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SPECTRAL SENSITIVITY ANALYSIS OF ELECTROMAGNETIC CAVITIES

Coordinatore: Ch.mo Prof. Giovanni Colombo

Supervisore: Ch.mo Prof. Pier Domenico Lamberti

Dottorando: Michele Zaccaron

Riassunto

In questa tesi si studia la dipendenza degli autovalori per equazioni di Maxwell stazionarie in cavità elettromagnetiche che fungono da conduttori perfetti in relazione a perturbazioni del dominio e del parametro di permittività. Sfruttando una tecnica generale sviluppata da P.D. Lamberti e M. Lanza de Cristoforis si provano risultati di analiticità per la dipendenza delle funzioni simmetriche di autovalori di molteplicità arbitraria; si dimostrano inoltre risultati alla Rellich-Nagy che descrivono il fenomeno di biforcazione corrispondente, e si applicano ad alcuni problemi di ottimizzazione di forma e di permittività. In particolare, si mostra che le palle sono forme critiche per perturbazioni isovolumetriche ed isoperimetriche del dominio. Inoltre, nell'ipotesi che una certa disuguaglianza di Gaffney uniforme valga, si provano risultati di stabilità per l'operatore *curl curl* assumendo restrizioni piuttosto deboli sulla robustezza delle perturbazioni. Si studia anche la validità della disuguaglianza di Gaffney, sfruttando risultati di V.G. Maz'ya e T.O. Shaposhnikova che si basano sui moltiplicatori di Sobolev.

La tesi è organizzata come segue. Il Capitolo 1 è dedicato ad alcuni preliminari. Si introducono spazi funzionali idonei e si fanno alcune considerazioni riguardo la disuguaglianza di Gaffney. Nel Capitolo 2 si considera il problema di Maxwell elettrico in una cavità elettromagnetica che funge da conduttore perfetto, e dove sia la permittività elettrica che la permeabilità magnetica sono state normalizzate. Si studia la dipendenza degli autovalori da perturbazioni di forma del dominio. Innanzitutto, si provano risultati di analiticità con conseguenti formule alla Hadamard per le funzioni elementari simmetriche degli autovalori. Dopodiché, si fornisce una caratterizzazione dei domini critici sotto il vincolo di volume o perimetro fissato, e si dimostra che le palle sono domini critici per entrambi i vincoli. Per concludere, si espongono formule note per gli autovalori e le autofunzioni nella palla, e si mostra che nella palla il primo autovalore ha molteplicità 3. Il Capitolo 3 è dedicato allo studio della stabilità spettrale del problema di Maxwell elettrico. Nella prima parte si introduce la trasformata di Piola di atlante e si sviluppano alcuni strumenti tecnici che servono a provare che, nell'ipotesi di una disuguaglianza di Gaffney uniforme, si ha la stabilità spettrale per famiglie di domini perturbati che convergono ad un dominio limite in una certa maniera. Nella seconda parte si tratta il caso

critico e, usando strumenti propri della teoria dell'omogeneizzazione, si mostra che la stabilità spettrale è ancora da aspettarsi quando la frontiera del dominio è soggetta ad una perturbazione di tipo oscillatorio. Nel Capitolo 4 si studia il comportamento qualitativo degli autovalori del problema di Maxwell elettrico rispetto a perturbazioni del parametro di permittività. Dapprima si prova che i singoli autovalori sono localmente Lipschitziani, e se ne dimostra la dipendenza continua anche rispetto alla topologia debole*. In seguito, si prova che le funzioni elementari simmetriche degli autovalori dipendono in modo reale analitico dal parametro di permittività, e si espongono formule per le loro derivate. Sfruttando tali formule, da un lato si esamina l'ottimizzazione delle funzioni simmetriche sotto un vincolo idoneo sulla permittività e si dimostra che esse non ammettono alcun punto di massimo o minimo locale vincolato. Dall'altro lato, si mostra che per una permittività generica tutti gli autovalori sono semplici. Infine, il Capitolo 5 è un'appendice dove vengono raccolti alcuni risultati noti in letteratura e che sono usati nel resto della tesi. In particolar modo, essa contiene una dimostrazione dettagliata di una decomposizione vettoriale di M.Sh. Birman e M.Z. Solomyak, ed un'esposizione di lemmi tecnici e teoremi di regolarità per il Laplaciano con condizioni al bordo di Dirichlet ripresi dalla monografia di V.G. Maz'ya e T.O. Shaposhnikova riguardante i moltiplicatori di Sobolev. Questi sono gli strumenti principali usati in una dimostrazione della disuguaglianza di Gaffney inserita nel Capitolo 1. Nell'appendice si fa un'attenta descrizione di tutte le costanti in gioco, e si giustifica la validità di disuguaglianze di Gaffney uniformi usate nel Capitolo 3.

Abstract

In this thesis we study the dependence of eigenvalues for stationary Maxwell's equations in perfectly conducting electromagnetic cavities upon perturbation of the domain and of the permittivity parameter. Using a general technique developed by P.D. Lamberti and M. Lanza de Cristoforis we provide analyticity results for the dependence of the elementary symmetric functions of eigenvalues of arbitrary multiplicity, as well as Rellich-Nagy-type results describing the corresponding bifurcation phenomenon, and we apply them to certain shape and permittivity optimization problems. In particular, we show that for isovolumetric or isoperimetric perturbations, balls are critical shapes. Moreover, under the validity of a uniform Gaffney inequality, we prove stability results for the *curl curl* operator imposing weak restrictions on the strength of the perturbations. We also analyse the validity of the Gaffney inequality, for which we exploit results of V. Maz'ya and T. Shaposhnikova based on Sobolev multipliers.

The thesis is organized as follows. Chapter 1 is dedicated to notation and some preliminaries. We introduce suitable function spaces and make some considerations about the Gaffney inequality. In Chapter 2 we consider the electric Maxwell problem in a cavity with perfect conductor boundary conditions, where both the electric permittivity and magnetic permeability have been normalized. We study the dependence of its eigenvalues upon variation of the shape of the cavity. First, we provide analyticity results with the appropriate Hadamard formulas for the elementary symmetric functions of the eigenvalues. We then characterize critical domains for such functions under constraint of fixed volume or fixed perimeter, and show that balls are critical shapes. Finally we provide known formulas for the eigenpairs in the case of a ball, showing that the first eigenvalue has multiplicity three. Chapter 3 is devoted to the study of the spectral stability the electric Maxwell problem. In the first part we introduce the Atlas Piola transform and develop some useful technical tools to show that, under the validity of a uniform Gaffney inequality, there occurs spectral stability for families of perturbed domains converging to a limit domain in a certain way. In the second part we treat the critical threshold case, and using machinery from homogenization theory we show that if a uniform Gaffney inequality holds, the spectral stability is still expected

when the boundary of a domain undergoes a perturbation of oscillatory type. In Chapter 4 we study the qualitative behaviour of the eigenvalues of the electric Maxwell problem upon perturbation of the permittivity parameter. We first prove the local Lipschitz continuity of single eigenvalues and we show that they depend continuously with respect to the weak* topology. Then, we prove that the elementary symmetric functions of the eigenvalues depend real-analytically on the permittivity and provide formulas for their derivatives. Making use of these formulas we first consider the optimization of the symmetric functions under a suitable constraint on the permittivity and we show that they do not admit any point of local minimum or maximum. Second, we show that for a generic permittivity all the eigenvalues are simple. Finally, Chapter 5 is an appendix including known results that are used in the rest of the thesis. Precisely, it contains a detailed proof of a vector decomposition by M.Sh. Birman and M.Z. Solomyak, and an exposition of technical lemmas and theorems on the regularity for the Dirichlet Laplacian taken from the monograph by V.G. Maz'ya and T.O. Shaposhnikova on Sobolev multipliers. These are the main tools used for a proof of the Gaffney inequality presented in Chapter 1. We make a careful description of all the constants involved, and we justify the validity of uniform Gaffney inequalities used in Chapter 3.

Introduction

Maxwell's equations are a fundamental cornerstone of physics, and the foundation of the classical theory of electromagnetism. They provide a mathematical model to comprehend the behaviour and evolution of electromagnetic phenomena, describing how the electric and magnetic field are generated and interact with each other. They read as follows

$$
\begin{aligned}
\frac{\partial \mathcal{B}}{\partial t} + \operatorname{curl} \mathcal{E} &= 0 && \text{(Faraday's Law of Induction)},\\
\frac{\partial \mathcal{D}}{\partial t} - \operatorname{curl} \mathcal{H} &= -\mathcal{J} && \text{(Ampère's Law)},\\
\operatorname{div} \mathcal{D} &= \rho && \text{(Gauss's Electric Law)},\\
\operatorname{div} \mathcal{B} &= 0 && \text{(Gauss's Magnetic Law)}.
\end{aligned}
$$

Here $\mathcal{E}$ and $\mathcal{D}$ denote the *electric field* and the *electric displacement*, while $\mathcal{H}$ and $\mathcal{B}$ denote the *magnetic field* and the *magnetic flux density*. Moreover, $\mathcal{J}$ is the *current density* and ρ stands for the *charge density* of the medium.

Under the assumption that the vector fields admit a Fourier transform in the time variable, or even if they behave periodically with respect to time with the same frequency ω, the above system simplifies into the following form, known as the "time-harmonic Maxwell's equations"

$$
\begin{aligned}
-\mathrm{i}\,\omega B + \operatorname{curl} E &= 0,\\
\mathrm{i}\,\omega D + \operatorname{curl} H &= \sigma E + J_e,\\
\operatorname{div} D &= \rho,\\
\operatorname{div} B &= 0,
\end{aligned}
\tag{0.0.1}
$$

which describes how the spatial parts (E, H) of electromagnetic waves behave if they are oscillating at frequency $\frac{\omega}{2\pi}$. Here we have also made the assumption that $J = \sigma E + J_e$ (Ohm's Law in a linear approximation), where J_ϵ is the *external current density* and σ the *conductivity* of the medium. The fields D and B are also called the *electric induction* and *magnetic induction* respectively.

The quantities D, E, B, H are linked by the following "constitutive relations":

$$D = \varepsilon E, \qquad B = \mu H. \tag{0.0.2}$$

The above identities, which specify the response of bound charge and current to the applied fields, are physically reasonable if we ignore ferro-electric and ferro-magnetic media and if the fields are relatively small. Here ε denotes the *electric permittivity* of the medium. It measures how easily charged particles can flow through a material, and is associated with the alignment of bound molecules electric dipole moments within the medium. Analogously, μ denotes the *magnetic permeability* of the medium. It measures how easily a magnetic field can pass through it and relates to the orientation of bound magnetic particles within the material. Note that from a mathematical point of view, the constitutive relations allow us to obtain a well-posed system of differential equations, as system (0.0.1) is undetermined.

For a large class of real-world materials, in particular for *linear anisotropic dielectric media* (the physical properties of these materials depend both on the on the point and the direction taken at that point), the permittivity and the permeability are modelled by second-rank tensors and are represented by 3×3-matrix valued maps. In the special case the quantities ε and μ are scalar, the material is called isotropic. If the material is also homogeneous, then ε and μ are constants. For further details on the physical aspect of Maxwell's equations we mention the text book [71], while for an introduction to modern mathematical methods in the theory of electromagnetism we refer to the extensive monographs [31, 74, 99, 106].

Depending on the physical model under consideration, various boundary conditions, radiation conditions at infinity and field continuity conditions can be coupled with Maxwell's differential equations. Therefore, various boundary value problems can be formulated, and can be tackled using integral methods and tools and techniques proper of potential theory, harmonic analysis, calculus of variations, functional analysis etc. Among them, of great importance is the one related to the *perfectly conducting cavity*, which models cases where the electric resistivity is negligible (for example in superconductors). The study of electromagnetic cavities has not only a pure interest for mathematicians and physicists, but has also several applications from an engineering point of view: for example in the design of cavity resonators or shielding structures for electronic circuits. We refer to [66, Ch. 10] for a detailed introduction to this field of investigation. We cite the well-known papers [38, 39, 40] by Costabel and Dauge, as well as the more recent papers [6, 16, 27, 37, 83, 101, 114].

In our mathematical setting cavities are understood as bounded connected open sets (domains) in the Euclidean space $\mathbb{R}^3$, and they are thought filled by a material which in general is inhomogeneous and anisotropic. The eigenfrequency problem in a bounded domain Ω in $\mathbb{R}^3$ consists in finding two non-zero eigenfields E, H and a non-zero eigenfrequency $\omega > 0$ (also called angular frequency or pulsation) such that the following coupled system of differential equations is satisfied:

$$\operatorname{curl} E - \mathrm{i}\,\omega\,\mu\,H = 0\,, \qquad \operatorname{curl} H + \mathrm{i}\,\omega\,\varepsilon\,E = 0 \quad \text{in } \Omega. \tag{0.0.3}$$

We assume that ε (the electric permittivity) and μ (the magnetic permeability) are represented by positive definite symmetric 3×3 matrices which depend on $x \in \Omega$ and whose entries are in $L^\infty(\Omega)$. In particular, since εE and μH are both curls, then necessarily

$$\operatorname{div} \varepsilon E = 0, \quad \operatorname{div} \mu H = 0 \qquad \text{in } \Omega.$$

Note that if we use the constitutive relations (0.0.2), system (0.0.3) corresponds to the time-harmonic Maxwell's equations (0.0.1) if we further assume that we are in a region free of charges ($\rho = 0$) and external currents ($J_e = 0$).

If the boundary of the cavity Ω represents perfectly conducting walls then we couple the Maxwell differential system with the following boundary conditions

$$\nu \times E = 0, \quad \nu \cdot \mu H = 0 \qquad \text{on } \partial\Omega, \tag{0.0.4}$$

where ν denotes the outer unit normal vector to the boundary of Ω. In other words, the electric field is normal at the boundary while the magnetic field is tangential.

The *cavity resonator problem* is to find non-zero frequencies ω and non-zero electromagnetic fields (E, H) such that

$$\begin{cases} \operatorname{curl} E - \mathrm{i}\,\omega\mu H = 0 & \text{in } \Omega, \\ \operatorname{curl} H + \mathrm{i}\,\omega\varepsilon E = 0 & \text{in } \Omega, \\ \operatorname{div} \varepsilon E = 0 \text{ and } \operatorname{div} \mu H = 0 & \text{in } \Omega, \\ \nu \times E = 0 \text{ and } \nu \cdot \mu H = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.5}$$

Applying the curl operator to the first two equations in (0.0.5) we get

$$\operatorname{curl} \mu^{-1} \operatorname{curl} E = \omega^2 \varepsilon E, \quad \operatorname{curl} \varepsilon^{-1} \operatorname{curl} H = \omega^2 \mu H \qquad \text{in } \Omega.$$

Hence, setting $\lambda = \omega^2 > 0$, we are presented with the two following eigenproblems:

$$(\text{electric Maxwell}) \begin{cases} \varepsilon^{-1} \operatorname{curl} \mu^{-1} \operatorname{curl} E = \lambda\, E & \text{in } \Omega, \\ \operatorname{div} \varepsilon E = 0 & \text{in } \Omega, \\ \nu \times E = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.6}$$

and

$$\text{(magnetic Maxwell)} \begin{cases} \mu^{-1} \operatorname{curl} \varepsilon^{-1} \operatorname{curl} H = \lambda\, H & \text{in } \Omega, \\ \operatorname{div} \mu H = 0 & \text{in } \Omega, \\ \nu \cdot \mu H = 0 & \text{on } \partial\Omega, \\ \nu \times \varepsilon^{-1} \operatorname{curl} H = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.7}$$

Note that solutions of (0.0.6) automatically satisfy the condition $\nu \cdot \operatorname{curl} E = 0$ on $\partial\Omega$, hence we have omitted it (cf. Lemma 1.3.5). Moreover, for $\lambda \neq 0$ the conditions of null divergence can also be omitted, but we conveniently leave it since it will be useful for clarity purposes. It is straightforward to see that the electric and the magnetic problems give rise to the same eigenvalues (see also [41, Lemma 2.1] and [115]), hence the term *Maxwell eigenvalues*. Finally, note that in general a monotonicity principle with respect to inclusion is not expected for the Maxwell eigenvalues (see Remark 2.1.7 and Counterexample 2.1.8).

In the first part of the thesis, our goal is to make a qualitative analysis of the Maxwell eigenvalues upon variation of the domain Ω. Since we are interested in the dependence upon the shape of the cavity, we simplify the problem with the normalization $\varepsilon = \mu = 1$ to obtain the following Maxwell electric eigenproblem:

$$\begin{cases} \operatorname{curl} \operatorname{curl} E = \lambda\, E & \text{in } \Omega, \\ \operatorname{div} E = 0 & \text{in } \Omega, \\ \nu \times E = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.8}$$

It is readily seen that the eigenvalues of (0.0.8) can be arranged in a non-decreasing, divergent sequence

$$0 < \lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \cdots \nearrow +\infty,$$

where each eigenvalue is repeated according to its multiplicity (that is, the dimension of the corresponding eigenspace). We study the dependence of $\lambda_n[\Omega]$ on the shape of Ω, and we aim at proving analyticity results for all eigenvalues, both simple and multiple, and we address an optimization problem concerning the role of balls in isovolumetric or isoperimetric domain perturbations.

We point out that the behaviour of the spectrum of problem (0.0.8) upon domain perturbation is not much discussed in the literature. We are aware only of the paper [72] by S. Jimbo which provides a Hadamard-type formula for the shape derivative of simple eigenvalues and quotes the book [70] by K. Hirakawa where an analogous but different formula is provided on the base of heuristic computations.

We also quote the recent papers [64, 65, 69] concerning the use of domain derivatives in inverse electromagnetic scattering.

On the contrary, the amount of scientific works concerning domain perturbations for other type of differential operators arising in elasticity theory, is vast and these types of issues have been largely investigated in the literature. We cite the classical book [68] by D. Henry, and [19, 67] for more information about the variational approach. We also mention the monograph [47] by Dalla Riva, Lanza de Cristoforis and Musolino for a functional analytic approach to stability problem. In particular, it appears that for classical boundary value problems involving rotational invariant operators, a kind of principle holds, namely, *all simple eigenvalues and the elementary symmetric functions of multiple eigenvalues depend real analytically on the domain, and balls are corresponding critical domains with respect to isovolumetric and isoperimetric domain perturbations*, see [22]. In this thesis we prove that the eigenvalue problem (0.0.8) obeys this principle, proving the analytic dependence and providing appropriate Hadamard-type formulas (see Theorem 2.3.5). From that, we show in Theorem 2.4.10 that an open set is a critical domain, under the volume constraint $\mathrm{Vol}(\Omega)$ = constant, for the elementary symmetric functions of the eigenvalues bifurcating from an eigenvalue λ of multiplicity m if and only if the following extra boundary condition is satisfied:

$$\sum_{l=1}^{m}\left(|E^{(l)}|^2-|H^{(l)}|^2\right)=c \text{ on } \partial\Omega, \tag{0.0.9}$$

where c is a constant, $E^{(l)}$, $l=1,\dots m$, is an orthonormal basis in $(L^2(\Omega))^3$ of the (electric) eigenspace associated with λ, and $H^{(l)}=-\mathrm{i}\,\mathrm{curl}\,E^{(l)}/\sqrt{\lambda}$ is the magnetic field associated with $E^{(l)}$ (cf. (0.0.3)). Similarly, in the case of isoperimetric domain perturbations and perimeter constraint $\mathrm{Per}(\Omega)$ = constant, the extra boundary condition reads

$$\sum_{l=1}^{m}\left(|E^{(l)}|^2-|H^{(l)}|^2\right)=c\,\mathcal{H}\ \text{ on } \partial\Omega, \tag{0.0.10}$$

where c is a constant and $\mathcal{H}$ is the mean curvature of $\partial\Omega$ (the sum of the principal curvatures). We then prove in Theorem 2.4.13 that both conditions (0.0.9) and (0.0.10) are satisfied if Ω is a ball. Note that this does not say that the ball is an extremizer among isovolumetric or isoperimetric shapes, since criticality is a more general property. It would be interesting to characterise all domains for which the above conditions are satisfied.

We remark that using the elementary symmetric functions of multiple eigenvalue, rather than the eigenvalues themselves, is quite natural since domain perturbations typically split the multiplicities of the eigenvalues and produce bifurcation phenomena responsible for corner points in the corresponding diagrams. Moreover, we believe that in the specific case of Maxwell equations, being able to deal with multiple eigenvalues is quite important since all eigenvalues of the Maxwell system in a ball are multiple.

We want to stress that our method is based on the theoretical results obtained in [78], which concerns the case of general families of compact self-adjoint operators in Hilbert spaces with variable inner product, and which was applied for the case of the Laplace operator with Dirichlet and Neumann boundary conditions in [79] and [80] respectively. The same method was further adopted in other works such as [21, 22, 24, 81, 87]. In particular we mention the work [20] by Buoso, where the author proves shape differentiability results for second order elliptic systems with constant coefficients satisfying the Legendre-Hadamard conditions, providing also optimality conditions within the class of isovolumetric shape perturbations. We also mention the paper [23] where the Reissner-Mindlin system is studied.

Here we consider a class of open sets $\Phi(\Omega)$ identified by a class of diffeomorphisms Φ defined on a fixed reference domain Ω. Then our problem is set on $\Phi(\Omega)$ and pulled-back to Ω by means of the covariant Piola transform associated with Φ(cf. (2.2.3)). This allows to recast the problem on Ω and to reduce it to the study of a curl-div problem with parameters depending on Φ, the eigenvalues of which are exactly $\lambda_n[\Phi(\Omega)]$. Then, passing to the analysis of the corresponding resolvents defined in $(L^2(\Omega))^3$ (with an appropriate equivalent inner product depending on Φ) we can prove our analyticity results for the maps $\Phi \mapsto \lambda_n[\Phi(\Omega)]$ and study their critical points under volume constraint $\mathrm{Vol}\,(\Phi(\Omega)) = \text{const.}$ or perimeter constraint $\mathrm{Per}\,\Phi(\Omega) = \text{const.}$

We also note that the families of compact self-adjoint operators under consideration are obtained by following the method of [39] which consists in adding the penalty term $-\tau\nabla \operatorname{div} E$ in the eigenvalue equation of (0.0.8), depending on an arbitrary positive number τ. Then it is enough to observe that the eigenvalues of the penalized problem are given by the union of the eigenvalues of problem (0.0.8) and the eigenvalues of the Dirichlet Laplacian $-\Delta$ in Ω, multiplied by τ, see [40, Theorem 1.1]. In particular, it follows that our analyticity result regarding problem (0.0.8) yields (in the case of regular domains) also the analyticity result proved in [79] for the eigenvalues of the Dirichlet Laplacian.

Besides the results described above, we would like to highlight two by-pass products of our analysis. First, in Theorem 2.3.17 we prove a Rellich-Nagy-type result describing the bifurcation phenomenon mentioned above. Namely, given an eigenvalue λ of multiplicity m, say $\lambda = \lambda_n = \cdots = \lambda_{n+m-1}$, and a perturbation of Ω of the form $\Phi_\epsilon(\Omega)$ with $\Phi_\epsilon(x) = x + \epsilon V(x)$ for all $x \in \Omega$ where V is a $\mathcal{C}^{1,1}$ vector field defined on $\overline{\Omega}$, we prove that the set of right derivatives at $\epsilon = 0$ of $\lambda_{n+k}[\Phi_\epsilon(\Omega)]$ for all $k = 0, \dots, m-1$ (which coincides with the set of left derivatives, although each right and left derivative may be different) are given by the eigenvalues of the matrix $(M_{i,j})_{i,j=1,\dots,m}$ where

$$M_{i,j} = \int_{\partial\Omega} \left(\lambda E^{(i)} \cdot E^{(j)} - \operatorname{curl} E^{(i)} \cdot \operatorname{curl} E^{(j)}\right) V \cdot \nu \, d\sigma \,, \tag{0.0.11}$$

and $E^{(i)}$, $i = 1, \dots m$, is a (real) orthonormal basis in $(L^2(\Omega))^3$ of the (electric) eigenspace associated with λ. In particular, if λ_n is a simple eigenvalue we get the Hadamard formula

$$\frac{d}{d\epsilon}\lambda_n[\Phi_\epsilon(\Omega)]\Big|_{\epsilon=0} = \int_{\partial\Omega} \left(\lambda|E|^2 - |\operatorname{curl} E|^2\right) V \cdot \nu \, d\sigma = \lambda \int_{\partial\Omega} \left(|E|^2 - |H|^2\right) V \cdot \nu \, d\sigma \,, \tag{0.0.12}$$

where E is an eigenvector normalized in $(L^2(\Omega))^3$ and $H = -\mathrm{i} \operatorname{curl} E/\sqrt{\lambda}$ as above.

Second, by using these formulas we prove a Rellich-Pohozaev formula for the Maxwell eigenvalues, namely any eigenvalue λ can be represented by the formula

$$\lambda = \frac{1}{2} \int_{\partial\Omega} \left(|\operatorname{curl} E|^2 - |\operatorname{curl} H|^2\right) x \cdot \nu \, d\sigma \,, \tag{0.0.13}$$

where E is any (electric) eigenvector associated with λ normalized in $(L^2(\Omega))^3$ and $H = -\mathrm{i} \operatorname{curl} E/\sqrt{\lambda}$ (see Theorem 2.3.19). In particular, we have the identity

$$\int_{\partial\Omega} \left(|H|^2 - |E|^2\right) x \cdot \nu \, d\sigma = 2. \tag{0.0.14}$$

We note that our formulas are proved under the assumption that the eigenvectors under consideration are of class H^2 (which means that they have square summable derivatives up to the second order) and that this assumption is satisfied if the corresponding domain is sufficiently regular, for example of class $C^{2,1}$ (see [5, 113]).

We then focus on the study of the spectral stability for problem (0.0.8). Given a fixed domain Ω, we consider a family $\{\Omega_\epsilon\}_{\epsilon>0}$ of perturbed domains converging to Ω as $\epsilon \to 0$. The convergence of Ω_ϵ to Ω will be described by means of a fixed atlas $\mathcal{A}$, that is a finite collection of rotated parallelepipeds V_j, $j = 1, \dots, s$ covering the domains under consideration and such that if V_j touches the boundaries of the domains then $\Omega \cap V_j$ and $\Omega_\epsilon \cap V_j$ are given by the subgraphs of two functions $g_j, g_{\epsilon,j}$ in two variables, say $\bar{x} = (x_1, x_2)$. Thus the convergence of Ω_ϵ to Ω is understood in terms of the convergence of $g_{\epsilon,j}$ to g_j as $\epsilon \to 0$.

Since we are dealing with a differential operator of the second order, it is not surprising that if $g_{\epsilon,j}$ converges uniformly to g_j together with its first and second derivatives as $\epsilon \to 0$ (in which case one talks of C^2-convergence) then we have spectral stability of the *curlcurl* operator, which means that the eigenvalues and eigenfunctions of problem (0.0.8) in Ω_ϵ converge to those in Ω as $\epsilon \to 0$. It is also not surprising that if $g_{\epsilon,j}$ converges uniformly to g_j together with its first derivatives and

$$\sup_{\epsilon>0}\sup_{\bar{x}\in\mathbb{R}^2}|D^2 g_{\epsilon,j}(\bar{x})| \neq \infty \tag{0.0.15}$$

then we have spectral stability again. These results are also immediate consequences of the results of the present thesis. The main question here is whether it is possible to relax condition (0.0.15). For example, if we assume that $g_{\epsilon,j}$ is of the form

$$g_{\epsilon,j} = \epsilon^{\alpha} b_j(\bar{x}/\epsilon) \tag{0.0.16}$$

where $\alpha > 0$ and b_j is a fixed C^2 function, condition (0.0.15) is encoded by the inequality $\alpha \geq 2$. In this model case, the question is whether one can get spectral stability for $\alpha < 2$. Note that a profile of the form (0.0.16) is typical in the study of boundary homogenization problems and thin domains, see for example [9, 10, 11, 12, 28, 30, 53, 54, 55].

This problem was solved for the biharmonic operator with intermediate boundary conditions (modelling an elastic hinged plate) in [11] where condition (0.0.15) is relaxed by introducing a suitable notion of weighted convergence which allows to prove spectral stability for $\alpha > 3/2$ in the model problem above. That condition is analogous to the one we use in (3.1.15). Namely we assume that for all $\epsilon > 0$ there exists $\kappa_\epsilon > 0$ such that

$$\begin{aligned}
&(i)\quad \kappa_\epsilon > \max_{j=1,\dots,s'} \left\|g_{\epsilon,j} - g_j\right\|_{L^\infty(W_j)};\\
&(ii)\quad \lim_{\epsilon\to 0}\kappa_\epsilon = 0;\\
&(iii)\quad \lim_{\epsilon\to 0}\frac{\max_{j=1,\dots,s'}\left\|D^\beta(g_{\epsilon,j} - g_j)\right\|_{L^\infty(W_j)}}{\kappa_\epsilon^{3/2-|\beta|}} = 0 \quad \text{for all } \beta\in\mathbb{N}^3 \text{ with } |\beta|\leq 2.
\end{aligned} \tag{0.0.17}$$

It is remarkable that the threshold $3/2$ is sharp since for $\alpha \leq 3/2$ spectral stability does not occur for the probelm discussed in [11] (in particular, it is proved in [11] that for $\alpha < 3/2$ a degeneration phenomenon occurs and for $\alpha = 3/2$ a strange term in the limit appears, as in many homogenization problems). An analogous trichotomy is found in [54] for the biharmonic operator subject to certain Steklov type boundary conditions. We also mention the paper [91] where a Babushka-type paradox, appearing in homogenization theory when thin circular plates are approximated by regular polygons, is studied on plates with convex holes.

We would like to point out that, throughout the whole thesis, a fundamental result we use in order to guarantee the validity of a number of facts is the celebrated Gaffney (or Gaffney-Friedrichs) inequality. In its standard form in the Euclidean space $\mathbb{R}^3$ it reads

$$\|u\|_{H^1(\Omega)^3} \leq C\left(\|u\|_{L^2(\Omega)^3} + \|\mathrm{curl}\, u\|_{L^2(\Omega)^3} + \|\mathrm{div}\, u\|_{L^2(\Omega)}\right) \tag{0.0.18}$$

for all vector fields $u \in L^2(\Omega)^3$ with distributional $\operatorname{curl} u \in L^2(\Omega)^3$ and $\operatorname{div} u \in L^2(\Omega)$, and such that they are normal to the boundary of Ω, namely $\nu \times u = 0$ on $\partial\Omega$ (ν is the outer unit normal to Ω). The Gaffney inequality traces back to the seminal works of Gaffney [58, 59, 60] and Friedrichs [57], and it is classical that it holds for domains of class C^2, or even piecewise $C^{1,1}$-boundaries (see [14, 15]). As we will show, the regularity of the boundary can be significantly weakened in order to include boundaries of class $C^{1,\beta}$ with $\beta > 1/2$ (see also [56, 102]).

In this thesis, specifically in Theorem 3.2.30, we prove that the relaxed convergence (0.0.17) guarantees the spectral stability of the *curlcurl* operator for problem (0.0.8). Our result requires that the Gaffney inequality (0.0.18) holds for all domains Ω_ϵ with a constant C independent of $\epsilon > 0$. Again, if one does not assume the validity of the uniform bound (0.0.15), then proving that a uniform Gaffney inequality holds is highly non-trivial. Here we manage to do this, by exploting the approach of [92, Ch. 14] based on the use of Sobolev multipliers and the notion of domains of class $M_2^{3/2}(\delta)$. In particular, if we assume that

$$|\nabla g_{\epsilon,j}(\bar{x}) - \nabla g_{\epsilon,j}(\bar{y})| \leq M|\bar{x} - \bar{y}|^\beta \tag{0.0.19}$$

for all $\bar{x}, \bar{y} \in \mathbb{R}^2$, with $\beta \in [1/2, 1]$ and M independent of ϵ, and we also assume that the sup-norms of functions $|\nabla g_{\epsilon,j}|$ are sufficienlty small, then our domains belong to the class $M_2^{3/2}(\delta)$ with δ small enough. This allows to apply [92, Thm. 14.5.1] which guarantees the validity of a uniform H^2- a priori estimate for the Dirichlet Laplacian which, in turn, is equivalent to the uniform Gaffney inequality.

In conclusion, the convergence of the domains Ω_ϵ to Ω in the sense of (0.0.17) combined with the validity of (0.0.19) and the smallness of the gradients of the profile functions $g_{\epsilon,j}$ guarantees the spectral stability of the *curlcurl* operator for problem (0.0.8). Note that, in principle, since Ω is of class C^1 one may think of choosing from the very beginning an atlas which guarantees that the gradients of the profile functions are as small as required (indeed, it is enough to adapt the atlas to the tangent planes of a sufficiently big number of boundary points of Ω). Then the convergence in the sense of (0.0.17) would imply the smallness of the gradients of the profile functions of Ω_ϵ as well.

By setting $\beta = \alpha - 1$, we deduce that a uniform Gaffney inequality holds for the model example provided by (0.0.16) if $\alpha \geq 3/2$. Moreover, if $\alpha > 3/2$ then spectral stability occurs for the same example since in this case also the convergence (0.0.17) occurs.

The case $\alpha \leq 3/2$ is more involved. In Section 3.4 of the thesis we also study the threshold case $\alpha = 3/2$ for the model example (0.0.16), managing to show that, still under the validity of a uniform Gaffney inequality, there still occurs the spectral stability. Moreover, we prove an analogous result for the magnetic counterpart of (0.0.8), namely for the following problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} H = \lambda\, H & \text{in } \Omega, \\ \operatorname{div} H = 0 & \text{in } \Omega, \\ \nu \cdot H = 0 & \text{on } \partial\Omega, \\ \nu \times \operatorname{curl} H = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.20}$$

In order to achieve these results we make a fundamental use of techniques from homogenization theory. In particular, we exploit the so-called *unfolding* method, introduced by Cioranescu, Damlamian and Griso in [34] (see also [35, 48]), which is in turn based on the *two-scale convergence*, one of the first methods developed in order to study homogenization and multi-scale problems. For a general reading on homogenization theory we refer to the monograph [36]. We also mention the survey work by Vainikko [111] where fundamental concepts about the convergence of operators are summarized, and the paper [9] where these concepts are applied to study the spectral behaviour of the biharmonic operator subject to homogeneous boundary conditions of Neumann type on dumbell domains.

We note that if $\alpha < 3/2$ one cannot expect a uniform Gaffney inequalities to hold, in particular because the regularity assumptions $C^{1,\beta}$ for any $\beta > 1/2$ is optimal for the validity of the Gaffney inequality itself, see [56, 102].

Finally, in the last part of this thesis we consider the following problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} E = \lambda\, \varepsilon\, E & \text{in } \Omega, \\ \operatorname{div} \varepsilon E = 0 & \text{in } \Omega, \\ \nu \times E = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.21}$$

involving a non-unitary permittivity parameter ε. To a certain extent, this study is inspired by the papers [76, 82] where the authors investigate the behaviour of the eigenvalues of the Laplacian and of a general elliptic operators upon perturbations of mass density.

Again, it is readily seen that the spectrum of problem (0.0.21) is discrete and made of (ε-dependent) eigenvalues that can be arranged in an infinite, divergent sequence

$$0 < \lambda_1[\varepsilon] \le \lambda_2[\varepsilon] \le \cdots \le \lambda_n[\varepsilon] \le \cdots \nearrow +\infty,$$

where each eigenvalue is repeated in accordance with its multiplicity. Here we consider permittivities ε as symmetric matrix-valued functions whose entries are in $W^{1,\infty}(\Omega)$. Moreover we assume that there exist a positive constant $m_\varepsilon > 0$ such that

$$\varepsilon\, \zeta \cdot \zeta \ge m_\varepsilon |\zeta|^2$$

for all $\zeta \in \mathbb{R}^3$ and almost all $x \in \Omega$. In Theorem 4.2.11 we prove that the eigenvalues $\lambda_n[\varepsilon]$ are locally Lipschitz continuous with respect to the parameter ε. Moreover, we show that they depend continuously also with respect to the weak* topology on $W^{1,\infty}(\Omega)$ induced by its ambient space $L^\infty(\Omega)^4$, see Theorem 4.3.6.

At this point we pass to consider higher regularity properties. Again, we follow the approach by Lamberti and Lanza de Cristoforis in [78, 79], and consider the elementary symmetric functions of the eigenvalues, since in general multiple eigenvalues may present bifurcation phenomena when the parameters are perturbed. We prove an analyticity result for those functions of the eigenvalues, and provide an explicit formula for their (Frèchet) differential with respect to ε (see Theorem 4.4.7). We also consider the case of one-parameter families of perturbations of ε and prove in Theorem 4.4.11 a Rellich-Nagy-type theorem which describes the bifurcation phenomenon of multiple eigenvalues, showing that all the branches of the eigenvalues splitting from a multiple eigenvalue of multiplicity m can be described by m real analytic functions of the scalar parameter of the family of perturbations, and provide formulas for their right and left derivatives.

As a first application of the above result, we consider a constrained optimization problem for the symmetric functions of the eigenvalues. Inspired by the analogous optimization problem for the eigenvalues of the Dirichlet Laplacian with mass density modeling a vibrating membrane with fixed mass (see [82]), we prove that the symmetric functions of Maxwell's eigenvalues have no critical points under the constraint that the integral of the permittivity is constant (cf. Theorem 4.5.1).

Finally, as a second application, in Section 4.6 we show that all the eigenvalues of problem (0.0.21) are simple for a generic permittivity. That is, in few words, given any permittivity ε it is always possible to find a small perturbation $\tilde{\varepsilon}$, as close to ε as desired, such that the eigenvalues $\{\lambda_j[\tilde{\varepsilon}]\}_{j\in\mathbb{N}}$ are all simple.

The thesis is organized as follows. Chapter 1 is dedicated to notation and some preliminaries. We introduce the appropriate function spaces suitable for the study of problems (0.0.8) and (0.0.21), and we make some considerations about the Gaffney inequality, presenting two known proofs. One is more direct and exploits L^2-estimates for first-order derivatives involving the curvature tensor $\mathfrak{B}$. The other one makes use of a special decomposition of vector fields, developed by Birman and Solomyak in [17], paired with regularity results for the Dirichlet Laplacian.

In Chapter 2 we consider problem (0.0.8) and study the dependence of its eigenvalues upon variation of the shape Ω. In the same spirit of [79], we provide analyticity results with the appropriate Hadamard formulas for the elementary symmetric functions of the eigenvalues. We then characterize critical domains for such functions under constraint of fixed volume or fixed perimeter, and show that balls are critical shapes. Then we provide known formulas for the eigenpairs in the case Ω is a ball, showing that the first eigenvalue has multiplicity three.

Chapter 3 is devoted to the study of the spectral stability of eigenvalues and eigenfunctions of problem (0.0.8). In the first part we introduce what we call “Atlas Piola transform" and develop some useful technical tools to show that, under the validity of a uniform Gaffney inequality, there occurs spectral stability for families of perturbed domains converging in the sense of (0.0.17). In the second part we treat the case with $\alpha = 3/2$, and using machinery from homogenization theory we show that if a uniform Gaffney inequality holds, the spectral stability is still expected when the boundary of a domain undergoes a perturbation of oscillatory type.

