{i\bar\jmath}V^i = \hat g^\perp_{i\bar\jmath}\bar V^{\bar\jmath} = 0$ if $V^i\partial_i \in TD$ and $\hat g^\perp_{i\bar\jmath}V^i\bar V^{\bar\jmath} = \hat g_{i\bar\jmath}V^i\bar V^{\bar\jmath}$ if $V^i\partial_i \in T_D^\perp$. Neglecting the flux contributions, the

D3-action becomes

$$
\begin{aligned}
S_{\rm D3} &= -\frac{2\pi}{\ell_{\rm s}^4}\int_{W\times\Sigma} {\rm d}^4\xi\sqrt{-\det\tilde{h}_{AB}}+\ldots\\
&= -\frac{\mathcal{T}_\Sigma}{V_\Sigma}\int_W {\rm d}^2\sigma{\rm d}^2u\sqrt{-\det\left[h_{\alpha\beta}+\frac{2\pi V_\Sigma}{\mathcal{T}_\Sigma}\left(\hat{g}^{\perp}_{i\bar{\jmath}}(Z,\bar{Z})\,\omega^i_I\bar{\varpi}^{\bar{\jmath}}_{\bar{J}}\partial_\alpha\varphi^I\partial_\beta\bar{\varphi}^{\bar{J}}+{\rm c.c.}\right)\right]\det(\hat{h}_{ab})}+\ldots\,.
\end{aligned}
\tag{A.2.6}
$$

Here we have introduced the volume V_Σ of Σ in string units, and the string tension

$$
\mathcal{T}_\Sigma = \frac{2\pi e^{2A}V_\Sigma}{\ell_{\rm s}^2}\,. \tag{A.2.7}
$$

We could now tune the bulk moduli so that, at energies $E\leq\Lambda\ll m_*$, we can expand the square root appearing in the second line of (A.2.6). We now argue that the validity of this property is preserved, or even improved, along the flow (2.2.9) generated by the EFT string. Indeed, we can make the identifications

$$
m_*^2=\frac{e^{2A}}{\ell_{\rm s}^2L_\perp^2}\qquad\text{and}\qquad \hat{g}^{\perp}_{i\bar{\jmath}}=L_\perp^2\,\hat{g}^{\perp}_{0i\bar{\jmath}}\,, \tag{A.2.8}
$$

where $L_\perp$ is a length scale (in string units) associated with the directions orthogonal to Σ. For instance, we may identify $L_\perp^4\sim V_D$, where D is some effective divisor with $D\cdot\Sigma\geq1$. This implies that $L_\perp^2$ scales as $\sqrt{\sigma}$ along the EFT string flow (2.2.9) and m_* can be identified with the lightest KK scale. Hence

$$
\frac{2\pi V_\Sigma}{\mathcal{T}_\Sigma}\hat{g}^{\perp}_{i\bar{\jmath}}=\frac{1}{m_*^2}\,\hat{g}^{\perp}_{0i\bar{\jmath}}\,. \tag{A.2.9}
$$

If we pick local inertial coordinates σ^α in which $h_{\alpha\beta}\simeq\eta_{\alpha\beta}$, at energy scales of order E the second term under the square root in the second line of (A.2.6) is of order $E^2/m_*^2\ll1$.

We can then expand the square root, getting

$$
S_{\rm D3}\simeq-\mathcal{T}_\Sigma\int_W{\rm d}^2\sigma\sqrt{-\det h_{\alpha\beta}}-\int_W{\rm d}^2\sigma\sqrt{-\det h_{\alpha\beta}}\,\mathcal{G}_{I\bar{J}}(\varphi,\bar{\varphi})h^{\alpha\beta}\partial_\alpha\varphi^I\partial_\beta\bar{\varphi}^{\bar{J}}+\ldots \tag{A.2.10}
$$

The first term is the standard contribution of the universal sector. The leading correction to takes the form of an NLSM, with target space metric

$$
\mathcal{G}_{I\bar{J}}\equiv2\pi\int_\Sigma{\rm d}^2u\sqrt{\det\hat{h}_\Sigma}\,g^{\perp}_{i\bar{\jmath}}(Z,\bar{Z})\,\omega^i_I\bar{\varpi}^{\bar{\jmath}}_{\bar{J}}\,. \tag{A.2.11}
$$

It remains to investigate the scaling of this target space metric along the EFT flow, which depends also on the scaling of $\hat{h}_\Sigma$. This can be understood by using the classification of possible EFT string flows provided in [71]. By adopting the terminology of this reference, one can distinguish three main cases labelled by an integer $q=0,1,2$. These are associated with so-called quasi-primitive EFT strings, which form the building blocks of EFT string limits; generalisations beyond these quasi-primitive limits are then possible along the lines of [71].

In limits of type $q = 0$, X can be regarded as a $\mathbb{P}^1$ fibration over a base two-fold B, and Σ can be identified with the $\mathbb{P}^1$ fiber. This EFT string is dual to a heterotic fundamental string moving in a Calabi-Yau which is elliptically fibered over B. In the dual heterotic description it is clear that an EFT string flow is compatible with a weakly coupled regime – see section 3.1.3. This can also be understood from the F-theory viewpoint, at least if we focus on the geometric moduli space $\mathcal{M}_{\text{geom}}$. Indeed, any vertical divisor in X has vanishing intersection with Σ and then, asymptotically, does not scale with σ. On the other hand, the base volume scales like σ. This implies that $\hat{h}_\Sigma$ scales like $\frac{1}{\sqrt{\sigma}}$, and then that $\mathcal{G}_{I\bar{J}}$ does not scale. This is consistent with the identification of $\mathcal{G}_{I\bar{J}}$ with the string frame metric of base B in the dual heterotic description.

In the case $q = 1$, X can be identified with the fibration of a surface S over a $\mathbb{P}^1$ base, and Σ is a movable curve inside the S fibre with $(\Sigma \cdot \Sigma)_S \geq 1$. In this case $S \cdot \Sigma = 0$ and then the volume of S does not scale. Consider now the effective divisor D_Σ of X obtained by fibering Σ - or in fact any curve in the fiber S with non-vanishing intersection with Σ - over the base $\mathbb{P}^1$. Hence $\Sigma \cdot D_\Sigma = (\Sigma \cdot \Sigma)_S \geq 1$ and the volume of D_Σ scales as σ along the EFT string flow. The analysis of [71] shows that V_Σ is constant and the base $\mathbb{P}^1$ volume scales as σ. From (A.2.11) we see that in this case $\mathcal{G}_{I\bar{J}}$ scales as σ. Note that this case corresponds to $w = 2$.

Finally, consider the case $q = 2$. This case describes a homogeneous decompactification. Hence, $V_\Sigma \sim \sqrt{\sigma}$ and $g^{\perp}_{i\bar{\jmath}} \sim \sqrt{\sigma}$ and then $\mathcal{G}_{I\bar{J}}$ scales as σ. Also in this case the scaling weight is $w = 2$.

In summary, we have found that

$$\mathcal{G}_{I\bar{J}} \sim \sigma^{w-1} \mathcal{G}^0_{I\bar{J}} \,. \tag{A.2.12}$$

Hence, if $w = 1$ we can either choose the NLSM to be weakly coupled or even exactly quantize it, as in perturbative superstring theory, since $g_{\text{s}} = e^\phi$ is automatically driven to zero. If $w > 1$, the EFT string does not uplift to a critical string but the corresponding flow automatically drives the bulk moduli to a large distance limit in which the $\mathcal{M}_{\text{geom}}$ sector of the NLSM is weakly coupled. By applying similar scaling arguments to the complete DBI-action, one can extend these conclusions to the twisted Wilson-line sector as well.

A.2.2 EFT strings in heterotic models

We now consider the EFT strings of the heterotic models discussed in Section 3.3.3.

Take first an EFT string that uplifts to a heterotic F1 string. Its flow is compatible with a weak coupling regime, since along it the (string frame) Kähler moduli do not change, while the ten-dimensional string coupling goes to zero [34]. Hence one may even exactly quantize it in superstring perturbation theory.

Let us then turn to an EFT string corresponding to an NS5-brane wrapping a nef divisor $D \subset X$. In this case g_{s} can increase along the EFT string flow – see [34] and below – and then it is convenient to uplift the model to an HW M-theory compactification on $I \times X$, with $I = S^1/\mathbb{Z}_2$. The NS5-brane uplifts to an M5-brane wrapping $\{\hat{y}\} \times D \subset I \times X$ and, as discussed in section 3.3.3, may require the insertion of open M2-branes. The M5's geometric moduli space can be identified with $I \times \mathcal{M}_D$, where $\mathcal{M}_D$ is the moduli space of $D \subset X$. Any nef divisor D is expected to be automatically effective, as conjectured in [39]. One can more precisely describe $\mathcal{M}_D$ as in [39, 118], but we will not need such a description.

We will focus on the NLSM description of the M5's geometric moduli space $I \times \mathcal{M}_D$, proceeding as for the D3-branes discussed in Appendix A.2.1. We introduce some complex coordinates z^i on X, φ^I on $\mathcal{M}_D$ and $\zeta^{\mathcal{I}}$ along D. $\mathcal{M}_D$ describes a family of local embeddings $\zeta^{\mathcal{I}} \mapsto Z^i(\zeta;\varphi)$ and a first-order infinitesimal deformation of D corresponds to an element of $T\mathcal{M}|_D \simeq H^0(ND)$, where $ND = TX|_D/TD \simeq \mathcal{O}_X(D)|_D$. The elements of a basis $\omega_I = \omega^i_I \partial_i$ of $H^0(ND)$ describe the deformations of the local embedding as in (A.2.1).

As in section 3.3.3, we set the M-theory Planck length $\ell_{\rm M} \equiv \ell_{\rm s}$ and parametrize $I \equiv S^1/\mathbb{Z}_2$ by a coordinate $y \simeq y + 2$, with $y \simeq -y$. We can then restrict to $y \in [0,1]$ with $y = 0, 1$ hosting the ten-dimensional E_8 gauge sectors. The eleven dimensional M-theory metric takes the form

$$\mathrm{d}s^2_{11} = e^{2A}\mathrm{d}s^2_4 + \ell^2_{\rm M}\left(\mathrm{d}\hat{s}^2_X + e^{\frac{4\phi}{3}}\,\mathrm{d}y^2\right) \tag{A.2.13}$$

with

$$e^{2A} = \frac{\ell^2_{\rm M} M^2_{\rm P} e^{-\frac{2\phi}{3}}}{4\pi \hat{V}_X}\,, \tag{A.2.14}$$

where $\hat{V}_X$ is the volume of X in eleven-dimensional Planck units. We can identify e^{ϕ} with the heterotic string coupling and $e^{\frac{2\phi}{3}}$ with the length of the HW interval. Here we are neglecting subleading backreaction effects due to non-trivial gauge bundles or G_4 field strength, and to bulk M5-branes, since they will not be relevant in the following.

An M5-brane has world-volume $\Gamma = W \times \{y = \hat{Y}\} \times D$, on which we introduce adapted real coordinates $\xi^A = (\sigma^\alpha, u^a)$ (where $u^a = u^a(\zeta,\bar{\zeta})$). The embedding is defined by $X^\mu(\sigma), \hat{Y}(\sigma)$, $Z^i(\zeta;\varphi(\sigma))$, and we can consider $X^\mu(\sigma), \hat{Y}(\sigma), \varphi^I(\sigma)$ as NLSM fields. By repeating the same steps followed in Appendix A.2.1, the geometrical sector supported by the M5-brane is described by the effective action

$$\begin{aligned} S_{\rm M5} &= -\frac{2\pi}{\ell^6_{\rm M}}\int_\Gamma \mathrm{d}^6\xi\sqrt{-\det g|_\Sigma} + (\text{flux terms}) \\ &= -\frac{\mathcal{T}_D}{\hat{V}_D}\int_\Gamma \mathrm{d}^6\xi\sqrt{-\det\left\{h_{\alpha\beta} + \frac{2\pi\hat{V}_D}{\mathcal{T}_D}\left[e^{\frac{4\phi}{3}}\partial_\alpha\hat{Y}\partial_\beta\hat{Y} + \left(\hat{g}^{\perp}_{i\bar{\jmath}}(Z,\bar{Z})\,\omega^i_I\bar{\omega}^{\bar{\jmath}}_{\bar{J}}\partial_\gamma\varphi^I\partial_\beta\bar{\varphi}^{\bar{J}} + \text{c.c.}\right)\right]\right\}} \\ &\qquad\qquad \cdot\sqrt{\det\hat{h}_{ab}} + (\text{flux terms})\,. \end{aligned} \tag{A.2.15}$$

Here we have introduced the projection of the Calabi-Yau metric to the orthogonal directions

$$\hat{g}^{\perp}_{i\bar{\jmath}} \equiv \hat{g}_{i\bar{\jmath}} - \hat{g}_{i\bar{k}}\,\hat{g}_{l\bar{\jmath}}\,\hat{h}^{\bar{\mathcal{I}}\mathcal{J}}\bar{\partial}_{\bar{\mathcal{I}}}\bar{Z}^{\bar{k}}\partial_{\mathcal{J}}Z^{l}\,, \tag{A.2.16}$$

the world-sheet metric $h_{\alpha\beta} = g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu$, the induced metric $\hat{h}_{ab} \equiv \hat{g}_{i\bar{\jmath}}(Z,\bar{Z})\partial_a Z^i \partial_b \bar{Z}^{\bar{\jmath}} +$ c.c. along D, the volume $\hat{V}_D$ of D in eleven-dimensional Planck units, and the EFT string tension:

$$\mathcal{T}_D = \frac{2\pi e^{2A}\hat{V}_D}{\ell^2_{\rm M}}\,. \tag{A.2.17}$$

We now note that the combination appearing in front of $(\partial Y)^2$ in (A.2.15) is given by

$$\frac{2\pi\hat{V}_D e^{\frac{4\phi}{3}}}{\mathcal{T}_D} = \ell^2_{\rm M} e^{\frac{4\phi}{3}} e^{-2A} \equiv \frac{1}{m_*^2}\,, \tag{A.2.18}$$

where m_* is the KK mass along I. As in the D3 case of Appendix A.2.1, we can write

$$\hat{g}^{\perp}_{i\bar{\jmath}} = L^2_{\perp}\,\hat{g}^{\perp}_{0i\bar{\jmath}}\,, \tag{A.2.19}$$

where $L_\perp$ is identified with the length scale (in eleven dimensional Planck units) of the directions in X transversal to D, and $g^{\perp}_{0i\bar{\jmath}}$ is at most of order $\mathcal{O}(1)$. By introducing the corresponding KK scale $m_{\rm KK} = e^A/(\ell_{\rm M} L_\perp)$, we obtain the relation

$$\frac{2\pi\hat{V}_D}{\mathcal{T}_D}\hat{g}^{\perp}_{i\bar{\jmath}} = \frac{1}{m^2_{\rm KK}}\hat{g}^{\perp}_{0i\bar{\jmath}}\,. \tag{A.2.20}$$

We can then expand the square root appearing in (A.2.15), since the second term appearing therein is at most of order $\mathcal{O}(\frac{E^2}{m_*^2}, \frac{E^2}{m^2_{\rm KK}})$, which is small in the EFT regime $E \leq \Lambda \ll \min\{m_*, m_{\rm KK}\}$.

