

On the improvement of Hölder seminorms in superquadratic Hamilton-Jacobi equations

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Abstract

We show in this paper that maximal L^q -regularity for time-dependent viscous Hamilton-Jacobi equations with unbounded right-hand side and superquadratic γ -growth in the gradient holds in the full range $q > (N + 2)\frac{\gamma-1}{\gamma}$. Our approach is based on new $\frac{\gamma-2}{\gamma-1}$ -Hölder estimates, which are consequence of the decay at small scales of suitable nonlinear space and time Hölder quotients. This is obtained by proving suitable oscillation estimates, that also give in turn some Liouville type results for entire solutions.

1 Introduction

The goal of the present paper is to address a problem of maximal L^q -regularity for semilinear equations of Hamilton-Jacobi (HJ) type

$$-\partial_t u - \Delta u + h(x, t)|Du|^\gamma = f(x, t) \quad \text{on } Q = \Omega \times (0, T) \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 1$, h is a continuous function which does not change sign (and which will be assumed here to be strictly positive), f is given in the Lebesgue space $L^q(Q)$, and the nonlinear gradient term is *superquadratic*, i.e. $\gamma > 2$. We will show that for strong solutions u in the parabolic Sobolev space $W_q^{2,1}$, it is possible to control (locally) the L^q -norms of $\partial_t u$, $D^2 u$ and $|Du|^\gamma$ with respect to the L^q -norm of f .

Equation (1.1) appears in several contexts, such as stochastic optimal control, where it is naturally backward in time. The regularity result we obtain here sits in between the realms of weak solutions (developed mostly below the so-called natural growth $\gamma = 2$), and viscosity solutions. It is impossible to summarize here the many, many contributions on the study of (1.1) and its quasi-linear generalizations, but it is rather well known how to deal with weak solutions, which involve the information $|Du|^\gamma \in L^1$, or viscosity ones, that structurally require f to be at least continuous. The problem of maximal regularity in L^q , which has a genuine flavor of *strong solutions*, has been recently considered in [5]. In the superquadratic case $\gamma > 2$, that will be addressed here, previous results are not complete, since maximal regularity is

expected to hold up to

$$q > q_0 = (N+2) \frac{\gamma-1}{\gamma},$$

while $q \geq (N+2) \frac{\gamma-1}{2}$ only has been covered in [5] (see also [12] for results on nonlocal HJ equations). Here, we deal with the full range $q > q_0$. This extension requires a substantial improvement of the methods developed previously.

There is an elliptic counterpart of maximal regularity for HJ equations, which has been addressed in [6, 8, 11, 13]. For $q > N(\gamma-1)/\gamma$, that was conjectured to hold by P.-L. Lions a decade ago, and it has been achieved using nonlinear methods (such as the Bernstein method) [6, 11, 13] or blow-up techniques [8]. The stationary problem has deep differences with respect to the parabolic one, and ideas developed for the former have by no means obvious adaptations to the latter. For example, blow-up techniques are based on the validity of Liouville theorems. For stationary HJ equations, these are known from the eighties [17, 19], and they are so powerful that they hold without any information on the growth of solutions at infinity. On the other hand, Liouville theorems for parabolic problems are known under rather restrictive conditions on the growth of the solutions [22], and they are even false for solutions having a linear behavior at infinity (see for example Remark 4.2). The problem is even more delicate beyond the natural growth $\gamma > 2$, which is addressed here, where the diffusive part of the equation is typically regarded as negligible (indeed, several “unnatural” phenomena which are typical of first-order equations arise, see e.g. [9, 21]). Heuristically, many difficulties come from the very different behavior at small scales of the two operators $-\partial_t u - \Delta u$ and $-\partial_t u - |Du|^\gamma$, and that is why one usually “chooses” between one of them, and regards the rest of the equation as a perturbation. Nevertheless, we wish to show that their effects are somehow intertwined at the α_0 -Hölder scale, where $\alpha_0 = \frac{\gamma-2}{\gamma-1}$.

Here, our way to maximal regularity goes indeed through the study of Hölder regularity, which is intimately connected with Liouville theorems, and in all of the three aspects our results are new, up to our knowledge. A few works, such as [3, 24], addressed the problem of Hölder regularity for parabolic problems with $\gamma > 2$ and unbounded f , although with a perturbative viewpoint which excluded possible regularization effects coming from the diffusion, and under the sign condition $f \geq 0$ (see Remark 6.2 on further comments on the issues related to the unboundedness of f from below). Let us also mention [1], where (first-order) counterexamples suggest that without any control on the modulus of continuity of h , there might be limitations on the Hölder regularity of solutions. Since our goal is to achieve α_0 -Hölder (which might be very close to Lipschitz, for large γ), we will admit estimates to depend on the modulus of continuity of h . First-order problems are also considered in [2], where Sobolev regularity of Du is studied with unbounded f ; it is worth noting that even without diffusion, the HJ equation works at different scales in space and time (that is why [2] uses techniques à la DiBenedetto, where cylinders depending on the solution itself are employed). Here, we also need to take care of the effects of the diffusion.

Let us now state the main result. Throughout the paper,

$$\gamma > 2, \quad h \in C(\overline{\Omega} \times [0, T]), \quad 0 < h_0 \leq h(x, t) \leq h_1.$$

Theorem 1.1. *Let $q_0 < q < N + 2$. For every $K > 0$ and $Q' \subset\subset Q$, there exists C depending on K, Q', Q, h, q_0 such that if $u \in W_q^{2,1}(Q)$ solves (1.1) in the strong sense, with $\|u\|_{L^\infty(Q)}, \|f\|_{L^q(Q)} \leq K$, then*

$$\|\partial_t u\|_{L^q(Q')} + \|D^2 u\|_{L^q(Q')} + \| |Du|^\gamma \|_{L^q(Q')} \leq C.$$

The statement is also true when $q \geq N + 2$, see Remark 6.1. As we previously announced, the proof of this result is a consequence of some Hölder regularity properties, which we state here as follows:

Theorem 1.2. *Let $q_0 < q < N + 2$. For every $K > 0$ and $Q' \subset\subset Q$, there exists C depending on K, Q', Q, h, q_0 such that if $u \in W_q^{2,1}(Q)$ solves (1.1) in the strong sense, with $\|u\|_{L^\infty(Q)}, \|f\|_{L^q(Q)} \leq K$, then*

$$\frac{|u(x, t) - u(x', t')|}{|x - x'|^\alpha + |t - t'|^{\alpha/2}} \leq C$$

for all $(x, t), (x', t') \in Q'$, where $\alpha = 2 - \frac{N+2}{q}$.

If $q = q_0$, then the same conclusion holds, but K might depend on f not only through its norm (in fact, K is independent of f whenever f varies in a uniformly integrable subset of $L^q(Q)$).

This theorem will be proven in two steps. First, for $q = q_0$ in Theorem 3.1. Then, for $q > q_0$ in Theorem 5.1. While the general strategy will be presented in detail in the second part of the introduction, we already mention that the first step is the core of the paper. When $q = q_0$, then $\alpha = \alpha_0 = \frac{\gamma-2}{\gamma-1}$. We may then regard Theorem 1.2 as a parabolic version of the α_0 -Hölder regularity obtained in [9] for stationary problems (actually for general divergence-form equations, for subsolutions, and up to the boundary...). The intermediate passage between the two steps $q = q_0$ and $q > q_0$ will be the Liouville Theorem 4.1.

The techniques used here have their roots in [5] (see also [4, 14]), in [8] and in [3]. However, new and crucial ingredients are introduced, such as the use of “nonlinear” Hölder seminorms, and the study of their decay (or improvement) in smaller and smaller cylinders. All the estimates obtained here are *local* in nature, but they could be extended up to the parabolic boundary of Q under suitable boundary conditions; we will not pursue their derivation here. Moreover, more general nonlinear functions of Du are admissible, see for instance Remark 3.4. On the other hand, the presence of (x, t) -dependent diffusions cause additional difficulties, and they will not be treated here.

Let us mention that we do not know whether maximal regularity holds below the exponent q_0 , as no counterexamples like in the stationary case are at this stage available. Nevertheless, we strongly believe that it is not possible to have maximal regularity below q_0 , and L^∞ estimates might even fail (f need not be bounded below here). Besides the fact that q_0 is the parabolic analogue of the sharp stationary exponent, q_0 is “critical” (with respect to scaling properties of (1.1)), and Hölder estimates “see” properties of f beyond its norm... Finally, we do not know at the moment whether it is possible or not to obtain *universal* estimates, as in parabolic problems with nonlinear zero-th order term [20], or for stationary HJ equations [8]; our Hölder and maximal regularity bounds depend indeed the L^∞ -norm of u .

Before entering into the details of the proofs, let us say a few words on the structure of the paper. Section 2 will be devoted to the proof of crucial oscillation estimates for HJ equations. In Section 3, these will be used to obtain the first step of α_0 -Hölder regularity, while in Section 4 we will show how to derive a Liouville type result. In Section 5 the full range of Hölder regularity will be obtained (that is, for any $\alpha > \alpha_0$), and in Section 6 we will finally reach the maximal regularity result. The Appendix A contains several useful results on linear equations which will be used thoroughly.

Strategy of the proofs. The first main idea to achieve $W_q^{2,1}$ estimates can be described as follows; standard parabolic embeddings read

$$W_q^{2,1} \hookrightarrow W_p^{1,0} \hookrightarrow C^{\alpha, \alpha/2},$$

where $\frac{1}{p} = \frac{1}{q} - \frac{1}{N+2}$ and $\alpha = 2 - \frac{N+2}{q}$ (for $q > \frac{N+2}{2}$). In fact, since we consider solutions in $W_q^{2,1}$, we look for a priori estimates (at all scales, from Hölder to $W_q^{2,1}$). If $q > q_0$, then a priori bounds can be obtained by following the arrows in the “opposite way”; if one is able indeed to control $|Du|^\gamma$ in L^q , then standard parabolic regularity yields $W_q^{2,1}$ estimates. Furthermore, if one has α -Hölder bounds, then an additional interpolation argument yields back $W_q^{2,1}$ estimates. This step is carried out by Proposition A.4. We stress that this is possible only if $q > q_0$. In other words, the ground α -Hölder control allows to treat the HJ equation in a perturbative manner, and the effect of the nonlinear Hamiltonian term $|Du|^\gamma$ can be absorbed by the usual gain of regularity coming from the linear part of the equation.

The second main ingredient is the improvement of the α_0 -Hölder bounds to α -Hölder bounds, which is done by Theorem 5.1. Here, we follow an idea developed in [8] for stationary problems: if there are sequences of solutions whose α -Hölder seminorm explodes, then a suitable rescaling argument allows to construct a nonconstant “entire” solution (on $\mathbb{R}^N \times (0, \infty)$), which is in contradiction with a Liouville Theorem, forcing in fact such a solution to be constant. There are a few points that are worth noticing: first, the rescaling here preserves the linear structure $((x, t) \mapsto (rx, r^2t))$, but the Hamiltonian might not vanish in the limit problem. For this reason we need a Liouville theorem for homogeneous HJ equations. Such a Liouville theorem cannot hold if the solution has linear growth in the x variable (see Remark 4.2), that is why we need to start with α_0 -Hölder bounds, that give in the limit solutions with sublinear growth.

The Liouville property (Theorem 4.1) and the α_0 -Hölder bounds (Theorem 3.1) are the main novelties in this paper, and they are deduced as follows. It is classical that Hölder regularity can be obtained whenever the oscillation of u in a cylinder is smaller (by a fixed factor) than the oscillation of u itself in a larger cylinder (at all scales). This principle, which has been used in the context of HJ equations in [3], typically leads to Hölder regularity with rather implicit exponent, which might be actually very close to zero. Here, we need the precise exponent α_0 (which approaches 1 as $\gamma \rightarrow \infty$). Therefore, we focus on the following principle:

*the α_0 -Hölder seminorm in a cylinder is strictly smaller than
the α_0 -Hölder seminorm in a larger cylinder,*

which can be seen as an “improvement of α_0 -Hölder” regularity.

Practically, we will rescale sequences of solutions having (by contradiction) growing α_0 -Hölder seminorm in a way that it is exactly 1 in a cylinder Q_0 of radius one, while the global α_0 -Hölder seminorm remains bounded by a constant. We then prove that the control on the α_0 -Hölder seminorm at the boundary of a *suitable* large cylinder Q_1 implies that the oscillation of u on Q_0 needs to be strictly smaller than 1, which is impossible. To perform this fundamental step, we will need to introduce a nonstandard “nonlinear” α_0 -Hölder seminorm, which will be denoted below by $\llbracket \cdot \rrbracket_{\alpha_0}$; indeed, if we rescale in a way that the full operator

$$-\partial_t u - \Delta u + |Du|^\gamma,$$

is preserved, then the standard α_0 -Hölder parabolic seminorm remains invariant. We will then rescale in a way that the transport part $-\partial_t u + |Du|^\gamma$ is kept unchanged, while the space-time Hölder quotients

$$\llbracket u \rrbracket_{\alpha_0} \approx \max \left\{ \frac{|u(x, t) - u(\bar{x}, t)|}{|x - \bar{x}|^\alpha}, \left(\frac{|u(x, t) - u(x, \bar{t})|}{|t - \bar{t}|^{\alpha/2}} \right)^{2/\gamma} \right\} \quad (1.2)$$

are renormalized. Note that in this way the rescaled equation has vanishing viscosity. The oscillation core estimate which is then used to conclude is contained in Proposition 2.1. Since the proof of the proposition is rather technical, we try to sketch here the main ideas involved (and briefly describe how to handle the vanishing viscosity, which deteriorates second-order regularization effects).

