On finite groups in which cyclic subgroups of the same order are conjugate

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In memoriam Maria Silvia Lucido (†2008)

Abstract

We consider finite groups G for which any two cyclic subgroups of the same order are conjugate in G. We prove various structure results and, in particular, any such group has at most one non-abelian composition factor, and this is isomorphic to $PSL(2, p^m)$, with m odd if p is odd, or to $Sz(2^{2m+1})$, or to one of the sporadic groups M_{11}, M_{23} or J_1.

Introduction.

In this paper we shall study the class of csc-groups in the following sense:

Definition. Let π be a set of prime numbers. A finite group G is called a csc_π-group if given two cyclic subgroups X, Y of G of the same order with $\pi(|X|) \subseteq \pi$, then there exists $g \in G$ such that $X = Y^g$. A finite group G is called a csc-group if G is a csc_π-group for $\pi = \pi(|G|)$.

Similar kinds of problems have often been object of investigation. For instance, in [5] Fitzpatrick, using the classification of finite simple groups, proved that if in a finite group G any two elements of the same order are conjugate, then G is isomorphic with the symmetric group S_n, with $n \in \{1, 2, 3\}$ (see also [4]). Then in [10] there is the classification of finite groups for which elements of the same order are conjugate or inverse-conjugate. Similar results, but concerning fusion in Aut G, have been obtained in [20], [11] and [12]. In [17], the author considers finite groups G for which any two isomorphic subgroups are conjugate in G.

The main result of the present paper is the following

Theorem. *Let G be a finite* csc*-group. Then*

$$F^*(G) = X_1 \times X_2 \times \ldots \times X_k$$

where the $|X_i|$'s for $i \in \{1, 2, \ldots, k\}$ are pairwise coprime and

(1) X_i *is a cyclic* p*-group;*

(2) X_i *is an elementary abelian* p*-group;*

(3) X_i *is a non-abelian* 2*-group such that* $\Omega_1(X_i)$ *and* $X_i/\Omega_1(G)$ *are elementary abelian and either* $|X_i| = |\Omega_1(X_i)|^2$ *or* $|X_i| = |\Omega_1(X_i)|^3$*;*

(4) $X_i \simeq \mathrm{PSL}(2, p^m)$ *or* $X_i \simeq \mathrm{SL}(2, p^m)$ *with* $p \neq 2$*,* $p^m > 3$ *and* m *odd;*

(5) $X_i \simeq \mathrm{PSL}(2, 2^m)$ *with* $2^m > 2$*;*

(6) X_i *is one of the sporadic groups* M_{11}*,* M_{23} *or* J_1*.*

Moreover, if P *is a Sylow* p*-subgroup of* $F^*(G)$ *and* P *is not cyclic, then* P *is a Sylow* p*-subgroup of* G*.*

We shall also determine further properties of csc-groups, giving a structure characterization in terms of certain minimal csc-subgroups.

The paper is structured as follows. In section 1 we introduce the notation and prove some preliminary results. In section 2 we deal with solvable csc-groups and Frobenius groups. In section 3 we classify the simple, almost-simple and quasisimple csc-groups. In section 4 we introduce the notion of monolithic csc-groups and determine the structure of the generalized Fitting subgroup for these groups. Finally in section 5 we deal with the general case.

All groups in this paper are meant to be finite. We shall make use of the Classification of Finite Simple Groups.

1 Notation and preliminary results.

We shall denote by $\mathbb{P}$ the set of prime numbers and by π a subset of $\mathbb{P}$, then we put $\pi' = \mathbb{P} \setminus \pi$. If $n \in \mathbb{N}$ with $n \geq 2$, we denote by $\pi(n)$ the set of primes dividing n.

A π-group is a group G such that $\pi(|G|) \subseteq \pi$. If G is a group, $O_\pi(G)$ is the largest normal subgroup of G which is a π-group. If $\pi = \{p\}$, we shall write $O_p(G)$ and $O_{p'}(G)$ instead of $O_\pi(G)$ and $O_{\pi'}(G)$ respectively. An element $g \in G$ is called a π-element if $\pi(|g|) \subseteq \pi$.

We denote by $\mathsf{Syl}_p(G)$ the set of Sylow p-subgroups of G. Also $E(G)$ denotes the subgroup of G generated by the quasisimple subnormal subgroups of G, $F^*(G) = F(G)E(G)$ is the generalized Fitting subgroup of G and $O_\infty(G)$ is the largest normal solvable subgroup of G (the solvable socle of G).

We denote by C_n the cyclic group of order n. For short we shall call *quaternions* the group of quaternions of order 8.

The following easy fact is essential for induction arguments on the order of G.

Lemma 1.1 *Let G be a csc_π-group, N a normal subgroup of G, $\overline{G} = G/N$. Let $\overline{x}, \overline{y} \in \overline{G}$ be elements of order $\overline{r}$ with $\pi_* = \pi(\overline{r}) \subseteq \pi$ and let x, y be preimages of $\overline{x}$, $\overline{y}$ in G such that x and y are π_*-elements. Then $|\langle x \rangle| = |\langle y \rangle|$.*

Proof. Let r_1 e r_2 be the orders of x and y respectively; by hypothesis, $\pi(r_1) = \pi(r_2) = \pi_* \subseteq \pi$. Let $m = (r_1, r_2)$, then $\overline{r}$ divides m and we may write $r_1 = ms_1$ and $r_2 = ms_2$ with $(s_1, s_2) = 1$. Suppose for a contradiction that $r_1 \neq r_2$; one shoud have $s_1 \neq s_2$. The subgroups $\langle x^{s_1} \rangle$ and $\langle y^{s_2} \rangle$ have the same order and, since $\pi(m) = \pi_* \subseteq \pi$, they are conjugate in G. But in $\overline{G}$ one has $|\langle \overline{x}^{s_1} \rangle| \neq |\langle \overline{y}^{s_2} \rangle|$, a contradiction. $\square$

Lemma 1.2 *Let G be a csc_π-group and let N be a normal subgroup of G. Then G/N is a csc_π-group.*

Proof. This follows immediately from Lemma 1.1. $\square$

Lemma 1.3 *Let G_1, G_2 be csc_π-groups. Then $G_1 \times G_2$ is a csc_π-group if and only if $\pi(|G_1|) \cap \pi(|G_2|) \cap \pi = \emptyset$.*

Proof. Sufficiency is clear. To prove necessity, assume for a contradiction that there exists a prime p in $\pi(|G_1|) \cap \pi(|G_2|) \cap \pi$. Let $\langle x_1 \rangle$ be a subgroup of order p of G_1 and $\langle x_2 \rangle$ a subgroup of order p of G_2. If $x_1 \in Z(G_1)$ then $\langle (x_1, 1) \rangle \leq Z(G_1 \times G_2)$ is not conjugate to $\langle (1, x_2) \rangle$; similarly x_2 is not in $Z(G_2)$. If we put $G = G_1 \times G_2$, we have $C_G(\langle (x_1, 1) \rangle) = C_{G_1}(x_1) \times G_2$, $C_G(\langle (1, x_2) \rangle) = G_1 \times C_{G_2}(x_2)$ and $C_G(\langle (x_1, x_2) \rangle) = C_{G_1}(x_1) \times C_{G_2}(x_2)$. In particular $\langle (x_1, x_2) \rangle$ has order p and is neither conjugate to $\langle (x_1, 1) \rangle$ nor to $\langle (1, x_2) \rangle$, a contradiction. $\square$

Lemma 1.4 *Let G be a* csc_π*-group and let $p \in \pi \cap \pi(|Z(G)|)$. Then the Sylow p-subgroups of G are cyclic or isomorphic to generalized quaternions.*

Proof. Let P be a Sylow p-subgroup of G and let x be an element of order p of $Z(G) \cap P$. Then $\langle x \rangle$ is the unique subgroup of order p of P and we conclude by 5.3.6 in [14]. □

Lemma 1.5 *Let G be a* csc_π*-group. Then $O_\pi(Z(G))$ is cyclic and $O_\pi(Z_2(G)) = O_\pi(Z(G))$.*

Proof. By Lemma 1.4, if $p \in \pi$ then $O_p(Z(G))$ is cyclic; it follows that $O_\pi(Z(G))$ is cyclic.

To prove the second statement, suppose for a contradiction that for a $p \in \pi$ there exists a p-element x of $O_p(Z_2(G))$ not lying in $O_\pi(Z(G))$. Let $y \in G$ with $[x, y] \neq 1$, then $x^y = xz$ for some $z \in O_p(Z(G))$. If the order of z is p^k we have $x^{y^{p^k}} = xz^{p^k} = x$, so that $y^{p^k} \in C_G(x)$; without loss of generality we may therefore assume that y is a p-element of G. Then $\langle x, y \rangle$ is a non-cyclic p-subgroup of G. By Lemma 1.4 we must have $p = 2$ and the Sylow 2-subgroups of G are isomorphic to generalized quaternions.

Let S be a Sylow 2-subgroup of G, and let $Z(S) = \langle z \rangle$; we have $|\langle z \rangle| = 2$ and, by hypothesis, $\langle z \rangle \leq Z(G)$. In $\overline{G} = G/\langle z \rangle$ we have $O_2(Z(\overline{G})) \neq 1$ so that, by Lemma 1.4, the Sylow 2-subgroups of $\overline{G}$ should be isomorphic to generalized quaternions. But $\overline{S} = S/\langle z \rangle$ is dihedral, a contradiction. □

2 Solvable csc_π-groups and Frobenius groups.

Lemma 2.1 *Let G be a solvable* csc_π*-group. Then for every $p \in \pi$, G has p-length at most* 1.

Proof. Let G be a counterexample of minimal order. Since every quotient of a csc_π-group is, by Lemma 1.1, a csc_π-group, we have $\ell_p(G) > 1$ and every proper quotient of G has p-length less or equal 1. By Proposition 9.3.8 in [14], $N = O_{p'p}(G)$ is an elementary abelian p-subgroup of G and there exists a subgroup H of G such that $G = NH$ and $N \cap H = \{1\}$. In H there is no subgroup of order p, since this then should be conjugate to every cyclic subgroup of N. Hence H is a p'-group and $G = O_{pp'}(G)$, a contradiction. □

Lemma 2.2 *Let G be a solvable* csc_π*-group, $p \in \pi$ and P a Sylow p-subgroup of G. Then one of the following holds:*

(1) P *is cyclic;*

(2) P *is elementary abelian;*

(3) $p = 2$*,* P *has class* 2*, exponent* 4 *and* $P' = \Phi(G) = Z(P) = \Omega_1(Z(P)) = \Omega_1(P)$*; moreover* $|P| = |Z(P)|^2$ *or* $|P| = |Z(P)|^3$.

Proof. Without loss of generality we may assume $P \trianglelefteq G$. Otherwise we consider $G/O_{p'}(G)$ (which is csc$_\pi$-group by Lemma 1.2) and use Lemma 2.1. We distinguish two cases

• P is abelian.