Hence, the leading contribution to the world-sheet action splits into a standard universal term and a NLSM term:

$$\begin{aligned} &-\mathcal{T}_D \int \mathrm{d}^2\sigma\sqrt{-\det\, h_{\alpha\beta}} - \pi e^{\frac{4\phi}{3}}\int \mathrm{d}^2\sigma\sqrt{-\det\, h_{\alpha\beta}}\, h^{\alpha\beta}\partial_\alpha \hat{Y}\partial_\beta\hat{Y} \\ &\qquad - \int \mathrm{d}^2\sigma\sqrt{-\det\, h_{\alpha\beta}}\, h^{\alpha\beta}\mathcal{G}_{I\bar{J}}(\varphi,\bar{\varphi})\partial_\alpha\varphi^I\partial_\beta\bar{\varphi}^{\bar{J}} + \ldots \end{aligned} \tag{A.2.21}$$

with

$$\mathcal{G}_{I\bar{J}} \equiv 2\pi\int_D \mathrm{d}^4 u\sqrt{\det \hat{h}_D}\,\hat{g}^{\perp}_{i\bar{\jmath}}(Z,\bar{Z})\,\omega^i_I\bar{\omega}^{\bar{\jmath}}_{\bar{J}}\,. \tag{A.2.22}$$

Note also that $\mathcal{G}_{I\bar{J}}$ approximately scales like $\hat{V}_X$ under a deformation of the internal space.

We can now discuss the scaling behaviour of the NLSM terms along the possible EFT string flows, following the classification of [34]. In order to lighten the discussion, we will assume that there are no background M5-branes. Their inclusion can be treated similarly.

As a preliminary step, we observe that combining (3.3.68) and (3.3.82a) (plus the absence of background M5s) we get

$$s^0 = \frac{1}{3!}e^{-2\phi}\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s}) + \frac{1}{2}p_a s^a\,, \tag{A.2.23}$$

and then

$$e^{2\phi} = \frac{\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})}{6(s^0 - \frac{1}{2}p_a s^a)} \qquad \text{and} \qquad \hat{V}_X = s^0 - \frac{1}{2}p_a s^a\,. \tag{A.2.24}$$

Furthermore, recalling (A.2.14) we can rewrite the microscopic mass scales m_* and $m_{\rm KK}$ as follows:

$$m_*^2 = \frac{M_{\rm P}^2 e^{-2\phi}}{4\pi\hat{V}_X} = \frac{3M_{\rm P}^2}{2\pi\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})} \quad , \quad m_{\rm KK}^2 = \frac{M_{\rm P}^2 e^{-\frac{2}{3}\phi}}{4\pi L_\perp^2\hat{V}_X} = \frac{6^{\frac{1}{3}}\,M_{\rm P}^2}{4\pi L_\perp^2[\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})]^{1/3}\hat{V}_X^{2/3}}\,. \tag{A.2.25}$$

Consider now an EFT string with charge vector $\mathbf{e} = (e^0, e^a)$. Without loss of generality, we can assume that $p_a e^a \geq 0$, since the case $p_a e^a \leq 0$ can be recovered by applying the symmetry (3.3.83). We will actually restrict to the case $p_a e^a > 0$, in which the EFT string detects the perturbative gauge sector.

From (3.3.90) we see that $D_{\mathbf{e}} \equiv e^a D_a$ must be nef and furthermore we must impose that $e^0 \geq p_a e^a$. We will then make the minimal choice $e^0 = p_a e^a$, since the extension of the following discussion to the case $e^0 > p_a e^a$ is immediate. Note in particular that the condition $e^0 = p_a e^a$ forces the internal volume $\hat{V}_X$ to asymptotically scale like

$$\hat{V}_X \simeq \frac{1}{2}p_a e^a \sigma\,, \tag{A.2.26}$$

under any EFT string flow of this class. This implies that $\mathcal{G}_{I\bar{J}}$ asymptotically scales like σ too, guaranteeing the weak-coupling description of the corresponding NLSM sector. These effects are induced by the inclusion of the higher derivative corrections and of the corresponding deformation (3.3.90) of the saxionic cone identified in [34], which was not taken into account in that paper. Hence, the discussion of [34] actually holds only if $p_a e^a = 0$, and we now revisit the three cases considered therein in the more general case $p_a e^a > 0$.

Case 1: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e}) > 0$

The first of (A.2.24) implies that $e^{2\phi} \simeq \frac{\kappa(\mathbf{e},\mathbf{e},\mathbf{e})}{3p_a e^a}\sigma^2$ asymptotically along the EFT string flow (2.2.9). We then see that the two NLSM terms appearing in (A.2.21) scale like $\sigma^{4/3}$ and σ respectively, hence naturally guaranteeing the weakly-coupled description. Furthermore, note that the length of the HW interval scales like $\sigma^{\frac{2}{3}}$, while the Calabi-Yau characteristic length scales like $\sigma^{\frac{1}{6}}$. Hence the EFT string backreaction generates a hierarchy and induces an intermediate decompactification to five dimensions. Furthermore $L_\perp^2 \sim \sigma^{1/3}$ and from (A.2.25) we get the asymptotic scalings $m_*^2 \sim M_{\rm P}^2\sigma^{-3}$ and $m_{\rm KK}^2 \sim M_{\rm P}^2\sigma^{-2}$, hence confirming the scaling weight $w = 3$ found in [34], but microscopically realizing it in a different way.

Case 2: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e}) = 0$ *but* $\kappa(\mathbf{e},\mathbf{e},\mathbf{e}') > 0$ *for some* $\mathbf{e}' \in \text{Nef}_{\mathbb{Z}}(X)$

In this case X can be realised as the T^2 fibration over a two-fold B and the divisor $D_{\mathbf{e}}$ is 'vertical'. Along the flow both $e^{2\phi}$ and $\mathcal{G}_{I\bar{J}}$ scale like σ. This implies that the two NLSM terms appearing in (A.2.21) scale like $\sigma^{2/3}$ and σ respectively, again justifying the weak-coupling assumption.

Note also that the effective curve $\mathcal{C} = D_{\mathbf{e}}^2$ is a multiple of the T^2 fibre and has constant string frame volume $\text{vol}(\mathcal{C}) = \kappa(\mathbf{e},\mathbf{e},\boldsymbol{s}) = \kappa(\mathbf{e},\mathbf{e},\boldsymbol{s}_0)$. In the M-theory frame, $\widehat{\text{vol}}(\mathcal{C}) = e^{-\frac{2\phi}{3}}\kappa(\mathbf{e},\mathbf{e},\boldsymbol{s}_0)$ and then $\mathcal{C}$ shrinks to zero-size as $e^{-2\phi/3} \sim \sigma^{-1/3}$. This implies that the M-theory base volume $\widehat{\text{vol}}(B)$ should diverge like $\sigma^{4/3}$. Hence, the corresponding length goes like $\sigma^{1/3}$, as the length of the HW interval. This means that the EFT string flow induces again a hierarchy of compactification scales, so that the configuration could be described by first reducing M-theory theory to nine dimensions along a T^2 fibre, and then by further compactifying along $B \times I$. Furthermore $L_\perp^2 \sim \sigma^{2/3}$ and (A.2.25) gives the asymptotic scalings $m_*^2 \sim M_{\text{P}}^2\sigma^{-2}$ and $m_{\text{KK}}^2 \sim M_{\text{P}}^2\sigma^{-2}$, confirming the scaling weight $w = 2$ found in [34], but again microscopically realizing it in a different way.

Case 3: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e}') = 0$ *for any* $\mathbf{e}' \in \text{Nef}_{\mathbb{Z}}(X)$

In this case $e^{2\phi}$ does not scale asymptotically along the EFT string flow, while as in the above cases $\mathcal{G}_{I\bar{J}}$ scales like σ, again allowing for a weakly-coupled NLSM description. The internal space X can be regarded as a T^4 or K3-fibration over $\mathbb{P}^1$. The divisor $D_{\mathbf{e}}$ is a multiple of the fibre and has constant string-frame volume $\frac{1}{2}\kappa(\mathbf{e},\boldsymbol{s}_0,\boldsymbol{s}_0)$. Hence $\hat{V}_{D_{\mathbf{e}}} = \frac{1}{2}e^{-\frac{4\phi}{3}}\kappa(\mathbf{e},\boldsymbol{s}_0,\boldsymbol{s}_0)$ is also constant. Since $\hat{V}_X$ diverges as σ, the volume of the $\mathbb{P}^1$ base should also diverge like σ, so that $L_\perp^2 \sim \sigma$. We then see that the EFT string flow induces a decompactification along the $\mathbb{P}^1$ base, while the other directions do not decompactify. In this case we get a scaling weight which is different from the one obtained in [34]. Indeed (A.2.25) now gives $m_*^2 \sim M_{\text{P}}^2\sigma^{-1}$ and $m_{\text{KK}}^2 \sim M_{\text{P}}^2\sigma^{-2}$, corresponding to a scaling weight $w = 2$, which is different from the value $w = 1$ found in [34].

In conclusion, in all cases a weakly-coupled NLSM description is allowed, if not even favored, by the EFT string flow.

A.3 Derivation of heterotic (s)axionic couplings

In this section, we recall the derivation of the threshold corrections to the (s)axionic couplings in both the $E_8 \times E_8$ and the $SO(32)$ heterotic string models in order to obtain the expression that we used in this thesis to check the validity of the EFT string constraints.

A.3.1 $E_8 \times E_8$ models

We begin by reviewing the derivation of the threshold corrections to the (s)axionic couplings in heterotic $E_8 \times E_8$ string models, referring to [138] and references therein for further details. Let us denote the two heterotic E_8 field strengths by $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ respectively. Our conventions for the traces are the same ones as described after (2.1.7) and in Footnote 1. Similarly, we denote the ten-dimensional curvature two-form by $\boldsymbol{R}$.

These objects enter the Bianchi identity

$$\mathrm{d}H_3 = -\frac{\ell_\mathrm{s}^2}{16\pi^2}\left[\mathrm{tr}(\boldsymbol{F}_1 \wedge \boldsymbol{F}_1) + \mathrm{tr}(\boldsymbol{F}_2 \wedge \boldsymbol{F}_2) + \mathrm{tr}(\boldsymbol{R} \wedge \boldsymbol{R})\right] + \ell_\mathrm{s}^2 \delta_4(\Gamma)\,, \tag{A.3.1}$$

where $\Gamma = \mathbb{R}^4 \times \mathcal{C}$ with $\mathcal{C} \equiv \bigcup_k \mathcal{C}^k$ denotes the overall world-volume of a set of NS5-branes, labelled by k, wrapped on irreducible internal curves $\mathcal{C}^k \subset X$.

We can now split $\boldsymbol{F}_{1,2}$ and $\boldsymbol{R}$ into external and internal contributions: $\boldsymbol{F}_{1,2} = F_{1,2} + \hat{F}_{1,2}$ and $\boldsymbol{R} = R + \hat{R}$. For simplicity, we assume that the possible non-trivial internal gauge bundles associated with $\hat{F}_{1,2}$ correspond to semi-simple sub-algebras of $\mathfrak{e}_8$. This leads to the following cohomological condition for the tadpole of the internal contribution,

$$\lambda(E_1) + \lambda(E_2) = c_2(X) - [\mathcal{C}], \tag{A.3.2}$$

where we have defined

$$\lambda(E) \equiv -\frac{1}{16\pi^2}\,\mathrm{tr}(\hat{F} \wedge \hat{F})\,. \tag{A.3.3}$$

We are interested in the higher curvature terms originating from the 10-dimensional Green-Schwarz counterterm [212, 213]

$$\Delta S_\mathrm{GS} = \frac{1}{\ell_\mathrm{s}^2}\int B_2 \wedge I_8\,, \tag{A.3.4}$$

written in terms of the anomaly polynomial

$$I_8 = \frac{1}{192(2\pi)^3}\Big[2(\mathrm{tr}\,\boldsymbol{F}_1^2)^2 + 2(\mathrm{tr}\,\boldsymbol{F}_2^2)^2 - 2\,\mathrm{tr}\,\boldsymbol{F}_1^2\,\mathrm{tr}\,\boldsymbol{F}_2^2 + (\mathrm{tr}\,\mathbf{F}_1^2 + \mathrm{tr}\,\boldsymbol{F}_2^2)\,\mathrm{tr}\,\boldsymbol{R}^2 + \mathrm{tr}\,\boldsymbol{R}^4 + \frac{1}{4}(\mathrm{tr}\,\boldsymbol{R}^2)^2\Big]\,, \tag{A.3.5}$$

and from the following two terms associated with the background NS5-branes,

$$\begin{aligned} S^{(1)} &= \frac{1}{192\pi} q_a \int_{M_4} a^a \left(\mathrm{tr}\,F_1^2 + \mathrm{tr}\,F_2^2 + \mathrm{tr}\,R^2\right)\,, \\ S^{(2)} &= \frac{1}{16\pi} \sum_k \int_{M_4} \tilde{a}^k \left(\mathrm{tr}\,F_1^2 - \mathrm{tr}\,F_2^2\right)\,. \end{aligned} \tag{A.3.6}$$

Here we are using a compact notation in which the wedge product is implicit, which we will often adopt also in the following. Furthermore we have introduced the intersection numbers

$$q_a \equiv \int_{\mathcal{C}} \omega_a = D_a \cdot \mathcal{C}\,, \tag{A.3.7}$$

and the NS5 axions defined as

$$\tilde{a}^k \equiv \int_{\mathcal{C}^k} \tilde{\mathcal{B}}_2^k\,, \tag{A.3.8}$$

where $\tilde{\mathcal{B}}_2^k$ is the self-dual gauge two-form living on the k-th NS5-brane wrapping the curve $\mathcal{C}^k$. The derivation of the corrections to the heterotic action from the 5-branes is particularly elegant in the framework of heterotic M-theory, where they can be attributed to the M5-brane sector. For further details on the derivation, we refer to [138].