As u , and its scaling w , solve an HJ equation, we have at our disposal a representation formula which arises in stochastic optimal control theory: if for simplicity h is constant, then

$$w(x, 0) = \inf_{b_s} \mathbb{E} \int_0^\tau \ell |b_s|^\gamma + f(X_s, s) ds + \mathbb{E} w(X_\tau, \tau),$$

where the infimum is taken among stochastic trajectories $dX_s = b_s ds + \sqrt{2\sigma} dB_s$ originating from $(x, 0)$ and τ is the first exit time from the cylinder $Q = B_R \times (0, T)$. In PDE terms, the previous formula reads

$$w(x, 0) = \iint_Q \ell |b|^\gamma m + f m \, dy \, ds + \int_{B_R} w(T) m(T) \, dy - \iint_{\partial B_R \times (0, T)} w Dm \cdot \nu \, dy \, ds, \quad (1.3)$$

where $b = -h\gamma|Dw|^{\gamma-2}Dw$, ν is the outward normal vector to ∂B_R , and

$$\begin{cases} \partial_s m - \sigma \Delta m - \operatorname{div}(bm) = 0 & \text{on } Q, \\ m|_{\partial B_R \times (0, T)} = 0, \quad m(0) = \delta_x. \end{cases} \quad (1.4)$$

Formula (1.3) has been used extensively to deduce regularity properties of w . Besides its interpretation in optimal control, it can be obtained using a sort of duality between (1.1) and (1.4); such a duality is at the core of the so-called (nonlinear) adjoint method, which allows to derive regularity of w by regularity properties of m . This strategy has been proposed in [10]

for HJ equations. In particular, Hölder seminorms of u are related to variations with respect to x in (1.3), that necessarily involve variations of m with respect to x , by the presence of $\iint fm$. Unfortunately, the only available information on m is a bound in L^{q_0} (see Lemma A.2); bounds on the derivatives of m are possible in smaller spaces only, and that is why Hölder and maximal regularity are not achieved up to $q = q_0$ in [5]. For the reader who is experienced with the adjoint method for HJ equations, the main challenge here is to “differentiate” u in (1.3) with respect to x without “differentiating” m in $\iint fm$! This is a substantial step which is developed in the present paper.

To compare $w(x, 0)$ and $w(x + h, 0)$, we just “bend” m to a new $\tilde{m}$, which starts from δ_{x+h} and coincides with m at final time. Such new $\tilde{m}$ is used in the representation formula for $w(x + h, 0)$ (since the new $\tilde{m}$ is “suboptimal”, the equality becomes an inequality in (1.3)), which is then compared with (1.3) for $w(x, 0)$. Note that $w(x + h, 0) - w(x, 0)$ is renormalized to be 1, and there is no need to divide by h . The discrepancies in the two formulas mainly involve the difference between the two Lagrangian terms $\iint |b|^{\gamma'} m$, the so-called running costs $\iint fm$, and the boundary contributions $\iint_{\partial B_R \times (0, T)} w Dm \cdot \nu$. These depend on the dimensions R, T of the large cylinder; loosely speaking, Lagrangian terms can be made small by choosing T large, and $\iint fm$ vanishes since the L^{q_0} -norm of f can be made small at small scales. The boundary terms are definitely the most delicate ones; the key point is to realize that $\iint_{\partial B_R \times (0, T)} Dm \cdot \nu$, which represent the total density leaving the cylinder from its lateral boundary, is small if R is large, and it can compensate w , which might be large (if the large cylinder is chosen carefully enough). Similar arguments allow to control the oscillation of w in time.

A few comments are now in order. As we said, the scaled operator has vanishing viscosity

$$-\partial_t u - \sigma \Delta u + |Du|^\gamma, \quad \sigma = o(1),$$

which allows to exploit the first order regularization effects, but of course brings a deterioration of linear regularity. Nevertheless, by the particular way in which space-time is scaled we can still use the (small) linear contribution: the deterioration of the L^{q_0} -norm of m is compensated by rapid vanishing of the L^{q_0} -norm of f , so that the whole term can $\iint fm$ be controlled in the scaling process. To carry out the whole procedure, we stress again that we need to work with ad-hoc Hölder quotients, and that both the linear and nonlinear part of the equation play a major role in regularization; the argument is by no means perturbative. Finally, the reader will see that we need an additional parameter z in the “nonlinear” Hölder seminorm to fine tune the argument. This is mainly because we are able to shrink the *time* Hölder seminorm from z to $z/2$ only if z is large, while z plays basically no role when shrinking seminorms in *space* (provided that cylinders are fat enough). Since our goal is to obtain local estimates, suitable distance functions will appear also in the computations, but this will be just an annoying technical point.

Notations.

- $q_0 = \frac{N+2}{\gamma}$, $\alpha_0 = \frac{\gamma-2}{\gamma-1}$. Throughout the paper, $\gamma > 2$, $q_0 \leq q < N + 2$ and $\alpha = 2 - \frac{N+2}{q}$, unless otherwise specified. Note that q is always larger than $1 + \frac{N}{2}$.

- $|\cdot|$ will denote the usual Euclidean distance in $\mathbb{R}^N$; $B_r(x) = \{y \in \mathbb{R}^N : |x - y| < r\}$. $d(x, \Omega)$ be the Euclidean distance from $x \in \mathbb{R}^N$ to the set $\Omega \subset \mathbb{R}^N$. For a cylinder $Q = \Omega \times (a, b)$, we will use two different parabolic distances from the (backward parabolic) boundary $\partial^+ Q = \partial\Omega \times (a, b) \cup \Omega \times \{b\}$:

$$d((x, t), \partial^+ Q) = d(x, \partial\Omega) + |b - t|^{1/2}, \quad \mathcal{d}_\alpha((x, t), \partial^+ Q) = d^\alpha(x, \partial\Omega) + |b - t|^{\alpha/\gamma}$$

($\mathcal{d}_\alpha$ will be often used without the subscript $\alpha \in (0, 1)$ for brevity).

- Parabolic cylinders: $Q_{r,t} = B_r(0) \times (0, t)$, $Q_r = B_r(0) \times (0, r^2)$.
- For $Q = \Omega \times (a, b)$, $0 < \alpha < 1$, $c > 0$, we define the usual parabolic Hölder seminorm

$$[u]_{\alpha;Q} = \sup_{(x,t),(\bar{x},\bar{t}) \in Q} \frac{|u(x,t) - u(\bar{x},\bar{t})|}{(|x - \bar{x}| + |t - \bar{t}|^{1/2})^\alpha},$$

and its wighted version

$$[u]_{\alpha;Q}^c = \sup_{(x,t),(\bar{x},\bar{t}) \in Q} [\min\{d((x,t), \partial^+ Q), d((\bar{x},\bar{t}), \partial^+ Q)\}]^c \frac{|u(x,t) - u(\bar{x},\bar{t})|}{(|x - \bar{x}| + |t - \bar{t}|^{1/2})^\alpha}.$$

Finally, for any $z > 0$,

$$\begin{aligned} \|u\|_{\alpha;Q}^x &= \sup_{(x,t),(\bar{x},\bar{t}) \in Q} \min\{\mathcal{d}_\alpha((x,t), \partial^+ Q), \mathcal{d}_\alpha((\bar{x},\bar{t}), \partial^+ Q)\} \frac{|u(x,t) - u(\bar{x},\bar{t})|}{|x - \bar{x}|^\alpha}, \\ \|u\|_{\alpha/2;Q}^t &= \sup_{(x,t),(\bar{x},\bar{t}) \in Q} [\min\{\mathcal{d}_\alpha((x,t), \partial^+ Q), \mathcal{d}_\alpha((\bar{x},\bar{t}), \partial^+ Q)\}]^{\gamma/2} \frac{|u(x,t) - u(\bar{x},\bar{t})|}{|t - \bar{t}|^{\alpha/2}}, \\ \|u\|_{\alpha,z;Q} &= \max \left\{ \|u\|_{\alpha;Q}^x, (z^{-1} \|u\|_{\alpha/2;Q}^t)^{2/\gamma} \right\}. \end{aligned}$$

- $\|\cdot\|_{L^q(X)}$ be the usual norm on the Lebesgue space $L^q(X)$. Parabolic Sobolev spaces:

$$W_q^{2,1}(X) = \{\text{measurable } u : \partial_t^r D_x^\beta u \in L^q(X) \text{ for any } |\beta| + 2r \leq 2\}.$$

- Recall that a set $\mathcal{F} \subset L^q(X)$ is L^q -uniformly integrable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f\|_{L^q(E \cap X)} < \varepsilon \quad \text{whenever } f \in \mathcal{F} \text{ and } \int_{E \cap X} 1 < \delta.$$

- $C, C_1, \dots$ will denote positive constants, whose value may vary from line to line. Unless otherwise specified, they may depend on N, α, γ, q, Q and h (in particular on h_0, h_1 and the modulus of continuity of h), but *not* on u and f .

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2 Oscillation estimates

This section is devoted to the proof of the oscillation estimates, which will be used to show that certain Hölder seminorms decay in smaller and smaller cylinders. These will be also crucial to deduce the Liouville theorem for entire solutions to HJ equations.

Throughout this section, $w \in W_q^{2,1}(Q_{R+1,\tau})$ (for some $q \geq q_0$) solves on $Q_{R+1,\tau}$ the differential inequalities

$$-\partial_s w - \sigma \Delta w + h_0 |Dw|^\gamma \leq g, \quad (2.1)$$

$$-\partial_s w - \sigma \Delta w + h_1 |Dw|^\gamma \geq g \quad (2.2)$$

in the strong sense, for a diffusion factor $0 \leq \sigma \leq 1$, and $0 < \underline{h} \leq h_0, h_1 \leq \bar{h}$. The function g is given in $L^{q_0}(Q_{R+1,\tau})$.

Proposition 2.1. *Assume that $R^2 \geq \sigma\tau$, and that for some $0 < \alpha < 1, z \geq 1$,*

$$\sup_{(y,s),(y',s') \in \bar{Q}_{R,\tau}} \frac{|w(y,s) - w(y',s')|}{|y - y'|^\alpha} \leq 3, \quad \sup_{(y,s),(y,s') \in \bar{Q}_{R,\tau}} \frac{|w(y,s) - w(y,s')|}{|s - s'|^{\alpha/2}} \leq 3^{\frac{\gamma}{2}} z.$$

Then there exist f^0, c_1, C_2, C_3 depending on $\underline{h}, \bar{h}, \alpha, q_0$, and $\mathcal{K} \geq 0$, such that if

$$\sigma^{-\gamma' \frac{N+1}{N+2}} \|g\|_{L^{q_0}(Q_{R,\tau})} \leq f^0, \quad z \frac{R^\alpha + \tau^{\frac{\alpha}{2}}}{R} \leq c_1, \quad (2.3)$$

then for any $|y_0| \leq 1$,

$$|w(0,0) - w(0,\tau)| + \mathcal{K} \leq C_2 \left(\tau^{\frac{\alpha}{2}} + \tau^{\frac{\alpha_0}{2}} + \tau^{\frac{\alpha}{\gamma - \alpha(\gamma-1)}} + \tau \frac{R^\alpha + \tau^{\frac{\alpha}{2}}}{R} z \right), \quad (2.4)$$

$$\begin{aligned} w(y_0,0) - w(0,0) &\leq C_3 \left(\frac{\mathcal{K}^{1/\gamma}}{\tau^{1/\gamma}} + \frac{1}{\tau^{\gamma'-1}} + \right. \\ &\quad \left. + \sigma^{-\gamma' \frac{N+1}{N+2}} (\mathcal{K} + \tau^{\alpha_0/2}) \|g\|_{L^{q_0}(Q_{R+1,\tau})} + \frac{\tau^{1/\gamma} \mathcal{K}^{1/\gamma'}}{R} + \frac{\tau}{R^2} + (\ell_0 - \ell_1) \mathcal{K} \right), \end{aligned} \quad (2.5)$$

where $\ell_i = \frac{h_i(\gamma-1)}{(h_i\gamma)^{\gamma'}}$, $i = 1, 2$

The proof will be obtained in several steps. In what follows, will always assume that $R^2 \geq \sigma\tau$.