Then P is homocyclic by Theorem VIII.5.8 (b) in [8]. If P is cyclic, then we are done. Let us assume that P is not cyclic, and let p^k be the exponent of P; then there exists $n \in \mathbb{N}$ with $n > 1$ such that $|P| = p^{kn}$. We show that $k = 1$. Assume for a contradiction that $k > 1$. The cyclic subgroups of order p^2 in P are $\frac{p^{2n}-p^n}{p(p-1)} = p^{n-1}\frac{p^n-1}{p-1}$ and are permuted transitively under the action of $H = G/C_G(\Omega_2(P))$; but this number is divisible by p since $n > 1$, while H is a p'-group: a contradiction.

• P is not abelian.

Then $p = 2$ by [15], [16]. If P has only one involution, then P is generalized quaternions of order 2^n say (see 5.3.6 in [14]). The condition that all subgroups of order 4 are conjugate in G, gives $n = 3$ and we are done. If P has more than one involution, then P is a Suzuki 2-group (following Definition VIII.7.1 in [8]), and by Theorem VIII.7.9 in [8] we conclude. □

Remark 2.3 Thompson (see Theorem IX.8.6 in [8]) proved that if a solvable group G is such that the Sylow 2-subgroups have more that one involution and all involutions in G are conjugate, then:

(a) the 2-length of G is 1;

(b) the Sylow 2-subgroups of G are homocyclic or Suzuki 2-groups.

On the other hand, Gaschütz and Yen (see Theorem IX.8.7 in [8]) proved that if G is a p-solvable group, where p is an odd prime divisor of $|G|$ and if the subgroups of order p of G are permuted transitively under the action of Aut G, then the p-length of G is 1. □

Lemma 2.1 could be obtained by the above mentioned results. We have given a direct short proof to make the paper as self contained as possible.

Remark 2.4 The Sylow 2-subgroups of $Sz(2^d)$ e di $PSU(3, 2^n)$ admit a solvable group of automorphisms which permutes transitively their involutions (see Remark XI.3.7.c in [9]).

Moreover

(a) Let S be a Sylow 2-subgroup of $Sz(2^d)$; then $|S| = 2^{2d}$ and $|\Omega_1(S)| = 2^d$ and there is an automorphism $\alpha \in \mathrm{Aut}(S)$ of order $2^d - 1$ which permutes transitively the involutions of S. The semidirect product $G = S\langle\alpha\rangle$ is a csc-group.

(b) Let S be a Sylow 2-subgroup of $\mathrm{PSU}(3, 2^n)$; then $|S| = 2^{3n}$ and $|\Omega_1(S)| = 2^n$ and there is an automorphism $\alpha \in \mathrm{Aut}(S)$ of order $2^n - 1$ which permutes transitively the involutions of S. The semidirect product $G = S\langle\alpha\rangle$ is not a csc-group (since $G/\Omega_1(S)$ is not a csc-group. However there exists $\beta \in \mathrm{Aut}(S)$ of order $2^n + 1$ such that $[\alpha, \beta] = 1$ and the semidirect product of S with the cyclic group $\langle\alpha, \beta\rangle$ is a csc-group.

We also observe that the Suzuki 2-groups S such that $|S| = |\Omega_1(S)|^2$ are classified in [8]: they are the groups $A(2^n, \theta)$ of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a^\theta \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b \in GF(2^n)$ and θ a non-trivial automorphism of odd order of $GF(2^n)$. In particolar not all these groups are Sylow subgroups of a simple Suzuki group. □

Let us consider the Galois group $\mathcal{G}$ of the field extension $GF(p^m)/GF(p)$; we have $\mathcal{G} = \langle \sigma \mid \sigma : GF(p^m) \to GF(p^m)\ ,\ x \mapsto x^p \rangle$ and $\mathcal{G}$ is cyclic of order m. We may consider the following transformation groups of $GF(p^m)$:

- $\mathrm{A}(p^m) = \{x \mapsto x + b \mid b \in V$, the *translation* group, isomorphic to the additive group of $GF(p^m)$;
- the *semilinear* group $\Gamma(p^m) = \{x \mapsto ax^\tau \mid a \in GF(p^m)^\sharp,\ \tau \in \mathcal{G}\}$;
- the subgroup $\Gamma_0(p^m) = \{x \mapsto ax \mid a \in GF(p^m)^\sharp\}$, normal in $\Gamma(p^m)$;

- the *semilinear affine* group

$$\mathrm{A}\Gamma(p^m) = \{x \mapsto ax^\tau + b \mid a \in GF(p^m)^\sharp, \tau \in \mathcal{G}, b \in GF(p^m)\}.$$

We note that the group $\Gamma(p^m)$ is metacyclic, since $\Gamma_0(p^m) \simeq GF(p^m)^\sharp$ and $\Gamma(p^m)/\Gamma_0(p^m) \simeq \mathcal{G}$ are cyclic of order $p^m - 1$ and m respectively.

Proposition 2.5 *Let G be a solvable $\mathsf{csc}_{\{\mathsf{p}\}}$-group with $O_{p'}(G) = \{1\}$. Let P be a Sylow p-subgroup of G, and suppose $P/\Phi(P)$ has order p^m. If $p^m \notin \{5^2, 7^2, 11^2, 23^2, 3^4\}$ then $G/\Phi(P)$ is isomorphic to a subgroup of $A\Gamma(p^m)$. Moreover, if $p \neq 2$ and $m > 1$, then $\Phi(P) = \{1\}$.*

Proof. Let P be a Sylow p-subgroup of G. Since $O_{p'}(G) = \{1\}$, by Lemma 2.1, we have $P \trianglelefteq G$ and $F(G) = P$. If P is cyclic, then G/P is isomorphic to a subgroup of Aut P and we are done.

Otherwise, by eventually considering the quotient $G/\Phi(P)$, we may assume that P is elementary abelian. Then $C_G(P) = P$ and $G = PH$ with $(|P|, |H|) = 1$; if $|P| = p^m$ we may consider H as a subgroup of $\mathrm{GL}(m, p)$. Let Z be the centre of $\mathrm{GL}(m, p)$, and let $\widehat{H} = HZ$. Since H permutes transitively the subgroups of order p of P, it follows that $\widehat{H}$ permutes transitively the elements of order p of P. Therefore, the group $\widehat{G} = P\widehat{H}$ is a solvable 2-transitive group. Such groups have been classified by Huppert (see Theorem XII.7.3 in [9]), and we conclude that $\widehat{G}$ is either a subgroup of the semilinear affine group $\mathrm{A}\Gamma(p^m)$, or p^m lies in $\{3^2, 5^2, 7^2, 11^2, 23^2, 3^4\}$. If $p^m = 3^2$, then $|\mathrm{Aut}\ P| = 2^4 \cdot 3$, and since P is a Sylow 3-subgroup of G, the order of H is a divisor of 16. But then G is isomorphic to a subgroup of $A\Gamma(3^2)$.

The last statement follows from Lemma 2.2. □

The following examples explain the structure of the *exceptional* solvable csc-groups appearing in the statement of Proposition 2.5.

Example 1. Let P be an elementary abelian group of order 5^2. There exists a subgroup H of $\mathrm{GL}(2, 5)$ with $H \simeq \mathrm{SL}(2, 3)$ such that the semidirect product $G = PH$ is a $\mathsf{csc}_{\{5\}}$-group. Such a G is a Frobenius group and turns out to be a csc-group.

Example 2. Let P be an elementary abelian group of order 7^2. There exists a subgroup H of $\mathrm{GL}(2, 7)$ with $H \simeq \mathrm{GL}(2, 3)$ such that the semidirect product $G = PH$ is a $\mathsf{csc}_{\{7\}}$-group. Such a G is a Frobenius group, but G is not a $\mathcal{K}_{\{2\}}$-group since H has a subgroup K of index 2 2 isomorphic to $\mathrm{SL}(2,3)$ and in $H \setminus K$ there are elements of order 2.

Example 3. Let P be an elementary abelian group of order 11^2. There exist subgroups H_1 and H_2 of $\mathrm{GL}(2,11)$ with $H_1 \simeq \mathrm{SL}(2,3)$ and $H_2 \simeq \mathrm{SL}(2,3) \times C_5$ such that the semidirect products $G_1 = PH_1$ and $G_2 = PH_2$ are $\mathsf{csc}_{\{11\}}$-groups. Such groups are Frobenius groups, and are both csc-groups.

Example 4. Let P be an elementary abelian group of order 23^2. There exist subgroups H_1 and H_2 of $\mathrm{GL}(2,23)$ with $H_1 \simeq \mathrm{GL}(2,3)$ e $H_2 \simeq \mathrm{GL}(2,3) \times C_{11}$ such that the semidirect products $G_1 = PH_1$ and $G_2 = PH_2$ are $\mathsf{csc}_{\{23\}}$-groups. Such groups are Frobenius groups, but are not $\mathsf{csc}_{\{2\}}$-groups.

Example 5. Let P be an elementary abelian group of order 3^4. There exist subgroups H_1, H_2 and H_3 of $\mathrm{GL}(4,3)$ of order $2^5 \cdot 5$, $2^6 \cdot 5$ and $2^7 \cdot 5$ respectively (such groups are explicitly described in Example XII.7.4 in [9]) such that the semidirect products $G_1 = PH_1$, $G_2 = PH_2$ and $G_3 = PH_3$ are $\mathsf{csc}_{\{3\}}$-gruppi. The structure of the Sylow 2-subgroups of H_1, H_2 and H_3 shows that G_1, G_2 and G_3 are neither $\mathsf{csc}_{\{2\}}$-groups nor Frobenius groups.

Corollary 2.6 *Let G be a solvable* csc*-group such that $O_{p'}(G) = \{1\}$ and let P be a Sylow p-subgroup of G. If $|P/\Phi(P)| \notin \{5^2, 11^2\}$, then G/P is isomorphic to a subgroup of $\Gamma(p^m)$, where $p^m = |P/\Phi(P)|$.*

Proof. This follows directly from Proposition 2.5 and the discussion in the above examples. □

Remark 2.7 Let G be a (Frobenius) sharply 2-transitive (here G is not necessarily assumed to be solvable) and let $|F(G)| = p^m$. If $p^m \notin \{7^2, 23^2\}$, then G is a csc-group.

Proof. Sharply 2-transitive groups have been classified by Zassenhaus (see Theorem XII.9.1, XII.9.4 in [9]). They are Frobenius groups, whose kernel P is an elementary abelian p-group and the action of G on P permutes transitively the elements of $P^\sharp$. The Frobenius complement in such groups is metacyclic, with 7 exceptions, 4 of which give rise to solvable groups (described in Examples $1, 2, 3, 4$), and the remaining are described in the following examples. □

Example 6. Let P be an elementary abelian group of order 11^2. There exists a subgroup H of $\mathrm{GL}(2, 11)$ with $H \simeq \mathrm{SL}(2, 5)$ such that the semidirect product $G = PH$ is a sharply 2-transitive group. One may check that G is a csc-group.