Under our assumptions on the internal bundles, we can split the field strengths as

$$\begin{aligned} &\mathrm{tr}\,\boldsymbol{F}_1^2 = \mathrm{tr}\,F_1^2 + \mathrm{tr}\,\hat{F}_1^2\,, \quad \mathrm{tr}\,\boldsymbol{F}_2^2 = \mathrm{tr}\,F_2^2 + \mathrm{tr}\,\hat{F}_2^2\,, \\ &\mathrm{tr}\,\boldsymbol{R}^2 = \mathrm{tr}\,R^2 + \mathrm{tr}\,\hat{R}^2\,, \quad \mathrm{tr}\,\boldsymbol{R}^4 = 0\,. \end{aligned} \tag{A.3.9}$$

Taking this into account and expanding the perturbative heterotic B-field as $B_2 = \ell_s^2 a^a \omega_a$, we arrive at the total threshold corrections for the gauge sector,

$$\begin{aligned} &-\frac{1}{8\pi}\int \left(\hat{a} + \frac{1}{2}p_a a^a - \frac{3}{8}q_a a^a - \frac{1}{2}\sum_k \tilde{a}^k\right)\mathrm{tr}(F_1 \wedge F_1) \\ &-\frac{1}{8\pi}\int \left(\hat{a} - \frac{1}{2}p_a a^a + \frac{1}{8}q_a a^a + \frac{1}{2}\sum_k \tilde{a}^k\right)\mathrm{tr}(F_2 \wedge F_2)\,. \end{aligned} \tag{A.3.10}$$

and the gravitational sector,

$$-\frac{1}{96\pi}\int \left(12\hat{a} + n_a a^a - \frac{3}{2}q_a a^a\right)\mathrm{tr}(R \wedge R)\,. \tag{A.3.11}$$

Here we have introduced the quantized constants

$$p_a \equiv -\int_X \omega_a \wedge \left[\lambda(E_2) - \frac{1}{2}c_2(X)\right]\,, \tag{A.3.12a}$$

$$n_a \equiv \frac{1}{2}\int_X \omega_a \wedge c_2(X)\,. \tag{A.3.12b}$$

After replacing $\hat{a}$ with a^0 defined as

$$\hat{a} = a^0 - \frac{1}{2}p_a a^a + \frac{3}{8}q_a a^a + \frac{1}{2}\sum_k \tilde{a}^k\,, \tag{A.3.13}$$

we can rewrite (A.3.10) and (A.3.11) in the form

$$-\frac{1}{8\pi}\int a^0\,\mathrm{tr}(F_1\wedge F_1)-\frac{1}{8\pi}\int\left(a^0-p_aa^a+\frac{1}{2}q_aa^a+\sum_k\tilde{a}^k\right)\mathrm{tr}(F_2\wedge F_2), \tag{A.3.14}$$

and

$$-\frac{1}{96\pi}\int\left(12a^0-6p_aa^a+n_aa^a+3q_aa^a+6\sum_k\tilde{a}^k\right)\mathrm{tr}(R\wedge R)\,, \tag{A.3.15}$$

respectively. We finally perform another shift, this time of the M5-brane axions, redefining

$$\tilde{a}^k\equiv a^k-\frac{1}{2\ell_\mathrm{s}^2}\int_{\mathcal{C}^k}B_2=a^k-\frac{1}{2}m^k_aa^a\qquad\text{with}\quad m^k_a\equiv D_a\cdot\mathcal{C}^k\,. \tag{A.3.16}$$

Note that, taking into account that

$$q_a=\sum_k m^k_a\,, \tag{A.3.17}$$

(A.3.16) implies that

$$\sum_k\tilde{a}^k=\sum_k a^k-\frac{1}{2}q_aa^a\,. \tag{A.3.18}$$

The saxionic counterpart of this redefinition reads

$$\tilde{s}^k=s^k-\frac{1}{2}m^k_as^a\,. \tag{A.3.19}$$

To understand the meaning of this linear redefinition, we may observe that it can be interpreted as a shift of what we mean by 'zero-position' of the background M5-branes on the HW interval and that the chiral fields t^k that contain the axions and saxions correspond to the chiral fields $-2\pi\mathrm{i}\Lambda_a$ in [138]. Hence, by adapting their (41) we identify

$$\tilde{s}^k=\lambda^k\int_{\Sigma^k}J=\lambda^km^k_as^a\quad(\text{no sum over }k)\,, \tag{A.3.20}$$

where λ^k represents the position of the M5 along the HW interval – denoted as λ_a in [138]. Now, as illustrated in Figure 2 therein, the values $\lambda^k=-\frac{1}{2}$ and $\lambda^k=\frac{1}{2}$ correspond to placing the k-th M5 on the left and right HW wall, respectively. That is, we can identify $\lambda^k=\hat{y}^k-\frac{1}{2}$, where $\hat{y}^k$ is the coordinate we used on the orbifold circle, with the property of being 0 on the left HW wall and 1 on the right one. In combination with (A.3.20), this implies that s^k introduced in (A.3.19) corresponds microscopically to

$$s^k=\hat{y}^k\int_{\Sigma^k}J=\hat{y}^km^k_as^a\quad(\text{no sum over }k)\,. \tag{A.3.21}$$

In terms of new M5 axions a^k introduced in (A.3.16), the axionic couplings take their final form

$$
\begin{aligned}
&-\frac{1}{8\pi}\int a^0\,\mathrm{tr}(F_1\wedge F_1)-\frac{1}{8\pi}\int\left(a^0-p_a a^a+\sum_k a^k\right)\mathrm{tr}(F_2\wedge F_2)\,,\\
&-\frac{1}{96\pi}\int\left(12a^0-6p_a a^a+n_a a^a+6\sum_k a^k\right)\mathrm{tr}(R\wedge R)\,.
\end{aligned}
\tag{A.3.22}
$$

The corresponding saxionic couplings then follow by supersymmetry:

$$
\begin{aligned}
&-\frac{1}{8\pi}\int s^0\,\mathrm{tr}(F_1\wedge *F_1)-\frac{1}{8\pi}\int\left(s^0-p_a s^a+\sum_k s^k\right)\mathrm{tr}(F_2\wedge *F_2)\,,\\
&-\frac{1}{96\pi}\int\left(12s^0-6p_a s^a+n_a s^a+6\sum_k s^k\right)\mathrm{tr}(R\wedge *R)\,.
\end{aligned}
\tag{A.3.23}
$$

A.3.2 $SO(32)$ models

The $SO(32)$ heterotic string models can be treated in a similar way to the $E_8\times E_8$ string. First, recall the form of the I_8 polynomial in the Green-Schwarz term needed for anomaly cancellation in this setting [212, 213]. Adapting it to our conventions, this is given by

$$
I_8=\frac{1}{192(2\pi)^3}\left[8\,\mathrm{tr}\,\boldsymbol{F}^4+\mathrm{tr}\,\boldsymbol{F}^2\,\mathrm{tr}\,\boldsymbol{R}^2+\mathrm{tr}\,\boldsymbol{R}^4+\frac{1}{4}(\mathrm{tr}\,\boldsymbol{R}^2)^2\right],
\tag{A.3.24}
$$

We proceed by splitting $\boldsymbol{F}$ and $\boldsymbol{R}$ in their internal and external components, according to the KK ansatz. This gives us

$$
\begin{aligned}
&\mathrm{tr}\,\boldsymbol{F}^2=\mathrm{tr}\,F^2+\mathrm{tr}\,\hat{F}^2\,,\\
&\mathrm{tr}\,\boldsymbol{R}^2=\mathrm{tr}\,R^2+\mathrm{tr}\,\hat{R}^2\,,\quad \mathrm{tr}\,\boldsymbol{R}^4=0\,.
\end{aligned}
\tag{A.3.25}
$$

In general, contrary to the $E_8\times E_8$ case, different breaking patterns for $SO(32)$ will result in different corrections to the 4d kinetic terms (see e.g. [214]). Here, for simplicity we assume an internal gauge bundle such that $\mathrm{tr}\,\boldsymbol{F}^4=0$. This choice leads to

$$
\begin{aligned}
I_8&=\frac{1}{192(2\pi)^3}\left[(\mathrm{tr}\,F^2+\mathrm{tr}\,\hat{F}^2)(\mathrm{tr}\,R^2+\mathrm{tr}\,\hat{R}^2)+\frac{1}{4}(\mathrm{tr}\,R^2+\mathrm{tr}\,\hat{R}^2)^2\right]\\
&\simeq\frac{1}{192(2\pi)^3}\left[\mathrm{tr}\,F^2\,\mathrm{tr}\,\hat{R}^2-\mathrm{tr}\,R^2\left(-\,\mathrm{tr}\,\hat{F}^2-\frac{1}{2}\,\mathrm{tr}\,\hat{R}^2\right)\right]\\
&=\frac{1}{96\pi}\left[\mathrm{tr}\,F^2c_2(X)-\mathrm{tr}\,R^2\left(\lambda(E)-\frac{1}{2}c_2(X)\right)\right]\\
&\simeq\frac{1}{96\pi}\left[\mathrm{tr}\,F^2c_2(X)-\frac{1}{2}\,\mathrm{tr}\,R^2c_2(X)\right],
\end{aligned}
\tag{A.3.26}
$$

where we ignored the terms that will not contribute to the 4d couplings we are interested in and the last steps are to be meant cohomologically and in absence of NS5-branes, having taken into account the tadpole condition $\lambda(E) = c_2(X)$.

Expanding the B_2 gauge two-form in the Green-Schwarz counterterm (A.3.4), we obtain the corrections to the axion couplings in absence of NS5-branes

$$-\frac{1}{8\pi}\int(\hat{a} - \frac{1}{6}n_a a^a)\,\mathrm{tr}(F\wedge F) - \frac{1}{96\pi}\int(12\hat{a} + n_a a^a)\,\mathrm{tr}(R\wedge R)\,. \tag{A.3.27}$$

If we now replace $\hat{a}$ with

$$a^0 \equiv \hat{a} - \frac{1}{6}n_a a^a\,, \tag{A.3.28}$$

we obtain the contribution

$$-\frac{1}{8\pi}\int a^0\,\mathrm{tr}(F\wedge F) - \frac{1}{96\pi}\int(12a^0 + 3n_a a^a)\,\mathrm{tr}(R\wedge R)\,. \tag{A.3.29}$$

By supersymmetry, the saxionic couplings are then given by

$$-\frac{1}{8\pi}\int s^0\,\mathrm{tr}(F\wedge *F) - \frac{1}{96\pi}\int(12s^0 + 3n_a s^a)\,\mathrm{tr}(R\wedge *R)\,. \tag{A.3.30}$$

We expect the gravitational couplings to be invariant under a transition that replaces some of gauge bundle by NS5-branes; the changes in the above computation due to the modification of the Bianchi identify should be counter-balanced by additional terms from NS5-branes. These are hard to compute directly in the heterotic frame, but are S-dual to curvature terms in the Chern-Simons actions of D5-branes in Type I string theory [215].

A.4 M5 instantons in $E_8 \times E_8$ heterotic models

In this appendix we discuss the structure of the saxionic cone (3.3.88) in terms of the Euclidean M5-brane instantons.

We start by considering a process in which the first four-dimensional gauge sector forms an elementary anti-self-dual BPS instanton of unit charge, $-\frac{1}{16\pi^2}\int \mathrm{tr}(F_1\wedge F_1) = 1$ – see Footnote 1. This instanton contributes to the amplitudes by an exponential factor $e^{2\pi i t^0} = e^{-2\pi s^0}e^{2\pi i a^0}$. By a small instanton transition, this gauge instanton can be continuously deformed into an M5-brane instanton wrapping X at $\hat{y} = 0$. Hence, by holomorphy, the M5-instanton contribution to the amplitudes should still be weighted by $e^{2\pi i t^0} = e^{-2\pi s^0}e^{2\pi i a^0}$ and then its Euclidean action should be given by $2\pi s^0$. By repeating the same argument with the second HW wall and taking into account (3.3.80), we conclude that an M5-brane instanton at $\hat{y} = 1$ should have Euclidean action $2\pi(s^0 - p_a s^a + q_a s^a)$. At first sight, these results may be puzzling, and one may naively be worried that if $(p_a - q_a)s^a > 0$ the M5 instanton on the first HW wall could slip to the second wall, and vice versa if $(p_a - q_a)s^a < 0$. But this would clearly appear in tension with the assumed BPS-ness of the M5 instanton, as we would expect that either the M5-instantons are stuck at the HW walls, or that their on-shell action should not change as we move them along the y direction.