In Chapter 4 we study the qualitative behaviour of the eigenvalues of problem (0.0.21) upon perturbation of the permittivity parameter ε. We first prove the local Lipschitz continuity of the single eigenvalues. We also show that they depend continuously with respect to the weak* topology on $W^{1,\infty}(\Omega)$. Then, using techniques adapted from [79], we prove that the elementary symmetric functions of the eigenvalues depend real-analytically on ε and provide formulas for their derivatives. Making use of these formulas we then prove a result concerning permittivity optimization, and the generic simplicity of the spectrum of problem (0.0.21).

Finally, Chapter 5 is an appendix including known results that are used in the rest of the thesis. Precisely, it contains a detailed proof of the Birman and Solomyak [17] vector decomposition, and an exposition of technical lemmas and theorems on the regularity for the Dirichlet Laplacian taken from the monograph [92] . These are the main tools used in the second proof of the Gaffney inequality present in Chapter 1. In this appendix we pay attention to the constants involved and make a careful description of all the quantities they depend on, ultimately allowing us to justify the uniform Gaffney inequalities used in Chapter 3.

The results in Chapter 2 on the qualitative analysis of the Maxwell eigenvalues upon domain perturbation have been published in [84] in collaboration with Prof. Pier Domenico Lamberti. The spectral stability results for $\alpha > 3/2$ present in Chapter 3 is included in the paper [85], also in collaboration with Prof. Pier Domenico Lamberti, and is currently submitted for publication in an international scientific journal. Finally, part of the results of Chapter 4 will be published in the forthcoming paper [90] in collaboration with Dr. Paolo Luzzini.

List of symbols

$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	natural numbers including 0
$\lvert\alpha\rvert, D^\alpha f$	multi-index notation
Df	Jacobian matrix of f
$D^2 f$	Hessian matrix of f
$f(z) = O(g(z))$ as $z \to 0$	big-O notation: $\limsup_{z\to 0} \frac{f(z)}{g(z)} < \infty$
$f(z) = o(g(z))$ as $z \to 0$	little-o notation: $\lim_{z\to 0} \frac{f(z)}{g(z)} = 0$
σ_m	surface area of the unit m-sphere
$d\sigma$	standard surface measure
$\mathcal{H}$	mean curvature of a surface
ν	outer unit normal vector field of a surface
$\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$	smooth functions with compact support in Ω
$\bar{\Omega}$	closure of the set Ω

$\mathbb{I}_N$	identity $N \times N$ matrix
$M^T, \mathbf{v}^T$	transpose of the matrix M, transpose of the vector $\mathbf{v}$ respectively
$M_{i,j}$	(i,j)-entry of the matrix M
$\delta_{i,j}$	Kronecker delta
ξ_{ijk}	Levi-Civita symbol
sgn	sign function
(a,b) or $]a,b[$	open interval
$[a,b]$	closed interval
$n!$ (factorial)	$(n-1)(n-2)\ldots 3\cdot 2\cdot 1$
$n!!$ (double factorial)	$n(n-2)(n-4)\ldots 4\cdot 2$ for even n $n(n-2)(n-4)\ldots 3\cdot 1$ for odd n
$\mathbb{R}^N_+$	(open) upper half-space $\mathbb{R}^N_+ = \{z=(x,y) : x \in \mathbb{R}^{N-1}, y > 0\}$

Contents

Chapter 1

Preliminaries

In this chapter we set the notation and introduce some known results which will be useful in the sequel

Let $N \geq 2$ be a natural number. If $\mathcal{X}$ is any Hilbert space of scalar functions with inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$, by $\mathcal{X}^N$ we denote the Hilbert space of vector-valued functions whose components belong to $\mathcal{X}$, endowed with inner product

$$\langle u, v \rangle_{\mathcal{X}^N} = \sum_{i=1}^{N} \langle u_i, v_i \rangle_{\mathcal{X}}$$

for all $u = (u_1, \dots, u_N), v = (v_1, \dots, v_N) \in \mathcal{X}^N$. Moreover, the Banach spaces we introduce shall be interpreted as vector spaces over the field of real numbers $\mathbb{R}$, since we will deal with self-adjoint operators. In this sense, e.g., if Ω is a measurable subset of $\mathbb{R}^3$ and $L^2(\Omega)$ is the standard Lebesgue space of square integrable real-valued functions, the space $L^2(\Omega)^3$ is endowed with the following inner product

$$\langle u, v \rangle_{L^2(\Omega)^3} = \int_\Omega u \cdot v \, dx = \int_\Omega (u_1 v_1 + u_2 v_2 + u_3 v_3) \, dx \qquad \forall u, v \in L^2(\Omega)^3.$$

Let Ω be an open set in $\mathbb{R}^N$. Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. By $W^{k,p}(\Omega)$ we denote the standard Sobolev spaces of functions in $L^p(\Omega)$ possessing weak derivatives in $L^p(\Omega)$ up to the order $k \in \mathbb{N}$. We denote the space $W^{k,2}(\Omega)$ also by $H^k(\Omega)$. If in addition $\Omega \subset \mathbb{R}^N$ is bounded and of class C^1, by $H^{1/2}(\partial\Omega)$ we denote the trace space of $H^1(\Omega)$.

As usual, we use the same symbol to denote weak and classical derivatives. Also, the terms weak and distributional will be interchangeable.

We define $\mathcal{D}(\Omega)$ to be the linear space of infinitely (smooth) differentiable functions with compact support in Ω. We shall often denote it also by $C_c^\infty(\Omega)$.

Then we set
$$\mathcal{D}(\bar{\Omega}) := \left\{ \varphi|_{\Omega} : \varphi \in \mathcal{D}(\mathbb{R}^N) \right\},$$
which is the space of smooth compactly supported functions restricted to Ω.

In this thesis we will repeatedly need to consider the (boundary) regularity of the set Ω. This means that Ω can be described in a neighbourhood of any point of its boundary as the subgraph of a sufficiently regular function g defined in a local system of orthogonal coordinates. The regularity of Ω depends on the regularity of the function g. Following [26] and [54], we find convenient to introduce the notion of *atlas*. Given a set $V \subset \mathbb{R}^N$ and a parameter $\rho > 0$, we write $V_\rho \colon = \left\{ x \in V : d(x, \partial V) > \rho \right\}$, where $d(x, S)$ is the Euclidean distance from x to the set S.

Definition 1.0.1. Let $N \geq 2$ be a natural number. Let $\rho > 0, s, s' \in \mathbb{N}, s' \leq s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids (i.e. rotations of rectangle parallelepipeds in $\mathbb{R}^N$) and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$. We say that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$.

A bounded domain Ω in $\mathbb{R}^N$ is said to be of class $C_M^{k,\gamma}(\mathcal{A})$ with $k \in \mathbb{N}$, $\gamma \in [0, 1]$ and $M > 0$ if it satisfies the following conditions:

(i) $\Omega \subset \bigcup_{j=1}^s (V_j)_\rho$ and $(V_j)_\rho \cap \Omega \neq \emptyset$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ for $j = 1, \ldots, s'$ and $V_j \cap \partial\Omega = \emptyset$ for $s' + 1 \leq j \leq s$;

(iii) for $j = 1, \ldots, s$ we have
$$r_j(V_j) = \left\{ x \in \mathbb{R}^N : a_{ij} < x_i < b_{ij}, i = 1, \ldots, N \right\},$$
for $j = 1, \ldots, s'$ we have
$$r_j(V_j \cap \Omega) = \left\{ x = (\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W_j, a_{Nj} < x_N < g_j(\bar{x}) \right\},$$
where $\bar{x} = (x_1, x_2)$,
$$W_j = \left\{ \bar{x} \in \mathbb{R}^{N-1}, a_{ij} < x_i < b_{ij}, i = 1, \ldots, N-1 \right\}$$
and the functions $g_j \in C^{k,\gamma}(\overline{W_j})$ for any $j = 1, \ldots, s'$. Moreover, for $j = 1, \ldots, s'$
$$a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho$$
for all $\bar{x} \in \overline{W_j}$.

(iv)

$$\sup_{|\alpha|\le k}\|D^\alpha g_j\|_{L^\infty(W_j)} + \sup_{|\alpha|=k}\sup_{\substack{\bar{x},\bar{y}\in W_j\\ \bar{x}\neq\bar{y}}}\frac{|D^\alpha g_j(\bar{x})-D^\alpha g_j(\bar{y})|}{|\bar{x}-\bar{y}|^\gamma}\le M$$

for $j = 1,\ldots,s'$.

We say that Ω is of class $C^{k,\gamma}(\mathcal{A})$ if it is of class $C^{k,\gamma}_M(\mathcal{A})$ for some $M > 0$; we say that Ω is of class $C^{k,\gamma}$ if it is of class $C^{k,\gamma}(\mathcal{A})$ for some atlas $\mathcal{A}$.

We shall often use the terminology “Lipschitz continuous" (or simply *Lipschitz*) instead of “of class $C^{0,1}$".

In the following sections we will introduce some useful tools, spaces and known results; we will mainly follow Chapters 2 and 3 of [62]. More details can also be found in [31], [49], [62], [106].

From now on, unless otherwise specified, the set Ω is assumed to be a bounded open set in $\mathbb{R}^3$.

1.1 The div operator

If $v = (v_1, v_2, v_3)$ is a smooth vector field defined in Ω, then the divergence of v is the (smooth) scalar function defined in Ω as follows

$$\operatorname{div} v = \sum_{i=1}^{3}\frac{\partial v}{\partial x_i}.$$

Using integration by parts we can define the more general notion of weak (or distributional) divergence for a more generic vector field, in the same way one can define the standard weak derivatives when dealing with Sobolev spaces. Let $v \in L^1_{loc}(\Omega)^3$; we say that the vector field v has a weak divergence if there exists a function $g \in L^1_{loc}(\Omega)$ such that the following integration by parts formula

$$\int_\Omega v\cdot\nabla\varphi\,dx = -\int_\Omega g\,\varphi\,dx$$

holds for all $\varphi \in \mathcal{D}(\Omega)$. The function g is called the *weak divergence* of v, and it is denoted by $\operatorname{div} v$. Thus we can write

$$\int_\Omega v\cdot\nabla\varphi\,dx = -\int_\Omega \operatorname{div} v\,\varphi\,dx \quad \text{for all } \varphi\in\mathcal{D}(\Omega).$$

Obviously, the weak divergence is uniquely defined up to sets of zero measure.

We can now introduce the space of square integrable vector fields with distributional divergence in $L^2(\Omega)$, namely

$$H(\operatorname{div},\Omega) := \left\{ v \in L^2(\Omega)^3 : \text{ there exists } \operatorname{div} v \in L^2(\Omega) \right\}.$$

This space becomes a Hilbert space when endowed with the following inner product:

$$\langle u, v\rangle_{\operatorname{div}} := \int_\Omega u \cdot v \, dx + \int_\Omega \operatorname{div} u \, \operatorname{div} v \, dx \quad \text{for all } u, v \in H(\operatorname{div},\Omega).$$

The norm associated with the inner product is denoted by $||u||_{H(\operatorname{div},\Omega)}$ and it is defined as follows

$$\|u\|_{H(\operatorname{div},\Omega)} := \left(\|u\|^2_{L^2(\Omega)^3} + \|\operatorname{div} u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

The following theorem provides a density result for the space $H(\operatorname{div},\Omega)$.

Theorem 1.1.1 ([62, Thm. 2.4])**.** *Let Ω be a Lipschitz open set in $\mathbb{R}^3$ (not necessarily bounded). Then the space $\mathcal{D}(\bar{\Omega})^3$ is dense in $H(\operatorname{div},\Omega)$.*

We also introduce the subspace $H_0(\operatorname{div},\Omega)$ defined as the closure in $H(\operatorname{div},\Omega)$ of smooth compactly supported vector fields, namely

$$H_0(\operatorname{div},\Omega) := \overline{\mathcal{D}(\Omega)^3}^{H(\operatorname{div},\Omega)}.$$

In order to better characterize the above defined space, we need to introduce the concept of *normal trace* for vector fields in $H(\operatorname{div},\Omega)$. By γ_n we denote the *normal trace operator*, which for smooth vector fields $\varphi \in \mathcal{D}(\bar{\Omega})^3$ is defined by $\gamma_n \varphi := \nu \cdot \varphi|_\Omega$. The following theorem allows us to define the normal trace for general vector fields in $H(\operatorname{div},\Omega)$.

Theorem 1.1.2. *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^3$. The mapping γ_n can be extended by continuity to a linear continuous mapping, still denoted γ_n, from $H(\operatorname{div},\Omega)$ to $H^{-1/2}(\partial\Omega)$. Moreover, the following Green's formula*

$$\int_\Omega v \cdot \nabla\varphi \, dx + \int_\Omega \operatorname{div} v \, \varphi \, dx = \langle \gamma_n v, \varphi \rangle_{\partial\Omega} \tag{1.1.3}$$

holds for all $v \in H(\operatorname{div},\Omega)$ and $\varphi \in H^1(\Omega)$. Here $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, which in the case $v \in H^1(\Omega)^3$ reads

$$\langle \gamma_n v, \varphi \rangle_{\partial\Omega} = \langle \nu \cdot v, \varphi \rangle_{\partial\Omega} = \int_{\partial\Omega} (\nu \cdot v) \, \varphi \, d\sigma.$$

Proof. To see the validity of the extension one can look up the proof of Thm. 2.5 of [62].

Formula (1.1.3) is easily deduced from the standard Gauss theorem for smooth functions and using the density of $\mathcal{D}(\bar{\Omega})$ in $H^1(\Omega)$ together with Theorem 1.1.1. □

For the sake of simplicity and to avoid confusion, we denote the normal component $\gamma_n v$ of v by $\nu \cdot v|_{\partial\Omega}$. Moreover, the notation for the restriction to $\partial\Omega$ of the function shall be often omitted, since it will be clear from the context.

If Ω is a bounded Lipschitz open set in $\mathbb{R}^3$, it turns out that the space $H_0(\mathrm{div}, \Omega)$ is exactly the kernel in $H(\mathrm{div}, \Omega)$ of the normal trace γ_n (see e.g. [62, Thm. 2.6]), so that we can write

$$H_0(\mathrm{div}, \Omega) = \left\{ v \in H(\mathrm{div}, \Omega) : \gamma_n v = \nu \cdot v|_{\partial\Omega} = 0 \right\}.$$

1.1.1 The case of non-unitary permittivity

Let $\varepsilon = (\varepsilon_{ij}(x))_{1 \le i,j \le 3}, x \in \Omega$ be a symmetric positive definite 3×3 real matrix-valued function, and let $m_\varepsilon, M_\varepsilon > 0$ be two positive constants such that

$$m_\varepsilon |\zeta|^2 \le \varepsilon(x)\zeta \cdot \zeta \le M_\varepsilon |\zeta|^2 \tag{1.1.4}$$

for all $(x, \zeta) \in \Omega \times \mathbb{R}^3$. By (1.1.4) it follows that the standard inner product of $L^2(\Omega)^3$ and

$$\int_\Omega \varepsilon u \cdot v \, dx \quad \text{for all } u, v \in L^2(\Omega)^3$$

give rise to equivalent norms in $L^2(\Omega)^3$.

A vector field $v \in L^1_{loc}(\Omega)^3$ has a *weak ε-divergence* if there exists a function in $L^1_{loc}(\Omega)$, denoted by $\mathrm{div}(\varepsilon v)$, such that

$$\int_\Omega \varepsilon u \cdot \nabla\varphi \, dx = -\int_\Omega \mathrm{div}(\varepsilon v)\varphi \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega)^3.$$

The set $H(\mathrm{div}\, \varepsilon, \Omega)$ is defined to be the space of all vector fields in $L^2(\Omega)^3$ with weak ε-divergence in $L^2(\Omega)^3$ and is endowed with the following inner product

$$\langle u, v \rangle_{\mathrm{div}\, \varepsilon} := \int_\Omega \varepsilon u \cdot v \, dx + \int_\Omega \mathrm{div}(\varepsilon u)\, \mathrm{div}(\varepsilon v) \, dx \qquad \text{for all } u, v \in H(\mathrm{div}\, \varepsilon, \Omega),$$

which makes it a Hilbert space.

1.2 The curl operator

Given a smooth vector field $v = (v_1, v_2, v_3)$ defined in Ω, the curl (or rotor) of v is the (smooth) vector field defined in Ω by

$$\operatorname{curl} v := \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

Using a more compact notation, the k-th component of the curl of v can be written as follows

$$(\operatorname{curl} v)_k = \sum_{i,j=1}^{3} \frac{\partial v_j}{\partial x_i}\, \xi_{ijk} = \sum_{i,j=1}^{3} \partial_i v_j\, \xi_{ijk}, \tag{1.2.1}$$

for any $k = 1, 2, 3$, where ξ_{ijk} is the Levi-Civita symbol defined by

$$\xi_{ijk} := \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ \ \ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } i = k. \end{cases}$$

Similarly to what we have done for the divergence, we can define the weak (or distributional) curl for a generic vector field. Given $v \in L^1_{loc}(\Omega)^3$, we say that that the vector field v has a weak curl if there exists a vector field $w \in L^1_{loc}(\Omega)^3$ such that the following integration by parts formula

$$\int_\Omega v \cdot \operatorname{curl} \varphi \, dx = \int_\Omega w \cdot \varphi \, dx$$

holds for all $\varphi \in \mathcal{D}(\Omega)^3$. The vector field w is called the *weak curl* of v, and it is denoted by $\operatorname{curl} v$. Thus we can write

$$\int_\Omega v \cdot \operatorname{curl} \varphi \, dx = \int_\Omega \operatorname{curl} v \cdot \varphi \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega)^3.$$

Again, the weak curl is uniquely defined up to sets of zero measure.

We now introduce the space $H(\text{curl}, \Omega)$, containing all vector fields in $L^2(\Omega)^3$ with distributional curl in $L^2(\Omega)^3$:

$$H(\text{curl}, \Omega) := \left\{ v \in L^2(\Omega) : \text{ there exists } \operatorname{curl} v \in L^2(\Omega)^3 \right\}.$$

If we endow it with the following inner product

$$\langle u, v \rangle_{\text{curl}} := \int_\Omega u \cdot v \, dx + \int_\Omega \operatorname{curl} u \cdot \operatorname{curl} v \, dx \quad \text{for all } u, v, \in H(\text{curl}, \Omega)$$

it becomes a Hilbert space, and the associated norm is

$$\|u\|_{H(\text{curl},\Omega)} = \left(\|u\|^2_{L^2(\Omega)^3} + \|\operatorname{curl} u\|^2_{L^2(\Omega)^3} \right)^{1/2}.$$

Similarly to the case of the divergence, we also have the following density result.

Theorem 1.2.2 ([62, Thm. 2.10]). *Let Ω be a Lipschitz open set in $\mathbb{R}^3$ (not necessarily bounded). Then the space $\mathcal{D}(\bar{\Omega})^3$ is dense in $H(\mathrm{curl}, \Omega)$.*

We define the space

$$H_0(\mathrm{curl}, \Omega) := \overline{\mathcal{D}(\Omega)^3}^{H(\mathrm{curl},\Omega)}$$

to be the closure in $H(\mathrm{curl}, \Omega)$ of smooth compactly supported vector fields. A useful characterization of this space is given by the following lemma.

Lemma 1.2.3 ([62, Lemma 2.4]). *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^3$. If a vector field $v \in H(\mathrm{curl}, \Omega)$ is such that*

$$\int_\Omega v \cdot \mathrm{curl}\, \varphi \, dx = \int_\Omega \mathrm{curl}\, v \cdot \varphi \, dx \quad \text{for all } \varphi \in \mathcal{D}(\bar{\Omega})^3,$$

then $v \in H_0(\mathrm{curl}, \Omega)$.

Remark 1.2.4. Assume Ω is an arbitrary open set in $\mathbb{R}^3$. Consider the densely defined unbounded operator $A : D(A) = H_0(\mathrm{curl}, \Omega) \subset L^2(\Omega)^3 \to L^2(\Omega)^3$, $Av = \mathrm{curl}\, v$ for any $v \in D(A)$. It is easy to see that the adjoint A^* has domain $D(A^*) = H(\mathrm{curl}, \Omega) \subset L^2(\Omega)^3$ and it is such that $A^* v = \mathrm{curl}\, v$ for any $v \in D(A^*)$. At this point, observing that $A^{**} = A$ since A is closed, one realizes that $H_0(\mathrm{curl}, \Omega)$ is exactly composed by those vector fields $v \in H(\mathrm{curl}, \Omega)$ such that

$$\int_\Omega v \cdot \mathrm{curl}\, u \, dx = \int_\Omega \mathrm{curl}\, v \cdot u \, dx \quad \text{for all } u \in H(\mathrm{curl}, \Omega).$$

Thus, Lemma 1.2.3 holds for any open set such that $\mathcal{D}(\bar{\Omega})^3$ is dense in $H(\mathrm{curl}, \Omega)$. However, for our purposes it is enough to assume that Ω has a Lipschitz boundary.

The following proposition provides another characterization of the space $H_0(\mathrm{curl}, \Omega)$ and is analogous to the well-known characterization of the Sobolev space $H^1_0(\Omega)$ (see e.g., [18, Prop. 9.18]). We include here two proofs: one makes use of Lemma 1.2.3, while the other one is based directly on the argument in [18, Prop. 9.18]. By v^0 we denote the extension-by-zero of a vector field v, that is

$$v^0 := \begin{cases} v & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus \Omega\, . \end{cases}$$

Proposition 1.2.5. *Let Ω be a bounded open set of class $C^{0,1}$ and $u \in H(\mathrm{curl}, \Omega)$. Then the following are equivalent:*

(i) $u \in H_0(\mathrm{curl}, \Omega)$;

(ii) *the function*

$$u^0 = \begin{cases} u & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus \Omega, \end{cases}$$

belongs to $H(\mathrm{curl}, \mathbb{R}^3)$.

In this case $\mathrm{curl}(u^0) = (\mathrm{curl}\, u)^0$.

Proof. The implication (i)⇒(ii) is trivial and does not need any regularity requirement on the boundary of Ω. Indeed, if $u \in H_0(\mathrm{curl}, \Omega)$ then there exists a sequence of $(C_c^\infty(\Omega))^3$ functions that converges strongly in $H(\mathrm{curl}, \Omega)$ to u. Then the same sequence converges strongly to the function u^0 in $H(\mathrm{curl}, \mathbb{R}^3)$, hence $u^0 \in H(\mathrm{curl}, \mathbb{R}^3)$.

The other implication is more involved. We first present a rather simple argument which exploits Lemma 1.2.3. Suppose that u^0 belongs to $H(\mathrm{curl}, \mathbb{R}^3)$. Hence, there exists $w \in L^2(\mathbb{R}^3)^3$ such that

$$\int_\Omega u \cdot \mathrm{curl}\, \varphi \, dx = \int_{\mathbb{R}^3} w \cdot \varphi \, dx \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^3)^3. \tag{1.2.6}$$

Since it holds in particular for all test functions $\varphi \in \mathcal{D}(\Omega)^3$, then necessarily $w = \mathrm{curl}\, u$ on Ω. On the other hand, since we can take any $\varphi \in \mathcal{D}(\mathbb{R}^3 \setminus \overline{\Omega})^3$, we see that $w = 0$ outside Ω. Thus we can rewrite (1.2.6) as follows

$$\int_\Omega u \cdot \mathrm{curl}\, \varphi \, dx = \int_\Omega \mathrm{curl}\, u \cdot \varphi \, dx \qquad \text{for all } \varphi \in \mathcal{D}(\bar{\Omega})^3. \tag{1.2.7}$$

By Lemma 1.2.3 it follows that $u \in H_0(\mathrm{curl}, \Omega)$.

It is worthy to note that in the above argument the difficulty is hidden in the proof of Lemma 1.2.3. □

Alternative proof of Proposition 1.2.5. The second argument which we propose here to prove the implication (ii)⇒(i) is adapted from the proof of Proposition 9.18 of [18]. Set

$$\mathbb{R}^3_- := \left\{ x = (\bar{x}, x_3) \in \mathbb{R}^2 \times \mathbb{R} : x_3 < 0 \right\},$$

$$Q := \left\{ x = (\bar{x}, x_3) \in \mathbb{R}^3 : |\bar{x}| < 1, |x_3| < 1 \right\}$$

and $Q_- := Q \cap \mathbb{R}^3_-$. By using a standard partition of unity and passing to local charts one is reduced to solve the following problem. Let $u \in L^2(Q_-)^3$ be such that the function

$$u^0 = \begin{cases} u & \text{if } x \in Q,\ x_3 < 0, \\ 0 & \text{if } x \in Q,\ x_3 > 0, \end{cases}$$

belongs to $H(\mathrm{curl}, Q)$; prove that

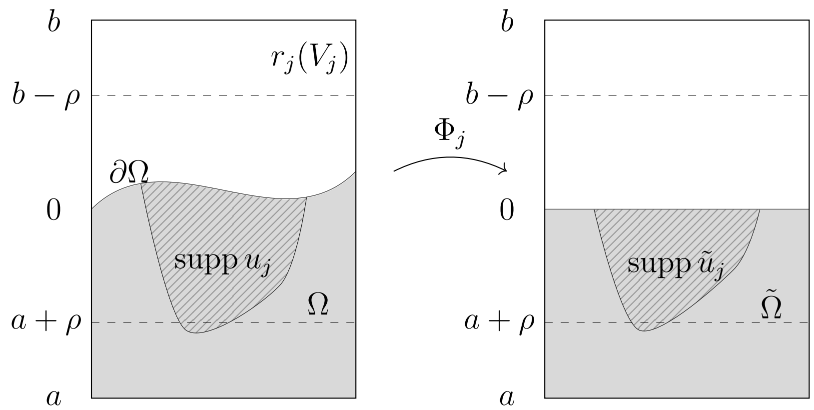
$$\beta u \in H_0(\mathrm{curl}, Q) \qquad \text{for all } \beta \in C_c^\infty(Q). \tag{1.2.8}$$

Indeed, following the notation used in Definition 1.0.1, let $\{\psi_j\}_{j=1}^s$ denote a partition of unity associated with the open cover $\{V_j\}_{j=1}^s$ of Ω. Write u as

$$u = \sum_{j=1}^s u\psi_j = \sum_{j=1}^s u_j,$$

with $u_j = u\,\psi_j \in H(\mathrm{curl}, \Omega)$. Then for every $j = 1, \dots, s$ we flatten the part of the boundary of Ω that affects ψ_j, i.e. the one relative to the cuboid V_j, through the local coordinate chart. Namely, we first straighten up the cuboid V_j using the rotation r_j, to get the new cuboid $r_j(V_j)$ and write $r_j(V_j \cap \Omega) = \big\{x = (\bar{x}, x_3) \in \mathbb{R}^3 : \bar{x} \in W_j, a_{3j} < x_3 < g_j(\bar{x})\big\}$. We assume without loss of generality that $a_{3j} + \rho < 0$, $b_{3j} - \rho > 0$, and g_j is a non-negative function of class $C^{0,1}(\overline{W}_j)$. We can then define a diffeomorphism of class $C^{0,1}$ that flattens the boundary, for example $\Phi_j : r_j(V_j \cap \Omega) \to r_j(V_j \cap \Omega)$, $\Phi_j(x) = (\bar{x}, x_3 - h_j(\bar{x}, x_3))$, with

$$h_j(\bar{x}, x_3) := \begin{cases} 0, & \text{if } |a_{3j} - x_3| \le \rho \text{ or } |b_{3j} - x_3| \le \rho, \\ g_j(\bar{x}) \left(\frac{x_3 - (a_{3j}+\rho)}{g_j(\bar{x}) - (a_{3j}+\rho)}\right)^2, & \text{if } a_{3j} + \rho < x_3 \le g_j(\bar{x}), \\ g_j(\bar{x}) \left(\frac{(b_{3j}-\rho) - x_3}{(b_{3j}-\rho) - g_j(\bar{x})}\right)^2, & \text{if } g_j(\bar{x}) < x_3 < b_{3j} - \rho. \end{cases}$$



Flattening of the boundary.

We can then work with the push-forward $\tilde{u}_j$ of u_j under the covariant Piola transform associated with the diffeomorphism $\Phi_j \circ r_j$ (cf. (2.2.3)). Observe that

$\tilde{u}_j \in H(\mathrm{curl}, r_j(V_j \cap \Omega))$ since $\Phi_j \circ r_j$ is of class $C^{0,1}$ (see Theorem 2.2.4 and Remark 2.2.8). Moreover, note that the push-forward of the extension-by-zero u_j^0 of u_j is exactly the extension-by-zero of the push-forward $\tilde{u}_j$ of u_j, and thus belongs to $H(\mathrm{curl}, \mathbb{R}^3$ by hypothesis. Hence, supposing that we managed to prove (1.2.8), we have that $\tilde{u}_j \in H_0(\mathrm{curl}, r_j(V_j \cap \Omega_j))$, and thus we find a sequence of functions in $C_c^\infty(r_j(V_j \cap \Omega_j))^3$ approximating $\tilde{u}_j$ in $H(\mathrm{curl}, (r_j(V_j \cap \Omega_j))$. Then, pulling back these sequence via the covariant Piola transform, we get a sequence $\{\zeta_n^j\}_{n\in\mathbb{N}} \subset H(\mathrm{curl}, \Omega)$ of functions approximating u_j in $H(\mathrm{curl}, \Omega)$, with compact support in Ω (to be precise in $V_j \cap \Omega$). We can do this for every $j = 1, \dots, s$ and finally, setting $\zeta_n := \sum_j \zeta_n^j$ for all $n \in \mathbb{N}$, we get that $\zeta_n \in H(\mathrm{curl}, \Omega)$ with compact support in Ω and

$$\zeta_n \xrightarrow[n\to\infty]{H(\mathrm{curl},\Omega)} \sum_{j=1}^{s} u_j = u. \tag{1.2.9}$$

Since $C_c^\infty(\bar{\Omega})^3$ is dense in $H(\mathrm{curl}, \Omega)$ (see Theorem 1.2.2) and ζ_n has compact support in Ω, the limit (1.2.9) is sufficient to guarantee that $u \in H_0(\mathrm{curl}, \Omega)$.

Let us prove (1.2.8). If $\rho \in C_c^\infty(\mathbb{R}^3)$ is a mollifier and $f \in H(\mathrm{curl}, \mathbb{R}^3)$, denoting with $\boldsymbol{\rho}$ the vector field (ρ, ρ, ρ), we have that $\boldsymbol{\rho} * f \in C^\infty(\mathbb{R}^3)^3 \cap H(\mathrm{curl}, \mathbb{R}^3)$ with $\mathrm{curl}(\boldsymbol{\rho} * f) = \boldsymbol{\rho} * \mathrm{curl}\, f$. We introduce now a sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^3)$ of mollifiers such that

$$\mathrm{supp}\, \rho_n \subset \left\{ x \in \mathbb{R}^3 : -\frac{1}{n} < x_3 < -\frac{1}{2n} \right\}.$$

For example we can set $\rho_n(x) = n^3\rho(nx)$, where the kernel of mollification ρ is such that $\mathrm{supp}\, \rho \subset \{x \in \mathbb{R}^3 : -1 < x_3 < -1/2\}$.

Let β be in $C_c^\infty(Q)$. By hypothesis $u^0 \in H(\mathrm{curl}, Q)$, and obviously $\mathrm{curl}\, u^0 = (\mathrm{curl}\, u)^0$. By an approximation argument (using Theorem 1.2.2), it is not difficult to see that the vector field βu^0 belongs to $H(\mathrm{curl}, Q)$ as well, with $\mathrm{curl}(\beta u^0) = \nabla\beta \times u^0 + \beta(\mathrm{curl}\, u)^0$. Note that since $\mathrm{supp}(\beta) \subset\subset Q$, then βu^0 extended by 0 outside Q belongs to $H(\mathrm{curl}, \mathbb{R}^3)$. We have that $\boldsymbol{\rho}_n * \beta u^0 \in C_c^\infty(\mathbb{R}^3)^3$ with $\mathrm{curl}(\boldsymbol{\rho}_n * (\beta u^0)) = \boldsymbol{\rho}_n * \mathrm{curl}(\beta u^0)$ for all $n \in \mathbb{N}$. Since βu^0 and $\mathrm{curl}(\beta u^0)$ are in $L^2(\mathbb{R}^3)^3$, then $\boldsymbol{\rho}_n * (\beta u^0) \to \beta u^0$ in $L^2(\mathbb{R}^3)^3$ and $\mathrm{curl}(\boldsymbol{\rho}_n * (\beta u^0)) \to \mathrm{curl}(\beta u^0)$ in $L^2(\mathbb{R}^3)^3$ (see, e.g., [18, Thm. 4.22]). Hence $\boldsymbol{\rho}_n * (\beta u^0) \to \beta u^0$ in $H(\mathrm{curl}, \mathbb{R}^3)$. Moreover

$$\mathrm{supp}(\boldsymbol{\rho}_n * \beta u^0) \subset \mathrm{supp}\, \boldsymbol{\rho}_n + \mathrm{supp}(\beta u^0) \subset\subset Q_-$$

for n large enough. It follows that for n large enough

$$\boldsymbol{\rho}_n * (\beta u^0) \in C_c^\infty(Q_-)^3$$

and thus $\beta u^0 \in H_0(\mathrm{curl}, Q_-)$. □

In order to further characterize the space $H_0(\operatorname{curl}, \Omega)$, we need to introduce the *tangential trace* of a vector field in $\mathbb{R}^3$. By γ_t we denote the *tangential trace operator* which for smooth vector fields $\varphi \in \mathcal{D}(\bar{\Omega})^3$ is defined by $\gamma_t \varphi := \nu \times \varphi|_{\partial\Omega}$. We have the following theorem.

Theorem 1.2.10 ([62, Thm. 2.11])**.** *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^3$. The mapping γ_t can be extended by continuity to a linear and continuous mapping, still denoted by γ_t, from $H(\operatorname{curl}, \Omega)$ to $H^{-1/2}(\partial\Omega)^3$. Moreover, the following Green's formula holds*

$$\int_\Omega \operatorname{curl} v \cdot \varphi \, dx - \int_\Omega v \cdot \operatorname{curl} \varphi \, dx = \langle \gamma_t v, \varphi \rangle_{\partial\Omega} \tag{1.2.11}$$

for all $v \in H(\operatorname{curl}, \Omega)$ and $\varphi \in H^1(\Omega)^3$. Here $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the pairing between $H^{-1/2}(\Omega)^3$ and $H^{1/2}(\Omega)^3$, which in the case $v \in H^1(\Omega)^3$ reads

$$\langle \gamma_t v, \varphi \rangle_{\partial\Omega} = \langle \nu \times v, \varphi \rangle_{\partial\Omega} = \int_{\partial\Omega} (\nu \times u) \cdot \varphi \, d\sigma.$$

For an easier reading, we shall denote $\gamma_t v$ by $\nu \times v|_{\partial\Omega}$, often omitting the restriction notation.

Therefore, in the case Ω is a bounded a Lipschitz open set in $\mathbb{R}^3$, by Lemma 1.2.3 and Theorem 1.2.10 one can deduce that $H_0(\operatorname{curl}, \Omega)$ is the kernel in $H(\operatorname{curl}, \Omega)$ of the map γ_t (cf. [62, Thm. 2.12]), i.e.

$$H_0(\operatorname{curl}, \Omega) = \operatorname{Ker}(\gamma_t) = \left\{ u \in H(\operatorname{curl}, \Omega) : \gamma_t v = \nu \times v|_{\partial\Omega} = 0 \right\}.$$

Proposition 1.2.12. *Let Ω be a bounded open set in $\mathbb{R}^3$. The following statements hold.*

(i) *If $v \in H(\operatorname{curl}, \Omega)$, then $\operatorname{curl} v \in H(\operatorname{div}, \Omega)$ and $\operatorname{div} \operatorname{curl} v = 0$ a.e. in Ω.*

(ii) *If $q \in H^1(\Omega)$, then $\nabla q \in H(\operatorname{curl}, \Omega)$ and $\operatorname{curl} \nabla q = 0$ a.e. in Ω. Moreover, if $q \in H_0^1(\Omega)$, then $\nabla q \in H_0(\operatorname{curl}, \Omega)$.*

Proof. Let us begin with the first point. Let $\varphi \in \mathcal{D}(\Omega)$ be a test function. Then, using formula (1.2.11) we get

$$\int_\Omega \operatorname{curl} v \cdot \nabla\varphi \, dx = \int_\Omega v \cdot \operatorname{curl}(\nabla\varphi) \, dx = 0,$$

where the last equality if trivial since the curl of a gradient is always zero. Hence, by definition, there exists the distributional divergence of $\operatorname{curl} v$, and it is equal to zero.

Now let us move to the second statement. Let now φ be a test function in $\mathcal{D}(\Omega)^3$. Then, using formula (1.1.3) we get

$$\int_\Omega \nabla q \cdot \operatorname{curl} \varphi \, dx = -\int_\Omega q \operatorname{div} \operatorname{curl} \varphi \, dx = 0,$$

where the last equality is due to the fact that the divergence of a curl is always zero. Thus we can conclude that there exists the distributional curl of ∇q, which is equal to zero in Ω.

Note that in both cases we have used results (formulas (1.1.3) and (1.2.11)) that require Ω to be Lipschitz. This is not a problem since φ is compactly supported in Ω.

Finally, if $q \in H_0^1(\Omega)$, then there exist a sequence of smooth functions $\{\varphi_n\}_{n\in\mathbb{N}}$ with compact support in Ω such that $\varphi_n \to q$ in $H^1(\Omega)$ as $n \to \infty$. Hence in particular $\nabla\varphi_n \to \nabla q$ in $L^2(\Omega)^3$ as $n \to \infty$. Since $\operatorname{curl} \nabla\varphi_n = 0$ for all $n \in \mathbb{N}$, we have that the sequence of compactly supported vector fields $\{\nabla\varphi_n\}_{n\in\mathbb{N}}$ converges to ∇q in $H(\operatorname{curl}, \Omega)$, hence by definition $\nabla q \in H_0(\operatorname{curl}, \Omega)$. □

If Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, its boundary $\partial\Omega$ can be decomposed into a finite number of connected components, which we denote by Γ_i, $0 \leq i \leq I$, with Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$. Hence

$$\partial\Omega = \bigsqcup_{i=0}^{I} \Gamma_i \,. \tag{1.2.13}$$

Theorem 1.2.14 ([62, Thm. 3.4]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. For $0 \leq i \leq I$ we denote by $\mathbb{1}_{\Gamma_i}$ the indicator function for Γ_i. A vector field $v \in L^2(\Omega)^3$ satisfies*

$$\operatorname{div} v = 0 \text{ in } \Omega, \quad \langle \gamma_n v, \mathbb{1}_{\Gamma_i} \rangle_{\partial\Omega} = 0 \text{ for } 0 \leq i \leq I$$

if and only if there exists a vector potential $\phi \in H^1(\Omega)^3$ such that

$$v = \operatorname{curl} \phi.$$

Furthermore, this vector potential can be taken divergence-free, i.e. such that $\operatorname{div} \phi = 0$ in Ω.

1.3 The spaces X_{N} and X_{T}

Unless otherwise indicated, in this section we assume Ω to be a bounded open set in $\mathbb{R}^3$ of class $C^{0,1}$.