Lemma 2.2. *There exists C_0 depending on α, q_0 such that*

$$w(0,0) - w(0,\tau) \leq C_0 \left(\tau^{\frac{\alpha}{2}} + \tau \frac{R^\alpha + \tau^{\frac{\alpha}{2}}}{R^2} z + \sigma^{-\gamma' \frac{N+1}{N+2}} \|g\|_{L^{q_0}(Q_{R,\tau})} \tau^{\alpha_0/2} \right).$$

Proof. Let μ be the solution of

$$\begin{cases} \partial_s \mu - \sigma \Delta \mu = 0 & \text{on } B_R \times (0, \tau) \\ \mu(y, s) = 0 & \text{on } \partial B_R \times (0, \tau) \\ \mu(0) = \delta_0 \end{cases}$$

(the existence of such μ and its regularity are detailed in Appendix A). Testing (2.1) by μ and the equation for μ by w , and integrating by parts on $Q_{R,\tau}$ yields

$$w(0,0) + h_0 \iint_{B_R \times (0,\tau)} |Dw|^\gamma \mu \, dy \, ds \leq \\ + \iint_{B_R \times (0,\tau)} g \mu \, dy \, ds + \int_{B_R} w(y,\tau) \mu(y,\tau) \, dy - \sigma \iint_{\partial B_R \times (0,\tau)} w D\mu \cdot \nu \, dy \, ds.$$

Since $\int_{B_R} \mu(y,\tau) - \sigma \iint_{\partial B_R \times (0,\tau)} D\mu \cdot \nu = 1$,

$$w(0,0) - w(0,\tau) \leq \int_{B_R} [w(y,\tau) - w(0,\tau)] \mu(y,\tau) \, dy \\ + \iint_{B_R \times (0,\tau)} g \mu \, dy \, ds - \sigma \iint_{\partial B_R \times (0,\tau)} [w - w(0,\tau)] D\mu \cdot \nu \, dy \, ds.$$

First, by the growth assumptions on w and Lemma A.1

$$\int_{B_R} [w(y,\tau) - w(0,\tau)] \mu(y,\tau) \, dy \leq 3 \int_{B_R} |y|^\alpha \mu(y,\tau) \leq 3C (\sigma\tau)^{\alpha/2}.$$

Secondly, applying Lemma A.2

$$\iint_{B_R \times (0,\tau)} g \mu \, dy \, ds \leq \|g\|_{L^{q_0}(Q_{R,\tau})} \|\mu\|_{L^{q'_0}(Q_{R,\tau})} \leq C \sigma^{-\gamma' \frac{N+1}{N+2}} \|g\|_{L^{q_0}(Q_{R,\tau})} \sigma^{\gamma'/2} \tau^{\alpha_0/2}.$$

Finally, again by the growth assumptions on w and (A.5)

$$\sigma \iint_{\partial B_R \times (0,\tau)} [w(y,s) - w(0,\tau)] (-D\mu(y,s) \cdot \nu(y)) \, dy \, ds \leq \\ \sigma 3^{\gamma/2} z (R^\alpha + \tau^{\frac{\alpha}{2}}) \iint_{\partial B_R \times (0,\tau)} -D\mu(y,s) \cdot \nu(y) \, dy \, ds \leq C z (R^\alpha + \tau^{\frac{\alpha}{2}}) \frac{\tau}{R^2},$$

which, together with the previous inequalities, yields the assertion. $\square$

Let us now define

$$b(y,s) = h_1 \gamma |Dw(y,s)|^{\gamma-2} Dw(y,s),$$

and m be the solution of the dual problem

$$\begin{cases} \partial_s m - \sigma \Delta m - \operatorname{div}(bm) = 0 & \text{on } B_R \times (0,\tau) \\ m(y,s) = 0 & \text{on } \partial B_R \times (0,\tau) \\ m(0) = \delta_0. \end{cases}$$

Lemma 2.3. *There exists $c_0, c_1, C_1, f^0 > 0$ depending on $\bar{h}, \alpha, q_0$ such that if*

$$\sigma^{-\gamma' \frac{N+1}{N+2}} \|g\|_{L^{q_0}(Q_{R,\tau})} \leq f^0, \quad z \frac{R^\alpha + \tau^{\frac{\alpha}{2}}}{R} \leq c_1, \quad (2.6)$$

then

$$w(0,0) - w(0,\tau) \geq c_0 \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds + \Phi,$$

where

$$\Phi \geq -C_1 \left(\tau^{\frac{\alpha}{2}} + \tau^{\frac{\alpha_0}{2}} + \tau^{\frac{\alpha}{\gamma-\alpha(\gamma-1)}} + \tau \frac{R^\alpha + \tau^{\frac{\alpha}{2}}}{R} z \right).$$

Proof. Testing (2.2) by m and the equation for m by w , and integrating by parts on $Q_{R,\tau}$ yields

$$\begin{aligned} w(0,0) &\geq h_1(\gamma-1) \iint_{B_R \times (0,\tau)} |Dw|^\gamma m \, dy \, ds + \\ &\quad + \iint_{B_R \times (0,\tau)} g m \, dy \, ds + \int_{B_R} w(y,\tau) m(y,\tau) \, dy - \sigma \iint_{\partial B_R \times (0,\tau)} w Dm \cdot \nu \, dy \, ds. \end{aligned} \quad (2.7)$$

whence

$$w(0,0) - w(0,\tau) \geq c_0 \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds + \Phi,$$

where $2c_0 = \frac{\bar{h}(\gamma-1)}{(\bar{h}\gamma)^{\gamma'}} \leq \frac{h_1(\gamma-1)}{(h_1\gamma)^{\gamma'}} =: \ell_1$, and

$$\begin{aligned} \Phi &= c_0 \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds + \int_{B_R} [w(y,\tau) - w(0,\tau)] m(y,\tau) \, dy \\ &\quad + \iint_{B_R \times (0,\tau)} g m \, dy \, ds - \sigma \iint_{\partial B_R \times (0,\tau)} [w - w(0,\tau)] Dm \cdot \nu \, dy \, ds. \end{aligned}$$

First, by the growth assumptions on w , inequality (A.6), and Young's inequality (notice that $\frac{\alpha}{\gamma} \left(\frac{\gamma'}{\alpha} \right)' = \frac{\alpha}{\gamma - \alpha(\gamma-1)}$),

$$\begin{aligned} \int_{B_R} [w(y,\tau) - w(0,\tau)] m(y,\tau) \, dy &\geq -3 \int_{B_R} |y|^\alpha m(y,\tau) \, dy \\ &\geq -C \tau^{\alpha/\gamma} \left(\iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds \right)^{\alpha/\gamma'} - C (\sigma \tau)^{\alpha/2} \\ &\geq -\frac{c_0}{3} \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds - C (\tau^{\alpha/2} + \tau^{\frac{\alpha}{\gamma - \alpha(\gamma-1)}}). \end{aligned}$$

Secondly, applying Lemma A.2 and assuming f^0 small enough (depending on c_0 and C below),

$$\begin{aligned} \iint_{B_R \times (0,\tau)} g m \, dy \, ds &\geq -\|g\|_{L^{q_0}(Q_{R,\tau})} \|m\|_{L^{q'_0}(Q_{R,\tau})} \geq -f^0 \sigma^{\gamma' \frac{N+1}{N+2}} \|m\|_{L^{q'_0}(Q_{R,\tau})} \\ &\geq -C f^0 \left(\iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dx \, dt + \sigma^{\gamma'/2} \tau^{\alpha_0/2} \right) \\ &\geq -\frac{c_0}{3} \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds - C \sigma^{\gamma'/2} \tau^{\alpha_0/2}. \end{aligned}$$

Finally, by the growth assumptions on w , (A.5) and Young's inequality,

$$\begin{aligned} \sigma \iint_{\partial B_R \times (0,\tau)} [w(y,s) - w(0,\tau)] (-Dm(y,s) \cdot \nu(y)) \, dy \, ds &\geq \\ &\geq -\sigma 3^{\gamma/2} z(R^\alpha + \tau^{\frac{\alpha}{2}}) \iint_{\partial B_R \times (0,\tau)} -Dm(y,s) \cdot \nu(y) \, dy \, ds \\ &\geq -C z(R^\alpha + \tau^{\frac{\alpha}{2}}) \left(\frac{\tau^{1/\gamma}}{R} \left(\iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dx \, dt \right)^{1/\gamma'} + \frac{\tau}{R^2} \right) \\ &\geq -\bar{C} z(R^\alpha + \tau^{\frac{\alpha}{2}}) \left(\frac{1}{R} \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dx \, dt + \frac{\tau}{R} + \frac{\tau}{R^2} \right) \\ &\stackrel{(2.6)}{\geq} -\frac{c_0}{3} \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds - \bar{C} z(R^\alpha + \tau^{\frac{\alpha}{2}}) \left(\frac{\tau}{R} + \frac{\tau}{R^2} \right), \end{aligned}$$

where c_1 in (2.6) is chosen to be $c_1 = c_0/(3\bar{C})$. The three estimates above yield the statement. $\square$

Corollary 2.4. *Under the assumption (2.6), for some c_0, C_2 depending on $\bar{h}, \alpha, q_0$,*

$$|w(0,0) - w(0,\tau)| + \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds \leq C_2 \left(\tau^{\frac{\alpha}{2}} + \tau^{\frac{\alpha_0}{2}} + \tau^{\frac{\alpha}{\gamma-\alpha(\gamma-1)}} + \tau^{\frac{R^\alpha + \tau^{\frac{\alpha}{2}}}{R}} z \right). \quad (2.8)$$

Proof. The previous lemma states that

$$c_0 \iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dy \, ds \leq w(0,0) - w(0,\tau) - \Phi,$$

hence by Lemma 2.2 one concludes (using that $R \geq 1$ and $\sigma^{-\gamma' \frac{N+1}{N+2}} \|g\|_{L^{q_0}(Q_{R,\tau})} \leq f^0$). $\square$

The previous corollary allows to control the oscillation of w in time. We now take care of the oscillation of w in space. Let y_0 be any unit vector, $R, \tau \geq 1$, $R^2 \geq \sigma\tau$ as before, and

$$\xi_s = \frac{\tau - s}{\tau} y_0.$$

Lemma 2.5. *The following inequality holds.*

$$\begin{aligned} w(y_0, 0) &\leq \ell_0 \iint_{B_R \times (0,\tau)} |b - \xi'_s|^{\gamma'} m \, dy \, ds + \iint_{B_R \times (0,\tau)} g(y + \xi_s, s) m(y, s) \, dy \, ds + \\ &\quad + \int_{B_R} w(y, \tau) m(y, \tau) \, dy - \sigma \iint_{\partial B_R \times (0,\tau)} w(y + \xi_s, s) Dm(y, s) \cdot \nu \, dy \, ds. \end{aligned} \quad (2.9)$$

Proof. Since

$$h_0 |p|^\gamma = \sup_{q \in \mathbb{R}^N} \left\{ p \cdot q - \ell_0 |q|^{\gamma'} \right\}, \quad \text{where } \ell_0 = \frac{h_0(\gamma-1)}{(h_0\gamma)^{\gamma'}},$$

for any vector field $\tilde{b}$ we have

$$-\partial_s w - \sigma \Delta w + Dw \cdot \tilde{b} - \ell_0 |\tilde{b}|^{\gamma'} \leq g \quad \text{on } Q_{R+1,\tau}. \quad (2.10)$$

Let us now choose

$$\tilde{b}(\tilde{y}, s) = b(\tilde{y} - \xi_s, s) - \xi'_s = h_1 \gamma |Dw(\tilde{y} - \xi_s, s)|^{\gamma-2} Dw(\tilde{y} - \xi_s, s) + \frac{y_0}{\tau} \quad \text{on } B_R(\xi_s) \times (0, \tau).$$

Test now the inequality (2.10) by $m(\tilde{y} - \xi_s, s)$, and integrate by parts on $B_R(\xi_s) \times (0, \tau) \ni (\tilde{y}, s)$ as follows:

$$\begin{aligned} - \iint_{B_R(\xi_s) \times (0,\tau)} \partial_s w(\tilde{y}, s) m(\tilde{y} - \xi_s, s) \, d\tilde{y} \, ds &= - \iint_{B_R \times (0,\tau)} \partial_s w(y + \xi_s, s) m(y, s) \, dy \, ds \\ &= - \iint_{B_R \times (0,\tau)} \partial_s (w(y + \xi_s, s)) m(y, s) \, dy \, ds + \iint_{B_R \times (0,\tau)} Dw(y + \xi_s, s) \cdot \xi'_s m(y, s) \, dy \, ds \\ &= \iint_{B_R \times (0,\tau)} w(y + \xi_s, s) \partial_s m(y, s) \, dy \, ds + \iint_{B_R \times (0,\tau)} Dw(y + \xi_s, s) \cdot \xi'_s m(y, s) \, dy \, ds \\ &\quad - \int_{B_R} w(y + \xi_\tau, \tau) m(y, \tau) \, dy + w(\xi_0, 0). \end{aligned}$$

Secondly,

$$\begin{aligned}
& - \iint_{B_R(\xi_s) \times (0, \tau)} \sigma \Delta w(\tilde{y}, s) m(\tilde{y} - \xi_s, s) d\tilde{y} ds = \\
& \quad = - \iint_{B_R \times (0, \tau)} \sigma \Delta w(y + \xi_s, s) m(y, s) dy ds \\
& \quad = - \iint_{B_R \times (0, \tau)} \sigma \Delta m(y, s) w(y + \xi_s, s) dy ds + \sigma \iint_{\partial B_R \times (0, \tau)} w(y + \xi_s, s) Dm \cdot \nu dy ds.
\end{aligned}$$

Regarding the third term,

$$\begin{aligned}
& \iint_{B_R(\xi_s) \times (0, \tau)} Dw(\tilde{y}, s) \cdot \tilde{b}(\tilde{y}, s) m(\tilde{y} - \xi_s, s) d\tilde{y} ds = \\
& \quad = \iint_{B_R \times (0, \tau)} Dw(y + \xi_s, s) \cdot [b(y, s) - \xi'_s] m(y, s) dy ds \\
& \quad = - \iint_{B_R \times (0, \tau)} w(y + \xi_s, s) \operatorname{div} \left(b(y, s) m(y, s) \right) dy ds \\
& \quad \quad - \iint_{B_R \times (0, \tau)} Dw(y + \xi_s, s) \cdot \xi'_s m(y, s) dy ds
\end{aligned}$$

