THE INFLUENCE OF DEPENDENCE ON DATA NETWORK MODELS OF BURSTINESS

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Abstract. We consider an infinite-source Poisson process to model end user inputs to a data network. We assume that the sources initiate transmissions according to a Poisson process and that transmission rates and durations are independent random variables. We analyze the traffic process that is obtained by discretizing time in slots of length δ and considering the quantity of transmitted data in adjacent time intervals. We study this discrete time process as the slot length δ goes to 0. This analysis extends and complements work in D'Auria and Resnick (2006) where an analogous model was studied which assumed independence of the transmission rates and the file sizes. It is striking that the two cases show rather different behaviour. While the cumulative input per slot in D'Auria and Resnick (2006) converges marginally to a normal distribution, in the model considered here we have an approximating distribution which is stable with infinite second moment. We also study dependence across time slots, characterize its slow rate of decay, and provide a detailed comparison of the two models.

1. Introduction

Many models have been proposed to explain empirically observed characteristics in collected data from networks such as the Internet. See e.g. Heath et al. (1998), Kaj and Taqqu (2004), Konstantopoulos and Lin (1998), Levy and Taqqu (2000), Maulik and Resnick, Mikosch et al. (2002), Taqqu et al. (1997). Some of these models attempt to reproduce the physical dynamics behind the measured data while others just try to match statistical characteristics. Data studies have shown that the network traffic typically has important features termed *invariants* or *stylized facts* (cf. D'Auria and Resnick (2006)). Here are some of these properties:

- Heavy tails abound (Leland et al. (1994), Willinger et al. (1998), Willinger and Paxson (1998), Willinger (1998)) for such things as file sizes (Arlitt and Williamson (1996), Resnick and Rootzén (2000)), transmission rates, transmission durations (Maulik et al. (2002), Resnick (2003)) or connection durations.
- The number of bits or packets per slot exhibits long range dependence across time slots (eg, Leland et al. (1993), Willinger et al. (1995)). There is also a perception of self-similarity as the width of the time slot varies across a range of time scales exceeding a typical round trip time.
- Network traffic is bursty with rare but influential periods of very high transmission rates punctuating typical periods of modest activity.

D'Auria and Resnick (2006) base an explanation of some of these features on an infinite source Poisson model in which it was assumed that each source had an associated random rate and file size which were independent. This paper complements this analysis by examining this model with a different assumption on the sources. Indeed we assume that each source has associated random rate and duration which are independent so that rate and file size are then dependent. There are valid statistical reasons for considering this alternative assumption. See Section 2 and Resnick (2006), Chapter 7.

Key words and phrases. Bursty traffic, M/G/∞ input model, infinite source Poisson model, network modelling, limit distributions, Lévy processes, Gaussian limits.

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For this model we are going to study the small scale asymptotic behaviour which means we measure the amount of data content that arrives in a time slot of length δ and we study its limit distribution as $\delta \to 0$ after a centering and scaling.

Section 2 contains more details on the description of the model as well as a discussion of scenarios where our independence assumption is appropriate. Section 3 derives the approximating stable distribution of cumulative input per time slot while Sections 4 and 5 describe dependence structure across time intervals. We provide detailed comparisons in Section 6 between the present model and the one in D'Auria and Resnick (2006) and highlight the impact differing dependence assumptions can have on tail heaviness and conclude in Section 7 with some final thoughts.

2. Model Description

The model for data traffic generation is a modification of the $M/G/\infty$ input or infinite source Poisson model. As in D'Auria and Resnick (2006), we assume the transmission rates are random. Here we assume transmission rate is independent of transmission duration whereas in D'Auria and Resnick (2006), we assumed file size was independent of transmission rate. We assume that a homogeneous Poisson process on $\mathbb{R}$ with points $\{\Gamma_k\}$ activates data transmission *sessions*. The parameter or rate of the Poisson process is $\lambda = \lambda(\delta)$, and to each transmission activation time Γ_k is associated a mark consisting of a triple (R_k, L_k, F_k). These three quantities have the following physical interpretations:

- R - Rate of the transmission,
- L - Duration of the transmission,
- F - Size of the transmitted file.

These three quantities are related by the relation $F = R \cdot L$.

We assume the marks $\{(R_k, L_k, F_k), -\infty < k < \infty\}$ are iid and independent of $\{\Gamma_k\}$. The univariate marginal distributions of the triple are

$$G(x) = P[F_1 \le x], \quad F_R(x) = P[R_1 \le x], \quad F_L(x) = P[L_1 \le x].$$

We suppose that all three distributions are heavy tailed

$$\bar{G}(x) = x^{-\alpha_F}\ell_F(x), \quad \bar{F}_R(x) = x^{-\alpha_R}\ell_R(x), \quad \bar{F}_L(x) = x^{-\alpha_L}\ell_L(x),$$

where ℓ_F, ℓ_R, ℓ_L are all slowly varying and we assume all three tail parameters satisfy

$$1 < \alpha_F,\ \alpha_R,\ \alpha_L < 2.$$

There is empirical evidence justifying these assumptions. See Azzouna et al. (2004), Campos et al. (2005), Guerin et al. (2003), Heffernan and Resnick (2005), Leland et al. (1994), Maulik et al. (2002), Park and Willinger (2000), Resnick (2003, 2004b), Riedi and Willinger (2000), Sarvotham et al. (2005), Willinger et al. (1995).

With these assumptions, the counting function of the points $\{(\Gamma_k, R_k, L_k, F_k)\}$

$$N = \sum_k \epsilon_{(\Gamma_k, R_k, L_k, F_k)} \tag{2.1}$$

on $\mathbb{R} \times [0,\infty)^3$ is a *Poisson random measure* with mean measure

$$\lambda ds P[(R_1, L_1, F_1) \in (dr, dl, du)] =: \mu^{\#}(ds, dr, dl, du). \tag{2.2}$$

See, for example, Kallenberg (1983), Neveu (1977), Resnick (1987, 1992, 2006).

For a time window of length δ, we will consider weak limits of the process

$$\mathbf{A}(\delta) := \{A(k\delta, (k+1)\delta], -\infty < k < \infty\} \tag{2.3}$$

as $\delta \downarrow 0$. Here $A(k\delta, (k+1)\delta]$ represents the total amount of work inputted to the system in the k-th time slot $(k\delta, (k+1)\delta]$. We will define this precisely for $k = 0$ and the definitions for the other values of k will be obvious by analogy.

Distinguish four disjoint regions in $\mathbb{R} \times [0,\infty)^3$ by a decomposition on arrival time of a session and its duration:

$$\begin{aligned}
\{>0,1\} =& \{(s,r,l,u) : 0 < s \le \delta,\ 0 < s+l \le \delta\},\\
\{>0,2\} =& \{(s,r,l,u) : 0 < s \le \delta,\ s+l > \delta\},\\
\{<0,1\} =& \{(s,r,l,u) : s < 0,\ 0 < s+l \le \delta\},\\
\{<0,2\} =& \{(s,r,l,u) : s < 0,\ s+l > \delta\}.
\end{aligned}$$

Region $\{>0,1\}$ corresponds to sessions which start and end in $(0,\delta]$ while the region $\{>0,2\}$ describes sessions starting in $(0,\delta]$ but ending subsequent to δ. Region $\{<0,1\}$ has sessions starting prior to time 0 and ending in $(0,\delta]$ while $\{<0,2\}$ has sessions initiated prior to 0 and ending subsequent to δ. See Figure 1.

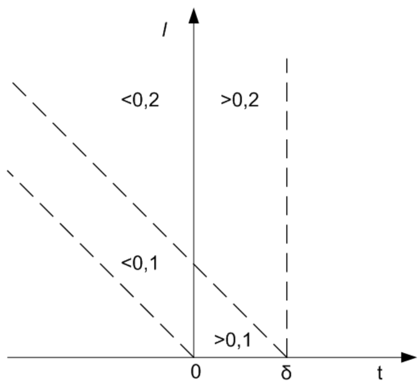

FIGURE 1. Four regions.

Corresponding to this decomposition of regions, if we restrict the Poisson random measure to the four regions, we get four independent Poisson processes:

$$N(\cdot \textstyle\bigcap \{>0,1\}),\ N(\cdot \bigcap \{>0,2\}),\ N(\cdot \bigcap \{<0,1\}),\ N(\cdot \bigcap \{<0,2\}), \tag{2.4}$$

and we use these to express $A(0,\delta) =: A(\delta)$ as the sum of four independent contributions:

$$A(\delta) = A^{>0,1}(\delta) + A^{>0,2}(\delta) + A^{<0,1}(\delta) + A^{<0,2}(\delta), \tag{2.5}$$

where

$$\begin{aligned}
A^{>0,1}(\delta) &= \sum_k R_k L_k 1_{[(\Gamma_k,R_k,L_k,F_k)\in\{>0,1\})]},\\
A^{>0,2}(\delta) &= \sum_k R_k (\delta - \Gamma_k) 1_{[(\Gamma_k,R_k,L_k,F_k)\in\{>0,2\})]},\\
A^{<0,1}(\delta) &= \sum_k R_k (L_k + \Gamma_k) 1_{[(\Gamma_k,R_k,L_k,F_k)\in\{<0,1\})]},\\
A^{<0,2}(\delta) &= \sum_k R_k \delta 1_{[(\Gamma_k,R_k,L_k,F_k)\in\{<0,2\})]}.
\end{aligned}$$

As a further notational device, we will adopt the convention that for a region $\mathcal{R}$ of the (s,r,l,u)-space, $A^{\mathcal{R}}(t_1,t_2]$ will denote the cumulative work inputted to the system in times $(t_1,t_2]$ from points $(\Gamma_k, R_k, L_k, F_k)$ in region $\mathcal{R}$.

We can represent the restrictions of N to each of the four regions as given in (2.4) as empirical measures of a Poisson number of iid points whose joint distributions are the mean measure $\mu^{\#}$ restricted to that region and normalized to be a probability measure. (See, for instance, Resnick (1992, page 341)). For example

$$N(\cdot \bigcap \{>0,1\}) = \sum_{k=1}^{P^{>0,1}(\delta)} \epsilon_{(\Gamma_k^{>0,1}, R_k^{>0,1}, L_k^{>0,1}, F_k^{>0,1})}$$

where $P^{>0,1}(\delta)$ is Poisson with parameter

$$\begin{aligned}\mu^{\#}(\{>0,1\}) &= \int_{\{>0,1\}} \lambda ds P[(R_1, L_1, F_1) \in (dr, dl, du)] \\ &= \int_0^{\delta} \lambda ds P[L_1 + s < \delta] = \int_0^{\delta} \lambda F_L(\delta - s) ds = \delta \hat{F}_L(\delta),\end{aligned}$$

(where $\hat{F}_L(x) = \int_0^x F_L(y)dy$) and $\{(\Gamma_k^{>0,1}, R_k^{>0,1}, L_k^{>0,1}, F_k^{>0,1})\}$ are iid with joint distribution

$$\frac{\mu^{\#}(\cdot \bigcap \{>0,1\})}{\mu^{\#}(\{>0,1\})}.$$

In what follows, we sometimes use the convention that $P^{\mathcal{R}}(\delta)$ is Poisson distributed with parameter equal to $\mu^{\#}(\mathcal{R})$, the mean measure of the region $\mathcal{R}$.

2.1. **Specifying dependence structure for** (R, L, F)**.** Depending on the dependence structure of the triple (R, L, F), it is possible to have different limit behaviour for $\mathbf{A}(\delta)$ in (2.3). We distinguish two cases that we denote by RL and RF:

- RL - the r.v.s R and L are independent (cf, Maulik et al. (2002));
- RF - the r.v.s R and F are independent (cf, D'Auria and Resnick (2006)).

We focus here on the RL model which assumes that R and L are independent. The model RF was thoroughly analyzed in D'Auria and Resnick (2006).

Modern network traffic is the superposition of heterogeneous applications. The assumption of independence of the transmission rates from the transmission durations is natural for some applications. Here we describe two possible situations.

The first scenario considers the transmission of streaming flows such as media streams (eg. video on-demand). Usually this kind of data is transmitted in real time. This means that the transmission durations approximately coincide with the time-length of the data content. For example, if we consider the transmission of a music song, such as by an internet radio broadcast, the transmission will last as long as the song duration. In some cases, like watching a movie on the Internet from a video on demand service, the transmission duration does not exactly coincide with the actual duration of the movie, due to buffering in the receiver necessitated by the need to prevent bad quality play due to high jitter in the transmission. Generally, since the required data for the movie greatly exceeds the buffer size, we can neglect the influence of the latter on the duration of the transmission. Therefore, if we consider the rate at which these transmissions take place, they depend on the sampling quality of the media stream and hence not its duration. Returning to the Internet radio example, the user can usually choose a download quality depending on the bandwidth of the Internet access. This choice affects the quality of the content, since it affects the total amount of data that will be transmitted, but it will not alter the duration of the transmission, which as previously noted, will depend of the duration of the content and/or the time the user wants to be connected to listen or watch. In the literature, these kinds of transmissions where the rate of the transmission depends on the sampling rate of the content, as well as on the compression scheme, are known as VBR transmissions (Heyman and Lakshman, 1996, Park and Willinger, 2000). VBR stays for Variable Bit Rate since usually the compression scheme implies non-constant transmission rates. By neglecting this feature and assuming for simplicity that the transmission rate stays constant all over the data transmission, we can consider the RL model a reasonable model for streaming–flow transmissions.