We introduce the spaces

$$X_{\mathrm{N}}(\Omega) := H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega),$$

and

$$X_{\mathrm{T}}(\Omega) := H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega),$$

or more explicitely

$$X_{\mathrm{N}}(\Omega) = \left\{ u \in L^2(\Omega)^3 : \mathrm{curl}\, u \in L^2(\Omega)^3, \mathrm{div}\, u \in L^2(\Omega) \text{ and } \nu \times u|_{\partial\Omega} = 0 \right\},$$

and

$$X_{\mathrm{T}}(\Omega) = \left\{ u \in L^2(\Omega)^3 : \mathrm{curl}\, u \in L^2(\Omega)^3, \mathrm{div}\, u \in L^2(\Omega) \text{ and } \nu \cdot u|_{\partial\Omega} = 0 \right\}.$$

Both spaces are endowed with the following inner product

$$\langle u, v \rangle_X := \int_\Omega u \cdot v \, dx + \int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, v \, dx + \int_\Omega \mathrm{div}\, u \;\mathrm{div}\, v \, dx,$$

for all $u, v \in H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$. The associated norm is

$$\|u\|_X = \left(\|u\|^2_{L^2(\Omega)^3} + \|\mathrm{curl}\, u\|^2_{L^2(\Omega)^3} + \|\mathrm{div}\, u\|^2_{L^2(\Omega)} \right)^{1/2},$$

and it will be denoted $||u||_{X_{\mathrm{N}}(\Omega)}$ or $||u||_{X_{\mathrm{T}}(\Omega)}$ depending on which space we are dealing with.

Under more regularity assumptions on Ω, we have density results also for these spaces.

Theorem 1.3.1 ([7, Lemmas 2.10 and 2.13]). *Let Ω be a bounded domain set in $\mathbb{R}^3$ of class $C^{1,1}$. Then the space $H^1(\Omega)^3 \cap X_{\mathrm{N}}(\Omega)$ is dense in the space $X_{\mathrm{N}}(\Omega)$, and the space $H^1(\Omega)^3 \cap X_{\mathrm{T}}(\Omega)$ is dense in $X_{\mathrm{T}}(\Omega)$.*

Following Section 1.1.1, we consider symmetric positive definite 3×3 real matrix-valued functions ε such that (1.1.4) holds, and introduce the space

$$X^{\varepsilon}_{\mathrm{N}}(\Omega) := H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}\, \varepsilon, \Omega).$$

We endow it with inner product

$$\langle u, v \rangle_{X^{\varepsilon}} := \int_\Omega \varepsilon u \cdot v \, dx + \int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, v \; + \int_\Omega \mathrm{div}(\varepsilon u) \;\mathrm{div}(\varepsilon v) \, dx$$

for all $u, v \in X^{\varepsilon}_{\mathrm{N}}(\Omega)$.

Finally, we set

$$X_{\mathrm{N}}(\operatorname{div} 0, \Omega) := \{u \in X_{\mathrm{T}}(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega\}$$

and

$$X_{\mathrm{N}}^{\varepsilon}(\operatorname{div} \varepsilon\, 0, \Omega) := \{u \in X_{\mathrm{T}}(\Omega) : \operatorname{div}(\varepsilon u) = 0 \text{ in } \Omega\}.$$

A key difference between the spaces $H^1(\Omega)^3$ and $H(\mathrm{curl}, \Omega)$ is that the embedding of $H(\mathrm{curl}, \Omega)$ or $H(\mathrm{div}, \Omega)$ into $L^2(\Omega)^3$ is not compact. Hence, the determination of suitable spaces that are compactly embedded in $L^2(\Omega)^3$ is an important issue.

If Ω is sufficiently regular, say of class $\mathcal{C}^{1,1}$, the spaces $X_{\mathrm{N}}(\Omega)$ and $X_{\mathrm{T}}(\Omega)$ are continuously embedded into the space $H^1(\Omega)^3$ of vector fields with components in the standard Sobolev space $H^1(\Omega)$. On the other hand, since $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, it follows that also $X_{\mathrm{N}}(\Omega)$ and $X_{\mathrm{T}}(\Omega)$ are compactly embedded into $L^2(\Omega)^3$.

We note that the compactness of the embeddings holds under weaker assumptions on the regularity of Ω. Indeed for a very large class of sets, for example for bounded domains satisfying the "restricted cone property" (see Definition 1.1 of [113]), the spaces $X_{\mathrm{N}}(\Omega)$ and $X_{\mathrm{T}}(\Omega)$ are compactly embedded in $L^2(\Omega)^3$.

The proof of following theorem can be found in [62, Lemma 3.4, Theorem 3.7].

Theorem 1.3.2. *The following statements hold.*

(i) *If Ω is a bounded, simply connected open set in $\mathbb{R}^3$ of class $C^{0,1}$ and $\partial\Omega$ has only one connected component then there exists $C > 0$ such that*

$$\|u\|_{L^2(\Omega)^3} \leq C \, \|\mathrm{curl} u\|_{L^2(\Omega)^3} ,$$

for all $u \in X_{\mathrm{N}}(\operatorname{div} 0, \Omega)$, and

$$\|u\|_{L^2(\Omega)^3} \leq C \, \left(\|\mathrm{curl} u\|_{L^2(\Omega)^3} + \|\mathrm{div} u\|_{L^2(\Omega)}\right) ,$$

for all $u \in X_{\mathrm{N}}(\Omega)$. Moreover, the embedding $X_{\mathrm{N}}(\Omega) \subset L^2(\Omega)^3$ is compact.

(ii) *If Ω is a bounded open set in $\mathbb{R}^3$ of class $C^{1,1}$ then $X_{\mathrm{N}}(\Omega)$ is continuously embedded into $H^1(\Omega)^3$, and there exists $C > 0$ such that the Gaffney inequality*

$$\|u\|_{H^1(\Omega)^3} \leq C \, \left(\|u\|_{L^2(\Omega)^3} + \|\mathrm{curl} u\|_{L^2(\Omega)^3} + \|\mathrm{div} u\|_{L^2(\Omega)}\right) , \qquad (1.3.3)$$

holds for all $u \in X_{\mathrm{N}}(\Omega)$. In particular, the embedding $X_{\mathrm{N}}(\Omega)) \subset L^2(\Omega)^3$ is compact.

A completely analogous result holds if we replace $X_{\mathrm{N}}(\Omega)$ by $X_{\mathrm{T}}(\Omega)$ in Theorem 1.3.2 (see [62, Lemma 3.6, Thm. 3.8]). Point (ii) of the above statement is particularly important for the study of boundary value problems for Maxwell's equations, and it will be discussed in the next section which focuses on the Gaffney inequality.

As one can expect, vector fields whose tangential and normal traces vanish, are null on the boundary.

Lemma 1.3.4 ([62, Lemma 2.5]). *Let Ω be a bounded open set in $\mathbb{R}^3$ of class $C^{0,1}$. The following equality holds algebraically and topologically:*

$$H_0^1(\Omega)^3 = H_0(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega).$$

At some point, we shall also need the following

Lemma 1.3.5. *Let Ω be a bounded open set in $\mathbb{R}^3$ of class $\mathcal{C}^{0,1}$ and $u \in H_0(\mathrm{curl}, \Omega)$. Then the normal trace of $\mathrm{curl}\, u$ exists and it is zero.*

Proof. By point (i) of Proposition 1.2.12 we have that $\mathrm{curl}\, u \in H(\mathrm{div}, \Omega)$ and $\mathrm{div}\,\mathrm{curl}\, u = 0$ in Ω. Hence, by Theorem 1.1.2, its normal trace $\gamma_n(\mathrm{curl}\, u)$ exists and belongs to $H^{-1/2}(\partial\Omega)$.

By integrating by parts using formulas (1.1.3) and (1.2.11), we get

$$\begin{aligned}\int_{\partial\Omega} (\nu \cdot \mathrm{curl}\, u)\varphi\, d\sigma &= \int_{\partial\Omega} (\nu \cdot \mathrm{curl}\, u)\varphi\, d\sigma - \int_{\Omega} \mathrm{div}\,\mathrm{curl}\, u\, \varphi\, dx \\ &= \int_{\Omega} \mathrm{curl}\, u \cdot \nabla\varphi\, dx = \int_{\Omega} u \cdot \mathrm{curl}\, \nabla\varphi\, dx + \int_{\partial\Omega} (\nu \times u) \cdot \nabla\varphi\, d\sigma = 0,\end{aligned}$$

for all $\varphi \in H^2(\Omega)$, where in the right-hand side of the first equality we have added the term $-\int_\Omega \mathrm{div}\,\mathrm{curl}\, u\, \varphi\, dx$, which is zero. Note that $\mathrm{curl}\, \nabla\varphi = 0$ in Ω (see point (ii) of Proposition 1.2.12) and $\int_{\partial\Omega} (\nu \times u) \cdot \nabla\varphi\, d\sigma = 0$ because $u \in H_0(\mathrm{curl}, \Omega)$ and thus has zero tangential trace. Hence by a standard approximation argument, we deduce that

$$\int_{\partial\Omega} (\nu \cdot \mathrm{curl}\, u)\, \varphi\, d\sigma = 0 \quad \text{for all } \varphi \in H^1(\Omega),$$

which proves that the normal trace of $\mathrm{curl}\, u$ is zero. □

A useful tool when dealing with vector fields is the celebrated Helmholtz decomposition, where any vector field in $L^2(\Omega)^3$ can be decomposed in the sum of a gradient (scalar potential) and a curl (vector potential).

Proposition 1.3.6 ([62, Coroll. 3.4]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then every vector field $v \in L^2(\Omega)^3$ can be orthogonally decomposed (in $L^2(\Omega)^3$) as follows*

$$v = \nabla q + \mathrm{curl}\, \phi,$$

where $q \in H^1(\Omega)/\mathbb{R}$ *is the unique solution of*

$$\int_\Omega \nabla q \cdot \nabla \eta \, dx = \int_\Omega v \cdot \nabla \eta \, dx \quad \text{for all } \eta \in H^1(\Omega)$$

and $\phi \in H^1(\Omega)^3$ *is such that* $\operatorname{curl} \phi \in H_0(\operatorname{div}, \Omega)$ *and* $\operatorname{div} \phi = 0$.

We also have this more specific decomposition.

Proposition 1.3.7. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^3$ *whose boundary has only one connected component. Then every function* $v \in X_{\mathrm{N}}(\Omega)$ *can be orthogonally decomposed as follows*

$$v = \nabla q + \operatorname{curl} \phi,$$

where the orthogonality is with respect to the inner product of $L^2(\Omega)^3$. *Here* $q \in H_0^1(\Omega)$, $\phi \in H^1(\Omega)^3$ *with* $\operatorname{div} \phi = 0$, *and they are such that both* ∇q *and* $\operatorname{curl} \phi$ *belong to* $X_{\mathrm{N}}(\Omega)$.

Proof. Let $v \in X_{\mathrm{N}}(\Omega)$. Let $q \in H_0^1(\Omega)$ be the (weak) solution to the following Dirichlet problem

$$\begin{cases} \Delta q = \operatorname{div} v \in L^2(\Omega), & \text{in } \Omega, \\ q = 0, & \text{on } \partial\Omega, \end{cases}$$

which in weak form reads as

$$-\int_\Omega \nabla q \cdot \nabla \varphi \, dx = \int_\Omega \operatorname{div} v \, \varphi \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

It is standard to see that $\nabla q \in X_{\mathrm{N}}(\Omega)$ and $\operatorname{div} \nabla q = \operatorname{div} v$. Set now $w := v - \nabla q$. By Proposition 1.2.12, we have that $w \in X_{\mathrm{N}}(\Omega)$ with $\operatorname{div} w = 0$ and $\operatorname{curl} w = \operatorname{curl} v$. Finally, by Theorem 1.2.14 there exists a divergence-free vector field $\phi \in H^1(\Omega)^3$ such that $w = \operatorname{curl} \phi$, hence the proof is concluded. $\square$

1.4 The Gaffney inequality

In this section we focus on the celebrated Gaffney (or Gaffney-Friedrichs) inequality, which was introduced in Theorem 1.3.2. The set Ω is assumed to be a bounded set in $\mathbb{R}^3$, with boundary of class at least $C^{0,1}$.

The inequality states that there exists a constant $C > 0$ dependingon Ω such that

$$\|u\|_{H^1(\Omega)^3} \leq C \left(\|u\|_{L^2(\Omega)^3} + \|\operatorname{curl} u\|_{L^2(\Omega)^3} + \|\operatorname{div} u\|_{L^2(\Omega)} \right) \tag{1.4.1}$$

for all vector fields $u \in L^2(\Omega)^3$ with distributional $\operatorname{curl} u \in L^2(\Omega)^3$ and $\operatorname{div} u \in L^2(\Omega)$, and having either vanishing tangential trace or vanishing normal trace on the boundary $\partial\Omega$. These estimates were proved, in the more general cases of Riemannian manifolds, first by Gaffney (see [58, 59, 60]) for manifolds without boundary, and by Friedrichs [57] for manifolds with boundary. We also recommend the book [108] by Schwarz , especially Section 2.1, and the papers [5, 45, 97, 101, 107].

We also mention the paper [38] by Costabel, where it is shown that if a vector field u has square integrable divergence and curl and is such that either the normal or the tangential component of u is square integrable over the boundary, then it belongs to the Sobolev space $H^{1/2}$ on the domain. In particular one recovers the compact embedding into L^2 of solution to Maxwell's equations (see also [7, Prop. 3.7]).

For the general case of a Gaffney inequality involving non-unitary permittivity parameters ε we refer to Prokhorov and Filonov [102, Thm. 1.1].

There exists a very close relation between Gaffney-type inequalities and a priori estimates for the Laplacian. Indeed, on bounded Lipschitz sets of $\mathbb{R}^3$ the two notions are equivalent.

We will focus on the case of vector fields with vanishing tangential trace: in this case we need to consider the Poisson problem for the Laplacian with Dirichlet boundary conditions. One of the two implications (namely, the validity of the Gaffney inequality implies the validity of the a priori estimate) is quite standard. The other one is a bit more involved and relies on a method by Birman and Solomyak [17].

In particular, as we shall see, the regularity assumptions on Ω in Theorem 1.3.2 part (ii) can be relaxed since the inequality holds for domains of class $C^{1,\alpha}$ with $\alpha \in]1/2, 1]$, but some care is required.

Before doing so, following [7], we present a direct proof of the Gaffney inequality under the assumption of $C^{1,1}$ regularity of the boundary. The idea is to explicitly write down the difference between the L^2-norm of the whole gradient and the L^2-norms of the curl and the divergence, and estimate it via the curvature tensor of the boundary.

1.4.1 Direct proof

In this section, we will assume Ω to be a bounded domain in $\mathbb{R}^3$ of class $C^{1,1}$.

Following Section 3.1 of [63], we introduce the curvature tensor $\mathfrak{B}$, which is nothing but the second fundamental form relative to the surface $\partial\Omega$. In the notation of Definition 1.0.1, the form $\mathfrak{B}$ coincides with the Hessian matrix of the local profile function of the boundary. Indeed, let $P \in \partial\Omega$ and let g be the function locally describing $\partial\Omega$ around P, so that there exists a cuboid V such that in the rotated coordinates

$$V \cap \Omega = \left\{ x = (\bar{y}, y_N) \in V : y_N \leq g(\bar{y}) \right\},$$

with $g(0) = P$ and $\nabla g(0) = 0$ (i.e. the new coordinates are chosen in such a way that the hyperplane $y_N = 0$ is tangent to $\partial\Omega$ at P). Then $\mathfrak{B}$ is the bilinear form that coincides with

$$\mathfrak{B}(\xi, \eta) = \sum \frac{\partial^2 g}{\partial y_k \partial y_i}(0)\zeta_k \eta_i,$$

for all ζ, η tangent vectors to $\partial\Omega$ at P. Note that this makes sense since we have assumed Ω to be of class $C^{1,1}$, hence the second derivatives exist for almost all $P \in \partial\Omega$.

It is worthy to note that if Ω has $C^{1,1}$ boundary, or equivalently the outer unit normal ν is $C^{0,1}$-regular, then

$$\operatorname{tr} \mathfrak{B} = -\operatorname{div} \nu = -2\mathcal{H}$$

a.e. on $\partial\Omega$, where $\mathcal{H}$ is the mean curvature of Ω. Moreover, observe that if Ω is $C^{1,1}$ then the form $\mathfrak{B}$ is uniformly bounded on $\partial\Omega$; in other words there exists a constant $C > 0$ independent of $P \in \partial\Omega$ such that

$$\big|\mathfrak{B}(\zeta, \eta)\big| \leq C|\zeta||\eta|$$

for all $\zeta, \eta \in \mathbb{R}^{N-1}$. We then have the following theorem.

Theorem 1.4.2 ([63, Thm. 3.1.1.1])**.** *Let Ω a bounded open set in $\mathbb{R}^3$ of class $C^{1,1}$. Then for all $v \in H^2(\Omega)^3$*

$$\begin{aligned}\int_\Omega |\operatorname{div} v|^2\, dx - \sum_{i,j=1}^N \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\, dx \\ = -2\int_{\partial\Omega} v_t \cdot \nabla_t(\nu \cdot v)\, d\sigma - \int_{\partial\Omega} \big(\mathfrak{B}(v_t, v_t) + (\operatorname{tr}\mathfrak{B})\,(\nu\cdot v)^2\big)\, d\sigma,\end{aligned}$$

where v_t and ∇_t denote the projection on the tangent hyperplane to $\partial\Omega$ of v and the gradient operator respectively, namely

$$v_t = v - (\nu \cdot v)\nu = \nu \times (v \times \nu), \quad \nabla_t f = \nabla f - \frac{\partial f}{\partial \nu}\, \nu.$$

From the previous theorem it is not difficult to deduce the following

Lemma 1.4.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set of class $C^{1,1}$. Then*

$$\|Dv\|^2_{L^2(\Omega)^{3\times 3}} = \|\operatorname{curl} v\|^2_{L^2(\Omega)^3} + \|\operatorname{div} v\|^2_{L^2(\Omega)^3} + \int_{\partial\Omega} (\operatorname{tr}\mathfrak{B})\,(\nu\cdot v)^2 d\sigma \qquad (1.4.4)$$

for all $v \in H^1(\Omega)^3 \cap X_{\mathrm{N}}(\Omega)$.

We can finally prove

Theorem 1.4.5 ([7, Thm 2.12]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set of class $C^{1,1}$. Then the Gaffney inequality (1.4.1) holds for all $u \in X_{\mathrm{N}}(\Omega)$.*

Moreover, if Ω is also convex, the stronger inequality

$$\|Dv\|^2_{L^2(\Omega)^3} \leq \|\operatorname{curl} v\|^2_{L^2(\Omega)^3} + \|\operatorname{div} v\|^2_{L^2(\Omega)^3}$$

holds for all $v \in X_{\mathrm{N}}(\Omega)$.

Proof. Let $v \in H^1(\Omega)^3 \cap X_{\mathrm{N}}(\Omega)$. Since under the assumptions for Ω, the curvature tensor $\mathfrak{B}$ is bounded, using [63, Thm. 1.5.1.10] we get that

$$\int_{\partial\Omega} (\operatorname{tr} \mathfrak{B})\, (\nu \cdot v)^2 d\sigma \leq c\|v\|^2_{L^2(\partial\Omega)^3} \leq \frac{1}{2}\|Dv\|^2_{L^2(\Omega)^{3\times 3}} + \tilde{c}\|v\|^2_{L^2(\Omega)^3}\,,$$

hence from (1.4.4) we recover the Gaffney inequality (1.4.1) for all $v \in H^1(\Omega)^3 \cap X_{\mathrm{N}}(\Omega)$.

From this, to deduce the validity for all $v \in X_{\mathrm{N}}(\Omega)$, one just needs to use the density result of Theorem 1.3.1.

Finally, if Ω is convex, it is sufficient to observe that the mean curvature $\mathcal{H}$ is non-negative. □

Using the same ideas, these results can be adapted to the case of $X_{\mathrm{T}}(\Omega)$ for vector fields with vanishing normal trace. Further details can be found in [7, Thm. 2.9].

1.4.2 Necessary condition for the validity of the Gaffney inequality

Recall that in general the permittivity tensor ε is represented by a symmetric 3×3 real matrix-valued function. We also assume that there exist two constants $m_\varepsilon, M_\varepsilon > 0$ for which

$$m_\varepsilon|\zeta|^2 \leq \varepsilon(x)\zeta \cdot \zeta \leq M_\varepsilon|\zeta|^2$$

for all $(x, \zeta) \in \Omega \times \mathbb{R}^3$.

In this section we will show that for any open set $\Omega \subset \mathbb{R}^3$, the H^2-regularity of the solution to the following classical elliptic problem

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla w) = f \in L^2(\Omega) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4.6}$$

is a necessary condition for the validity of the Gaffney inequality. The above problem has to be interpreted in the following weak variational sense:

$$\int_\Omega (\varepsilon \nabla w) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \text{for all } \varphi \in H^1_0(\Omega), \tag{1.4.7}$$

in the unknown $w \in H^1_0(\Omega)$.

Define

$$\mathfrak{D}(\varepsilon) := \left\{ w \in H^1_0(\Omega) : \exists f \in L^2(\Omega) \text{ such that } w \text{ satisfies (1.4.7) } \forall \varphi \in H^1_0(\Omega) \right\}$$

and

$$X^\varepsilon_{\mathrm{N}}(\Omega) := H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}\, \varepsilon, \Omega).$$

Of course, when $\varepsilon = \mathbb{I}_3$ is the 3×3 identity matrix we recover the standard Dirichlet Laplacian, and the space $X_{\mathrm{N}}(\Omega)$.

We have the following

Proposition 1.4.8. *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^3$. Suppose that $X^\varepsilon_{\mathrm{N}}(\Omega) \subset H^1(\Omega)^3$ and there exists a constant $C = C(\Omega) > 0$ such that*

$$\|u\|_{H^1(\Omega)^3} \le C \left(\|u\|_{L^2(\Omega)^3} + \|\mathrm{curl}\, u\|_{L^2(\Omega)^3} + \left\|\mathrm{div}(\varepsilon u)\right\|_{L^2(\Omega)} \right) \tag{1.4.9}$$

for all $u \in X^\varepsilon_{\mathrm{N}}(\Omega)$. Then $\mathfrak{D}(\varepsilon) \subset H^2(\Omega)$, and if w is a solution of problem (1.4.6) with datum f, we have

$$\|w\|_{H^2(\Omega)^3} \le C \|f\|_{L^2(\Omega)}\,, \tag{1.4.10}$$

where $C > 0$ is a constant depending only on ε and Ω.

Proof. Consider a solution $w \in \mathfrak{D}(\varepsilon)$ to problem (1.4.6), and take $u = \nabla w$. Then, by definition, $u \in H(\mathrm{div}\, \varepsilon, \Omega)$ and $\mathrm{div}(\varepsilon u) = f$. Moreover, recalling Proposition 1.2.12, we have that u also belongs to $H_0(\mathrm{curl}, \Omega)$ and $\mathrm{curl}\, u = 0$. Thus $u \in X^\varepsilon_{\mathrm{N}}(\Omega)$. Hence using inequality (1.4.9) we have that $u = \nabla w$ is in $H^1(\Omega)^3$ and

$$\|\nabla w\|_{H^1(\Omega)^3} \le C \left(\|\nabla w\|_{L^2(\Omega)^3} + \|f\|_{L^2(\Omega)} \right). \tag{1.4.11}$$

Moreover, using Hölder ([18, Thm. 4.6]) and Poincaré ([18, Coroll. 9.19]) inequalities, and denoting with $c_{\mathcal{P}} = c_{\mathcal{P}}(\Omega) > 0$ the Poincaré constant, we have that

$$\begin{aligned} m_\varepsilon \|\nabla w\|^2_{L^2(\Omega)} \le \int_\Omega \varepsilon \nabla w \cdot \nabla w \, dx = \int_\Omega f\, w \, dx &\le \|f\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \\ &\le c_{\mathcal{P}} \|f\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)}\,, \end{aligned}$$

which in turn implies that

$$\|\nabla w\|_{L^2(\Omega)} \le \frac{c_{\mathcal{P}}}{m_\varepsilon} \|f\|_{L^2(\Omega)}\,. \tag{1.4.12}$$

Finally, combining (1.4.11) and (1.4.12) we obtain (1.4.10). $\square$

The converse statement, that is the sufficiency of the H^2-regularity for the validity of the Gaffney inequality, is not so obvious. Some notable work in this direction has been done by Saranen in [107] in the case of bounded convex domains, and by Birman and Solomyak in [17] for regions with Lipschitz continuous boundaries. We also note the work by Filonov and Prokhorov [102], where the authors prove the validity of the more general Gaffney inequality (1.4.9) involving the permittivity parameter for a large class of domains, which includes convex domains or in general domains satisfying the exterior ball condition, with the additional assumptions of ε to be in $W^{1,3}(\Omega)$.

Following the approach of [17], we will present now a proof in the case of unitary permittivity $\varepsilon = \mathbb{I}_3$. Then, we will draw our attention to some results of Maz'ya and Shaposhnikova in [92] concerning the regularity of elliptic problems. We will see that in domains with $C^{3/2+\delta}$-Hölder continuous boundary the H^2-regularity for solutions of the Dirichlet Laplacian is valid, and a consequent a priori estimate holds. In this regard, an interesting read is [56], where it is shown that the exponent $3/2$ is in some sense sharp for the argument presented by Birman and Solomyak in [17] when dealing with the Gaffney inequality by means of the Dirichlet Laplacian.

1.4.3 H^2-regularity approach

Let Ω be a bounded Lipschitz open set in $\mathbb{R}^3$. Following [17] we introduce the spaces

$$H^1_{\mathrm{N}}(\Omega) := X_{\mathrm{N}}(\Omega) \cap H^1(\Omega)^3$$

and

$$E(\Omega) := \left\{ \nabla\varphi : \varphi \in H^1_0(\Omega), \Delta\varphi \in L^2(\Omega) \right\}.$$

The following theorem was proved by Birman and Solomyak in [17]: it allows to split a vector field in $X_{\mathrm{N}}(\Omega)$ in a "good part" in $H^1_{\mathrm{N}}(\Omega)$, and a "bad part" in $E(\Omega)$ (not necessarily divergence-free, thus we cannot really talk about the theorem below as a Helmholtz decomposition). Note that $E(\Omega) \subset H(\mathrm{div}, \Omega)$. A detailed proof can be found in the appendix of the thesis.

Theorem 1.4.13 ([17, Thm. 4.1])**.** *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^3$. Then there exist two linear continuous operators P and Q*

$$P : X_{\mathrm{N}}(\Omega) \to \left(H^1_{\mathrm{N}}(\Omega), \, \|\cdot\|_{H^1(\Omega)^3} \right), \qquad Q : X_{\mathrm{N}}(\Omega) \to \left(E(\Omega), \, \|\cdot\|_{H(\mathrm{div},\Omega)} \right),$$

such that $u = Pu + Qu$ for any $u \in X_{\mathrm{N}}(\Omega)$. In particular there exists a constant $C_{BS} > 0$ such that

$$\|Pu\|^2_{H^1(\Omega)} + \|Qu\|^2_{L^2(\Omega)} + \|\mathrm{div}\, Qu\|^2_{L^2(\Omega)} \le C_{BS} \|u\|^2_{X_{\mathrm{N}}(\Omega)} \tag{1.4.14}$$

for all $u \in X_{\mathrm{N}}(\Omega)$.

Then we can state the following

Theorem 1.4.15. *Let Ω be a bounded domain in $\mathbb{R}^3$ of class $C^{0,1}_M(\mathcal{A})$. Then the Gaffney inequality (1.4.1) holds for all $u \in X_{\mathrm{N}}(\Omega)$ and a constant $C > 0$ independent of u if and only if (the weak, variational) solutions $\varphi \in H^1_0(\Omega)$ to the Poisson problem*

$$\begin{cases} -\Delta\varphi = f, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4.16}$$

satisfy the a priori estimate

$$\|\varphi\|_{H^2(\Omega)} \leq \tilde{C}\|f\|_{L^2(\Omega)} \tag{1.4.17}$$

for all $f \in L^2(\Omega)$ and a constant $\tilde{C} > 0$ independent of f. In particular, the constants C and $\tilde{C}$ depend on each other, M and $\mathcal{A}$.

Proof. One implication was already proved in Proposition 1.4.8.

Conversely, assume now that the a priori estimate (1.4.17) holds, and let $u \in X_{\mathrm{N}}(\Omega)$. Let P and Q be the two operators introduced by Theorem 1.4.13. A careful inspection of the proof of Theorem 1.4.13 reveals that C_{BS} depends only on M, $\mathcal{A}$. By definition, $Qu = \nabla\varphi$ with $\varphi \in H^1_0(\Omega)$ and $\Delta\varphi \in L^2(\Omega)$. Since we have assumed that (1.4.17) holds, then

$$\|\varphi\|_{H^2(\Omega)} \leq \tilde{C}\|\Delta\varphi\|_{L^2(\Omega)} = \tilde{C}\|\operatorname{div} Qu\|_{L^2(\Omega)} \leq \tilde{C}C_{BS}\|u\|_{X_{\mathrm{N}}(\Omega)}$$

Thus, since $\|Qu\|_{H^1(\Omega)^3}$ is obviously controlled by $\|\varphi\|_{H^2(\Omega)}$, by (1.4.14) we deduce that

$$\|u\|_{H^1(\Omega)^3} \leq \|Pu\|_{H^1(\Omega)^3} + \|Qu\|_{H^1(\Omega)^3} \leq C\|u\|_{X_{\mathrm{N}}(\Omega)},$$

for all $u \in X_{\mathrm{N}}(\Omega)$, and (1.4.1) is proved. □

Example 1.4.18. Let $N \geq 2$. Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 such that around a boundary point (identified here with the origin) is described by the subgraph $x_N < g(\bar{x})$ of the C^1 function defined by

$$g(x_1, \dots, x_{N-1}) = |x_1|/\log|x_1|$$

It is proved in [92, §14.6.1] that for this domain the a priori estimate (1.4.17) does not hold. Thus, by Theorem 1.4.15 it follows that not even the Gaffney inequality holds for this domain for $N = 3$.

Theorem 1.4.15 highlights the importance of proving the a priori estimate (1.4.17) and getting information on the constant $\tilde{C}$. We do this by following the approach of Maz'ya and Shaposhnikova [92] and using the notion of domains Ω

with boundaries $\partial\Omega$ of class $M^{\frac{3}{2}}(\delta)$. We re-formulate the definition in Maz'ya and Shaposhnikova [92, § 14.3.1] by using the atlas classes. Here we can treat the general case of domains in $\mathbb{R}^N$ with $N \geq 2$.

Note that in this section, following [92] we find it convenient to assume directly that the functions g_j describing the boundary of Ω as in Definition 1.0.1 are extended to the whole of $\mathbb{R}^{N-1}$ and belong to the corresponding function spaces defined on $\mathbb{R}^{N-1}$.

Definition 1.4.19. Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and $\delta > 0$. We say that a bounded domain Ω in $\mathbb{R}^N$ is of class $M^{\frac{3}{2}}(\delta, \mathcal{A})$ if Ω is of class $C^{0,1}(\mathcal{A})$ and the gradients ∇g_j of the functions g_j describing the boundary of Ω as in Definition 1.0.1 belong to the space $MW_2^{1/2}(\mathbb{R}^{N-1})$ of Sobolev multipliers with

$$\|\nabla g_j\|_{MW_2^{1/2}(\mathbb{R}^{N-1})} \leq \delta \tag{1.4.20}$$

for all $j = 1, \dots s'$. We say that a bounded domain Ω in $\mathbb{R}^N$ is of class $M^{\frac{3}{2}}(\delta)$ if it is of class $M^{\frac{3}{2}}(\delta, \mathcal{A})$ for some atlas $\mathcal{A}$.

Recall that $MW_2^{1/2}(\mathbb{R}^{N-1}) = \{f \in W_{2,loc}^{1/2}(\mathbb{R}^{N-1}) :\ f\varphi \in W_2^{1/2}(\mathbb{R}^{N-1})$ for all $\varphi \in W_2^{1/2}(\mathbb{R}^{N-1})\}$ and that $\|f\|_{MW_2^{1/2}(\mathbb{R}^{N-1})} = \sup\{\|f\varphi\|_{W_2^{1/2}(\mathbb{R}^{N-1})} : \|\varphi\|_{W_2^{1/2}(\mathbb{R}^{N-1})} = 1\}$, where $W_2^{1/2}(\mathbb{R}^{N-1})$ denotes the standard Sobolev space with fractional order of smoothness $1/2$ and index of summability 2 (for simplicity, in (1.4.20) we use the the same symbol for the norm of a vector field).

Remark 1.4.21. We note that by [92, Thm. 4.1.1], there exists $c > 0$ depending only on N such that the functions g_j in Definition 1.4.19 satisfy the estimate $\|\nabla g_j\|_\infty \leq c\|\nabla g_j\|_{MW_2^{1/2}(\mathbb{R}^{N-1})} \leq c\delta$, see also [92, Thm. 14.6.4]. Thus if Ω is of class $M^{\frac{3}{2}}(\delta, \mathcal{A})$ then it is also of class $C_M^{0,1}(\mathcal{A})$ with $M = c\delta$.

The following theorem is a reformulation of [92, Thm. 14.5.1]

Theorem 1.4.22. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. If Ω is a bounded domain of class $M^{\frac{3}{2}}(\delta, \mathcal{A})$ for some δ sufficiently small (depending only on N) then the a priori estimate* (1.4.17) *holds for some constant $\tilde{C}$ depending only on N and $\mathcal{A}$.*

By [92, Corollaries 14.6.1, 14.6.2] it is possible to prove the following theorem based on the condition (1.4.25) from [92, (14.6.9)] . Here, by a refinement of an atlas $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$, we mean an atlas of the type $\widetilde{\mathcal{A}} = (\tilde{\rho}, \tilde{s}, \tilde{s}', \{\tilde{V}_j\}_{j=1}^{\tilde{s}}, \{\tilde{r}_j\}_{j=1}^{\tilde{s}})$ where $\tilde{\rho} \leq \rho$, $s \leq \tilde{s}$, $s' \leq \tilde{s}'$, $\cup_{j=1}^{\tilde{s}} \tilde{V}_j = \cup_{j=1}^{s} V_j$, $\{\tilde{r}_j\}_{j=1}^{\tilde{s}} \subset \{r_j\}_{j=1}^{s}$, which can be thought as an atlas constructed from $\mathcal{A}$ by replacing each cuboid $V_j = r_j(W_j \times]a_{N,j}, b_{N,j}[)$ by a finite number of cuboids of the form $\widetilde{V}_{j,l} = r_j(\widetilde{W}_{j,l} \times]a_{N,j}, b_{N,j}[)$, $l = 1, \dots m_j$, where $W_j = \cup_{l=1}^{m_j} \widetilde{W}_{j,l}$.

Theorem 1.4.23. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let Ω be a bounded domain of class $C^{0,1}_M(\mathcal{A})$. Let ω be a (non-decreasing) modulus of continuity for the gradients ∇g_j of the functions g_j describing the boundary of Ω, that is*

$$|\nabla g_j(\bar{x}) - \nabla g_j(\bar{y})| \leq \omega(|\bar{x}-\bar{y}|) \tag{1.4.24}$$

for all $\bar{x}, \bar{y} \in \mathbb{R}^{N-1}$. Assume that there exists $D > 0$ such that the function ω satisfies the following condition[1]

$$\int_0^\infty \left(\frac{\omega(t)}{t}\right)^2 dt \leq D\,. \tag{1.4.25}$$

Then there exists $C > 0$ depending only on N, $\mathcal{A}$, D such that if $M \leq C\delta$ then, possibly replacing the atlas $\mathcal{A}$ with a refinement of $\mathcal{A}$, Ω is of class of class $M^{\frac{3}{2}}(\mathcal{A},\delta)$.

Proof. We begin with the case $N \geq 3$. By [92, Cor. 14.6.1] there exists $c > 0$ depending only on N such that if $x = (\bar{x}, g_j(\bar{x})) \in \partial\Omega$ is any point of the boundary represented in local charts by a profile function g_j and the following inequality

$$\lim_{\rho\to 0}\left(\sup_{E\subset B_\rho(\bar{x})} \frac{\|D_{3/2}(g_j,B_\rho)\|_{L^2(E)}}{|E|^{\frac{N-2}{2(N-1)}}} + \|\nabla g_j\|_{L^\infty(B_\rho(\bar{x}))}\right) \leq c\delta \tag{1.4.26}$$

is satisfied, then, possibly replacing the atlas $\mathcal{A}$ with a refinement of its, Ω is of class $M^{\frac{3}{2}}(\delta,\mathcal{A})$. Here $|E|$ denotes the $N-1$-dimensional Lebesgue measure of the set E,

$$D_{3/2}(g_j,B_\rho)(\bar{x}) = \left(\int_{B_\rho(\bar{x})} |\nabla g_j(\bar{x}) - \nabla g_j(\bar{y})|^2 |\bar{x}-\bar{y}|^{-N} d\bar{y}\right)^{1/2},$$

and $B_\rho(\bar{x})$ the ball in $\mathbb{R}^{N-1}$ of radius ρ and centre $\bar{x}$. We refer to [92, §14.7.2] for the local characterization of the boundaries of domains of class $M^{\frac{3}{2}}(\delta,\mathcal{A})$.

We have

$$\begin{aligned}\int_E\int_{B_\rho(\bar{x})} |\nabla g_j(\bar{x}) - \nabla g_j(\bar{y})|^2 |\bar{x}-\bar{y}|^{-N} d\bar{y}d\bar{x} &\leq \int_E\int_{B_\rho(\bar{x})} \frac{\omega^2(|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|^N} d\bar{y}d\bar{x} \\ = \int_E\int_{B_\rho(0)} \frac{\omega^2(|\bar{h}|)}{|\bar{h}|^N} d\bar{h}d\bar{x} &= \sigma_{N-2}|E| \int_0^\rho \left|\frac{\omega(t)}{t}\right|^2 dt \leq \sigma_{N-2}D|E| \end{aligned} \tag{1.4.27}$$

Here σ_m denotes the m-dimensional measure of the m-dimensional unit sphere. Thus

$$\frac{\|D_{3/2}(g_j,B_\rho)\|_{L^2(E)}}{|E|^{\frac{N-2}{2(N-1)}}} \leq (\sigma_{N-2}D)^{1/2}|E|^{\frac{1}{2(N-1)}} = O(\rho^{1/2}),$$

[1] Here only the integrability at zero really matters and one could consider integrals defined in a neigborhood of zero.

hence

$$\frac{\|D_{3/2}(g_j, B_\rho)\|_{L^2(E)}}{|E|^{\frac{N-2}{2(N-1)}}} \leq c\delta, \tag{1.4.28}$$

provided ρ is sufficiently small. Thus, inequality (1.4.26) follows if we assume directly that $\|\nabla g_j\|_{L^\infty(B_\rho)} \leq c\delta$.