The key point is that in presence of a non-vanishing G_4 flux along X, the Euclidean M5-branes sitting at intermediate positions $y_{\rm E5} \in (0,1)$ are not isolated but must be rather connected to one of the HW walls by open Euclidean M2-branes. Indeed recall that, in general, if the boundary of one or more open M2-branes contains a two-cycle Σ supported on the M5 world-volume, they contribute as follows to the Bianchi identity of the M5 self-dual three-form $\tilde{\mathcal{H}}_3$:

$$\mathrm{d}\tilde{\mathcal{H}}_3 = \ell_{\rm M}^{-3} G_4|_{\rm M5} - \delta^4(\Sigma)\,. \tag{A.4.1}$$

This implies the cohomological constraint

$$\ell_{\rm M}^{-3}[G_4]|_{\rm M5} = [\Sigma]\,. \tag{A.4.2}$$

In other words, if $[G_4]_{\rm M5}$ is non-trivial in cohomology, then the M5-brane must host a homologically non-trivial component Σ of the M2 boundary, fixed by (A.4.2).

In the present setting, (3.3.86b) implies that the constants p_a defined in (3.3.76) can be alternatively identified with

$$p_a = \frac{1}{\ell_{\rm M}^3} \lim_{y\to 1^-} \int_{\{y\}\times D_a} G_4\,, \tag{A.4.3}$$

and then measure the flux quanta of G_4 close to the second HW wall. As we move to the left and we cross a given subset of background M5-branes, the G_4 flux quanta change to

$$\frac{1}{\ell_{\rm M}^3}\int_{\{y\}\times D_a} G_4 = p_a - \sum_{k|\hat{y}^k>y} m_a^k \quad \Rightarrow \quad \frac{1}{\ell_{\rm M}^3} \lim_{y\to 0^+} \int_{\{y\}\times D_a} G_4 = p_a - q_a\,. \tag{A.4.4}$$

Combined with (A.4.2), this implies that

$$\int_\Sigma J = \frac{1}{\ell_{\rm M}^3}\int_{\{y_{\rm E5}\}\times X} J\wedge G_4 = p_a s^a - \sum_{k|\hat{y}^k>y_{\rm E5}} m_a^k s^a\,. \tag{A.4.5}$$

In our setting, BPS Euclidean M2-branes extend along the HW interval and wrap an effective (possibly reducible) curve $\mathcal{C}\subset X$. Considering BPS Euclidean M2-branes ending on the M5-instanton from the left or from the right (in the y direction), we can then identify their boundary with $\Sigma = \mathcal{C}$ or $\Sigma = -\mathcal{C}$, respectively. Since we necessarily have $\int_{\mathcal{C}} J > 0$, (A.4.2) and (A.4.5) imply that these open M2-branes should end on the M5-instanton from the left if $p_a s^a - \sum_{k|\hat{y}^k>y_{\rm E5}} m_a^k s^a$ is positive, and from the right if it is negative. Note that $m_a^k s^a = \int_{\mathcal{C}^k} J \geq 0$ and then the combination $p_a s^a - \sum_{k|\hat{y}^k>y_{\rm E5}} m_a^k s^a$ increases as the M5 instanton crosses the background M5-branes from the left.

Consider for example the case in which $(p_a - q_a)s^a > 0$. In this case an isolated M5 instanton sitting on the first HW wall can be moved away from it to a more general position $y_{\rm E5} > 0$, but this process will generate Euclidean M2-branes stretching between the HW wall, the M5 instanton and the intermediate background M5-branes. Figure 3.1 in Section 3.3.3 illustrates the case with a single background M5-brane wrapping the curve $\mathcal{C}_{\rm M5} \simeq q_a \Sigma^a$ (where $\Sigma^a \cdot D_b = \delta^a_b$) at $\hat{y} \leq y_{\rm E5}$, and the open Euclidean M2-branes wrapping the curves $\mathcal{C}_{\rm E2} \simeq (p_a - q_a)\Sigma^a$ and $\mathcal{C}'_{\rm E2} \simeq p_a \Sigma^a$ respectively. The total action does not change from the initial value $2\pi s^0$, but is the sum of the contributions of the different Euclidean branes.

Let us order the background M5-branes so that $\hat{y}^k < \hat{y}^{k+1}$. We can write the contribution to the total instanton action coming from the open M2-branes as

$$S^{\rm inst}_{\rm M2} = 2\pi \Big[y_{\rm E5} \Big(p_a - \sum_{k|\hat{y}^k > y_{\rm E5}} m^k_a \Big) s^a - \sum_{k|\hat{y}^k < y_{\rm E5}} s^k \Big] . \tag{A.4.6}$$

Hence, the contribution to the instanton action coming from the M5 at $y_{\rm E5}$ must be given by

$$S^{\rm inst}_{\rm M5} = S^{\rm inst} - S^{\rm inst}_{\rm M2} = 2\pi \Big[s^0 - y_{\rm E5} \Big(p_a - \sum_{k|\hat{y}^k > y_{\rm E5}} m^k_a \Big) s^a + \sum_{k|\hat{y}^k < y_{\rm E5}} s^k \Big] . \tag{A.4.7}$$

It is easy to check that $S^{\rm inst}_{\rm M2}$ and $S^{\rm inst}_{\rm M5}$ are continuous in $y_{\rm E5}$ and that, as expected, the latter reduces to $2\pi(s^0 - p_a s^a + \sum_k s^k)$ in the limit $y_{\rm E5} \to 1$.

Note that, on the other hand, if $(p_a - q_a)s^a > 0$ an isolated M5 instanton sitting on the second HW cannot move from it, since it would require the presence of open M2-branes ending on it from the left. If instead $p_a s^a < 0$ the role of the two HW walls is inverted: an isolated M5 instanton on the first HW will remain stuck on it, while an isolated M5 instanton on the second HW will be free to move away from it, forming a composite M5/M2 instanton. Indeed, this is expected from the $\mathbb{Z}_2$ symmetry (3.3.83).

There could also be intermediate cases in which $p_a s^a > 0$ and $(p_a - q_a)s^a < 0$. In these cases the isolated M5 instantons sitting in both HW walls should be trapped, while there could be composite mobile M5/M2 instantons. It would be interesting to study better the transitions between these various possibilities as we move in the Kähler cone, and in particular the connection with the bundle stability walls. But for the purposes of the present thesis, we just need to observe that the positivity of the M5-brane action (A.4.7) is guaranteed if we impose the conditions $s^0 > 0$ and $s^0 - p_a s^a + \sum_k s^k > 0$.

Appendix B

B.1 Basic considerations on wormholes and their regime of validity

B.1.1 2-derivative solution and estimation of the cut-off

Within a two-derivative approximation, explicit wormhole solutions have been known since the seminal work [201]. Extensions with several axions, or their dual two-forms, or axions plus additional scalars have subsequently been constructed by many authors [5–7].

In the absence of scalar potentials, wormholes are built out of two asymptotically flat Euclidean geometries glued together at the point of maximal curvature, the wormhole throat. In the two-derivative approximation, the metric of half a wormhole, or semi-wormholes, is

$$\mathrm{d}s^2 = \frac{1}{1-\frac{L^4}{r^4}}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2, \tag{B.1.1}$$

where $r \in [L, \infty)$ and the characteristic length scale L is determined in terms of the wormhole charges and the fundamental parameters of the theory. For example, purely axionic solutions have $L^4 = q_i(f^{-2})^{ij}q_j/(24\pi^4 M_{\mathrm{P}}^2)$, with $(f^2)_{ij}$ the axions' decay constants, q_i the axionic charges, and M_{P} the reduced Planck scale. While model-dependent, L always increases in the regime of large wormhole charge.

We have argued below eq. (4.0.4) that a measure of the strength of wormholes in the semi-classical approximation is provided by 1/2 of the wormhole action. Given the symmetry of such a configuration, to obtain $S_{\mathrm{wh}}/2$ we might as well decide to compute the action of a semi-wormhole. The explicit result in the case of purely axionic solutions is $S_{\mathrm{wh}}/2 = 3\pi^3 M_{\mathrm{P}}^2 L^2$. [1] Superficially, one might think these wormhole solutions have validity in the regime $2\pi^2 M_{\mathrm{P}}^2 L^2 \gg$

[1] Up to a different convention for the Planck scale, our semi-wormhole action agrees with the one first derived in [201]. We also agree with [7] provided no Gibbons-Hawking term is added at the wormhole throat (recall we are interested in 1/2 of the wormhole action and in such a calculation the Gibbons-Hawking term does not contribute to the on-shell action because the geometry of a full wormhole is asymptotically flat). We also agree with the result shown in eq.(12) of [203], up to a typo in the last equality. The expression for half a wormhole presented in [206, 207], however, includes a 1/2 in front and restricts the integration over half wormhole, and thus effectively corresponds to the action of a fourth of a wormhole.

1 of large action, or equivalently large charge or small axion decay constants. But in fact this is in general a very optimistic bound.

The truncation at the two-derivative level is justified only if interactions of higher dimensions lead to small corrections. One expects the latter operators to be suppressed by inverse powers of the cutoff Λ, via an expansion of the form ∂^2/Λ^2. Observing that the maximal gradient in the leading order wormhole solutions is set by $\partial^2 \sim 1/L^2$, we see that to retain perturbative control on our analysis it is necessary to impose $(\Lambda L)^2 \gtrsim 1$. On purely dimensional grounds, the action of a wormhole at the 2-derivative level scales as $S_{\rm wh} \sim 2\pi^2 M_{\rm P}^2 L^2$. Reliable solutions of the truncated action must thus satisfy [201]

$$S_{\rm wh} \gtrsim 2\pi^2 \frac{M_{\rm P}^2}{\Lambda^2}. \tag{B.1.2}$$

The effective field theory has no information on Λ, but one can identify an upper bound on such a quantity requiring a well defined perturbative expansion. In a perturbative effective field theory with N light degrees of freedom, the physical cutoff must necessarily be smaller than the *low-energy species scale*

$$\Lambda^2 \ll \Lambda^2_{\rm low-sp} \equiv \frac{16\pi^2 M_{\rm P}^2}{N}. \tag{B.1.3}$$

The reason is that the effective coupling of matter to gravity, namely $g^2_{\rm EFT} \equiv p^2/M_{\rm P}^2$ with $p^2 < \Lambda^2$ the typical momentum transfer, must satisfy $g^2_{\rm EFT} N/16\pi^2 \ll 1$ in order to retain perturbative control. This consistency requirement translates into (B.1.3). We call $\Lambda_{\rm low-sp}$ the low-energy species scale to distinguish it from the species scale that contains information about the tower of states beyond the 4-dimensional effective field theory, which we discuss in Section 4.2.3.

The low-energy species scale $\Lambda_{\rm low-sp}$ is not a physical mass, but rather the highest possible energy at which the effective field theory can be treated perturbatively. The true regime of validity of the effective field theory is instead determined by the first physical threshold beyond the effective theory, i.e. the mass of the first excited state. It is the latter that we must identify with the UV cutoff Λ. According to the estimate (B.1.2) we therefore anticipate that, in a theory with $N \gg 1$ low-energy degrees of freedom, reliable wormhole solutions should have actions that scale at least as $S_{\rm wh} \gg N$, and their impact should be suppressed by

$$e^{-S_{\rm wh}/2} \lesssim e^{-\pi^2 M_{\rm P}^2/\Lambda^2} \ll e^{-\mathcal{O}(N)}. \tag{B.1.4}$$

We will provide evidence of this scaling in various ways in the rest of this thesis.

B.1.2 The Gauss-Bonnet correction

Our estimate (B.1.4) of the wormhole action was based on dimensional grounds and the implicit assumption that the uncontrollable higher derivative terms can modify the wormhole configuration. There are potentially two exceptions to this expectation: the Gauss-Bonnet operator

$$E_{\rm GB} \equiv \frac{1}{32\pi^2}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right) \tag{B.1.5}$$

and the Pontryagin operator $E_{\rm P} \equiv \epsilon^{cdef} R^{a}{}_{bef} R^{b}{}_{acd}$. Being total derivatives, they do not alter the equations of motion and therefore do not affect the wormhole solution. Furthermore, their coefficients are a priori unconstrained by our perturbative considerations because they do not describe graviton vertices. Let us therefore analyze them in some detail.

Consider the Gauss-Bonnet term first. Despite it being a total derivative, and similarly to the Einstein-Hilbert term, its variational problem is not well posed unless an appropriate Hawking-Gibbons term Q is included. Subtracting the flat space contribution at the boundary so as to have a vanishing action in Euclidean space, the complete Gauss-Bonnet term reads

$$S_{\rm GB} = -\gamma \int_{\mathcal{M}} \sqrt{g}\, E_{\rm GB} - \gamma \int_{\partial\mathcal{M}} \sqrt{h}(Q - Q_0)\,, \tag{B.1.6}$$

with γ an unknown coefficient. After the boundary term Q is included the Gauss-Bonnet term acquires topological significance. More precisely, [216]

$$\begin{aligned}\chi(\mathcal{M}) &= b_0(\mathcal{M}) - b_1(\mathcal{M}) + b_2(\mathcal{M}) - b_3(\mathcal{M}) - b_4(\mathcal{M}) \\ &= \int_{\mathcal{M}} \sqrt{g} E_{\rm GB} + \int_{\partial\mathcal{M}} \sqrt{h} Q\end{aligned} \tag{B.1.7}$$

is the Euler number (or characteristic), which counts the alternating sum of the ranks of the homology groups of $\mathcal{M}$: $b_k = \dim H_k(\mathcal{M})$. Viewing Euclidean space $\mathbb{E}_4$ as a ball with asymptotic S^3 boundary, we get $\chi(\mathbb{E}_4) = 1$, a value entirely determined by the boundary term. [2] As a result eq.(B.1.6) becomes

$$S_{\rm GB} = -\gamma[\chi(\mathcal{M}) - \chi(\mathbb{E}_4)] = -\gamma[\chi(\mathcal{M}) - 1]\,. \tag{B.1.8}$$

This expression reproduces the topological contribution considered in [201]. A configuration of $n_{\rm wh}$ wormholes have topology characterized by $\chi(\mathcal{M}) - 1 = -n_{\rm wh}$. In practice, this reveals the effective field theory contains a non-perturbative coupling

$$e^{-\gamma n_{\rm wh}} \tag{B.1.9}$$

[2] We may adopt an alternatively asymptotic compactification prescription for the asymptotically flat space which would give different $\chi(\mathcal{M})$ and $\chi(\mathbb{E}_4)$. For instance, adding just one asymptotic point, $\mathbb{E}_4$ becomes S^4 and we would get $\chi(\mathbb{E}_4) = 2$. Nevertheless, the combination $\chi(\mathcal{M}) - \chi(\mathbb{E}_4)$, and hence $S^{(\rm E)}_{\rm GB}$, does not change.

suppressing wormhole insertions (provided $\gamma > 0$). Knowledge of the coefficient γ of the Gauss-Bonnet term offers another, independent, rationale to quantify the impact of wormholes.