The second scenario is peer-to-peer (P2P) networks (see Pandurangan et al. (2003) and Tanenbaum (1996)). A typical P2P network is composed of a collection of users that are simultaneously on-line sharing resources. Usually users dynamically connect to and disconnect from the P2P network so that the size and type of data the network holds change continuously in time. Viewed from the point of view of one particular user, the network contains content that is always changing, and files the user wishes to download are alternatively present and absent. Often it is the case that P2P users open two different communication channels, one for uploading and one for downloading, and for each channel they specify the maximum allowed transmission bandwidth. Usually this specification is done only once at the beginning of the connection and is never modified. Therefore since the choice of the maximum bandwidth does not depend on the subsequently transferred data, it seems a natural assumption to consider the chosen maximum allowed rates independent of the content and duration of the data transmissions. In addition, due to the large size and high fluctuations of the population comprising the P2P network, it often happens that the upload channel is always fully utilized while the utilization of the download channel is always fluctuating depending on the state of the network and therefore the availability of the desired content. This means that the download rates are fluctuating while the upload rates are constant and equal to the maximum allowed upload rates. Now assuming that users connect to the P2P network according to a Poisson process, we can associate them with the sources of our infinite source Poisson model. According to this association we can consider a source transmission as the total data transfer that one user has transmitted by the upload channel. Therefore the transmission durations L will be given by the lifetime of a user on the P2P network, while the transmission rates R are given by the maximum upload bandwidth. In this setting it seems natural to assume R and L are independent so that the model RL is appropriate.

Undoubtedly in practice, it may not be true that R and L are actually independent but rather satisfy some form of asymptotic independence. However, assuming asymptotic independence rather than full independence would lead to unacceptable complications in the analysis and proofs without changing conclusions and thus, at this stage, choosing full independence of L and R is an appropriate modeling assumption.

2.2. **The model RL.** We assume that the rates of transmissions are independent of transmission durations. The file sizes are computed by the relation $F = LR$. From Breiman's theorem (Breiman, 1965), this means that the distribution tail of the random variable F is given by

$$\bar{G}(u) \sim \begin{cases} \mathbb{E}(R^{\alpha_L})\bar{F}_L(u), & \text{if } \alpha_R > \alpha_L; \\ \mathbb{E}(L^{\alpha_R})\bar{F}_R(u), & \text{if } \alpha_R < \alpha_L. \end{cases} \tag{2.6}$$

The case $\alpha_R = \alpha_L$ is of somewhat less interest in applied probability; this case could be handled by a refinement of Breiman's theorem which proceeds under the condition that $P[R > x] = o(P[L > x])$ (or vice versa); this result is given in Embrechts and Goldie (1980). A product result in Cline (1983), quoted in Davis and Resnick (1986, page 542), of a slightly different character describes the case where $R \stackrel{d}{=} L$, R, L independent, $P[R > x]$ is regularly varying with index $-\alpha$, and $E(R^\alpha) = \infty$.

By using the property that the random variables R and L are heavy-tailed, we derive the tail behaviour of the random variables $A^{\mathcal{R}}(\delta)$ with $\mathcal{R} \in \big\{\{<0,1\}; \{<0,2\}; \{>0,1\}; \{>0,2\}\big\}$; that is, $\mathcal{R}$ is one of the four regions shown in Figure 1. For a fixed $\delta > 0$, as $x \to \infty$, the tails satisfy

$$\frac{P\big[A^{\mathcal{R}}(\delta) > x\big]}{\bar{F}_R(x)} \sim \begin{cases} \lambda \int_{s=0}^{\delta} \Big(\int_{l=0}^{s} l^{\alpha_R} F_L(dl)\Big) ds & \mathcal{R} = \{>0,1\}; \\ \\ \lambda \int_0^{\delta} s^{\alpha_R} \bar{F}_L(s)\, ds & \mathcal{R} = \{>0,2\} \text{ or } \{<0,1\}; \\ \\ \lambda E(L) \bar{F}_L^{(0)}(\delta) \delta^{\alpha_R} & \mathcal{R} = \{<0,2\}. \end{cases}$$

The tails of all the regions are regularly varying with index $-\alpha_R$.

We give a sample calculation which explains how to obtain the last relation about the tails of $A^{\mathcal{R}}$ for the case $\mathcal{R} = \{>0,1\}$. The methodology is similar to the one used in Section 3. We have

$$\frac{P\big[A^{>0,1}(\delta) > x\big]}{\bar F_R(x)} \sim \frac{\mathbb{E}(P^{>0,1}(\delta))P\big[F_0^{>0,1} > x\big]}{\bar F_R(x)} = \frac{1}{\bar F_R(x)} \iiint_{\substack{0<s<\delta\\ l<\delta-s\\ r\,l>x}} \lambda ds F_R(dr) F_L(dl)$$

$$= \lambda \int_{s=0}^{\delta} \int_{l=0}^{\delta-s} \frac{\bar F_R(l^{-1}x)}{\bar F_R(x)} F_L(dl)\, ds \overset{x\to\infty}{\to} \lambda \int_{s=0}^{\delta} \int_{l=0}^{s} l^{\alpha_R} F_L(dl) ds. \tag{2.7}$$

Since our limiting procedure will shrink the observation window $(0,\delta]$, there is no hope to get a weak limit in (2.3) unless we increase the arrival rate $\lambda = \lambda(\delta)$ of sessions. We adopt a heavy traffic limit theorem philosophy and imagine moving through a family of models indexed by δ as $\delta \downarrow 0$. A convenient and effective choice of λ is

$$\lambda(\delta) = \frac{1}{\delta \bar F_R(\delta^{-1})}. \tag{2.8}$$

Using assumption (2.8), the behaviour of the rv's $A^{(\cdot)}(\delta)$ is as follows:

- $A^{<0,1}(\delta)$, suitably centered, converges weakly to a stable random variable $X_{\alpha_R}^{<0,1}$ with infinite second moment and index $\alpha_R \in (1,2)$.
- $A^{<0,2}(\delta)$ does not converge weakly without scaling; with centering and scaling it converges to a stable random variable $X_{\alpha_R}^{<0,2}$ with index α_R. We also note that if we suitably decompose the region $\{<0,2\}$ into two subregions, according to whether the transmission rate is small or large, we can have convergence in the region where the rate is small to a Gaussian random variable but the required scaling is of smaller order compared with the scaling yielding $X_{\alpha_R}^{<0,2}$;
- $A^{>0,1}(\delta)$ is negligible in the limit under suitable conditions;
- $A^{>0,2}(\delta)$ is equal in distribution to $A^{<0,1}(\delta)$.

2.3. **Symbol finder.** For convenience and reference, we list some symbols and concepts frequently used.

F_R, F_L, G	The distributions of rate, duration and file size.
$\bar F$	For a distribution function $F(x)$, $\bar F = 1 - F$ is the distribution tail.
$F^{(0)}$	For a distribution F with finite mean μ, $F^{(0)}(x) = \int_0^x \mu^{-1} \bar F(s) ds$.
$RV_{-\alpha}$	The class of regularly varying functions with index $-\alpha$.
$l(x)$	A slowly varying function: $\lim_{t\to\infty} l(tx)/l(t) = 1$ for $x > 0$.
$\epsilon_x(\cdot)$	The point probability measure putting all mass at the point x.
RL	Model where $R \perp\!\!\!\perp L$; that is R and L are independent.
RF	Model where $R \perp\!\!\!\perp F$; that is R and F are independent.
δ	Time slot width.
λ	The Poisson rate of connection arrivals $\lambda = \lambda(\delta)$.
$\mu^{\#}(ds, dr, dl, du)$	$\lambda ds P[(R_1, L_1, F_1) \in (dr, dl, du)]$.
$P^{\mathcal{R}}(\delta)$	A Poisson random variable with mean $\mu^{\#}(\mathcal{R})$.
$A^{\mathcal{R}}(I)$	Cumulative work inputted into the system in time interval I by points with characteristics in $\mathcal{R}$.
$\mu_\delta(dr)$	$\frac{F_R(\delta^{-1}dr)}{\bar F_R(\delta^{-1})}$.

3. Limits for cumulative input $A(\delta)$

Here we analyze cumulative input in $[0,\delta]$ by analyzing the four pieces separately in the decomposition (2.5).

3.1. Region $\{>0,2\}$.

Recall this is the region contributing input in $(0,\delta]$ from sessions initiated in $(0,\delta]$ but terminating after δ.

3.1.1. *Characteristic function.* For $\theta\in\mathbb{R}$, we compute

$$\begin{aligned}
\mathbb{E}\Big(e^{i\theta A^{>0,2}(\delta)}\Big) =&\mathbb{E}\exp\{i\theta \sum_{i=1}^{P^{>0,2}(\delta)} R_i^{>0,2}(\delta-\Gamma_i^{>0,2})\}\\
=&\exp\{\mathbb{E}\big(P^{>0,2}(\delta)\big)\mathbb{E}\big[e^{i\theta R_1^{>0,2}(\delta-\Gamma_1^{>0,2})}-1\big]\}\\
=&\exp\Big\{\iiint_{\substack{0<s<\delta\\ s+l>\delta\\ r>0}}\Big(e^{i\theta r(\delta-s)}-1\Big)\lambda ds F_R(dr)F_L(dl)\Big\}\\
=&\exp\Big\{\iiint_{\substack{0<s<\delta\\ l>s\\ r>0}}\Big(e^{i\theta rs}-1\Big)\lambda ds F_R(dr)F_L(dl)\Big\}\\
=&\exp\Big\{\int_{s=0}^{\delta}\int_{r=0}^{\infty}(e^{i\theta rs}-1)\bar F_L(s)F_R(dr)\lambda ds\Big\}\\
=&\exp\Big\{\int_{r=0}^{\infty}\int_{s=0}^{r\delta}(e^{i\theta s}-1)\bar F_L(r^{-1}s)\lambda r^{-1}dsF_R(dr)\Big\}\\
=&\exp\Big\{\int_{s=0}^{\infty}\int_{r=\delta^{-1}s}^{\infty}(e^{i\theta s}-1)\bar F_L(r^{-1}s)\lambda r^{-1}F_R(dr)ds\Big\}\\
=&\exp\Big\{\lambda\delta\bar F_R(\delta^{-1})\int_{s=0}^{\infty}(e^{i\theta s}-1)\int_{r=s}^{\infty}\bar F_L(\delta r^{-1}s)r^{-1}\frac{F_R(\delta^{-1}dr)}{\bar F_R(\delta^{-1})}ds\Big\}
\end{aligned}$$

and finally using $\lambda=\frac{1}{\delta\bar F_R(\delta^{-1})}$, we get

$$=\exp\Big\{\int_{s=0}^{\infty}(e^{i\theta s}-1)\left(\int_{r=s}^{\infty}\bar F_L(\delta r^{-1}s)r^{-1}\mu_\delta(dr)\right)ds\Big\}$$

where

$$\mu_\delta(dr):=\frac{F_R(\delta^{-1}dr)}{\bar F_R(\delta^{-1})}.$$

Write

$$\nu_\delta^{>0,2}(ds)=(\nu_\delta^{>0,2})'(s)ds=\int_{r=s}^{\infty}\bar F_L(\delta r^{-1}s)r^{-1}\mu_\delta(dr)ds \tag{3.1}$$

and we obtain

$$\mathbb{E}\Big(e^{i\theta A^{>0,2}(\delta)}\Big)=\exp\Big\{\int_{s=0}^{\infty}(e^{i\theta s}-1)\nu_\delta^{>0,2}(ds)\Big\}. \tag{3.2}$$

3.1.2. *Properties of $\nu_\delta^{>0,2}$.*

Proposition 1. *As $\delta\to0$,*

$$\nu_\delta^{>0,2}\xrightarrow{v}\nu_0^{>0,2},$$

on $(0,\infty]$; that is, we have vague convergence to the limit measure $\nu_0^{>0,2}$, which is a Lévy measure with density

$$\frac{\alpha_R}{1+\alpha_R}x^{-\alpha_R-1}.$$

Proof. The proof is similar to Proposition 1 of D'Auria and Resnick (2006). Observe that for $s \geq 0$,

$$(\nu_\delta^{>0,2})'(s) = \int_{r=s}^{\infty} r^{-1}\bar{F}_L(\delta r^{-1}s)\mu_\delta(dr) \leq s^{-1}\mu_\delta(s,\infty]$$

and by Potter's bounds (Bingham et al. (1987), de Haan (1970), Resnick (1987), Seneta (1976)), for some small η, all $s \geq 1$, some $c > 0$, and for all sufficiently small δ, we have the upper bound

$$\leq cs^{-(\alpha_R-\eta)-1}$$

which is integrable with respect to Lebesgue measure on any neighborhood of ∞. Hence, by dominated convergence, for $x > 0$,

$$\nu_\delta^{>0,2}(x,\infty] = \int_x^\infty (\nu_\delta^{>0,2})'(s)ds \to \int_x^\infty \int_s^\infty r^{-1}\alpha_R r^{-\alpha_R-1}drds$$

$$=\nu_0^{>0,2}(x,\infty] = \frac{\alpha_R}{1+\alpha_R}\int_x^\infty s^{-\alpha_R-1}ds = \frac{x^{-\alpha_R}}{1+\alpha_R}. \tag{3.3}$$

To check $\nu_0^{>0,2}$ is a Lévy measure, note

$$\int_0^1 s^2\nu_0^{>0,2}(ds) = \frac{\alpha_R}{1+\alpha_R}\int_0^1 s^2 s^{-\alpha_R-1}ds < \infty$$

since $1 < \alpha_R < 2$. □

3.1.3. *Weak limit for $A^{>0,2}(\delta)$.* Now we use (3.2) and write

$$\mathbb{E}\exp\{i\theta(A^{>0,2}(\delta) - \int_0^1 s\nu_\delta^{>0,2}(ds))\} = \exp\Big\{\int_1^\infty (e^{i\theta s}-1)\nu_\delta^{>0,2}(ds) + \int_0^1 (e^{i\theta s}-1-i\theta s)\nu_\delta^{>0,2}(ds)\Big\}. \tag{3.4}$$

The two integrals on the right in (3.4) each converge when $\delta \to 0$.