Adding up the three previous inequalities, one finds the equation for m which is being multiplied by $w(y + \xi_s, s)$, and the two terms $\iint Dw(y + \xi_s, s) \cdot \xi'_s m$ cancel out, so

$$\begin{aligned}
& - \int_{B_R} w(y + \xi_\tau, \tau) m(y, \tau) dy + w(\xi_0, 0) + \sigma \iint_{\partial B_R \times (0, \tau)} w(y + \xi_s, s) Dm \cdot \nu dy ds + \\
& \quad - \ell_0 \iint_{B_R(\xi_s) \times (0, \tau)} |\tilde{b}|^{\gamma'} m(\tilde{y} - \xi_s, s) d\tilde{y} ds \leq \iint_{B_R(\xi_s) \times (0, \tau)} g m(\tilde{y} - \xi_s, s) d\tilde{y} ds.
\end{aligned}$$

After a further change of variables $y = \tilde{y} - \xi_s$, and the fact that $\xi_\tau = 0$ and $\xi_0 = y_0$ we obtain the assertion. $\square$

Lemma 2.6. Let $\mathcal{K} = \iint_{B_R \times (0, \tau)} |b|^{\gamma'} m dy ds$. Then, for some C_3 depending on $\underline{h}, \alpha, q_0$,

$$\begin{aligned}
w(y_0, 0) - w(0, 0) \leq C_3 \left(\frac{\mathcal{K}^{1/\gamma}}{\tau^{1/\gamma}} + \frac{1}{\tau^{\gamma'-1}} + \sigma^{-\gamma'} \frac{N+1}{N+2} (\mathcal{K} + \tau^{\alpha_0/2}) \|g\|_{L^{q_0}(Q_{R+1, \tau})} + \right. \\
\left. + \frac{\tau^{1/\gamma} \mathcal{K}^{1/\gamma'}}{R} + \frac{\tau}{R^2} + (\ell_0 - \ell_1) \mathcal{K} \right), \quad (2.11)
\end{aligned}$$

where $\ell_i = \frac{h_i(\gamma-1)}{(h_i \gamma)^{\gamma'}}$, $i = 1, 2$.

Proof. Recall (2.7), which reads

$$\begin{aligned}
w(0, 0) \geq \ell_1 \mathcal{K} + \iint_{B_R \times (0, \tau)} g m dy ds + \\
+ \int_{B_R} w(y, \tau) m(y, \tau) dy - \sigma \iint_{\partial B_R \times (0, \tau)} w Dm \cdot \nu dy ds.
\end{aligned}$$

Subtracting (2.9) and the previous equality yields

$$\begin{aligned} w(y_0, 0) - w(0, 0) &\leq \ell_0 \iint_{B_R \times (0, \tau)} [|b - \xi'_s|^{\gamma'} - |b|^{\gamma'}] m \, dy \, ds \\ &\quad + \iint_{B_R \times (0, \tau)} [g(y + \xi_s, s) - g(y, s)] m(y, s) \, dy \, ds \\ &\quad - \sigma \iint_{\partial B_R \times (0, \tau)} [w(y + \xi_s, s) - w(y, s)] Dm(y, s) \cdot \nu \, dy \, ds \\ &\quad + (\ell_0 - \ell_1) \mathcal{K}. \end{aligned}$$

To estimate the first term, we make use of the inequality (A.11), that $|\xi'| = \tau^{-1}$ and the Hölder inequality:

$$\begin{aligned} \ell_0 \iint_{B_R \times (0, \tau)} [|b - \xi'_s|^{\gamma'} - |b|^{\gamma'}] m \, dy \, ds &\stackrel{(A.11)}{\leq} \\ &\leq C |\xi'_s| \iint_{B_R \times (0, \tau)} |b|^{\gamma'-1} m \, dy \, ds + C |\xi'_s|^{\gamma'} \iint_{B_R \times (0, \tau)} m \, dy \, ds \\ &\leq \frac{C}{\tau} \left(\iint_{B_R \times (0, \tau)} |b|^{\gamma'} m \, dy \, ds \right)^{\frac{1}{\gamma'}} \left(\iint_{B_R \times (0, \tau)} m \, dy \, ds \right)^{\frac{1}{\gamma'}} + \frac{C}{\tau^{\gamma'}} \iint_{B_R \times (0, \tau)} m \, dy \, ds \\ &\stackrel{(A.2)}{\leq} \frac{C \mathcal{K}^{1/\gamma}}{\tau^{1/\gamma}} + \frac{C}{\tau^{\gamma'-1}}. \end{aligned}$$

As for the second term, Lemma (A.2) implies

$$\begin{aligned} \iint_{B_R \times (0, \tau)} [g(y + \xi_s, s) - g(y, s)] m(y, s) \, dy \, ds &\leq 2 \|g\|_{L^{q_0}(Q_{R+1, \tau})} \|m\|_{L^{q'_0}(Q_{R, \tau})} \\ &\leq C \sigma^{-\gamma' \frac{N+1}{N+2}} \|g\|_{L^{q_0}(Q_{R+1, \tau})} \left(\mathcal{K} + \sigma^{\gamma'/2} \tau^{\alpha_0/2} \right) \end{aligned}$$

Finally, by the growth assumptions on w ,

$$\begin{aligned} \sigma \iint_{\partial B_R \times (0, \tau)} [w(y + \xi_s, s) - w(y, s)] (-Dm \cdot \nu) \, dy \, ds &\leq \\ &\leq 3\sigma \iint_{\partial B_R \times (0, \tau)} (-Dm \cdot \nu) \, dy \, ds \stackrel{(A.5)}{\leq} \frac{3\tau^{1/\gamma}}{R} \mathcal{K}^{1/\gamma'} + \frac{C\tau}{R^2}, \end{aligned}$$

and by collecting the three estimate we conclude. $\square$

To conclude the proof of Proposition 2.1, it just suffices to collect the estimates (2.8) and (2.11).

3 α_0 -Hölder regularity

In this section we prove the α_0 -Hölder estimates, which are at the base of further regularity results.

Theorem 3.1. *Let $\mathcal{F} \subset L^{q_0}(Q)$ be a uniformly integrable set in $L^{q_0}(Q)$ and assume that $\|u\|_{L^\infty(Q)} \leq K$. Then there exists C, z depending on $\mathcal{F}, K, Q, h, q_0$ such that*

$$\llbracket u \rrbracket_{\alpha_0, z; Q} \leq C.$$

1. Set-up of the contradiction argument. Fix $z > 0$, which will be specified below. Assume by contradiction that for some sequence $\{f_n\} \subset \mathcal{F}$ and $\{u_n\}$ solving (1.1),

$$\|u\|_{\alpha_0, z; Q} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Setting $L_n := \|u\|_{\alpha_0, z; Q}/2$, we will distinguish two cases:

$$\begin{aligned} \text{(a)} \quad & 2L_n = \|u\|_{\alpha_0; Q}^x \geq (z^{-1} \|u\|_{\alpha_0/2; Q}^t)^{\frac{2}{\gamma}}, \\ \text{(b)} \quad & 2L_n = (z^{-1} \|u\|_{\alpha_0/2; Q}^t)^{\frac{2}{\gamma}} \geq \|u\|_{\alpha_0; Q}^x. \end{aligned}$$

In case (a), there are sequences $\{(x_n, \bar{t}_n)\}, \{(\bar{x}_n, \bar{t}_n)\} \subset Q$ such that for every n , $x_n \neq \bar{x}_n$ and

$$L_n \leq \min\{\mathcal{d}((x_n, \bar{t}_n), \partial Q), \mathcal{d}((\bar{x}_n, \bar{t}_n), \partial Q)\} \frac{|u_n(x_n, \bar{t}_n) - u(\bar{x}_n, \bar{t}_n)|}{|x_n - \bar{x}_n|^{\alpha_0}} \leq 2L_n.$$

If we assume w.l.o.g. that $\mathcal{d}((x_n, \bar{t}_n), \partial Q) \geq \mathcal{d}((\bar{x}_n, \bar{t}_n), \partial Q)$ and $u(\bar{x}_n, \bar{t}_n) = 0$, setting

$$d_n = d(\bar{x}_n, \partial \Omega), \quad M_n = |u_n(x_n, \bar{t}_n)|, \quad r_n = |x_n - \bar{x}_n|,$$

then the previous inequalities read

$$L_n \leq \left(d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}}\right) \frac{M_n}{r_n^{\alpha_0}} \leq 2L_n. \quad (3.1)$$

In the other case (b), there are sequences $\{(\bar{x}_n, t_n)\}, \{(\bar{x}_n, \bar{t}_n)\} \subset Q$ such that for every n , $t_n > \bar{t}_n$ and

$$L_n \leq z^{-\frac{2}{\gamma}} \min\{\mathcal{d}((\bar{x}_n, t_n), \partial Q), \mathcal{d}((\bar{x}_n, \bar{t}_n), \partial Q)\} \frac{|u(\bar{x}_n, t_n) - u(\bar{x}_n, \bar{t}_n)|^{\frac{2}{\gamma}}}{|t_n - \bar{t}_n|^{\frac{\alpha_0}{\gamma}}} \leq 2L_n.$$

If we let

$$d_n = d(\bar{x}_n, \partial \Omega), \quad M_n = \frac{|u(\bar{x}_n, t_n)|}{z}, \quad r_n = (t_n - \bar{t}_n)^{\frac{1}{\gamma}} M_n^{\frac{\gamma-1}{\gamma}},$$

we have (as before $u(\bar{x}_n, \bar{t}_n) = 0$)

$$L_n \leq \left(d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}}\right) \frac{M_n^{\frac{2}{\gamma} + \alpha_0 \frac{\gamma-1}{\gamma}}}{r_n^{\alpha_0}} = \left(d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}}\right) \frac{M_n}{r_n^{\alpha_0}} \leq 2L_n. \quad (3.2)$$

In *both* cases, since $M_n \leq \|u_n\|_{\infty} \leq K$ and Q is bounded, we infer from (3.1) and (3.2) that, as $n \rightarrow \infty$,

$$\frac{d_n}{r_n} \rightarrow +\infty, \quad r_n \rightarrow 0, \quad \frac{M_n^{\gamma-1}}{r_n^{\gamma-2}} = \left(\frac{M_n}{r_n^{\alpha_0}}\right)^{\gamma-1} \rightarrow +\infty, \quad (3.3)$$

as well as

$$\begin{aligned} (T - \bar{t}_n) \frac{M_n^{\gamma-1}}{r_n^{\gamma}} &\geq (T - \bar{t}_n) \frac{M_n^{\frac{\gamma}{\alpha_0}}}{r_n^{\gamma}} K^{-\frac{2}{\alpha_0}} \rightarrow +\infty, \\ \frac{M_n^{\frac{\alpha_0}{\gamma}}}{r_n^{\alpha_0}} (d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}}) &\geq \frac{M_n}{r_n^{\alpha_0}} (d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}}) K^{\frac{\alpha_0}{\gamma}-1} \rightarrow +\infty. \end{aligned} \quad (3.4)$$

2. Scaling. We now define the blow-up sequence

$$w_n(y, s) = \frac{1}{M_n} u_n \left(\bar{x}_n + r_n y, \bar{t}_n + \frac{r_n^\gamma}{M_n^{\gamma-1}} s \right),$$

$$(y, s) \in \frac{\Omega - \bar{x}_n}{r_n} \times \left(-\bar{t}_n \frac{M_n^{\gamma-1}}{r_n^\gamma}, (T - \bar{t}_n) \frac{M_n^{\gamma-1}}{r_n^\gamma} \right) = Q_n.$$

In case (a), let y_0 be such that $x_n = \bar{x}_n + r_n y_0$. By the definition of r_n , $|y_0| = 1$. Recall also that $u(\bar{x}_n, \bar{t}_n) = 0$, so

$$|w_n(y_0, 0) - w_n(0, 0)| = \frac{1}{M_n} |u_n(x_n, \bar{t}_n) - u_n(\bar{x}_n, \bar{t}_n)| = 1. \quad (3.5)$$

On the other hand, in case (b),

$$|w_n(0, 1) - w_n(0, 0)| = \frac{1}{M_n} |u_n(\bar{x}_n, t_n)| = z. \quad (3.6)$$

The sequence w_n solves the following HJ equation

$$-\partial_s w_n - \sigma_n \Delta w_n + h \left(\bar{x}_n + r_n y, \bar{t}_n + \frac{r_n^\gamma}{M_n^{\gamma-1}} s \right) |Dw_n|^\gamma = g_n \quad \text{on } Q_n, \quad (3.7)$$

where

$$\sigma_n = \frac{r_n^{\gamma-2}}{M_n^{\gamma-1}}, \quad g_n(y, s) = \frac{r_n^\gamma}{M_n^\gamma} f_n \left(\bar{x}_n + r_n y, \bar{t}_n + \frac{r_n^\gamma}{M_n^{\gamma-1}} s \right)$$

Note that $\sigma_n \rightarrow 0$ (see (3.3)). Moreover, for any $R, \tau > 0$, a straightforward computation yields

$$\|g_n\|_{L^{q_0}(Q_{R,\tau})} = \sigma_n^{\gamma' \frac{N+1}{N+2}} \|f_n\|_{L^{q_0}(\tilde{Q}_{n,R,\tau})}, \quad \tilde{Q}_{n,R,\tau} = B_{Rr_n}(\bar{x}_n) \times (\bar{t}_n, \bar{t}_n + r_n^\gamma M_n^{1-\gamma} \tau)$$

Therefore, since f_n is uniformly integrable in $L^{q_0}(Q)$,

$$\|f_n\|_{L^{q_0}(\tilde{Q}_{n,R,\tau})} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (3.8)$$

because $r_n, r_n^\gamma M_n^{1-\gamma} \rightarrow 0$ in view of (3.3), (3.4). Since h is continuous on $\bar{Q}$, for any $\varepsilon > 0$ (smaller than $h_0/2$), for n large enough we have on Q_n

$$\begin{aligned} -\partial_s w_n - \sigma_n \Delta w_n + (h_n - \varepsilon) |Dw_n|^\gamma &\leq g_n, \\ -\partial_s w_n - \sigma_n \Delta w_n + (h_n + \varepsilon) |Dw_n|^\gamma &\geq g_n, \end{aligned} \quad (3.9)$$

where $h_0 \leq h_n = h(\bar{x}_n, \bar{t}_n) \leq h_1$.