Example 7. Let P be an elementary abelian group of order 29^2. There exist subgroups H_1 and H_2 of $\mathrm{GL}(2, 23)$ with $H_1 \simeq \mathrm{SL}(2, 5) \times C_7$ and $H_2 \simeq \mathrm{SL}(2, 5)$ such that the semidirect products $G_1 = PH_1$ and $G_2 = PH_2$ are Frobenius groups (G_1 is sharply 2-transitive) One may check that G_1 are G_2 are csc-groups.

Example 8. Let P be an elementary abelian group of order 59^2. There exist subgroups H_1 and H_2 of $\mathrm{GL}(2, 59)$ with $H_1 \simeq \mathrm{SL}(2, 5) \times C_{29}$ and $H_2 \simeq \mathrm{SL}(2, 5)$ such that the semidirect products $G_1 = PH_1$ and $G_2 = PH_2$ are Frobenius groups (G_1 is sharply 2-transitive) One may check that G_1 are G_2 are csc-groups.

There is a further *exceptional* case which is not a sharply 2-transitive group, e which therefore does not appear in Zassenhaus' list. Such a group comes from Hering's list in [6], classifying 2-transitive groups of affine type (see also Remark XII.7.5 in [9]).

Example 9. Let P be an elementary abelian group of order 19^2. There exist subgroups H_1, H_2 and H_3 of $\mathrm{GL}(2, 19)$ with $H_1 \simeq \mathrm{SL}(2, 5)$, $H_2 \simeq \mathrm{SL}(2, 5) \times C_3$ and $H_3 \simeq \mathrm{SL}(2, 5) \times C_9$, such that $G_1 = PH_1$, $G_2 = PH_2$ and $G_3 = PH_3$ are csc_{19}-groups. One may check that G_1 is a Frobenius group and a csc-group. On the other hand, from the structure of H_2 and H_3 it follows that G_2 and G_3 are neither csc_3-groups nor Frobenius groups. We observe that G_3 is a 2-transitive group.

A set of generators for Frobenius complements of the groups described in Examples 1-4 and 6-8 in terms of matrices of $M_2(GF(p))$ ($p \in \{5, 7, 11, 29, 59\}$) is given in Remark XII.9.5 in [9]. For completeness we observe that the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 16 & 17 \end{pmatrix}$$

with coefficients in $GF(19)$ generate the complement H_1 of the csc-group G_1 of Example 9.

Corollary 2.8 *Let G be a solvable* csc*-group, $p \in \pi(|G|)$. If P is a Sylow p-subgroup of G and P is neither cyclic, nor quaternions, then P is normal in G.*

Proof. We argue by induction on the order of G. We distinguish two cases.

- $O_p(G) \neq \{1\}$.

If in $\overline{G} = G/O_p(G)$, $\overline{P}$ is neither cyclic nor quaternions, by induction $\overline{P} \trianglelefteq \overline{G}$, so that $P \trianglelefteq G$. Suppose P is elementary abelian; then if $P/O_p(G)$ is cyclic, there exists $x \in G \setminus O_p(G)$ of order p, and for every $y \in O_p(G)$ with $y \neq 1$, the subgroups $\langle x \rangle$ and $\langle y \rangle$ can not be conjugate in G, a contradiction.

Hence $p = 2$ and suppose P has the structure as in Lemma 2.2 (3). If $P/O_p(G)$ is cyclic of order 4 or quaternions, there would be involutions in both $O_2(G)$ and $G \setminus O_2(G)$, a contradiction. If $P/O_2(G)$ is cyclic of order 2, then there are elements of order 4 both in $O_2(G)$ and in $G\ O_2(G)$, again a contradiction. Hence in this case $P \trianglelefteq G$.

- $O_p(G) = \{1\}$, so that $(|F(G)|, p) = 1$.

If $F(G)$ is cyclic, then $G/F(G)$ is abelian, hence cyclic being a csc-group. If $F(G) = Q_8 \times C_n$ (n odd) then, since G is a csc-group, $G/F(G)$ must be a $2'$-group, and again, from the structure of $\mathrm{Aut}(Q_8 \times C_n)$ (and since in a solvable group $C_G(F(G)) \leq F(G)$), $G/F(G)$ is abelian, and then cyclic. In both cases we have shown that $G/F(G)$ is cyclic, and this is not possible by the hypothesis.

So let us suppose that a Sylow q-subgroup Q of $F(G)$ is neither cyclic nor quaternions. Then by induction and the above reasoning, we deduce $Q \in \mathsf{Syl}_q(G)$. We may assume, up to considering the quotient $G/\Phi(Q)$, that Q is elementary abelian. Then $G/C_G(Q)$ has the structure described in Proposition 2.5; in particular P centralizes Q. We may therefore consider G/Q; proceeding in this way, after a finite number of steps we are reduced to the case when $F(G)$ is cyclic. This is again a contradiction.

Hence p divides $|F(G)|$ and $P \trianglelefteq G$. □

Corollary 2.9 *Let G be a solvable* csc*-group. Then the derived length of G is at most 4.*

Proof. We argue by induction on the order of G. If G has at least 2 minimal normal subgroups N_1 and N_2, the induction hypothesis applied to G/N_1 and to G/N_2 gives that both G/N_1 and G/N_2 have derived length at most 4, and the same holds for G, since $N_1 \cap N_2 = \{1\}$.

Let us suppose that G has a unique minimal normal subgroup N. Then N is a p-group for a certain prime p, so that $F(G)$ is also a p-group. Let P be a Sylow p-subgroup of G; then $F(G) \leq P$ and since $F(G)$ is neither cyclic nor quaternions, also P is neither cyclic nor quaternions. By Corollary 2.8 we have $P \trianglelefteq G$, so that $F(G) = P$. By Proposition 2.5 and Corollary 2.9 we have the following cases

- $F(G)$ is elementary abelian of order p^m and G is isomorphic to a subgroup of $A\Gamma(p^m)$. In this case we conclude by observing that $A\Gamma(p^m)$ has derived length 3.
- $F(G)$ is a 2-group with structure as in Lemma 2.2 and $G/\Phi(F(G))$ is isomorphic to a subgroup of $A\Gamma(2^m)$ where $|F(G)/\Phi(F(G))| = 2^m$. Then $G^{(3)} \leq \Phi(F(G))$ and since $\Phi(F(G))$ is abelian, also in this case we are done.
- $|F(G)| = 5^2$ and $G/F(G) \simeq \mathrm{SL}(2,3)$. Then G satisfies the thesis (see Example 1).
- $|F(G)| = 11^2$ e $G/F(G) \simeq \mathrm{SL}(2,3)$ or $G/F(G) \simeq \mathrm{SL}(2,3) \times C_5$. Then G satisfies the thesis (see Example 3). □

3 Simple, almost-simple and quasisimple csc-groups.

We observe that if G is a csc-group, then G has at most $\phi(n)$ conjugacy classes of elements of order n, where ϕ denotes Euler's function. In particular G has a unique class of involutions; we shall also use the fact that $\phi(3) = 2$ and $\phi(4) = 2$. We start by giving the list of the simple groups with only one class of involutions. This may be found in [19], here we present a more detailed statement:

Proposition 3.1 *The non-abelian simple groups with precisely one class of involutions are those in the following List (A)*

(a) Groups of Lie type in odd characteristic

(a1) $PSL(2,q)$, $q > 3$;

(a2) $PSL(3,q)$;

(a3) $PSL(4,q)$, $q \equiv 5 \mod 8$;

(a4) $PSU(3,q)$;

(a5) $PSU(4,q)$, $q \equiv 3 \mod 8$;

(a6) ${}^3D_4(q)$;

(a7) $G_2(q)$;

(a8) ${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$.

(b) Groups of Lie type in characteristic 2

(b1) $PSL(2,q)$, $q > 2$;

(b2) $PSL(3,q)$;

(b3) $PSU(3,q)$, $q > 2$;

(b4) $Sz(q) = {}^2B_2(q)$, $q = 2^{2m+1}$, $m \geq 1$.

(c) Alternating groups $\mathbb{A}_n$, $5 \leq n \leq 7$.

(d) Sporadic groups

$$M_{11}, \quad M_{22}, \quad M_{23}, \quad J_1, \quad J_3, \quad McL, \quad Ly, \quad O'N, \quad Th = F_3.$$

□

In the next two theorems we determine the simple and almost-simple groups which are csc-groups. We shall prove them at the same time. Let us introduce the following list of simple groups

$$List(B)$$

(a) Groups of Lie type in odd characteristic

(a1) $PSL(2,q)$, $q > 3$, $q = p^m$, m odd;

(b) Groups of Lie type in characteristic 2 (b1) $PSL(2,q)$, $q > 2$;

(b2) $PSL(3,2)$;

(b3) $Sz(q) = {}^2B_2(q)$, $q = 2^{2m+1}$, $m \geq 1$.

(c) Alternating group $\mathbb{A}_5$;

(d) Sporadic groups M_{11}, M_{23}, J_1.

Note that $PSL(3,2) \simeq PSL(2,7)$, $\mathbb{A}_5 \simeq PSL(2,4) \simeq PSL(2,5)$. Therefore a simple group is in the List(B) if and only if it is isomorphic to one in the following List(C):

(i) $PSL(2,q)$, $q \geq 4$, $q = p^m$, m odd if p odd;

(ii) $Sz(q)$, $q = 2^{2m+1}$, $m \geq 1$.

(iii) M_{11}, M_{23}, J_1

Lemma 3.2 *The finite simple groups which are* csc*-groups are those in List(*B*).*

Lemma 3.3 *Let* $S < G \leq$ *Aut* S*, with* S *simple non-abelian. Then* G *is* csc*-group if and only if* G *is of the form* $G = S : \langle\psi\rangle$ *where* S *is isomorphic to* $PSL(2,q)$*,* $q \geq 4$*,* $q = p^m$*,* m *odd if* p *odd, or to* $Sz(2^{2m+1})$*,* $m \geq 1$*, and* ψ *is a field automorphism of* S *of order coprime to* $|S|$*.*

Note that in particular if G is almost simple with socle S, and G is a csc-group, then S is a csc-group. To prove Theorem 3.3, we shall use the following result.

Lemma 3.4 *[19, Lemma 2] Let* S *be a simple group with at least 2 conjugacy classes of involutions. Then not all involutions in* S *are conjugate in Aut* S*.* □

Proof of Lemmas 3.2, 3.3. Assume $S \leq G \leq$ Aut S, with S simple non-abelian. If G is a csc-group, then, by Lemma 3.4, S has only one class of involutions hence, by Proposition 3.1, S is in List(A). For every S in List(A) we determine in which cases an almost-simple group G with socle S is a csc-group. For root subgroups we use the notation in [3].

Groups of Lie type in odd characteristic

- $S = G_2(q)$, $q = p^f$ odd. Here S has two subgroups of order p which are not conjugate in Aut S. In fact, if α and β are orthogonal roots, with β long, then $x_\alpha(1)x_\beta(1)$ and $x_\beta(1)$ have centralizers of different order in S, hence $\langle x_\alpha(1)x_\beta(1)\rangle$ and $\langle x_\beta(1)\rangle$, which have order p, are not conjugate in Aut S.