Proposition 2. *As $\delta \to 0$:*

$$\int_1^\infty (e^{i\theta s}-1)\nu_\delta^{>0,2}(ds) \to \int_1^\infty (e^{i\theta s}-1)\nu_0^{>0,2}(ds) \tag{3.5}$$

$$\int_0^1 (e^{i\theta s}-1-i\theta s)\nu_\delta^{>0,2}(ds) \to \int_0^1 (e^{i\theta s}-1-i\theta s)\nu_0^{>0,2}(ds). \tag{3.6}$$

Therefore, as $\delta \to 0$

$$A^{>0,2}(\delta) - \int_0^1 s\nu_\delta^{>0,2}(ds) \Rightarrow X_{\alpha_R}^{>0,2},$$

where the limit random variable is spectrally positive stable with index α_R with with Lévy measure $\nu_0^{>0,2}$ given in (3.3) *and characteristic function given by the right side of* (3.4) *with $\nu_\delta^{>0,2}$ replaced by $\nu_0^{>0,2}$.*

Proof. The convergence in (3.5) follows from standard weak convergence since the integrand is bounded and continuous and

$$\frac{\nu_\delta^{>0,2}(\cdot)}{\nu_\delta^{>0,2}(1,\infty]} \Rightarrow \frac{\nu_0^{>0,2}(\cdot)}{\nu_0^{>0,2}(1,\infty]}$$

weakly as probability measures on $(1,\infty]$.

For the proof of (3.6), observe that

$$|e^{i\theta s}-1-i\theta s|(\nu_\delta^{>0,2})'(s) \leq \frac{\theta^2 s^2}{2}s^{-1}\mu_\delta(s,\infty]$$

$$\leq cs\frac{\bar{F}_R(\delta^{-1}s)}{\bar{F}_R(\delta^{-1})} = c\frac{V(\delta^{-1}s)}{V(\delta^{-1})}$$

where $V(s) = s\bar{F}_R(s)$ is regularly varying with index $-\alpha_R+1$. Now as $\delta \to 0$,

$$|e^{i\theta s}-1-i\theta s|(\nu_\delta^{>0,2})'(s) \to |e^{i\theta s}-1-i\theta s|(\nu_0^{>0,2})'(s)$$

and

$$\frac{V(\delta^{-1}s)}{V(\delta^{-1})} \to s^{-\alpha_R+1}.$$

Furthermore, by Karamata's theorem,

$$\int_0^1 \frac{V(\delta^{-1}s)}{V(\delta^{-1})}ds \to \int_0^1 s^{-\alpha_R+1}ds = \frac{1}{2-\alpha_R}.$$

The desired result now follows from Pratt's lemma (Pratt (1960) or Resnick (1998, page 164)) since Pratt's lemma may be applied to both the real and imaginary parts of

$$\big(e^{i\theta s} - 1 - i\theta s\big)(\nu_\delta^{>0,2})'(s)$$

to get convergence to the limit after integrating on $[0,1]$. □

3.2. **Region** $\{<0,1\}$**.** In this region the contribution to $A(0,\delta]$ is given by the sessions initiated before 0 and terminating in $(0,\delta]$. In D'Auria and Resnick (2006, Proposition 6) it is proven that this contribution is identical in distribution to the one of region $\{>0,2\}$. So we have the following

Proposition 3. *We have*

$$A^{<0,1}(\delta) \stackrel{d}{=} A^{>0,2}(\delta) \tag{3.7}$$

and therefore, as $\delta \to 0$,

$$A^{<0,1}(\delta) - \int_0^1 s\nu_\delta^{<0,1}(ds) \Rightarrow X_{\alpha_R}^{<0,1},$$

where $\nu_\delta^{<0,1} = \nu_\delta^{>0,2}$ *and* $X_{\alpha_R}^{<0,1} \stackrel{d}{=} X_{\alpha_R}^{>0,2}$ *with the quantities indexed by '*$>0,2$*' defined as in Proposition 2.*

3.3. **Region** $\{>0,1\}$**.** The region $\{>0,1\}$ is a small region decreasing in size with δ and should not make a contribution to the overall traffic. Reasonable conditions which assure this include regular variation of $F_L(x)$ at zero. This includes distributions $F_L(x)$ satisfying

$$F_L(x) \sim x^\beta, \quad x \downarrow 0;\ \beta > 0,$$

and hence distributions F_L with densities $F_L'(x)$ satisfying

$$F_L'(x) \sim cx^{\beta-1}, \quad x \downarrow 0,\ \beta > 0.$$

Densities which look like Gamma densities near 0 are appropriate. In particular, our assumptions allow $\beta = 1$ as is the case for the standard exponential density.

We start analysis of this region by observing that, without loss of generality, we can assume that $F_L(0) = 0$. Indeed if $F_L(0) = a$ with $0 < a < 1$, we can define $F_L^{\#}(x) = \frac{F_L(x)-a}{1-a}$ and decompose the Poisson arrival process in two processes: the first with rate $(1-a)\lambda$ and the second with arrival rate $a\lambda$. The first arrival process has associated sessions whose lengths are always different from zero and the second has sessions always of null lengths. The contribution of the second process to cumulative loads is always 0, so the original arrival process will contribute to the process $A(\delta)$ as the first component with the modified arrival rate.

Assuming $F_L(x)$ is regularly varying at 0 is equivalent to supposing $W := L^{-1}$ has a distribution tail which is regularly varying at ∞ with index $-\alpha_W < 0$. To roughly estimate the traffic contribution of the region $\{>0,1\}$, we compute the mean of $A^{>0,1}(\delta)$. We have

$$\begin{aligned} E(A^{>0,1}(\delta)) =& E(P^{>0,1}(\delta))E\big(R^{>0,1}L^{>0,1}\big) = \iiint_{\substack{0<s<\delta \\ r>0, l\le\delta-s}} rl\lambda ds F_R(dr)F_L(dl) \\ =& E(R)\lambda \int_{s=0}^{\delta}\int_{l=0}^{s} lF_L(dl)ds = E(R)\lambda\int_{s=0}^{\delta} E(L1_{[L\le s]})ds \\ =& E(R)\lambda\int_{s=\delta^{-1}}^{\infty} E\big(L1_{[L^{-1}\ge s]}\big)\frac{ds}{s^2}. \end{aligned} \tag{3.8}$$

Now suppose,

$$P[W > x] = \bar{F}_W(x) \sim x^{-\alpha_W} L(x), \quad x \to \infty.$$

Then a variant of Karamata's theorem (Resnick (2006, Exercise 2.5)) implies that

$$E\big(L1_{[L^{-1} \geq s]}\big) \sim \frac{\alpha_W}{1+\alpha_W} s^{-1} \bar{F}_W(s), \quad (s \to \infty),$$

and by Karamata's theorem, as $\delta \to 0$,

$$E(A^{>0,1}(\delta)) \sim E(R) \frac{\alpha_W}{(1+\alpha_W)(2+\alpha_W)} \frac{\bar{F}_W(\delta^{-1})}{\delta^{-1}\bar{F}_R(\delta^{-1})}.$$

Thus, if we assume F_L is regularly varying at 0 or equivalently that $\bar{F}_W$ is regularly varying at ∞, we get that $E(A^{>0,1}(\delta)) \to 0$, as $\delta \to 0$ iff

$$\lim_{x \to \infty} \frac{\bar{F}_W(x)}{x\bar{F}_R(x)} = 0.$$

This implies $A^{>0,1}(\delta) \stackrel{L_1}{\to} 0$ and hence $\{>0,1\}$ gives a negligible contribution to cumulative work. A sufficient condition is that

$$1 + \alpha_W > \alpha_R. \tag{3.9}$$

One reasonable circumstance where (3.9) holds is the following. Suppose F_L has a density $F_L'(x)$ which converges to a limit at zero: $F_L'(x) \to F_L'(0) \in (0, \infty)$, as $x \to 0$. The standard exponential density satisfies this condition. Then, as $l \to 0$,

$$F_L(l) \sim lF_L'(0)$$

and as $x \to \infty$,

$$\bar{F}_W(x) = P[L^{-1} > x] = P[L < \frac{1}{x}] \sim F_L'(0)\frac{1}{x},$$

and so $\bar{F}_W(x) \in RV_{-1}$ and $\alpha_W = 1$. So due to the condition $1 < \alpha_R < 2$, (3.9) is satisfied.

For the Boston University data set, we were curious to see what the left tail behaviour of L turned out to be. QQ plots of the log-transformed data plotted against theoretical quantiles of the exponential distribution are given in Figure 2 and show that a reasonable estimate for α_W is 4.5.

Henceforth, we assume $A^{>0,1}(\delta) \stackrel{L_1}{\to} 0$, so that we can neglect the asymptotic contribution to loading from region $\{>0,1\}$.

3.4. **The contribution of the region** $\{<0,2\}$**.** We divide the region $\{<0,2\}$ into two regions $\{<0,2^-\}$ and $\{<0,2^+\}$ defined in the following way:

$$\{<0,2^-\} = \{(s,r,l,u) \in \{<0,2\} : 0 \leq r < 1/\delta\}, \quad \{<0,2^+\} = \{<0,2\} \setminus \{<0,2^-\}.$$

The reason for this further splitting is due to the fact we are looking for a region that gives asymptotically a Gaussian contribution. We show that $\{<0,2^-\}$ is that region. In addition, we show that the region $\{<0,2^+\}$ dominates in the limit as $\delta \to 0$ so that in the limit the Gaussian component disappears.

3.4.1. *Characteristic function of* $A^{<0,2^-}(\delta)$. Since

$$A^{<0,2^-}(\delta) = \sum_{i=1}^{P^{<0,2^-}(\delta)} R_i^{<0,2^-}\delta, \tag{3.10}$$

the characteristic function of $A^{<0,2^-}(\delta)$ is computed as follows. For $\theta \in \mathbb{R}$,

$$\begin{aligned} Ee^{i\theta A^{<0,2^-}(\delta)} &= \exp\{E(P^{<0,2^-}(\delta))E[e^{i\theta R_1^{<0,2^-}\delta} - 1]\} \\ &= \exp\{\iiint_{\substack{s<0,0\leq r<1/\delta \\ l>|s|+\delta}} (e^{i\theta r\delta} - 1)\lambda ds F_L(dl) F_R(dr)\} \\ &= \exp\{\lambda \int_{s>\delta}\int_{l>s} F_L(dl) ds \int_{r=0}^{1/\delta} (e^{i\theta r\delta} - 1) F_R(dr)\} \end{aligned}$$

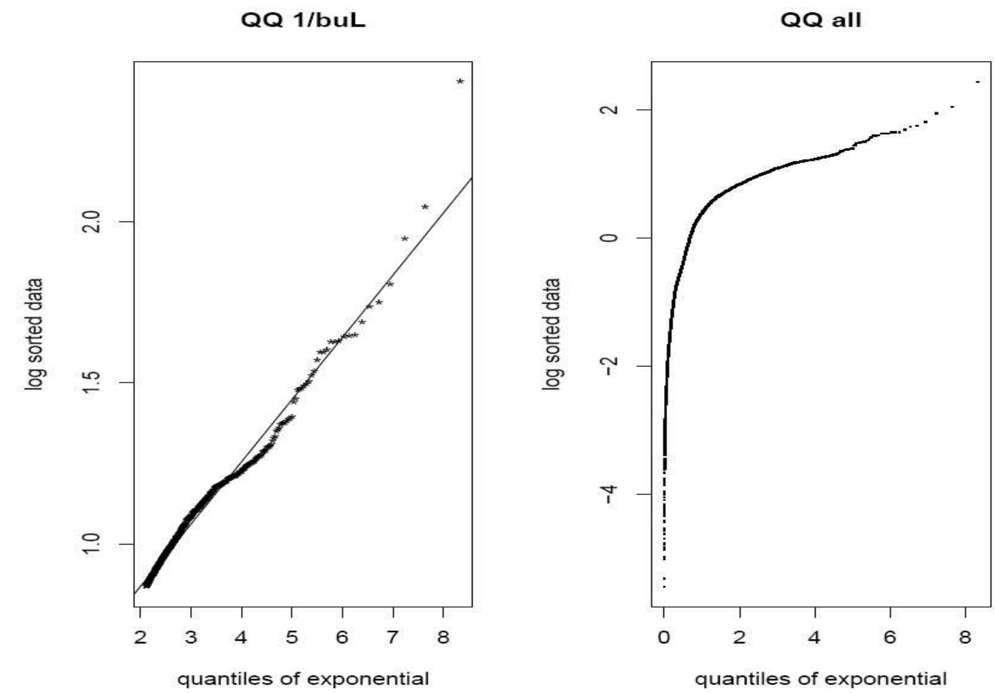

FIGURE 2. QQ plot of the 500 largest order statistics (left) of 1/buL the corresponding plot (right) for the whole data set. The slope estimator applied to the left plot gives an estimate of α_W of 4.5.

