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On the orthogonality of generalized eigenspaces for the Ornstein–Uhlenbeck operator

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Abstract. We study the orthogonality of the generalized eigenspaces of an Ornstein–Uhlenbeck operator $\mathscr{L}$ in $\mathbb{R}^N$, with drift given by a real matrix B whose eigenvalues have negative real parts. If B has only one eigenvalue, we prove that any two distinct generalized eigenspaces of $\mathscr{L}$ are orthogonal with respect to the invariant Gaussian measure. Then we show by means of two examples that if B admits distinct eigenvalues, the generalized eigenspaces of $\mathscr{L}$ may or may not be orthogonal.

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1. Introduction. In this note, we discuss the orthogonality of the generalized eigenspaces associated to a general Ornstein–Uhlenbeck operator $\mathscr{L}$ in $\mathbb{R}^N$.

Recently, the authors started studying some harmonic analysis issues in a nonsymmetric Gaussian context [1–3]. In particular, the Ornstein–Uhlenbeck semigroup $\left(\mathscr{H}_t\right)_{t>0}$ generated by $\mathscr{L}$ is not assumed to be self-adjoint in $L^2(\gamma_\infty)$; here γ_∞ denotes the unique invariant probability measure under the action of the semigroup, and will be specified later.

In this general framework, the Ornstein–Uhlenbeck operator $\mathscr{L}$ admits a complete system of generalized eigenfunctions; see [8]. But without self-adjointness, the orthogonality of distinct eigenspaces of $\mathscr{L}$ is not guaranteed. In fact, while the kernel of $\mathscr{L}$ is always orthogonal to the other generalized eigenspaces of $\mathscr{L}$ in $L^2(\gamma_\infty)$, the question of orthogonality between generalized

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eigenspaces associated to nonzero eigenvalues is more delicate. As expected, the spectral properties of B play a prominent role here. Indeed, we prove in Section 3 that if B has a unique eigenvalue, then any two generalized eigenfunctions of $\mathscr{L}$ corresponding to different eigenvalues are orthogonal in $L^2(\gamma_\infty)$.

Then in Sections 4 and 5, we exhibit two examples showing, respectively, that if B admits two distinct eigenvalues, the generalized eigenspaces associated to $\mathscr{L}$ may or may not be orthogonal. The last section also contains a result which relates the orthogonality of the eigenspaces of $\mathscr{L}$ to that of the eigenspaces of the drift matrix, under some restrictions.

In the following, the symbol I_k will denote the identity matrix of size k, and we omit the subscript when the size is obvious. We will write $\langle\cdot,\cdot\rangle$ for scalar products both in $\mathbb{R}^N$ and in $L^2(\gamma_\infty)$. By $\mathbb{N}$ we mean $\{0,1,\dots\}$.

2. The Ornstein–Uhlenbeck operator. In this section, we specify the definition of the Ornstein–Uhlenbeck operator $\mathscr{L}$ and recall some known facts concerning its spectrum.

We consider the Ornstein–Uhlenbeck semigroup $\left(\mathscr{H}_t^{Q,B}\right)_{t>0}$, given for all bounded continuous functions f in $\mathbb{R}^N$, $N\geq 1$, and all $t>0$ by the Kolmogorov formula

$$\mathscr{H}_t^{Q,B} f(x) = \int f(e^{tB}x-y)\,d\gamma_t(y)\,, \quad x\in\mathbb{R}^N\,,$$

(see [6] and [7, Theorem 9.1.1]). Here B is a real $N\times N$ matrix whose eigenvalues have negative real parts, and Q is a real, symmetric, and positive-definite $N\times N$ matrix. Then we introduce the covariance matrices

$$Q_t = \int\limits_0^t e^{sB}\,Q\,e^{sB^*}\,ds\,, \qquad t\in(0,+\infty],$$

all symmetric and positive definite. Finally, the normalized Gaussian measures γ_t are defined for $t\in(0,+\infty]$ by

$$d\gamma_t(x) = (2\pi)^{-\frac{N}{2}}(\det Q_t)^{-\frac12}e^{-\frac12\langle Q_t^{-1}x,x\rangle}\,dx.$$

As mentioned above, γ_∞ is the unique invariant probability measure of the Ornstein–Uhlenbeck semigroup.