- $S = {}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$. If P, P_1 are distinct Sylow 3-subgroups of G, then $P \cap P_1 = 1$ ([18], Theorem (2)). Moreover there are subgroups X, Y of P of order 3 such that $X \leq Z(P)$ and $Y \not\leq Z(P)$. Hence X and Y are not conjugate in Aut S.

- $S = {}^3D_4(q)$, $q = p^m$, p odd (note that $S \leq P\Omega_8^+(q^3)$). We have Aut $S = S : \langle\varphi\rangle$, where φ is a field automorphism of order $3m$. The elements $x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$ and $x_{\alpha_2}(k)$ for $k \in \mathbb{F}_{q^3}^*$ are not conjugate in Aut S (since they are not conjugate in $P\Omega^+(q^3)$ and φ is a field automorphism), hence $\langle x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)\rangle$ and $\langle x_{\alpha_2}(1)\rangle$, which have order p, are not conjugate in Aut S.

- $S = PSU(4,q)$, $q \equiv 3 \mod 8$, $q = p^m$, p odd (note that $S \leq PSL(4,q^2)$). We have Aut $S = PGU(4,q) : \langle\varphi\rangle$, where φ is a field automorphism of order $2m$. The elements $x_{\alpha_1}(1)x_{\alpha_3}(1)$ and $x_{\alpha_2}(k)$ for $k \in \mathbb{F}_{q^2}^*$ are not conjugate in Aut S (since they are not conjugate is $PGL(4,q)$ and φ is a field automorphism), hence $\langle x_{\alpha_1}(1)x_{\alpha_3}(1)\rangle$ and $\langle x_{\alpha_2}(1)\rangle$, which have order p, are not conjugate in Aut S.

- $S = PSL(4,q)$, $q \equiv 5 \mod 8$, $q = p^m$, p odd. We have Aut $S = PGL(4,q) : \langle\varphi\rangle : \langle\delta\rangle$, where φ is a field automorphism of order m, δ is the graph automorphism. The elements $x_{\alpha_1}(1)x_{\alpha_3}(1)$ and $x_{\alpha_2}(k)$ for $k \in \mathbb{F}_q^*$ are not conjugate in Aut S (since they are not conjugate is $PGL(4,q)$ and $\langle\varphi,\delta\rangle$ fixes the set $\{x_{\alpha_2}(k) \mid k \in \mathbb{F}_q^*\}$), hence $\langle x_{\alpha_1}(1)x_{\alpha_3}(1)\rangle$ and $\langle x_{\alpha_2}(1)\rangle$, which have order p, are not conjugate in Aut S.

- $S = PSL(3,q)$, $q = p^m$, p odd. We have Aut $S = PGL(3,q) : \langle\varphi\rangle : \langle\delta\rangle$, where φ is a field automorphism of order m, δ is the graph automorphism. The elements $x_{\alpha_1}(1)x_{\alpha_2}(1)$ and $x_{\alpha_1}(k)$ for $k \in \mathbb{F}_q^*$ are not conjugate in Aut S (since $x_{\alpha_1}(1)x_{\alpha_2}(1)$ is regular, so that it lies in a unique Sylow p-subgroup of S, while $x_{\alpha_1}(k)$ is not regular), hence $\langle x_{\alpha_1}(1)x_{\alpha_2}(1)\rangle$ and $\langle x_{\alpha_1}(1)\rangle$, which have order p, are not conjugate in Aut S.

- $S = PSU(3,q)$, $q = p^m$, p odd (note that $S \leq PSL(3,q^2)$). We have Aut $S = PGU(3,q) : \langle\varphi\rangle$, where φ is a field automorphism of order $2m$. In S there are regular and non-regular unipotent elements of order p. Then, as for $PSL(3,q)$, we conclude that there are subgroups X, Y of order p of S which are not conjugate in Aut S.

- $S = PSL(2,q)$, $q = p^m$, p odd. We have Aut $S = PGL(2,q) : \langle \varphi \rangle$, where φ is a field automorphism of order m. By [7], Satz 8.5, the groups $PSL(2,q)$ are groups with partition and the Sylow r-subgroups are cyclic for $r \neq 2, p$. Assume X and Y are cyclic subgroups of S of the same order k. Then k divides only one among p, $\frac{q-1}{2}$, $\frac{q+1}{2}$ (which are pairwise coprime), so that X and Y are conjugate in S unless $k = p$.

So assume $k = p$. Then S has only one class of subgroups of order p if and only if the subgroups of order p in a Sylow p-subgroup P of S are conjugate in $N(P)$ (since two distinct Sylow p-subgroups intersect trivially). We may assume P are the unitriangular upper matrices, and H are the triangular matrices in S. Then $N(P) = HP$, and the unipotent elements in P fall in 2 classes under the action of H. Therefore there is a unique class of subgroups of order p in P under H if and only if $\mathbb{F}_q^* = (\mathbb{F}_q^*)^2\, \mathbb{F}_p^*$, i.e. if and only if m is odd.

Therefore the finite simple group $PSL(2,q)$, with $q = p^m$, p odd is a csc-group if and only if m is odd. We note that the group $PSL(2,3) \simeq A_4$ is a csc-group. Hence we may state that the groups $PSL(2,p^m)$ with odd p are csc-groups if and only if m is odd.

Now assume $S < G \leq$ Aut S. We determine in which cases G is a csc-group. We have Out $S \simeq C_2 \times C_m$. By [13], Aut S splits over S if and only if $(\frac{p^m-1}{2}, 2, m) = 1$, i.e. if and only if m is odd. So let us first assume m is odd. Then we know that $PSL(2,p^{m'})$ is a csc-group for each divisor m' of m. Then we use

Proposition 3.5 *Let $S = PSL(2,q)$, $q = p^m$, p odd, m odd, $q > 3$, $S < G \leq$ Aut S. Then G is a csc-group if and only if $G = S : \langle \psi \rangle$, where ψ is the field automorphism of order k (hence $k|m$), where $(|S|, k) = 1$.*

Proof. Assume G is a csc-group. Since Out S is cyclic, and the only subgroup of order 2 in Out S corresponds to $PGL(2,q)$ which always splits over S, we must have $G \cap PGL(2,q) = 1$, so that $G = S : \langle \psi \rangle$, where $\psi = \varphi^{m/k}$. Moreover we must have $(|S|, k) = 1$. On the other hand, if $G = S : \langle \psi \rangle$, with $\psi = \varphi^{m/k}$, with $(|S|, k) = 1$, then G is a csc-group, since $C_S(\varphi^{m/h}) \simeq PSL(2,p^h)$ is a csc-group for each divisor h of m. □

To conclude the case $S = PSL(2, p^m)$, assume finally m is even, $m = 2n$. Then we have seen that S is not a csc-group, since there are 2 classes of subgroups of order p. We show that there are no groups G with socle S which are csc-groups. In this case Aut S does not split over S, and Out $S \simeq C_2 \times C_{2k}$. Let τ be the involution in $\langle\varphi\rangle$. Suppose for a contradiction that such a G exists. We note that $G \cap PGL(2, q) = G \cap (S : \langle\tau\rangle) = 1$, since $PGL(2, q)$ splits over S, $PGL(2) = S : \langle\sigma\rangle$. There exists $\alpha \in G$ such that

$$\langle\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\rangle^{\alpha} = \langle\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}\rangle$$

where $\mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$, and α must be of the form $\alpha = i_s\sigma\varphi^i$, for a certain $s \in S$, $\varphi^i \neq 1$ (otherwise $PGL(2, q) = S\langle\alpha\rangle \leq G$, a contradiction). Hence $S < S\langle\sigma\varphi^i\rangle \leq G$. Let $[S\langle\sigma\varphi^i\rangle : S] = 2^a f$, with odd f, $a \geq 1$. By taking the f-power of $\sigma\varphi^i$ we get $\sigma\varphi^j \in S\langle\sigma\varphi^i\rangle$, with $[S\langle\sigma\varphi^j\rangle : S] = 2^a$. However, if $a > 1$, the minimal subgroup of $S\langle\sigma\varphi^j\rangle/S$ is $S : \langle\tau\rangle/S$, so that $G \geq S : \langle\tau\rangle$, a contradiction. Hence $[S\langle\sigma\varphi^j\rangle : S] = 2$, i.e. $S\langle\sigma\varphi^j\rangle = S\langle\sigma\tau\rangle$ (which does not split over S). We prove that there exists an element of order 4 in $S\langle\sigma\tau\rangle \setminus S$, so in $G \setminus S$ there is an element of order 4. This is a contradiction, since S always has elements of order 4, m being even.

To show that in $S\langle\sigma\tau\rangle \setminus S$ there is an element of order 4, it is enough to exhibit an element $\delta \in PGL(2, q) \setminus S$ such that $\delta\delta^\tau$ has order 2 (if $\beta = i_\delta\tau$ then $\beta^2 = i_{\delta\delta^\tau}$).

Suppose $p^n \equiv 3 \bmod 4$, $2^h \| (p^n + 1)$ and take $u \in \mathbb{F}_q^*$ of order 2^{h+1}. Let

$$\delta = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in PGL(2, q) \setminus S$$

Then

$$\delta\delta^\tau = \begin{pmatrix} u^{1+p^n} & 0 \\ 0 & 1 \end{pmatrix}$$

Since $(u^{1+p^n})^2 = 1$, while $u^{1+p^n} \neq 1$, it follows that $\delta\delta^\tau$ has order 2.

Suppose $p^n \equiv 1 \bmod 4$, $2^h \| (p^n - 1)$ and take $u \in \mathbb{F}_q^*$ of order 2^{h+1}. Let

$$\delta = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} \in PGL(2, q) \setminus S$$

Then

$$\delta\delta^\tau = \begin{pmatrix} u & 0 \\ 0 & u^{p^n} \end{pmatrix} \in PGL(2, q) \setminus S$$

Then $u^2 = u^{2p^n}$, while $u \neq u^{p^n}$, hence $\delta\delta^\tau$ has order 2.

We now deal with simple groups of Lie type in characteristic 2.

- $S = {}^2B_2(q)$, $q = 2^{2m+1}$, $m \geq 1$ ($S \leq Sp_4(q)$). We have Aut $S = S : \langle\varphi\rangle$, where φ is a field automorphism of order $2m+1$. We can deal with this case in the same way as in case $PSL(2,q)$, since S is a group with partition. However, in this case the cyclic subgroups of order 4 are all conjugate in S, since there are 2 classes of elements of order 4, and x is not conjugate to x^{-1} if x has order 4 (the Sylow 2-subgroup has exponent 4). Hence S is a csc-group. We observe that also ${}^2B_2(2) \simeq 5:4$, which is solvable, is a csc-group.

If $S < G \leq$ Aut S, then G is a csc-group if and only if $G = S : \langle\psi\rangle$, with $\psi = \varphi^{(2m+1)/k}$, with $(|S|, k) = 1$ (since $C_S(\psi) \simeq {}^2B_2(2^{2m+1)/k})$ is a csc-group even when it is not simple).