$$
\begin{aligned}
&= \exp\{\lambda \int_{s>\delta} \bar{F}_L(s)ds \int_{r=0}^{1/\delta} (e^{i\theta r\delta} - 1)F_R(dr)\} \\
&= \exp\{\lambda \bar{F}_R(\delta^{-1})E(L)\bar{F}_L^{(0)}(\delta) \int_{r=0}^{1} (e^{i\theta r} - 1)\mu_\delta(dr)\} \\
&= \exp\{\lambda \bar{F}_R(\delta^{-1}) \int_0^1 (e^{i\theta r} - 1)\nu_\delta^{<0,2^-}(dr)\}.
\end{aligned}
$$

where

$$\nu_\delta^{<0,2^-}(dr) = E(L)\bar{F}_L^{(0)}(\delta)\mu_\delta(dr).$$

Finally substituting $\lambda = \frac{1}{\delta \bar{F}_R(\delta^{-1})}$ we have

$$Ee^{i\theta A^{<0,2^-}(\delta)} = \exp\{\delta^{-1} \int_0^1 (e^{i\theta r} - 1)\nu_\delta^{<0,2^-}(dr)\}. \tag{3.11}$$

3.4.2. *Gaussian limit for* $A^{<0,2^-}(\delta)$. For fixed $\delta > 0$, the quantity

$$m(\delta) := \delta^{-1} \int_0^1 r\, \nu_\delta^{<0,2^-}(dr) \tag{3.12}$$

is finite, since

$$m(\delta) \le E(L)\delta^{-1} \int_0^1 1\, \mu_\delta(dr) = E(L)\delta^{-1} \frac{F_R(\delta^{-1})}{\bar{F}_R(\delta^{-1})} < \infty.$$

Define

$$a(\delta) := \Big(\delta^{-1}\int_0^1 r^2\,\nu_\delta^{<0,2^-}(dr)\Big)^{1/2}. \tag{3.13}$$

Note that as $\delta\to 0$, we have $a(\delta)\to\infty$ since by Fatou's lemma,

$$\lim_{\delta\to 0}\delta a^2(\delta) = \lim_{\delta\to 0} E(L)\bar F_L^{(0)}(\delta)\int_0^1 r^2\mu_\delta(dr) \ge E(L)\int_0^1 r^2\alpha_R r^{-\alpha_R-1}dr = E(L)\frac{\alpha_R}{2-\alpha_R}. \tag{3.14}$$

Now we use (3.11) and write

$$\begin{aligned} &E\exp\{i\theta[A^{<0,2^-}(\delta)-m(\delta)]/a(\delta)\}\\ &=\exp\{\int_0^1 \delta^{-1}(e^{ia^{-1}(\delta)\theta r}-1)\nu_\delta^{<0,2^-}(dr) - i\frac{\theta}{a(\delta)}\int_0^1 \delta^{-1} r\nu_\delta^{<0,2^-}(dr)\}\\ &=\exp\{\int_0^1 \delta^{-1}(e^{ia^{-1}(\delta)\theta r}-1-i\frac{\theta}{a(\delta)}r)\nu_\delta^{<0,2^-}(dr)\}, \end{aligned}$$

and the exponent in the last expression converges to $-\theta^2/2$ since

$$\begin{aligned} &\Big|\int_0^1 \delta^{-1}(e^{ia^{-1}(\delta)\theta r}-1-i\frac{\theta}{a(\delta)}r)\nu_\delta^{<0,2^-}(dr)+\frac{\theta^2}{2}\Big|\\ &\qquad=\Big|\int_0^1 \delta^{-1}\big(e^{ia^{-1}(\delta)\theta r}-1-i\frac{\theta}{a(\delta)}r-\frac{1}{2}(\frac{i\theta r}{a(\delta)})^2\big)\nu_\delta^{<0,2^-}(dr)\Big|\\ &\qquad\le\frac{\delta^{-1}}{a^3(\delta)}\int_0^1\frac{1}{3!}|\theta|^3 r^3\nu_\delta^{<0,2^-}(dr) \end{aligned}$$

and

$$\frac{\delta^{-1}}{a^3(\delta)}\int_0^1 r^3\nu_\delta^{<0,2^-}(dr)\le\frac{\delta^{-1}}{a^3(\delta)}\int_0^1 r^2\nu_\delta^{<0,2^-}(dr)=\frac{1}{a(\delta)}\to 0.$$

We summarize the previous result by the following Proposition.

Proposition 4. *With $m(\delta)$ defined by (3.12) and $a(\delta)$ given by (3.13), we have*

$$\frac{A^{<0,2^-}(\delta)-m(\delta)}{a(\delta)}\Rightarrow N^{<0,2^-}\sim N(0,1)$$

as $\delta\to 0$.

3.4.3. *Characteristic function of $A^{<0,2^+}(\delta)$.* We have

$$A^{<0,2^+}(\delta)=\sum_{i=1}^{P^{<0,2^+}(\delta)} R_i^{<0,2^+}\delta. \tag{3.15}$$

Define

$$\nu_\delta^{<0,2^+}(dr)=E(L)\bar F_L^{(0)}(\delta)\mu_{\delta/b(\delta)}(dr),$$

where

$$\mu_{\delta/b(\delta)}(dr):=\frac{F_R(\delta^{-1}b(\delta)dr)}{\bar F_R(\delta^{-1}b(\delta))},$$

with $b(\delta)$ satisfying the relation

$$\gamma(\delta):=\frac{\bar F_R(\delta^{-1}b(\delta))}{\delta\bar F_R(\delta^{-1})}\to 1. \tag{3.16}$$

In Section 3.4.4 below, we study the function $b(\delta)$ and show that

$$b(\delta)=\Big(\frac{1}{\delta}\Big)^{1/\alpha_R}\ell(\frac{1}{\delta})\to\infty\quad(\delta\downarrow 0), \tag{3.17}$$

for a function ℓ which is slowly varying at ∞.

Now define

$$n(\delta) := b(\delta)\gamma(\delta)\int_{1/b(\delta)}^{1} r\nu_\delta^{<0,2^+}(dr).$$

The characteristic function of $\frac{A^{<0,2^+}(\delta)-n(\delta)}{b(\delta)}$ is computed as follows. For $\theta \in \mathbb{R}$,

$$\begin{aligned}
Ee^{i\theta\frac{A^{<0,2^+}(\delta)-n(\delta)}{b(\delta)}} &= \exp\{E(P^{<0,2^+}(\delta))E[e^{i\frac{\theta}{b(\delta)}R_1^{<0,2^+}\delta}-1] - i\theta\frac{n(\delta)}{b(\delta)}\} \\
&= \exp\{\iiint_{\substack{s<0,r\geq 1/\delta \\ l>|s|+\delta}} (e^{i\frac{\theta}{b(\delta)}r\delta}-1)\lambda ds F_L(dl)F_R(dr) - i\theta\frac{n(\delta)}{b(\delta)}\} \\
&= \exp\{\lambda\int_{s>\delta}\int_{l>s} F_L(dl)ds\int_{r=1/\delta}^{\infty}(e^{i\theta r\frac{\delta}{b(\delta)}}-1)F_R(dr) - i\theta\frac{n(\delta)}{b(\delta)}\} \\
&= \exp\{\lambda\int_{s>\delta}\bar{F}_L(s)ds\int_{r=1/\delta}^{\infty}(e^{i\theta r\frac{\delta}{b(\delta)}}-1)F_R(dr) - i\theta\frac{n(\delta)}{b(\delta)}\} \\
&= \exp\{\lambda\bar{F}_R(\delta^{-1}b(\delta))E(L)\bar{F}_L^{(0)}(\delta)\int_{r=1/b(\delta)}^{\infty}(e^{i\theta r}-1)\mu_{\delta/b(\delta)}(dr) - i\theta\frac{n(\delta)}{b(\delta)}\} \\
&= \exp\{\lambda\bar{F}_R(\delta^{-1}b(\delta))\int_{1/b(\delta)}^{\infty}(e^{i\theta r}-1)\nu_\delta^{<0,2^+}(dr) - i\theta\frac{n(\delta)}{b(\delta)}\}
\end{aligned}$$

and using $\lambda = 1/(\delta\bar{F}_R(\delta^{-1}))$

$$\begin{aligned}
&= \exp\{\gamma(\delta)\Big[\int_{1/b(\delta)}^{1}(e^{i\theta r}-1-i\theta r)\nu_\delta^{<0,2^+}(dr) + \int_1^\infty (e^{i\theta r}-1)\nu_\delta^{<0,2^+}(dr)\Big]\} \\
&\to \exp\{\int_0^1 (e^{i\theta r}-1-i\theta r)\nu(dr) + \int_1^\infty (e^{i\theta r}-1)\nu(dr)\}.
\end{aligned}$$

where $\nu := \nu_0^{<0,2^+}$ is a Lévy measure with density $E(L)\alpha_R x^{-\alpha_R-1}$.

We summarize the previous result by the following Proposition.

Proposition 5. *As $\delta \to 0$*

$$\frac{A^{<0,2+}(\delta)-n(\delta)}{b(\delta)} \Rightarrow X_{\alpha_R}^{<0,2^+},$$

where the limit random variable is stable with Lévy measure ν with density $E(L)\alpha_R x^{-\alpha_R-1}$.

3.4.4. *On the function $b(\delta)$.* The definition of $b(\cdot)$ in (3.16) is related to the concept of *conjugate inverses* of regularly varying functions. See Bingham et al. (1987). The relationship in (3.16) can be rephrased as follows: Define

$$t=\delta^{-1}, \quad V(t)=\frac{1}{1-F_R(t)}, \quad h(\frac{1}{\delta})=b(\delta)$$

and we require

$$\frac{tV(t)}{V(th(t))} \to 1, \quad t\to\infty,$$

or

$$\frac{V(th(t))}{tV(t)} \to 1, \quad t\to\infty.$$

Now define

$$c(t)=th(t)$$

and we need

$$V\circ c(t) \sim tV(t),$$

so an obvious solution is

$$c(t) \sim V^{\leftarrow}(tV(t)).$$

Therefore

$$h(t) = \frac{c(t)}{t} = \frac{V^{\leftarrow}(tV(t))}{t}, \quad t \to \infty.$$

Since

(1) $tV(t) \in RV_{\alpha_R+1}$,
(2) $V^{\leftarrow} \in RV_{1/\alpha_R}$ and therefore $V^{\leftarrow}(tV(t)) \in RV_{(1+\alpha_R)/\alpha_R} = RV_{1+1/\alpha_R}$,
(3) $h(t) = V^{\leftarrow}(tV(t))/t \in RV_{1/\alpha_R+1-1} = RV_{1/\alpha_R}$

Therefore

$$b(\delta) = h(\frac{1}{\delta}) = \Big(\frac{1}{\delta}\Big)^{1/\alpha_R}\ell(\frac{1}{\delta}),$$

for a function ℓ which is slowly varying at ∞.

The connection to conjugate pairs of slowly varying functions (Bingham et al., 1987, Section 1.5.7) is as follows: Two slowly varying functions (ℓ, ℓ_*) are conjugate pairs if as $x \to \infty$,

$$\ell(x)\ell_*(x\ell(x)) \to 1 \quad \text{and} \quad \ell_*(x)\ell(x\ell_*(x)) \to 1. \tag{3.18}$$

Given the functions $V, V^{\leftarrow}$, we may write

$$V(t) \sim t^{\alpha_R}\ell(t^{\alpha_R}) \quad \text{and} \quad V^{\leftarrow}(t) \sim t^{1/\alpha_R}\ell_*^{1/\alpha_R}(t),$$

for a conjugate pair (ℓ, ℓ_*). This representation is possible (Bingham et al., 1987, Proposition 1.5.15) because

$$V \circ V^{\leftarrow}(t) \sim t, \quad V^{\leftarrow} \circ V(t) \sim t,$$

re-expresses (3.18). This allows the following expression for h in terms of (ℓ, ℓ_*):

$$\begin{aligned} h(t) =& \frac{1}{t}V^{\leftarrow}\big(tV(t)\big) \sim \frac{1}{t}\big(tV(t)\big)^{1/\alpha_R}\ell_*^{1/\alpha_R}\big(tV(t)\big) \\ =& t^{1/\alpha_R}\Big(\ell^{1/\alpha_R}(t^{\alpha_R})\ell_*^{1/\alpha_R}\big(t^{1+\alpha_R}\ell(t^{\alpha_R})\big)\Big). \end{aligned}$$

The expression in the big parentheses is slowly varying and expressed in terms of (ℓ, ℓ_*).