The Ornstein–Uhlenbeck operator is the infinitesimal generator of the semigroup $\left(\mathscr{H}_t^{Q,B}\right)_{t>0}$, and it is explicitly given by

$$\mathscr{L}^{Q,B} f(x) = \frac12\operatorname{tr}\left(Q\nabla^2 f\right)(x) + \langle Bx,\nabla f(x)\rangle\,, \qquad f\in\mathscr{S}(\mathbb{R}^N),$$

where ∇ is the gradient and ∇^2 the Hessian.

By convention, we abbreviate $\mathscr{H}_t^{Q,B}$ and $\mathscr{L}^{Q,B}$ to $\mathscr{H}_t$ and $\mathscr{L}$, respectively. We can thus write $\mathscr{H}_t = e^{t\mathscr{L}}$.

In [8, Theorem 3.1], it is verified that the spectrum of $\mathscr{L}$ is the set

$$\left\{\sum_{j=1}^r n_j\lambda_j\ :\ n_j\in\mathbb{N}\right\}, \tag{1}$$

where $\lambda_1, \dots, \lambda_r$ are the eigenvalues of the drift matrix B. In particular, 0 is an eigenvalue of $\mathscr{L}$, and the corresponding eigenspace $\ker \mathscr{L}$ is one-dimensional and consists of all constant functions, as proved in [8, Section 3].

We also recall that, given a linear operator T on some L^2 space, a number $\lambda \in \mathbb{C}$ is a generalized eigenvalue of T if there exists a nonzero $u \in L^2$ such that $(T - \lambda I)^k\, u = 0$ for some positive integer k. Then u is called a generalized eigenfunction, and those u span the generalized eigenspace corresponding to λ. As already recalled, it is known from [8, Section 3] that the Ornstein–Uhlenbeck operator $\mathscr{L}$ admits a complete system of generalized eigenfunctions, that is, the linear span of the generalized eigenfunctions is dense in $L^2(\gamma_\infty)$. It is also known that all generalized eigenfunctions of $\mathscr{L}$ are polynomials, see [7, Theorem 9.3.20].

2.1. Use of Hermite polynomials. As proved in [9], a suitable linear change of coordinates in $\mathbb{R}^N$ makes $Q = I$ and Q_∞ diagonal. When applying this, we adhere to the notation introduced in [4, Lemma 1], where also the following facts can be found. Let $\mathbf{H}_n$ denote the space of Hermite polynomials of degree n in these coordinates, adapted by means of a dilation to γ_∞ in the sense that the $\mathbf{H}_n$ are mutually orthogonal in $L^2(\gamma_\infty)$ (they are called $H_{\lambda,k}$ in [4]). The classical Hermite expansion (called the Itô-Wiener decomposition in [4]) says that $L^2(\gamma_\infty)$ is the closure of the direct sum of the $\mathbf{H}_n$; we refer to [10, p. 64] for a proof in dimension one and note that the extension to higher dimension is trivial. In other words, we can decompose any function $u \in L^2(\gamma_\infty)$ as

$$u = \sum_j u_j \tag{2}$$

with $u_j \in \mathbf{H}_j$ and convergence in $L^2(\gamma_\infty)$. Further, each $\mathbf{H}_n$ is invariant under $\mathscr{L}$; see [4, Proposition 1].

The Hermite decomposition implies, in particular, that each generalized eigenfunction of $\mathscr{L}$ with a nonzero eigenvalue is orthogonal to the space of constant functions, that is, to the kernel of $\mathscr{L}$. Anyway, we provide here a proof of this fact which is independent of Hermite polynomials.

Lemma 2.1. *Let $\lambda \neq 0$. If $u \in L^2(\gamma_\infty)$ and $(\mathscr{L} - \lambda)^k\, u = 0$ for some $k \in \{1, 2, \dots\}$, then $\int u\, d\gamma_\infty = 0$.*

Proof. The implication is trivial if we set $k = 0$, so assume it holds for some $k \geq 0$ and that $(\mathscr{L} - \lambda)^{k+1}\, u = 0$.