- $S = PSL(3,q)$, $q = 2^m$. Let r be a prime divisor of $q-1$, and assume $r \neq 3$. If

$$x = \text{diag}(\alpha, \alpha, \alpha^{-2}) \quad , \quad y = \text{diag}(\alpha, 1, \alpha^{-1})$$

then x and y have order r and act in a different way on the projective plane. In particular $\langle x\rangle$ and $\langle y\rangle$ are not conjugate in Aut S. We are left with $PSL(3,2)$ and $PSL(3,4)$.

We have $PSL(3,2) \simeq PSL(2,7)$ which is a csc-group, while Aut $(PSL(3,2)) \simeq PGL(2,7)$ is not a csc-group.

Finally $PSL(3,4)$ is not a csc-group since it has 3 classes of elements of order 3. Moreover if $S < G \leq$ Aut S, then G splits over S, since Aut S splits over S, and G is not a csc-group, since in S there are elements of order 2 and 3.

- $S = PSL(2,q)$, $q = 2^m$, $m \geq 2$. Here we argue as in the case q odd. However here the Sylow 2-subgroup is elementary abelian, so that S is a csc-group. We note that $PSL(2,2) \simeq S_3$ is a csc-group. Since Aut $S = S : \langle\varphi\rangle$, where φ is a field automorphism, then $S < G \leq$ Aut S is a csc-group if and only if $G = S : \langle\psi\rangle$, ψ in $\langle\varphi\rangle$ of order k, with $(|\,S\,|, k) = 1$ (since $C_S(\psi) \simeq PSL(2, 2^{m/k})$ is a csc-group).

- $PSU(3,q)$, $q = 2^m$, $m \geq 2$ (note that $S \leq PSL(3,q^2)$). We have Aut $S = PGU(3,q) : \langle\varphi\rangle$, where φ is a field automorphism of order $2m$.

Let r be a primitive prime divisor of $2^{2m}-1$: r exists if $m = 2$ or $m \geq 4$. Then r divides 2^m+1, and $r \neq 3$, since 3 divides 2^2-1. S contains a copy of $C_r \times C_r$: for a suitable basis of the 3-dimensional vector space over $\mathbb{F}_{q^2}$, the non-singular Hermitian scalar product can be represented by the identity matrix. Therefore the elements

$$x = \mathrm{diag}(\alpha, \alpha, \alpha^{-2}) \quad , \quad y = \mathrm{diag}(\alpha, 1, \alpha^{-1})$$

for $\alpha \in \mathbb{F}_{q^2}^*$ of order r are in S, and act in different way on the projective plane over $\mathbb{F}_{q^2}$. In particular $\langle x \rangle$ and $\langle y \rangle$ are not conjugate in Aut S. We are left with $PSU(3, 8)$. In this case there are elements of order 3 in S with centralizers of different orders.

Alternating groups. We have $A_5 \simeq PSL(2, 5)$, $A_6 \simeq PSL(2, 9)$, so that A_5 is a csc-group, $S_5 =$ Aut A_5 is not a csc-group. If G is such that $A_6 \leq G \leq$ Aut A_6, then G is not a csc-group; A_7 is not a csc-group since it has 2 elements of order 3 with centralizers of different order, S_7 is not a csc-group, since it splits over A_7.

Sporadic groups. By [1], the groups M_{11}, M_{23}, J_1 are csc-groups, while M_{22}, ON, Ly, Th are not csc-groups (since they contain elements of order 4 with centralizers of different orders) and J_3, McL are not csc-groups (since they contain elements of order 3 with centralizers of different orders). If $S < G \leq$ Aut S, then S is not a csc group, since G splits over S (and $[G : S] = 2$).

The proof of Lemmas 3.2 and 3.3 are completed. □

Lemma 3.6 *Let G be a quasisimple* csc*-group which is not simple. Then $G \simeq \mathrm{SL}(2, p^m)$ with $p \neq 2$ and m odd.*

Proof. The group $\mathrm{SL}(2, p^m)$ is certainly a csc-group if $\mathrm{PSL}(2, p^m)$ is. The groups $\mathrm{SL}(2, 2^n)$ and $\mathrm{Sz}(2^{2n+1})$ with $n \geq 2$ do not admit central extensions, and the same holds for M_{11}, M_{23} and J_1 (see [1]). Again using [1] one can check that no non-trivial central extension of $\mathrm{Sz}(8)$ is a csc-group. □

4 Monolithic csc-groups.

We introduce the following

Definition 4.1 *A* csc*-group is called* csc*-monolithic (or, simply, monolithic) if either $F^*(G)$ is a p-group or $F^*(G) = E(G)$.*

Lemma 4.2 *Let G be a group with a normal elementary abelian subgroup N of order p^{2m} and such that G/N is isomorphic to a subgroup of $\Gamma\mathrm{L}(2, p^{2m})$ containing $\mathrm{SL}(2, p^{2m})$. If $N = C_G(N)$ and if the action induced by G on N is the natural one, then G is not a $\mathsf{csc_p}$-group.*

Proof. Let G_0 be the normal subgroup of G containing N such that $G_0/N \simeq \mathrm{SL}(2, p^{2m})$. It is enough to show that there exists an element of order p in $G_0 \setminus N$. We distinguish two cases.

- $p \neq 2$.

Let z be an element of order 2 of G_0 and let $\overline{G} = G/N$. Then $Z(\overline{G}) = \langle \overline{z} \rangle$ and z induces inversion on N. Let $\overline{x}$ be an element of order $2p$ of $\overline{G}$ and let x be a preimage of $\overline{x}$ in G. Then $x^{2p} \in N$ and x has order either $2p$ or $2p^2$. If the order of x is $2p$, then $x^2 \in G \setminus N$ is an element of order p.

If x has order $2p^2$, then x^{2p} would be a non-trivial element of N and as such it should be inverted by x^{p^2}, a contradiction (this argument shows that the Sylow p-subgroups of G have exponent p).

- $p = 2$.

Let x be an element of order 3 of G_0 and let $\overline{x}$ be the corresponding element of $\overline{G} = G/N$. The minimal polynomial of $\overline{x}$ as an element of $\mathrm{SL}(2, 2^{2m})$ is $T^2 + T + 1$; hence $C_N(x) = \{1\}$, for every $y \in N$ we have $yy^xy^{x^2} = 1$ and, in particular, $\langle y, y^x \rangle$ is a x-invariant subgroup of N.

In G_0/N there exists an element $\overline{a}$ of order 2 inverting $\overline{x}$. Let a be a preimage of $\overline{a}$ in G which is a 2-element. Let us fix $y \in C_N(a)$ such that $y \neq 1$: then $(y^x)^a = (y^a)^{x^{-1}} = y^{x^2} \in \langle y, y^x \rangle$. Therefore, if $T = \langle y, x, a \rangle$ and $L = \langle y, y^x \rangle$ it follows that L is an elementary abelian normal subgroup of order 4 of T, such that $T/L \simeq S_3$.

As x is an element of order 3 of T acting fixed-point-freely on L, it follows that $T \simeq S_4$. In particular in T there exists an element b of order 2 inverting x. Certainly $b \notin N$. □

Lemma 4.3 *Let G be a group with a normal elementary abelian subgroup N of order 3^n. Assume that*

(i) $G/N \simeq \mathrm{SL}(2, q)$ *with q odd;*

(ii) the involutions of G induce inversion on N.

Then G is not a csc$_3$*-group.*

Proof. It is enough to exhibit an element of order 3 in $G \setminus N$. Let $\overline{G} = G/N$, $\overline{x} \in \overline{G}$ be an element of order 6 (such an element exists since the order of $\mathrm{SL}(2, q)$ is divisible by 3 and $Z(\overline{G})$ has order 2) and let x be a preimage of $\overline{x}$ in G. Then $x^6 \in N$ and x has order 6 or 18. If $|x| = 6$, then x^2 is an element of order 3 not in N. If $|x| = 18$, then x^6 would be a non-trivial element of N and as such, it would be inverted by x^9, a contradiction (this argument shows that the Sylow 3-subgroups of G have exponent 3). □

Lemma 4.4 *Let G be a* csc*-group and suppose $F^*(G)$ is a p-group. Then $G/\Phi(F^*(G))$ is isomorphic to a subgroup of the semilinear affine group $A\Gamma(p^m)$ where $p^m = |F^*(G)/\Phi(F^*(G))|$ or G is a Frobenius group with kernel $F^*(G)$ which is elementary abelian of order p^2 and one (and only one) of the following holds:*

(i) $p = 5$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 3)$*;*

(ii) $p = 11$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 3)$*;*

(iii) $p = 11$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 3) \times C_5$*;*

(iv) $p = 11$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 5)$*;*

(v) $p = 19$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 5)$*;*

(vi) $p = 29$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 5)$*;*

(vii) $p = 29$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 5) \times C_7$*;*

(viii) $p = 59$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 5)$*;*

(ix) $p = 59$ *and* $G/F^*(G) \simeq \mathrm{SL}(2, 5) \times C_{29}$.

Moreover $F^*(G)$ is a Sylow p-subgroup of G whose structure is described in Lemma 2.2.

Proof. We may assume, up to considering $G/\Phi(F^*(G))$, that $F^*(G)$ is elementary abelian, of order p^m say. One then has $C_G(F^*(G)) = F^*(G)$ and $\overline{G} = G/F^*(G)$ acts on $F^*(G)$ as a subgroup of $GL(m,p)$. Since $\overline{G}$ permutes transitively the subgroups of order p of $F^*(G)$, we may obtain a group $\widetilde{G} = \overline{G}Z$ (where Z is the center of $GL(m,p)$) which permutes transitively the elements of order p of $F^*(G)$, and such that $\overline{G} \trianglelefteq \widetilde{G}$.

We may apply the already mentioned classification theorem by Hering ([6], see also Remark XII.7.5 in [9]), to conclude that for $\widetilde{G}$ there are the following possibilities:

(1) There exist $h, k \in \mathbb{N}$ with $m = kh$ and $\mathrm{SL}(k,p^h) \leq \widetilde{G} \leq \Gamma L(k,p^h)$. Since $\overline{G}$ is normal in $\widetilde{G}$ we must have $\mathrm{SL}(k,p^h) \leq \overline{G} \leq \Gamma L(k,p^h)$. On the other hand, $\overline{G}$ is a csc-group, so that, by Lemma 3.2, $k = 2$ and Lemma 4.2 allows to exclude this case.

(2) There exists $h, k \in \mathbb{N}$ with $m = kh$, $\widetilde{G} \simeq Sp(k,p^h)$. Then also $\overline{G} \simeq Sp(k,p^h)$, hence, by Lemma 3.2, $k = 2$ and Lemma 4.2 allows to exclude this case.

(3) We have $p = 2$, $m = 6h$ and $\widetilde{G} \simeq G_2(2^h)$. Then also $\overline{G} \simeq G_2(2^h)$, but, by Lemmas 3.2, 3.3, $G_2(2^h)$ is not a csc-group and this case is excluded.

(4) $\widetilde{G}$ contains a normal extraspecial subgroup of order 2^{m+1}. If $m = 2$, then $p \in \{3, 5, 7, 11, 23\}$ and, by Proposition 2.5 and the observations in Examples 1, 2, 3 and 4 we are in one of the cases (i), (ii) or (iii). If $m > 2$, then $m = 4$ and $p = 3$: this case can not occur due to the observations in Example 5.