3.5. **Discussion and summary.** We summarize the contributions of the four regions to cumulative traffic in $(0, \delta)$.

(1) For the region $\{>0, 2\}$, we have, as $\delta \to 0$,

$$X^{>0,2}(\delta) := A^{>0,2}(\delta) - \int_0^1 s\nu_\delta^{>0,2}(ds) \Rightarrow X_{\alpha_R}^{>0,2},$$

a spectrally positive, stable random variable with index α_R and with Lévy measure $\nu_0^{>0,2}$ given in (3.3).

(2) For the region $\{>0, 1\}$, we have under suitable conditions on F_L that

$$A^{>0,1}(\delta) \xrightarrow{L_1} 0.$$

The contribution of this region in negligible in the limit.

(3) For the region $\{<0, 2^-\}$, we have

$$X^{<0,2^-}(\delta) := \frac{A^{<0,2^-}(\delta) - m(\delta)}{a(\delta)} \Rightarrow N^{<0,2^-} \sim N(0,1).$$

(4) For the region $\{<0, 2^+\}$, we have

$$X^{<0,2^+}(\delta) := \frac{A^{<0,2^+}(\delta) - n(\delta)}{b(\delta)} \Rightarrow X_{\alpha_R}^{<0,2^+} =: X_{\alpha_R}^{<0,2}.$$

a spectrally positive, stable random variable with index α_R and Lévy measure ν with density $E(L)\alpha_R x^{-\alpha_R-1}$ on $(0, \infty]$.

(5) For the region $\{<0,1\}$, we have

$$A^{<0,1}(\delta) \stackrel{d}{=} A^{>0,2}(\delta),$$

so

$$X^{<0,1}(\delta) := A^{<0,1}(\delta) - \int_0^1 s\nu_\delta^{>0,2}(ds) \Rightarrow X_{\alpha_R}^{<0,1} \stackrel{d}{=} X_{\alpha_R}^{>0,2}.$$

We may thus write

$$\begin{aligned} A(\delta) = & X^{>0,2}(\delta) + \int_0^1 s\nu_\delta^{>0,2}(ds) + A^{>0,1}(\delta) \\ & + a(\delta)X^{<0,2^-}(\delta) + m(\delta) + b(\delta)X^{<0,2^+}(\delta) + n(\delta) \\ & + X^{<0,1}(\delta) + \int_0^1 s\nu_\delta^{>0,2}(ds). \end{aligned}$$

We conclude

$$A(\delta) - m(\delta) - n(\delta) - 2\int_0^1 s\nu_\delta^{>0,2}(ds) = X^{>0,2}(\delta) + A^{>0,1}(\delta) + a(\delta)X^{<0,2^-}(\delta) + b(\delta)X^{<0,2^+}(\delta), \tag{3.19}$$

where the summands on the right are independent and

$$\begin{aligned} & X^{<0,1}(\delta) \stackrel{d}{=} X^{>0,2}(\delta) \Rightarrow X_{\alpha_R}^{>0,2} && \text{(spectrally positive, stable, index } \alpha_R) \\ & A^{>0,1}(\delta) \stackrel{L_1}{\to} 0, && \text{(negligible)} \\ & X^{<0,2^+}(\delta) \Rightarrow X_{\alpha_R}^{<0,2^+} && \text{(spectrally positive, stable, index } \alpha_R) \\ & X^{<0,2^-}(\delta) \Rightarrow N^{<0,2^-} && \text{(normal).} \end{aligned}$$

Also,

$$\frac{A(\delta) - d(\delta)}{b(\delta)} \Rightarrow X_{\alpha_R}^{<0,2} \tag{3.20}$$

that is stable with index α_R, and where

$$d(\delta) := m(\delta) + n(\delta) + 2\int_0^1 s\nu_\delta^{>0,2}. \tag{3.21}$$

Using (3.17) and (3.14), we see that

$$\begin{aligned} \lim_{\delta\to 0} \frac{b(\delta)}{a(\delta)} \leq & (const) \lim_{\delta\to 0} \frac{\delta^{-1/\alpha_R} l(1/\delta)}{1/\delta^{1/2}} \\ = & (const) \lim_{\delta\to 0} \delta^{1/2-1/\alpha_R} l(1/\delta) = 0, \end{aligned}$$

since $1/2 < 1/\alpha_R < 1$. So our final traffic representation for cumulative traffic on a small interval for the RL model is

$$\begin{aligned} A(0,\delta] - d(\delta) \stackrel{d}{=} & \Big(X_{\alpha_R}^{>0,2} + o_p(1)\Big) + \Big(X_{\alpha_R}^{<0,1} + o_p(1)\Big) \\ & + \Big(X_{\alpha_R}^{<0,2+} + o_p(1)\Big) b(\delta) + \Big(N^{<0,2-}(0,1) + o_p(1)\Big) o(b(\delta)). \end{aligned}$$

The model RL predicts that cumulative traffic over a small time interval is approximated by a stable random variable. We separated the region $\{<0,2\}$ into two parts in order to find a region capable of giving a normal limit. In actual measurements (Sarvotham et al., 2005), a component termed the β-traffic is observed which seems well approximated by a Gaussian distribution. At first, this observation seems to discourage the use of the RL model since the alternate RF model does give a normal approximation. However, as previously mentioned in Section 2, the RL model seems appropriate for specific Internet applications. This suggests that by looking at these specific kinds of data flows one could expect to find at high aggregation levels behaviour different from Gaussian.

4. Dependence structure: asymptotic distributions

We now analyze the weak limits of the stochastic process

$$\mathbf{A}(\delta) := \{A(k\delta,(k+1)\delta], -\infty < k < \infty\}$$

defined in (2.3). We will see that the $\mathbb{R}^\infty$ family indexed by δ converges to a limiting stable sequence

$$\boldsymbol{X}_\infty = \{X_\infty(k), -\infty < k < \infty\}$$

with

$$P[X_\infty(0) = X_\infty(k)] = 1.$$

The price paid for letting $\delta \to 0$ is thus a limit sequence with degenerate dependence structure. The consequence of sampling at too high a frequency (using economic terminology) is perfect dependence.

Before starting the analysis, we state the following lemma that will considerably simplify subsequent computations. Its proof follows by the same computations done for Proposition 5.

Lemma 1. *Let $\mathcal{R} = \mathcal{R}(\delta)$ be a subset of $\{<0,2\}$ of the form $\mathcal{R}(\delta) = \{(s,r,l,u) \in \{<0,2\} : (s,l) \in \mathcal{B}^{\mathcal{R}}(\delta)\}$ where $\mathcal{B}^{\mathcal{R}}(\delta)$ is a Borel subset of $\mathbb{R}\times\mathbb{R}_+$, and define*

$$A^{\mathcal{R}}(\delta) = \sum_{i=1}^{P^{\mathcal{R}}(\delta)} R_i^{\mathcal{R}}\delta.$$

Then, as $\delta \to 0$,

$$\frac{A^{\mathcal{R}}(\delta) - \frac{|\mathcal{B}^{\mathcal{R}}(\delta)|}{E(L)\bar{F}_L^{(0)}(\delta)}(n(\delta)+m(\delta))}{b(\delta)} \Rightarrow \frac{|\mathcal{B}^{\mathcal{R}}(0)|}{E(L)}X,$$

where $|\mathcal{B}^{\mathcal{R}}(\delta)|$ is the measure of the set $\mathcal{B}^{\mathcal{R}}(\delta)$ under the measure $ds \times F_L(dl)$, $|\mathcal{B}^{\mathcal{R}}(0)| = \lim_{\delta\to 0}|\mathcal{B}^{\mathcal{R}}(\delta)| \geq 0$ that is assumed to exist, and the limit random variable is stable with Lévy measure $\frac{|\mathcal{B}^{\mathcal{R}}(0)|}{E(L)}\nu$ whose density is $\frac{|\mathcal{B}^{\mathcal{R}}(0)|}{E(L)}\alpha_R x^{-\alpha_R-1}$.

Corollary 1. *Lemma 1 implies that if $|\mathcal{B}^{\mathcal{R}}(\delta)| \to 0$ as $\delta \to 0$ then*

$$A^{\mathcal{R}}(\delta) - \frac{|\mathcal{B}^{\mathcal{R}}(\delta)|}{E(L)\bar{F}_L^{(0)}(\delta)}(n(\delta)+m(\delta)) = o_p(b(\delta)).$$

4.1. Convergence of finite dimensional distributions of $\{A(i\delta,(i+1)\delta], i \geq 1\}$.

In this section we prove the following result.

Proposition 6. *For any non-negative integer k, as $\delta \to 0$, we have in $\mathbb{R}^{k+1}$,*

$$\frac{1}{b(\delta)}\begin{pmatrix} A(0,\delta] - d(\delta) \\ A(\delta,2\delta] - d(\delta) \\ \vdots \\ A(k\delta,(k+1)\delta] - d(\delta)\end{pmatrix} \Rightarrow \begin{pmatrix} X_\infty(0) \\ X_\infty(1) \\ \vdots \\ X_\infty(k)\end{pmatrix} \tag{4.1}$$

where

$$b(\delta) = \left(\frac{1}{\delta}\right)^{1/\alpha_R}\ell(\frac{1}{\delta}) \to \infty \quad (\delta \downarrow 0), \tag{4.2}$$

$$d(\delta) = 2\int_0^1 v\int_{r=v}^\infty \bar{F}_L(\delta r^{-1}v)r^{-1}\mu_\delta(dr)dv + b(\delta)\gamma(\delta)E(L)\bar{F}_L^{(0)}(\delta)\int_0^1 r\mu_{\delta b^{-1}(\delta)}(dr), \tag{4.3}$$

and $X_\infty(i)$ for $0 \leq i \leq k$ are each stable with Lévy measure ν whose density is $E(L)\alpha_R x^{-\alpha_R-1}$. In addition $P[X_\infty(i) = X_\infty(j)] = 1$ for $0 \leq i,j \leq k$.

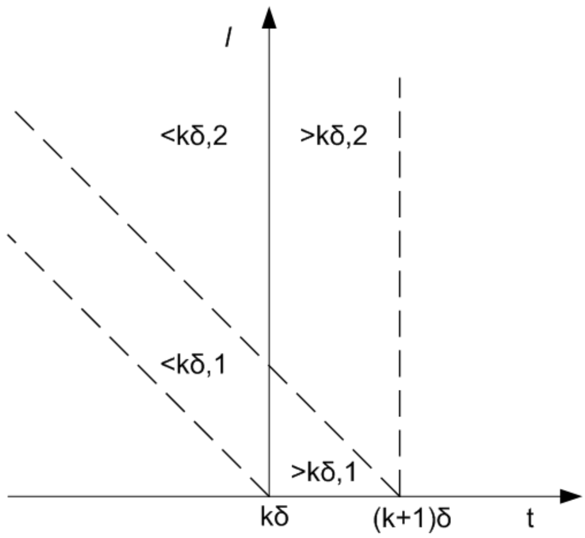

FIGURE 3. Four regions for analyzing contributions in k-th slot.

Remark 1. *The function $d(\delta)$ is the same function defined in (3.21). In equation (4.3) we have used the fact that*

$$m(\delta)+n(\delta)=\delta^{-1}\int_0^{b(\delta)} r\nu_\delta^{<0,2^-}(dr)=b(\delta)\gamma(\delta)\int_0^1 r\nu_\delta^{<0,2^+}(dr)$$

that is straightforward to check.

Proof. Along with the regions $\{<0,1\},\{<0,2\},\{>0,1\},\{>0,2\}$ used to analyze the convergence in distribution of $A(0,\delta]$, we need the analogously defined regions $\{<k\delta,1\},\{<k\delta,2\},\{>k\delta,1\},\{>k\delta,2\}$, where for example

$$\begin{aligned}\{<k\delta,2\}=&\{(s,r,l,u):s<k\delta,s+l>(k+1)\delta\}\\ \{>k\delta,2\}=&\{(s,r,l,u):k\delta<s<(k+1)\delta,s+l>(k+1)\delta\}.\end{aligned}$$

See Figure 3.

Additionally, for analyzing dependence between $A(0,\delta]$ and $A(k\delta,(k+1)\delta]$, we will need the regions $\mathcal{R}_{11},\mathcal{R}_{12},\mathcal{R}_{21},\mathcal{R}_{22}$ which contain points (Γ_k,R_k,L_k,F_k) contributing to both $A(0,\delta]$ as well as $A(k\delta,(k+1)\delta]$. (See the left graphic in Figure 4.) In particular, points in $\mathcal{R}_{22}=\{<0,2\}\cap\{<k\delta,2\}$ contribute

$$A^{\mathcal{R}_{22}}=\sum_{k:(\Gamma_k,R_k,L_k,F_k)\in\mathcal{R}_{22}} R_k\delta$$

to both $A(0,\delta]$ and $A(k\delta,(k+1)\delta]$ which shows a high degree of dependence is expected.