In both cases, we have the following control on Hölder seminorms of w_n .

Lemma 3.2. *For any $R, \tau > 1$, we have for n large enough that $\bar{Q}_{R,\tau} = \bar{B}_R \times [0, \tau] \subset Q_n$ and*

$$\sup_{(y,s),(y',s') \in \bar{Q}_{R,\tau}} \frac{|w_n(y,s) - w_n(y',s')|}{|y - y'|^{\alpha_0}} \leq 3, \quad \sup_{(y,s),(y,s') \in \bar{Q}_{R,\tau}} \frac{|w_n(y,s) - w_n(y,s')|}{|s - s'|^{\alpha_0/2}} \leq 3^{\frac{\gamma}{2}} z.$$

Proof. First,

$$d(0, \partial\Omega_n) = d\left(0, \frac{\partial\Omega - \bar{x}_n}{r_n}\right) = \frac{1}{r_n} d(\bar{x}_n, \partial\Omega) \rightarrow +\infty$$

by (3.3). Moreover, (3.4) guarantees that $(T - \bar{t}_n) \frac{M_n^{\gamma-1}}{r_n^\gamma} \rightarrow +\infty$, therefore $\overline{B_R} \times [0, \tau] \subset Q_n$ whenever n is large.

Let now $(y, s), (y', s) \in \overline{Q}_{R, \tau}$ and assume that $\vartheta((y', s), \partial Q_n) \geq \vartheta((y, s), \partial Q_n)$. Then, writing

$$(x, t) = \left(\bar{x}_n + r_n y, \bar{t}_n + \frac{r_n^\gamma}{M_n^{\gamma-1}} s\right), \quad (x', t) = \left(\bar{x}_n + r_n y', \bar{t}_n + \frac{r_n^\gamma}{M_n^{\gamma-1}} s\right)$$

we get

$$\frac{|u_n(x, t) - u_n(x', t)|}{|x - x'|^{\alpha_0}} \leq \frac{[u]_{\alpha_0; Q}^x}{d^{\alpha_0}(x, \partial\Omega) + |T - t|^{\frac{\alpha_0}{\gamma}}} \leq \frac{2L_n}{d^{\alpha_0}(x, \partial\Omega) + |T - t|^{\frac{\alpha_0}{\gamma}}}. \quad (3.10)$$

Now,

$$\begin{aligned} d_n^{\alpha_0} + (T - \bar{t}_n)^{\frac{\alpha_0}{\gamma}} &\leq d^{\alpha_0}(x, \partial\Omega) + (T - t)^{\frac{\alpha_0}{\gamma}} + |x - \bar{x}_n|^{\alpha_0} + |t - \bar{t}_n|^{\frac{\alpha_0}{\gamma}} \\ &\leq d^{\alpha_0}(x, \partial\Omega) + (T - t)^{\frac{\alpha_0}{\gamma}} + r_n^{\alpha_0} R^{\alpha_0} + \frac{r_n^{\alpha_0}}{M_n^{\frac{\gamma}{\gamma}}} \tau^{\frac{\alpha_0}{\gamma}} \\ &\leq d^{\alpha_0}(x, \partial\Omega) + (T - t)^{\frac{\alpha_0}{\gamma}} + C \frac{r_n^{\alpha_0}}{M_n^{\frac{\gamma}{\gamma}}}, \end{aligned} \quad (3.11)$$

for some C depending on K, R, τ . Therefore,

$$\begin{aligned} \frac{|w_n(y, s) - w_n(y', s)|}{|y - y'|^{\alpha_0}} &= \frac{|u_n(x, t) - u_n(x', t)|}{|x - x'|^{\alpha_0}} \frac{r_n^{\alpha_0}}{M_n} \stackrel{(3.1), (3.10)}{\leq} \frac{2L_n}{d^{\alpha_0}(x, \partial\Omega) + |T - t|^{\frac{\alpha_0}{\gamma}}} \frac{d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}}}{L_n} \\ &\stackrel{(3.11)}{\leq} 2 \left(1 - C \frac{r_n^{\alpha_0}}{M_n^{\frac{\gamma}{\gamma}} (d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}})}\right)^{-1} \leq 3, \end{aligned}$$

the last inequality being true for large n .

Similarly, $(y, s), (y, s') \in \overline{Q}_{R, \tau}$ be such that $\vartheta((y, s'), \partial Q_n) \geq \vartheta((y, s), \partial Q_n)$. Then, for

$$(x, t) = \left(\bar{x}_n + r_n y, \bar{t}_n + \frac{r_n^\gamma}{M_n^{\gamma-1}} s\right), \quad (x, t') = \left(\bar{x}_n + r_n y, \bar{t}_n + \frac{r_n^\gamma}{M_n^{\gamma-1}} s'\right)$$

we have

$$\frac{|u_n(x, t) - u_n(x, t')|}{|t - t'|^{\frac{\alpha_0}{2}}} \leq \frac{[u_n]_{\frac{\alpha_0}{2}; Q}^t}{[d^{\alpha_0}(x, \partial\Omega) + |T - t|^{\frac{\alpha_0}{\gamma}}]^{\frac{\gamma}{2}}} \leq z 2^{\frac{\gamma}{2}} \frac{L_n^{\frac{\gamma}{2}}}{[d^{\alpha_0}(x, \partial\Omega) + |T - t|^{\frac{\alpha_0}{\gamma}}]^{\frac{\gamma}{2}}}. \quad (3.12)$$

Thus,

$$\begin{aligned}
\frac{|w_n(y, s) - w_n(y, s')|}{|s - s'|^{\frac{\alpha_0}{2}}} &= \frac{|u_n(x, t) - u_n(x, t')|}{|t - t'|^{\frac{\alpha_0}{2}}} \frac{r_n^{\frac{\alpha_0}{2}}}{M_n^{\frac{\gamma}{2}}} \\
&\stackrel{(3.1), (3.12)}{\leq} \frac{2^{\frac{\gamma}{2}} z L_n^{\frac{\gamma}{2}}}{[d^{\alpha_0}(x, \partial\Omega) + |T - t|^{\frac{\alpha_0}{\gamma}}]^{\frac{\gamma}{2}}} \frac{[d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}}]^{\frac{\gamma}{2}}}{L_n^{\frac{\gamma}{2}}} \\
&\stackrel{(3.11)}{\leq} 2^{\frac{\gamma}{2}} z \left(1 - C \frac{r_n^{\alpha_0}}{M_n^{\frac{\alpha_0}{\gamma}} (d_n^{\alpha_0} + |T - \bar{t}_n|^{\frac{\alpha_0}{\gamma}})} \right)^{-\frac{\gamma}{2}} \leq 3^{\frac{\gamma}{2}} z.
\end{aligned}$$

□

3. Conclusion. We now have to choose z, τ, R, ε , and pick n large enough so that a contradiction is reached. To avoid circular dependencies, the general idea will be to choose first z large, then τ , then R large enough, and finally ε small. The smallness of $\|f_n\|$ will also play a crucial role.

In view of Lemma 3.2, we can employ Proposition 2.1 with $\alpha = \alpha_0$ (note that in this case $\frac{\alpha_0}{\gamma - \alpha_0(\gamma - 1)} = \frac{\alpha_0}{2}$ and $\sigma_n^{-\gamma' \frac{N+1}{N+2}} \|g_n\|_{L^{q_0}(Q_{R,\tau})} = \|f_n\|_{L^{q_0}(\tilde{Q}_{n,R,\tau})}$), which yields

$$|w_n(0, 0) - w_n(0, \tau)| + \mathcal{K} \leq C_2 \left(\tau^{\frac{\alpha_0}{2}} + \tau \frac{R^{\alpha_0} + \tau^{\frac{\alpha_0}{2}}}{R} z \right) \quad (3.13)$$

and

$$\begin{aligned}
|w_n(y_0, 0) - w_n(0, 0)| &\leq C_3 \left(\frac{\mathcal{K}^{1/\gamma}}{\tau^{1/\gamma}} + \frac{1}{\tau^{\gamma'-1}} + (\mathcal{K} + \tau^{\alpha_0/2}) \|f_n\|_{L^{q_0}(\tilde{Q}_{n,R+2,\tau})} \right. \\
&\quad \left. + \frac{\tau^{1/\gamma} \mathcal{K}^{1/\gamma'}}{R} + \frac{\tau}{R^2} + \eta_n \mathcal{K} \right), \quad (3.14)
\end{aligned}$$

provided that

$$\overline{B_{R+2}} \times [0, \tau] \subset Q_n, \quad \|f_n\|_{L^{q_0}(\tilde{Q}_{n,R,\tau})} \leq f^0, \quad R^2 \geq \sigma_n \tau, \quad z \frac{R^{\alpha_0} + \tau^{\frac{\alpha_0}{2}}}{R} \leq c_1,$$

where, since $h_n - \varepsilon \geq h_0/2$,

$$\eta_n = \frac{(\gamma - 1)}{\gamma^{\gamma'}} \left(\frac{1}{(h_n - \varepsilon)^{\gamma'}} - \frac{1}{(h_n + \varepsilon)^{\gamma'}} \right) \leq C_4 \varepsilon.$$

Here f^0, c_1, C_2, C_3 are universal constants. The first and the second condition are always satisfied for large n (see Lemma 3.2 and (3.8)). Note that the absolute value of $|w_n(y_0, 0) - w_n(0, 0)|$ appears in (3.14); this is a consequence of (2.5), and (2.5) again if one exchanges the roles of 0 and y_0 (which can be done eventually enlarging the domain from $Q_{R,\tau}$ to $Q_{R+1,\tau}$). The third and the fourth condition will be verified below.

We first aim to rule out case (b). To do so, we fix

$$z = 4C_2, \quad \tau = 1$$

and $R > 1$ large enough so that

$$4C_2 \frac{R^{\alpha_0} + 1}{R} \leq \min\{1, c_1\},$$

hence (3.13) applies, yielding

$$|w_n(0,0) - w_n(0,1)| \leq C_2 \left(1 + 4C_2 \frac{R^{\alpha_0} + 1}{R}\right) \leq 2C_2 = \frac{z}{2}.$$

For large n , $Q_{R,1} \subset \tilde{Q}_n$, hence (3.6) is impossible.

We now treat case (a), with the same choice of z as before, but possibly with larger τ, R . First, pick $\tau \geq 1$ so that

$$C_2 \tau^{\frac{\alpha_0}{2}} \leq \left(\frac{1}{8C_3}\right)^\gamma \tau, \quad \frac{1}{\tau^{\gamma'-1}} \leq \frac{1}{8C_3},$$

and R large such that

$$C_2 \left(\tau \frac{R^{\alpha_0} + \tau^{\frac{\alpha_0}{2}}}{R} z \right) \leq \left(\frac{1}{8C_3}\right)^\gamma \tau, \quad \left(2 \left(\frac{1}{8C_3}\right)^\gamma \tau\right)^{1/\gamma'} \frac{\tau^{1/\gamma}}{R} \leq \frac{1}{8C_3}, \quad \frac{\tau}{R^2} \leq \frac{1}{8C_3}.$$

Then, n be large so that

$$\left(2 \left(\frac{1}{8C_3}\right)^\gamma \tau + \tau^{\alpha_0/2}\right) \|f_n\|_{L^{q_0}(\tilde{Q}_{n,R+2,\tau})} \leq \frac{1}{8C_3}, \quad \text{and } Q_{R+2,\tau} \subset Q_n.$$

With these choices, (3.13) reads

$$\mathcal{K} \leq C_2 \left(\tau^{\frac{\alpha_0}{2}} + \tau \frac{R^{\alpha_0} + \tau^{\frac{\alpha_0}{2}}}{R} z \right) \leq 2 \left(\frac{1}{8C_3}\right)^\gamma \tau,$$

whence

$$\frac{\mathcal{K}^{1/\gamma}}{\tau^{1/\gamma}} \leq \frac{2^{1/\gamma}}{8C_3}, \quad \frac{\tau^{1/\gamma} \mathcal{K}^{1/\gamma'}}{R} \leq \frac{1}{8C_3}, \quad (\mathcal{K} + \tau^{\alpha_0/2}) \|f_n\|_{L^{q_0}(\tilde{Q}_{n,R+2,\tau})} \leq \frac{1}{8C_3}.$$

We can eventually pick $\varepsilon > 0$ small (and consequently increase n , if needed) so that

$$\eta_n \mathcal{K} \leq 2C_4 \left(\frac{1}{8C_3}\right)^\gamma \tau \varepsilon \leq \frac{1}{8C_3}.$$

Finally, we plug the previous inequalities into (3.14) to get

$$|w_n(y_0,0) - w_n(0,0)| \leq \frac{2^{1/\gamma} + 5}{8} \leq \frac{7}{8},$$

contradicting (3.5), so case (a) is also impossible, and this proves Theorem 3.1.