In the case $N = 2$, by [92, Cor. 14.6.1] it suffices to replace (1.4.26) by the following inequality

$$\lim_{\rho\to 0}\left(\sup_{E\subset B_\rho(\bar{x})} \|D_{3/2}(g_j, B_\rho)\|_{L^2(E)}|\log|E||^{1/2} + \|\nabla g_j\|_{L^\infty(B_\rho(\bar{x}))}\right) \leq c\delta \tag{1.4.29}$$

and use the same argument as above. □

By combining Theorems 1.4.22 and 1.4.23, we deduce the validity of the following result

Corollary 1.4.30. *Under the same assumptions of Theorem 1.4.23, there exists $\tilde{C} > 0$ depending only on N, $\mathcal{A}$, D such that if $M < \tilde{C}^{-1}$ then the a priori estimate (1.4.17) holds.*

Finally, by Theorem 1.4.15 and Corollary 1.4.30 we deduce the following result ensuring the validity of uniform Gaffney inequality that can be used in our spectral stability results.

Corollary 1.4.31. *Under the same assumptions of Theorem 1.4.23, there exists $C > 0$ depending only on N, $\mathcal{A}$, D such that if $M < C^{-1}$ then the Gaffney inequality (1.4.1) holds.*

Chapter 2

Shape perturbation

In this chapter we study the dependence of the eigenvalues of the electric Maxwell problem (2.1.1) upon the variation of the domain Ω. The main theorem is Theorem 2.3.5, which provides a real-analytic result for the elementary symmetric functions of the eigenvalues as well as formulas for their Hadamard derivatives. Then, we prove a Rellich-Nagy-type theorem (see Theorem 2.3.17) and a Rellich-Pohozaev formula for the eigenvalues (cf. Theorem 2.3.19) . In Theorem 2.4.10 we provide a characterization for critical domains under the constraint of fixed volume and fixed perimeter, and in Theorem 2.4.13 we show that balls satisfy both conditions. Finally, in Section 2.5 we provide known formulas for the eigenvalues and eigenvectors of problem (2.1.1) in the case Ω is a ball, showing that the first Maxwell eigenvalue in the ball has multiplicity 3.

Note that in the present chapter we consider row vectors.

2.1 Main problem

Let Ω be a bounded Lipschitz domain (connected open set) in $\mathbb{R}^3$. Consider the following boundary value problem for the curl curl operator subject to electric boundary conditions:

$$\begin{cases} \operatorname{curl}\operatorname{curl} u = \lambda u, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \times u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

Using formula (1.2.11), it is easy to see that the weak formulation of problem (2.1.1) is

$$\int_\Omega \operatorname{curl} u \cdot \operatorname{curl} \varphi \, dx = \lambda \int_\Omega u \cdot \varphi \, dx, \quad \text{for all } \varphi \in X_{\mathrm{N}}(\operatorname{div} 0, \Omega), \tag{2.1.2}$$

in the unknowns $u \in X_{\mathrm{N}}(\operatorname{div} 0, \Omega)$ and $\lambda \in \mathbb{R}$. Observe that $\lambda \geq 0$, as one can see by testing u in equation (2.1.2).

Since for our purposes we prefer to work in the space $X_{\mathrm{N}}(\Omega)$ rather than in the space $X_{\mathrm{N}}(\operatorname{div} 0, \Omega)$, following [39, 40], we introduce a penalty term in the equation and we replace problem (2.1.2) with the problem

$$\int_\Omega \operatorname{curl} u \cdot \operatorname{curl} \varphi \, dx + \tau \int_\Omega \operatorname{div} u \operatorname{div} \varphi \, dx = \lambda \int_\Omega u \cdot \varphi \, dx, \quad \text{for all } \varphi \in X_{\mathrm{N}}(\Omega), \quad (2.1.3)$$

in the unknowns $u \in X_{\mathrm{N}}(\Omega)$ and $\lambda \in \mathbb{R}$. Here $\tau > 0$ is any fixed positive real number.

Since the eigenvalues λ of (2.1.1) coincide with the square of the angular frequency $\omega > 0$ appearing in the time-harmonic Maxwell system (0.0.3), we are interested only in the positive eigenvalues. Still, the zero eigenspace $K_{\mathrm{N}}(\Omega)$ of problem (2.1.3) (and problem (2.1.2)) has an interest of its own. It is composed by all those vector fields in $X_{\mathrm{N}}(\Omega)$ whose curl and divergence vanish in Ω, namely

$$K_{\mathrm{N}}(\Omega) = \left\{ u \in X_{\mathrm{N}}(\Omega) : \operatorname{curl} u = 0 \text{ and } \Omega, \operatorname{div} u = 0, \text{in } \Omega \right\}.$$

Note that in literature a vector field which is both irrotational (i.e. curl-free) and solenoidal (i.e. divergence-free) is called a *Laplacian vector field*, due to the vector calculus identity

$$\boldsymbol{\Delta} v \equiv \nabla \operatorname{div} v - \operatorname{curl} \operatorname{curl} v,$$

where $\boldsymbol{\Delta}$ denotes the vector Laplacian. Therefore, $K_{\mathrm{N}}(\Omega)$ is the set of all Laplacian vector fields whose tangential trace vanish on the boundary of Ω.

Since Ω is a bounded Lipschitz domain then its boundary $\partial\Omega$ can be decomposed into a finite number of connected components, which we will denote by Γ_i, $0 \leq i \leq I$, with Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$. Hence

$$\partial\Omega = \bigsqcup_{i=0}^{I} \Gamma_i \, .$$

In this case, Proposition 3.18 of [7] provides a characterization of the space $K_{\mathrm{N}}(\Omega)$: it has dimension exactly equal to I, and it is spanned by gradients of the harmonic functions which are zero on Γ_0 and constant on the components $\Gamma_i, 1 \leq i \leq I$. In particular, if the domain Ω is such that the boundary has only one connected component, the kernel $K_{\mathrm{N}}(\Omega)$ becomes trivial and the bilinear form on the left hand-side of formula (2.1.3) is coercive on $X_{\mathrm{N}}(\Omega)$, becoming equivalent to the standard inner product of $X_{\mathrm{N}}(\Omega)$.

A similar result exists for the case of Laplacian vector fields with vanishing normal trace. For example one can look at Proposition 3.14 of [7], where a characterization of the set

$$K_{\mathrm{T}}(\Omega) := \left\{ u \in X_{\mathrm{T}}(\Omega) : \operatorname{curl} u = 0 \text{ and } \operatorname{div} u = 0 \text{ in } \Omega \right\}$$

in terms of gradients of harmonic functions, satisfying Neumann-type boundary conditions on $\partial\Omega$, is given. In this case, the dimension of the space coincides with the first Betti number of the set Ω, and the space $K_{\mathrm{T}}(\Omega)$ itself is isomorphic to the first cohomolgy space of Ω. The vector fields in $K_{\mathrm{N}}(\Omega)$ and $K_{\mathrm{T}}(\Omega)$ are also known as *Dirichlet fields* and *Neumann fields*, respectively. For more general details on these kernels we refer to Chapter IX-A, Section 3 of [49].

It is obvious that the solutions of problem (2.1.2) are the divergence free solutions of (2.1.3). On the other hand, it is also not difficult to see that the solutions of problem (2.1.3) which are not divergence free are given by the vector fields $u = \nabla f$ of the gradients of the solutions f to the Helmohltz equation with Dirichlet boundary conditions, that is

$$\begin{cases} -\Delta f = \frac{\lambda}{\tau} f, & \text{in } \Omega, \\ f = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1.4}$$

In fact, we have the following result from [40]

Lemma 2.1.5. *Let Ω be a bounded domain in $\mathbb{R}^3$ of class $C^{0,1}$. A vector field $u \in X_{\mathrm{N}}(\Omega)$ is a solution of problem (2.1.3) with $\mathrm{div} u = 0$ if and only if $u \in X_{\mathrm{N}}(\mathrm{div}\, 0, \Omega)$ is a solution of problem (2.1.2). Moreover, a vector field $u \in X_{\mathrm{N}}(\Omega)$ with $\mathrm{div} u \neq 0$ is a solution of problem (2.1.3) if and only if $u = \nabla f$ where $f \in H^1_0(\Omega)$ is a solution of problem (2.1.4). In particular, the set of eigenvalues of problem (2.1.3) are given by the union of the set of eigenvalues of problem (2.1.2) and the set of eigenvalues of the Dirichlet Laplacian in Ω multiplied by τ.*

In view of the previous lemma, in order to distinguish the solutions arising from the original Maxwell system from the spurious solutions associated with the Helmohltz equation, we give the following definition.

Definition 2.1.6. We say that a couple (u, λ) in $X_{\mathrm{N}}(\Omega) \times \mathbb{R}$ is a (electric) Maxwell eigenpair if $\lambda \neq 0$ and (u, λ) is an eigenpair of equation (2.1.3) with $\mathrm{div}\, u = 0$ in Ω, in which case u is called a (electric) Maxwell eigenvector and λ a Maxwell eigenvalue.

A first important remark on the Maxwell eigenvalues is the following one.

Remark 2.1.7. In general there is no monotonicity principle with respect to inclusion for the Maxwell eigenvalues.

The following example shows that in general one should not expect a monotonicity principle with respect to inclusion for the Maxwell eigenvalues. Here we consider rectangular parallelepipeds.

Counterexample 2.1.8. Let $\ell_1, \ell_2, \ell_3 > 0$ be three positive real numbers and consider the following rectangular parallelepiped

$$\Omega := (0, \ell_1) \times (0, \ell_2) \times (0, \ell_3).$$

The Maxwell eigenvalues in Ω are given by

$$\left(\frac{k_1 \pi}{\ell_1}\right)^2 + \left(\frac{k_2 \pi}{\ell_2}\right)^2 + \left(\frac{k_3 \pi}{\ell_3}\right)^2,$$

counted one time for all integers $k_1, k_2, k_3 \geq 0$ with exactly one index $j \in \{1, 2, 3\}$ such that $k_j = 0$, and counted twice for all integers $k_1, k_2, k_3 \geq 1$ (see [41] for more details, especially Theorem 3.6 and Section 6).

The smallest eigenvalue λ_1 is given by the case $k_1, k_2, k_3 \geq 0$ with $k_j = 0$ for the index $j \in \{1, 2, 3\}$ corresponding to the smallest side ℓ_j of the parallelepiped Ω. Without loss of generality we can suppose that $\ell_1 > \ell_2 > \ell_3$. In this case λ_1 is given by $k_1 = k_2 = 1$ and $k_3 = 0$, and it is equal to

$$\lambda_1 = \left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2}\right) \pi^2.$$

If we consider the halved parallelepiped

$$\hat{\Omega} := (0, \ell_1/2) \times (0, \ell_2/2) \times (0, \ell_3/2),$$

which is obviously contained in Ω, and denote the smallest Maxwell eigenvalue in $\hat{\Omega}$ by $\hat{\lambda}_1$, then

$$\hat{\lambda}_1 = 4 \left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2}\right) \pi^2,$$

which is strictly larger than λ_1.

Now we consider a parallelepiped $\tilde{\Omega}$ and, with a slight abuse of notation, by $\tilde{\ell}_i, i = 1, 2, 3$ we denote both its sides and their respective length. We suppose that $\tilde{\Omega} \subset \Omega$, with $\tilde{\ell}_1 > \ell_1$ and $\tilde{\ell}_2 = \ell_2$. Note that this is always possible, and one can modify the parallelepiped $\tilde{\Omega}$ in order to tweak the length of the bigger side $\tilde{\ell}_1$ as close as desired to $\sqrt{\ell_1^2 + \ell_3^2}$, and making $\tilde{\ell}_3$ small enough to guarantee the inclusion of $\tilde{\Omega}$ in Ω (see Figure 2.1).

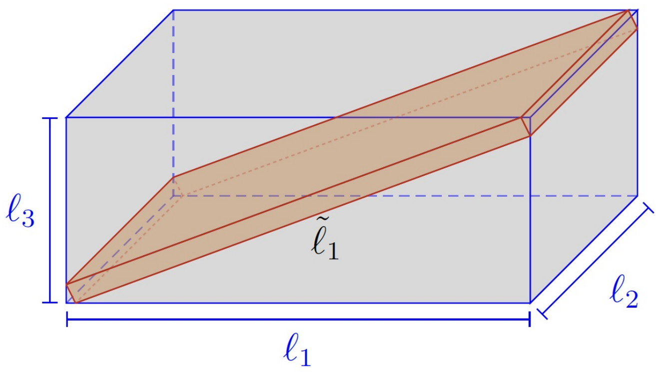


Figure 2.1: The parallelepipeds Ω (in gray) and $\tilde{\Omega}$ (in red).

Then the smallest Maxwell eigenvalue $\tilde{\lambda}_1$ in $\tilde{\Omega}$ is equal to

$$\tilde{\lambda}_1 = \left(\frac{1}{\tilde{\ell}_1^2} + \frac{1}{\ell_2^2}\right)\pi^2,$$

which is strictly smaller than λ_1.

We have thus found two subsets $\hat{\Omega}, \tilde{\Omega} \subset \Omega$ such that

$$\tilde{\lambda}_1 < \lambda_1 < \hat{\lambda}_1.$$

We note that for the second Maxwell eigenvalues λ_2 and $\tilde{\lambda}_2$ in Ω and $\tilde{\Omega}$ respectively, we have that

$$\lambda_2 = \left(\frac{1}{\ell_1^2} + \frac{1}{\ell_3^2}\right)\pi^2 < \left(\frac{1}{\tilde{\ell}_1^2} + \frac{1}{\tilde{\ell}_3^2}\right)\pi^2 = \tilde{\lambda}_2,$$

provided $\tilde{\ell}_3$ is small enough.

Remark 2.1.9. In this chapter, it will be understood that the value of τ in (2.1.3) is fixed. It is important to note that in applying our results one is free to choose $\tau > 0$ in order to avoid the overlapping of Maxwell and Helmholtz eigenvalues. In fact, since the set of eigenvalues of problem (2.1.3) are given by the union of the set of eigenvalues of problem (2.1.2) and the set of eigenvalues of the Dirichlet Laplacian in Ω multiplied by τ, one cannot exclude that a Maxwell eigenvalue could coincide with an eigenvalue of the Dirichlet Laplacian multiplied by some $\tau \in]0, \infty[$. However, if λ is a fixed Maxwell eigenvalue it is possible to choose $\tau \in]0, \infty[$ such that $\lambda \neq \tau\vartheta$ for all eigenvalues ϑ of the Dirichlet Laplacian, in other words one can choose τ in order to avoid 'resonance'. It is also useful to recall that the eigenvalues of the Dirichlet Laplacian depend with continuity upon sufficiently regular perturbations of Ω, as those considered in this thesis (see e.g., [79]), hence it is possible to avoid 'resonance' around a fixed Maxwell eigenvalue $\lambda(\Omega)$, possibly multiple, and all those eigenvalues bifurcating from it when Ω is slightly perturbed. This can obviously be done also to avoid resonances for a finite subset of of Maxwell eigenvalues.

We now describe a standard procedure that allows us to recast the eigenvalue problem (2.1.3) as an eigenvalue problem for a compact self-adjoint operator in Hilbert space. We consider the operator T from $X_{\mathrm{N}}(\Omega)$ to its dual $(X_{\mathrm{N}}(\Omega))'$ defined by the pairing

$$< Tu, \varphi >= T[u][\varphi] = T[\varphi][u] := \int_{\Omega} \mathrm{curl} u \cdot \mathrm{curl} \varphi \, dx + \tau \int_{\Omega} \mathrm{div} u \, \mathrm{div} \varphi \, dx, \quad (2.1.10)$$

for all $u, \varphi \in X_{\mathrm{N}}(\Omega)$. Then, we consider the map J from $L^2(\Omega)^3$ to $(X_{\mathrm{N}}(\Omega))'$ defined by the pairing

$$< Ju, \varphi >= J[u][\varphi] := \int_{\Omega} u \cdot \varphi \, dx,$$

for all $u \in L^2(\Omega)^3$ and $\varphi \in X_{\mathrm{N}}(\Omega)$. Note that the operator J is essentially the standard inner product $\langle \cdot, \cdot \rangle_{L^2(\Omega)^3}$ of $L^2(\Omega)^3$. Moreover, observe that by the Riesz Theorem the operator $T + J$ is a homeomorphism from $X_{\mathrm{N}}(\Omega)$ to its dual.

Lemma 2.1.11. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. The operator S from $L^2(\Omega)^3$ to itself defined by*

$$Su = \iota \circ (T + J)^{-1} \circ J$$

where ι denotes the embedding of $X_{\mathrm{N}}(\Omega)$ into $L^2(\Omega)^3$, is a non-negative self-adjoint operator in $L^2(\Omega)^3$. Moreover, λ is an eigenvalue of problem (2.1.3) if and only if $\mu = (\lambda + 1)^{-1}$ is an eigenvalue of the operator S, the eigenfunctions being the same.

Proof. Let $u, \varphi \in L^2(\Omega)^3$. We want to show that

$$\langle Su, \varphi \rangle_{L^2(\Omega)^3} = \langle u, S\varphi \rangle_{L^2(\Omega)^3}.$$

Using the natural symmetry present in the definition of the operators T and J we have that

$$\begin{aligned}
\langle Su, \varphi \rangle_{L^2(\Omega)^3} &= J[\varphi] \left[(T + J)^{-1} \circ J[u] \right] \\
&= \left((T + J) \circ (T + J)^{-1} \circ J \right) [\varphi] \left[(T + J)^{-1} \circ J[u] \right] \\
&= (T + J) \left[(T + J)^{-1} \circ J[\varphi] \right] \left[(T + J)^{-1} \circ J[u] \right] \\
&= (T + J) \left[(T + J)^{-1} \circ J[u] \right] \left[(T + J)^{-1} \circ J[\varphi] \right] \\
&= J[u] \left[(T + J)^{-1} \circ J[\varphi] \right] = \langle u, S\varphi \rangle_{L^2(\Omega)^3}.
\end{aligned}$$

Hence S is self-adjoint.

To see that S is non-negative, if $u \in L^2(\Omega)^3 \setminus \{0\}$ then the function $(T+J)^{-1} \circ J[u]$ is not zero, and consequently by the above formula we have that

$$(Su, u)_{L^2(\Omega)^3} = (T + J)\left[(T + J)^{-1} \circ J[u]\right]\left[(T + J)^{-1} \circ J[u]\right] > 0. \quad (2.1.12)$$

This actually proves that S is positive-definite.

Finally, suppose that λ is an eigenvalue of problem (2.1.3) with associated eigenfunction $u \in X_{\mathrm{N}}(\Omega)$. This exactly means that $T[u] = \lambda J[u]$. Hence $(\lambda+1)Su = u$, meaning that u is an eigenfunction of S with associated eigenvalue $\mu = (\lambda+1)^{-1}$. Viceversa, if $Su = \mu u$, by the same reasoning one deduces that $T[u] = \lambda J[u]$, with $\lambda = \mu^{-1} - 1$. Observe that μ cannot be zero by formula (2.1.12). □

If the space $X_{\mathrm{N}}(\Omega)$ is compactly embedded into $L^2(\Omega)^3$, that is, ι is a compact map, then the operator S is compact, hence the spectrum $\sigma(S)$ of S can be represented as $\sigma(S) = \{0\} \cup \{\mu_n(\Omega)\}_{n\in\mathbb{N}}$ where $\mu_n(\Omega)$, $n \in \mathbb{N}$ is a decreasing sequence of positive eigenvalues of finite multiplicity, which converges to zero. Accordingly, the eigenvalues of problem (2.1.3) are given by the sequence $\lambda_n(\Omega)$, $n \in \mathbb{N}$ defined by $\lambda_n(\Omega) = \mu_n^{-1}(\Omega) - 1$. As customary, we agree to repeat each eigenvalue in the sequence as many times as its multiplicity. Thus, we have the following result where formula (2.1.14) can be proved by applying the classical Min-Max Principle (see, e.g., Section 4.5 of [50]) to the operator S.

Theorem 2.1.13. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ such that the embedding $X_{\mathrm{N}}(\Omega) \subset L^2(\Omega)^3$ is compact. The eigenvalues of problem (2.1.3) have finite multiplicity and are given by a divergent sequence $\lambda_n(\Omega)$, $n \in \mathbb{N}$ which can be represented by means of the following min-max formula:*

$$\lambda_n(\Omega) = \min_{\substack{V \subset X_{\mathrm{N}}(\Omega)\\ \dim V = n}} \max_{u \in V\setminus\{0\}} \frac{\int_\Omega \left(|\mathrm{curl} u|^2 + \tau |\mathrm{div} u|^2\right) dx}{\int_\Omega |u|^2\, dx}. \quad (2.1.14)$$

2.2 Domain transplantation

Given a bounded Lipschitz domain Ω in $\mathbb{R}^3$, we consider problem (2.1.3) on a class of domains $\Phi(\Omega)$ obtained as diffeomorphic images of Ω. Namely, we consider the family of diffeomorphisms

$$\mathcal{A}_\Omega := \left\{\Phi \in C^{1,1}(\overline{\Omega}, \mathbb{R}^3) : \Phi \text{ is injective, } \det D\Phi(x) \neq 0\ \forall x \in \overline{\Omega}\right\}. \quad (2.2.1)$$

Here $C^{1,1}(\overline{\Omega}, \mathbb{R}^3)$ is the Banach space of $C^{1,1}$ functions from $\overline{\Omega}$ to $\mathbb{R}^3$ endowed with its standard norm defined by $\|\Phi\|_{C^{1,1}} = \|\Phi\|_\infty + \|\nabla\Phi\|_\infty + |\nabla\Phi|_{0,1}$ for all $\Phi \in C^{1,1}(\overline{\Omega}, \mathbb{R}^3)$, where $|\cdot|_{0,1}$ denotes the Lipschitz seminorm. We first make some observations regarding this set of admissible diffeomorphisms and their properties.

Lemma 2.2.2. *Let Ω be a bounded domain in $\mathbb{R}^3$. Then the following statements hold:*

(i) *If Ω is sufficiently regular, say of class C^1, then the set $\mathcal{A}_\Omega$ defined in (2.2.1) is open in $C^{1,1}(\overline{\Omega}, \mathbb{R}^3)$.*

(ii) *If $\Phi \in \mathcal{A}_\Omega$, then $\Phi(\Omega)$ is a bounded domain in $\mathbb{R}^3$, such that $\partial\Phi(\overline{\Omega}) = \Phi(\partial\Omega) = \partial\Phi(\Omega)$, and $\Phi(\Omega)$ is the interior of $\Phi(\overline{\Omega})$. The map Φ is a homeomorphism of $\overline{\Omega}$ onto $\overline{\Phi(\Omega)}$.*

(iii) *Let $k \in \mathbb{N}$, $\alpha \in [0, 1]$. Let Ω be of class $C^{k,\alpha}$ and $\Phi \in C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^3) \cap \mathcal{A}_\Omega$. Then $\Phi(\Omega)$ is of class $C^{k,\alpha}$.*

For more details on this type of diffeomorphisms and for a proof of the lemma as well, one can consult [78], [79] and [86].

In order to study problem (2.1.3) on $\Phi(\Omega)$, it is convenient to pull it back to Ω by means of a change of variables. As is known, in order to transform the curl in a natural way and preserve the boundary conditions in (2.1.1), it is necessary to pull back any vector field v defined on $\Phi(\Omega)$ to the vector field u defined on Ω by means of the covariant Piola transform (see e.g., [98]) defined by

$$u(x) = \big((v \circ \Phi) D\Phi\,\big)(x), \quad \text{for all } x \in \Omega. \tag{2.2.3}$$

By setting

$$y = \Phi(x), \quad \text{for all } x \in \Omega,$$

equality (2.2.3) can be rewritten in the form

$$v(y) = \big(u (D\Phi)^{-1}\big) \circ \Phi^{(-1)}(y) = \Big(u \circ \Phi^{(-1)}\Big)\, D(\Phi^{(-1)})(y), \quad y \in \Phi(\Omega).$$

Note that in the sequel we shall often use the following notation

$$\partial_j u_i(x) = \frac{\partial u_i}{\partial x_j}(x) \quad \text{and} \quad \partial'_a v_b(y) = \frac{\partial v_b}{\partial y_a}(y).$$

Then we have the following known result, which can be found for example in [98, Corollary 3.58]. For the convenience of the reader, we include a proof (which differs from that of [98, Corollary 3.58]). Note that the assumption $\Phi \in C^{1,1}$ can be relaxed, but some care is required, see Remark 2.2.8.

Theorem 2.2.4. *Let Ω be a bounded domain in $\mathbb{R}^3$ and $\Phi \in \mathcal{A}_\Omega$. Then a function v belongs to $H(\mathrm{curl}, \Phi(\Omega))$ ($H_0(\mathrm{curl}, \Phi(\Omega))$, respectively), if and only if the function u defined by (2.2.3) belongs to $H(\mathrm{curl}, \Omega)$ ($H_0(\mathrm{curl}, \Omega)$, respectively), in which case*

$$(\mathrm{curl}_y\, v(y)) \circ \Phi = \frac{\mathrm{curl}_x\, u(x)\, \big(D\Phi(x)\big)^T}{\det\big(D\Phi(x)\big)}\,. \tag{2.2.5}$$

Proof. Assume for the time being that u is a vector field of class $C^{0,1}$. The chain rule yields

$$
\begin{aligned}
\frac{\partial v_b}{\partial y_a}(y) &= \frac{\partial[(u_i \circ \Phi^{(-1)})\partial'_b \Phi_i^{(-1)}]}{\partial y_a}(y) \\
&= \frac{\partial u_i}{\partial x_j}(\Phi^{(-1)}(y))\, \frac{\partial \Phi_j^{(-1)}}{\partial y_a}(y)\, \frac{\partial \Phi_i^{(-1)}}{\partial y_b}(y) + u_i(\Phi^{(-1)}(y)) \frac{\partial^2 \Phi_i^{(-1)}}{\partial y_a \partial y_b}(y).
\end{aligned}
$$

Note that summation symbols are omitted here and in the sequel. Recall that the c-component of the curl of v is given by

$$
\left[\operatorname{curl}_y v(y)\right]_c = \partial'_a v_b(y)\, \xi_{abc}\,,
$$

where ξ_{abc} is the Levi-Civita symbol defined by

$$
\xi_{abc} = \begin{cases} +1 & \text{if } (a,b,c) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (a,b,c) \text{ is an odd permutation of } (1,2,3), \\ \ \ 0 & \text{if } a=b, \text{ or } b=c, \text{ or } a=c. \end{cases}
$$

Then

$$
\begin{aligned}
\left[\operatorname{curl}_y v(y)\right]_c = {} & \partial_j u_i(\Phi^{(-1)}(y))\, \partial'_a \Phi_j^{(-1)}(y)\, \partial'_b \Phi_i^{(-1)}(y)\, \xi_{abc} \\
& + u_i(\Phi^{(-1)}(y)) \partial'_a \partial'_b \Phi_i^{(-1)}(y)\, \xi_{abc}.
\end{aligned}
$$

Since $\xi_{abc} = -\xi_{bac}$ we have that for all $i = 1,2,3$

$$
\sum_{a,b=1}^{3} \partial'_a \partial'_b \Phi_i^{(-1)}(y)\, \xi_{abc} = 0.
$$

Thus

$$
\begin{aligned}
\left[\operatorname{curl}_y v(y) \left(D\Phi^{(-1)}(y)\right)^T\right]_k &= \left[\operatorname{curl}_y v(y)\right]_c \frac{\partial \Phi_k^{(-1)}}{\partial y_c}(y) \\
&= \partial_j u_i(\Phi^{(-1)}(y))\, \partial'_a \Phi_j^{(-1)}(y)\, \partial'_b \Phi_i^{(-1)}(y)\, \partial'_c \Phi_k^{(-1)}(y)\, \xi_{abc} \\
&= \partial_j u_i(\Phi^{(-1)}(y))\, \xi_{jik} \det\left(D\Phi^{(-1)}(y)\right) \\
&= \left(\operatorname{curl}_x u(\Phi^{(-1)})\right)_k \det\left(D\Phi^{(-1)}(y)\right),
\end{aligned}
$$

where we have used the fact that $\partial'_a F_j\, \partial'_b F_i\, \partial'_c F_k\, \xi_{abc} = \xi_{jik} \det(DF)$, for any vector field F of class C^1, which follows from the definition of determinant of a 3×3 matrix.

Therefore

$$\operatorname{curl}_y v(y) = \operatorname{curl}_x u(\Phi^{(-1)}(y)) \left(D\Phi^{(-1)}(y)\right)^{-T} \det \left(D\Phi^{(-1)}(y)\right),$$

and formula (2.2.5) follows.

We now prove the validity of formula (2.2.5) in the weak sense. We begin with proving that if $v \in H(\operatorname{curl}, \Phi(\Omega))$ then the distributional curl of the function u defined above belongs to $L^2(\Omega)^3$ and satisfies formula (2.2.5). To do so, it suffices to prove that for

$$\int_\Omega u\,(\operatorname{curl}\varphi)^T dx = \int_\Omega (\operatorname{curl}_y v(y))(\Phi(x)) \left(D\Phi(x)\right)^{-T} \det D\Phi(x) \varphi^T(x)\, dx, \quad (2.2.6)$$

for all $\varphi \in C_c^\infty(\Omega)$. Following formula (2.2.3), we define a function ψ on $\Phi(\Omega)$ by setting

$$\varphi(x) = \left((\psi \circ \Phi) D\Phi\right)(x)\,. \quad (2.2.7)$$

By formula (2.2.5) we get

$$\operatorname{curl}_x \varphi(x) = (\operatorname{curl}_y \psi(y))(\Phi(x)) \left(D\Phi(x)\right)^{-T} \det D\Phi(x)$$

and this implies that

$$\begin{aligned}
\int_\Omega u\,(\operatorname{curl}\varphi)^T dx &= \int_\Omega u \left(D\Phi(x)\right)^{-1} (\operatorname{curl}_y \psi(y))^T(\Phi(x)) \det D\Phi(x)\, dx \\
&= \int_\Omega v(\Phi(x))(\operatorname{curl}_y \psi(y))^T(\Phi(x)) \det D\Phi(x)\, dx \\
&= \int_{\Phi(\Omega)} v(y)(\operatorname{curl}_y \psi(y))^T \operatorname{sgn}(\det D\Phi^{(-1)}(y))\, dy \\
&= \int_{\Phi(\Omega)} (\operatorname{curl}_y v(y))\psi^T \operatorname{sgn}(\det D\Phi^{(-1)}(y))\, dy \\
&= \int_\Omega (\operatorname{curl}_y v(y))(\Phi(x)) \left(D\Phi(x)\right)^{-T} \det D\Phi(x)\varphi^T(x)\, dx,
\end{aligned}$$

as required. In the same way, one can prove that if $u \in H(\operatorname{curl}, \Omega)$ then the distributional curl of the function v belongs to $L^2(\Phi(\Omega))^3$, which completes the first part of the proof.

In order to prove that v belongs to $H_0(\operatorname{curl}, \Phi(\Omega))$ if and only u belongs to $H_0(\operatorname{curl}, \Omega)$ one can use the definition of these spaces and smooth vector fields compactly supported in Ω, for which formula (2.2.5) is valid, and conclude by density arguments. $\square$

Remark 2.2.8. Theorem 2.2.4 holds also under weaker assumptions on the diffeomorphism Φ. Namely, assume that $\Phi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ is injective and that $\det D\Phi(x)$ is different from zero for all $x \in \overline{\Omega}$. Then the thesis of Theorem 2.2.4 holds. Indeed, given any smooth domain U with $\overline{U} \subset \Omega$, one can find an approximating sequence Φ_n, $n \in \mathbb{N}$ of smooth functions converging to Φ in $C^1(\overline{U}, \mathbb{R}^3)$. This can be done by using standard mollifiers. Then, since the set of functions $C^1(\overline{U}, \mathbb{R}^3)$ which are injective and such that $\det D\Phi(x) \neq 0$ $\forall x \in \overline{U}$, is an open set in $C^1(\overline{U}, \mathbb{R}^3)$ (cfr., [86, Lemma 5.2]), it follows that $\Phi_n \in \mathcal{A}_U$ for all n sufficiently large, hence Theorem 2.2.4 is applicable to Φ_n. Passing to the limit as $n \to \infty$ we get the validity of formula (2.2.5) in U, and since U is arbitrary, formula (2.2.5) holds also in the whole of Ω. The preservation of the spaces easily follows by formula (2.2.5) itself and changing variables in integrals.

Formula (2.2.5) also suggests that the regularity assumptions could be further relaxed to include the case of bi-Lipschitz transformations as in the case of the classical chain rule for functions in the Sobolev space (see e.g., [105, p. 23]). However, we shall not discuss this issue here.

Remark 2.2.9. The fact that v belongs to $H_0(\mathrm{curl}, \Phi(\Omega))$ if and only if the function u defined by (2.2.3) belongs to $H_0(\mathrm{curl}, \Omega)$ as stated in Theorem 2.2.4 has an immediate explanation by using traces in the classical sense as follows. It is not difficult to realise that the unit outer normals to $\partial\Omega$ and $\partial\Phi(\Omega)$ satisfy the relation

$$\nu_{\partial\Phi(\Omega)} \circ \Phi = \pm \frac{\nu_{\partial\Omega}(D\Phi)^{-1}}{\left|\nu_{\partial\Omega}(D\Phi)^{-1}\right|}.$$

Then, using the fact that $aM \times bM = \det(\mathrm{M})(\mathrm{a} \times \mathrm{b})(M)^{-1}$ for all vectors $a, b \in \mathbb{R}^3$ and for all invertible matrices $M \in GL_3(\mathbb{R})$, we immediately deduce that

$$v \times \nu_{\partial\Phi(\Omega)} = 0 \text{ on } \partial\Phi(\Omega) \ \text{ if and only if } \ u \times \nu_{\partial\Omega} = 0 \text{ on } \partial\Omega, \tag{2.2.10}$$

for all vector fields admitting boundary values in the classical sense.

In order to transplant problem (2.1.3) from $\Phi(\Omega)$ to Ω we also need a formula for the transformation of the divergence under the action of the pull-back operator defined in (2.2.3). Note that in this case we need to assume more regularity on the pulled-back function because the covariant Piola transform does not behave well for the divergence, contrary to the curl, and consequently the formula for the transformed divergence is more involved. Namely, we will need the function to have square integrable first order partial derivatives.

Theorem 2.2.11. *Let Ω be a bounded domain in $\mathbb{R}^3$ and $\Phi \in \mathcal{A}_\Omega$. Then a function v belongs to $H^1(\Phi(\Omega))^3$ if and only if the function u defined by (2.2.3) belongs to $H^1(\Omega)^3$, in which case*

$$(\mathrm{div}_y\, v) \circ \Phi(x) = \frac{\mathrm{div}_x \left[u(x)(D\Phi(x))^{-1}(D\Phi(x))^{-T} \det(D\Phi(x))\right]}{\det(D\Phi(x))}. \tag{2.2.12}$$

Proof. The first part of the statement is standard and can be proved by using the chain rule and changing variables in integrals. We now provide a proof of formula (2.2.12) in the same spirit of the one carried out for the curl. To simplify notation, we set $M = D\Phi^{(-1)}$, so that $\partial'_a = M_{i,a}\partial_i$, where $M_{i,a} = \partial\Phi_i^{(-1)}/\partial y_a$. Note that $M_{j,a} = \sum_{m,k=1}^3 M_{j,m}M_{k,m}(M^{-1})_{a,k}$ simply because $M_{j,a} = (MM^TM^{-T})_{j,a}$. Since $v_a = (u_j \circ \Phi^{(-1)})\, \partial'_a\Phi_j^{(-1)} = (u_j \circ \Phi^{(-1)})M_{j,a}$ we have that

$$
\begin{aligned}
\operatorname{div}_y v = \partial'_a v_a &= \partial'_a[\left(u_j \circ \Phi^{(-1)}\right) M_{j,a}] \\
&= \partial'_a[(u_j \circ \Phi^{(-1)})M_{j,m}M_{k,m}(M^{-1})_{a,k}] \\
&= \partial'_a[(u_j \circ \Phi^{(-1)})(\partial'_m\Phi_j^{(-1)})(\partial'_m\Phi_k^{(-1)})((\partial_k\Phi_a) \circ \Phi^{(-1)})] \\
&= \partial'_a[P_k\, Q_{ka}] = (\partial'_a P_k)Q_{ka} + P_k(\partial'_a Q_{ka})
\end{aligned}
$$

where we have set $P_k = \sum_{j,m=1}^3 (u_j \circ \Phi^{(-1)})(\partial'_m\Phi_j^{(-1)})(\partial'_m\Phi_k^{(-1)}) \det^{-1}(D\Phi^{(-1)})$ and $Q_{ka} = ((\partial_k\Phi_a) \circ \Phi^{(-1)}) \det(D\Phi^{(-1)})$.

We claim that $\sum_{a=1}^3 \partial'_a Q_{ka} = 0$. Indeed, if by C we denote the cofactor matrix of M, we have that (see [93], p.12)

$$
C_{k,a} = \frac{1}{2}\sum_{n,m,i,j=1}^{3} \xi_{anm}\, \xi_{kij} M_{i,n}M_{j,m}\,,
$$

hence

$$
\begin{aligned}
\partial'_a\left((M^{-1})_{a,k}\det(D\Phi^{(-1)})\right) &= \partial'_a(C_{k,a}) = \frac{1}{2}\partial'_a(\xi_{anm}\,\xi_{kij}M_{i,n}M_{j,m}) \\
&= \frac{1}{2}\xi_{kij}M_{j,m}\left(\partial'_a M_{i,n}\right)\xi_{anm} + \frac{1}{2}\xi_{anm}M_{i,n}\left(\partial'_a M_{j,m}\right)\xi_{kij}.
\end{aligned}
$$

Moreover

$$
\sum_{a,n=1}^{3} \xi_{anm}\,\partial'_a M_{i,n} = \sum_{a,n=1}^{3} \xi_{anm}\,\partial'_n M_{i,a} = \sum_{a,n=1}^{3} \xi_{nam}\,\partial'_a M_{i,n} = -\sum_{a,n=1}^{3} \xi_{anm}\,\partial'_a M_{i,n}.
$$

Thus $\sum_{a,n=1}^3 \xi_{anm}\,\partial'_a M_{i,n} = 0$. Similarly $\sum_{a,m=1}^3 \xi_{anm}\,\partial'_a M_{j,m} = 0$ and the claim is proved.

Then

$$
\begin{aligned}
\operatorname{div}_y v &= (\partial'_a P_k) Q_{ka} = \det\left(D\Phi^{(-1)}\right)(M)^{-1}_{a,k}\partial'_a P_k \\
&= \det\left(D\Phi^{(-1)}\right)(M)^{-1}_{a,k} M_{i,a}[\partial_i (P_k \circ \Phi)] \circ \Phi^{(-1)} \\
&= \det\left(D\Phi^{(-1)}\right)\delta_{i,k}[\partial_i (P_k \circ \Phi)] \circ \Phi^{(-1)} \\
&= \det\left(D\Phi^{(-1)}\right)\partial_i \left[\frac{\left(u_j \circ \Phi^{(-1)}\right)(\partial'_m \Phi^{(-1)}_j)(\partial'_m \Phi^{(-1)}_i)}{\det\left(D\Phi^{(-1)}\right)} \circ \Phi \right] \circ \Phi^{(-1)} \\
&= \left[\frac{\partial_i \left[u_j \left((D\Phi)^{-1}(D\Phi)^{-T}\right)_{j,i} \det(D\Phi) \right]}{\det(D\Phi)} \right] \circ \Phi^{(-1)} \\
&= \left[\frac{\partial_i \left[\left(u(D\Phi)^{-1}(D\Phi)^{-T}\right)_i \det(D\Phi) \right]}{\det(D\Phi)} \right] \circ \Phi^{(-1)} \\
&= \frac{\operatorname{div}_x \left[u(D\Phi)^{-1}(D\Phi)^{-T} \det(D\Phi) \right]}{\det(D\Phi)} \circ \Phi^{(-1)},
\end{aligned}
$$

hence formula (2.2.12) is proved. □

From now on until the end of the chapter, we assume Ω to be a bounded domain in $\mathbb{R}^3$ of class $C^{1,1}$, unless explicitly stated otherwise. Recall that thanks to the Gaffney inequality (cf. Theorem 1.3.2),under these assumptions the space $X_{\mathrm{N}}(\Omega)$ is continuously embedded into $H^1(\Omega)^3$. Moreover, by Statement (iii) of Lemma 2.2.2, if $\Phi \in \mathcal{A}_\Omega$, then $\Phi(\Omega)$ is also of class $C^{1,1}$.