By applying Lemma 1 we have

$$\frac{A^{\mathcal{R}_{22}}-\frac{|\mathcal{B}^{\mathcal{R}_{22}}(\delta)|}{E(L)\bar{F}_L^{(0)}(\delta)}(n(\delta)+m(\delta))}{b(\delta)}\Rightarrow X_{\alpha_R}^{\mathcal{R}_{22}} \tag{4.4}$$

as $\delta\to 0$ with $X_{\alpha_R}^{\mathcal{R}_{22}}$ stable with Lévy measure ν.

As for the other regions (see the right side of Figure 4), set

$$\mathcal{R}^{<0,(\delta,(k+1)\delta]}=\{(s,r,l,u):s<0,\delta<|s|+l\le(k+1)\delta\}$$

and write

$$A(0,\delta]=A^{>0,1}(0,\delta]+A^{>0,2}(0,\delta]+A^{<0,1}(0,\delta]+A^{<0,2}(0,\delta]$$

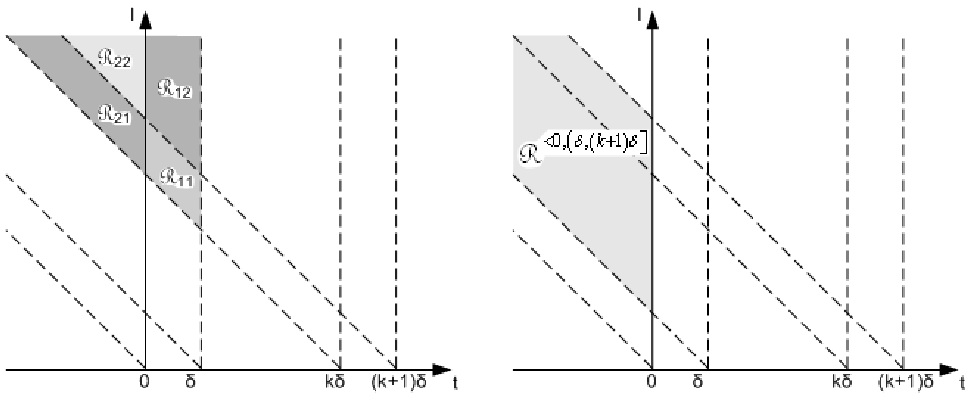

FIGURE 4. Regions for dependence analysis.

$$=A^{>0,1}(0,\delta]+A^{>0,2}(0,\delta]+A^{<0,1}(0,\delta]+A^{\mathcal{R}^{<0,(\delta,(k+1)\delta]}}(0,\delta]+A^{R_{22}}(0,\delta]. \tag{4.5}$$

To analyze region $\mathcal{R}^{<0,(\delta,(k+1)\delta]}$, observe that we have

$$|\mathcal{B}^{\mathcal{R}^{<0,(\delta,(k+1)\delta]}}(\delta)|=E(L)\Big(\bar F_L^{(0)}(\delta)-\bar F_L^{(0)}((k+1)\delta)\Big)\to 0.$$

Hence, by applying Corollary 1, we have that

$$A^{\mathcal{R}^{<0,(\delta,(k+1)\delta]}}(0,\delta]-(1-\frac{|\mathcal{B}^{\mathcal{R}_{22}}|}{E(L)\bar F_L^{(0)}(\delta)})(m(\delta)+n(\delta))=o_p(b(\delta)).$$

Therefore, we conclude

$$A(0,\delta]-d(\delta)=A^{\mathcal{R}_{22}}(0,\delta]-d(\delta)+o_p(b(\delta)). \tag{4.6}$$

Likewise, we consider $A(i\delta,(i+1)\delta]$ for $1\le i\le k$. We set

$$\mathcal{R}^{<0,((i+1)\delta,(k+1)\delta]}=\{(s,r,l,u):s<0,(i+1)\delta<s+l<(k+1)\delta\}$$
$$\mathcal{R}^{(0,i\delta],((i+1)\delta,\infty]}=\{(s,r,l,u):0<s\le i\delta;s+l>(i+1)\delta\}$$

(see Figure 5) and write

$$\begin{aligned}A(i\delta,(i+1)\delta]=&A^{\mathcal{R}^{>i\delta,1}}(i\delta,(i+1)\delta]+A^{\mathcal{R}^{>i\delta,2}}(i\delta,(i+1)\delta]+A^{\mathcal{R}^{<i\delta,1}}(i\delta,(i+1)\delta]\\&+\Big[A^{\mathcal{R}^{<0,((i+1)\delta,(k+1)\delta]}}(i\delta,(i+1)\delta]+A^{\mathcal{R}_{22}}(i\delta,(i+1)\delta]+A^{\mathcal{R}^{(0,i\delta],((i+1)\delta,\infty]}}(i\delta,(i+1)\delta]\Big].\end{aligned}$$

Again by Corollary 1 we have

$$A^{\mathcal{R}^{<0,((i+1)\delta,(k+1)\delta]}}(i\delta,(i+1)\delta]-(m(\delta)+n(\delta))\frac{|\mathcal{B}^{\mathcal{R}^{<0,((i+1)\delta,(k+1)\delta]}}(\delta)|}{E(L)\bar F_L^{(0)}(\delta)}=o_p(b(\delta))$$
$$A^{\mathcal{R}^{(0,i\delta],((i+1)\delta,\infty]}}(i\delta,(i+1)\delta]-(m(\delta)+n(\delta))\frac{|\mathcal{B}^{\mathcal{R}^{(0,i\delta],((i+1)\delta,\infty]}}(\delta)|}{E(L)\bar F_L^{(0)}(\delta)}=o_p(b(\delta))$$

Therefore, keeping in mind that

$$A^{\mathcal{R}_{22}}(k\delta,(k+1)\delta]=A^{\mathcal{R}_{22}}(i\delta,(i+1)\delta]=A^{\mathcal{R}_{22}}(0,\delta],$$

and that

$$|\mathcal{B}^{\mathcal{R}^{(0,i\delta],((i+1)\delta,\infty]}}(\delta)|+|\mathcal{B}^{\mathcal{R}^{<0,((i+1)\delta,(k+1)\delta]}}(\delta)|+|\mathcal{B}^{\mathcal{R}_{22}}(\delta)|=E(L)\bar F_L^{(0)}(\delta)$$

we have

$$A(i\delta,(i+1)\delta]-d(\delta)=A^{R_{22}}(0,\delta]-d(\delta)+o_p(b(\delta)). \tag{4.7}$$

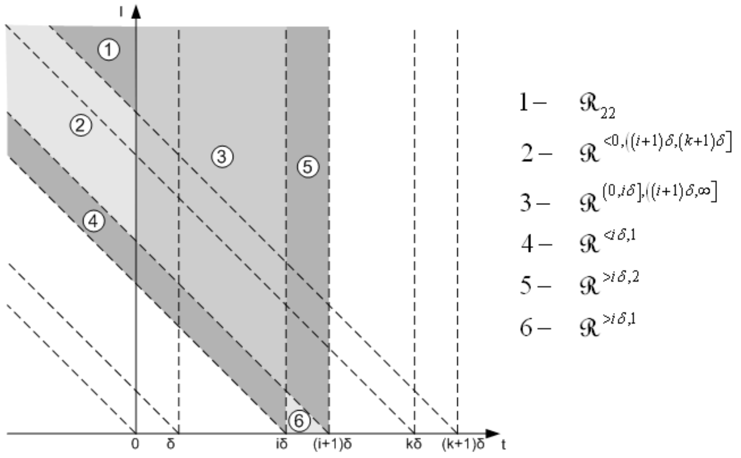

FIGURE 5. Regions for dependence analysis.

We thus have that

$$\begin{pmatrix} A(0,\delta]-d(\delta) \\ A(\delta,2\delta]-d(\delta) \\ \vdots \\ A(k\delta,(k+1)\delta]-d(\delta) \end{pmatrix} = \begin{pmatrix} A^{R_{22}}(0,\delta]-d(\delta) \\ A^{R_{22}}(0,\delta]-d(\delta) \\ \vdots \\ A^{R_{22}}(0,\delta]-d(\delta) \end{pmatrix} + \begin{pmatrix} o_p(b(\delta)) \\ o_p(b(\delta)) \\ \vdots \\ o_p(b(\delta)) \end{pmatrix}$$

and the conclusion of Proposition 6 follows. □

4.2. **Asymptotic distribution over intervals at fixed distance apart.** The previous section discusses dependence over successive slots of length δ. The asymptotic statement in Proposition 6 leads to a degenerate limit because $\delta \downarrow 0$ shrinks the distance between $A(0,\delta]$ and $A(k\delta,(k+1)\delta]$. Here we investigate $(A(0,\delta], A(t,t+\delta])$ for $t > \delta$ with t fixed and find that as $\delta \downarrow 0$, this vector is asymptotically stable with a nondegenerate dependence structure.

Proposition 7. *Suppose $t > 0$ is fixed. As $\delta \downarrow 0$,*

$$\frac{1}{b(\delta)} \begin{pmatrix} A(0,\delta]-d(\delta) \\ A(t,t+\delta]-d(\delta) \end{pmatrix} \Rightarrow \begin{pmatrix} X_1 + X_t \\ X_2 + X_t \end{pmatrix} \tag{4.8}$$

where $d(\delta)$ is given by (4.3) in Proposition 6, X_1, X_2, X_t are independent, stable random variables with Lévy measures respectively

$$\nu_1(dx) = \nu_2(dx) = E(L)F_L^{(0)}(t)\alpha_R x^{-\alpha_R-1}dx$$
$$\nu_t(dx) = E(L)\bar{F}_L^{(0)}(t)\alpha_R x^{-\alpha_R-1}dx.$$

Proof. As in the proof of Proposition 6 we decompose

$$A(0,\delta] = A^{<0,(\delta,t+\delta]}(\delta) + A^{<0,(t+\delta,\infty]}(\delta) + o_p(b(\delta)),$$
$$A(t,t+\delta] = A^{(0,t],(t+\delta,\infty]}(t,t+\delta] + A^{<0,(t+\delta,\infty]}(t,t+\delta] + o_p(b(\delta)),$$

and keep in mind that

$$A^{<0,(t+\delta,\infty]}(\delta) = A^{<0,(t+\delta,\infty]}(t,t+\delta]$$

and

$$A^{<0,(\delta,t+\delta]}(\delta) \overset{d}{=} A^{(0,t],(t+\delta,\infty]}(t,t+\delta].$$

Then we apply Lemma 1 by noticing that

$$|\mathcal{B}^{<0,(\delta,t+\delta]}(0)| = E(L)F_L^{(0)}(t)$$

and

$$|\mathcal{B}^{<0,(t+\delta,\infty]}(0)| = E(L)\bar{F}_L^{(0)}(t).$$

□

5. Dependence structure: Extremal dependence analysis

Correlations are not defined for either the pair $(A(0,\delta], A(k\delta,(k+1)\delta])$ or for the limits (X_1+X_t, X_2+X_t) in Proposition 7. This precludes a conventional discussion of long range dependence of variables lagged by k or t. In such circumstances, alternatives such as *covariation* (Samorodnitsky and Taqqu (1994)) for stable processes or the *extremal dependence measure* (Campos et al. (2005), Resnick (2004a)) for regularly varying processes attempt to provide a numerical summary of dependence.

The extremal dependence measure (EDM) is defined for two non-negative random variables whose joint distribution is multivariate regularly varying on the cone $[0,\infty)^2$ with limit measure ν and angular probability measure S. (This terminology is reviewed in Resnick (2006, Chapter 6).)

For a random vector $\boldsymbol{Z} = (Z^{(1)}, Z^{(2)})$, this means there must exist a normalizing sequence $b_n \to \infty$ and a Radon measure ν on $[0,\infty]^2 \setminus \{\boldsymbol{0}\}$ such that

$$nP[\frac{\boldsymbol{Z}}{b_n} \in \cdot] \xrightarrow{v} \nu(\cdot), \tag{5.1}$$

where $\xrightarrow{v}$ denotes vague convergence on $[0,\infty]^2 \setminus \{\boldsymbol{0}\}$. The limit measure ν has the property that the polar coordinate transformation converts ν into a product measure,

$$\nu\{\boldsymbol{x} \in [0,\infty]^2 \setminus \{\boldsymbol{0}\} : r(\boldsymbol{x}) > r_0, \theta(\boldsymbol{x}) \in \cdot\} = cr_0^{-\alpha}S(\cdot),$$

for some $c > 0$, some index $\alpha > 0$ and S a probability measure on $[0,\pi/2]$. Here $(r(\boldsymbol{x}), \theta(\boldsymbol{x}))$ are the usual polar coordinates of $\boldsymbol{x}$ and S is called the angular measure.

The *extremal dependence measure* (EDM) of the two random variables $(Z^{(1)}, Z^{(2)})$ is

$$EDM(Z^{(1)}, Z^{(2)}) := 1 - \frac{\int_0^{\pi/2}\left(\theta - \frac{\pi}{4}\right)^2 S(d\theta)}{(\pi/4)^2} = \frac{1}{\left(\frac{\pi}{4}\right)^2}\int_0^{\frac{\pi}{2}} \theta\left(\frac{\pi}{2} - \theta\right) S(d\theta). \tag{5.2}$$

The EDM has some of the desirable properties possessed by correlation (Resnick, 2004a). The EDM is 0 if $(Z^{(1)}, Z^{(2)})$ are independent or asymptotically independent and the EDM is 1 if $(Z^{(1)}, Z^{(2)})$ are comonotone, $P[Z^{(1)} = Z^{(2)}] = 1$, or if asymptotic full dependence holds. Another useful property which we need is summarized next.