Then

$$\mathscr{L}(\mathscr{L} - \lambda)^k\, u = \lambda(\mathscr{L} - \lambda)^k\, u,$$

and thus for any $t > 0$,

$$e^{t\mathscr{L}}\, (\mathscr{L} - \lambda)^k\, u = e^{t\lambda}(\mathscr{L} - \lambda)^k\, u.$$

These operators commute, so

$$(\mathscr{L} - \lambda)^k\, e^{t\mathscr{L}}\, u = (\mathscr{L} - \lambda)^k\, e^{t\lambda} u,$$

that is,

$$(\mathscr{L} - \lambda)^k\, (e^{t\mathscr{L}}\, u - e^{t\lambda} u) = 0.$$

The induction assumption now implies that

$$\int (e^{t\mathscr{L}} u - e^{t\lambda} u)\, d\gamma_\infty = 0.$$

Since γ_∞ is invariant under the semigroup, this means that

$$\int u\, d\gamma_\infty = e^{t\lambda} \int u\, d\gamma_\infty$$

for all $t > 0$. Thus the integral vanishes. □

3. The case when B has only one eigenvalue.

Proposition 3.1. *If the drift matrix B has only one eigenvalue, then any two generalized eigenfunctions of $\mathscr{L}$ with different eigenvalues are orthogonal with respect to γ_∞.*

Let λ be the unique eigenvalue of B, which is necessarily real and negative. We first state a lemma and use it to prove the proposition. Recall that any generalized eigenfunction of $\mathscr{L}$ is a polynomial.

Lemma 3.2. *Let u be a generalized eigenfunction of $\mathscr{L}$ which is a polynomial of degree $n \geq 0$. Then the corresponding eigenvalue is $n\lambda$.*

Proof of Proposition 3.1. Let u be a generalized eigenfunction of $\mathscr{L}$, thus satisfying $(\mathscr{L} - \mu)^k\, u = 0$ for some $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$. Applying the coordinates from Subsection 2.1, we can decompose u as in (2), where the sum is now finite. Since then

$$\sum_j (\mathscr{L} - \mu)^k u_j = 0$$

and each term here is in the corresponding $\mathbf{H}_j$, all the terms are 0. But this is compatible with Lemma 3.2 only if there is only one nonzero term in the decomposition of u. Thus $u \in \mathbf{H}_n$, where n is the polynomial degree of u.

Lemma 3.2 then implies that two generalized eigenfunctions with different eigenvalues are of different degrees and thus belong to different $\mathbf{H}_n$. The desired orthogonality now follows from that of the $\mathbf{H}_n$. □

Proof of Lemma 3.2. Let u be a generalized eigenfunction of $\mathscr{L}$ of polynomial degree n. We denote the corresponding eigenvalue by μ. Decomposing u as in (2), we see that this sum is for $j \leq n$ and that the term u_n is nonzero and a generalized eigenfunction of $\mathscr{L}$ with eigenvalue μ. For some m, the function $(\mathscr{L} - \mu)^m u_n$ will then be an eigenfunction with the same eigenvalue. This function is in $\mathbf{H}_n$ and thus a polynomial of degree n. As a result, we can assume that u is actually an eigenfunction of $\mathscr{L}$ when proving the lemma.

We now choose coordinates in $\mathbb{R}^N$ that give a Jordan decomposition of B. This means that $B = \lambda I + R$, where $R = (R_{i,j})$ is a matrix with nonzero entries only in the first subdiagonal. More precisely, $R_{i,i-1} = 1$ for $i \in P$, where P is a subset of $\{2, \ldots, N\}$, and all other entries of R vanish.

We write $\mathscr{L} = \mathscr{S} + \mathscr{B}$, where

$$\mathscr{B} f(x) = \langle Bx, \nabla f(x) \rangle,$$

and $\mathscr{S}$ is the remaining, second-degree part of $\mathscr{L}$. Notice that, when applied to polynomials, $\mathscr{B}$ preserves the degree whereas $\mathscr{S}$ decreases it by 2. So if v is the nth-degree part of u, we must have $\mathscr{B}v = \mu v$.