When discussing wormholes, on the other hand, the Pontryagin term is always irrelevant because of topological considerations. To start, $E_{\rm P} \equiv \epsilon^{cdef} R^a{}_{bef} R^b{}_{acd} = 0$ on the simplest wormhole solution (B.1.1). Yet, even allowing potential corrections beyond the two-derivative approximation, the topological structure of an arbitrary wormhole configuration is such that the integral $\int \sqrt{g} E_{\rm P}$ is exactly zero. We can therefore safely ignore this term in our discussion.

B.2 Perturbative deviation from homogeneous solution

Writing $\ell_i = \ell_i^0 + \delta\ell_i$, where $\ell_i^0 = q_i \tilde{\ell}$, and expanding the action (4.4.12) up to quadratic order in the fluctuation, we have

$$\begin{aligned} \delta_2 S &= \pi \int \mathrm{d}\tau\; \mathcal{G}^{ij}(\mathbf{q}) \frac{1}{\tilde{\ell}^2} \left\{ \delta\dot{\ell}_i \delta\dot{\ell}_j - \frac{4}{\tilde{\ell}} \dot{\tilde{\ell}}\, \delta\ell_i \delta\dot{\ell}_j + \frac{3}{\tilde{\ell}^2} \left(2 - \frac{\tilde{\ell}^2}{\tilde{\ell}_*^2} \right) \delta\ell_i \delta\ell_j \right\} \\ &= \pi \int \mathrm{d}\tau\; \mathcal{G}^{ij}(\mathbf{q}) \left\{ \frac{\delta\dot{\ell}_i \delta\dot{\ell}_j}{\tilde{\ell}^2} + \left[1 - \sin^2\left(\frac{\tau}{\tilde{\ell}_*}\right) \right] \frac{\delta\ell_i \delta\ell_j}{\tilde{\ell}^4} \right\} \end{aligned} \tag{B.2.1}$$

Since we are only interested in the equations of motion for $\delta\ell_i$, we have ignored all boundary terms. By setting

$$\delta\ell_i = \tilde{\ell} f_i \,, \tag{B.2.2}$$

after one integration by parts and discarding again boundary terms, this quadratic action can be rewritten as

$$\delta_2 S = \pi \int \mathrm{d}\tau\; \mathcal{G}^{ij}(\mathbf{q}) \left\{ \dot{f}_i \dot{f}_j + \frac{2}{\tilde{\ell}^2} f_i f_j \right\} . \tag{B.2.3}$$

The equations of motion for f_i are

$$\ddot{f}_i = \frac{2}{\tilde{\ell}^2} f_i = \frac{2}{\tilde{\ell}^2 \cos^2\left(\frac{\tau}{\tilde{\ell}_*}\right)} f_i \,. \tag{B.2.4}$$

From (B.2.2) and (4.5.7) it is easy to see that the initial condition $\delta\dot{\ell}_i(0) = 0$ corresponds to $\dot{f}_i(0) = 0$. The general solution of (B.2.4) satisfying this initial condition is[3]

$$f_i(\tau) = f_{*i} \Big[1 + \frac{\tau}{\tilde{\ell}_*} \tan\left(\frac{\tau}{\tilde{\ell}_*}\right) \Big] \quad \Leftrightarrow \quad \delta\ell_i = f_{*i}\, \tilde{\ell}_* \Big[\cos\left(\frac{\tau}{\tilde{\ell}_*}\right) + \frac{\tau}{\tilde{\ell}_*} \sin\left(\frac{\tau}{\tilde{\ell}_*}\right) \Big] \,, \tag{B.2.5}$$

where $f_{*i} = f_i(0)$ is the integration constant representing the value of $f_i(\tau)$ at the throat. We emphasize that the relative fluctuations $f_i(\tau)$, rather then $\delta\ell_i(\tau)$, must be small in order for the perturbative expansion to make sense. This certainly requires that $f_{*i} \ll 1$, and that the

[3] The most general solution of (B.2.4) contains also a term proportional to $\tan\left(\frac{\tau}{\tilde{\ell}_*}\right)$, which however violates the initial condition $\dot{f}_i(0) = 0$.

$f_i(\tau)$ remains small along the flow.

As a consistency check, if we choose $f_{*i} = \tilde{f}_* q_i$ the solution (B.2.5) coincides with the first order term appearing in the $\tilde{f}_* \ll 1$ expansion of (4.5.7) with $\tilde{\ell}_* \to \tilde{\ell}'_* = \tilde{\ell}_*(1+\tilde{f}_*)$, as expected. This also implies that without loss of generality we can impose some transversality condition on f_{*i}, as for instance

$$\mathcal{G}^{ij}(\mathbf{q}) f_{*i} q_j = 0\,. \tag{B.2.6}$$

This condition is useful because it implies that the deformation changes the wormhole radius (4.4.19) only to second order in f_{*i}:

$$L^4 = L^4_{(0)}\,(1+\delta) \qquad \text{with} \qquad \delta = \frac{6}{n}\mathcal{G}^{ij}(\mathbf{q}) f_{*i} f_{*j} \tag{B.2.7}$$

Recalling (4.4.23), this implies τ_∞ is only modified by second order corrections too. Hence, to first order, we can still use (4.5.6) to express $\tilde{\ell}_*$ in terms of τ_∞, getting

$$f_i(\tau) = f_{*i}\left[1 + \frac{\pi}{2}\sqrt{\frac{3}{n}}\frac{\tau}{\tau_\infty}\tan\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\frac{\tau}{\tau_\infty}\right)\right]. \tag{B.2.8}$$

We see that the deviation grows towards $\tau = \tau_\infty$, with maximum relative gain reading

$$\frac{f_{\infty i}}{f_{*i}} = 1 + \frac{\pi}{2}\sqrt{\frac{3}{n}}\tan\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right) \simeq \{\infty, 7.37, 4.29, 3.24, 2.70\} \tag{B.2.9}$$

for $n = 3, 4, 5, 6, 7$. For $n \geq 4$ the perturbative expansion makes sense everywhere for sufficiently small f_{*i}. For instance, $f_i(\tau) < 0.1$ can be guaranteed along the entire wormhole by choosing $f_i^* < 0.014$ for $n = 3$, or $f_i^* < 0.03$ for $n = 6$. On the other hand, the general arguments of Section 4.5.2 suggest that in a complete non-perturbative treatment it is sufficient to pick moderately larger f_i^* to allow for a much larger range of f_i^∞, as confirmed by the numerical evidence of Figs. 4.3, 4.4, 4.5, 4.6.

As expected, if $n = 3$ the solution (B.2.8) diverges at infinite radial distance and thus, for any arbitrarily small initial values f_{*i}, the perturbative expansion breaks down at some finite radius. Nevertheless, as discussed in Section 4.5.3, it can still make sense in presence of some type of IR cutoff.

B.3 Other tests of the species scale bound

B.3.1 Another model: $\mathbb{P}^1$ fibration over $\mathbb{F}_n$

Another class of simple models is obtained by choosing X to be a $\mathbb{P}^1$ fibration over the Hirzebruch surface $\mathbb{F}_n$, which in turn can be described as a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$, specified by the integer $n \geq 0$. This model has been recently discussed in a closely related framework by [71], to which we refer for more details. For our purposes it is sufficient to restrict to a $\mathbb{P}^1$ fibration over $\mathbb{F}_n$ specified a single non-negative integers $h \in \mathbb{Z}_{\geq 0}$,[4] and to identify the relevant cones of divisors and curves and their intersection numbers.

The cone of effective divisors is simplicial and is generated by three effective divisors E^a, $a = 1, 2, 3$. These three effective divisors can be roughly regarded as twisted products of the possible pairs of the three $\mathbb{P}^1$'s involved in the geometry. We can also introduce a basis of nef divisors D^a, with respect to which

$$E^1 = D^1 - nD^2 \,, \quad E^2 = D^2 \,, \quad E^3 = D^3 - hD^1 \,. \tag{B.3.1}$$

The divisors D^a generate all the other nef divisors, as well as the Kähler cone. Hence, by using these divisors in the expansion (4.3.1) the Kähler cone corresponds to $v_1, v_2, v_3 > 0$. The triple intersections are given by the coefficients of

$$\mathcal{I}(X) = D^1D^2D^3 + n(D^1)^2D^3 + hnD^1(D^3)^2 + hD^2(D^3)^2 + h^2n(D^3)^3 \,. \tag{B.3.2}$$

By using the expansion $\boldsymbol{\ell} = \ell_a D^a$, the kinetic potential (4.3.13) becomes

$$\mathcal{F}_{\rm K} = \log \kappa(\boldsymbol{\ell}, \boldsymbol{\ell}, \boldsymbol{\ell}) = \log\left(6\ell_1\ell_2\ell_3 + 3n\ell_1^2\ell_3 + 3hn\ell_1\ell_3^2 + 3h\ell_2\ell_3^2 + h^2n\ell_3^3\right) \,. \tag{B.3.3}$$

The condition that J belongs to the Kähler cone is equivalent to $\ell_a > 0$. In order to understand the complete (dual) saxionic domain, we have to consider the saxions $s^a = \frac{3\kappa^{abc}\ell_b\ell_c}{2\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell})}$, which identify the $\mathbb{R}$-effective curves:

$$\boldsymbol{s} = s^a\Sigma_a = \frac{3\,\boldsymbol{\ell}\cdot\boldsymbol{\ell}}{2\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell})} \,, \tag{B.3.4}$$

where Σ_a are effective curves

$$\Sigma_1 = E^2 \cdot E^3 \quad , \quad \Sigma_2 = E^1 \cdot E^3 \quad , \quad \Sigma_3 = E^1 \cdot E^2 \,, \tag{B.3.5}$$

which generate the whole cone of effective curves $\text{Eff}_1(X)$ and are dual to the nef divisors D^1: $D^a \cdot \Sigma_b = \delta^a_b$. On the other hand, the saxionic cone can be identified with the cone of movable curves – see (4.3.3) – which is generated by the (effective) movable curves

$$\hat{\Sigma}_1 = D^2 \cdot D^3 = \Sigma_1 + h\Sigma_3 \,, \quad \hat{\Sigma}_2 = D^1 \cdot D^3 = n\Sigma_1 + \Sigma_2 + hn\Sigma_3 \,, \quad \hat{\Sigma}_3 = D^1 \cdot D^2 = \Sigma_3 \,, \tag{B.3.6}$$

which are dual to the effective divisors E^a: $\hat{\Sigma}_a \cdot E^b = \delta^b_a$. Hence, if we use the expansion $\boldsymbol{s} = \hat{s}^a\hat{\Sigma}_a$, the saxionic cone is defined by the positivity conditions $\hat{s}^a > 0$. By using (B.3.1)

[4] Following [124], the $\mathbb{P}^1$ fibration can be defined in terms of a line bundle $\mathcal{T} = hd_1 + pd_2$, where d_1, d_2 are elementary nef divisors over $\mathbb{F}_n$. In particular, their intersection numbers are given by the coefficients of $\mathcal{I}(\mathbb{F}_n) = nd_1^2 + d_1d_2$. In the notation of [71], our integers (h, p) correspond to (s, t). Here we are restricting to fibrations with $p = 0$.

and (B.3.4) we can compute the components $\hat{s}^a = E^a \cdot \boldsymbol{s} = \frac{3E^a\cdot\boldsymbol{\ell}\cdot\boldsymbol{\ell}}{2\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell})}$, getting

$$\hat{s}^1 = \frac{6\ell_2\ell_3}{2\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell})} \quad , \quad \hat{s}^2 = \frac{3(2\ell_1\ell_3 + h\,\ell_3^2)}{2\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell})} \quad , \quad \hat{s}^3 = \frac{3(2\ell_1\ell_2 + n\,\ell_1^2)}{2\kappa(\boldsymbol{\ell},\boldsymbol{\ell},\boldsymbol{\ell})}\,. \tag{B.3.7}$$

The saxionic cone condition $\hat{s}^a > 0$ is clearly satisfied if $\ell_a > 0$.[5]

The $\mathcal{P}_{\rm K}$ boundaries can again be characterized in terms of tensionless string limits. The set $\mathcal{C}_{\rm S}^{\rm EFT}$ of EFT string charges is generated by the movable curves (B.3.6) and hence, in the basis Σ_a, we identify the following corresponding tensions:

$$\mathcal{T}_{\hat{\Sigma}_1} = M_{\rm P}^2(\ell_1 + h\ell_3)\,, \quad \mathcal{T}_{\hat{\Sigma}_2} = M_{\rm P}^2(n\ell_1 + \ell_2 + hn\ell_3)\,, \quad \mathcal{T}_{\hat{\Sigma}_3} = M_{\rm P}^2\ell_3\,. \tag{B.3.8}$$

We then see that, assuming $n, h > 0$, $\mathcal{T}_{\hat{\Sigma}_3} = 0$ on the two-dimensional boundary component $\{\ell_3 = 0\}$, $\mathcal{T}_{\hat{\Sigma}_1} = 0$ on the one dimensional boundary component $\{\ell_1 = \ell_3 = 0\}$, while $\mathcal{T}_{\hat{\Sigma}_2} = 0$ at the tip $\{\ell_1 = \ell_2 = \ell_3 = 0\}$. These boundary components are at infinite distance. On the other hand, the BPS but non-EFT strings of charges Σ_1 and Σ_2 have tensions $\mathcal{T}_{\Sigma_1} = M_{\rm P}^2\ell_1$ and $\mathcal{T}_{\Sigma_2} = M_{\rm P}^2\ell_2$, which vanish on the boundaries $\ell_1 = 0$ and $\ell_2 = 0$, respectively, which are at finite field distance (if $\ell_3 > 0$). More precisely, they correspond to the finite distance Δ boundaries $\hat{s}^3 = 0$ and $\hat{s}^1 = 0$, respectively, while $\ell_3 \to 0$ corresponds to the infinite distance limit $\hat{s}^3 \to \infty$. Viceversa, a limit $\hat{s}^2 \to 0$ (with $\hat{s}^1, \hat{s}^3$) fixed corresponds to a limit $\ell_2 \to \infty$. So, as in subsection 4.3.1, while the saxionic convex hull is simply given by $\hat{\Delta}_\alpha = \{\tilde{s}^a \geq \frac{1}{\alpha}\}$, its dual saxionic counterpart $\hat{\mathcal{P}}_\alpha$ is more complicated – see figure B.1.