Remark 3.3 (On the role of h). The constant C in Theorem 3.1 depends on h just through its lower and upper bounds h_0, h_1 , and its modulus of continuity on $\overline{Q}$.

Remark 3.4 (On the assumptions). The proof of Theorem 3.1 can be carried out analogously for solutions of

$$-\partial_t u - \Delta D^2 u + H(x, t, Du) = 0$$

under the following assumption on H : for some $0 < h_0 \leq h_1$, for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that if $|Q| < \delta$, then

$$(h_Q - \varepsilon)|\xi|^\gamma - f_Q(x, t) \leq H(x, t, \xi) \leq (h_Q + \varepsilon)|\xi|^\gamma + f_Q(x, t) \quad \text{on } Q \times \mathbb{R}^N$$

for some $h_0 \leq h_Q \leq h_1$ and $\|f_Q\|_{L^{q_0}(Q)} \leq \varepsilon$. Indeed, this is enough to guarantee (3.8), (3.9), and the rest of the proof remains unchanged. For example,

$$H(x, t, \xi) = h|\xi|^\gamma + b(x, t) \cdot \xi,$$

where b varies in a set of uniformly L^{N+2} -integrable function, satisfies the previous assumption. Note finally that we actually need (3.9) for *some* ε small enough, hence discontinuous h are allowed, provided that the local oscillation is smaller than some universal constant $\varepsilon_0 > 0$.

4 A Liouville theorem

Let $q > q_0$, $v \in W_{q,loc}^{2,1}(\mathbb{R}^N \times (0, \infty))$ be a strong solution of

$$-\partial_t v - \Delta v + h|Dv|^\gamma = 0 \quad \text{on } \mathbb{R}^N \times (0, \infty), \quad (4.1)$$

$h \neq 0$. The purpose of this section is to prove the following Liouville-type theorem for “far future” solutions v (which would be just “ancient” solutions if the equation was forward in time). Note that a standard bootstrap procedure shows that v is actually smooth, up to C^∞ .

Theorem 4.1. *Suppose that for some $0 < \alpha < 1$,*

$$\sup_{(x,t) \neq (x',t') \in \mathbb{R}^N \times (0,\infty)} \frac{|v(x,t) - v(x',t')|}{|x - x'|^\alpha + |t - t'|^{\alpha/2}} < \infty.$$

Then, v is constant on $\mathbb{R}^N \times (0, \infty)$.

Proof. The linear case $h = 0$ is proved in [23, Theorem 1.2(b)]. Otherwise, w.l.o.g. we may assume that $h > 0$ (if $h < 0$ it is sufficient to replace v by $-v$). We may also assume that

$$\sup_{(y,s),(y',s') \in \mathbb{R}^N \times (0,\infty)} \frac{|v(y,s) - v(y',s')|}{|y - y'|^\alpha} \leq 3, \quad \sup_{(y,s),(y,s') \in \mathbb{R}^N \times (0,\infty)} \frac{|v(y,s) - v(y,s')|}{|s - s'|^{\alpha/2}} \leq 3,$$

which can be always verified after dividing v by a suitable positive constant (and changing h accordingly).

Then, using Proposition 2.1 with $g \equiv 0$ and $h_0 = h_1 = h$,

$$\begin{aligned} v(x_0, 0) - v(0, 0) &\leq C_3 \left(\frac{\mathcal{K}^{1/\gamma}}{\tau^{1/\gamma}} + \frac{1}{\tau^{\gamma'-1}} + \frac{\tau^{1/\gamma} \mathcal{K}^{1/\gamma'}}{R} + \frac{\tau}{R^2} \right), \\ \mathcal{K} &\leq C_2 \left(\tau^{\frac{\alpha}{2}} + \tau^{\frac{\alpha_0}{2}} + \tau^{\frac{\alpha}{\gamma-\alpha(\gamma-1)}} + \tau \frac{R^\alpha + \tau^{\frac{\alpha}{2}}}{R} \right), \end{aligned}$$

for any $R \geq R_0 = R_0(\tau)$ and $|x_0| \leq 1$. If we let $R \rightarrow +\infty$, we obtain

$$v(x_0, 0) - v(0, 0) \leq C_3 \left(\left(\frac{\tau^{\frac{\alpha}{2}} + \tau^{\frac{\alpha_0}{2}} + \tau^{\frac{\alpha}{\gamma-\alpha(\gamma-1)}}}{\tau} \right)^{1/\gamma} + \frac{1}{\tau^{\gamma'-1}} \right).$$

Since $\alpha/2, \alpha_0/2, \alpha/(\gamma - \alpha(\gamma - 1)) < 1$, as $\tau \rightarrow \infty$ we conclude that $v(x_0, 0) - v(0, 0) \leq 0$ for any $|x_0| \leq 1$. By exchanging the roles of x_0 and 0 we get that $v(\cdot, 0)$ is constant. The same argument applies to $v(\cdot, t)$ for any $t > 0$, which allows to conclude that $v(\cdot, t)$ is constant for any $t > 0$. Since now $\partial_t v \equiv 0$ on $\mathbb{R}^N \times (0, \infty)$, the conclusion is reached. $\square$

Remark 4.2 (On the behavior of v). We note that the main requirement on v is its behavior as $(|x|, t)$ goes to infinity. We could indeed get the same result assuming only, for $0 < \alpha < 1$ and $\beta > 0$,

$$\sup_{(x,t),(x',t') \in \mathbb{R}^N \times (0,\infty)} \frac{|v(x,t) - v(x',t')|}{|x - x'|^\alpha + |t - t'|^\beta + 1} < \infty,$$

using the arguments developed in the previous section. Note that the statement would be false if $\alpha = 1$ was allowed, just consider $v(x, t) = ht + x$.

Remark 4.3 (On the role of the diffusion). Since the right-hand side in (4.1) is zero, there is no real need of the regularizing effect of a nondegenerate diffusion. In other words, we can replace Δv by $\text{tr } AD^2 v$, $A \geq 0$, and the proof of the Liouville theorem works, at least *formally*. To make it rigorous, one has then to be careful on the notion of solution for (4.1), and the dual Fokker-Planck equation which is used in Proposition 2.1.

5 α -Hölder regularity

Theorem 5.1. *Assume that $\|u\|_{L^\infty(Q)}, [u]_{\alpha_0; Q}, \|f\|_{L^q(Q)} \leq K$. Then, there exists C depending on K, Q, h, q, N such that*

$$[u]_{\alpha; Q}^{\alpha - \alpha_0} \leq C,$$

where $\alpha = 2 - \frac{N+2}{q}$.

1. Set-up of the contradiction argument. Assume by contradiction that for some sequence $\{f_n\}$ and $\{u_n\}$ solving (1.1),

$$[u]_{\alpha; Q}^{\alpha - \alpha_0} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Setting $L_n := [u]_{\alpha; Q}^{\alpha-\alpha_0}/2$ there are sequences $\{(x_n, t_n)\}, \{(\bar{x}_n, \bar{t}_n)\} \subset Q$ such that for every n , $(x_n, t_n) \neq (\bar{x}_n, \bar{t}_n)$ and

$$L_n \leq [\min\{d((x, t), \partial Q), d((\bar{x}, \bar{t}), \partial Q)\}]^{\alpha-\alpha_0} \frac{|u(x, t) - u(\bar{x}, \bar{t})|}{(|x - \bar{x}| + |t - \bar{t}|^{\frac{1}{2}})^{\alpha}} \leq 2L_n.$$

If we assume w.l.o.g. that $d((x_n, t_n), \partial Q) \geq d((\bar{x}_n, \bar{t}_n), \partial Q)$ and $u(\bar{x}_n, \bar{t}_n) = 0$, setting

$$d_n = d((\bar{x}_n, \bar{t}_n), \partial Q), \quad M_n = |u_n(x_n, t_n)|, \quad r_n = |x_n - \bar{x}_n| + |t_n - \bar{t}_n|^{\frac{1}{2}}.$$

the previous inequality reads

$$L_n \leq d_n^{\alpha-\alpha_0} \frac{M_n}{r_n^{\alpha}} \leq 2L_n. \quad (5.1)$$

Since $\frac{M_n}{r_n^{\alpha_0}} \leq K$ and Q is bounded, we infer from (5.1) that, as $n \rightarrow \infty$,

$$\frac{d_n}{r_n} \rightarrow +\infty, \quad r_n \rightarrow 0, \quad \frac{r_n^{\alpha}}{M_n} \rightarrow 0. \quad (5.2)$$

2. Scaling. We now define the blow-up sequence

$$w_n(y, s) = \frac{1}{M_n} u_n(\bar{x}_n + r_n y, \bar{t}_n + r_n^2 s), \quad (y, s) \in \frac{\Omega - \bar{x}_n}{r_n} \times \left(-\frac{\bar{t}_n}{r_n^2}, \frac{T - \bar{t}_n}{r_n^2}\right) = Q_n.$$

Let (y_0, s_0) be such that $(x_n, t_n) = (\bar{x}_n + r_n y_0, \bar{t}_n + r_n^2 s_0)$. By the definition of r_n , $|y_0| + |s_0|^{1/2} = 1$. Recall also that $u(\bar{x}_n, \bar{t}_n) = 0$, so

$$|w_n(y_0, s_0) - w_n(0, 0)| = \frac{1}{M_n} |u_n(x_n, t_n) - u_n(\bar{x}_n, \bar{t}_n)| = 1. \quad (5.3)$$

The sequence w_n solves the following HJ equation

$$-\partial_s w_n - \Delta w_n + \theta_n h(\bar{x}_n + r_n y, \bar{t}_n + r_n^2 s) |Dw_n|^{\gamma} = g_n \quad \text{on } Q_n, \quad (5.4)$$

where

$$\theta_n = \frac{M_n^{\gamma-1}}{r_n^{\gamma-2}}, \quad g_n(y, s) = \frac{r_n^2}{M_n} f_n(\bar{x}_n + r_n y, \bar{t}_n + r_n^2 s)$$

Note that by the assumptions,

$$\theta_n^{\frac{1}{\gamma-1}} = \frac{|u_n(x_n, t_n) - u_n(\bar{x}_n, \bar{t}_n)|}{(|x_n - \bar{x}_n| + |t_n - \bar{t}_n|^{\frac{1}{2}})^{\alpha_0}} \leq K. \quad (5.5)$$

Moreover, for any $R, \tau > 0$, a straightforward computation yields

$$\|g_n\|_{L^q(Q_{R,\tau})} = \frac{r_n^{\frac{2-\frac{N+2}{q}}{q}}}{M_n} \|f_n\|_{L^q(\tilde{Q}_{n,R,\tau})}, \quad \tilde{Q}_{n,R,\tau} = B_{Rr_n}(\bar{x}_n) \times (\bar{t}_n, \bar{t}_n + r_n^2 \tau).$$

Therefore, since $\|f_n\|_q \leq K$ and $\frac{r_n^{\frac{2-\frac{N+2}{q}}{q}}}{M_n} = \frac{r_n^{\alpha}}{M_n} \rightarrow 0$ by (5.2), then

$$\|g_n\|_{L^q(Q_{R,\tau})} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5.6)$$

Lemma 5.2. For any $R > 1$, we have for n large enough that $\overline{Q}_R = \overline{B}_R \times [0, R^2] \subset Q_n$ and

$$\sup_{(y,s),(y',s') \in \overline{Q}_R} \frac{|w_n(y,s) - w_n(y',s')|}{(|y - y'| + |s - s'|^{1/2})^\alpha} \leq 3.$$

Proof. First,

$$d((0,0), \partial^+ Q_n) = d\left(0, \frac{\partial\Omega - \bar{x}_n}{r_n}\right) + \frac{(T - \bar{t}_n)^{1/2}}{r_n} = \frac{d_n}{r_n} \rightarrow +\infty$$

by (5.2).