(5) We have $\widetilde{G}^{(\infty)} \simeq \mathrm{SL}(2,5)$, where $\widetilde{G}^{(\infty)}$ denotes the last term of the derived series of $\widetilde{G}$. Then also $\overline{G}^{(\infty)} \simeq \mathrm{SL}(2,5)$ and $p^m \in \{3^4, 11^2, 19^2, 29^2, 59^2\}$. We have to exclude the case $p^m = 3^4$ by Lemma 4.3, while the other possibilities give rise to one of the cases (iv) - (ix) (see Examples 6, 7, 8 e 9).

(6) We have $\widetilde{G} \simeq A_6$ and $p^m = 2^4$; this case can not occur since A_6 is not a csc-group.

(7) We have $\widetilde{G} \simeq A_7$ and $p^m = 2^4$; this case can not occur since A_7 is not a csc-group.

(8) We have $\widetilde{G} \simeq \mathrm{SL}(2,13)$ and $p^m = 3^6$; this case can not occur by Lemma 4.3.

(9) We have $\widetilde{G} \simeq PSU(3,3^2)$ and $p^m = 2^6$; this case can not occur since $PSU(3,3^2)$ is not a csc-group.

We prove the last statement. If G is solvable, since by hypothesis $F^*(G) = F(G)$ is a p-group, by Lemma 2.1, it follows that $F(G)$ is a Sylow p-subgroup of G. If G is not solvable, then G is isomorphic to one of the groups in (iv) – (ix) and from a direct inspection it follows that again $F^*(G) = F(G)$ is a Sylow p-subgroup of G.

If $p \neq 2$, then one can show that $F^*(G)$ is cyclic or elementary abelian using the same arguments used in the proof of Lemma 2.2. If $p = 2$, then $G/F(G)$ has odd order, so that G is solvable; by Lemma 2.2, one concludes that $F(G)$ has the structure stated in that lemma. □

We have therefore proved

Proposition 4.5 *Let G be a monolithic* csc*-group. Then either $F^*(G) = F(G)$ and G has the structure described in Lemma 4.4, or $F^*(G)$ is a simple or quasisimple group as described in Lemmas 3.2, 3.6 and $G/F^*(G)$ is cyclic of order coprime to the order of $F^*(G)$.* □

5 The general case.

In this section we prove the Theorem stated in the Introduction. We begin by showing that in csc-group at most one composition factor is non-abelian.

Lemma 5.1 *Let $S_1, S_2, \ldots, S_n$ be non-abelian simple groups, $n \geq 2$. Then the direct product $S = S_1 \times S_2 \times \cdots \times S_n$ is not isomorphic to a normal subgroup of a* csc*-group G.*

Proof. By Feit-Thompson's Theorem, there exists an involution $x_i \in S_i$ for each $i = 1, \ldots, n$. The subgroups $\langle (x_1, x_2, \ldots, x_{n-1}, 1) \rangle$ and $\langle (x_1, x_2, \ldots, x_{n-1}, x_n) \rangle$ have the same order, but centralizers of different order in S, hence they are not conjugate in G. □

Lemma 5.2 *Let G be a* csc*-group. Then at most one composition factor of G is non-abelian.*

Proof. If G is solvable, then the result is clear. Let us suppose that G is non-solvable. By Lemma 1.2, without loss of generality we may assume $O_\infty(G) = \{1\}$ and that $F^*(G) = E(G)$ is a direct product of simple groups. By Lemma 5.1, $F^*(G)$ is simple and, since $C_G(F^*(G)) \leq F^*(G)$, we must have $C_G(F^*(G)) = \{1\}$. Then $G/F^*(G)$ is isomorphic to a subgroup of Out $F^*(G)$ which, by the classification of finite simple groups, is solvable. □

Lemma 5.3 *Let G be a* csc*-group with $E(G) \neq \{1\}$. Then $F^*(G) = O_{2'}(F(G)) \times E(G)$, $(|O_{2'}(F(G))|, |E(G)|) = 1$ and $O_2(F(G))$ has order 1 or 2.*

Proof. It is well known that $F^*(G)$ is a central product of $F(G)$ and $E(G)$ (see 31.12 in [2]).

Let us first consider the case $Z(E(G)) = \{1\}$; then $E(G)$ is simple, and it is the unique non-abelian composition factor of G. In this case we clearly have $F^*(G) = F(G) \times E(G)$. If $p \in \pi(F(G)) \cap \pi(E(G))$, then, taken $x \in F(G)$ and $y \in E(G)$ both of order p, we would have $C_G(x)$ non-solvable since it contains $E(G)$, while $C_G(y)$ is solvable, $E(G)$ being the unique non-abelian composition factor of G. Therefore $\langle x \rangle$ e $\langle y \rangle$ are not conjugate in G, a contradiction.

If $Z(E(G)) \neq \{1\}$ then, by Lemma 3.6, $|Z(E(G))| = 2$ and we conclude by considering $G/Z(E(G))$. □

Lemma 5.4 *Let G be a* csc*-group and let P be a non-cyclic Sylow subgroup of $F(G)$. Then P is a (normal) Sylow subgroup of G and P has the structure described in Lemma 2.2. Moreover, $G/PC_G(P)$ is isomorphic to a subgroup of $\Gamma(p^m)$ where $p^m = |P/\Phi(P)|$ or P is abelian, with $|P| \in \{5^2, 11^2, 19^2, 29^2, 59^2\}$ and $G/C_G(P)$ has the structure of one of the Frobenius complements described in Lemma 4.4 (i) – (ix).*

Proof. We argue by induction on the order of G. If G is monolithic, we conclude by Lemma 4.4 and Proposition 4.5. So let N be a minimal normal p'-subgroup of G and let $P_1 \in \mathsf{Syl}_p(G)$. If $\overline{G} = G/N$, then by induction, we have $\overline{P}_1 \trianglelefteq \overline{G}$ and $\overline{P}_1$ has the structure described in Lemma 2.2. Since $\overline{P}_1 \simeq P_1$, also P_1 has the structure described in Lemma 2.2. If $P = P_1$ we are done. We show that if we suppose that $P \neq P_1$, then we get a contradiction. We distinguish two cases:

- P_1 is elementary abelian. Let $x \in P_1 \setminus P$ and let y be a non-trivial element of P; clearly the subgroups $\langle x \rangle$ e $\langle y \rangle$ have the same order and are not conjugate in G, a contradiction.

- $p = 2$ and P_1 has the structure described in Lemma 2.2 (3). If $P_1 \setminus P$ contains an involution, or if P contains an element of order 4 we may conclude by an argument similar to the one used in the previous case. So let $P = \Omega_1(P_1) = Z(P_1)$. Then P centralizes N and the elementary abelian 2-group P_1/P acts faithfully on N; since $|P_1/P| > 2$ we can not have $N = E(G)$; hence $F^*(G) = F(G)$ and N is a minimal normal q-subgroup of G, N is not cyclic, where $q \in \pi(G) \setminus \{2\}$. If $Q \in \mathsf{Syl}_q(F(G))$ then, since Q is not cyclic, from the previous case we get that Q is an elementary abelian Sylow q-subgroup of G. By induction, $\overline{G}/C_{\overline{G}}(\overline{Q})$ has the required structure, in particular the Sylow 2-subgroups of $\overline{G}/C_{\overline{G}}(\overline{Q})$ can not be elementary abelian, a contradiction.

Hence $P \in \mathsf{Syl}_p(G)$; to prove the last statement, we may assume, up to considering $G/\Phi(P)$, that P is elementary abelian. Then $G/C_G(P)$ is a csc-group, and a p'-group permuting transitively the cyclic subgroups of P. Again we conclude by using the above mentioned classification theorem by Hering ([6]), and the proof of Lemma 4.4. □

We remark that due to Lemmas 3.2, 3.3, 3.6, 4.4, 5.3, 5.4, the Theorem stated in the Introduction is proved.

To obtain a more detailed classification of csc-groups, we shall use the following definition and the forthcoming notation.

Definition 5.5 *A* csc*-group G is called minimal in one of the following cases.*

(1) $F^*(G) = F(G)$ *and the following conditions hold*

(i) $F(G)$ *contains a unique non-cyclic Sylow p-subgroup P,*

(ii) every proper normal subgroup of G containing P in not a csc*-group,*

(iii) $\pi(O_{p'}(G)) \subseteq \pi(G/F(G))$*;*

(2) $G = E(G)$*;*

(3) G *is metacyclic.*

Let p be a prime and let

$$p - 1 = \prod_{q \in \pi(p-1)} q^{\alpha(q)}$$

be the factorization of $p - 1$. If $m \in \mathbb{N}^*$, we put

$$\rho_p(m) = \pi(p-1) \cap \pi((p^m - 1)/(p-1))$$

and

$$\epsilon_p(m) = \prod_{q \in \rho_p(m)} q^{\alpha(q)} \quad , \quad \delta_p(m) = \frac{p^m - 1}{p - 1} \cdot \epsilon_p(m).$$

We remark that $\epsilon_2(m) = 1$ and that, in general, $\epsilon_p(m)$ depends on $m \bmod p - 1$.

Lemma 5.6 *Let G be a subgroup of $A\Gamma(p^m)$ containing the Fitting subgroup P of $A\Gamma(p^m)$ and let us write $G = PH$ with $H \leq \Gamma(p^m)$. Then G is a minimal* csc*-group if and only if G is a Frobenius group with complement of order $\delta_p(m)$. Moreover, if $Z = Z(\Gamma(p^m))$, then $P\widehat{H}$ is a sharply 2-transitive group, where $\widehat{H} = HZ$,*

Proof. Let $\pi = \pi(H)$ and for a fixed $g \in P^\sharp$ let $C = C_H(g)$. Let $\Gamma(p^m) = \Gamma_0(p^m)\langle\alpha\rangle$ with $|\langle\alpha\rangle| = m$ and $H_0 = H \cap \Gamma_0(p^m)$.

Suppose for a contradiction that there exists $q \in \pi \setminus \pi(H_0)$. Then a Sylow q subgroup Q of H is conjugate in $\Gamma(p^m)$ to a Sylow q-subgroup of $\langle\alpha\rangle$. In particolar Q is cyclic and $C_P(Q) \neq \{1\}$; hence we may assume $Q \leq C$. Since H_0 and H/H_0 are cyclic, if T is a Hall q'-subgroup of H then $T \trianglelefteq H$ and $H = TQ$. Therefore T permutes transitively the subgroups of order p of P and since T is clearly a csc-group, PT turns out to be a csc-group, a contradiction to minimality of G.

Therefore $\pi(H_0) = \pi$; if $C \neq \{1\}$ and if $r \in \pi(C)$ then there exists an element c of order r in C and an element x of order r in H_0. It follows that $\langle c\rangle$ and $\langle x\rangle$ are subgroups of order r of H, not conjugate in H, and this is a contradiction, since H is a csc-group. Hence $C = \{1\}$ and PH is a Frobenius group.