The anti-canonical divisor is

$$\begin{aligned} \overline{K}_X &= 2E^3 + (2+h)E^1 + (2+n+nh)E^2 \\ &= 2D^3 + (2-h)D^1 + (2-n)D^2\,. \end{aligned} \tag{B.3.9}$$

Hence from (4.4.3) and (4.3.5) we get

$$\gamma(s) = \pi\left[2\tilde{s}^3 + (2+h)\tilde{s}^1 + (2+n+nh)\tilde{s}^2\right]\,. \tag{B.3.10}$$

Hence

$$\gamma(s)|_{\hat{\Delta}_\alpha} \geq \frac{\pi(6+n+h+nh)}{\alpha}\,, \tag{B.3.11}$$

which is again stronger than (4.2.18) with $N = 3$. If for instance $n, h \geq 1$ and $\alpha \leq \frac{1}{10}$, we get the lower bound $\gamma(s)|_{\hat{\Delta}_\alpha} > 282$.

[5]Note that, if we consider fibrations of the kind described in footnote 4 with $p > 0$, in general the image of the cone $\{\ell_a > 0\}$ under the map (B.3.7) does not cover the entire saxionic cone Δ – see [71] – as discussed in general in Section 4.3.1. This issue is absent if we set $p = 0$.

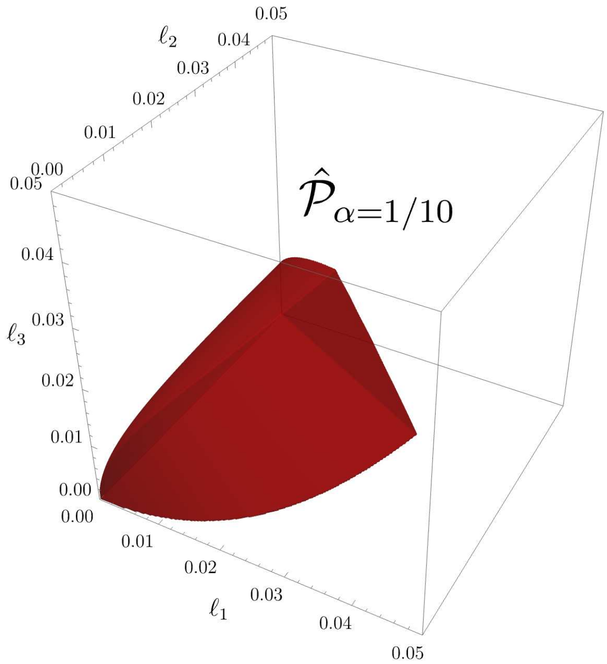


Figure B.1: Dual saxionic convex hull $\hat{\mathcal{P}}_\alpha$ for the F-theory model $\mathbb{P}^1$ over $\mathbb{F}_n$. The plot has been drawn with the reference value $\alpha = 1/10$. Note that the valid region is the red one (as opposed to figure 4.1b where the valid region is the one not colored).

B.3.2 Energy scales in the $\mathbb{P}^1$ over $\mathbb{F}_n$ model

In the F-theory model of $\mathbb{P}^1$ fibration over $\mathbb{F}_n$ of the previous section B.3.1 we have seen that we have the following set of EFT string tensions, each associated to a generator of the cone of movable curves

$$\mathcal{T}_{\hat{\Sigma}_1} = M_{\rm P}^2(\ell_1 + h\ell_3)\,, \quad \mathcal{T}_{\hat{\Sigma}_2} = M_{\rm P}^2(n\ell_1 + \ell_2 + hn\ell_3)\,, \quad \mathcal{T}_{\hat{\Sigma}_3} = M_{\rm P}^2\ell_3\,. \tag{B.3.12}$$

It is clear that, assuming $n, h > 0$, $\mathcal{T}_{\hat{\Sigma}_3}$ is always the lowest of the three in the controlled regime and can thus be identified with the upper bound (4.2.23)

$$\Lambda^2_{\rm max} = \mathcal{T}_* = M_{\rm P}^2\ell_3\,. \tag{B.3.13}$$

If $\ell_3 \ll \ell_1, \ell_2$ and hence $\mathcal{T}_* \ll \mathcal{T}_{\hat{\Sigma}_1}, \mathcal{T}_{\hat{\Sigma}_2}$, we know the limit corresponds to the weakly coupling limit of a critical string in the dual heterotic frame, whose tension given by $\mathcal{T}_*$ can be seen as the species scale, so that (4.2.23) is satisfied and saturated.

This time we have two different regimes to look at, the one with $\mathcal{T}_* \simeq \mathcal{T}_{\hat{\Sigma}_1}$ and the one with $\mathcal{T}_* \simeq \mathcal{T}_{\hat{\Sigma}_2}$, and check (4.2.23).

In the first case, $\mathcal{T}_* \simeq \mathcal{T}_{\hat{\Sigma}_1}$, we should have $\mathcal{T}_{\hat{\Sigma}_1}/\mathcal{T}_* \simeq 1$, but also

$$\frac{\mathcal{T}_{\hat{\Sigma}_1}}{\mathcal{T}_*} = \frac{\ell_1}{\ell_3} + h > h\,. \tag{B.3.14}$$

Hence, this is possible only if $h \sim \mathcal{O}(1)$ and $\ell_1/\ell_3 \lesssim 1$. We must also remember that to be sure to be in a weakly coupled EFT description we have to impose $\mathcal{T}_{\hat{\Sigma}_1}, \mathcal{T}_{\hat{\Sigma}_2} \leq M_{\rm P}^2$. In this situation these are $\ell_1 + h\ell_3 \leq 1$ and $n\ell_1 + \ell_2 + hn\ell_3 \leq 1$, therefore they imply $\ell_3 \lesssim \frac{1}{hn}$, $\ell_1 \lesssim \frac{1}{n}$ and $\ell_2 \lesssim 1$. Thanks to these upper bounds, we can see that the square of the 10-dimensional Planck scale, given by

$$\Lambda_{\rm QG}^2 = M_{\rm P}^2 \sqrt{2\ell_1\ell_2\ell_3 + n\ell_1^2\ell_3 + hn\ell_1\ell_3^2 + h\ell_2\ell_3^2 + \frac{1}{3}h^2 n\ell_3^3}\,, \tag{B.3.15}$$

is indeed constrained from above or at best of the same order of $\mathcal{T}_*$, neglecting an irrelevant $\mathcal{O}(1)$ factor.

In the second case, $\mathcal{T}_* \simeq \mathcal{T}_{\hat{\Sigma}_2}$, we can repeat the same line of reasoning. We must have $\mathcal{T}_{\hat{\Sigma}_2}/\mathcal{T}_* \simeq 1$, but we know that

$$\frac{\mathcal{T}_{\hat{\Sigma}_2}}{\mathcal{T}_*} = n\frac{\ell_1}{\ell_3} + \frac{\ell_2}{\ell_3} + hn > hn\,. \tag{B.3.16}$$

Hence, we need $h, n \sim \mathcal{O}(1)$, together with $\ell_1/\ell_3 \lesssim 1/n$ and $\ell_2/\ell_3 \lesssim 1$. We have again the same conditions as before the EFT control, i.e. $\mathcal{T}_{\hat{\Sigma}_1}, \mathcal{T}_{\hat{\Sigma}_2} \leq M_{\rm P}^2$, giving us $\ell_3 \lesssim \frac{1}{hn}$, $\ell_1 \lesssim \frac{1}{n}$ and $\ell_2 \lesssim 1$. Therefore, we can even in this case bound from above the scale (B.3.15) with $\mathcal{T}_*$, up to a $\mathcal{O}(1)$ factor, confirming our conjecture of section 4.2.3 also in this example.

B.3.3 $N = 2$ models in 4d Type IIA

Let us recall some generalities that will be useful for computing and comparing the species scale in various models and EFT string limits, in particular the case of $N = 2$ models in 4d in type IIA and the heterotic models with $p_a e^a = 0$. The universal saxion s^0 is given by

$$s^0 = \frac{1}{6} e^{-2\phi} \kappa(\boldsymbol{s}, \boldsymbol{s}, \boldsymbol{s}) = e^{-2\phi} V_X\,, \tag{B.3.17}$$

where ϕ is the 10d dilaton and V_X is the volume of the Calabi-Yau X in string units as seen from the 10d string frame metric, that reads

$$\mathrm{d}s^2 = e^{2A}\mathrm{d}s_4^2 + \ell_s^2 \mathrm{d}s_X^2\,, \tag{B.3.18}$$

with

$$e^{2A} = \frac{\ell_s^2 M_{\rm P}^2 e^{2\phi}}{4\pi V_X} = \frac{\ell_s^2 M_{\rm P}^2}{4\pi s^0}\,, \tag{B.3.19}$$

fixed by taking $\mathrm{d}s_4^2$ to be the 4d Einstein frame metric. We can distinguish from the start the two notable cases of infinite distance limits: the one where the s^0 is taken to diverge along the flow of σ as $s^0(\sigma) = s_0^0 + \sigma$ with the Kähler saxions, and hence V_X, kept fixed, that is a weakly string coupling limit as we can see from (B.3.17); and the ones where instead we keep s^0 fixed, that will represent strong coupling limits.

In the first case of weak coupling, that is of 10d dilaton becoming small, the EFT string limit represents a 10-dimensional fundamental string becoming tensionless, and its corresponding tension measured from the 4d Einstein frame is

$$\mathcal{T}_* = \frac{2\pi e^{2A}}{\ell_s^2} = M_{\rm P}^2 \ell_0 = \frac{M_{\rm P}^2}{2s^0} \tag{B.3.20}$$

It is easy to see that the square of the mass scale of the lightest KK tower is proportional to such tension. In fact, it is given by

$$m_*^2 = \frac{e^{2A}}{R_*^2} = \frac{\ell_s^2 \mathcal{T}_*}{2\pi R_*^2}\,, \tag{B.3.21}$$

with R_*^2 being the largest compactification length-scale measured in the string frame. But, since in this infinite distance limit we keep these length scale fixed, this will scale with σ exactly like the tension, realizing the $w = 1$ situation of [64]. In this case the quantum gravity species scale is just the string scale and no other check is needed.

In the second class of cases, the EFT string comes from NS5-branes wrapped on internal 4-cycles that are nef divisors. The tension associated to an EFT string from the divisor $\boldsymbol{e} = [D] = e^a[D_a]$ is given by

$$\mathcal{T}_{\boldsymbol{e}} = M_{\rm P}^2 e^a \ell_a = \frac{3M_{\rm P}^2 \kappa(\boldsymbol{e},\boldsymbol{s},\boldsymbol{s})}{2\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})}\,. \tag{B.3.22}$$

Keeping s^0 fixed has the effect of making $e^{2\phi}$ diverge as $\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})$, as we can see from (B.3.17). New light degrees of freedom can appear from strong string coupling effect, in particular the scale m_* is now given by the KK scale along the circle S^1 that is becoming large in the type IIA to M-theory limit, or equivalently the mass of the D0-branes. Taking into account the Einstein frame rescaling, we have

$$m_*^2 = \frac{2\pi e^{2A} e^{-2\phi}}{\ell_s^2} = \frac{3M_{\rm P}^2}{\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})}\,. \tag{B.3.23}$$

It is interesting also to look at the mass of the states coming from D2-branes wrapped on a 2-cycle $\mathcal{C}$ that might become relevant in the problem of evaluating the scales of the setting

$$m_{\rm D2}^2 = \frac{2\pi e^{2A} e^{-2\phi} V_{\mathcal{C}}^2}{\ell_s^2} = \frac{3M_{\rm P}^2 V_{\mathcal{C}}^2}{\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})}\,. \tag{B.3.24}$$

Which towers of states should be considered in computing the quantum gravity scale depends on the intersection properties of the nef divisor associated to the charge $\mathbf{e}$ and the associated saxion flow $\boldsymbol{s} = \boldsymbol{s}_0 + \mathbf{e}\sigma$. As remarked in [64] and [70], these can be divided into three possibilities.