Let now $(y,s), (y',s') \in \overline{Q}_R$ and assume that $d((y',s'), \partial^+ Q_n) \geq d((y,s), \partial^+ Q_n)$. Then, writing

$$(x,t) = (\bar{x}_n + r_n y, \bar{t}_n + r_n^2 s), \quad (x',t') = (\bar{x}_n + r_n y', \bar{t}_n + r_n^2 s')$$

we get

$$\frac{|u_n(x,t) - u_n(x',t')|}{(|x - x'| + |t - t'|^{1/2})^\alpha} \leq \frac{[u]_{\alpha;Q}^x}{d^{\alpha-\alpha_0}((x,t), \partial^+ Q)} \leq \frac{2L_n}{d^{\alpha-\alpha_0}((x,t), \partial^+ Q)}. \quad (5.7)$$

Now,

$$\begin{aligned} d_n = d((\bar{x}_n, \bar{t}_n), \partial^+ Q) &\leq d(x, \partial\Omega) + (T - t)^{1/2} + |x - \bar{x}_n| + |t - \bar{t}_n|^{1/2} \\ &\leq d(x, \partial\Omega) + (T - t)^{1/2} + 2r_n R \\ &= d((x,t), \partial^+ Q) + 2r_n R. \end{aligned} \quad (5.8)$$

Therefore,

$$\begin{aligned} \frac{|w_n(y,s) - w_n(y',s')|}{(|y - y'| + |s - s'|^{1/2})^\alpha} &= \frac{|u_n(x,t) - u_n(x',t')|}{(|x - x'| + |t - t'|^{1/2})^\alpha} \frac{r_n^\alpha}{M_n} \stackrel{(5.1),(5.7)}{\leq} \frac{2L_n}{d^{\alpha-\alpha_0}((x,t), \partial^+ Q)} \frac{d_n^{\alpha-\alpha_0}}{L_n} \\ &\stackrel{(5.8)}{\leq} 2 \left(1 - 2R \frac{r_n}{d_n}\right)^{-(\alpha-\alpha_0)} \leq 3, \end{aligned}$$

the last inequality being true for large n . $\square$

Lemma 5.3. For any $R > 1$, there exists C such that

$$\|\partial_t w_n\|_{q;Q_R} + \|D^2 w_n\|_{q;Q_R} \leq C.$$

for n large enough.

Proof. First, n need to be large so that $Q_{2R} \subset Q_n$ (which is possible in view of the previous lemma). Moreover, by (5.6), again for large n one has $\|g_n\|_{L^q(Q_{2R})} \leq 1$. Since w_n solves the HJ equation (5.4),

$$|-\partial_s w_n - \Delta w_n| = |-\theta_n h(\bar{x}_n + r_n y, \bar{t}_n + r_n^2 s)| |Dw_n|^\gamma + g_n \stackrel{(5.5)}{\leq} h_1 K^{\gamma-1} |Dw_n|^\gamma + 1,$$

the conclusion follows by invoking Proposition A.4. $\square$

3. The limit problem. We now pass to the limit in the equation (5.4). Limits below are up to subsequences, and they are constructed via a standard diagonal procedure. First, by Lemmata 5.2 and 5.3, w_n has (local) weak limits in $W_q^{2,1}$ on $\mathbb{R}^N \times (0, \infty)$.

We may assume that $\bar{x}_n$ and θ_n converge to $\bar{x}_\infty$ and θ_∞ respectively. Note that standard parabolic embeddings guarantee that Dw_n converge (locally) strongly in L^p for any $p < \frac{(N+2)q}{N+2-q}$. Therefore, as $\gamma q < \frac{(N+2)q}{N+2-q}$,

$$\theta_n h(\bar{x}_n + r_n y, \bar{t}_n + r_n^2 s) |Dw_n|^\gamma \rightarrow \theta_\infty h(\bar{x}_\infty, \bar{t}_\infty) |Dw|^\gamma$$

locally strongly in L^q , and the same holds for g_n (see (5.6)). Finally, w_n converges locally uniformly to some w which cannot be constant on Q_2 in view of (5.3).

Hence, the limit $w \in W_{q,loc}^{2,1}(\mathbb{R}^N \times (0, \infty))$ is a nonconstant, strong solution of

$$-\partial_s w - \Delta w + \theta_\infty h(\bar{x}_\infty, \bar{t}_\infty) |Dw|^\gamma = 0 \quad \text{on } \mathbb{R}^N \times (0, \infty).$$

It also satisfies

$$\sup_{(y,s),(y',s')} \frac{|w(y,s) - w(y',s')|}{(|y - y'| + |s - s'|^{\frac{1}{2}})^\alpha} \leq 3,$$

hence the Liouville Theorem 4.1 applies, contradicting the fact that w cannot be constant.

6 Proof of Theorem 1.1 and final remarks

We now conclude with the proof the main theorem on maximal regularity, and list a few remarks.

Proof of Theorem 1.1. Fix any Q'', Q''' such that $Q' \subset\subset Q'' \subset\subset Q''' \subset\subset Q$. Note that since $q > q_0$ and Q is bounded, functions satisfying $\|f\|_{q;Q} \leq K$ belong to a uniformly integrable set of $L^{q_0}(Q)$. Therefore, Theorem 3.1 yields the existence of C such that

$$\frac{|u(x,t) - u(\bar{x},t)|}{|x - \bar{x}|^\alpha} + \left(\frac{|u(x,t) - u(x,\bar{t})|}{|t - \bar{t}|^{\alpha/2}} \right)^{2/\gamma} \leq C$$

for any $(x,t), (x,\bar{t}), (\bar{x},t) \in Q'''$. Hence Theorem 5.1 applies on Q''' , and for some $C' > 0$,

$$[u]_{\alpha;Q''} \leq C'.$$

Finally, one achieves the desired estimate by Proposition A.4. $\square$

Remark 6.1 ($q \geq N+2$ and Lipschitz estimates). If $q \geq N+2$, then the same conclusion of Theorem 1.1 holds using a standard bootstrap argument. Choose indeed any $(N+2)\frac{\gamma+1}{\gamma} < \tilde{q} < N+2$, and apply Theorem 1.1 to control u in $W_{\tilde{q}}^{2,1}$. Therefore, by Sobolev embeddings one has a bound of $|Du|^\gamma$ in L^p , where $1/p = 1/\tilde{q} - 1/(N+2)$, and $p > N+2$. Hence, linear parabolic regularity allows to control u in $W_{\min\{p,q\}}^{2,1}$ on a smaller set, which in turn yields *Lipschitz* estimates. A further application of parabolic regularity gives finally bounds in $W_q^{2,1}$.

Note that this extends Lipschitz estimates to the full range $q > N + 2$. Previously, these were known [4] up to $q > \frac{N+2}{2(\gamma'-1)}$ (when $\gamma > 3$).

Remark 6.2 (On the sign of f). Since f can be unbounded, its sign plays a major role. If one has in mind that f appears in the running cost of an optimal control problem, then optimal trajectories are pushed away by *positive* singular points of f , while they are strongly attracted by *negative* ones. Heuristically, the former affects the value function u less than the latter. If one assumes $f \geq 0$, then it is shown in [3] that

$$f \in L^q, \quad q > 1 + \frac{N}{\gamma}$$

is sufficient for u to be Hölder continuous, even in first-order problems. Note that if $\gamma > 2$, then $1 + \frac{N}{\gamma}$ is always smaller than our threshold q_0 , and it even improves as γ increases.

Here, we do not assume any sign condition on f . It is not clear whether the critical integrability q_0 is related just to the behavior of the negative part of f , but we believe that for some $f \notin L^{q_0}$, u is bounded from below. This is strictly related to exponent q'_0 appearing in (A.7), which is “natural” by invariance properties of the Fokker-Planck equation, but its sharpness is not clear, as no counterexamples below q'_0 are available yet.

Not that if no sign condition is imposed on f , then our results hold also for the so-called “repulsive” case $h < 0$. By the methods developed here, it is not clear if they can be extended to arbitrary (possibly vanishing) h .

A Miscellaneous estimates

We first start with some estimates on solutions to Fokker-Planck equations. Let $\sigma > 0$, b a measurable vector field such that $b \in L^{N+2}(Q_{R,\tau})$ and $\operatorname{div} b \in L^{\frac{N+2}{2}}(Q_{R,\tau})$. Below, m be a solution of

$$\begin{cases} \partial_t m - \sigma \Delta m - \operatorname{div}(bm) = 0 & \text{on } B_R \times (0, \tau) \\ m(x, t) = 0 & \text{on } \partial B_R \times (0, \tau) \\ m|_{t=0} = \delta_0 \end{cases} \quad (\text{A.1})$$

Such a solution can be constructed by approximation, that is, taking limits of classical solutions with smooth approximating drifts b_k and smooth initial data $m_k(0)$. Due to the singular nature of $m(0)$, $m \in C([0, \tau]; \mathcal{M}(\Omega))$ is a distributional solution. Furthermore, m is on $Q_{R,\tau}$ a strong solution, since classical parabolic Calderón-Zygmund estimates apply away from $t = 0$. In particular, $m \in W_{\tilde{q}}^{2,1}(B_R \times (\tau', \tau))$ for any $0 < \tau' < \tau$ and $1 < \tilde{q} < (N+2)/2$ (see for example [15, Theorem IV.9.1]). Moreover, $m \in L^{q_0}(Q_{R,\tau})$, since the estimate (A.7) below “sees” only the L^1 -norm of the initial datum. Finally, the trace of $m(t)$ on ∂B_R is well defined for a.e. t , and it belongs to $W_{\tilde{q}}^{2-1/\tilde{q}}(\partial B_R)$.

We list several properties of m . First, $-Dm \cdot \nu \geq 0$ a.e. on $\partial B_R \times (0, \tau)$. Indeed, m is a (limit of) non-negative solution vanishing on the lateral boundary $\partial B_R \times (0, \tau)$, hence $Dm \cdot \zeta \leq 0$ on

$\partial B_R \times (0, \tau)$ for any $\zeta \cdot \nu > 0$, and clearly $\nu \cdot \nu > 0$. Moreover, for a.e. $s \in (0, \tau]$,

$$\int_{B_R} m(x, s) \, dx \leq \int_{B_R} m(x, s) \, dx - \sigma \iint_{\partial B_R \times (0, s)} Dm \cdot \nu \, dx \, dt = 1. \quad (\text{A.2})$$

Note that m is the density of a killed Brownian particle starting from $x = 0$ at time $t = 0$, on the domain B_R). The term $-\sigma \iint Dm \cdot \nu$ represents the probability that the particle has reached ∂B_R before $t = \tau$. We will estimate below such probability, as a corollary of the following lemma.

Lemma A.1. *Assume that $|b|m \in L^1(Q_{R,\tau})$, and let $\alpha \in (0, 1)$. Then, for some C depending on α ,*

$$\int_{B_R} |x|^\alpha m(x, \tau) \, dx \leq C \left(\iint_{B_R \times (0, \tau)} |b|m \, dx \, dt \right)^\alpha + C (\sigma\tau)^{\alpha/2}.$$

Proof. Let $\chi \in C^\infty([0, +\infty); [0, +\infty))$ be such that $\chi(r) \equiv 1$ for all $r \geq 1$ and $\chi(0) = 0$. For $0 < y \leq R$, test the equation for m by $\chi_y(x) = \chi(|x|/y)$ to get

$$\begin{aligned} \int_{B_R} \chi_y(x) m(x, \tau) \, dx - \sigma \iint_{\partial B_R \times (0, \tau)} \chi_y Dm \cdot \nu \, dx \, dt &= \\ &= \sigma \iint_{B_R \times (0, \tau)} m \Delta \chi_y \, dx \, dt - \iint_{B_R \times (0, \tau)} mb \cdot D\chi_y \, dx \, dt, \end{aligned}$$

whence

$$\int_{|x| \geq y} m(x, \tau) \, dx - \sigma \iint_{\partial B_R \times (0, \tau)} Dm \cdot \nu \, dx \, dt \leq \frac{C\sigma\tau}{y^2} + \frac{C}{y} \iint_{B_R \times (0, \tau)} |b|m \, dx \, dt. \quad (\text{A.3})$$

In particular, by (A.2), setting $k_1 = C\sigma\tau$ and $k_2 = C \iint_{B_R \times (0, \tau)} |b|m$,

$$\int_{|x| \geq y} m(x, \tau) \, dx \leq \min \left\{ \frac{k_1}{y^2} + \frac{k_2}{y}, 1 \right\}. \quad (\text{A.4})$$

Now,

$$\begin{aligned} \int_{B_R} |x|^\alpha m(x, \tau) \, dx &= \alpha \int_{B_R} \int_0^{|x|} y^{\alpha-1} m(x, \tau) \, dy \, dx = \alpha \int_0^R y^{\alpha-1} \int_{|x| \geq y} m(x, \tau) \, dx \, dy \\ &\stackrel{(\text{A.4})}{\leq} \alpha \int_0^R y^{\alpha-1} \min \{ k_1 y^{-2}, 1 \} \, dy + \alpha \int_0^R y^{\alpha-1} \min \{ k_2 y^{-1}, 1 \} \, dy \\ &\leq k_1 \alpha \int_{k_1^{1/2}}^{+\infty} y^{\alpha-3} \, dy + \alpha \int_0^{k_1^{1/2}} y^{\alpha-1} \, dy + k_2 \alpha \int_{k_2}^{+\infty} y^{\alpha-2} \, dy + \alpha \int_0^{k_2} y^{\alpha-1} \, dy \\ &= \alpha \frac{k_1^{\alpha/2}}{2-\alpha} + k_1^{\alpha/2} + \alpha \frac{k_2^\alpha}{1-\alpha} + k_2^\alpha. \end{aligned}$$