We fix $\Phi \in \mathcal{A}_\Omega$ and consider equation (2.1.3) on $\Phi(\Omega)$, that is

$$
\begin{aligned}
&\int_{\Phi(\Omega)} \operatorname{curl} v \cdot \operatorname{curl} \psi \, dy \\
&\quad + \tau \int_{\Phi(\Omega)} \operatorname{div} v \operatorname{div} \psi \, dx = \lambda \int_{\Phi(\Omega)} v \cdot \psi \, dx \quad \text{for all } \psi \in X_{\mathrm{N}}(\Phi(\Omega)),
\end{aligned}
\tag{2.2.13}
$$

in the unknowns $v \in X_{\mathrm{N}}(\Phi(\Omega))$ and $\lambda \in \mathbb{R}$. Let u be the function defined in Ω by formula (2.2.3) and, analogously, φ the function defined by (2.2.7). By changing variables in (2.2.13) and using Theorem 2.2.4 and Theorem 2.2.11 we get

$$\int_\Omega \frac{\operatorname{curl} u\,(D\Phi)^T D\Phi(\operatorname{curl}\varphi)^T}{|\det(D\Phi)|}dx$$
$$+\tau\int_\Omega \frac{\operatorname{div}_x\big(u(D\Phi)^{-1}(D\Phi)^{-T}\det(D\Phi)\big)\operatorname{div}_x\big(\varphi(D\Phi)^{-1}(D\Phi)^{-T}\det(D\Phi)\big)}{|\det(D\Phi)|}\,dx$$
$$=\lambda\int_\Omega u(D\Phi)^{-1}(D\Phi)^{-T}\varphi^T|\det D\Phi|\,dx\,. \tag{2.2.14}$$

By construction we have that $v\in X_{\mathrm{N}}(\Phi(\Omega))$ is a solution to (2.2.13) associated with an eigenvalue λ if and only if $u=(v\circ\Phi)D\Phi\in X_{\mathrm{N}}(\Omega)$ is a solution to (2.2.14), the eigenvalue being the same. Thus, instead of studying problem (2.2.13) in the varying domain $\Phi(\Omega)$, we can study problem (2.2.14) where the unknown $u\in X_{\mathrm{N}}(\Omega)$ and the test functions $\varphi\in X_{\mathrm{N}}(\Omega)$ are defined on the fixed domain Ω.

It is clear that the natural L^2-space for problem (2.2.14) is the usual L^2-space endowed with the inner product $\langle\cdot,\cdot\rangle_\Phi$ defined by

$$\langle u,\varphi\rangle_\Phi:=\int_\Omega u(D\Phi)^{-1}(D\Phi)^{-T}\varphi^T|\det D\Phi|\,dx\,, \tag{2.2.15}$$

for all $u,\varphi\in L^2(\Omega)^3$, which is equivalent to the usual one (cf. Theorem 3.18 of [79]). We denote by $L^2_\Phi(\Omega)$ the space $L^2(\Omega)^3$ endowed with inner product (2.2.15). As we have done for equation (2.1.3) we recast problem (2.2.14) as a problem for a compact self-adjoint operator. To do so, we consider the operator T_Φ from the space $X_{\mathrm{N}}(\Omega)$ to its dual by setting $\langle T_\Phi u,\varphi\rangle$ equal to the left-hand side of equation (2.2.14). In the same way, we define the operator J_Φ from $L^2_\Phi(\Omega)$ to the dual of $X_{\mathrm{N}}(\Omega)$ by setting $\langle J_\Phi u,\varphi\rangle$ equal to the right-hand side of equation (2.2.14) divided by λ. Obviously, the operator J_Φ can be also interpreted as a coercive bilinear form from $L^2(\Omega)^3\times L^2(\Omega)^3$ to $\mathbb{R}$, equivalent to the inner product $\langle\cdot,\cdot\rangle_\Phi$ of $L^2_\Phi(\Omega)$. We will often use the notation $T_\Phi[u][\varphi]$ and $J_\Phi[u][\varphi]$ instead of $\langle T_\Phi u,\varphi\rangle$ and $\langle J_\Phi u,\varphi\rangle$ respectively.

We have the following result, whose proof can be carried out in a similar way to that one of Lemma 2.1.11.

Lemma 2.2.16. *Let Ω be a bounded domain in $\mathbb{R}^3$. The operator S_Φ from $L^2_\Phi(\Omega)$ to itself defined by*

$$S_\Phi u=\iota\circ(T_\Phi+J_\Phi)^{-1}\circ J_\Phi$$

where ι denotes the embedding of $X_{\mathrm{N}}(\Omega)$ into $L^2_\Phi(\Omega)$, is a non-negative self-adjoint operator in $L^2_\Phi(\Omega)$. Moreover, λ is an eigenvalue of problem (2.2.14) if and only if $\mu=(\lambda+1)^{-1}$ is an eigenvalue of the operator S_Φ, the eigenfunctions being the same.

Clearly, if the space $X_{\mathrm{N}}(\Omega)$ is compactly embedded into $L^2(\Omega)^3$ then the operator S_Φ is compact and its spectrum is given by $\sigma(S_\Phi) = \{0\} \cup \{\mu_n(\Phi)\}_{n\in\mathbb{N}}$ where

$$\mu_n[\Phi] = (\lambda_n[\Phi] + 1)^{-1}$$

and $\lambda_n[\Phi] := \lambda_n(\Phi(\Omega))$ are the eigenvalues of problem (2.2.13).

2.3 Analyticity results and Hadamard-type formulas

Given a finite non-empty set of indices $F \subset \mathbb{N}$, we consider

$$\mathcal{A}_\Omega[F]\colon = \left\{\Phi \in \mathcal{A}_\Omega : \lambda_j[\Phi] \neq \lambda_l[\Phi],\ \forall j \in F, l \in \mathbb{N} \setminus F\right\}$$

and the elementary symmetric functions of the corresponding eigenvalues

$$\Lambda_{F,s}[\Phi] = \sum_{\substack{j_1,\dots,j_s\in F\\ j_1<\dots<j_s}} \lambda_{j_1}[\Phi]\cdots\lambda_{j_s}[\Phi], \qquad s = 1,\dots,|F|\,. \tag{2.3.1}$$

It is also convenient to consider

$$\hat{\Lambda}_{F,s}[\Phi] = \sum_{\substack{j_1,\dots,j_s\in F\\ j_1<\dots<j_s}} (\lambda_{j_1}[\Phi]+1)\cdots(\lambda_{j_s}[\Phi]+1), \tag{2.3.2}$$

for all $\Phi \in \mathcal{A}_\Omega[F]$ and to note that

$$\Lambda_{F,s}[\Phi] = \sum_{k=0}^{s}(-1)^{s-k}\binom{|F|-k}{s-k}\hat{\Lambda}_{F,k}[\Phi], \tag{2.3.3}$$

where we have set $\Lambda_{F,0} = \hat{\Lambda}_{F,0} = 1$.

Finally, we set

$$\Theta_\Omega[F]\colon = \left\{\Phi \in \mathcal{A}_\Omega[F] : \lambda_j[\Phi] \text{ have a common value } \lambda_F[\Phi]\ \forall j \in F\right\}.$$

Then, we can state the main theorem of this chapter, the proof of which is also based on Lemma 2.3.7 below. Concerning the assumption on the summability of the second order derivatives of the eigenvectors, we include the following remark.

Remark 2.3.4. If Ω is of class $C^{2,1}$ (which means that locally at the boundary Ω can be described by the subgraphs of functions of class C^2 with Lipschitz continuous second order derivatives), hence in particular if Ω is of class C^3, then the eigenvectors of problem (2.1.2) belong to the standard Sobolev space $H^2(\Omega)$ of functions in $L^2(\Omega)$ with weak derivatives up to the second order in $L^2(\Omega)$. See e.g., [112] or the more recent paper [5].

Theorem 2.3.5. *Let Ω be a bounded domain in $\mathbb{R}^3$ of class $C^{1,1}$. Let F be a finite non-empty subset of $\mathbb{N}$. Then $\mathcal{A}_\Omega[F]$ is an open set in $C^{1,1}(\overline{\Omega},\mathbb{R}^3)$ and $\Lambda_{F,s}[\Phi]$ depends real-analytically on $\Phi \in \mathcal{A}_\Omega[F]$.*

Suppose in addition that $\tilde{\Phi} \in \Theta_\Omega[F]$ is fixed. Assume that $\lambda_F[\tilde{\Phi}]$ is a Maxwell eigenvalue and $\tilde{E}^{(1)},\dots,\tilde{E}^{(|F|)} \in X_{\mathrm{N}}(\mathrm{div}0,\tilde{\Phi}(\Omega))$ is an orthonormal basis of Maxwell eigenvectors for the corresponding eigenspace, where the orthonormality is taken in $L^2(\tilde{\Phi}(\Omega))^3$, and assume that those eigenvectors belong to $H^2(\tilde{\Phi}(\Omega))$. Then for any $s \in \{1,\dots,|F|\}$, we have

$$
\begin{aligned}
d|_{\Phi=\tilde{\Phi}}(\Lambda_{F,s})[\Psi] = &\binom{|F|-1}{s-1}(\lambda_F[\tilde{\Phi}])^{s-1} \\
&\cdot \sum_{l=1}^{|F|}\int_{\partial\tilde{\Phi}(\Omega)}\left(\lambda_F[\tilde{\Phi}]\left|\tilde{E}^{(l)}\right|^2 - \left|\mathrm{curl}\,\tilde{E}^{(l)}\right|^2\right)\left((\Psi\circ\tilde{\Phi}^{(-1)})\cdot\nu\right)d\sigma
\end{aligned}
\tag{2.3.6}
$$

for all $\Psi \in C^{1,1}(\overline{\Omega},\mathbb{R}^3)$.

Proof. Recall that $\mu_j[\Phi] = (\lambda_j[\Phi]+1)^{-1}$, $j \in \mathbb{N}$, are the eigenvalues of the operator S_Φ, hence the set $\mathcal{A}_\Omega[F]$ coincides with the set $\{\Phi \in \mathcal{A}_\Omega : \mu_j[\Phi] \neq \mu_l[\Phi],\ \forall j \in F, l \in \mathbb{N}\setminus F\}$. Moreover, S_Φ is a compact self-adjoint operator in $L^2_\Phi(\Omega)$. Since both the operator S_Φ and the inner product $\langle\cdot,\cdot\rangle_\Phi$ defined in (2.2.15) depend real-analytically on Φ, being obtained by compositions and inversions of real-analytic maps (such as linear and multilinear continuous maps), we can apply the general result [79, Thm. 2.30]. Thus, the set $\left\{\Phi \in \mathcal{A}_\Omega : \mu_j[\Phi] \neq \mu_l[\Phi],\ \forall j \in F, l \in \mathbb{N}\setminus F\right\}$ is open in $C^{1,1}(\overline{\Omega},\mathbb{R}^3)$ as required, and the functions

$$
M_{F,s}[\Phi] := \sum_{\substack{j_1,\dots,j_s\in F\\ j_1<\dots<j_s}} \mu_{j_1}[\Phi]\cdots\mu_{j_s}[\Phi],
$$

depend real-analytically on $\Phi \in \mathcal{A}_\Omega[F]$. Since

$$
\hat{\Lambda}_{F,s}[\Phi] = \frac{M_{F,|F|-s}[\Phi]}{M_{F,|F|}[\Phi]}, \qquad s = 1,\dots,|F|,
$$

we deduce that $\hat{\Lambda}_{F,s}[\Phi]$ depend real-analytically on $\Phi \in \mathcal{A}_\Omega[F]$. Finally, by formula (2.3.3) we conclude that $\Lambda_{F,s}[\Phi]$ depend real-analytically on $\Phi \in \mathcal{A}_\Omega[F]$.

We now prove formula (2.3.6). We set $\tilde{u}^{(l)} = (\tilde{E}^{(l)}\circ\tilde{\Phi})D\tilde{\Phi}$ for all $l = 1,\dots,|F|$ and we observe that $\tilde{u}^{(l)}$, $l \in 1,\dots,|F|$ is an orthonormal basis in $L^2_{\tilde{\Phi}}(\Omega)$ of the eigenspace associated with the eigenvalue $\mu_F[\tilde{\Phi}] := (\lambda_F[\tilde{\Phi}]+1)^{-1}$ of the operator $S_{\tilde{\Phi}}$. By applying [79, Thm. 2.30], we have that

$$
d|_{\Phi=\tilde{\Phi}}M_{F,s}[\Psi] = \binom{|F|-1}{s-1}(\lambda_F[\tilde{\Phi}]+1)^{1-s}\sum_{l=1}^{|F|}\langle d|_{\Phi=\tilde{\Phi}}\,S_\Phi[\Psi][\tilde{u}^{(l)}],\tilde{u}^{(l)}\rangle_{\tilde{\Phi}},
$$

where we recall that $\langle\cdot,\cdot\rangle_{\tilde\Phi}$ is the inner product of $L^2_{\tilde\Phi}(\Omega)$ (cf. (2.2.15)). Therefore, by Lemma 2.3.7

$$\begin{aligned}
&d|_{\Phi=\tilde\Phi}(\hat\Lambda_{F,s})[\Psi]\\
&= \{d|_{\Phi=\tilde\Phi}M_{F,|F|-s}[\Phi][\Psi]M_{F,|F|}[\tilde\Phi] - M_{F,|F|-s}[\tilde\Phi]\, d|_{\Phi=\tilde\Phi}M_{F,|F|}[\Phi][\Psi]\}\\
&\quad\cdot(\lambda_F[\tilde\Phi]+1)^{2|F|}\\
&= \left[\binom{|F|-1}{|F|-s-1}(\lambda_F[\tilde\Phi]+1)^{s+1-2|F|} - \binom{|F|}{s}\binom{|F|-1}{|F|-1}\right.\\
&\quad\left.\cdot(\lambda_F[\tilde\Phi]+1)^{s+1-2|F|}\right](\lambda_F[\tilde\Phi]+1)^{2|F|}\sum_{l=1}^{|F|}\langle d|_{\Phi=\tilde\Phi}\, S_\Phi[\Psi][\tilde u^{(l)}],\tilde u^{(l)}\rangle_{\tilde\Phi}\\
&= (\lambda_F[\tilde\Phi]+1)^{s-1}\binom{|F|-1}{s-1}\\
&\quad\cdot\sum_{l=1}^{|F|}\int_{\partial\tilde\Phi(\Omega)}\left(\lambda_F[\tilde\Phi]\left|\tilde E^{(l)}\right|^2-\left|\operatorname{curl}\tilde E^{(l)}\right|^2\right)\left((\Psi\circ\tilde\Phi^{(-1)})\cdot\nu\right)d\sigma.
\end{aligned}$$

Thus, using (2.3.3), we get

$$\begin{aligned}
&d|_{\Phi=\tilde\Phi}(\Lambda_{F,s})[\Psi]\\
&= \sum_{k=1}^{s}(-1)^{s-k}(\lambda_F[\tilde\Phi]+1)^{k-1}\binom{|F|-k}{s-k}\binom{|F|-1}{k-1}\\
&\quad\cdot\sum_{l=1}^{|F|}\int_{\partial\tilde\Phi(\Omega)}\left(\lambda_F[\tilde\Phi]\left|\tilde E^{(l)}\right|^2-\left|\operatorname{curl}\tilde E^{(l)}\right|^2\right)\left((\Psi\circ\tilde\Phi^{(-1)})\cdot\nu\right)d\sigma\\
&= \binom{|F|-1}{s-1}\sum_{k=0}^{s-1}\binom{s-1}{k}(\lambda_F[\tilde\Phi]+1)^{k}(-1)^{s-k-1}\\
&\quad\cdot\sum_{l=1}^{|F|}\int_{\partial\tilde\Phi(\Omega)}\left(\lambda_F[\tilde\Phi]\left|\tilde E^{(l)}\right|^2-\left|\operatorname{curl}\tilde E^{(l)}\right|^2\right)\left((\Psi\circ\tilde\Phi^{(-1)})\cdot\nu\right)d\sigma\\
&= \binom{|F|-1}{s-1}(\lambda_F[\tilde\Phi])^{s-1}\\
&\quad\cdot\sum_{l=1}^{|F|}\int_{\partial\tilde\Phi(\Omega)}\left(\lambda_F[\tilde\Phi]\left|\tilde E^{(l)}\right|^2-\left|\operatorname{curl}\tilde E^{(l)}\right|^2\right)\left((\Psi\circ\tilde\Phi^{(-1)})\cdot\nu\right)d\sigma
\end{aligned}$$

which proves formula (2.3.6). □

Lemma 2.3.7. *Let Ω be a bounded domain in $\mathbb{R}^3$ of class $C^{1,1}$ and $\tilde\Phi\in\mathcal{A}_\Omega$. Let $\tilde v,\tilde w\in X_{\mathrm{N}}(\operatorname{div}0,\tilde\Phi(\Omega))$ be two Maxwell eigenvectors associated with a Maxwell* *eigenvalue* $\tilde\lambda$. *Assume that* $\tilde v, \tilde w \in H^2(\tilde\Phi(\Omega))$. *Let* $\tilde u = (\tilde v \circ \tilde\Phi) D\tilde\Phi$, $\tilde\varphi = (\tilde w \circ \tilde\Phi) D\tilde\Phi$. *Then*

$$\begin{aligned} &\langle d|_{\Phi=\tilde\Phi} S_\Phi[\Psi][\tilde u], \tilde\varphi\rangle_{\tilde\Phi} \\ &\qquad = (\tilde\lambda+1)^{-2} \int_{\partial\tilde\Phi(\Omega)} \Big(\operatorname{curl}\tilde v \cdot \operatorname{curl}\tilde w - \tilde\lambda\, \tilde v\cdot\tilde w\Big)\Big((\Psi\circ\tilde\Phi^{(-1)})\cdot\nu\Big)\, d\sigma\,, \end{aligned} \tag{2.3.8}$$

for all $\Psi \in C^{1,1}(\overline\Omega, \mathbb{R}^3)$.

Proof. To shorten our notation, we set $\Upsilon_\Phi = T_\Phi + J_\Phi$. Since $J_{\tilde\Phi}[\tilde u] = (\tilde\lambda+1)^{-1}\Upsilon_{\tilde\Phi}[\tilde u]$, $J_{\tilde\Phi}[\tilde\varphi] = (\tilde\lambda+1)^{-1}\Upsilon_{\tilde\Phi}[\tilde\varphi]$ (cf. Lemma 2.2.16), and $\Upsilon_{\tilde\Phi}$ is symmetric, we have that

$$\begin{aligned} &\langle d|_{\Phi=\tilde\Phi} S_\Phi[\Psi][\tilde u], \tilde\varphi\rangle_{\tilde\Phi} \\ &= \langle \iota\circ\Upsilon_{\tilde\Phi}^{-1}\circ d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u], \tilde\varphi\rangle_{\tilde\Phi} + \langle \iota\circ d|_{\Phi=\tilde\Phi}\Upsilon_\Phi^{-1}[\Psi]\circ J_{\tilde\Phi}[\tilde u], \tilde\varphi\rangle_{\tilde\Phi} \\ &= J_{\tilde\Phi}[\tilde\varphi]\Big[\iota\circ\Upsilon_{\tilde\Phi}^{-1}\circ d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u]\Big] + J_{\tilde\Phi}[\tilde\varphi]\left[\iota\circ d|_{\Phi=\tilde\Phi}\Upsilon_\Phi^{-1}[\Psi]\circ J_{\tilde\Phi}[\tilde u]\right] \\ &= (\tilde\lambda+1)^{-1}\Upsilon_{\tilde\Phi}[\tilde\varphi] \\ &\quad \cdot\Big[\Upsilon_{\tilde\Phi}^{-1}\circ d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u] - \Upsilon_{\tilde\Phi}^{-1}\circ d|_{\Phi=\tilde\Phi}\Upsilon_\Phi[\Psi]\circ\Upsilon_{\tilde\Phi}^{-1}\circ J_{\tilde\Phi}[\tilde u]\Big] \\ &= (\tilde\lambda+1)^{-1}\Upsilon_{\tilde\Phi} \\ &\quad \cdot\Big[\Upsilon_{\tilde\Phi}^{-1}\circ d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u] - \Upsilon_{\tilde\Phi}^{-1}\circ d|_{\Phi=\tilde\Phi}\Upsilon_\Phi[\Psi]\circ\Upsilon_{\tilde\Phi}^{-1}\circ(\tilde\lambda+1)^{-1}\Upsilon_{\tilde\Phi}[\tilde u]\Big][\tilde\varphi] \\ &= (\tilde\lambda+1)^{-1}\Big(d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u][\tilde\varphi] - (\tilde\lambda+1)^{-1} d|_{\Phi=\tilde\Phi}\Upsilon_\Phi[\Psi][\tilde u][\tilde\varphi]\Big) \\ &= (\tilde\lambda+1)^{-1}\Big(\tilde\lambda(\tilde\lambda+1)^{-1}\, d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u][\tilde\varphi] - (\tilde\lambda+1)^{-1} d|_{\Phi=\tilde\Phi} T_\Phi[\Psi][\tilde u][\tilde\varphi]\Big) \\ &= (\tilde\lambda+1)^{-2}\Big(\tilde\lambda\, d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u][\tilde\varphi] - d|_{\Phi=\tilde\Phi} T_\Phi[\Psi][\tilde u][\tilde\varphi]\Big)\,. \end{aligned} \tag{2.3.9}$$

We begin by computing the second term in the right-hand side of (2.3.9). Obviously, we have that

$$\begin{aligned} &d|_{\Phi=\tilde\Phi} T_\Phi[\Psi][\tilde u][\tilde\varphi] \\ &\quad = \int_\Omega \operatorname{curl}\tilde u\,\big(d|_{\Phi=\tilde\Phi} G_\Phi[\Psi]\big)(\operatorname{curl}\tilde\varphi)^T\, dx + \tau\, d|_{\Phi=\tilde\Phi}\left(\int_\Omega \frac{N(\Phi,\tilde u,\tilde\varphi)}{|\det(D\Phi)|}\, dx\right)[\Psi], \end{aligned}$$

where we have set $G_\Phi = \dfrac{(D\Phi)^T D\Phi}{|\det(D\Phi)|}$ and $N(\Phi,\tilde u,\tilde\varphi) = N(\Phi,\tilde u) N(\Phi,\tilde\varphi)$ with

$$N(\Phi,\eta) = \operatorname{div}_x\Big(\eta(D\Phi)^{-1}(D\Phi)^{-T}\det(D\Phi)\Big)$$

for any vector field η. To shorten our notation, we also set $\zeta = \Psi \circ \tilde{\Phi}^{(-1)}$. By a change of variables one can see that

$$
\begin{aligned}
&\int_\Omega \operatorname{curl} \tilde{u}\, d|_{\Phi=\tilde{\Phi}} G_\Phi[\Psi]\, (\operatorname{curl} \tilde{\varphi})^T\, dx = \\
&= \int_\Omega \operatorname{curl} \tilde{u}\, \frac{(D\Psi)^T D\tilde{\Phi} + (D\tilde{\Phi})^T D\Psi}{\left|\det(D\tilde{\Phi})\right|}\, (\operatorname{curl} \tilde{\varphi})^T\, dx \\
&\quad - \int_\Omega \operatorname{curl} \tilde{u}\, \frac{(D\tilde{\Phi})^T D\tilde{\Phi}\, d|_{\Phi=\tilde{\Phi}} \left|\det(D\Phi)\right| [\Psi]}{\left(\det(D\tilde{\Phi})\right)^2}\, (\operatorname{curl} \tilde{\varphi})^T\, dx \\
&= \int_{\tilde{\Phi}(\Omega)} \operatorname{curl} \tilde{v} \left(D\zeta + (D\zeta)^T\right) (\operatorname{curl} \tilde{w})^T\, dy - \int_{\tilde{\Phi}(\Omega)} \operatorname{curl} \tilde{v}\, (\operatorname{curl} \tilde{w})^T \operatorname{div}(\zeta)\, dy.
\end{aligned}
$$

Now by standard calculus we have

$$
\begin{aligned}
&d|_{\Phi=\tilde{\Phi}} \left(\int_\Omega \frac{N(\Phi, \tilde{u}, \tilde{\varphi})}{\left|\det(D\Phi)\right|}\, dx \right) [\Psi] = \int_\Omega \frac{d}{dt} \left(\frac{N(\tilde{\Phi} + t\Psi, \tilde{u}, \tilde{\varphi})}{\left|\det(D(\tilde{\Phi} + t\Psi)\right|} \right)\Bigg|_{t=0} dx \\
&= \int_\Omega \frac{\frac{d}{dt}(N(\tilde{\Phi} + t\Psi, \tilde{u}, \tilde{\varphi})\big|_{t=0}\, \left|\det(D\tilde{\Phi})\right| - N(\tilde{\Phi}, \tilde{u}, \tilde{\varphi})\, d|_{\Phi=\tilde{\Phi}} \left|\det(D\Phi)\right| [\Psi]}{(\det(D\tilde{\Phi}))^2}\, dx \\
&= \int_\Omega \Bigg(\frac{d}{dt}(N(\tilde{\Phi} + t\Psi, \tilde{u})\big|_{t=0} N(\tilde{\Phi}, \tilde{\varphi}) \\
&\qquad + N(\tilde{\Phi}, \tilde{u}) \frac{d}{dt}(N(\tilde{\Phi} + t\Psi, \tilde{\varphi})\big|_{t=0} \Bigg) \left|\det D\tilde{\Phi}\right|^{-1} dx \\
&\qquad\qquad - \int_{\tilde{\Phi}(\Omega)} \operatorname{div}_y(\tilde{v}) \operatorname{div}_y(\tilde{w}) \operatorname{div}_y(\zeta)\, dy = 0.
\end{aligned}
$$

Here we have used the fact that

$$
\begin{aligned}
\frac{N(\tilde{\Phi}, \tilde{u})(x)}{\det(D\tilde{\Phi}(x))} &= \frac{\operatorname{div}_x \left[\tilde{\varphi}(x)(D\tilde{\Phi}(x))^{-1}(D\tilde{\Phi}(x))^{-T} \det(D\tilde{\Phi}(x))\right]}{\det(D\tilde{\Phi}(x))} \\
&= \operatorname{div}_y \tilde{w}(\tilde{\Phi}(x)) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{N(\tilde{\Phi}, \tilde{u})(x)}{\det(D\tilde{\Phi}(x))} &= \frac{\operatorname{div}_x \left[\tilde{u}(x)(D\tilde{\Phi}(x))^{-1}(D\tilde{\Phi}(x))^{-T} \det(D\tilde{\Phi}(x))\right]}{\det(D\tilde{\Phi}(x))} \\
&= \operatorname{div}_y \tilde{v}(\tilde{\Phi}(x)) = 0.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
d|_{\Phi=\tilde\Phi} T_\Phi[\Psi][\tilde u][\tilde\varphi] = & \int_{\tilde\Phi(\Omega)} \operatorname{curl}\tilde v \left(D\zeta + (D\zeta)^T\right) (\operatorname{curl}\tilde w)^T\,dy \\
& - \int_{\tilde\Phi(\Omega)} \operatorname{curl}\tilde v\,(\operatorname{curl}\tilde w)^T \operatorname{div}(\zeta)\,dy.
\end{aligned} \tag{2.3.10}
$$

We now compute the first term in the right-hand side of (2.3.9). We claim that

$$
d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u][\tilde\varphi] = -\int_{\tilde\Phi(\Omega)} \tilde v \left(D\zeta + D\zeta^T\right) \tilde w^T dy + \int_{\tilde\Phi(\Omega)} \tilde v \tilde w^T \operatorname{div}\zeta dy, \tag{2.3.11}
$$

where we recall that $\zeta = \Psi \circ \tilde\Phi^{(-1)}$. Indeed, it is easy to see that

$$
\left[\left[d|_{\Phi=\tilde\Phi}(\det(D\Phi))[\Psi]\right] \circ \tilde\Phi^{(-1)}\right] \det D\tilde\Phi^{(-1)} = \operatorname{div}\left(\Psi\circ\tilde\Phi^{(-1)}\right). \tag{2.3.12}
$$

Moreover, since

$$
J_\Phi[\tilde u][\tilde\varphi] = \int_\Omega \tilde u\, R_\Phi\, \tilde\varphi^T |\det D\Phi|\,dx,
$$

where $R_\Phi = (D\Phi)^{-1}(D\Phi)^{-T}$, then

$$
\begin{aligned}
& d|_{\Phi=\tilde\Phi} J_\Phi[\Psi][\tilde u][\tilde\varphi] \\
& = \int_\Omega \tilde u\, d|_{\Phi=\tilde\Phi} R_\Phi[\Psi]\, \tilde\varphi^T |\det D\Phi|\,dx + \int_\Omega \tilde u\, R_{\tilde\Phi}\, \tilde\varphi^T d|_{\Phi=\tilde\Phi} |\det D\Phi|\,[\Psi]\,dx.
\end{aligned} \tag{2.3.13}
$$

Note that

$$
\left[d|_{\Phi=\tilde\Phi} R_\Phi[\Psi]\right] \circ \tilde\Phi^{(-1)} = -\, D\tilde\Phi^{(-1)} \left[D(\Psi\circ\tilde\Phi^{(-1)}) + (D(\Psi\circ\tilde\Phi^{(-1)}))^T\right] (D\tilde\Phi^{(-1)})^T. \tag{2.3.14}
$$

Therefore by equalities (2.3.12), (2.3.13), (2.3.14) and a change of variables, equality (2.3.11) follows.

We now consider the first term in the right-hand side of (2.3.11). By integrating by parts we obtain that

$$
\begin{aligned}
& -\int_{\tilde\Phi(\Omega)} \tilde v \left(D\zeta + (D\zeta)^T\right) \tilde w^T\,dy = -\int_{\tilde\Phi(\Omega)} \tilde v_i(\partial_j\zeta_i)\tilde w_j\,dy - \int_{\tilde\Phi(\Omega)} \tilde v_i(\partial_i\zeta_j)\tilde w_j\,dy \\
& = \int_{\tilde\Phi(\Omega)} (\partial_j\tilde v_i)\tilde w_j\zeta_i\,dy + \int_{\tilde\Phi(\Omega)} (\tilde v\cdot\zeta)\operatorname{div}(\tilde w)\,dy - \int_{\partial\tilde\Phi(\Omega)} (\tilde v\cdot\zeta)(\tilde w\cdot\nu)\,d\sigma \\
& \quad + \int_{\tilde\Phi(\Omega)} (\tilde w\cdot\zeta)\operatorname{div}(\tilde v)\,dy + \int_{\tilde\Phi(\Omega)} (\partial_i\tilde w_j)\tilde v_i\zeta_j\,dy - \int_{\partial\tilde\Phi(\Omega)} (\tilde v\cdot\nu)(\tilde w\cdot\zeta)\,d\sigma \\
& = \int_{\tilde\Phi(\Omega)} (\partial_j\tilde v_i)\tilde w_j\zeta_i\,dy + \int_{\tilde\Phi(\Omega)} (\partial_i\tilde w_j)\tilde v_i\zeta_j\,dy + \int_{\tilde\Phi(\Omega)} (\tilde v\cdot\zeta)\operatorname{div}(\tilde w)\,dy \\
& \quad + \int_{\tilde\Phi(\Omega)} (\tilde w\cdot\zeta)\operatorname{div}(\tilde v)\,dy - 2\int_{\partial\tilde\Phi(\Omega)} (\tilde v\cdot\tilde w)(\zeta\cdot\nu)\,d\sigma,
\end{aligned}
$$

where we have used the fact that $\tilde{v} \times \nu = 0 = \tilde{w} \times \nu$ hence

$$(\tilde{v} \cdot \nu)(\tilde{w} \cdot \zeta) = (\tilde{v} \cdot \zeta)(\tilde{w} \cdot \nu) = (\tilde{v} \cdot \nu)(\tilde{w} \cdot \nu)(\zeta \cdot \nu) = (\tilde{v} \cdot \tilde{w})(\zeta \cdot \nu)$$

on $\partial\tilde{\Phi}(\Omega)$. Now, since

$$\sum_{i,j=1}^{3} (\partial_j \tilde{v}_i)\tilde{w}_j \zeta_i - (\partial_i \tilde{v}_j)\tilde{w}_j \zeta_i = \operatorname{curl} \tilde{v} \cdot (\tilde{w} \times \zeta)$$

and

$$\sum_{i,j=1}^{3} (\partial_i \tilde{w}_j)\tilde{v}_i \zeta_j - (\partial_j \tilde{w}_i)\tilde{v}_i \zeta_j = \operatorname{curl} \tilde{w} \cdot (\tilde{v} \times \zeta),$$

we deduce that

$$\begin{aligned}
&\int_{\tilde{\Phi}(\Omega)} (\partial_j \tilde{v}_i)\tilde{w}_j \zeta_i \, dy + \int_{\tilde{\Phi}(\Omega)} (\partial_i \tilde{w}_j)\tilde{v}_i \zeta_j \, dy \\
&= \int_{\tilde{\Phi}(\Omega)} (\partial_j \tilde{v}_i)\tilde{w}_j \zeta_i \, dy + \int_{\tilde{\Phi}(\Omega)} (\partial_i \tilde{w}_j)\tilde{v}_i \zeta_j \, dy - \int_{\tilde{\Phi}(\Omega)} (\partial_i \tilde{v}_j)\tilde{w}_j \zeta_i \, dy \\
&\quad - \int_{\tilde{\Phi}(\Omega)} (\partial_j \tilde{w}_i)\tilde{v}_i \zeta_j \, dy + \int_{\tilde{\Phi}(\Omega)} \nabla(\tilde{v} \cdot \tilde{w}) \cdot \zeta \, dy \\
&= \int_{\tilde{\Phi}(\Omega)} \operatorname{curl} \tilde{v} \cdot (\tilde{w} \times \zeta) \, dy + \int_{\tilde{\Phi}(\Omega)} \operatorname{curl} \tilde{w} \cdot (\tilde{v} \times \zeta) \, dy + \int_{\tilde{\Phi}(\Omega)} \nabla(\tilde{v} \cdot \tilde{w}) \cdot \zeta \, dy.
\end{aligned}$$

Thus, since for any triple of vectors $a, b, c \in \mathbb{R}^3$ we have that $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$, we get

$$\begin{aligned}
&- \int_{\tilde{\Phi}(\Omega)} \tilde{v} \left(D\zeta + (D\zeta)^T \right) \tilde{w}^T \, dy \\
&= \int_{\tilde{\Phi}(\Omega)} \tilde{v} \cdot (\zeta \times \operatorname{curl} \tilde{w}) \, dy + \int_{\tilde{\Phi}(\Omega)} \tilde{w} \cdot (\zeta \times \operatorname{curl} \tilde{v}) \, dy + \int_{\tilde{\Phi}(\Omega)} \nabla(\tilde{v} \cdot \tilde{w}) \cdot \zeta \, dy \\
&\quad + \int_{\tilde{\Phi}(\Omega)} (\tilde{v} \cdot \zeta) \operatorname{div} \tilde{w} \, dy + \int_{\tilde{\Phi}(\Omega)} (\tilde{w} \cdot \zeta) \operatorname{div} \tilde{v} \, dy - 2 \int_{\partial\tilde{\Phi}(\Omega)} (\tilde{v} \cdot \tilde{w})(\zeta \cdot \nu) \, d\sigma.
\end{aligned}$$

Since v and w satisfy the equation in (2.1.3) on $\tilde{\Phi}(\Omega)$, we have

$$\begin{aligned}
&\int_{\tilde{\Phi}(\Omega)} \tilde{v} \cdot (\zeta \times \operatorname{curl} \tilde{w}) \, dy \\
&= \lambda^{-1} \int_{\tilde{\Phi}(\Omega)} \operatorname{curl} \operatorname{curl} \tilde{v} \cdot (\zeta \times \operatorname{curl} \tilde{w}) \, dy - \lambda^{-1} \tau \int_{\tilde{\Phi}(\Omega)} \nabla \operatorname{div} \tilde{v} \cdot (\zeta \times \operatorname{curl} \tilde{w}) \, dy
\end{aligned}$$

$$
\begin{aligned}
&= -\lambda^{-1}\int_{\tilde\Phi(\Omega)} (\operatorname{curl}\tilde w\cdot\operatorname{curl}\tilde v)\operatorname{div}(\zeta)\,dy + \lambda^{-1}\int_{\tilde\Phi(\Omega)} \operatorname{curl}\tilde v D\zeta\,(\operatorname{curl}\tilde w)^T\,dy \\
&\quad - \lambda^{-1}\int_{\tilde\Phi(\Omega)} \operatorname{curl}\tilde v D(\operatorname{curl}\tilde w)\,\zeta^T\,dy - \lambda^{-1}\tau\int_{\tilde\Phi(\Omega)} \nabla\operatorname{div}\tilde v\cdot(\zeta\times\operatorname{curl}\tilde w)\,dy \\
&\quad + \lambda^{-1}\int_{\partial\tilde\Phi(\Omega)} (\zeta\cdot\nu)(\operatorname{curl}\tilde w\cdot\operatorname{curl}\tilde v)\,d\sigma - \lambda^{-1}\int_{\partial\tilde\Phi(\Omega)} (\zeta\cdot\operatorname{curl}\tilde v)(\nu\cdot\operatorname{curl}\tilde w)\,d\sigma,
\end{aligned}
\tag{2.3.15}
$$

where, in order to compute the boundary integrals, we have used the following formula

$$
\begin{aligned}
&(\nu\times\operatorname{curl}\tilde v)\cdot(\zeta\times\operatorname{curl}\tilde w) \\
&\quad = \xi_{ijk}\nu_j(\operatorname{curl}\tilde v)_k\xi_{ilm}\zeta_l(\operatorname{curl}\tilde w)_m = (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})\nu_j\zeta_l(\operatorname{curl}\tilde v)_k(\operatorname{curl}\tilde w)_m \\
&\quad = (\zeta\cdot\nu)(\operatorname{curl}\tilde w\cdot\operatorname{curl}\tilde v) - (\nu\cdot\operatorname{curl}\tilde w)(\zeta\cdot\operatorname{curl}\tilde v).
\end{aligned}
$$

Note that the last boundary term in formula (2.3.15) vanishes by Lemma 1.3.5. Then

$$
\begin{aligned}
&-\tilde\lambda\int_{\tilde\Phi(\Omega)} \tilde v\left(D\zeta + (D\zeta)^T\right)\tilde w^T\,dy \\
&= -2\int_{\tilde\Phi(\Omega)} (\operatorname{curl}\tilde w\cdot\operatorname{curl}\tilde v)\operatorname{div}\zeta\,dy + \int_{\tilde\Phi(\Omega)} \operatorname{curl}\tilde w\left(D\zeta+(D\zeta)^T\right)(\operatorname{curl}\tilde v)^T\,dy \\
&\quad - \int_{\tilde\Phi(\Omega)} \nabla(\operatorname{curl}\tilde w\cdot\operatorname{curl}\tilde v)\cdot\zeta\,dy - \tau\int_{\tilde\Phi(\Omega)} \nabla\operatorname{div}\tilde v\cdot(\zeta\times\operatorname{curl}\tilde w)\,dy \\
&\quad - \tau\int_{\tilde\Phi(\Omega)} \nabla\operatorname{div}\tilde w\cdot(\zeta\times\operatorname{curl}\tilde v)\,dy + 2\int_{\partial\tilde\Phi(\Omega)} (\zeta\cdot\nu)(\operatorname{curl}\tilde w\cdot\operatorname{curl}\tilde v)\,d\sigma \\
&\quad + \tilde\lambda\int_{\tilde\Phi(\Omega)} \nabla(\tilde v\cdot\tilde w)\cdot\zeta\,dy + \tilde\lambda\int_{\tilde\Phi(\Omega)} (\tilde w\cdot\zeta)\operatorname{div}\tilde v\,dy + \tilde\lambda\int_{\tilde\Phi(\Omega)} (\tilde v\cdot\zeta)\operatorname{div}\tilde w\,dy \\
&\quad - 2\tilde\lambda\int_{\partial\tilde\Phi(\Omega)} (\tilde v\cdot\tilde w)(\zeta\cdot\nu)\,d\sigma \\
&= -\int_{\tilde\Phi(\Omega)} (\operatorname{curl}\tilde w\cdot\operatorname{curl}\tilde v)\operatorname{div}\zeta\,dy + \int_{\tilde\Phi(\Omega)} \operatorname{curl}\tilde w\left(D\zeta+(D\zeta)^T\right)(\operatorname{curl}\tilde v)^T\,dy \\
&\quad - \tau\int_{\tilde\Phi(\Omega)} \nabla\operatorname{div}\tilde v\cdot(\zeta\times\operatorname{curl}\tilde w)\,dy - \tau\int_{\tilde\Phi(\Omega)} \nabla\operatorname{div}\tilde w\cdot(\zeta\times\operatorname{curl}\tilde v)\,dy \\
&\quad - \tilde\lambda\int_{\tilde\Phi(\Omega)} (\tilde v\cdot\tilde w)\operatorname{div}\zeta\,dy + \tilde\lambda\int_{\tilde\Phi(\Omega)} (\tilde w\cdot\zeta)\operatorname{div}\tilde v\,dy + \tilde\lambda\int_{\tilde\Phi(\Omega)} (\tilde v\cdot\zeta)\operatorname{div}\tilde w\,dy \\
&\quad - \tilde\lambda\int_{\partial\tilde\Phi(\Omega)} (\tilde v\cdot\tilde w)(\zeta\cdot\nu)\,d\sigma + \int_{\partial\tilde\Phi(\Omega)} (\operatorname{curl}\tilde w\cdot\operatorname{curl}\tilde v)(\zeta\cdot\nu)\,d\sigma.
\end{aligned}
\tag{2.3.16}
$$

Hence, by (2.3.9), (2.3.10) and (2.3.16) we get

$$
\begin{aligned}
&(\tilde\lambda+1)^2\langle d|_{\Phi=\tilde\Phi} S_\Phi[\Psi][\tilde u], \tilde\varphi\rangle_{\tilde\Phi} \\
&\quad = -\tilde\lambda \int_{\tilde\Phi(\Omega)} \tilde v \left(D\zeta + (D\zeta)^T\right) \tilde w^T\, dy + \tilde\lambda \int_{\tilde\Phi(\Omega)} \tilde v \tilde w^T \operatorname{div}(\zeta)\, dy \\
&\quad - \int_{\tilde\Phi(\Omega)} \operatorname{curl} \tilde v \left(D\zeta + (D\zeta)^T\right) (\operatorname{curl} \tilde w)^T\, dy + \int_{\tilde\Phi(\Omega)} \operatorname{curl} \tilde v (\operatorname{curl} \tilde w)^T \operatorname{div}(\zeta)\, dy \\
&\quad = \int_{\partial\tilde\Phi(\Omega)} \left(\operatorname{curl} \tilde v \cdot \operatorname{curl} \tilde w - \tilde\lambda\, \tilde v \cdot \tilde w\right) (\zeta\cdot\nu)\, d\sigma \\
&\quad - \tau \int_{\tilde\Phi(\Omega)} \nabla \operatorname{div} \tilde v \cdot (\zeta \times \operatorname{curl} \tilde w)\, dy - \tau \int_{\tilde\Phi(\Omega)} \nabla \operatorname{div} \tilde w \cdot (\zeta \times \operatorname{curl} \tilde v)\, dy \\
&\quad + \tilde\lambda \int_{\tilde\Phi(\Omega)} (\tilde w\cdot\zeta) \operatorname{div}(\tilde v)\, dy + \tilde\lambda \int_{\tilde\Phi(\Omega)} (\tilde v\cdot\zeta) \operatorname{div}(\tilde w)\, dy \\
&\quad = \int_{\partial\tilde\Phi(\Omega)} \left(\operatorname{curl} \tilde v \cdot \operatorname{curl} \tilde w - \tilde\lambda\, \tilde v \cdot \tilde w\right) (\zeta\cdot\nu)\, d\sigma,
\end{aligned}
$$

and the proof is complete. □

In the case of domain perturbations depending real analytically on one scalar parameter, it is possible to prove a Rellich-Nagy-type theorem and describe all the eigenvalues splitting from a multiple eigenvalue of multiplicity m by means of m real-analytic functions. One can also check [73] for more details, specifically Theorem 3.9.