Proposition 8. *Suppose $\boldsymbol{Z}_1, \boldsymbol{Z}_2$ are two independent random vectors of dimension 2 satisfying* (5.1) *with the same scaling sequence $\{b_n\}$. Then with obvious notation regarding subscripting we have the EDM of $\boldsymbol{Z}_1 + \boldsymbol{Z}_2$ satisfies*

$$\begin{aligned} EDM(Z_1^{(1)} + Z_2^{(1)}, Z_1^{(2)} + Z_2^{(2)}) =& \Big(\frac{\nu_1\{\boldsymbol{x} : \|\boldsymbol{x}\| > 1\}}{\nu_1\{\boldsymbol{x} : \|\boldsymbol{x}\| > 1\} + \nu_2\{\boldsymbol{x} : \|\boldsymbol{x}\| > 1\}}\Big) EDM(Z_1^{(1)}, Z_1^{(2)}) \\ &+ \Big(\frac{\nu_2\{\boldsymbol{x} : \|\boldsymbol{x}\| > 1\}}{\nu_1\{\boldsymbol{x} : \|\boldsymbol{x}\| > 1\} + \nu_2\{\boldsymbol{x} : \|\boldsymbol{x}\| > 1\}}\Big) EDM(Z_2^{(1)}, Z_2^{(2)}). \end{aligned} \tag{5.3}$$

Proof. The limit measure for $\boldsymbol{Z}_1 + \boldsymbol{Z}_2$ is $\nu_1 + \nu_2$ (Resnick (2006, Proposition 7.4, Section 7.3)) and therefore the angular measure of $\nu_1 + \nu_2$ can be written as

$$\frac{(\nu_1 + \nu_2)\big(\{\boldsymbol{x} : r(\boldsymbol{x}) > 1, \theta(\boldsymbol{x}) \in \cdot\}\big)}{(\nu_1 + \nu_2)\big(\{\boldsymbol{x} : r(\boldsymbol{x}) > 1\}\big)}$$

The rest follows from algebra. □

Remark 2. One practical implication of Proposition 8 is that if we can decompose a bivariate vector into an independent sum, then any summand which possesses independent or asymptotically independent components will not contribute to the EDM and can be neglected.

5.1. **The extremal dependence measure of** (X_1+X_t, X_2+X_t)**.** To see how an EDM calculation works, consider the stable vector

$$(X_1 + X_t, X_2 + X_t) = (X_1, X_2) + (X_t, X_t).$$

Its two dimensional Lévy measure is $\nu_1 + \nu_2$ where ν_1 is the Lévy measure of (X_1, X_2) and ν_2 is the Lévy measure of (X_t, X_t). Since (X_1, X_2) has independent components, its two dimensional Lévy measure concentrates on the axes and will contribute zero to the overall EDM of the sum. The measure ν_2 concentrates on the diagonal $\{(x, x) : x > 0\}$ and puts mass $E(L)\bar{F}_L^{(0)}(t)\nu_{\alpha_R}(dx)$ along the diagonal. The angular measure S corresponding to $\nu_1 + \nu_2$ has the form

$$S(d\theta) = c'\epsilon_0(d\theta) + c'\epsilon_{\pi/2}(d\theta) + c_t\epsilon_{\pi/4}(d\theta),$$

where to make S a probability measure we require

$$2c' + c_t = 1.$$

Proposition 8 implies that the constant c_t is given by

$$c_t := \nu_2\{\boldsymbol{x} : \|\boldsymbol{x}\| \geq 1\} = (const)\bar{F}_L^{(0)}(t).$$

Therefore, the EDM of $(X_1 + X_t, X_2 + X_t)$, denoted $EDM(t)$, is

$$\begin{aligned} EDM(t) =& 1 - \frac{\int_0^{\pi/2}\left(\theta - \frac{\pi}{4}\right)^2 S(d\theta)}{(\pi/4)^2} \\ =& 1 - \frac{2c'\left(\frac{\pi}{4}\right)^2}{(\pi/4)^2} = 1 - 2c' = c_t \\ =& (const)\bar{F}_L^{(0)}(t). \end{aligned}$$

Thus $EDM(t)$ decays as $ct^{-(\alpha_L - 1)}\ell_L(t)$ for some $c > 0$.

5.2. **The extremal dependence measure of** $(A(0, \delta], A(k\delta, (k+1)\delta])$ **as a function of** k**.** In this section we fix δ and compute an asymptotic form for the EDM of the bivariate random vector $(A(0, \delta], A(k\delta, (k+1)\delta])$ and show a power law decay as $k \to \infty$.

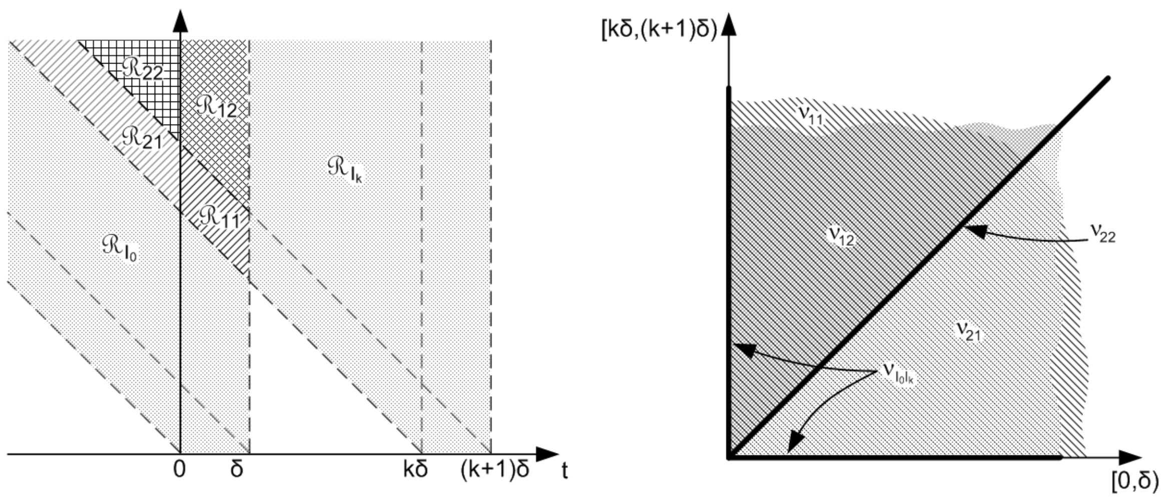

FIGURE 6. Regions and Lévy measures.

We first decompose the components of $(A(0,\delta], A(k\delta,(k+1)\delta])$ in the following terms that refer to the various regions depicted in Figure 6:

$$A(0,\delta] = A_0^{\mathcal{R}_{I_0}} + A_0^{\mathcal{R}_{11}} + A_0^{\mathcal{R}_{12}} + A_0^{\mathcal{R}_{21}} + A_0^{\mathcal{R}_{22}}$$
$$A(k\delta,(k+1)\delta] = A_k^{\mathcal{R}_{I_k}} + A_k^{\mathcal{R}_{11}} + A_k^{\mathcal{R}_{12}} + A_k^{\mathcal{R}_{21}} + A_k^{\mathcal{R}_{22}}$$

where

$$A_0^{\mathcal{R}_{11}} = \sum_{i=1}^{P^{\mathcal{R}_{11}}} R_i^{\mathcal{R}_{11}}(\delta - \Gamma_i^{\mathcal{R}_{11}}); \quad A_k^{\mathcal{R}_{11}} = \sum_{i=1}^{P^{\mathcal{R}_{11}}} R_i^{\mathcal{R}_{11}}(L_i^{\mathcal{R}_{11}} + \Gamma_i^{\mathcal{R}_{11}} - k\delta)$$
$$A_0^{\mathcal{R}_{12}} = \sum_{i=1}^{P^{\mathcal{R}_{12}}} R_i^{\mathcal{R}_{12}}(\delta - \Gamma_i^{\mathcal{R}_{12}}); \quad A_k^{\mathcal{R}_{12}} = \sum_{i=1}^{P^{\mathcal{R}_{12}}} R_i^{\mathcal{R}_{12}}\delta$$
$$A_0^{\mathcal{R}_{21}} = \sum_{i=1}^{P^{\mathcal{R}_{21}}} R_i^{\mathcal{R}_{21}}\delta; \quad A_k^{\mathcal{R}_{21}} = \sum_{i=1}^{P^{\mathcal{R}_{21}}} R_i^{\mathcal{R}_{21}}(L_i^{\mathcal{R}_{21}} + \Gamma_i^{\mathcal{R}_{21}} - k\delta)$$
$$A_0^{\mathcal{R}_{22}} = \sum_{i=1}^{P^{\mathcal{R}_{22}}} R_i^{\mathcal{R}_{22}}\delta; \quad A_k^{\mathcal{R}_{22}} = \sum_{i=1}^{P^{\mathcal{R}_{22}}} R_i^{\mathcal{R}_{22}}\delta$$

This allows us to decompose the bivariate random vector in the following way

$$(A(0,\delta], A(k\delta,(k+1)\delta]) = (A_0^{\mathcal{R}_{I_0}}, A_k^{\mathcal{R}_{I_k}}) + (A_0^{\mathcal{R}_{11}}, A_k^{\mathcal{R}_{11}}) + (A_0^{\mathcal{R}_{12}}, A_k^{\mathcal{R}_{12}}) + (A_0^{\mathcal{R}_{21}}, A_k^{\mathcal{R}_{21}}) + (A_0^{\mathcal{R}_{22}}, A_k^{\mathcal{R}_{22}}), \tag{5.4}$$

where all the random vectors on the right side are compound Poisson, and independent of each other. Furthermore, using the same argument as the one used to prove equality in distribution of the random variables $A^{>0,2}(\delta)$ and $A^{<0,1}(\delta)$, we have $A_0^{\mathcal{R}_{I_0}} \stackrel{d}{=} A_k^{\mathcal{R}_{I_k}}$ and $(A_0^{\mathcal{R}_{12}}, A_k^{\mathcal{R}_{12}}) \stackrel{d}{=} (A_k^{\mathcal{R}_{21}}, A_0^{\mathcal{R}_{21}})$.

We claim

(1) The dominant contribution to EDM is from $(A_0^{\mathcal{R}_{22}}, A_k^{\mathcal{R}_{22}})$, which as a function of k decays as a constant times $\bar{F}_L^{(0)}(k)$.
(2) The contribution of $(A_0^{\mathcal{R}_{I_0}}, A_k^{\mathcal{R}_{I_k}})$ to EDM can be neglected because the random vector consists of independent components.
(3) The contribution of $(A_0^{\mathcal{R}_{21}}, A_k^{\mathcal{R}_{21}})$ to EDM is of lower order when $k \to \infty$. This is also true for $(A_0^{\mathcal{R}_{12}}, A_k^{\mathcal{R}_{12}})$ because of the distributional identities.

5.2.1. *Contribution to EDM from* $(A_0^{\mathcal{R}_{22}}, A_k^{\mathcal{R}_{22}})$. This is a relatively easy calculation since

$$A_0^{\mathcal{R}_{22}} = A_k^{\mathcal{R}_{22}} = \sum_{i=1}^{\mathcal{P}^{\mathcal{R}_{22}}} R_1^{\mathcal{R}_{22}}\delta,$$

and therefore, using (5.3), the contribution to EDM from $(A_0^{\mathcal{R}_{22}}, A_k^{\mathcal{R}_{22}})$ is proportional to $\nu_{22}\{\boldsymbol{x} : \|\boldsymbol{x}\| \geq 1\}$, where ν_{22} is the limit measure of $(A_0^{\mathcal{R}_{22}}, A_k^{\mathcal{R}_{22}})$. For some constant $c > 0$ (not necessarily the same with each use), and b_n satisfying $n\bar{F}_R(b_n) \to 1$, we have

$$\begin{aligned}
\nu_{22}\{\boldsymbol{x} : \|\boldsymbol{x}\| \geq 1\} &= \lim_{n\to\infty} nP[\|(A_0^{\mathcal{R}_{22}}, A_k^{\mathcal{R}_{22}})\|/b_n \geq 1] \\
&= \lim_{n\to\infty} cnP[\sum_{i=1}^{\mathcal{P}^{\mathcal{R}_{22}}} R_i^{\mathcal{R}_{22}}\delta > b_n] = \lim_{n\to\infty} cE(\mathcal{P}^{\mathcal{R}_{22}})nP[R_1^{\mathcal{R}_{22}}\delta > b_n] \\
&= \lim_{n\to\infty} cE(\mathcal{P}^{\mathcal{R}_{22}})P[R_1^{\mathcal{R}_{22}}\delta > b_n]/\bar{F}_R(b_n) = c\iiint_{\substack{r\delta > b_n,\, s<0 \\ s+l>(k+1)\delta}} \lambda ds \frac{F_R(dr)}{\bar{F}_R(b_n)} F_L(dl) \\
&= c\lambda \frac{\bar{F}_R(x/\delta)}{\bar{F}_R(b_n)} \iint_{\substack{s<0 \\ l>(k+1)\delta - s}} F_L(dl)ds = c\bar{F}_L^{(0)}((k+1)\delta).
\end{aligned}$$

Our conclusion is that, as $k \to \infty$, the contribution to the composite EDM will be proportional to $\bar{F}_L^{(0)}(k)$.