We let $\mathscr{B}$ act on a monomial x^α, where $\alpha \in \mathbb{N}^N$ is a multiindex of length $|\alpha| = n$, getting

$$\begin{aligned}\mathscr{B}x^\alpha &= \sum_j \lambda x_j \frac{\partial x^\alpha}{\partial x_j} + \sum_{i\in P} x_{i-1} \frac{\partial x^\alpha}{\partial x_i} \\ &= \lambda \sum_j \alpha_j x^\alpha + \sum_{i\in P} \alpha_i \frac{x_{i-1}}{x_i} x^\alpha = \lambda n\, x^\alpha + \sum_{i\in P} \alpha_i\, x^{\alpha^{(i)}},\end{aligned}$$

where $\alpha^{(i)} = \alpha + e_{i-1} - e_i$ for $i \in P$. Here $\{e_j\}_{j=1}^n$ denotes the standard basis in $\mathbb{R}^N$. Thus the restriction of $\mathscr{B}$ to the space of homogeneous polynomials of degree n is given as $\lambda n I + \mathscr{R}$, where $\mathscr{R}$ is the linear operator that maps x^α to $\sum_{i\in P} \alpha_i\, x^{\alpha^{(i)}}$.

We claim that the only eigenvalue of $\mathscr{R}$ is 0. If so, the only eigenvalue of the restriction of $\mathscr{B}$ mentioned above is λn, which would prove the lemma since $\mathscr{B}v = \mu v$.

In order to prove this claim, we define for any $\alpha \in \mathbb{N}^N$ with $|\alpha| = n$,

$$V(\alpha) = \sum_1^N j\alpha_j.$$

Clearly $V(\alpha^{(i)}) = V(\alpha) - 1$. We select a basis in the linear space of all homogeneous polynomials of degree n consisting of all monomials x^α with $|\alpha| = n$, enumerated in such a way that V is nondecreasing. The definition of $\mathscr{R}$ now shows that its matrix with respect to this basis is upper triangular with zeros on the diagonal. The claim follows, and so does the lemma. $\square$

4. *B* has two distinct eigenvalues: a first example. The following example shows that the generalized eigenspaces of the Ornstein–Uhlenbeck operator may be orthogonal even in the case when B has more than one eigenvalue. We show that $\mathscr{L}$, while not being self-adjoint, is normal; then the orthogonality of its eigenspaces follows from the spectral theorem.
In two dimensions, we let

$$Q = I_2 \quad\text{and}\quad B = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \tag{3}$$

whose eigenvalues are $-1 \pm i$.
One finds that

$$e^{sB} = e^{-s} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$$

and

$$e^{sB}\, e^{sB^*} = e^{-2s}\, I_2,$$

so that

$$Q_\infty = \frac{1}{2} I_2, \qquad\qquad Q_\infty^{-1} = 2\, I_2.$$

We write

$$B = -I_2 + R, \qquad \text{where} \qquad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $R = -R^*$, [8, Proposition 2.1] implies that $\mathscr{L}$ is normal (observe that $I_2 = \frac{1}{2} D_{1/\lambda}$ in the notation of [8]).
However, we give below a brief, direct proof of this fact, independent of the change of variables adopted in [8,9]. In the following, we write

$$\mathscr{L} = \mathscr{L}^0 + \mathscr{R},$$

where

$$\mathscr{L}^0 = \frac{1}{2}\Delta - \langle x, \nabla \rangle$$

is the standard Ornstein–Uhlenbeck operator and so self-adjoint in $L^2(\gamma_\infty)$. Further,

$$\mathscr{R} = \langle Rx, \nabla \rangle$$

is seen to be an antisymmetric operator in $L^2(\gamma_\infty)$. This leads to

$$[\mathscr{L}, \mathscr{L}^*] = [\mathscr{L}^0 + \mathscr{R}, \mathscr{L}^0 - \mathscr{R}] = -\mathscr{L}^0\mathscr{R} + \mathscr{R}\mathscr{L}^0 - \mathscr{L}^0\mathscr{R} + \mathscr{R}\mathscr{L}^0 = 2[\mathscr{R}, \mathscr{L}^0].$$