The minimality condition on G and elementary arithmetic considerations give $|H| = \delta_p(m)$ (in fact $O_\pi(Z) \leq H$).

We have $Z = O_\pi(Z) \times O_{\pi'}(Z)$ and $\widehat{H} = H \times O_{\pi'}(Z)$; since H permutes transitively the subgroups of order p of P, we get that $\widehat{H}$ permutes transitively the elements of $P^\sharp$. Moreover, $C_{\widehat{H}}(g) = C = \{1\}$ and due to the fact that $|\widehat{H}| = p^m - 1$, $P\widehat{H}$ is a sharply 2-transitive group. $\square$

Remark 5.7 Let G be a subgroup of $A\Gamma(p^m)$ containing the Fitting subgroup of $A\Gamma(p^m)$ and let $\pi = \pi((p^m - 1))$. If G is a csc-group and if H is a Hall π-subgroup of G then, from the proof of Lemma 5.6, it follows that $|H|$ divides $p^m - 1$.

We introduce the following classes of groups

- $\mathcal{A}(p^m)$: the class of subgroups of $A\Gamma(p^m)$ considered in Lemma 5.6.

- $\mathcal{S}(2^m)$: the class of groups having a normal subgroup P of order 2^{2m} with $P' = \Omega_1(P) = Z(P) = \Phi(G)$ of order 2^m extended by a cyclic or metacyclic group of order $2^m - 1$ permuting transitively the involutions of P' and of P/P'. If $G \in \mathcal{S}(2^m)$ and $P = F(G)$, then $G/P' \in \mathcal{A}(2^m)$.

- $\mathcal{U}(2^m)$: the class of groups having a normal subgroup P of order 2^{3m} with $P' = \Omega_1(P) = Z(P) = \Phi(G)$ of order 2^m extended by a cyclic or metacyclic group of order $2^{2m} - 1$ permuting transitively the involutions of P' and of P/P'. If $G \in \mathcal{S}(2^m)$ and $P = F(G)$, then $G/P' \in \mathcal{A}(2^{2m})$. Moreover $|C_G(P')| = 2^m(2^m + 1)$.

- $\mathcal{B}(p^2)$: the class of Frobenius groups with elementary abelian kernel of order p^2 and complement isomorphic to $\mathrm{SL}(2,3)$ with $p \in \{5, 11\}$.

- $\mathcal{C}(p^2)$: the class of Frobenius groups with elementary abelian kernel of order p^2 and complement isomorphic to $\mathrm{SL}(2,5)$ with $p \in \{11, 19, 29, 59\}$.

It is straightforward to verify that all groups in the above classes are minimal csc-groups, with trivial center. We shall refer to these as to (minimal) csc-groups of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}, \mathcal{U}$ and we shall say that a csc-group is of type $\mathcal{E}$ if it is simple or quasisimple (as described in Lemmas 3.2, 3.6).

Remark 5.8 The order of a p-complement of a group G in one of the classes $\mathcal{A}(p^m)$, $\mathcal{S}(2^m)$, $\mathcal{U}(p^m)$, $\mathcal{B}(p^2)$ or $\mathcal{C}(59^2)$ is $\delta_p(m)$ and it is therefore the least possible. On the other hand, if $G \in \mathcal{C}(p^2)$ with $p \in \{11, 19, 29\}$ then a p-complement of G has order $5 \cdot \delta_{11}(2)$, $3 \cdot \delta_{19}(2)$ and $2 \cdot \delta_{29}(2)$ respectively.

Remark 5.9 Among the groups of type $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{S}$, $\mathcal{U}$ only those of type $\mathcal{U}$ are not Frobenius groups, and only those of type $\mathcal{C}$ are not solvable.

Remark 5.10 Each class $\mathcal{A}(p^m)$, $\mathcal{B}(p^2)$, $\mathcal{C}(p^2)$, $\mathcal{S}(2^m)$, $\mathcal{U}(2^m)$ clearly contains, up to isomorphisms, a finite number of groups. Moreover the following classes contain just one group:

- $\mathcal{A}(p^m)$ if $(m, p^m - 1) = 1$;

- $\mathcal{S}(2^m)$ if $(m, 2^m - 1) = 1$;
- $\mathcal{U}(2^m)$ if $(m, 2^m - 1) = 1$;
- $\mathcal{B}(p^2)$;
- $\mathcal{C}(p^2)$.

Lemma 5.11 *Let G be a minimal* csc*-group. If $G \neq E(G)$ and if P is the unique non-cyclic Sylow p-subgroup of $F(G)$ then, if $\overline{G} = G/O_{p'}(F(G))$, one of the following holds:*

(1) $F(\overline{G})$ is an elementary abelian p-group of order p^m and $\overline{G}$ belongs to one of the classes $\mathcal{A}(p^m)$, $\mathcal{B}(p^2)$ or $\mathcal{C}(p^2)$;

(2) $F(\overline{G})$ is a non-abelian 2-group; in this case, if $|F(\overline{G})'| = 2^m$ then $\overline{G}$ belongs to one of the classes $\mathcal{S}(2^m)$ or $\mathcal{U}(2^m)$.

Proof. The proof follows immediately from the definition of minimal csc-group and Lemmas 5.4, 5.6. □

Remark 5.12 It is not difficult to show that under the hypothesis and with the notation of Lemma 5.11, if $\overline{G} \in \mathcal{B}(p^2)$, then $O_{p'}(G)$ is a (possibly trivial) 3-group and if $\overline{G} \in \mathcal{C}(p^2)$ then $O_{p'}(G) = \{1\}$.

Lemma 5.13 *Let G be a* csc*-group with $E(G) \neq \{1\}$. Then $E(G)$ is a Hall subgroup of G and a minimal* csc*-group. Moreover there exists in G a complement R of $E(G)$ which is a solvable* csc*-group (and such that $(|E(G)|, |R|) = 1$).*

Proof. We prove that $E(G)$ is a Hall subgroup of G by induction on the order of $G/E(G)$.

By Lemma 5.3, $F^*(G) = O_{2'}(F(G)) \times E(G)$. If $O_{2'}(F(G)) \neq \{1\}$ we conclude by the induction hypothesis applied to $G/O_{2'}(F(G))$. On the other hand, if $O_{2'}(F(G)) = \{1\}$, then $F^*(G) = E(G)$ is simple (we may in fact without loss of generality assume $Z(E(G)) = \{1\}$ by considering $G/Z(E(G))$) and we conclude by Lemma 3.3.

The existence of R follows by the Schur-Zassenhaus theorem. Morever R is solvable by Lemma 5.2, and a csc-group since $R \simeq G/E(G)$. □

Lemma 5.14 *Let G be a* csc*-group and let P be a non-cyclic Sylow p-subgroup of $F(G)$. Then there exists a Hall subgroup H of G containing P, such that H is a minimal* csc*-group. Any subgroup of G with the same properties is conjugate to H in G.*

Proof. We make induction on the order of G. If $E(G) \neq \{1\}$ then, by Lemma 5.13, in G there exists a complement R of $E(G)$. Applying to R the inductive hypothesis, and observing that all complements of $E(G)$ are conjugate in G, we conclude.

We may therefore assume $F^*(G) = F(G)$ and write $F(G) = P \times T$. If $T = \{1\}$ then G is a monolithic csc-group, and we are done by Lemma 4.4. So let $T \neq \{1\}$, and put $\overline{G} = G/T$. Then $\overline{P} \in \mathsf{Syl}_p(F(\overline{G}))$ (in fact $P \in \mathsf{Syl}_p(G)$ by Lemma 5.4). By the inductive hypothesis applied to $\overline{G}$ we get that $\overline{P}$ is contained in a Hall subgroup $\overline{H}$ of $\overline{G}$, such that $\overline{H}$ is a csc-group, and every pair of such subgroups are conjugate in $\overline{G}$. Let K be the preimage of $\overline{H}$ in G and let $\pi = \pi(\overline{H})$. Since $O_{\pi'}(T) \trianglelefteq K$ and $(|O_{\pi'}(T)|, |K/O_{\pi'}(T)|) = 1$ we may apply the Schur-Zassenhaus theorem to conclude that K contains a Hall π-subgroup H and that every Hall π-subgroup of K is conjugate to H. Since $P \trianglelefteq G$ and $p \in \pi$, clearly $P \trianglelefteq H$; but $\overline{H}$ is a Hall π-subgroup of $\overline{G}$, so that H is a Hall π-subgroup of G. Moreover, by what have been said above, any Hall π-subgroup of G is conjugate to H.

If $q \in \pi \setminus \{p\}$ and if $Q \in \mathsf{Syl}_q(F(H))$ then, since q divides the order of H/T, Q must be cyclic. Therefore P is the unique non-cyclic Sylow subgroup of $F(H)$ and by construction H is a minimal csc-group. □

Definition 5.15 *If G is a* csc*-gruppo, we denote by $\pi_{\mathsf{csc}}(G)$ (or simply by π_{csc}) the set of primes $p \in \pi(G)$ such that the Sylow p-subgroup of $F(G)$ is not cyclic. We denote by $\mathsf{H}_p(G)$ (or simply by H_p) one of the Hall subgroups of G described in Lemma 5.14.*

Lemma 5.16 *Let G be a* csc*-group and let $p \in \pi_{\mathsf{csc}}(G)$. If $P \in \mathsf{Syl}_p(F(G))$, then $C_G(P)\mathsf{H}_p$ is normal in G.*

Proof. We argue by induction on the order of G. We may assume, up to considering $G/\Phi(P)$, that P is elementary abelian, of order p^m say. Let $\overline{G} = G/C_G(P)$, and let $\overline{\mathsf{H}}_p$ be the image of H_p in $\overline{G}$. It is enough to show that $\overline{\mathsf{H}}_p \trianglelefteq \overline{G}$. We distinguish three cases:

- $\mathsf{H}_p/O_{p'}(\mathsf{H}_p) \in \mathcal{A}(p^m)$. In this case $\overline{G}$ is isomorphic to a subgroup of $\Gamma(p^m)$ and since $\overline{\mathsf{H}}_p$ contains a Hall $\pi(\frac{p^m-1}{p-1})$-subgroup of $\overline{G}$ we easily obtain $\overline{\mathsf{H}}_p \trianglelefteq \overline{G}$.

- $\mathsf{H}_p/O_{p'}(\mathsf{H}_p) \in \mathcal{B}(p^2)$. In this case $\overline{\mathsf{H}}_p \simeq \mathrm{SL}(2,3)$ is a direct factor of $\overline{G}$.