Case 1: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e})>0$

In this case we obtain $\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})\simeq\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{e})\sigma^3$ and $\kappa(\boldsymbol{e},\boldsymbol{s},\boldsymbol{s})\simeq\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{e})\sigma^2$, hence $\mathcal{T}_{\boldsymbol{e}}\simeq\frac{3M_{\rm P}^2}{2\sigma}$ and $m_*^2\simeq\frac{3M_{\rm P}^2}{\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{e})\sigma^3}$. Therefore it is a $w=3$ EFT string limit. The KK mass scale is $m_{KK}=\frac{e^A}{\ell_s(V_X)^{\frac{1}{6}}}$ and it is parametrically higher than m_*. The same occurs for m_{D2} for any internal 2-cycle. We can compute the quantum gravity scale using only the KK species coming from the tower of D0-branes with mass m_*. If we do that, we get $\Lambda_{\rm QG}^2\simeq\frac{\sqrt[3]{3}M_{\rm P}^2}{\sqrt[3]{\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{e})\sigma}}$, that is manifestly less than $\mathcal{T}_{\boldsymbol{e}}$.

Case 2: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e})=0$ *but* $\kappa(\mathbf{e},\mathbf{e},\mathbf{e}')>0$ *for some* $\mathbf{e}'\in{\rm Nef}_{\mathbb{Z}}(X)$

In this limit, the CY base X can be seen as a T^2 fibration over a twofold, with the non-trivial curve $\mathcal{C}=\boldsymbol{e}\cdot\boldsymbol{e}$ being a multiple of the T^2 fiber. We have $\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})\simeq3\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s}_0)\sigma^2$ and $\kappa(\boldsymbol{e},\boldsymbol{s},\boldsymbol{s})\simeq2\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s}_0)\sigma$, hence $\mathcal{T}_{\boldsymbol{e}}\simeq\frac{M_{\rm P}^2}{\sigma}$ and $m_*^2\simeq\frac{M_{\rm P}^2}{\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s}_0)\sigma^2}$, realizing a $w=2$ EFT string limit. The volume of $\mathcal{C}$ in the string frame is constant along this flow, indeed we have $\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s})=\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s}_0)$. Therefore the CY KK scale comes from the 4-dimensional base of the fibration becoming large, that scales as $m_{KK}^2\simeq\frac{M_{\rm P}^2}{s^0\sigma}$, again much heavier than m_*. But this time the mass of the wrapped D2-branes is given by $m_{D2}^2\simeq\frac{M_{\rm P}^2\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s}_0)}{\sigma^2}$, the same rate of m_*. The evaluation of the species scale must be performed by taking into account both those towers as multiplicative species, as described by [217]. The result is a $\Lambda_{\rm QG}^2$ scaling exactly like $\mathcal{T}_{\boldsymbol{e}}$ and being strictly lower than it if we consider the $O(1)$ factor coming from the actual number of states below $\Lambda_{\rm QG}$ not being just $N_{D0}N_{D2}=\frac{\Lambda_{\rm QG}^2}{m_*m_{D2}}$.

Case 3: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e}')=0$ *for any* $\mathbf{e}'\in{\rm Nef}_{\mathbb{Z}}(X)$

In this case we have $\boldsymbol{e}\cdot\boldsymbol{e}=0$ and in this limit the CY can be seen as a K3 or T^4 fibration over a $\mathbb{P}^1$. We have $\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})\simeq3\kappa(\boldsymbol{e},\boldsymbol{s}_0,\boldsymbol{s}_0)\sigma$ and $\kappa(\boldsymbol{e},\boldsymbol{s},\boldsymbol{s})=\kappa(\boldsymbol{e},\boldsymbol{s}_0,\boldsymbol{s}_0)$. Hence m_*^2 scales like the tension and this is a $w=1$ EFT string limit. There exist a dual description in which this is a tensionless critical string limit. Thus, like in the case where s^0 was not fixed, the quantum gravity species scale will be automatically given by the string scale.

B.3.4 EFT string limits in heterotic models

Consider an EFT string in a heterotic setting with charge vector $\boldsymbol{e}=(e^0,e^a)$. We must have $e^0\geq p_ae^a$ to have a good EFT string charge and we will make the minimal choice $e^0=p_ae^a>0$ in this section. With $p_a\neq0$, the definition of the universal saxion has to be changed as

$$s^0=\frac{1}{6}e^{-2\phi}\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})+\frac{1}{2}p_as^a\,, \tag{B.3.25}$$

and we can define $\hat{V}_X=s^0-\frac{1}{2}p_as^a$. Notice that, with the choice $e^0=p_ae^a>0$, we have $\hat{V}_X\simeq\frac{1}{2}p_ae^a\sigma$ for any asymptotic EFT string limit of the class that we want to study. Again we can divide these EFT string limits according to the triple intersection number of the associated nef divisor $D_{\boldsymbol{e}}$.

Case 1: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e})>0$

In this case we obtain $\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})\simeq\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{e})\sigma^3$ and $\kappa(\boldsymbol{e},\boldsymbol{s},\boldsymbol{s})\simeq\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{e})\sigma^2$, hence $e^{2\phi}\simeq\frac{\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{e})}{3p_a e^a}$, $\mathcal{T}_{D_{\boldsymbol{e}}}\simeq\frac{5M_{\mathrm{P}}^2}{2\sigma}$ and $m_*^2\sim M_{\mathrm{P}}^2\sigma^{-3}$. Therefore it is a $w=3$ EFT string limit. The KK mass scale goes like $m_{KK}\sim M_{\mathrm{P}}^2\sigma^{-1}$ and it is parametrically higher than m_*. Also m_{D2} is parametrically higher for any internal 2-cycle. We can hence compute the quantum gravity scale using only the KK species coming from the tower of D0-branes with mass m_*. If we do that, we get $\Lambda_{\mathrm{QG}}^2\sim\frac{M_{\mathrm{P}}^2}{\sigma}$, that is asymptotically less than or equal to $\mathcal{T}_{D_{\boldsymbol{e}}}$.

Case 2: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e})=0$ *but* $\kappa(\mathbf{e},\mathbf{e},\mathbf{e}')>0$ *for some* $\mathbf{e}'\in\mathrm{Nef}_{\mathbb{Z}}(X)$

In this limit, the CY base X can be seen as a T^2 fibration over a twofold, with the non-trivial curve $\mathcal{C}=\boldsymbol{e}\cdot\boldsymbol{e}$ being a multiple of the T^2 fiber. We have $\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})\simeq 3\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s}_0)\sigma^2$ and $\kappa(\boldsymbol{e},\boldsymbol{s},\boldsymbol{s})\simeq 2\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s}_0)\sigma$, hence $\mathcal{T}_{D_{\boldsymbol{e}}}\simeq\frac{2M_{\mathrm{P}}^2}{\sigma}$. The volume of $\mathcal{C}$ in the string frame is constant along this flow, indeed we have $\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s})=\kappa(\boldsymbol{e},\boldsymbol{e},\boldsymbol{s}_0)$. The D0-brane mass and CY KK scalings have been computed in detail in [65] and are given by $m_*^2\sim m_{KK}^2\sim\frac{M_{\mathrm{P}}^2}{\sigma^2}$, corresponding to a $w=2$ EFT string limit. As in the $N=2$ case, the presence of two or more mass towers becoming massless with the same scaling $\sim\sigma^{-2}$ is enough to ensure the species scale is asymptotically lower to or equal to $\mathcal{T}_{D_{\boldsymbol{e}}}$.

Case 3: $\kappa(\mathbf{e},\mathbf{e},\mathbf{e}')=0$ *for any* $\mathbf{e}'\in\mathrm{Nef}_{\mathbb{Z}}(X)$

In this case we have $\boldsymbol{e}\cdot\boldsymbol{e}=0$ and in this limit the CY can be seen as a K3 or T^4 fibration over a $\mathbb{P}^1$. We have $\kappa(\boldsymbol{s},\boldsymbol{s},\boldsymbol{s})\simeq 3\kappa(\boldsymbol{e},\boldsymbol{s}_0,\boldsymbol{s}_0)\sigma$ and $\kappa(\boldsymbol{e},\boldsymbol{s},\boldsymbol{s})=\kappa(\boldsymbol{e},\boldsymbol{s}_0,\boldsymbol{s}_0)$, moreover the dilaton is constant along the flow in this particular limit. m_*^2 here scales like the tension but this is not a $w=1$ EFT string limit as one could naively think. Indeed the detailed analysis in [65] shows that in this case we have $m_{KK}^2\sim\frac{M_{\mathrm{P}}^2}{\sigma^2}$, thus this is actually a $w=2$ EFT string limit. What is becoming large is the $\mathbb{P}^1$ base of the K3 or T^4 fibration, thus there will be two separate KK towers of states becoming massless with the same scaling for m_{KK}. The calculation of the species scale is analogous to the one in B.3.3, hence again a species scale asymptotically lower than or equal to the square root of the EFT string tension.

B.4 GB correction to on-shell action

In this appendix we show that discuss the GB term contribution to the on-shell effective action of the homogeneous wormholes is proportional to the BPS on-shell action (4.4.35), up to a coefficient dependent on n and $\mathbf{q}$. More precisely, this additional contribution gives we are working in PT

$$S_{\mathrm{GB}}|_{\mathrm{hw}}=\frac{\pi\tilde{C}_i\mathcal{F}^i(\mathbf{q})}{8\tilde{\ell}_*}\int_0^{\frac{\pi}{2}}\mathrm{d}x\frac{\cos^3 x}{\cos\left(\sqrt{\frac{3}{n}}\,x\right)}\qquad\text{(B.4.1)}$$

Combining it with (4.5.20), we get

$$(S+S_{\rm GB})|_{\rm hw}=2\pi B_{n,\mathbf{q}}\,\langle\mathbf{q},\boldsymbol{s}_\infty\rangle\,, \tag{B.4.2}$$

with

$$B_{n,\mathbf{q}}\equiv\sin\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right)+\frac{\tilde{C}_i\mathcal{F}^i(\mathbf{q})}{8n}\cos\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right)\int_0^{\frac{\pi}{2}}\mathrm{d}x\frac{\cos^3 x}{\cos\left(\sqrt{\frac{3}{n}}\,x\right)}\,, \tag{B.4.3}$$

where the second term comes from the GB term. Notice that $\mathcal{F}^i(\lambda\mathbf{q})=\frac{1}{\lambda}\mathcal{F}^i(\mathbf{q})$, which shows that $\mathcal{F}^i(\mathbf{q})$ and hence the GB contribution is smaller for larger charges and fixed asymptotic $\boldsymbol{s}_\infty$, as expected. In order to get an idea on (B.4.2), let us assume that $\tilde{\mathbf{C}}\in\mathcal{C}_{\rm I}^{\rm EFT}$ and pick charges $q_i=\frac{1}{3}\tilde{C}_i$ (which would for instance be allowed by the quantization conditions in the F-theory models), so that $\tilde{C}_i\mathcal{F}^i(\mathbf{q})=3n$. With this specific choice $B_{4,\mathbf{q}}\simeq 1.036$, $B_{5,\mathbf{q}}\simeq 1.032$, $B_{6,\mathbf{q}}\simeq 1.015$ and $B_{7,\mathbf{q}}\simeq 0.993$. So we see that for $n=4,5,6$ the GB term is already sufficient to make $(S+S_{\rm GB})|_{\rm hw}$ larger than the BPS action. On the one hand, this clearly shows how higher-derivative terms can drastically change the relation between the on-shell half-wormhole action and the corresponding BPS action. On the other hand, one should not take (B.4.2) too seriously, since there could be other additional terms coming from other four-derivative terms involving the saxions.

In the marginally degenerate case $n=3$, by adopting the $n\to 3(1+\frac{4\epsilon}{\pi})$ regularization described in Section 4.5.3, the equation (B.4.3) gives the following leading contributions

$$B_{n,\mathbf{q}}=1+\frac{\pi\tilde{C}_i\mathcal{F}^i(\mathbf{q})}{96}\,\epsilon+\mathcal{O}(\epsilon^2) \tag{B.4.4}$$

B.5 Universal wormholes with factorized metrics

In this appendix we study more in detail the possible restrictions on the throat value $\boldsymbol{\ell}_*$ imposed by the global existence of the wormhole solution. A similar discussion in a slightly different context and conventions can be found in [218]. Assuming a Kähler potential of the form (4.1.3), we already know that the allowed $\boldsymbol{\ell}_*$ form a cone.

If we have multiple decoupled sectors one can get larger families of explicit wormhole solutions. Suppose

$$\mathcal{F}(\ell^{(1)},\ell^{(2)})=\mathcal{F}_{(1)}(\ell^{(1)})+\mathcal{F}_{(2)}(\ell^{(2)})\,. \tag{B.5.1}$$

In this case the equations of motion for the saxions in the two sectors are completely decoupled. Furthermore $\mathbf{q}=(\mathbf{q}^{(1)},\mathbf{q}^{(2)})$ and

$$V_{\mathbf{q}}(\boldsymbol{\ell})=V^{(1)}_{\mathbf{q}^{(1)}}(\boldsymbol{\ell}^{(1)})+V^{(2)}_{\mathbf{q}^{(2)}}(\boldsymbol{\ell}^{(2)})\,. \tag{B.5.2}$$

From (4.4.23) it follows that

$$\frac{1}{(\tau_\infty)^2} = \frac{1}{(\tau_\infty^{(1)})^2} + \frac{1}{(\tau_\infty^{(2)})^2}\,. \tag{B.5.3}$$

This shows that the total τ_∞ is generically smaller than both $\tau_\infty^{(1)}$ and $\tau_\infty^{(2)}$. Hence, the flow in the two sectors must roll along a shorter path in order to completely traverse the entire wormhole, and this can help the existence of a global solution.