If we write ∂_i for ∂_{x_i}, $i = 1, 2$, this amounts to

$$2\left[x_2\,\partial_1 - x_1\,\partial_2\,,\ \frac{1}{2}\Delta - x_1\,\partial_1 - x_2\,\partial_2\right].$$

A straightforward computation shows that this vanishes, and so $\mathscr{L}$ is normal. The spectral theorem for normal operators now implies the following result.

Proposition 4.1. *With $N = 2$, let Q and B be as in (3). Then each generalized eigenfunction of $\mathscr{L}$ is an eigenfunction. Moreover, any two eigenfunctions of $\mathscr{L}$ with different eigenvalues are orthogonal with respect to γ_∞.*

5. *B* has two distinct eigenvalues: a second example. In this section, we exhibit a class of drift matrices B with two different eigenvalues (which, in contrast to those in the example in Section 4, are real), but such that the generalized eigenspaces associated to the corresponding Ornstein–Uhlenbeck operator $\mathscr{L}$ are not orthogonal.

In $\mathbb{R}^2$, we consider $Q = I_2$ and

$$B = \begin{pmatrix} -a + d & 0 \\ c & -a - d \end{pmatrix}, \tag{4}$$

with $a > d > 0$ and $c \neq 0$. To compute the exponential of sB, we write $B = -aI + M$, where

$$M = \begin{pmatrix} d & 0 \\ c & -d \end{pmatrix}.$$

Since $MM = d^2 I$, we get for $s > 0$,

$$\exp(sB) = e^{-as}\,\left(\cosh(sd)\,I + d^{-1}\,\sinh(sd)\,M\right).$$

This leads to

$$\exp(sB)\,\exp(sB^*) = e^{-2as}\begin{pmatrix} e^{2sd} & \frac{c}{d}\,e^{sd}\sinh(sd) \\ \frac{c}{d}\,e^{sd}\sinh(sd) & \frac{c^2}{d^2}\,\sinh^2(sd)+e^{-2sd}\end{pmatrix}.$$

Integrating this matrix over $0<s<\infty$, we obtain

$$Q_\infty = \begin{pmatrix} \frac{1}{2(a-d)} & \frac{c}{4a(a-d)} \\ \frac{c}{4a(a-d)} & \frac{c^2}{4a(a-d)(a+d)}+\frac{1}{2(a+d)}\end{pmatrix},$$

and so

$$\frac{1}{2}\,Q_\infty^{-1} = \frac{1}{c^2+4a^2}\begin{pmatrix} 2a[c^2+2a(a-d)] & -2ac(a+d) \\ -2ac(a+d) & 4a^2(a+d)\end{pmatrix}.$$

The invariant measure γ_∞ is thus proportional to

$$\exp\left(-\frac{2a[c^2+2a(a-d)]}{c^2+4a^2}\,x_1^2+\frac{4ac(a+d)}{c^2+4a^2}\,x_1x_2-\frac{4a^2(a+d)}{c^2+4a^2}\,x_2^2\right)dx$$
$$=\exp\big(-(a-d)\,x_1^2\big)\,\exp\left(-\frac{a+d}{c^2+4a^2}\,(cx_1-2ax_2)^2\right)dx.$$

Writing $z_1=\sqrt{a-d}\,x_1$ and $z_2=\sqrt{\frac{a+d}{c^2+4a^2}}\,\big(2ax_2-cx_1\big)$ and recalling that γ_∞ is a probability measure, we see that

$$d\gamma_\infty = \pi^{-1}\exp\big(-z_1^2-z_2^2\big)\,dz.$$

To find some eigenfunctions of $\mathscr{L}$, we consider polynomials in x_1, x_2 of degree 2. One finds that

$$v_1 = x_1^2-\frac{1}{2(a-d)},$$
$$v_2 = x_1^2-\frac{2d}{c}\,x_1x_2-\frac{1}{2a},$$
$$v_3 = x_1^2-\frac{4d}{c}\,x_1x_2+\frac{4d^2}{c^2}\,x_2^2-\frac{c^2+4d^2}{2c^2(a+d)}$$

are eigenfunctions, with eigenvalues $-2(a-d)$, $-2a$, and $-2(a+d)$, respectively.