- $\mathsf{H}_p/O_{p'}(\mathsf{H}_p) \in \mathcal{C}(p^2)$. Also in this case $\overline{\mathsf{H}}_p \simeq \mathrm{SL}(2,5)$ is a direct factor of $\overline{G}$. □

Lemma 5.17 *Let G be a* csc*-group and let $p, q \in \pi_{\mathsf{csc}}$ with $p \neq q$. Then $(|\mathsf{H}_p|, |\mathsf{H}_q|) = 1$ and $\mathsf{H}_p\mathsf{H}_q = \mathsf{H}_q\mathsf{H}_p$ is a Hall subgroup of G.*

Proof. We argue by induction on the order of G. Let $P \in \mathsf{Syl}_p(F(G))$ and $Q \in \mathsf{Syl}_q(F(G))$; by Lemma 5.4, $P \in \mathsf{Syl}_p(G)$ and $Q \in \mathsf{Syl}_q(G)$. Let R be a Hall $\{p,q\}'$-subgroup of $F^*(G)$ (see Lemma 5.3); we have $[P,R] = \{1\}$ and $[Q,R] = \{1\}$, and if $R \neq \{1\}$ we conclude by considering G/R.

Therefore we may assume $F^*(G) = P \times Q$ and that P and Q are elementary abelian (otherwise we conclude by considering $G/\Phi(P)$ or $G/\Phi(Q)$). Then we have $C_{\mathsf{H}_p}(P) = P$ and $C_{\mathsf{H}_q}(Q) = Q$, so that H_p and H_q are monolithic csc-groups. We observe that if $|P| = p^m$ and $|Q| = q^n$, then the number of cyclic subgroups of order pq of $P \times Q$ is $\nu = \frac{p^m-1}{p-1} \cdot \frac{q^n-1}{q-1}$.

Let us first consider the case when $\mathsf{H}_p \in \mathcal{A}(p^m)$ and $\mathsf{H}_q \in \mathcal{A}(q^n)$. By Lemma 5.4, $G/C_G(P)$ is isomorphic to a subgroup of $\Gamma(p^m)$ and $\mathsf{H}_pC_G(P) \simeq \mathsf{H}_p/P$ is a Hall subgroup of $G/C_G(P)$ which is normal by Lemma 5.16. Therefore $|G/C_G(P)| = r \cdot |\mathsf{H}_p/P|$ and, by Lemma 4.4 and Remark 5.7 r divides $(p-1)m$ (and clearly $(|\mathsf{H}_p|, r) = 1$); we may write $r = r_1r_2$ with $\pi(r_1) \subseteq \pi(|\mathsf{H}_p|)$ and $(r_1, r_2) = 1$. Similarly we get $|G/C_G(Q)| = s \cdot |\mathsf{H}_q/Q|$ and $s = s_1s_2$ with $\pi(s_1) \subseteq \pi(|\mathsf{H}_q|)$ and $(s_1, s_2) = 1$. Hence $|G/F(G)| = r_2 \cdot s_2 \cdot |\mathsf{H}_p| \cdot |\mathsf{H}_q|$. Moreover, it easy to show that $G/\mathsf{H}_pC_G(P)$ is cyclic, so that there is a (normal) subgroup N_1 in G of index r_2 containing H_p, H_q and $C_G(P)$; since G permutes transitively the cyclic subgroups of order pq of $P \times Q$ and $(r_2, \nu) = 1$, also N_1 has this property. Similarly there exists a (normal) subgroup N_2 in G of index s_2 containing H_p, H_q and $C_G(Q)$, and satisfying the same property. If we put $N = N_1 \cap N_2$, then $F(G) \leq N$ and $|G/F(G)| = |\mathsf{H}_p/P| \cdot |\mathsf{H}_q/Q|$, so that $|G| = |\mathsf{H}_p| \cdot |\mathsf{H}_q|$. The fact that $|\mathsf{H}_p/P| = \epsilon_p(m) \cdot \frac{p^m-1}{p-1}$ and $|\mathsf{H}_q/Q| = \epsilon_q(n) \cdot \frac{q^n-1}{q-1}$ shows that $\pi(|\mathsf{H}_p/P|) \cap \pi(|\mathsf{H}_q/Q|) = \emptyset$ and therefore $\pi(|\mathsf{H}_p|) \cap \pi(|\mathsf{H}_q|) = \emptyset$.

If one of H_p and H_q belongs to one of the classes $\mathcal{B}$ or $\mathcal{C}$, then it is clear that the other must lie in the class $\mathcal{A}$ and, arguing as before (and taking into account Remark 5.8) we conclude that $\pi(|\mathsf{H}_p|) \cap \pi(|\mathsf{H}_q|) = \emptyset$.

Due to the fact that $|N| = |\mathsf{H}_p| \cdot |\mathsf{H}_q|$, it follows that $N = \mathsf{H}_p\mathsf{H}_q$ and then $\mathsf{H}_p\mathsf{H}_q$ is a (Hall) subgroup of G. □

Let G be a csc-group. If $\pi_{\mathsf{csc}}(G) = \{p_1, p_2, \cdots, p_t\}$, then we denote by $\mathsf{H}(G)$ or simply by H the subgroup $\mathsf{H}_{p_1}\mathsf{H}_{p_2}\cdots\mathsf{H}_{p_t}$ of G. Clearly $\mathsf{H}(G)$ is a Hall subgroup of G and it is easy to check that it is a csc-group.

Lemma 5.18 *Let G be a* csc*-group and let $\mathsf{H} = \mathsf{H}(G)$. Then there exists a (possibly trivial) Hall subgroup H_0 of G such that:*

(i) $(|\mathsf{H}|, |\mathsf{H}_0|) = 1$ *and* $(|E(G)|, |\mathsf{H}_0|) = 1$*;*

(ii) $G = \mathsf{H}_0\mathsf{H}E(G)$*;*

(iii) H_0 *is a cyclic or metacyclic* csc*-group.*

Proof. To prove the statement, it is enough to show that if we put $\rho = \pi(|\mathsf{H}|) \cup \pi(|E(G)|)$, then there exists a Hall ρ'-subgroup in G. We argue by induction on the order of G.

If $E(G) \neq \{1\}$, then, by Lemma 5.13, $E(G)$ is a Hall subgroup of G, and we may write $G = E(G)R$ with $E(G) \cap R = \{1\}$: then we conclude by considering R. We may therefore assume $E(G) = \{1\}$ (and $F^*(G) = F(G)$). Let $\sigma = \pi(|F(G)|) \setminus \pi(|\mathsf{H}|)$ and let T be a Hall σ'-subgroup of $F(G)$. If $T = \{1\}$, then all Sylow subgroups of $F(G)$ are cyclic, G is metacyclic, and we are done. Otherwise, by the inductive hypothesis applied to $\overline{G} = G/T$, there exists a Hall subgroup $\overline{\mathsf{H}}_0$ in $\overline{G}$ with the required properties.

If $\overline{G} = \overline{\mathsf{H}}_0$, then G is solvable and certainly in G there is a Hall ρ'-subgroup. Otherwise, let H be the preimage of $\overline{\mathsf{H}}_0$ in G. Then H is solvable, and therefore H has a Hall ρ'-subgroup H_0; we conclude by observing that $\rho' \subseteq \pi(|H|)$, so that H_0 is a Hall ρ'-ssubgroup of G. □

From the previous lemmas, we obtain the following characterization of csc-groups:

Proposition 5.19 *Let G be a* csc*-group. Then G is the product of its Hall minimal* csc*-subgroups. Moreover, among these factors, at most one is non-soluble, and at most one is cyclic or metacyclic.*

□

We note that from Proposition 5.19 it follows for instance that if G is a csc-group, then the Sylow subgroups of $G/F^*(G)$ are cyclic or quaternions.

We conclude with a series of examples.

Example 10. Let A be an elementary abelian group of order 4 and $B \in \mathcal{A}(5^3)$ (B is a Frobenius group with elementary abelian kernel of order 5^3 and complement of order 31). Let $\langle x \rangle$ be a cyclic group of order 3 and let x act on A in such a way that $A\langle x \rangle \in \mathcal{A}(2^2)$ and on B so that $B\langle x \rangle$ is (isomorphic) to a subgroup of $A\Gamma(5^3)$. We have $\mathsf{H}_2(G) = A\langle x \rangle$ and $\mathsf{H}_5(G) = B$; note that $\mathsf{H}_2(G)$ is not normal in G but $C_G(A)\mathsf{H}_2(G) \trianglelefteq G$.

Example 11. Let A be a Frobenius group with elementary abelian kernel of order 5^2 and complement isomorphic to quaternions. Let B be a Frobenius group with elementary abelian kernel of order 11^3 and complement cyclic of order $7 \cdot 19$. If $\langle x \rangle$ is of order 3^n, with $n \geq 1$, we can make x act on A so that $A\langle x \rangle/\langle x^3 \rangle \in \mathcal{B}(5^2)$. We make x act on B so that $B\langle x \rangle/\langle x^3 \rangle$ is isomorphic to a subgroup of $A\Gamma(11^3)$, and we consider the semidirect product $G_1 = (A \times B)\langle x \rangle$. It easily follows that G_1 is a csc-group, $Z(G_1) = \langle x^3 \rangle$, $\mathsf{H}_5(G_1) = A\langle x \rangle$ and $\mathsf{H}_{11}(G_1) = B$; note that if $n \geq 2$, then $(|F(G)|, |G/F(G)|) = 3 \neq 1$.

If we let x act trivially on B (keeping the same action of x on A as above), then we may construct another group $G_2 = (A \times B)\langle x \rangle$. Obviously $|G_1| = |G_2|$, but $G_1 \not\simeq G_2$.

Example 12. We give an example of a group which is not a csc-group. Let A be elementary abelian of order 8, B a Frobenius group with elementary abelian kernel of order 11^3 and complement of order 19 and let $\langle x \rangle$ be of order 7. We make x act on A so that $A\langle x \rangle \in \mathcal{A}(2^3)$ and on B so that $B\langle x \rangle \in \mathcal{A}(11^3)$. Then $G = (A \times B)\langle x \rangle$ permutes transitively the cyclic subgroups of order 2 of $O_2(G)$ and the cyclic subgroups of order 11 of $O_{11}(G)$. However G is not a csc-group, since it does not permute transitively the $7^2 \cdot 19$ cyclic subgroups of order 22 of $F(G)$.

Example 13. Let $A \in \mathcal{C}(29^2)$ and let B be a Frobenius group with elementary abelian kernel of order 11^3 and complement of order 19. If $\langle x \rangle$ is of order 7, we can make x act on A so that $A\langle x \rangle$ is a Frobenius group, and on B so that $B\langle x \rangle \in \mathcal{A}(11^3)$. Then $G = (A \times B)\langle x \rangle$ is a non-solvable csc-group. We have $F^*(G) = F(G)$, $\mathsf{H}_{29}(G) = A$ and $\mathsf{H}_{11}(G) = B\langle x \rangle$.

Example 14. Let $A \simeq \mathrm{Sz}(8)$ and let $B \in \mathcal{A}(11^3)$. Let $\langle x \rangle$ be of order 3 and let x act on A as a (field) automorphism and on B so that $B\langle x \rangle$ is (isomorphic to) a subgroup of $A\Gamma(11^3)$. Then $G = (A \times B)\langle x \rangle$ is a non-solvable csc-group. We have $E(G) = A$, $F^*(G) = E(G) \times O_{11}(G)$; moreover $\mathsf{H}_{11}(G) = B$ and $\mathsf{H}_0(G) = \langle x \rangle$.

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