Consider now the particular case in which $\mathcal{F}^{(1)}$ takes the form (4.1.4). In the first sector we can restrict to a profile $\boldsymbol{\ell}^{(1)}(\tau) = \mathbf{q}^{(1)}\tilde{\ell}^{(1)}(\tau)$, as in section 4.5.1. In particular

$$\tilde{\ell}^{(1)}(\tau) = \tilde{\ell}_*^{(1)} \cos\left(\frac{\pi}{2}\sqrt{\frac{3}{n_1}}\,\frac{\tau}{\tau_\infty^{(1)}}\right) \tag{B.5.4}$$

where n_1 is the degrees of $\tilde{P}^{(1)}(\ell^{(1)})$ and

$$\tau_\infty^{(1)} = \frac{\pi}{2}\sqrt{\frac{3}{n_1}}\tilde{\ell}_*^{(1)}\,. \tag{B.5.5}$$

The profile (B.5.4) is everywhere regular only if $\tau_\infty^{(1)} > \sqrt{\frac{3}{n_1}}\,\tau_\infty$, which by (B.5.3) becomes

$$\left(\frac{\tau_\infty^{(1)}}{\tau_\infty^{(2)}}\right)^2 > \frac{3-n_1}{n_1}\,. \tag{B.5.6}$$

By using (B.5.5) and the more general (4.4.23) for the second sector, we get the condition

$$\left(\tilde{\ell}_*^{(1)}\right)^2 > \frac{3-n_1}{4|V^{(2)}_{\mathbf{q}^{(2)}}(\ell_*^{(2)})|}\,. \tag{B.5.7}$$

Hence, if $n_1 < 3$, for fixed $\boldsymbol{\ell}_*^{(2)}$ the initial value $\tilde{\ell}_*^{(1)}$ defining the flow in the first sector is not arbitrary, but must satisfy the lower bound (B.5.7).

Of course, this is not sufficient to guarantee the existence of an admissible profile in the second sector too, which similarly depends on $\tilde{\ell}_*^{(1)}$ too. These mutual constraints become more explicit if we assume that $\mathcal{F}^{(1)}$ takes the form (4.1.4) and we restrict to a profile $\boldsymbol{\ell}^{(2)}(\tau) = \mathbf{q}^{(2)}\tilde{\ell}^{(2)}(\tau)$, too. By inverting the roles of the two sectors in (B.5.7), one then gets the the condition for the global existence of the second profile. Taking into account (4.5.5), we conclude that the wormhole is globally well defined if and only if $\tilde{\ell}_*^{(1)}$ and $\tilde{\ell}_*^{(2)}$:

$$\left(\frac{\tilde{\ell}_*^{(1)}}{\tilde{\ell}_*^{(2)}}\right)^2 > \frac{3-n_1}{n_2} \quad , \quad \left(\frac{\tilde{\ell}_*^{(2)}}{\tilde{\ell}_*^{(1)}}\right)^2 > \frac{3-n_2}{n_1}\,. \tag{B.5.8}$$

Note that these conditions identify a (possibly empty) cone. If we choose $\tilde{\ell}_*^{(1)} = \tilde{\ell}_*^{(2)}$, as in the universal solutions of section 4.5.1, the conditions (B.5.8) reduce to

$$n = n_1 + n_2 > 3\,, \tag{B.5.9}$$

as expected. This necessary condition must actually hold more generically, as can be seen by taking the product of the two conditions (B.5.8), and is equivalent to requiring that the right-hand sides of (B.5.8) are smaller than 1. As another consistency checks, if $n_1, n_2 \geq 3$, they are always satisfied. On the other hand, (B.5.8) is generically non-trivial. If for instance $n_1 = n_2 = 2$, then we get the cone identified by $\tilde{\ell}_*^{(1)} < \sqrt{2}\,\tilde{\ell}_*^{(2)}$ and $\tilde{\ell}_*^{(2)} < \sqrt{2}\,\tilde{\ell}_*^{(1)}$. If instead for instance $n_1 = 1$ and $n_2 = 3$, then they reduce to $\tilde{\ell}_*^{(1)} > \sqrt{\frac{2}{3}}\,\tilde{\ell}_*^{(2)}$.

Note, however,that these constraints on $\tilde{\ell}_*^{(1)}$ and $\tilde{\ell}_*^{(2)}$ do not restrict $\tilde{\ell}_\infty^{(1)}$ and $\tilde{\ell}_\infty^{(2)}$. Let us set

$$x_* \equiv \frac{\tilde{\ell}_*^{(1)}}{\tilde{\ell}_*^{(2)}} \tag{B.5.10}$$

so that (B.5.8) correspond to $x_*^2 > \frac{3-n_1}{n_2}$ and $\frac{1}{x_*^2} > \frac{3-n_2}{n_1}$. One can obtain the relation

$$x_\infty \equiv \frac{\tilde{\ell}_\infty^{(1)}}{\tilde{\ell}_\infty^{(2)}} = x_* \frac{\cos\left(\frac{\pi}{2}\sqrt{\frac{3}{n_1+n_2x_*^2}}\right)}{\cos\left(\frac{\pi}{2}\sqrt{\frac{3x_*^2}{n_1+n_2x_*^2}}\right)}\,. \tag{B.5.11}$$

Hence, assuming that $n_1, n_2 \leq 3$, for $x_*^2 \to \frac{3-n_1}{n_2}$ we have $x_\infty \to 0$, while for $1/x_*^2 \to \frac{3-n_2}{n_1}$ we have $x_\infty \to \infty$. Hence, all the possible asymptotic values can be obtained.

It is also interesting to observe that in the critical case $n = n_1 + n_2 = 3$, the bounds (B.5.8) become $x_* < 1$ and $x_* > 1$, respectively. Hence, we can at best saturate and only marginally violate them by picking $\tilde{\ell}_*^{(1)} = \tilde{\ell}_*^{(2)}$. This marginal violation is still physically acceptable, since the degeneration happens at infinite spatial distance. On the other hand, if we choose $\tilde{\ell}_*^{(1)} \neq \tilde{\ell}_*^{(2)}$, one of the two bounds is necessarily non-marginally violated (while the other is satisfied), hence giving a non-physical configuration. This would suggest that the only marginally sensible configuration is the one with $\tilde{\ell}_*^{(1)} = \tilde{\ell}_*^{(2)}$ and then $\tilde{\ell}^{(1)}(\tau) = \tilde{\ell}^{(2)}(\tau)$ for any $\tau \in (0, \tau_\infty)$. Consider, however, a slightly regularized choice, in which $n = 3 + \epsilon$ and $n_1 < 3$, so that $n_2 = 3 + \epsilon - n_1$. The bound (B.5.8) can now be written as $x_*^2 > \frac{3-n_1}{3-n_1+\epsilon}$ and $x_*^2 < \frac{n_1}{n_1-\epsilon}$ and then identifies a finite allowed interval for x, of order ϵ around 1. Note, however, that this infinitesimal interval for x_* maps to the infinite interval $x_\infty \in (0, \infty)$ at spatial infinity. By taking the limit $\epsilon \to 0$ we then gets flows with $\tilde{\ell}^{(1)}(\tau) \neq \tilde{\ell}^{(2)}(\tau)$ even if $\tilde{\ell}_*^{(1)} = \tilde{\ell}_*^{(2)}$.

This discussion can be readily generalized to the case of a higher number of sectors of the form (4.1.4): $\mathcal{F}(\ell^{(1)}, \ell^{(2)}, \ldots) = \sum_a \mathcal{F}_{(a)}(\ell^{(a)})$. In particular, the existence bounds (B.5.8) become

$$\frac{1}{(\ell_*^{(a)})^2} < \frac{1}{3}\sum_b \frac{n_b}{(\ell_*^{(b)})^2}\,. \tag{B.5.12}$$

Multiplying them by n_a and summing over their index a, we get again the necessary condition $\sum_a n_a > 3$.

As above, we can define $x_*^{ab} \equiv \frac{\ell_*^{(a)}}{\ell_*^{(b)}}$ and try to repeat the previous discussion to see whether all the possible asymptotic values x_∞^{ab} can be reached from allowed initial conditions. Let us assume N decoupled sectors and focus on x_*^{aN}. If we have $n_a < 3$ there will be a non-trivial lower bound on x_*^{aN}, similarly if $n_N < 3$ there will be an upper bound on it as well.

$$\frac{3-n_a}{\sum_{b\neq a} n_b (x_*^{bN})^{-2} + n_N} < (x_*^{aN})^2 < \frac{n_a}{3 - \sum_{b\neq a} n_b (x_*^{bN})^{-2} - n_N}\,. \tag{B.5.13}$$

For this to be well defined we need to impose

$$\frac{3-n_a}{\sum_{b\neq a} n_b (x_*^{bN})^{-2} + n_N} \leq \frac{n_a}{3 - \sum_{b\neq a} n_b (x_*^{bN})^{-2} - n_N} \Rightarrow \sum_{b\neq a} n_b (x_*^{bN})^{-2} + n_N + n_a \geq 3 \tag{B.5.14}$$

Notice that these bounds identify an open set around 1 that degenerates into just $x_*^{aN} = 1$ when the condition above saturates the inequality. If $N > 3$, a single $x_*^{bN} \leq 1$ with $b \neq a$ is enough to ensure the set is non-degenerate.

The asymptotic value of $(x_\infty^{aN})^2$ is given by

$$(x_\infty^{aN})^2 = (x_*^{aN})^2 \frac{\cos^2\left(\frac{\pi}{2}\sqrt{\frac{3}{n_a + (x_*^{aN})^2 \sum_{b\neq a} n_b (x_*^{bN})^{-2}}}\right)}{\cos^2\left(\frac{\pi}{2}\sqrt{\frac{3}{n_a (x_*^{aN})^{-2} + \sum_{b\neq a} n_b (x_*^{bN})^{-2}}}\right)}\,. \tag{B.5.15}$$

As $(x_*^{aN})^2$ approaches the saturation of its bound, that is $(x_*^{aN})^2 \to \frac{3-n_a}{\sum_{b\neq a} n_b (x_*^{bN})^{-2}}$, the asymptotic value will go to zero, provided that the argument of the cosine at denominator does not go to $\pi/2$ as well. This can happen if we have

$$n_a (x_*^{aN})^{-2} + \sum_{b\neq a} n_b (x_*^{bN})^{-2} > 3\,. \tag{B.5.16}$$

Therefore, in the considered limit, when

$$\sum_{b\neq a} n_b (x_*^{bN})^{-2} > 3 - n_a \Rightarrow n_a + n_N + \sum_{b\neq a,N} n_b (x_*^{bN})^{-2} > 3\,, \tag{B.5.17}$$

and this can be satisfied if we have $N \geq 4$ and if we are allowed to take at least one of the $(x_*^{bN})^{-2} > 1$, and it could be marginally violated in the case $N = 3$ with $n_1 = n_2 = n_3 = 1$. Indeed, the bound on the generic x_*^{bN} allows us to choose it equal to $1 - \epsilon < 1$ when $N \geq 4$ in the saturation limit of x_*^{aN}

$$\begin{aligned}\frac{3}{(x_*^{bN})^2} &< \frac{n_a}{(x_*^{aN})^2} + \frac{n_b}{(x_*^{bN})^2} + \sum_{c\neq a,b,N} \frac{n_c}{(x_*^{cN})^2} + n_N \\ \Rightarrow (x_*^{bN})^2 &> \frac{3 - n_a - n_b}{\sum_{c\neq a,b,N} n_c (x_*^{cN})^{-2} + n_N} \,.\end{aligned} \tag{B.5.18}$$

Being all the ns positive integers, it is clear the lower bound on x_*^{bN} is a number strictly lower than 1. The same argument can be repeated for each x_*^{aN} to conclude that we can reach any $(x_\infty^{1N}, \cdots, x_\infty^{N-1,N}) \in (0,1)^{N-1}$.

The opposite saturation limit would be when the argument of the cosine at denominator goes to $\pi/2$, that is $(x_*^{aN})^2 \to \frac{n_a}{3-\sum_{b\neq a} n_b (x_*^{bN})^{-2}}$. In this limit the asymptotic value $(x_\infty^{aN})^2$ will diverge, provided that the argument of the cosine at numerator does not vanish. But, again as above, this condition amounts to

$$\frac{3}{n_a + \frac{n_a}{3-\sum_{b\neq a} n_b (x_*^{bN})^{-2}} \sum_{b\neq a} n_b (x_*^{bN})^{-2}} < 1 \Rightarrow \sum_{b\neq a,N} n_b (x_*^{bN})^{-2} > 3 - n_a - n_N \,, \tag{B.5.19}$$

exactly as in (B.5.17). Hence, we are again sure this can be done if we have at least one $x_*^{bN} = 1$, and notice that in this case we are not allowed to take it strictly lower than 1. Since all this discussion was done for a generic x_*^{aN}, this means we are sure to be able to explore all the asymptotic values $x_\infty^{aN} \in (0, \infty)$ by starting from the allowed initial conditions.

We are left with the question about the behavior of the x_∞^{bN} with $b \neq a$ when $x_\infty^{aN} \to \infty$. For each of them, it is still true that the argument in the cosine at denominator is approaching $\pi/2$, so we have to check what the argument at numerator is doing in the same saturation regime. In particular it will be given by

$$\frac{\tau_\infty^{(b)}}{\tilde{\ell}_*^{(b)}} = \frac{\pi}{2}\sqrt{\frac{3}{n_b + (x_*^{bN})^2 (n_a (x_*^{aN})^{-2} + \sum_{c\neq a,b,N} n_c (x_*^{cN})^{-2} + n_N)}} \to \frac{\pi}{2x_*^{bN}} \,, \tag{B.5.20}$$

therefore whether x_∞^{bN} will be finite or infinite in this limit depends on whether x_*^{bN} is, respectively, equal to or greater than 1. One might worry that taking each $x_*^{bN} > 1$ risks to be inconsistent with the condition (B.5.17), and hence that the region where all the asymptotic values become big is not accessible. However, there is no problem for $N \geq 4$. Indeed, in this situation we can always satisfy such condition by taking $x_*^{bN} = 1 + \epsilon_1$ and $x_*^{cN} = 1 + \epsilon_2$, with $b, c \neq a, N$, such that $n_b (x_*^{bN})^{-2} + n_c (x_*^{cN})^{-2} > 1$.

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