Theorem 2.3.17. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{1,1}$. Let $\tilde\Phi \in \mathcal{A}_\Omega$ and $\{\Phi_\epsilon\}_{\epsilon\in\mathbb{R}} \subset \mathcal{A}_\Omega$ be a family depending real-analytically on ϵ such that $\Phi_0 = \tilde\Phi$. Assume that $\tilde\lambda$ is a Maxwell eigenvalue on $\tilde\Phi(\Omega)$ of multiplicity m, $\tilde\lambda = \lambda_n[\tilde\Phi] = \cdots = \lambda_{n+m-1}[\tilde\Phi]$ for some $n \in \mathbb{N}$ and that $\tilde E^{(1)}, \dots, \tilde E^{(m)} \in X_{\mathrm{N}}(\operatorname{div} 0, \tilde\Phi(\Omega))$ is an orthonormal basis of the eigenspace of $\tilde\lambda$, the orthonormality being taken in $L^2(\tilde\Phi(\Omega))^3$. Moreover, assume that $\tilde E^{(k)} \in H^2(\tilde\Phi(\Omega))$ for all $k = 1, \dots, m$.*

Then there exists an open interval I containing zero and m real-analytic functions $g_1, \dots, g_m$ defined on I such that $\{\lambda_n[\Phi_\epsilon], \dots, \lambda_{n+m-1}[\Phi_\epsilon]\} = \{g_1(\epsilon), \dots, g_m(\epsilon)\}$ coincide for all $\epsilon \in I$.

Furthermore, the derivatives $g_1'(0), \dots, g_m'(0)$ of the functions $g_1, \dots, g_m$ at zero coincide with the eigenvalues of the matrix

$$
\left(\int_{\partial\tilde\Phi(\Omega)} \left(\tilde\lambda\, \tilde E^{(i)} \cdot \tilde E^{(j)} - \operatorname{curl} \tilde E^{(i)} \cdot \operatorname{curl} \tilde E^{(j)}\right) \zeta\cdot\nu\, d\sigma \right)_{i,j=1,\dots,m} \tag{2.3.18}
$$

where $\zeta = \dot\Phi_0 \circ \tilde\Phi^{(-1)}$, and $\dot\Phi_0$ denotes the derivative at zero of the map $\epsilon \mapsto \Phi_\epsilon$.

Proof. First of all, we note that by our assumptions, $\tilde{\lambda}$ does not coincide with any of the eigenvalues of the Dirichlet Laplacian in Ω multiplied by τ, see Remark 2.1.9, and by the well-known continuity of the eigenvalues of the Dirichlet Laplacian, this implies that for all ϵ in a sufficiently small neighbourhood of zero, the eigenvalues $\{\lambda_n[\Phi_\epsilon], \dots, \lambda_{n+m-1}[\Phi_\epsilon]\}$ satisfy the same property. By applying [79, Theorem 2.27, Corollary 2.28] to the family of operators S_{Φ_ϵ} we deduce that there exists an open interval I containing zero and m real-analytic functions $\tilde{g}_1, \dots, \tilde{g}_m$ from I to $\mathbb{R}$ such that $\{(1+\lambda_n[\Phi_\epsilon])^{-1}, \dots, (1+\lambda_{n+m-1}[\Phi_\epsilon])^{-1}\} = \{\tilde{g}_1(\epsilon), \dots, \tilde{g}_m(\epsilon)\}$ for all $\epsilon \in I$; moreover, the derivatives of those functions at zero are given by the eigenvalues of the matrix

$$\left(\langle d|_{\Phi=\tilde{\Phi}} S_\Phi[\dot{\Phi}_0][\tilde{u}^{(i)}], \tilde{u}^{(j)}\rangle_{\tilde{\Phi}}\right)_{i,j=1,\dots,m},$$

where $\tilde{u}^{(k)} = (\tilde{E}^{(k)} \circ \tilde{\Phi})D\tilde{\Phi}$ for all $k = 1, \dots, m$. The proof follows by setting $g_k(\epsilon) := \dfrac{1}{\tilde{g}_k(\epsilon)} - 1$, $k = 1, \dots, m$, and using Lemma 2.3.7. ☐

We conclude this section by proving an immediate consequence of our results, namely the Rellich-Pohozaev identity for Maxwell eigenvalues.

Theorem 2.3.19 (Rellich-Pohozaev Identity)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$ of class $C^{1,1}$. Let λ be a Maxwell eigenvalue and $E \in X_{\mathrm{N}}(\mathrm{div}, 0, \Omega)$ a corresponding non-trivial eigenvector normalized in $L^2(\Omega)^3$. Assume that $E \in H^2(\Omega)$. Then*

$$\lambda = \frac{1}{2}\int_{\partial\Omega} \left(|\mathrm{curl}\, E|^2 - \lambda|E|^2\right)(x \cdot \nu)\, d\sigma\,.$$

Proof. Assume that $\lambda = \lambda_n(\Omega) = \dots = \lambda_{n+m-1}(\Omega)$ is a Maxwell eigenvalue with multiplicity m (with the understanding that the corresponding m-dimensional eigenspace is made of Maxwell eigenvectors, see Remark 2.1.9). We consider a family of dilations $(1+\epsilon)\Omega$ of Ω which can viewed as a family of diffeomorphisms $\Phi_\epsilon = Id + \epsilon\, Id$, $\epsilon \in \mathbb{R}$, where Id denotes the identity map. It is obvious that $\lambda_{n+i}[\Phi_\epsilon] = (1+\epsilon)^{-2}\lambda$ for all $i = 0, \dots, m-1$. In particular, the domain perturbation under consideration preserves the multiplicity of λ and the matrix (2.3.18) is a multiple of the identity. By differentiating with respect to ϵ and applying Theorem 2.3.17 with $\zeta = Id$ we obtain

$$\frac{d}{d\epsilon}\lambda_n[\Phi_\epsilon]\bigg|_{\epsilon=0} = \int_{\partial\Omega} \left(\lambda\left|E(x)\right|^2 - \left|\mathrm{curl}\, E(x)\right|^2\right)\left(x \cdot \nu(x)\right)\, d\sigma(x), \tag{2.3.20}$$

where the given normalized eigenvector E is serving as an element of the orthonormal basis of the eigenspace. If we differentiate the equality $\lambda_n[\Phi_\epsilon] = (1+\epsilon)^{-2}\lambda$ with respect to ϵ, we obtain

$$\frac{d}{d\epsilon}\lambda_n[\Phi_\epsilon]\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\left((1+\epsilon)^{-2}\lambda\right)\bigg|_{\epsilon=0} = -2\lambda. \tag{2.3.21}$$

By combining (2.3.20) and (2.3.21) we conclude. □

2.4 Criticality for symmetric functions of the eigenvalues

In this section we apply the previous results in order to find a characterization of critical points, thus including maxima or minima, for the symmetric function of the eigenvalues, upon volume or perimeter constraint for the perturbed domains. We denote by $\mathcal{V}[\Phi]$ the measure of $\Phi(\Omega)$, that is

$$\mathcal{V}[\Phi]: \ = \int_{\Phi(\Omega)} dx = \int_{\Omega} |\det D\Phi| \ dx, \tag{2.4.1}$$

and by $\mathcal{P}[\Phi]$ the perimeter of $\Phi(\Omega)$ that is

$$\mathcal{P}[\Phi] := \int_{\partial\Phi(\Omega)} d\sigma = \int_{\partial\Omega} \left|\nu(D\Phi)^{-1}\right| |\det D\Phi| \, d\sigma \, . \tag{2.4.2}$$

We are interested in extremum problems of the type

$$\min_{\mathcal{V}[\Phi]=\text{const.}} \Lambda_{F,s}[\Phi] \quad \text{or} \quad \max_{\mathcal{V}[\Phi]=\text{const.}} \Lambda_{F,s}[\Phi], \tag{2.4.3}$$

as well as problems of the type

$$\min_{\mathcal{P}[\Phi]=\text{const.}} \Lambda_{F,s}[\Phi] \quad \text{or} \quad \max_{\mathcal{P}[\Phi]=\text{const.}} \Lambda_{F,s}[\Phi]. \tag{2.4.4}$$

It is convenient to recall the following lemma from [77], where $\mathcal{H} = \mathrm{div}\nu$ denotes the mean curvature of Ω, that is, the sum of the principal curvatures.

Lemma 2.4.5. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{1,1}$. Then the maps $\mathcal{V}$, $\mathcal{P}$ from $\mathcal{A}_\Omega$ to $\mathbb{R}$ defined in (2.4.1),(2.4.2) are real-analytic. Moreover, the differentials of $\mathcal{V}$ and $\mathcal{P}$ at any point $\tilde{\Phi} \in \mathcal{A}_\Omega$ are given by the formulas*

$$d|_{\Phi=\tilde{\Phi}}\mathcal{V}[\Phi][\Psi] = \int_{\partial\tilde{\Phi}(\Omega)} (\Psi \circ \tilde{\Phi}^{(-1)}(x)) \cdot \nu(x) \, d\sigma, \tag{2.4.6}$$

and

$$d|_{\Phi=\tilde{\Phi}}\mathcal{P}[\Phi][\Psi] = \int_{\partial\tilde{\Phi}(\Omega)} \mathcal{H}(x) \, (\Psi \circ \tilde{\Phi}^{(-1)}(x)) \cdot \nu(x) d\sigma, \tag{2.4.7}$$

for all $\Psi \in C^{1,1}(\overline{\Omega}, \mathbb{R}^3)$.

For $\alpha \in]0,+\infty[$, we set

$$V(\alpha) := \left\{\Phi \in \mathcal{A}_\Omega : \mathcal{V}(\Phi) = \alpha\right\}, \quad P(\alpha) := \left\{\Phi \in \mathcal{A}_\Omega : \mathcal{P}(\Phi) = \alpha\right\}.$$

Keeping in mind the Lagrange Multiplier Theorem (which holds also in infinite dimensional spaces), we note that if $\tilde{\Phi} \in \mathcal{A}_\Omega[F]$ is a minimizer/maximizer in (2.4.3) or (2.4.4) respectively, then it is a critical point for the function $\Phi \mapsto \Lambda_{F,s}[\Phi]$ under the constraint $\Phi \in V(\tilde{\alpha})$ or the constraint $\Phi \in P(\tilde{\beta})$ respectively, where $\tilde{\alpha} = \mathcal{V}(\tilde{\Phi})$ and $\tilde{\beta} = \mathcal{P}(\tilde{\Phi})$, which means that

$$\operatorname{Ker} d|_{\Phi=\tilde{\Phi}} \mathcal{V}[\Phi] \subset \operatorname{Ker} d|_{\Phi=\tilde{\Phi}} \Lambda_{F,s}[\Phi], \tag{2.4.8}$$

or

$$\operatorname{Ker} d|_{\Phi=\tilde{\Phi}} \mathcal{P}[\Phi] \subset \operatorname{Ker} d|_{\Phi=\tilde{\Phi}} \Lambda_{F,s}[\Phi], \tag{2.4.9}$$

respectively.

The following theorem provides a characterization of those points $\tilde{\Phi} \in \mathcal{A}_\Omega[F]$ satisfying (2.4.8) or (2.4.9).

Theorem 2.4.10. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{1,1}$. Let F be a non-empty finite subset of $\mathbb{N}$ and $\tilde{\alpha} \in]0,+\infty[$. The following statements hold:*

(i) *Assume that $\tilde{\Phi} \in \Theta_\Omega[F] \cap V(\tilde{\alpha})$ is such that $\lambda_j[\tilde{\Phi}]$ are Maxwell eigenvalues with common value $\lambda_F[\tilde{\Phi}]$ for all $j \in F$. Assume that $\tilde{E}^{(1)}, \dots, \tilde{E}^{(|F|)} \in X_{\mathrm{N}}(\operatorname{div} 0, \tilde{\Phi}(\Omega))$ is an orthonormal basis of the eigenspace corresponding to $\lambda_F[\tilde{\Phi}]$ and that those eigenvectors belong to $H^2(\tilde{\Phi}(\Omega))$.*

For $s = 1, \dots, |F|$ the function $\tilde{\Phi}$ is a critical point for $\Lambda_{F,s}$ with volume constraint $\Phi \in V(\tilde{\alpha})$ (that is, condition (2.4.8) is satisfied) if and only if there exists a constant $c \in \mathbb{R}$ such that

$$\sum_{l=1}^{|F|} \left(\lambda_F[\tilde{\Phi}] \left|\tilde{E}^{(l)}\right|^2 - \left|\operatorname{curl} \tilde{E}^{(l)}\right|^2 \right) = c, \qquad \text{on } \partial\tilde{\Phi}(\Omega). \tag{2.4.11}$$

(ii) *Assume that $\tilde{\Phi} \in \Theta_\Omega[F] \cap P(\tilde{\alpha})$ is such that $\lambda_j[\tilde{\Phi}]$ are Maxwell eigenvalues with common value $\lambda_F[\tilde{\Phi}]$ for all $j \in F$. Assume that $\tilde{E}^{(1)}, \dots, \tilde{E}^{(|F|)} \in X_{\mathrm{N}}(\operatorname{div} 0, \tilde{\Phi}(\Omega))$ is an orthonormal basis of the eigenspace corresponding to $\lambda_F[\tilde{\Phi}]$ and that those eigenvectors belong to $H^2(\tilde{\Phi}(\Omega))$. For $s = 1, \dots, |F|$ the function $\tilde{\Phi}$ is a critical point for $\Lambda_{F,s}$ with perimeter constraint $\Phi \in P(\tilde{\alpha})$ (that is, condition (2.4.9) is satisfied) if and only if there exists a constant $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} \left(\lambda_F[\tilde{\Phi}] \left|\tilde{E}^{(l)}\right|^2 - \left|\operatorname{curl} \tilde{E}^{(l)}\right|^2 \right) = c\mathcal{H}, \qquad \text{on } \partial\tilde{\Phi}(\Omega). \tag{2.4.12}$$

Proof. It suffices to observe that by standard linear algebra condition (2.4.8) or condition (2.4.9) is satisfied if and only if there exists a constant $l \in \mathbb{R}$ such that $d|_{\Phi=\tilde{\Phi}}\Lambda_{F,s}[\Phi] = l\, d|_{\Phi=\tilde{\Phi}}\mathcal{V}[\Phi]$ or $d|_{\Phi=\tilde{\Phi}}\Lambda_{F,s}[\Phi] = l\, d|_{\Phi=\tilde{\Phi}}\mathcal{P}[\Phi]$ respectively . By formulas (2.3.6), (2.4.6), (2.4.7) and the Fundamental Lemma of the Calculus of Variations, we conclude. □

In the next theorem, we show that balls are critical domains.

Theorem 2.4.13. *Let Ω be a bounded domain in $\mathbb{R}^3$ of class $C^{1,1}$. Let $\tilde{\Phi} \in \mathcal{A}_\Omega$ be such that $\tilde{\Phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be a Maxwell eigenvalue in $\tilde{\Phi}(\Omega)$ with an eigenspace of dimension m in $X_{\mathrm{N}}(\mathrm{div}\, 0, \tilde{\Phi}(\Omega))$. Assume that $\lambda_{n-1}[\tilde{\Phi}(\Omega)] < \lambda_n[\tilde{\Phi}(\Omega)] = \cdots = \lambda_{n+m-1}[\tilde{\Phi}(\Omega)] < \lambda_{n+m}[\tilde{\Phi}(\Omega)]$ for some $n \in \mathbb{N}$, and let $F = \{n, \dots, n+m-1\}$. Then $\tilde{\Phi}$ is both a critical point for $\Lambda_{F,s}$ with volume constraint $\Phi \in V(\mathcal{V}(\tilde{\Phi}))$ and a critical point for $\Lambda_{F,s}$ with perimeter constraint $\Phi \in P(\mathcal{P}(\tilde{\Phi}))$, for all $s = 1, \dots, |F|$.*

Proof. Conditions (2.4.11) and (2.4.12) are satisfied thanks to Lemma 2.4.14 below. □

We highlight the fact that the proof of the next lemma is based on the rotational invariance of the curlcurl operator which means that for any orthogonal matrix $A \in O_3(\mathbb{R})$ we have

$$\mathrm{curl}\,\mathrm{curl}(E(Ax)A) = \big((\mathrm{curl}\,\mathrm{curl}\,E)(Ax)\big)\, A$$

where E is any vector filed defined on a radially symmetric domain, an eigenvector in our case, see below. We refer e.g., to [20, Definition 3] for the a definition of rotational invariance for a large class of partial differential operators. We also note that a similar method has been used in [23, Lemma 5.3, Theorem 5.4] for the Reissner-Mindlin system.

Lemma 2.4.14. *Let B be a ball in $\mathbb{R}^3$ centered at zero. Let λ be a Maxwell eigenvalue in B with an eigenspace of dimension m in $X_{\mathrm{N}}(\mathrm{div}\, 0, B)$ and let $E^{(1)}, \dots, E^{(m)}$ be a corresponding orthonormal basis. Then, the functions*

$$\sum_{l=1}^{m} \left|E^{(l)}\right|^2, \quad \sum_{l=1}^{m} \left|\mathrm{curl}\, E^{(l)}\right|^2$$

are radial.

Proof. Let E be an eigenvector of problem (2.1.1) with eigenvalue λ. Take an orthogonal matrix $A \in O_3(\mathbb{R})$ and consider the vector field u defined by $u = (E \circ A)A$. Then the Jacobian matrix Du of u is

$$Du(x) = A^T DE(Ax)\, A.$$

Note that in this proof, for simplicity, we denote by Ax the row vector $(Ax^T)^T$ which is identified with the image of x via the linear transformation associated with the matrix A. Thus

$$\operatorname{div} u\ (x) = \operatorname{Tr}(Du(x)) = \operatorname{Tr}(A^T DE(Ax)\, A) = \operatorname{Tr}(DE(x)) = \operatorname{div} E\ (Ax)\,.$$

Moreover, we have that

$$\begin{aligned}\Delta u_i(x) &= \frac{\partial}{\partial x_k}\frac{\partial u_i}{\partial x_k}(x) = \frac{\partial}{\partial x_k}[A_{rk}(\partial_r E_j)(Ax)A_{ji}] \\ &= A_{rk}(\partial_s\partial_r E_j)(Ax)A_{sk}A_{ji} = (\partial_s\partial_r E_j)(Ax)\delta_{rs}A_{ji} = \partial_r^2 E_j(Ax)A_{ji} \\ &= \big[\Delta E(Ax)\, A\big]_i\,.\end{aligned}$$

Therefore the vector laplacian of u satisfies

$$\Delta u(x) = \Delta E(Ax)\, A\,.$$

Finally, we get that

$$\begin{aligned}\operatorname{curl}\operatorname{curl} u(x) = D \operatorname{div} u(x) - \Delta u(x) &= \big[(D\operatorname{div} E)(Ax) - (\Delta E)(Ax)\big]\, A \\ &= \big[(\operatorname{curl}\operatorname{curl} E)(Ax)\big]\, A = \lambda E(Ax)A = \lambda u(x).\end{aligned}$$

This proves that if $E^{(1)}, \dots, E^{(m)}$ is an orthonormal basis of the eigenspace associated with the eigenvalue λ, then $\Big\{u^{(j)} = (E^{(j)} \circ A)A : j = 1, \dots, m\Big\}$ is another orthonormal basis for the eigenspace associate with λ. Since both $\{E^{(j)}\}_{j=1,\dots,m}$ and $\{u^{(j)}\}_{j=1,\dots,m}$ are orthonormal bases, then there exists $R[A] \in O_m(\mathbb{R})$ with matrix $(R_{ij}[A])i, j = 1, \dots,\, m$ such that

$$u^{(j)} = \sum_{l=1}^{m} R_{jl}[A]E^{(l)}.$$

Therefore

$$\begin{aligned}\sum_{j=1}^{m}\Big|E^{(j)}\Big|^2 \circ A = \sum_{j=1}^{m}\Big|\Big(E^{(j)} \circ A\Big)\, A\Big|^2 &= \sum_{j=1}^{m}\Big|u^{(j)}\Big|^2 \\ &= \sum_{j=1}^{m}\left(\sum_{l=1}^{m} R_{jl}[A]E^{(l)}\right)\cdot\left(\sum_{h=1}^{m} R_{jh}[A]E^{(h)}\right) \\ &= \sum_{j=1}^{m}\sum_{l,h=1}^{m} R_{jl}[A]R_{jh}[A](E^{(l)}\cdot E^{(h)}) = \sum_{l=1}^{m}\Big|E^{(l)}\Big|^2,\end{aligned}$$

which proves that $\sum_{l=1}^m \left|E^{(l)}\right|^2$ is a radial function.

Note that $\operatorname{curl} u^{(j)} = \sum_{l=1}^m R_{jl}[A] \operatorname{curl} E^{(l)}$. By formula (2.2.5) we have that

$$\operatorname{curl} E \circ A = \frac{\operatorname{curl} u\, A^T}{\det A}.$$

Thus

$$\begin{aligned}
\sum_{j=1}^m \left|\operatorname{curl} E^{(j)}\right|^2 \circ A &= \sum_{j=1}^m \left(\operatorname{curl} E^{(j)} \circ A\right) \cdot \left(\operatorname{curl} E^{(j)} \circ A\right) \\
&= \sum_{j=1}^m \left(\operatorname{curl} u^{(j)}\, A^T\right) \cdot \left(\operatorname{curl} u^{(j)}\, A^T\right) \frac{1}{\det(A)^2} = \sum_{j=1}^m \left|\operatorname{curl} u^{(j)}\right|^2 \\
&= \sum_{j=1}^m \left(\sum_{l=1}^m R_{jl}[A] \operatorname{curl} E^{(l)}\right) \cdot \left(\sum_{h=1}^m R_{jh}[A] \operatorname{curl} E^{(h)}\right) \\
&= \sum_{l=1}^m \delta_{lh} \operatorname{curl} E^{(l)} \cdot \operatorname{curl} E^{(h)} = \sum_{l=1}^m \left|\operatorname{curl} E^{(l)}\right|^2,
\end{aligned}$$

which proves that $\sum_{l=1}^m |\operatorname{curl} E^{(l)}|^2$ is also a radial function. □

2.5 Maxwell eigenfunctions on the ball

Throughout this section, by B we denote the ball in $\mathbb{R}^3$ of radius $R > 0$ centred at zero. We use the spherical coordinates $(\rho, \theta, \varphi) \in [0, R] \times [0, \pi] \times [0, 2\pi]$ where θ is the polar angle ($\theta = 0$ at the north pole) and φ is the azimuthal angle. It is also convenient to use the standard local orthonormal base $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}})$ canonically associated with (ρ, θ, φ), namely

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
\hat{\boldsymbol{\theta}} &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\
\hat{\boldsymbol{\varphi}} &= (-\sin\varphi, \cos\varphi, 0).
\end{aligned}$$

The eigenpairs (λ, u) of problem (2.1.1) in B can be expressed in terms of the Riccati-Bessel functions ψ_n and the spherical harmonics Y_n^m. Recall that $\psi_n(z) = \sqrt{\frac{\pi z}{2}} J_{n+\frac{1}{2}}(z)$, where $J_{n+\frac{1}{2}}$ denote the Bessel functions of the first kind and half-integer order, and that the functions ψ_n satisfy the differential equation

$$z^2 f''(z) + (z^2 - n(n+1)) f(z) = 0.$$

Recall also that the spherical harmonics $Y_n^m(\theta,\varphi)$, with $|m|\le n$, are eigenfunctions of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$ on the unit sphere $\mathbb{S}^2$, namely

$$\Delta_{\mathbb{S}^2} Y_n^m + n(n+1)Y_n^m = 0.$$

For more details on these functions we refer to [1].

It it known (see e.g. [41, §8] or [66]) that the eigenpairs (λ, u) of problem (2.1.1) in B are given by the union of the two families

$$\left\{k_{nh}^2, \operatorname{curl}[Y_n^m(\theta,\varphi)\psi_n(k_{nh}\rho)\hat{\boldsymbol{\rho}}]\right\}_{nmh}, \left\{(k'_{nl})^2, \operatorname{curl}\operatorname{curl}[Y_n^m(\theta,\varphi)\psi_n(k'_{nl}\rho)\hat{\boldsymbol{\rho}}]\right\}_{nml} \tag{2.5.1}$$

where $n,h,l\in\mathbb{N}$, $m\in\mathbb{Z}$ with $|m|\le n$. Here k_{nh}, $h\in\mathbb{N}$ denote the positive zeros of the function $k\mapsto\psi_n(kR)$, arranged in increasing order and k'_{nl}, $l\in\mathbb{N}$ denote the positive zeros of the function $k\mapsto\psi'_n(kR)$, arranged in the same way.

Now, we compute explicitly the eigenvectors in (2.5.1). Recalling the formula $\operatorname{curl}(q\hat{\boldsymbol{\rho}}) = \nabla q\times\hat{\boldsymbol{\rho}}$ we have that

$$\begin{aligned}&\operatorname{curl}[Y_n^m(\theta,\varphi)\psi_n(k_{nh}\rho)\hat{\boldsymbol{\rho}}]\\ &\quad=\frac{1}{\rho}\left(\frac{1}{\sin\theta}\partial_\varphi\left(Y_n^m(\theta,\varphi)\right)\psi_n(k_{nh}\rho)\hat{\boldsymbol{\theta}} - \partial_\theta\left(Y_n^m(\theta,\varphi)\right)\psi_n(k_{nh}\rho)\hat{\boldsymbol{\varphi}}\right).\end{aligned} \tag{2.5.2}$$

Observe that the vector in (2.5.2) is zero if and only if $n=0$. Similarly, using the formula $\operatorname{curl}\operatorname{curl}(q\hat{\boldsymbol{\rho}}) = \nabla(\partial_\rho q) - \rho\Delta(\frac{q}{\rho})\hat{\boldsymbol{\rho}}$ and the fact that

$$-\rho\Delta(Y_n^m(\theta,\varphi)\psi_n(k\,\rho)/\rho) = k^2 Y_n^m(\theta,\varphi)\psi_n(k\,\rho),$$

see [41], we have that

$$\begin{aligned}&\operatorname{curl}\operatorname{curl}[Y_n^m(\theta,\varphi)\psi_n(k'_{nl}\rho)\hat{\boldsymbol{\rho}}]\\ &\quad=(k'_{nl})^2 Y_n^m(\theta,\varphi)\left[\psi''_n(k'_{nl}\rho)+\psi_n(k'_{nl}\rho)\right]\hat{\boldsymbol{\rho}} + \frac{1}{\rho}k'_{nl}\partial_\theta\left(Y_n^m(\theta,\varphi)\right)\psi'_n(k'_{nl}\rho)\hat{\boldsymbol{\theta}}\\ &\qquad\frac{1}{\rho\sin\theta}k'_{nl}\partial_\varphi\left(Y_n^m(\theta,\varphi)\right)\psi'_n(k'_{nl}\rho)\hat{\boldsymbol{\varphi}} = \frac{1}{\rho}\left(\frac{n(n+1)}{\rho}Y_n^m(\theta,\varphi)\psi_n(k'_{nl}\rho)\hat{\boldsymbol{\rho}}\right.\\ &\qquad\left.+\,k'_{nl}\partial_\theta\left(Y_n^m(\theta,\varphi)\right)\psi'_n(k'_{nl}\rho)\hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta}k'_{nl}\partial_\varphi\left(Y_n^m(\theta,\varphi)\right)\psi'_n(k'_{nl}\rho)\hat{\boldsymbol{\varphi}}\right).\end{aligned} \tag{2.5.3}$$

Again, this vector is zero if and only if $n=0$. Then, we are ready to prove the following theorem. Recall that the Riccati-Bessel function ψ_1 is given by $\psi_1(z) = \frac{\sin z}{z} - \cos z$.

Theorem 2.5.4. *The first Maxwell eigenvalue in a ball of radius R centred at zero is $(k'_{11})^2$ where k'_{11} is the first positive zero of the derivative of the rescaled Riccati-Bessel function $k \mapsto \psi'_1(Rk)$. Its multiplicity is three and the corresponding Electric eigenspace is generated by the three Electric fields*

$$\begin{aligned} E^{(m)}(\rho,\theta,\varphi) = \frac{1}{\rho}\Bigg(&\frac{2}{\rho}Y_1^m(\theta,\varphi)\psi_1(k'_{11}\rho)\hat{\boldsymbol{\rho}} + k'_{11}\partial_\theta\left(Y_1^m(\theta,\varphi)\right)\psi'_1(k'_{11}\rho)\hat{\boldsymbol{\theta}} \\ &+\frac{1}{\sin\theta}k'_{11}\partial_\varphi\left(Y_1^m(\theta,\varphi)\right)\psi'_1(k'_{11}\rho)\hat{\boldsymbol{\varphi}}\Bigg), \end{aligned} \tag{2.5.5}$$

for $m=-1,0,1$. The associated magnetic fields are given by

$$\begin{aligned} H^{(m)}(\rho,\theta,\varphi) &= -\frac{\mathrm{i}}{k'_{11}}\mathrm{curl}E^{(m)}(\rho,\theta,\varphi) \\ &= \frac{\mathrm{i}k'_{11}}{\rho}\left(\frac{1}{\sin\theta}\partial_\varphi\left(Y_1^m(\theta,\varphi)\right)\psi_1(k'_{11}\rho)\hat{\boldsymbol{\theta}} - \partial_\theta\left(Y_1^m(\theta,\varphi)\right)\psi_1(k'_{11}\rho)\hat{\boldsymbol{\varphi}}\right). \end{aligned} \tag{2.5.6}$$

Proof. Recall that $\psi_n(z) = zj_n(z)$ where j_n are the spherical Bessel functions of the first kind. Due to the above observations, we need to find the smallest positive number $\bar{z} > 0$ such that there exists $n \geq 1$ with either $\psi_n(\bar{z}) = 0$ or $\psi'_n(\bar{z}) = 0$; the first eigenvalue would then be $(\bar{z}/R)^2$. Observe that the positive zeros of ψ_n coincide with the zeros of j_n. First, we recall a useful result about the zeros of the spherical Bessel functions and their derivatives. Denote by $a_{n,s}$ and by $a'_{n,s}$ the s-th positive zero of the function j_n and j'_n respectively, for all $n \in \mathbb{N}$; then we have the following interlacing relations:

$$a_{n,1} < a_{n+1,1} < a_{n,2} < a_{n+1,2} < a_{n,3} < \cdots$$

and

$$a'_{n,1} < a'_{n+1,1} < a'_{n,2} < a'_{n+1,2} < a'_{n,3} < \cdots.$$

For a proof of these relations we refer to [89]. From this we can easily deduce that for each $s \in \mathbb{N}$, the sequences $\{a_{n,s}\}_{n=1}^\infty$ and $\{a'_{n,s}\}_{n=1}^\infty$ are strictly monotonically increasing.

Observe that since the functions ψ_n are smooth and $\psi_n(0) = 0$ for all $n \in \mathbb{N}$, the number $\bar{z}$ we are looking for is the first positive zero of ψ'_n for some $n \in \mathbb{N}$. We claim that it is the first positive zero of the function

$$\psi'_1(z) = \frac{\cos z}{z} + \sin z - \frac{\sin z}{z^2},$$

i.e. $\bar{z} \sim 2.74 \pm 0.01$. To prove this, note that

$$\psi'_n(z) = j_n(z) + zj'_n(z). \tag{2.5.7}$$

Since

$$j_n(z) = z^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(2m+2n+1)!!} \left(\frac{z^2}{2}\right)^m ,$$

then $j_n(0) = 0$ and $j_n(z) > 0$ for all $n \in \mathbb{N}$ and for all z between zero and $a_{n,1}$. Then by (2.5.7), we have that $\psi_n'(z) > 0$ for all $z \in]0, a_{n,1}'[$. Due to the monotonicity of the sequence $a_{n,1}'$, $n \in \mathbb{N}$, it is then sufficient to prove that $\bar{z} \in]0, a_{2,1}'[$, because in this way the first positive zero of all other functions $\psi_n', n \geq 2$, will be necessarily larger. Since $a_{2,1}' \sim 3.34 \pm 0.01$, the claim is proved, and the first eigenvalue is $k_{11}' = (\bar{z}/R)^2$, where $\bar{z}$ is the first positive zero of the function ψ_1'. The eigenvectors and their curls are computed by using formulas (2.5.2) and (2.5.3) above and it is easily seen that the multiplicity is three.

□

Chapter 3

Stability

In this chapter we study the spectral stability for the electric Maxwell problem. Note that we will actually study the more general (penalized) problem (3.0.1), already introduced in its weak formulation in the previous chapter. In Definition 3.1.12 we introduce the “Atlas Piola transform" and in Theorem 3.1.16 we prove some properties. In Section 3.2 we prove the main result, concerning the spectral stability (cf. Theorem 3.2.30). Then, in Section 3.3 we show that under the hypotheses on the domains involved and on their rate of convergence (cf. (3.1.15)), a uniform Gaffney inequality is valid, which in turn justifies the previous result on spectral stability. Finally, in Section 3.4 we study the critical threshold case $\alpha = 3/2$, which is not covered by the previous theorems. Here α stands for the rate of oscillation of the perturbed boundary (see also (3.3.3)). We prove a stability result for the electric case (see Theorem 3.4.57) and, for the sake of completeness, we also provide an analogous result for the magnetic case (see Theorem 3.4.109).

Consider the following penalized boundary value problem for the curl curl operator subject to electric boundary conditions:

$$\begin{cases} \mathrm{curl}\,\mathrm{curl} u - \tau \nabla \mathrm{div} u = \lambda u, & \text{in } \Omega, \\ \mathrm{div} u = 0, & \text{on } \partial\Omega, \\ \nu \times u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.0.1}$$

where $\tau > 0$ is any fixed positive real number.