Substituting k_1, k_2 gives the assertion. $\square$

As a consequence (A.2) and (A.3) (with $y = R$), we have

$$-\sigma \iint_{\partial B_R \times (0, \tau)} Dm \cdot \nu \, dx \, dt \leq C \frac{\tau^{1/\gamma}}{R} \left(\iint_{B_R \times (0, \tau)} |b|^{\gamma'} m \, dx \, dt \right)^{1/\gamma'} + C \frac{\sigma\tau}{R^2} \quad (\text{A.5})$$

and by the previous lemma,

$$\int_{B_R} |x|^\alpha m(x, \tau) \, dx \leq C \tau^{\alpha/\gamma} \left(\iint_{B_R \times (0, \tau)} |b|^{\gamma'} m \, dx \, dt \right)^{\alpha/\gamma'} + C (\sigma\tau)^{\alpha/2} \quad (\text{A.6})$$

Lemma A.2. Suppose that $R^2 \geq \tau\sigma$. There exists C depending on q_0 such that

$$\sigma^{\gamma' \frac{N+1}{N+2}} \|m\|_{L^{q'_0}(Q_{R,\tau})} \leq C \left(\iint_{B_R \times (0,\tau)} |b|^{\gamma'} m \, dx \, dt + \sigma^{\gamma'/2} \tau^{\alpha_0/2} \right). \quad (\text{A.7})$$

Proof. We start with the case $\sigma = R = 1$, following the same lines as [7, Proposition 2.4], that is

$$\begin{cases} \partial_t \tilde{m} - \Delta \tilde{m} - \operatorname{div}(\tilde{b} \tilde{m}) = 0 & \text{on } B_1 \times (0, \bar{s}) \\ m(z, s) = 0 & \text{on } \partial B_1 \times (0, \bar{s}) \\ m(0) = \delta_0, \end{cases} \quad (\text{A.8})$$

for any $\bar{s} \in (0, 1]$. We might assume that $\tilde{m}$ is smooth, the general case follows by approximation. Let ζ be the solution of the dual problem

$$\begin{cases} -\partial_t \zeta - \Delta \zeta = \tilde{m}^{q'_0-1} & \text{on } B_1 \times (0, \bar{s}) \\ \zeta(z, s) = 0 & \text{on } \partial B_1 \times (0, \bar{s}) \\ \zeta(z, \bar{s}) = 0 & \text{on } B_1. \end{cases}$$

We now employ L^{q_0} -parabolic regularity, and embedding theorems of $W_{q_0}^{2,1}(Q_{1,\bar{s}})$ into $W_q^{1,0}(Q_{1,\bar{s}})$ and $C^{\alpha_0/2}(0, \bar{s}; C^{\alpha_0}(B_1))$, where $q^{-1} = q_0^{-1} - (N+2)^{-1}$ and $\alpha_0 = 2 - \frac{N+2}{q_0} = 2 - \gamma'$ (see [15, Theorem IV.9.1] and [15, Lemma II.3.3], recall that $\frac{N+2}{2} < q_0 < N+2$), to get

$$|\zeta(0,0)| \leq C' \bar{s}^{\alpha_0/2} \|\zeta\|_{W_{q_0}^{2,1}(Q_{1,\bar{s}})} \leq C \bar{s}^{\alpha_0/2} \|\tilde{m}^{q'_0-1}\|_{L^{q_0}(Q_{1,\bar{s}})} = C \bar{s}^{\alpha_0/2} \|\tilde{m}\|_{L^{q'_0}(Q_{1,\bar{s}})}^{q'_0-1}, \quad (\text{A.9})$$

$$\|D\zeta\|_{L^q(Q_{1,\bar{s}})} \leq C' \|\zeta\|_{W_{q_0}^{2,1}(Q_{1,\bar{s}})} \leq C \|\tilde{m}^{q'_0-1}\|_{L^{q_0}(Q_{1,\bar{s}})} = C \|\tilde{m}\|_{L^{q'_0}(Q_{1,\bar{s}})}^{q'_0-1}. \quad (\text{A.10})$$

Note that C, C' do not depend on $\bar{s} \leq 1$. Then, by duality between $\tilde{m}$ and ζ ,

$$\iint_{Q_{1,\bar{s}}} \tilde{m}^{q'_0} = |\zeta(0,0)| - \iint_{Q_{1,\bar{s}}} \tilde{b} \tilde{m} \cdot D\zeta.$$

Hence, by Hölder inequality ($1 = \frac{1}{\gamma'} + \frac{1}{\gamma q'_0} + \frac{1}{q}$),

$$\begin{aligned} \|\tilde{m}\|_{L^{q'_0}(Q_{1,\bar{s}})}^{q'_0} &\leq C \|\tilde{b} \tilde{m}^{1/\gamma'}\|_{L^{\gamma'}(Q_{1,\bar{s}})} \|\tilde{m}^{1/\gamma}\|_{L^{\gamma q'_0}(Q_{1,\bar{s}})} \|D\zeta\|_{L^q(Q_{1,\bar{s}})} + |\zeta(0,0)| \\ &\stackrel{(\text{A.9})}{\leq} C \left(\left(\iint_{Q_{1,\bar{s}}} |\tilde{b}|^{\gamma'} \tilde{m} \right)^{1/\gamma'} \|\tilde{m}\|_{L^{q'_0}(Q_{1,\bar{s}})}^{1/\gamma} + \bar{s}^{\alpha_0/2} \right) \|\tilde{m}\|_{L^{q'_0}(Q_{1,\bar{s}})}^{q'_0-1}, \end{aligned}$$

and the desired inequality (A.7) follows by means of Young's inequality.

The general case, for m solving (A.1), can be obtained by scaling: setting $\tilde{m}(z, s) = R^N m(Rz, R^2 \sigma^{-1} s)$, $\tilde{b}(z, s) = R \sigma^{-1} b(Rz, R^2 \sigma^{-1} s)$, and $\bar{s} = \tau R^{-2} \sigma \leq 1$, then (A.8) holds. Therefore, by a straightforward computation

$$\begin{aligned} \sigma^{1/q'_0} R^{N(1-1/q'_0)-2/q'_0} \|m\|_{L^{q'_0}(Q_{R,\tau})} &= \|\tilde{m}\|_{L^{q'_0}(Q_{1,\bar{s}})} \leq C \left(\iint_{Q_{1,\bar{s}}} |\tilde{b}|^{\gamma'} \tilde{m} \, dz \, ds + \bar{s}^{\alpha_0/2} \right) = \\ &= C \left(\frac{R^{\gamma'-2}}{\sigma^{\gamma'-1}} \iint_{Q_{R,\tau}} |b|^{\gamma'} m \, dx \, dt + \frac{\sigma^{\alpha_0/2}}{R^{\alpha_0}} \tau^{\alpha_0/2} \right), \end{aligned}$$

which yields the conclusion, since $N(1 - 1/q'_0) - 2/q'_0 + 2 - \gamma' = 0$, $\alpha_0 = 2 - \gamma'$ and $\alpha_0/2 + \gamma' - 1 = \gamma'/2$. $\square$

We now state a property of $\zeta \mapsto |\zeta|^{\gamma'}$.

Lemma A.3. *For any $\gamma' \in (1, 2)$, there exists C depending on γ' such that*

$$|\zeta + \xi|^{\gamma'} - |\zeta|^{\gamma'} \leq C(|\zeta|^{\gamma'-1}|\xi| + |\xi|^{\gamma'}) \quad (\text{A.11})$$

for all $\zeta, \xi \in \mathbb{R}^N$.

Proof. Since $\mathbb{R}^n \ni p \mapsto |p|^{\gamma'}$ is convex,

$$|\zeta + \xi|^{\gamma'} - |\zeta|^{\gamma'} \leq \gamma' |\zeta + \xi|^{\gamma'-2} (\zeta + \xi) \cdot \xi.$$

If $|\zeta| \leq |\xi|$,

$$|\zeta + \xi|^{\gamma'} - |\zeta|^{\gamma'} \leq \gamma' |\zeta + \xi|^{\gamma'-1} |\xi| \leq \gamma' 2^{\gamma'-1} |\xi|^{\gamma'}.$$

Else, $|\zeta| > |\xi|$; since $(0, \infty) \ni t \mapsto t^{\gamma'-1}$ is concave, and $(0, \infty) \ni t \mapsto t^{\gamma'-2}$ is decreasing,

$$\begin{aligned} |\zeta + \xi|^{\gamma'} - |\zeta|^{\gamma'} &\leq \gamma' (|\zeta + \xi|^{\gamma'-1} - |\zeta|^{\gamma'-1}) |\xi| + \gamma' |\zeta|^{\gamma'-1} |\xi| \\ &\leq \gamma' (\gamma' - 1) |\zeta|^{\gamma'-2} (|\zeta + \xi| - |\zeta|) |\xi| + \gamma' |\zeta|^{\gamma'-1} |\xi| \\ &\leq \gamma' (\gamma' - 1) |\zeta|^{\gamma'-2} |\xi|^2 + \gamma' |\zeta|^{\gamma'-1} |\xi| \\ &\leq \gamma' (\gamma' - 1) |\xi|^{\gamma'} + \gamma' |\zeta|^{\gamma'-1} |\xi|. \end{aligned}$$

$\square$

We finally present a crucial result on the $W_q^{2,1}$ regularity of solutions to HJ equations enjoying suitable Hölder bounds. This is a parabolic analogue of [8, Proposition 3.3].

Proposition A.4. *Let $R > 0$, $g \in L^q(Q_{2R})$ and $v \in W_q^{2,1}(Q_{2R})$ be such that $v(0) = 0$,*

$$\|g\|_{L^q(Q_{2R})} + [v]_{\alpha; Q_{2R}} \leq c_1 \quad |-\partial_t v - \Delta v| \leq c_2 |Dv|^\gamma + g$$

a.e. in Q_{2R} , for some $c_1, c_2 > 0$, and $\alpha + \frac{N+2}{q} = 2$. Then, there exists K depending on c_1, c_2, R, N, q such that

$$\|\partial_t v\|_{L^q(Q_R)} + \|D^2 v\|_{L^q(Q_R)} \leq K.$$

Moreover, the same statement holds if Q_R and Q_{2R} are replaced by cylinders $Q' \subset\subset Q$ respectively.

Proof. First, since $[v]_{\alpha; Q_{2R}} \leq c_1$ and $v(0) = 0$,

$$\|v\|_{L^q(Q_{2R})} \leq c_1 C R^{\alpha + \frac{N+2}{q}} = c_1 C R^2. \quad (\text{A.12})$$

Let $R \leq \rho \leq 2R$ and $0 < \sigma < 1$. Arguing as in the proof of [16, Theorem 7.22], and using Hölder's inequality and (A.12)

$$\begin{aligned} \|D^2 v\|_{L^q(Q_{\sigma\rho})} &\leq \frac{C}{(1-\sigma)^2 R^2} \left(R^2 \|c_2 |Dv|^\gamma + g\|_{L^q(Q_\rho)} + \|v\|_{L^q(Q_\rho)} \right) \\ &\leq \frac{C}{(1-\sigma)^2} \left(R^{\gamma - \frac{N+2}{q}(\gamma-1)} \|Dv\|_{p; Q_\rho}^\gamma + 1 \right). \end{aligned} \quad (\text{A.13})$$

where $\frac{1}{p} = \frac{1}{q} - \frac{1}{N+2}$. By the Gagliardo-Nirenberg interpolation inequality [18], there exists C (independent of ρ) such that for a.e. $t \in (0, \rho^2)$,

$$\|Dv(t)\|_{p;B_\rho} \leq C \left(\|D^2v(t)\|_{L^q(B_\rho)}^a [v(t)]_{\alpha;B_\rho}^{1-a} + [v(t)]_{\alpha;B_\rho} \right), \quad \text{where } a = 1 - \frac{q}{N+2} < \frac{1}{\gamma}.$$

Since $ap = q$, if we raise the previous inequality to the power p and integrate in time we obtain

$$\|Dv\|_{p;Q_\rho}^p \leq C(\|D^2v\|_{L^q(Q_\rho)}^q + R^2).$$

If $\|D^2v\|_{L^q(Q_R)}^q \leq R^2$, then there is nothing else to prove, since $\|D^2v\|_{q,Q_R}$ is bounded, and a control on $\|\partial_t v\|_{q,Q_R}$ follows immediately by integrating the differential inequality for v . Otherwise, plugging the resulting inequality into (A.13) implies

$$\|D^2v\|_{L^q(Q_{\sigma\rho})} \leq \frac{C}{(1-\sigma)^2} \left(R^{\gamma - \frac{N+2}{q}(\gamma-1)} \|D^2v\|_{L^q(Q_\rho)}^{a\gamma} + 1 \right). \quad (\text{A.14})$$

As before, if $R^{\gamma - \frac{N+2}{q}(\gamma-1)} \|D^2v\|_{L^q(Q_R)}^{a\gamma} \leq 1$, then we are done. Otherwise, (A.14) yields, for every $R \leq \rho \leq 2R$ and $0 < \sigma < 1$,

$$\|D^2v\|_{L^q(Q_{\sigma\rho})} \leq \frac{E}{(1-\sigma)^2} \|D^2v\|_{L^q(Q_\rho)}^{a\gamma}, \quad (\text{A.15})$$

where $E = CR^{\gamma - \frac{N+2}{q}(\gamma-1)}$, and C is independent of ρ, σ . We may then conclude following exactly the same lines as in the proof of [8, Proposition 3.3]; the statement with Q', Q follows by a standard covering argument. $\square$

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