5.2.2. *The contribution of* $(A_0^{\mathcal{R}_{21}}, A_k^{\mathcal{R}_{21}})$ *to EDM is of lower order when* $k \to \infty$. We start with a marginal calculation analogous to the one just completed: As $x \to \infty$,

$$\begin{aligned} P[A_0^{\mathcal{R}_{21}} > x] \sim & \iiint_{\substack{s<0, r\delta > x \\ k\delta < s+l \leq (k+1)\delta}} \lambda ds F_R(dr) F_L(dl) \\ =& \lambda \bar{F}_R(x/\delta) \int_{s=-\infty}^{0} \int_{l=k\delta - s}^{(k+1)\delta - s} F_L(dl) \lambda ds \end{aligned}$$

and changing variables again $-s \mapsto s$ yields

$$= E(L) \bar{F}_R(s/\delta) \lambda \big[\bar{F}_L^{(0)}(k\delta) - \bar{F}_L^{(0)}((k+1)\delta) \big].$$

As $k \to \infty$, because of the difference, this is asymptotic to a constant times $\bar{F}_L(k)$ which is of lower order than $\bar{F}_L^{(0)}$.

For the EDM analysis of $(A_0^{\mathcal{R}_{21}}, A_k^{\mathcal{R}_{21}})$, note

$$A_0^{\mathcal{R}_{21}} \geq A_k^{\mathcal{R}_{21}}. \tag{5.5}$$

Referring back to (5.1), let b_n be the appropriate scaling constant (which could be the quantile function of F_R). Then from the definition (5.2) and the ordering (5.5) we have

$$\begin{aligned} EDM(A_0^{\mathcal{R}_{21}}, A_k^{\mathcal{R}_{21}}) =& \int_0^{\pi/2} \theta(\pi/2 - \theta) \lim_{n \to \infty} nP[\sqrt{(A_0^{\mathcal{R}_{21}})^2 + (A_k^{\mathcal{R}_{21}})^2} > b_n, \Theta \in d\theta] \\ \leq & (const) nP[(A_0^{\mathcal{R}_{21}})^2 + (A_k^{\mathcal{R}_{21}})^2 > b_n^2] \\ \leq & nP[\sqrt{2} A_0^{\mathcal{R}_{21}} > b_n] \\ \sim & (const) \big[\bar{F}_L^{(0)}(k\delta) - \bar{F}_L^{(0)}((k+1)\delta) \big]. \end{aligned}$$

Therefore, as $k \to \infty$

$$EDM(A_0^{\mathcal{R}_{21}}, A_k^{\mathcal{R}_{21}}) = o(\bar{F}_L^{(0)}(k)).$$

5.2.3. *Conclusion.* We have the following conclusion. The EDM satisfies as $k \to \infty$,

$$EDM(A(0,\delta], A(k\delta, (k+1)\delta]) \sim (const) \bar{F}_L^{(0)}(k) = (const) k^{-(\alpha_L - 1)} \ell_L(k).$$

So α_R controls the heaviness of tails of cumulative input and α_L controls dependence decay for fixed δ, as $k \to \infty$.

6. Comparison of the RL and RF models.

In this section we compare the two models, RL and RF, in order to emphasize their differences. As noted in Section 2, these two models make different assumptions about the joint distribution of the mark vector (R, L, F) even though in both cases $F = LR$. The RL model assumes the components R and L independent while the RF chooses R and F as independent.

6.1. **Asymptotic distribution of** $A(0, \delta)$**.** For each of the two models, Table 1 summarizes the result about the limit distribution of the random variable $A(\delta)$ and its components $A^{\mathcal{R}}(\delta)$ relative to the regions $\mathcal{R} \in \big\{\{>0, 2\}; \{>0, 1\}; \{<0, 2\}; \{<0, 1\}\big\}$. For reading Table 1, recall

$$P[F \leq x] = G(x), \quad P[L \leq x] = F_L(x), \quad P[R \leq x] = F_R(x),$$

and the tails of the three distributions are regularly varying with parameters $\alpha_F, \alpha_L, \alpha_R$ respectively, all assumed strictly between 0 and 1. Further, $\mu_\delta(dr) := \frac{F_R(\delta^{-1} dr)}{\bar{F}_R(\delta^{-1})}$ and

$$\bar{G}_0(x) := \int_x^\infty \bar{G}(u) du / E(F); \quad \bar{F}_L^{(0)}(x) := \int_x^\infty \bar{F}_L(u) du / E(L).$$

The table uses the following additional notation:

$$\nu^{\delta}_{\rm RL}(ds) := \Big(\int_s^\infty \bar F_L(\delta r^{-1}s) r^{-1}\mu_\delta(dr)\Big)ds; \qquad \nu^{\delta}_{\rm RF}(ds) := \Big(\bar G(s)\int_{r=s}^\infty r^{-1}\mu_\delta(dr)\Big)ds;$$

$$n_{\rm RL}(\delta) := \delta^{-1}\int_\delta^\infty \bar F_L(l)dl \int_0^{b_{\rm RL}(\delta)} r\mu_\delta(dr) \qquad n_{\rm RF}(\delta) := E(F)\int_0^1 \bar G_0(r)\mu_\delta(dr);$$

$$d_{\rm RL}(\delta) := n_{\rm RL}(\delta) + 2\int_0^1 s\,\nu^\delta_{\rm RL}(ds) \qquad d_{\rm RF}(\delta) := n_{\rm RF}(\delta) + 2\int_0^1 s\,\nu^\delta_{\rm RF}(ds);$$

$$b_{\rm RL}(\delta) := \Big(\frac{1}{\delta}\Big)^{1/\alpha_R}\ell(\frac{1}{\delta}); \qquad b_{\rm RF}(\delta) := \Big(E(F)\int_0^1 r\bar G_0(r)\mu_\delta(dr)\Big)^{1/2}$$

$\mathcal{R}$	RL model	RF model
$\{>0,1\}$	negligible (under conditions)	$A^{>0,1}(\delta) \Rightarrow X^{>0,1}_{\rm RF}$, id, $P[X^{>0,1}_{\rm RF} > x] \in RV_{-(\alpha_R+\alpha_F)}$
$\{<0,2\}$	$\frac{A^{<0,2}(\delta)-n_{\rm RL}(\delta)}{b_{\rm RL}(\delta)} \Rightarrow X^{<0,2}_{\rm RL}$, stable$(\alpha_R)$	$\frac{A^{<0,2}(\delta)-n_{\rm RF}(\delta)}{b_{\rm RF}(\delta)} \Rightarrow N^{<0,2}_{\rm RF} \sim N(0,1)$
$\{>0,2\}$	$A^{>0,2}(\delta) - \int_0^1 s\nu^\delta_{\rm RL}(ds) \Rightarrow X^{>0,2}_{\rm RL}$, stable$(\alpha_R)$	$A^{>0,2}(\delta) - \int_0^1 s\,\nu^\delta_{\rm RF}(ds) \Rightarrow X^{>0,2}_{\rm RF}$, id, $P[X^{>0,2}_{\rm RF} > x] \in RV_{-(\alpha_R+\alpha_F)}$
$\{<0,1\}$	$A^{<0,1}(\delta) \overset{d}{=} A^{>0,2}(\delta)$	$A^{<0,1}(\delta) \overset{d}{=} A^{>0,2}(\delta)$
all	$\frac{A(\delta)-d_{\rm RL}(\delta)}{b_{\rm RL}(\delta)} \Rightarrow X^{<0,2}_{\rm RL}$, stable$(\alpha_R)$	$\frac{A(\delta)-d_{\rm RF}(\delta)}{b_{\rm RF}(\delta)} \Rightarrow N^{<0,2}_{\rm RF} \sim N(0,1)$

TABLE 1. Comparison of the RL and RF models: weak limit of $A(\delta)$. Note *id* stands for "infinitely divisible".

The comparison table shows that in both models, $A(0,\delta]$ converges weakly after centering and scaling. In the RF model, the weak limit is a normal random variable while in the RL model, the weak limit is a heavy tailed stable random variable with index α_R.

For both models, the main component in $A(0,\delta]$ comes from region $\{<0,2\}$. We need to understand the difference in treatment of this region by the two models. Sources that contribute in this region are ones whose durations are relatively long, since they start from the past and continue past δ and their contributions are given by $R\delta$ in each case. In the *RF* model, R and F are independent and it is therefore unlikely they are both large. The relationship $L = F/R$ means that long L may be associated with small R and large F. To have a contribution of a long L limits the values of R in such a way that the central limit theorem holds. In the RL model, on the other hand, the independence of R and L makes it unlikely that R and L are both large but a large value of R makes cumulative input in $(0,\delta]$ asymptotically stable with index α_R while a large value of L induces dependence measured by decay governed by α_L.

For both models, we have always assumed that $\lambda(\delta) = \frac{1}{\delta \bar F_R(\delta^{-1})}$. If we compare the normalizing functions $b_{\rm RF}(\delta)$ and $b_{\rm RL}(\delta)$, we have as $\delta \to 0$ that

$$b_{\rm RF}(\delta) = o\big(b_{\rm RL}(\delta)\big).$$

This means that if we construct a combined model that mixes the *RF* and the *RL* model, we would get that its limit behaviour coincides with that of the *RL* model. In order to get in the limit a linear combination of a normal distribution and an infinite divisible distribution, the Poisson intensity functions $\{\lambda(\delta)\}$ would have to grow at different speeds.

6.2. **Dependence structure.** If we look at the dependence structure, we have, in the limit, the effect of high frequency sampling for both models. For any nonnegative integer k, and $h \in \{RL, RF\}$, as $\delta \to 0$ we have

$$\frac{1}{b_h(\delta)} \begin{pmatrix} A(0,\delta] - d_h(\delta) \\ A(\delta, 2\delta] - d_h(\delta) \\ \vdots \\ A(k\delta, (k+1)\delta] - d_h(\delta) \end{pmatrix} \Rightarrow \begin{pmatrix} X_h^{<0,2}(0) \\ X_h^{<0,2}(1) \\ \vdots \\ X_h^{<0,2}(k) \end{pmatrix}$$

in $\mathbb{R}^{k+1}$, with $P[X_h^{<0,2}(i) = X_h^{<0,2}(j)] = 1$ for $0 \le i, j \le k$. If $h = RF$, the limit is Gaussian. If $h = RL$, the limit is stable.

The time slots $(i\delta, (i+1)\delta]$, $i = 0, \dots, k$ are at a distance from each other which converges to 0 as $\delta \to 0$. For time slots $(0, \delta], (t, t+\delta]$ at a minimum distance t apart, we have for fixed $t > 0$, with $h \in \{RF, RL\}$, as $\delta \downarrow 0$, that

$$\frac{1}{b_h(\delta)} \begin{pmatrix} A(0,\delta] - d_h(\delta) \\ A(t, t+\delta] - d_h(\delta) \end{pmatrix} \Rightarrow \begin{pmatrix} X_h^{<0,2}(0) \\ X_h^{<0,2}(t) \end{pmatrix}. \tag{6.1}$$

When $h = RL$, the limit is the dependent stable vector given in Proposition 7; see (4.8). When $h = RF$, the limit is a dependent Gaussian pair (D'Auria and Resnick, 2006).

Assessing decay of dependence requires different techniques for the two models. For the RF model, tails are relatively thin allowing traditional correlation techniques to be used to claim long range dependence. For the heavy tailed RL model, correlations do not exist and an alternative technique based on the EDM is used to show slow decay of dependence at power law rate. These results are summarized in Table 2 which uses the notation for the limit in (6.1).

	RL model	RF model
$t \to \infty$	$\mathrm{EDM}(X_{\mathrm{RL}}^{<0,2}(0), X_{\mathrm{RL}}^{<0,2}(t)) \sim ct^{-(\alpha_L - 1)}\ell_L(t)$	$\mathrm{Cov}(X_{\mathrm{RF}}^{<0,2}(0), X_{\mathrm{RF}}^{<0,2}(t)) \sim ct^{-(\alpha_F - 1)}\ell_F(t)$
$k \to \infty$, fixed δ	$\mathrm{EDM}(A(0,\delta], A(k\delta, (k+1)\delta]) \sim ck^{-(\alpha_L - 1)}\ell_L(k)$	$\mathrm{Cov}(A(0,\delta], A(k\delta, (k+1)\delta]) \sim ck^{-(\alpha_F - 1)}\ell_F(k)$

TABLE 2. Comparison of the RL and RF models: dependence structure.

7. FINAL THOUGHTS

Heterogeneous traffic comprising different types of applications may behave differently than more homogeneous traffic. In particular, it looks sensible to decompose traffic into classes of fairly homogeneous applications and to study each separately seeking statistical differences in their characteristics. Our results suggest that tail behaviour, dependence structure and approximating distributions may depend on the statistical characteristics of each application component of network traffic.

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