The weak formulation of problem (3.0.1) is the following (cf. (2.1.3)):

$$\int_\Omega \mathrm{curl} u \cdot \mathrm{curl} \varphi \, dx + \tau \int_\Omega \mathrm{div} u \, \mathrm{div} \varphi \, dx = \lambda \int_\Omega u \cdot \varphi \, dx, \quad \text{for all } \varphi \in X_{\mathrm{N}}(\Omega), \tag{3.0.2}$$

in the unknowns $u \in X_{\mathrm{N}}(\Omega)$ and $\lambda \in \mathbb{R}$.

As we have done in the previous chapter, we recast the eigenvalue problems under consideration in the form of eigenvalue problems for compact self-adjoint operators and this can be done by passing to the analysis of the corresponding resolvent operators. Hence we recall the following operator T from $X_{\mathrm{N}}(\Omega)$ to its dual $(X_{\mathrm{N}}(\Omega))'$

$$< Tu, \varphi >= \int_{\Omega} \mathrm{curl} u \cdot \mathrm{curl} \varphi \, dx + \tau \int_{\Omega} \mathrm{div} u \, \mathrm{div} \varphi \, dx \quad \text{for all } u, \varphi \in X_{\mathrm{N}}(\Omega),$$

and the map J from $L^2(\Omega)^3$ to $(X_{\mathrm{N}}(\Omega))'$

$$< Ju, \varphi >= \int_{\Omega} u \cdot \varphi \, dx \quad \text{for all } u \in L^2(\Omega)^3, \varphi \in X_{\mathrm{N}}(\Omega).$$

Observe that the operator $T + J$ is a homeomorphism from $X_{\mathrm{N}}(\Omega)$ to its dual by the Riesz Theorem. Then we have the following (cf. (2.1.11))

Lemma 3.0.3. *If Ω is a bounded domain in $\mathbb{R}^3$ such that the embedding ι of $X_{\mathrm{N}}(\Omega)$ into $L^2(\Omega)^3$ is compact , then the operator S_Ω from $L^2(\Omega)^3$ to itself defined by*

$$S_\Omega u := \iota \circ (T + J)^{-1} \circ J$$

is a non-negative compact self-adjoint operator in $L^2(\Omega)^3$ whose eigenvalues μ are related to the eigenvalues λ of problem (3.0.2) by the equality $\mu = (\lambda + 1)^{-1}$.

By the previous lemma and standard spectral theory it follows that the eigenvalues of problem (3.0.2) can be represented by the sequence $\lambda_n(\Omega)$, $n \in \mathbb{N}$ defined by $\lambda_n(\Omega) = \mu_n^{-1}(\Omega) - 1$. Moreover, the classical Min-Max Principle yields the following variational representation

$$\lambda_n(\Omega) = \min_{\substack{V \subset X_{\mathrm{N}}(\Omega) \\ \dim V = n}} \max_{u \in V \setminus \{0\}} \frac{\int_\Omega \left(|\mathrm{curl} u|^2 + \tau |\mathrm{div} u|^2 \right) dx}{\int_\Omega |u|^2 \, dx}. \tag{3.0.4}$$

3.1 A Piola-type approximation of the identity

As we have seen in Chapter 2, given two domains Ω and $\tilde{\Omega}$ in $\mathbb{R}^3$ and a diffeomorphism $\Phi : \tilde{\Omega} \to \Omega$ of class $C^{1,1}$, the standard way to pull-back vector fields from $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\tilde{\Omega})$ consists in using the (covariant) Piola transform defined by

$$u(x) = \big((v \circ \Phi) D\Phi \big) (x), \quad \text{for all } x \in \tilde{\Omega}, \tag{3.1.1}$$

for all $v \in X_{\mathrm{N}}(\Omega)$. By Theorem 2.2.4 we have that $v \in H_0(\mathrm{curl}, \Omega)$ if and only if $u \in H_0(\mathrm{curl}, \tilde{\Omega})$, in which case we have

$$(\mathrm{curl}\, v) \circ \Phi = \frac{\mathrm{curl}\, u \, (D\Phi)^T}{\det (D\Phi)}. \tag{3.1.2}$$

Moreover, note that by Theorem 2.2.11, for functions u, v in H^1 we also have

$$(\operatorname{div} v) \circ \Phi = \frac{\operatorname{div}\left[u(D\Phi)^{-1}(D\Phi)^{-T}\det(D\Phi)\right]}{\det(D\Phi)}, \tag{3.1.3}$$

and in this case $v \in X_{\mathrm{N}}(\Omega) \cap H^1(\Omega)^3$ if and only if $u \in X_{\mathrm{N}}(\tilde{\Omega}) \cap H^1(\tilde{\Omega})^3$. Unfortunately, given two domains Ω and $\tilde{\Omega}$, in general it is not possible to define explicitly a diffeomorphism between Ω and $\tilde{\Omega}$ (even if it is known a priori that the two domains are diffeomorphic). Nevertheless, it is important for our purposes to define an operator which allows to pass from $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\tilde{\Omega})$ as the Piola transform does. This can be done by assuming that Ω and $\tilde{\Omega}$ belongs to the same atlas class and using a partition of unity in order to paste together Piola transforms defined locally, as described in the following. Note that the specific choice of local Piola transforms reflects our need for a transformation close to the identity.

Let $\mathcal{A}$ be a fixed atlas in $\mathbb{R}^3$ and let $\Omega, \tilde{\Omega}$ be two domains of class $C^{1,1}(\mathcal{A})$. Let $g_j, \tilde{g}_j$ be the profile functions of Ω and $\tilde{\Omega}$ as in Definition 1.0.1.Assume that $k > 0$ is such that

$$k > \max_{j=1,\dots,s'} \|\tilde{g}_j - g_j\|_\infty, \text{ and } \tilde{g}_j - k > a_{3,j} + \rho, \ \forall j = 1, \dots, s'. \tag{3.1.4}$$

For any $j = 1, \dots, s'$ we set

$$\hat{g}_j := \tilde{g}_j - k \tag{3.1.5}$$

and we define the map $h_j : r_j(\overline{\tilde{\Omega} \cap V_j}) \to \mathbb{R}$

$$h_j(\bar{x}, x_3) := \begin{cases} 0, & \text{if } a_{3j} \le x_3 \le \hat{g}_j(\bar{x}), \\ (\tilde{g}_j(\bar{x}) - g_j(\bar{x}))\left(\frac{x_3 - \hat{g}_j(\bar{x})}{\tilde{g}_j(\bar{x}) - \hat{g}_j(\bar{x})}\right)^3, & \text{if } \hat{g}_j(\bar{x}) < x_3 \le \tilde{g}_j(\bar{x}), \end{cases} \tag{3.1.6}$$

and the map

$$\Phi_j : r_j(\overline{\tilde{\Omega} \cap V_j}) \to r_j(\overline{\Omega \cap V_j}), \qquad \Phi_j(\bar{x}, x_3) := (\bar{x}, x_3 - h_j(\bar{x}, x_3)). \tag{3.1.7}$$

Note that Φ_j coincides with the identity map on the set

$$K_j := \left\{(\bar{x}, x_3) \in W_j \times]a_{3j}, b_{3j}[\, : a_{3j} < x_3 < \hat{g}_j(\bar{x})\right\}. \tag{3.1.8}$$

Finally, if $s' + 1 \le j \le s$ we define $\Phi_j : r_j(\overline{V_j}) \to r_j(\overline{V_j})$ to be the identity map.

Observe that since $h_j \in C^{1,1}(r_j(\overline{\tilde{\Omega} \cap V_j}))$, then Φ_j is of class $C^{1,1}$, and so is the following map

$$\Psi_j : \overline{\tilde{\Omega} \cap V_j} \to \overline{\Omega \cap V_j}, \qquad \Psi_j := r_j^{-1} \circ \Phi_j \circ r_j. \tag{3.1.9}$$

An easy computation shows that if

$$k > \frac{3}{\alpha} \max_{j=1,\dots,s'} \|\tilde{g}_j - g_j\|_\infty \tag{3.1.10}$$

for some constant $\alpha \in]0,1[$ then

$$0 < 1-\alpha \le \det(D\Psi_j(x)) \le 1+\alpha \quad \text{for any } x \in \tilde{\Omega} \cap V_j. \tag{3.1.11}$$

Let $\{\psi_j\}_{j=1}^s$ be a C^∞-partition of unity associated with the open cover $\{V_j\}_{j=1}^s$ of the compact set $\overline{\cup_{j=1}^s (V_j)_\rho}$ that is $0 \le \psi_j \le 1$, $\operatorname{supp}(\psi_j) \subset V_j$ for all $j = 1,\dots,s$, and $\sum_{j=1}^s \psi_j \equiv 1$ in $\overline{\cup_{j=1}^s (V_j)_\rho}$, in particular also in $\overline{\Omega \cup \tilde{\Omega}}$. Note that this is a partition of unity is independent of $\Omega, \tilde{\Omega}$ in the atlas class under consideration.

Since for any $\varphi \in X_{\mathrm{N}}(\Omega)$ we have $\varphi = \sum_{j=1}^s \varphi_j$ where $\varphi_j := \psi_j \varphi$, then it is natural to give the following (note that here we consider open sets of class $C^{1,1}$ hence the spaces X_{N} are embedded into H^1).

Definition 3.1.12. Let $\mathcal{A}$ be an atlas in $\mathbb{R}^3$ and $\Omega, \tilde{\Omega}$ be two domains of class $C^{1,1}(\mathcal{A})$. Assume that $k > 0$ satisfies (3.1.4), and $\{\psi_j\}_{j=1}^s$ is a partition of unity as above. The *Atlas Piola transform* from Ω to $\tilde{\Omega}$, with parameters $\mathcal{A}$, k, and $\{\psi_j\}_{j=1}^s$, is the map from $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\tilde{\Omega})$ defined by

$$\mathcal{P}\varphi := \sum_{j=1}^{s'} \tilde{\varphi}_j + \sum_{j=s'+1}^{s} \varphi_j \tag{3.1.13}$$

for all $\varphi \in X_{\mathrm{N}}(\Omega)$, where

$$\tilde{\varphi}_j(x) := \begin{cases} (\varphi_j \circ \Psi_j(x)) D\Psi_j(x), & \text{if } x \in \tilde{\Omega} \cap V_j, \\ 0, & \text{if } x \in \tilde{\Omega} \setminus V_j, \end{cases} \tag{3.1.14}$$

for any $j = 1,\dots,s'$.

This Atlas Piola transform will be used in the following for a family $\Omega_\epsilon, \epsilon > 0$ of domains of class $C^{1,1}(\mathcal{A})$, converging in some sense to a domain Ω of class $C^{1,1}(\mathcal{A})$. In this case, Ω_ϵ will play the role of the domain $\tilde{\Omega}$ and the corresponding transformation will allow us to pass from $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\Omega_\epsilon)$.

Given a family of domains Ω_ϵ, $\epsilon > 0$, and a fixed domain Ω, all of class $C^{1,1}(\mathcal{A})$, we shall denote by $g_{\epsilon,j}$ and g_j the corresponding profile functions (defined on W_j) of Ω_ϵ and Ω respectively, as in Definition 1.0.1. Following [11, 54], we use a notion of convergence for the open sets Ω_ϵ to Ω, which is expressed in terms of convergence of the the profile functions $g_{\epsilon,j}$ to g_j. Namely, we assume that for any $\epsilon > 0$ there exists $\kappa_\epsilon > 0$ such that for any $j \in \{1, \dots, s'\}$

$$
\begin{aligned}
&(i)\quad \kappa_\epsilon > \max_{j=1,\dots,s'} \left\| g_{\epsilon,j} - g_j \right\|_{L^\infty(W_j)};\\
&(ii)\quad \lim_{\epsilon\to 0} \kappa_\epsilon = 0;\\
&(iii)\quad \lim_{\epsilon\to 0} \frac{\max_{j=1,\dots,s'} \left\| D^\beta (g_{\epsilon,j} - g_j) \right\|_{L^\infty(W_j)}}{\kappa_\epsilon^{3/2-|\beta|}} = 0 \quad \text{for all} \quad \beta \in \mathbb{N}^3 \text{ with } |\beta| \le 2.
\end{aligned}
\tag{3.1.15}
$$

Note that if every function $g_{\epsilon,j}$ converges to g_j uniformly together with the first order derivatives and

$$
\sup_{\epsilon>0} \sup_{\bar{x}\in\mathbb{R}^2} |D^2 g_{\epsilon,j}(\bar{x})| < \infty
$$

is satisfied (in particular, if the second order derivatives of $g_{\epsilon,j}$ converge uniformly to those of g_j), then conditions (3.1.15) are fulfilled, see [11].

We now fix a partition of unity $\{\psi_j\}_{j=1}^s$ associated with the covering of cuboids of the atlas $\mathcal{A}$ as above, and independent of Ω_ϵ and Ω. We also choose $k = 6\kappa_\epsilon$ and we denote by $\mathcal{P}_\epsilon$ the Atlas Piola transform from Ω to Ω_ϵ (with parameters $\mathcal{A}$, k, $\{\psi_j\}_{j=1}^s$). Note that conditions (3.1.4), (3.1.10) (3.1.11) are satisfied with $\alpha = 1/2$ if ϵ is sufficiently small.

In the following, we shall denote by $\hat{g}_{\epsilon,j}$, $h_{\epsilon,j}$, $\Phi_{\epsilon,j}$, $K_{\epsilon,j}$, $\Psi_{\epsilon,j}$, $\tilde{\varphi}_{\epsilon,j}$ all quantities defined in (3.1.5), (3.1.6), (3.1.7), (3.1.8), (3.1.9), (3.1.14) respectively, with $\tilde{\Omega} = \Omega_\epsilon$ and $k = 6\kappa_\epsilon$.

Then we can prove the following theorem. We note that in the proof, some technical issues related to pasting together functions defined in different charts are treated in the spirit of the arguments used in [54] for the Sobolev spaces $H^2(\Omega)$.

Theorem 3.1.16. *Let $\Omega_\epsilon, \epsilon > 0$, and Ω be bounded domains of class $C^{1,1}(\mathcal{A})$. Assume that Ω_ϵ converges to Ω as $\epsilon \to 0$ in the sense of (3.1.15). Let $\mathcal{P}_\epsilon$ be the Atlas Piola transform from Ω to Ω_ϵ defined for ϵ sufficiently small as above. Then the following statements hold:*

(i) *for any $\varepsilon > 0$ the function $\mathcal{P}_\epsilon$ maps $X_{\mathrm{N}}(\Omega)$ to $X_{\mathrm{N}}(\Omega_\epsilon)$ with continuity;*

(ii) *for any compact set $\mathcal{K}$ contained in Ω there exists $\epsilon_{\mathcal{K}} > 0$ such that*

$$
(\mathcal{P}_\epsilon \varphi)(x) = \varphi(x), \quad \forall x \in \mathcal{K} \tag{3.1.17}
$$

for all $\epsilon \in]0, \epsilon_{\mathcal{K}}[$ and $\varphi \in X_{\mathrm{N}}(\Omega)$;

(iii) *the limit*

$$
\|\mathcal{P}_\epsilon \varphi\|_{X_{\mathrm{N}}(\Omega_\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} \|\varphi\|_{X_{\mathrm{N}}(\Omega)}, \tag{3.1.18}
$$

holds for all $\varphi \in X_{\mathrm{N}}(\Omega)$;

(iv) *the limit*

$$\|\mathcal{P}_\epsilon\varphi - \varphi\|_{X(\Omega_\epsilon\cap\Omega)} \xrightarrow[\epsilon\to 0]{} 0, \tag{3.1.19}$$

holds for all $\varphi \in X_{\mathrm{N}}(\Omega)$.

Proof. Let $\varphi \in X_{\mathrm{N}}(\Omega)$ be fixed. Note that Ω is of class $C^{1,1}$ hence the Gaffney inequality holds and $\varphi \in H^1(\Omega)^3$. Moreover, $\varphi_j \in X_{\mathrm{N}}(\Omega) \cap H^1(\Omega)^3$ for all $j = 1, \dots, s'$ hence $\tilde{\varphi}_{\epsilon,j}$ belongs to $X_{\mathrm{N}}(\Omega_\epsilon) \cap H^1(\Omega_\epsilon)^3$ for all $j = 1, \dots, s'$. It follows that $\mathcal{P}_\epsilon\varphi \in X_{\mathrm{N}}(\Omega_\epsilon)$. The continuity of the operator $\mathcal{P}_\epsilon$ will be addressed at the end of the proof.

Let us proceed and prove (ii). For any fixed compact set $\mathcal{K}$ contained in Ω, since $\hat{g}_{\epsilon,j}$ converges uniformly to g_j, we have

$$\mathcal{K} \cap V_j \subset r_j^{-1}(K_{\epsilon,j})$$

for all $j = 1, \dots, s'$ and ϵ sufficiently small; this, combined with the fact that $\Phi_{\epsilon,j}$ coincides with the identity on $K_{\epsilon,j}$, it follows that $\tilde{\varphi}_{\epsilon,j} = \varphi_j$ on $\mathcal{K}$ for all ϵ sufficiently small and (3.1.17) follows.

We now prove statement (iii). We have to prove the following limiting relations:

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} |\mathcal{P}_\epsilon\varphi|^2 = \int_\Omega |\varphi|^2, \tag{3.1.20}$$

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} |\operatorname{curl} \mathcal{P}_\epsilon\varphi|^2 = \int_\Omega |\operatorname{curl} \varphi|^2, \tag{3.1.21}$$

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} |\operatorname{div} \mathcal{P}_\epsilon\varphi|^2 = \int_\Omega |\operatorname{div} \varphi|^2. \tag{3.1.22}$$

We begin by proving (3.1.20). To see this, it just suffices to show that

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \tilde{\varphi}_{\epsilon,j} \cdot \tilde{\varphi}_{\epsilon,h} = \int_\Omega \varphi_j \cdot \varphi_h, \tag{3.1.23}$$

and

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \tilde{\varphi}_{\epsilon,j} \cdot \varphi_i = \int_\Omega \varphi_j \cdot \varphi_i \tag{3.1.24}$$

for any $j, h \in \{1, \dots, s'\}$ and $i \in \{s'+1, \dots, s\}$. We will only show (3.1.23), since the computations to prove (3.1.24) are similar. We will first see that

$$\lim_{\epsilon\to 0} \int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\tilde{\varphi}_{\epsilon,j}\right|^2 = 0. \tag{3.1.25}$$

Notice that for any $j \in \{1, \dots, s'\}$ we have $\left|(\Omega \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j})\right| \to 0$ as ϵ goes to 0. Moreover, if $w \in \mathbb{R}^3$ is a vector, then $\left|wD\Psi_{\epsilon,j}\right| = \left|wD\Phi_{\epsilon,j}\right| \le C|w|$ for some

$C > 0$ independent of ϵ, since

$$D\Phi_{\epsilon,j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\partial h_{\epsilon,j}}{\partial x_1} & -\frac{\partial h_{\epsilon,j}}{\partial x_2} & 1-\frac{\partial h_{\epsilon,j}}{\partial x_3} \end{pmatrix}$$

and the first derivatives of $h_{\epsilon,j}$ are all bounded due to the hypothesis on the functions $g_{\epsilon,j}$ (see also (3.1.40)). Note that here and in what follows, by c we denote a constant independent of ϵ which may vary from line to line. Then by using also (3.1.11), we have

$$\begin{aligned} \int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\tilde{\varphi}_{\epsilon,j}\right|^2 dy &= \int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|(\varphi_j\circ\Psi_{\epsilon,j})D\Psi_{\epsilon,j}\right|^2 dy \\ &\le c\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\varphi_j\circ\Psi_{\epsilon,j}\right|^2 dy \\ &= c\int_{(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \frac{\left|\varphi_j\right|^2}{\left|\det(D\Psi_{\epsilon,j})\circ\Psi_{\epsilon,j}^{(-1)}\right|} dx \\ &\le c\int_{(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\varphi_j\right|^2 dx \underset{\epsilon\to 0}{\longrightarrow} 0. \end{aligned}$$

By (3.1.25) we deduce that

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left|\tilde{\varphi}_{\epsilon,j}\right|^2 = \int_\Omega \left|\varphi_j\right|^2. \tag{3.1.26}$$

Indeed, since $\Psi_{\epsilon,j}$ is the identity on $r_j^{-1}(K_{\epsilon,j})\subset\Omega\cap\Omega_\epsilon$, using (3.1.25) yields

$$\int_{\Omega_\epsilon}\left|\tilde{\varphi}_{\epsilon,j}\right|^2 = \int_{r_j^{-1}(K_{\epsilon,j})}\left|\tilde{\varphi}_{\epsilon,j}\right|^2 + \int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|\tilde{\varphi}_{\epsilon,j}\right|^2 \underset{\epsilon\to 0}{\longrightarrow} \int_{\Omega\cap V_j}\left|\varphi_j\right|^2 = \int_\Omega\left|\varphi_j\right|^2.$$

Observe now that

$$\begin{aligned} \int_{\Omega_\epsilon}\tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h} &= \int_{\Omega_\epsilon\cap V_j\cap V_h}\tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h} \\ &= \int_{r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h})}\tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h} + \int_{(\Omega_\epsilon\cap V_j\cap V_h)\setminus(r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h}))}\tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h}. \end{aligned} \tag{3.1.27}$$

It is obvious that

$$\lim_{\epsilon\to 0}\int_{r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h})}\tilde{\varphi}_{\epsilon,j}\cdot\tilde{\varphi}_{\epsilon,h} = \lim_{\epsilon\to 0}\int_{r_j^{-1}(K_{\epsilon,j})\cap r_h^{-1}(K_{\epsilon,h})}\varphi_j\cdot\varphi_h = \int_\Omega\varphi_j\cdot\varphi_h. \tag{3.1.28}$$

Here and in the following we will make use of the identity

$$(\Omega_\epsilon \cap V_j \cap V_h) \setminus (r_j^{-1}(K_{\epsilon,j}) \cap r_h^{-1}(K_{\epsilon,h})) = \\ [(\Omega_\epsilon \cap V_j \cap V_h) \setminus r_j^{-1}(K_{\epsilon,j})] \cup [(r_j^{-1}(K_{\epsilon,j}) \setminus r_h^{-1}(K_{\epsilon,h})) \cap V_h]. \tag{3.1.29}$$

Observe that by (3.1.25), (3.1.26) we get

$$\left| \int_{(\Omega_\epsilon \cap V_j \cap V_h)\setminus r_j^{-1}(K_{\epsilon,j})} \tilde{\varphi}_{\epsilon,j} \cdot \tilde{\varphi}_{\epsilon,h} \right| \\ \leq \left(\int_{(\Omega_\epsilon \cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} |\tilde{\varphi}_{\epsilon,j}|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_\epsilon \cap V_h} |\tilde{\varphi}_{\epsilon,h}|^2 \right)^{\frac{1}{2}} \xrightarrow[\epsilon \to 0]{} 0, \tag{3.1.30}$$

and

$$\left| \int_{(r_j^{-1}(K_{\epsilon,j})\setminus r_h^{-1}(K_{\epsilon,h}))\cap V_h} \tilde{\varphi}_{\epsilon,j} \cdot \tilde{\varphi}_{\epsilon,h} \right| \\ \leq \left(\int_{r_j^{-1}(K_{\epsilon,j})} |\tilde{\varphi}_{\epsilon,j}|^2 \right)^{\frac{1}{2}} \left(\int_{(\Omega_\epsilon \cap V_h)\setminus r_h^{-1}(K_{\epsilon,h})} |\tilde{\varphi}_{\epsilon,h}|^2 \right)^{\frac{1}{2}} \xrightarrow[\epsilon \to 0]{} 0. \tag{3.1.31}$$

Hence, by formula (3.1.29), we see that the second term of the sum in (3.1.27) vanishes as ϵ goes to zero, hence we deduce the validity of (3.1.23) from (3.1.27) and (3.1.28).

We now prove (3.1.21). Again, we need to check that

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \operatorname{curl} \tilde{\varphi}_{\epsilon,j} \cdot \operatorname{curl} \tilde{\varphi}_{\epsilon,h} = \int_\Omega \operatorname{curl} \varphi_j \cdot \operatorname{curl} \varphi_h \tag{3.1.32}$$

for any $j, h \in \{1, \dots, s'\}$. Note that

$$D\Phi_{\epsilon,j}^{(-1)} = \frac{1}{\det(D\Phi_{\epsilon,j})} \begin{pmatrix} 1 - \frac{\partial h_{\epsilon,j}}{\partial x_3} & 0 & 0 \\ 0 & 1 - \frac{\partial h_{\epsilon,j}}{\partial x_3} & 0 \\ \frac{\partial h_{\epsilon,j}}{\partial x_1} & \frac{\partial h_{\epsilon,j}}{\partial x_2} & 1 \end{pmatrix} \circ \Phi_{\epsilon,j}^{(-1)},$$

and recall that $\Psi_{\epsilon,j} = r_j^{-1} \circ \Phi_{\epsilon,j} \circ r_j$. Moreover, by (3.1.2) we have

$$\operatorname{curl} \tilde{\varphi}_{\epsilon,j} = \left(\operatorname{curl} \varphi_j \circ \Psi_{\epsilon,j}\right) (D\Psi_{\epsilon,j})^{-T} \det D(\Psi_{\epsilon,j}) \quad \text{on } \Omega_\epsilon \cap V_j \tag{3.1.33}$$

so that

$$\left|\operatorname{curl} \tilde{\varphi}_{\epsilon,j}\right| \leq c \left|\operatorname{curl} \varphi_j \circ \Psi_{\epsilon,j}\right|.$$

Then, with computations analogous to those performed above, we get

$$\lim_{\epsilon\to 0}\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})}\left|\mathrm{curl}\,\tilde{\varphi}_{\epsilon,j}\right|^2 = 0. \tag{3.1.34}$$

It is also obvious that

$$\lim_{\epsilon\to 0}\int_{r_j^{-1}(K_{\epsilon,j})}\left|\mathrm{curl}\,\tilde{\varphi}_{\epsilon,j}\right|^2 = \int_\Omega\left|\mathrm{curl}\,\varphi_j\right|^2 \tag{3.1.35}$$

and thus

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\left|\mathrm{curl}\,\tilde{\varphi}_{\epsilon,j}\right|^2 = \int_\Omega\left|\mathrm{curl}\,\varphi_j\right|^2. \tag{3.1.36}$$

By using the same argument above, formula (3.1.29) together with the new identities (3.1.34), (3.1.35) and (3.1.36), we obtain (3.1.32) .

Finally, we prove (3.1.22). To do so, we need to prove that

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\mathrm{div}\,\tilde{\varphi}_{\epsilon,j}\,\mathrm{div}\,\tilde{\varphi}_{\epsilon,h} = \int_\Omega\mathrm{div}\,\varphi_j\,\mathrm{div}\,\varphi_h \tag{3.1.37}$$

for any $j,h\in\{1,\dots,s'\}$. Here and in the rest of the proof, the vectors under consideration will be represented as follows: $\varphi_j=(\varphi_j^1,\varphi_j^2,\varphi_j^3)$ and $\Psi_{\epsilon,j}=(\Psi^1_{\epsilon,j},\Psi^2_{\epsilon,j},\Psi^3_{\epsilon,j})$.

Since $\varphi\in X_{\mathrm{N}}(\Omega)$ and the Gaffney inequality holds on Ω, it follows that $\varphi\in H^1(\Omega)^3$. Thus, recalling that $\tilde{\varphi}_{\epsilon,j}(x)=(\varphi_j\circ\Psi_{\epsilon,j}(x))D\Psi_{\epsilon,j}(x)$ for all $x\in\Omega_\epsilon\cap V_j$, it is possible to apply the chain rule and obtain that

$$\mathrm{div}(\tilde{\varphi}_{\epsilon,j}) = \sum_{m,n,i=1}^{3}\underbrace{\left(\frac{\partial\varphi_j^m}{\partial x_n}(\Psi_{\epsilon,j})\frac{\partial\Psi^n_{\epsilon,j}}{\partial x_i}\frac{\partial\Psi^m_{\epsilon,j}}{\partial x_i}\right)}_{\text{type A}} + \sum_{m,i=1}^{3}\underbrace{\varphi_j^m(\Psi_{\epsilon,j})\frac{\partial^2\Psi^m_{\epsilon,j}}{\partial x_i^2}}_{\text{type B}} \quad \text{in } \Omega_\epsilon\cap V_j. \tag{3.1.38}$$

where the terms in the first sum are called of type A and the others are called terms of type B.

Recall that $h_{\epsilon,j}$ are the functions in (3.1.6) used to define the diffeomorphisms $\Phi_{\epsilon,j}$. We note that by the Leibniz rule we have

$$D^\alpha h_{\epsilon,j}(x) = \sum_{0\le\gamma\le\alpha}\binom{\alpha}{\gamma}D^\gamma\left(g_{\epsilon,j}(\bar{x})-g_j(\bar{x})\right)D^{\alpha-\gamma}\left(\frac{x_3-\hat{g}_{\epsilon,j}(\bar{x})}{g_{\epsilon,j}(\bar{x})-\hat{g}_{\epsilon,j}(\bar{x})}\right)^3$$

for all $x=(\bar{x},x_3)\in W_j\times]a_{3j},b_{3j}[$ with $\hat{g}_{\varepsilon,j}(x)<x_3<g_{\varepsilon,j}(x)$. Moreover by standard calculus

$$\left|D^{\alpha-\gamma}\left(\frac{x_3-\hat{g}_{\epsilon,j}(\bar{x})}{g_{\epsilon,j}(\bar{x})-\hat{g}_{\epsilon,j}(\bar{x})}\right)^3\right| \le \frac{c}{\left|g_{\epsilon,j}(\bar{x})-\hat{g}_{\epsilon,j}(\bar{x})\right|^{|\alpha|-|\gamma|}} \le \frac{c}{\kappa_\epsilon^{|\alpha|-|\gamma|}}. \tag{3.1.39}$$

Therefore

$$\left\|D^{\alpha}h_{\epsilon,j}\right\|_{\infty} \leq c \sum_{0\leq\gamma\leq\alpha} \frac{\left\|D^{\gamma}(g_{\epsilon,j}-g_j)\right\|_{\infty}}{\kappa_{\epsilon}^{|\alpha|-|\gamma|}} \tag{3.1.40}$$

for all $\epsilon > 0$ sufficiently small. It follows by the definitions of $\Psi_{\epsilon,j}, \Phi_{\epsilon,j}$, by (3.1.40) and part (iii) of condition (3.1.15), that for all $m, i = 1,2,3$

$$\left\|\frac{\partial^2 \Psi^m_{\epsilon,j}}{\partial x_i^2}\right\|_{L^{\infty}(\Omega_\epsilon \cap V_j)} = o(\kappa_\epsilon^{-1/2}), \quad \text{as } \epsilon \to 0. \tag{3.1.41}$$

We claim that

$$\lim_{\epsilon\to 0} \int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\operatorname{div}\tilde{\varphi}_{\epsilon,j}\right|^2 = 0. \tag{3.1.42}$$

To prove that, we analyse first the terms of type A in (3.1.38). By changing variables in integrals we get:

$$\begin{aligned}
&\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\frac{\partial\varphi_j^m}{\partial x_n}\circ\Psi_{\epsilon,j}\right|^2 \left|\frac{\partial\Psi^n_{\epsilon,j}}{\partial x_i}\right|^2 \left|\frac{\partial\Psi^m_{\epsilon,j}}{\partial x_i}\right|^2 dy \\
&\leq c\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\frac{\partial\varphi_j^m}{\partial x_n}\circ\Psi_{\epsilon,j}\right|^2 dy \\
&= c\int_{(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\frac{\partial\varphi_j^m}{\partial x_n}\right|^2 \frac{1}{\left|\det(D\Psi_{\epsilon,j})\circ\Psi^{(-1)}_{\epsilon,j}\right|} dx \\
&\leq c\int_{(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\frac{\partial\varphi_j^m}{\partial x_n}\right|^2 dx \xrightarrow[\epsilon\to 0]{} 0.
\end{aligned} \tag{3.1.43}$$

We now consider the terms of type B. By setting $\eta_j(z) := \varphi_j(r_j^{-1}(z))$ and recalling

(3.1.41) we have that

$$
\begin{aligned}
&\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left| \varphi_j^m(\Psi_{\epsilon,j}) \frac{\partial^2 \Psi^m_{\epsilon,j}}{\partial x_i^2} \right|^2 dy \\
&\quad \le \left\| \frac{\partial^2 \Psi^m_{\epsilon,j}}{\partial x_i^2} \right\|^2_{L^\infty(\Omega_\epsilon\cap V_j)} \int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left| \varphi_j(\Psi_{\epsilon,j}) \right|^2 dy \\
&\quad = o(\kappa_\epsilon^{-1}) \int_{(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left| \varphi_j \right|^2 \frac{1}{\left| \det(D\Psi_{\epsilon,j}) \circ \Psi^{(-1)}_{\epsilon,j} \right|} dx \\
&\quad \le o(\kappa_\epsilon^{-1}) \int_{(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left| \varphi_j(x) \right|^2 dx \\
&\quad = o(\kappa_\epsilon^{-1}) \int_{r_j(\Omega\cap V_j)\setminus K_{\epsilon,j}} \left| \eta_j(z) \right|^2 dz \\
&\quad = o(\kappa_\epsilon^{-1}) \int_{W_j} \left(\int_{\hat{g}_{\epsilon,j}(\bar{z})}^{g_j(\bar{z})} \left| \eta_j(\bar{z}, z_3) \right|^2 dz_3 \right) d\bar{z} \\
&\quad \le o(\kappa_\epsilon^{-1}) \int_{W_j} \left| g_j(\bar{z}) - \hat{g}_{\epsilon,j}(\bar{z}) \right| \left\| \eta_j(\bar{z}, \cdot) \right\|^2_{L^\infty(a_{3j}, g_j(\bar{z}))^3} d\bar{z} \\
&\quad \le o(\kappa_\epsilon^{-1}) \left\| g_j - \hat{g}_{\epsilon,j} \right\|_{L^\infty(W_j)} \int_{W_j} \left\| \eta_j(\bar{z}, \cdot) \right\|^2_{H^1(a_{3j}, g_j(\bar{z}))^3} d\bar{z} \\
&\quad \le o(\kappa_\epsilon^{-1})\, \kappa_\epsilon \left\| \eta_j \right\|^2_{H^1(r_j(\Omega\cap V_j))^3} \xrightarrow[\epsilon\to 0]{} 0.
\end{aligned}
\tag{3.1.44}
$$

Here we have used the following one dimensional embedding estimate for functions in Sobolev space (see e.g., Burenkov [25]):

$$
\|f\|_{L^\infty(a,b)} \le c \|f\|_{H^1(a,b)}
$$

for all $f \in H^1(a,b)$, where the constant $c = c(d)$ is uniformly bounded for $|b-a| > d$. We conclude that (3.1.42) holds.

Using (3.1.42), the fact that $\Psi_{\epsilon,j}$ in $r_j^{-1}(K_{\epsilon,j})$ coincides with the identity and that $\left| (\Omega \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j}) \right| \to 0$ as ϵ goes to 0, we deduce that

$$
\lim_{\epsilon\to 0} \int_{r_j^{-1}(K_{\epsilon,j})} \left| \operatorname{div} \tilde{\varphi}_{\epsilon,j} \right|^2 = \int_\Omega \left| \operatorname{div} \varphi_j \right|^2, \tag{3.1.45}
$$

and

$$
\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \left| \operatorname{div} \tilde{\varphi}_{\epsilon,j} \right|^2 = \int_\Omega \left| \operatorname{div} \varphi_j \right|^2. \tag{3.1.46}
$$

With (3.1.42), (3.1.45) and (3.1.46) in mind, in order to prove (3.1.37), it suffices to reproduce the same argument used before starting from (3.1.27) combined with formula (3.1.29). We omit the details. Thus statement (iii) is proved.

The proof of statement (iv) follows by the same considerations above. First of all, for any $j = 1, \dots, s'$ the function $\tilde{\varphi}_{\epsilon,j}$ coincides with φ_j on $r_j^{-1}(K_{\epsilon,j})$. Thus $\mathcal{P}_\epsilon \varphi = \varphi$ on $(\cup_{j=1,\dots,s'} r_j^{-1}(K_{\epsilon,j})) \cup (\cup_{j=s'+1,\dots,s} V_j)$. It follows that

$$\|\mathcal{P}_\epsilon \varphi - \varphi\|_{X(\Omega_\epsilon \cap \Omega)} \leq \|\mathcal{P}_\epsilon \varphi - \varphi\|_{X(\cup_{j=1,\dots,s'} (\Omega_\epsilon \cap V_j) \setminus r_j^{-1}(K_{\epsilon,j}))}$$

This combined with by the limiting relations (3.1.25), (3.1.34) and (3.1.42) yields the validity of statement (iv).

We finish by proving the continuity of the operator $\mathcal{P}_\epsilon : X_{\mathrm{N}}(\Omega) \to X_{\mathrm{N}}(\Omega_\epsilon)$, where $\epsilon > 0$ is fixed. In order to do that we will show that if a sequence $\{\varphi_k\}_{k \in \mathbb{N}} \subset X_{\mathrm{N}}(\Omega)$ converges to φ in $X_{\mathrm{N}}(\Omega)$, then $\{\mathcal{P}_\epsilon \varphi_k\}_{k \in \mathbb{N}}$ converges to $\mathcal{P}_\epsilon \varphi$ in $X_{\mathrm{N}}(\Omega_\epsilon)$ as k goes to infinity. This in turn boils down to show that

$$\mathcal{P}_\epsilon \varphi_k \xrightarrow[k \to \infty]{} \mathcal{P}_\epsilon \varphi, \quad \operatorname{curl} \mathcal{P}_\epsilon \varphi_k \xrightarrow[k \to \infty]{} \operatorname{curl} \mathcal{P}_\epsilon \varphi, \quad \operatorname{div} \mathcal{P}_\epsilon \varphi_k \xrightarrow[k \to \infty]{} \operatorname{div} \mathcal{P}_\epsilon \varphi$$

in $L^2(\Omega_\epsilon)^3$. We will prove only the first limit since the other limiting relations can be proved in the same way.