Any two of these polynomials turn out not to be orthogonal with respect to the invariant measure, as follows by straightforward computations. We sketch one example.

One simply multiplies v_1 and v_3 and rewrites the product in terms of z_1 and z_2. Doing so, one can neglect all terms of odd order in z_1 or z_3, when integrating with respect to γ_∞. Writing "odd" for such terms, we find that the product is

$$\frac{1}{a^2}\,z_1^4+\frac{d^2(c^2+4a^2)}{a^2c^2(a^2-d^2)}\,z_1^2z_2^2-\left[\frac{c^2+4d^2}{2c^2(a^2-d^2)}+\frac{1}{2a^2}\right]z_1^2$$
$$-\frac{d^2(c^2+4a^2)}{2a^2c^2(a^2-d^2)}\,z_2^2+\frac{c^2+4d^2}{4c^2(a^2-d^2)}+\text{odd}.$$

Integrating and simplifying, we get

$$\int v_1 v_3 \, d\gamma_\infty = \frac{1}{2a^2} > 0,$$

so v_1 and v_3 are not orthogonal.

Remark 5.1. Let now $d = a/2$ in this example. Then the fourth-degree polynomial

$$v_4 = x_1^4 - \frac{6}{a} x_1^2 + \frac{3}{a^2}$$

is an eigenfunction of $\mathscr{L}$ with eigenvalue $-2a$, like v_2. Thus eigenfunctions of different polynomial degrees can have the same eigenvalue. This shows that for an eigenfunction u, the sum in (2) may consist of more than one term, and a (generalized) eigenspace need not be contained in one $\mathbf{H}_n$.

The eigenvalues of the matrix B defined in (4) are $-a \pm d$, and it is easily seen that the corresponding eigenspaces are not orthogonal in $\mathbb{R}^2$. This turns out to be related to the non-orthogonality of the eigenspaces of $\mathscr{L}$, at least in two dimensions, in the following way.

Proposition 5.2. *Let $N = 2$ and $Q = I$, and assume that B has two different, real eigenvalues. Then the generalized eigenspaces of $\mathscr{L}$ are orthogonal in $L^2(\gamma_\infty)$ if and only if the two eigenspaces of B are orthogonal in $\mathbb{R}^2$.*

Proof. To begin with, we consider a coordinate change $\widetilde{x} = Hx$, where H is an orthogonal matrix. Simple computations show that the operator $\mathscr{L}^{Q,B}$ is transformed to $\mathscr{L}^{\widetilde{Q},\widetilde{B}}$ in the new coordinates, with $\widetilde{Q} = HQH^*$ and $\widetilde{B} = HBH^*$; cf. [9, p. 474]. In our case, $\widetilde{Q} = Q = I$. The eigenvalues of B and the angle between its eigenvectors will not change.

To prove the proposition, assume first that the (real) eigenvectors of B are orthogonal in $\mathbb{R}^2$. Then B is symmetric since it can be diagonalized by means of an orthogonal change of coordinates as just described. This implies that $\mathscr{L}$ is symmetric ([7, Proposition 9.3.10]), so that the orthogonality of its eigenspaces is trivial.

Next, we assume that the eigenvectors of B are not orthogonal in $\mathbb{R}^2$. By Schur's decomposition theorem (see [5, Theorem 2.3.1]), there exists an orthogonal change of coordinates which makes B lower triangular, though not diagonal. We are thus in the situation described in (4). As we have seen, some eigenspaces of $\mathscr{L}$ are then not orthogonal with respect to the invariant measure. □

We finally remark that the "if" part of this proposition easily extends to arbitrary dimension N. Then it is assumed that B has N different, real eigenvalues with mutually orthogonal eigenspaces.

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