Consider

$$\begin{aligned} \int_{\Omega_\epsilon} |\mathcal{P}_\epsilon \varphi_k - \mathcal{P}_\epsilon \varphi|^2 \, dx &= \int_{\Omega_\epsilon} \left| \sum_{j=1}^{s'} \left((\tilde{\varphi}_k)_{\epsilon,j} - \tilde{\varphi}_{\epsilon,j} \right) + \sum_{j=s'+1}^{s} \left((\varphi_k)_j - \varphi_j \right) \right|^2 dx \\ &\leq 2s \sum_{j=1}^{s'} \int_{\Omega_\epsilon} \left| (\tilde{\varphi}_k)_{\epsilon,j} - \tilde{\varphi}_{\epsilon,j} \right|^2 dx + 2s \sum_{j=s'+1}^{s} \int_{\Omega_\epsilon} \left| (\varphi_k)_j - \varphi_j \right|^2 dx. \end{aligned} \tag{3.1.47}$$

By changing variables in the integral we have that

$$\begin{aligned} \int_{\Omega_\epsilon} \left| (\tilde{\varphi}_k)_{\epsilon,j} - \tilde{\varphi} \right|^2 dx &= \int_{\Omega_\epsilon} \left| \left(\left((\varphi_k)_j - \varphi_j \right) \circ \Psi_{\epsilon,j} \right) D\Psi_{\epsilon,j} \right|^2 dx \\ &\leq C \int_{\Omega} \left| (\varphi_k)_j - \varphi_j \right|^2 dx, \end{aligned}$$

where $C > 0$ is independent of $k \in \mathbb{N}$. Moreover, since $0 \leq \psi_j \leq 1$,

$$\int_{\Omega_\epsilon} \left| (\varphi_k)_j - \varphi_j \right|^2 dx = \int_{\Omega_\epsilon} |\varphi_k - \varphi|^2 \left| \psi_j \right|^2 dx \leq \|\varphi_k - \varphi\|^2_{L^2(\Omega_\epsilon)^3} \xrightarrow[k \to \infty]{} 0$$

for any $j = 1, \dots, s$. Hence we proved that the right-hand side of (3.1.47) goes to 0 as k goes to infinity.

The cases for the curl and the divergence are similar: one just needs to make use of formulas (3.1.33), (3.1.38) and remember that thanks to the Gaffney inequality, which holds in Ω, the sequence φ_k converges to φ in $H^1(\Omega)^3$. We skip the details. $\square$

3.2 Spectral stability

Let Ω_ϵ, $\epsilon > 0$, and Ω be bounded domains of class $C^{1,1}(\mathcal{A})$. For simplicity, it is convenient to set $\Omega_0 = \Omega$. In this section, we prove that if Ω_ϵ converges to Ω as $\epsilon \to 0$ in the sense of (3.1.15), and a uniform Gaffney inequality holds on the domains Ω_ϵ then we have spectral stability for the curlcurl operator defined on the domains Ω_ϵ with respect to the reference domain Ω. By uniform Gaffney inequality, we mean that the spaces $X_{\mathrm{N}}(\Omega_\epsilon)$ are embedded into $H^1(\Omega_\epsilon)^3$ and there exists a positive constant C independent of ϵ such that

$$\|u\|_{H^1(\Omega_\epsilon)^3} \leq C\|u\|_{X_{\mathrm{N}}(\Omega_\epsilon)}\,, \tag{3.2.1}$$

for all $u \in X_{\mathrm{N}}(\Omega_\epsilon)$ and $\epsilon > 0$. (Note that by Theorem 1.3.2, for every $\epsilon > 0$ there exists a positive constant C_ϵ, possibly depending on ϵ, such that (3.2.1) holds, but here we need a constant independent of ϵ).

To do so, for any $\epsilon \geq 0$, we denote by S_ϵ the operator S_{Ω_ϵ} from $L^2(\Omega_\epsilon)$ to itself defined in Lemma 3.0.3. Recall that if $f_\epsilon \in L^2(\Omega_\epsilon)^3$ is the datum of the following Poisson problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} v_\epsilon - \tau\nabla \operatorname{div} v_\epsilon + v_\epsilon = f_\epsilon, & \text{in } \Omega_\epsilon,\\ \operatorname{div} v_\epsilon = 0, & \text{on } \partial\Omega_\epsilon,\\ v_\epsilon \times \nu = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{3.2.2}$$

then the unique solution $v_\epsilon \in X_{\mathrm{N}}(\Omega_\epsilon)$ is precisely $S_\epsilon f_\epsilon$, that is $v_\epsilon = S_\epsilon f_\epsilon$. Recall that τ is a fixed positive constant (one could normalise it by setting $\tau = 1$ but we prefer to keep it as it is also with reference to other papers where it is important to have the possibility to use different values of τ, see for example Remark 2.1.9). In this section we prove that S_ϵ compactly converges to S_0 as $\epsilon \to 0$, and this implies spectra stability. This has to be understood in the following sense.

We denote by $E = \{E_\epsilon\}_{\epsilon>0}$ the family of the extension-by-zero operators $E_\epsilon : L^2(\Omega)^3 \to L^2(\Omega_\epsilon)^3$ defined by

$$E_\epsilon\varphi = \varphi^0 = \begin{cases} \varphi, & \text{if } x \in \Omega_\epsilon \cap \Omega,\\ 0, & \text{if } x \in \Omega_\epsilon \setminus \Omega, \end{cases} \tag{3.2.3}$$

for all $\varphi \in L^2(\Omega)^3$. Note that under our assumptions we have that for all $\varphi \in L^2(\Omega)^3$

$$\lim_{\epsilon\to 0}\|E_\epsilon\varphi\|_{L^2(\Omega_\epsilon)^3} = \|\varphi\|_{L^2(\Omega)^3}\,,$$

since $\left|\Omega \setminus (\Omega_\epsilon \cap \Omega)\right| \to 0$ as ϵ goes to 0. We recall the following definition from [111], see also [8] and [29].

Definition 3.2.4. Let $u_\epsilon \in L^2(\Omega_\epsilon)$, for $\epsilon > 0$, be a family of functions. We say that u_ϵ E-converges to $u_0 \in L^2(\Omega)$ as $\epsilon \to 0$ and we write $u_\epsilon \xrightarrow{E} u_0$ if

$$\|u_\epsilon - E_\epsilon u_0\|_{L^2(\Omega_\epsilon)} \to 0, \quad \text{as} \quad \epsilon \to 0\,.$$

We also say that S_ϵ E-converges to S_0 as $\epsilon \to 0$ and we write $S_\epsilon \xrightarrow{EE} S_0$ if for any family of functions $f_\epsilon \in L^2(\Omega_\epsilon)$, we have

$$f_\epsilon \xrightarrow{E} f_0 \quad \Longrightarrow \quad S_\epsilon f_\epsilon \xrightarrow{E} S_0 f_0\,.$$

Finally, we say that S_ϵ E-compact converges to S_0 as $\epsilon \to 0$ and we write $S_\epsilon \xrightarrow{C} S_0$ if $S_\epsilon \xrightarrow{EE} S_0$ and for any family of functions $f_\epsilon \in L^2(\Omega_\epsilon)$, with $\|f_\epsilon\|_{L^2(\Omega_\epsilon)} = 1$ and any sequence of positive numbers ϵ_n with $\epsilon_n \to 0$, there exists a subsequence ϵ_{n_k} and $u \in L^2(\Omega)$ such that $S_{\epsilon_{n_k}} f_{n_k} \xrightarrow{E} u$.

The following theorem from [111, Thm. 6.3] holds.

Theorem 3.2.5. *If $S_\epsilon \xrightarrow{C} S_0$ then the eigenvalues of the operator S_ϵ converge to the eigenvalues of the operator S_0, and the eigenfunctions of the operator S_ϵ E-converge to the eigenfunctions of the operator S_0 as $\epsilon \to 0$.*

If we denote by $\mu_n(\epsilon)$, $n \in \mathbb{N}$ the sequence of eigenvalues of S_ϵ and by $u_n(\epsilon)$, $n \in \mathbb{N}$ a corresponding orthonormal sequence of eigenfunctions, then the stability of eigenvalues and eigenfunctions stated above has to be interpreted in the following sense:

(i) For every $n \in \mathbb{N}$ we have $\mu_n(\epsilon) \to \mu_n(0)$ as $\epsilon \to 0$.

(ii) For any sequence ϵ_k, $k \in \mathbb{N}$, converging to zero there exists an orthonormal sequence of eigenfunctions $u_n(0)$, $n \in \mathbb{N}$ in $L^2(\Omega)^3$ associated with $\mu_n(0)$, $n \in \mathbb{N}$ such that, possibly passing to a subsequence of ϵ_k, $u_n(\epsilon_k) \xrightarrow{E} u_n(0)$.

(iii) Given m eigenvalues $\mu_n(0), \dots, \mu_{n+m-1}(0)$ with

$$\mu_n(0) \neq \mu_{n-1}(0) \quad \text{and} \quad \mu_{n+m-1}(0) \neq \mu_{n+m}(0)$$

and corresponding orthonormal eigenfunctions $u_n(0), \dots, u_{n+m-1}(0)$, there exist m orthonormal generalized eigenfunctions (i.e. linear combinations of eigenfunctions) $v_n(\epsilon), \dots, v_{n+m-1}(\epsilon)$ associated with $\mu_n(\epsilon), \dots, \mu_{n+m-1}(\epsilon)$ such that $v_{n+i}(\epsilon) \xrightarrow{E} u_{n+i}(0)$ for all $i = 0, 1, \dots, m-1$.

Recall that μ is an eigenvalue of S_ϵ if and only if $\lambda = \mu^{-1}$ is an eigenvalue of the problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} v_\epsilon - \tau\nabla \operatorname{div} v_\epsilon + v_\epsilon = \lambda v_\epsilon, & \text{in } \Omega_\epsilon, \\ \operatorname{div} v_\epsilon = 0, & \text{on } \partial\Omega_\epsilon, \\ v_\epsilon \times \nu = 0, & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{3.2.6}$$

and that the corresponding eigenfunctions are the same. Note that the eigenvalues of (3.2.6) differ from those of (3.0.1) just by a translation. Thus, studying the stability of eigenvalues and eigenfunctions of the problem (3.2.6) or (3.0.1), is equivalent to studying the spectral stability of the family of operators S_ϵ. To do so, we recall that the weak formulation of problem (3.2.2) reads as follows: find $v_\epsilon \in X_{\mathrm{N}}(\Omega_\epsilon)$ such that

$$\int_{\Omega_\epsilon} v_\epsilon \cdot \eta \, dx + \int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \eta \, dx + \tau \int_{\Omega_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \eta \, dx = \int_{\Omega_\epsilon} f_\epsilon \cdot \eta \, dx \tag{3.2.7}$$

for all $\eta \in X_{\mathrm{N}}(\Omega_\epsilon)$.

Suppose that for every $\epsilon > 0$ we have that $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \le C$ for some $C > 0$. Then, setting $\eta = v_\epsilon$ in (3.2.7) and observing that $\int_{\Omega_\epsilon} f_\epsilon \cdot v_\epsilon \, dx \le \frac{1}{2}\int_{\Omega_\epsilon} |f_\epsilon|^2 \, dx + \frac{1}{2}\int_{\Omega_\epsilon} |v_\epsilon|^2 \, dx$, we get

$$\frac{1}{2}\int_{\Omega_\epsilon} |v_\epsilon|^2 \, dx + \int_{\Omega_\epsilon} |\operatorname{curl} v_\epsilon|^2 \, dx + \tau \int_{\Omega_\epsilon} |\operatorname{div} v_\epsilon|^2 \, dx \le \frac{1}{2}\int_{\Omega_\epsilon} |f_\epsilon|^2 \, dx.$$

This in turn implies that for all $\epsilon > 0$

$$\begin{aligned} \|v_\epsilon\|_{X_{\mathrm{N}}(\Omega_\epsilon)} &= \left(\|v_\epsilon\|^2_{L^2(\Omega_\epsilon)^3} + \|\operatorname{curl} v_\epsilon\|^2_{L^2(\Omega_\epsilon)^3} + \|\operatorname{div} v_\epsilon\|^2_{L^2(\Omega_\epsilon)} \right)^{1/2} \\ &\le c\|f_\epsilon\|_{L^2(\Omega_\epsilon)} = O(1). \end{aligned} \tag{3.2.8}$$

In order to prove the E-convergence of the operators S_ϵ, it is necessary to consider the limit of functions v_ϵ. We note that if $\Omega \subset \Omega_\epsilon$ for all $\epsilon > 0$ then it would suffice to consider the restriction of v_ϵ to Ω and pass to the weak limit in Ω. Otherwise, it is convenient to extend functions v_ϵ to the whole of $\mathbb{R}^3$. To do so, we observe that by the uniform Gaffney inequality combined with inequality (3.2.8) it follows that $\|v_\epsilon\|_{H^1(\Omega_\epsilon)^3}$ is uniformly bounded. Moreover, the domains Ω_ϵ belongs to the same Lipschitz class $C^{0,1}_M(\mathcal{A})$ for some $M > 0$ hence the functions v_ϵ can be extended to the whole of $\mathbb{R}^3$ with a uniformly bounded norm, see e.g., [25]. Thus, in the sequel we shall directly make the following assumption:

$$v_\epsilon \in H^1(\mathbb{R}^3)^3 \cap X_{\mathrm{N}}(\Omega_\epsilon), \quad \sup_{\epsilon > 0} \|v_\epsilon\|_{\in H^1(\mathbb{R}^3)^3} \neq \infty \,. \tag{3.2.9}$$

Thus the family $\{v_\epsilon|_\Omega\}_{\epsilon>0}$ is bounded in $H(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$ and we can extract a sequence $\{v_{\epsilon_n}|_\Omega\}_{n\in\mathbb{N}}$, with $\epsilon_n \to 0$ as n goes to ∞, such that

$$v_{\epsilon_n}|_\Omega \underset{n\to\infty}{\rightharpoonup} v \qquad \text{weakly in } H(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega) \tag{3.2.10}$$

for some $v \in H(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$. It turns out that v preserves the boundary conditions as the following lemma clarifies.

Lemma 3.2.11. *Assume that for some $\epsilon_n > 0$ with $\epsilon_n \to 0$, there exists $v \in H(\mathrm{curl},\Omega)\cap H(\mathrm{div},\Omega)$ such that $\{v_{\epsilon_n}|_\Omega\}_{n\in\mathbb{N}}$ weakly converges to v in $H(\mathrm{curl},\Omega)\cap H(\mathrm{div},\Omega)$. Then $v\in X_{\mathrm{N}}(\Omega)$.*

Proof. To prove that $v\in X_{\mathrm{N}}(\Omega)$ we just need to make sure that $v\in H_0(\mathrm{curl},\Omega)$. Since $v_{\epsilon_n}\in H_0(\mathrm{curl},\Omega_\epsilon)$ for all $n\in\mathbb{N}$, by Proposition 1.2.5 we know that the extension-by-zero $v^0_{\epsilon_n}$ of v_{ϵ_n} belongs to $H(\mathrm{curl},\mathbb{R}^3)$ for all $n\in\mathbb{N}$. By the reflexivity of $H(\mathrm{curl},\mathbb{R}^3)$ and the boundedness of the sequence $\left\{v^0_{\epsilon_n}\right\}_{n\in\mathbb{N}}$, we deduce that possibly passing to a subsequence, there exists a function $\tilde{v}\in H(\mathrm{curl},\mathbb{R}^3)$ such that $v^0_{\epsilon_n}\rightharpoonup\tilde{v}$ weakly in $H(\mathrm{curl},\mathbb{R}^3)$ as n goes to ∞. It suffices to show that $\tilde{v}=v^0$. Since $v^0_{\epsilon_n}$ is equal to zero outside of Ω_{ϵ_n}, it is clear that $\tilde{v}=0$ a.e. in $\mathbb{R}^3\setminus\Omega$. Moreover, since $v_{\epsilon_n}|_\Omega\rightharpoonup v,\tilde{v}$ in $H(\mathrm{curl},\Omega)$, we have that $v=\tilde{v}$ a.e. in Ω. Thus the extension by zero of v to the whole of $\mathbb{R}^3$ is precisely $\tilde{v}$ and belongs to $H(\mathrm{curl},\mathbb{R}^3)$. Using Proposition 1.2.5 again, we see that $v\in H_0(\mathrm{curl},\Omega)$. $\square$

Lemma 3.2.12. *Assume that condition (3.1.15) and the uniform Gaffney inequality (3.2.1) hold. For any $\epsilon>0$ let $f_\epsilon\in L^2(\Omega_\epsilon)^3$. Suppose that $\sup_{\epsilon>0}\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3}\neq\infty$ and that the extension-by-zero of the functions f_ϵ converge weakly in $L^2(\Omega)^3$ to some function $f\in L^2(\Omega)^3$ as $\epsilon\to 0$. For all $\epsilon>0$, let $v_\epsilon := S_\epsilon f_\epsilon$ the (unique) weak solution in $X_{\mathrm{N}}(\Omega_\epsilon)$ of (3.2.7) with datum f_ϵ. Assume (3.2.9) and suppose that $v_\epsilon\rightharpoonup v$ weakly in $H(\mathrm{curl},\Omega)\cap H(\mathrm{div},\Omega)$ to some $v\in H(\mathrm{curl},\Omega)\cap H(\mathrm{div},\Omega)$. Then $v=S_0 f$.*

Proof. First of all, we note that by Lemma 3.2.11 the function v belongs to $X_{\mathrm{N}}(\Omega)$. Define for $u,w\in H(\mathrm{curl},\Omega_\epsilon)\cap H(\mathrm{div},\Omega_\epsilon)$

$$Q_{\Omega_\epsilon}(u,w) := \int_{\Omega_\epsilon} u\cdot w\,dx + \int_{\Omega_\epsilon}\mathrm{curl}\,u\cdot\mathrm{curl}\,w\,dx + \tau\int_{\Omega_\epsilon}\mathrm{div}\,u\cdot\mathrm{div}\,w\,dx,$$

which is equivalent to the inner product of the space $H(\mathrm{curl},\Omega_\epsilon)\cap H(\mathrm{div},\Omega_\epsilon)$. The square of the induced norm will be denoted by $Q_{\Omega_\epsilon}(\cdot)$. Note that since v_ϵ is the solution with datum f_ϵ, then we have that

$$Q_{\Omega_\epsilon}(v_\epsilon,\eta) = \int_{\Omega_\epsilon} f_\epsilon\cdot\eta$$

for all $\eta \in X_{\mathrm{N}}(\Omega_\epsilon)$.

Let φ be any function in $X_{\mathrm{N}}(\Omega)$ and let $\mathcal{P}_\epsilon\varphi$ the Atlas Piola trasform of φ. Since $\mathcal{P}_\epsilon\varphi \in X_{\mathrm{N}}(\Omega_\epsilon)$, we deduce that

$$Q_{\Omega_\epsilon}(v_\epsilon, \mathcal{P}_\epsilon\varphi) = \int_{\Omega_\epsilon} f_\epsilon \cdot \mathcal{P}_\epsilon\varphi \tag{3.2.13}$$

for all $\epsilon > 0$. We now show that

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} f_\epsilon \cdot \mathcal{P}_\epsilon\varphi = \int_\Omega f \cdot \varphi \tag{3.2.14}$$

and

$$\lim_{\epsilon\to 0} Q_{\Omega_\epsilon}(v_\epsilon, \mathcal{P}_\epsilon\varphi) = Q_\Omega(v, \varphi) \tag{3.2.15}$$

In order to prove the first limit, it suffices to prove that

$$\int_{\Omega_\epsilon\cap V_j} f_\epsilon \cdot \tilde{\varphi}_{\epsilon,j} \underset{\epsilon\to 0}{\longrightarrow} \int_{\Omega\cap V_j} f \cdot \varphi_j \tag{3.2.16}$$

for any $j = 1, \dots, s'$, where $\tilde{\varphi}_{\epsilon,j}$ is defined in (3.1.14) (with $\tilde{\Omega}$ replaced by Ω_ϵ), since it is obvious that

$$\int_{\Omega_\epsilon\cap V_j} f_\epsilon \cdot \varphi_j \underset{\epsilon\to 0}{\longrightarrow} \int_{\Omega\cap V_j} f \cdot \varphi_j \tag{3.2.17}$$

for any $j = s'+1, \dots, s$. We have that

$$\int_{\Omega_\epsilon\cap V_j} f_\epsilon \cdot \tilde{\varphi}_{\epsilon,j} = \int_{r_j^{-1}(K_{\epsilon,j})} f_\epsilon \cdot \varphi_j + \int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} f_\epsilon \cdot \tilde{\varphi}_{\epsilon,j}. \tag{3.2.18}$$

Obviously

$$\lim_{\epsilon\to 0} \int_{r_j^{-1}(K_{\epsilon,j})} f_\epsilon \cdot \varphi_j = \int_{\Omega\cap V_j} f \cdot \varphi_j \tag{3.2.19}$$

since the extension-by-zero of the functions f_ϵ weakly converge to f in $L^2(\Omega)^3$, $\sup_{\epsilon>0}\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} < \infty$ and $\left|(\Omega\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})\right|$ goes to 0 as $\epsilon \to 0$. Meanwhile

$$\left|\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} f_\epsilon \cdot \tilde{\varphi}_{\epsilon,j}\right| \leq \|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \left(\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|\tilde{\varphi}_{\epsilon,j}\right|^2\right)^{\frac{1}{2}} \underset{\epsilon\to 0}{\longrightarrow} 0 \tag{3.2.20}$$

by (3.1.25) and the hypothesis that $\sup_{\epsilon>0}\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \neq \infty$. From (3.2.18), (3.2.19) and (3.2.20) we immediately deduce (3.2.16). Hence we have proved (3.2.14).

Let us now focus on (3.2.15). We write

$$
\begin{aligned}
Q_{\Omega_\epsilon}(v_\epsilon, \mathcal{P}_\epsilon\varphi) &= Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) + Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) \\
&= Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi - \varphi) + Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \varphi) + Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) \\
&= Q_{\Omega}(v_\epsilon, \varphi) - Q_{\Omega\setminus\Omega_\epsilon}(v_\epsilon, \varphi) + Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi - \varphi) + Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi)
\end{aligned} \tag{3.2.21}
$$

By the weak convergence of v_ϵ to v we have that

$$
Q_\Omega(v_\epsilon, \varphi) \to Q_\Omega(v, \varphi), \quad \text{as } \epsilon \to 0. \tag{3.2.22}
$$

By the Cauchy-Schwarz inequality and Theorem 3.1.16, (iv) we get that

$$
Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi-\varphi) \leq (Q_{\Omega_\epsilon\cap\Omega}(v_\epsilon))^{1/2}(Q_{\Omega_\epsilon\cap\Omega}(\mathcal{P}_\epsilon\varphi-\varphi))^{1/2} \to 0, \ \text{ as } \epsilon \to 0. \tag{3.2.23}
$$

Similarly,

$$
Q_{\Omega\setminus\Omega_\epsilon}(v_\epsilon, \varphi) \leq (Q_{\Omega\setminus\Omega_\epsilon}(v_\epsilon))^{1/2}(Q_{\Omega\setminus\Omega_\epsilon}(\varphi))^{1/2} \to 0 \ \text{ as } \epsilon \to 0. \tag{3.2.24}
$$

Finally,

$$
Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon, \mathcal{P}_\epsilon\varphi) \leq (Q_{\Omega_\epsilon\setminus\Omega}(v_\epsilon))^{1/2}(Q_{\Omega_\epsilon\setminus\Omega}(\mathcal{P}_\epsilon\varphi))^{1/2} \to 0 \ \text{ as } \epsilon \to 0, \tag{3.2.25}
$$

since by (3.1.25), (3.1.34) and (3.1.42) it follows that $Q_{\Omega_\epsilon\setminus\Omega}(\mathcal{P}_\epsilon\varphi) \to 0$ as $\epsilon \to 0$. By combining (3.2.22)-(3.2.25), we deduce that limit (3.2.15) holds.

In conclusion, by using the limiting relations (3.2.14) and (3.2.15) in equation (3.2.13) we conclude that

$$
Q_\Omega(v, \varphi) = \int_\Omega f \cdot \varphi
$$

which means exactly that v is the solution in $X_{\mathrm{N}}(\Omega)$ of the given problem with datum $f \in L^2(\Omega)$, as required.

□

Remark 3.2.26. A careful inspection of the proof of Lemma 3.2.12 reveals that the uniform Gaffney inequality has been used only to prove the limiting relations (3.2.22) and (3.2.24) since the functions v_ϵ are required here to be defined on Ω and to have uniformly bounded norms. This problem does not occur if $\Omega \subset \Omega_\epsilon$ in which case only the Gaffney inequality in Ω is necessary. However, the uniform Gaffney inequality will be used in an essential way in the following statements also in the particular case $\Omega \subset \Omega_\epsilon$

In the next lemma we prove that S_ϵ E-converges to S_0 as $\epsilon \to 0$.

Lemma 3.2.27. *Assume that condition* (3.1.15) *and the uniform Gaffney inequality* (3.2.1) *hold. Let* $f_\epsilon \in L^2(\Omega_\epsilon)^3, \epsilon > 0$ *be such that* $f_\epsilon \xrightarrow[\epsilon\to 0]{E} f \in L^2(\Omega)^3$ *for some function* $f \in L^2(\Omega)^3$. *Set* $v_\epsilon := S_\epsilon f_\epsilon$ *and* $v := S_0 f$. *Then* $v_\epsilon \xrightarrow[\epsilon\to 0]{E} v$, *hence* $S_\epsilon \xrightarrow[\epsilon\to 0]{EE} S_0$.

Proof. Since $f_\epsilon \xrightarrow[\epsilon\to 0]{E} f$, then $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \le C$ for all $\epsilon > 0$ sufficiently small and consequently $\|v_\epsilon\|_{X_{\mathrm{N}}(\Omega_\epsilon)}$ is uniformly bounded with respect to ϵ, as shown in (3.2.8). By the uniform Gaffney inequality it follows that also $\|v_\epsilon\|_{H^1(\Omega_\epsilon)^3}$ is uniformly bounded. In particular

$$\lim_{\epsilon\to 0}\|v_\epsilon\|_{L^2(\Omega_\epsilon\setminus\Omega)^3} = 0$$

because $\left|\Omega_\epsilon \setminus \Omega\right| \to 0$ as ϵ goes to 0. This can be proved using the same argument used for (3.1.44) as follows:

$$\begin{aligned}
&\int_{(\Omega_\epsilon\cap V_j)\setminus r_j^{-1}(K_{\epsilon,j})} \left|v_\epsilon(x)\right|^2 dx = \int_{r_j(\Omega_\epsilon\cap V_j)\setminus K_{\epsilon,j}} \left|v_\epsilon \circ r_j^{-1}(z)\right|^2 dz \\
&= \int_{W_j} \left(\int_{\hat{g}_{\epsilon,j}(\bar{z})}^{g_{\epsilon,j}(\bar{z})} \left|v_\epsilon \circ r_j^{-1}(\bar{z}, z_3)\right|^2 dz_3 \right) d\bar{z} \\
&\le \int_{W_j} \left|g_{\epsilon,j}(\bar{z}) - \hat{g}_{\epsilon,j}(\bar{z})\right| \left\|v_\epsilon \circ r_j^{-1}(\bar{z}, \cdot)\right\|^2_{L^\infty(a_{3j}, g_{\epsilon,j}(\bar{z}))^3} d\bar{z} \\
&\le \left\|g_{\epsilon,j} - \hat{g}_{\epsilon,j}\right\|_{L^\infty(W_j)} \int_{W_j} \left\|v_\epsilon \circ r_j^{-1}(\bar{z}, \cdot)\right\|^2_{H^1(a_{3j}, g_{\epsilon,j}(\bar{z}))^3} d\bar{z} \\
&\le \kappa_\epsilon \left\|v_\epsilon \circ r_j^{-1}\right\|^2_{H^1(r_j(\Omega_\epsilon\cap V_j))^3} \xrightarrow[\epsilon\to 0]{} 0.
\end{aligned}$$

Hence to prove that $v_\epsilon \xrightarrow[\epsilon\to 0]{E} v$ we just have to show that

$$\lim_{\epsilon\to 0}\left\|v_\epsilon|_\Omega - v\right\|_{L^2(\Omega)^3} = 0. \tag{3.2.28}$$

Recall that $\{v_\epsilon|_\Omega\} \subset H^1(\Omega)^3$ is bounded in H^1-norm. Select now a sequence $\{v_{\epsilon_n}\}_{n\in\mathbb{N}}$ from the family. By the compact embedding of $H^1(\Omega)^3$ into $L^2(\Omega)^3$ we have that, up to choosing a subsequence, $v_{\epsilon_n}|_\Omega \to v^*$ strongly in $L^2(\Omega)^3$ and $v_{\epsilon_n}|_\Omega \rightharpoonup v^*$ weakly in $H^1(\Omega)^3$ for some $v^* \in H^1(\Omega)^3$. By Lemma 3.2.12 we have that $v^* = S_0 f = v \in X_{\mathrm{N}}(\Omega)$. This shows that for any extracted sequence of the family $\{v_\epsilon|_\Omega - v\}_{\epsilon>0}$, there exist a subsequence such that $\left\|v_{\epsilon_{n_k}}|_\Omega - v\right\|_{L^2(\Omega)^3} \xrightarrow[k\to\infty]{} 0$. Thus we can conclude that $\left\|v_\epsilon|_\Omega - v\right\|_{L^2(\Omega)^3} \xrightarrow[\epsilon\to 0]{} 0$, which is exactly (3.2.28). □

Remark 3.2.29. The hypothesis of Lemma 3.2.27 concerning the functions f_ϵ can be weakened to only require that $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3}$ are uniformly bounded and that the extension-by-zero of f_ϵ (restricted to Ω) weakly converges to f in $L^2(\Omega)^3$ as ϵ goes to 0, as it can be easily seen from the proof, which is a weaker assumption than $f_\epsilon \xrightarrow[\epsilon\to 0]{E} f$.

Finally we can state and prove the main theorem of this chapter.

Theorem 3.2.30. *Let $\mathcal{A}$ be an atlas and $\{\Omega_\epsilon\}_{\epsilon>0}$ be a family of bounded domains of class $C^{1,1}(\mathcal{A})$ converging to a bounded domain Ω of class $C^{1,1}(\mathcal{A})$ as $\epsilon \to 0$, in the sense that condition (3.1.15) holds. Suppose that the uniform Gaffney inequality (3.2.1) holds. Then $S_\epsilon \xrightarrow{C} S_0$ as $\epsilon \to 0$. In particular, spectral stability occurs: the eigenvalues of the operator S_ϵ converge to the eigenvalues of the operator S_0, and the eigenfunctions of the operator S_ϵ E-converge to the eigenfunctions of the operator S_0 as $\epsilon \to 0$.*

Proof. By Lemma 3.2.27 we have that $S_\epsilon \xrightarrow[\epsilon\to 0]{EE} S$. Now, suppose that we are given a family of data $\{f_\epsilon\}_{\epsilon>0}$ such that $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \le 1$ for all $\epsilon > 0$, and extract a sequence $\{f_{\epsilon_n}\}_{n\in\mathbb{N}}$ from it. We have to show that we can always find a subsequence $\epsilon_{n_k} \to 0$ and a function $v \in L^2(\Omega)^3$ such that

$$S_{\epsilon_{n_k}} f_{\epsilon_{n_k}} \xrightarrow[k\to\infty]{E} v. \tag{3.2.31}$$

Possibly passing to a subsequence, we can find a function f to which the restriction to Ω of the extension-by-zero of $\{f_{\epsilon_n}\}_{n\in\mathbb{N}}$ weakly converge in $L^2(\Omega)^3$. Setting $v := S_0 f$, we can apply Lemma 3.2.27 and Remark 3.2.29 to find out that (3.2.31) holds.

Finally, the spectral stability is a consequence of the compact convergence of compact operators as stated in Theorem 3.2.5. ☐

3.3 Uniform Gaffney Inequalities and applications to families of oscillating boundaries

In this section we want to apply the results regarding the Gaffney inequality presented in Section 1.4 of Chapter 1. It is clear that in order to apply Theorem 1.4.23 and Corollaries 1.4.30, 1.4.31, it suffices to assume that the gradients ∇g_j of the functions g_j describing the boundary of a domain Ω as in Definition 1.0.1 are of class $C^{0,\beta}$ with $\beta \in]1/2, 1]$, that is

$$|\nabla g_j(\bar{x}) - \nabla g_j(\bar{y})| \le K|\bar{x} - \bar{y}|^\beta \tag{3.3.1}$$

for some positive constant K and all $\bar{x}, \bar{y} \in W_j$, and that the functions g_j have sufficiently small Lipschitz constants. As we have already mentioned, in principle, the second condition is not a big obstruction to the application of these results, since for a domain of class C^1 one can find a sufficiently refined atlas, adapted to the tangent planes of a finite number of boundary points, such that the C^1 norms, hence the Lipschitz constants, of the profile functions g_j are arbitrarily close to zero. Thus, we can apply our results to uniform classes of domains of class $C^{1,\beta}$ since condition (1.4.25) would be satisfied exactly because $\beta > 1/2$ (as we have said, here what matters is the behaviour of the modulus of continuity $\omega(t)$ for t close to zero and one can assume directly that $\omega(t)$ is constant for t big enough).

Thus, we can prove the following result. Note that here the domains Ω_ϵ are assumed to be of class $C^{1,1}$ and that they belong to the uniform class $C^{1,\beta}_K(\mathcal{A})$ with $K > 0$ fixed, which in particular implies the validity of (3.3.1) for all functions $g_{\epsilon,j}$ and all $\epsilon > 0$. (Recall that the operators S_ϵ are defined in the beginning of Section 3.2.)

Theorem 3.3.2. *Let $\mathcal{A}$ be an atlas and $\{\Omega_\epsilon\}_{\epsilon>0}$ be a family of bounded domains of class $C^{1,1}(\mathcal{A})$ converging to a bounded domain Ω of class $C^{1,1}(\mathcal{A})$ as $\epsilon \to 0$, in the sense that condition (3.1.15) holds. Suppose that Ω is of class $C^{0,1}_M(\mathcal{A})$ with M small enough as in Corollary 1.4.31. Suppose also that all domains Ω_ϵ are of class $C^{1,\beta}_L(\mathcal{A})$ with the same parameters $\beta \in]1/2, 1]$ and $L > 0$. Then the uniform Gaffney inequality (3.2.1) holds provided ϵ is small enough. Moreover, $S_\epsilon \xrightarrow{C} S$ as $\epsilon \to 0$. In particular, spectral stability occurs: the eigenvalues of the operator S_ϵ converge to the eigenvalues of the operator S_0, and the eigenfunctions of the operator S_ϵ E-converge to the eigenfunctions of the operator S_0 as $\epsilon \to 0$.*

Proof. Since Ω_ϵ converges to Ω as $\epsilon \to 0$ in the sense that condition (3.1.15) holds, it follows that the gradients of the functions $g_{\epsilon,j}$ describing the boundary of Ω_ϵ converge uniformly to the gradients of the functions g_j describing the boundary of Ω. Thus, Ω_ϵ is of class $C^{0,1}_M(\mathcal{A})$ provided ϵ is small enough. By the discussion above, Corollary 1.4.31 is applicable and the uniform Gaffney inequality (3.2.1) holds provided ϵ is small enough. Then the last part of the statement follows by Theorem 3.2.30. □

A prototype for the classes of domains under discussion is given by domains designed by profile functions often used in homogenization theory, in particular in the study of thin domains. Namely, assume that one of the profile functions $g_{\epsilon,j}$, call it g_ϵ, is of the form

$$g_\epsilon(\bar{x}) = \epsilon^\alpha b(\bar{x}/\epsilon) \tag{3.3.3}$$

for some function b of class $C^{1,1}(\mathbb{R}^{N-1})$ and $\alpha > 0$, and assume that the gradient of b is bounded. If $\omega_{\nabla b}$ is a (non-decreasing) modulus of continuity of ∇b then we have

$$|\nabla g_\epsilon(\bar{x}) - \nabla g_\epsilon(\bar{y})| = \epsilon^{\alpha-1}|\nabla b(\bar{x}/\epsilon) - \nabla b(\bar{y}/\epsilon)| \le \epsilon^{\alpha-1}\omega_{\nabla b}\left(\frac{\bar{x}-\bar{y}}{\epsilon}\right),$$

hence the function ω to be considered in (1.4.24) is given by $\omega(t) = \epsilon^{\alpha-1}\omega_{\nabla b}(t/\epsilon)$. Observe that

$$\int_0^\infty \left(\frac{\omega(t)}{t}\right)^2 dt = \epsilon^{2\alpha-2}\int_0^\infty \left(\frac{\omega_{\nabla b}(t/\epsilon)}{t}\right)^2 dt = \epsilon^{2\alpha-3}\int_0^\infty \left(\frac{\omega_{\nabla b}(s)}{s}\right)^2 ds\,. \quad (3.3.4)$$

Moreover, since b is assumed to be of class $C^{1,1}$, we have that $\omega_{\nabla b}(t) \le ct$ for t in a neighborhood of zero. Thus, if $\alpha \ge 3/2$ and ϵ_0 is any fixed positive constant, it follows that that

$$\sup_{\epsilon\in]0,\epsilon_0]} \epsilon^{2\alpha-3}\int_0^\infty \left(\frac{\omega_{\nabla b}(s)}{s}\right)^2 ds \neq \infty\,. \quad (3.3.5)$$

Since the gradient of g_ϵ is arbitrarily close to zero for ϵ sufficiently small, we have that Theorem 1.4.23 and Corollaries 1.4.30, 1.4.31 are applicable and the Gaffney inequality (3.2.1) holds for all ϵ sufficiently small, with a constant $C > 0$ independent of ϵ. The same arguments can be applied to families of profile functions of the type

$$g_\epsilon(\bar{x}) = \epsilon^\alpha b(\bar{x}/\epsilon)\psi(\bar{x})$$

where b is as above and ψ is a fixed $C^{1,1}$ function with bounded gradient. Thus, we can state the following stability result concerning a local perturbation of a domain Ω.

Theorem 3.3.6. *Let W be a bounded open rectangle in $\mathbb{R}^2$, $b \in C^{1,1}(\mathbb{R}^2)$ with bounded gradient, $b \ge 0$, and $\psi \in C_c^{1,1}(W)$, $\alpha \in]3/2, 2]$. Assume that Ω and Ω_ϵ, $\epsilon > 0$ are domains of class $C^{1,1}$ in $\mathbb{R}^3$ satisfying the following condition:*

(i) $\Omega \cap (W\times]-1,1[) = \{(\bar{x}, x_3) \in \mathbb{R}^3 : \ \bar{x} \in W,\ -1 < x_3 < 0\}$;

(ii) $\Omega_\epsilon \cap (W\times]-1,1[) = \{(\bar{x}, x_3) \in \mathbb{R}^3 : \ \bar{x} \in W,\ -1 < x_3 < \epsilon^\alpha b(\bar{x}/\epsilon)\psi(\bar{x})\}$ *where* $b \in C^{1,1}(\mathbb{R}^2)$ *has bounded gradient, and* $\psi \in C_c^{1,1}(W)$;

(iii) $\Omega \setminus (W\times]-1,1[) = \Omega_\epsilon \setminus (W\times]-1,1[)$;

Then the family $\{\Omega_\epsilon\}_{\epsilon>0}$ converges to Ω in the sense that condition (3.1.15) holds. Moreover, the uniform Gaffney inequality (3.2.1) holds and $S_\epsilon \xrightarrow{C} S$ as $\epsilon \to 0$. In particular, spectral stability occurs: the eigenvalues of the operator S_ϵ converge to the eigenvalues of the operator S_0, and the eigenfunctions of the operator S_ϵ E-converge to the eigenfunctions of the operator S_0 as $\epsilon \to 0$.

Proof. By assumptions, the domains Ω and Ω_ϵ belong to the same atlas class $C^{1,1}(\mathcal{A})$ for a suitable atlas $\mathcal{A}$, and $W\times]-1,1[$ is one of the local charts of $\mathcal{A}$. In particular, the profile functions describing the boundaries of Ω and Ω_ϵ in that chart are given by $g(\bar{x}) = 0$ and $g_\epsilon = \epsilon^\alpha b(\bar{x}/\epsilon)\psi(\bar{x})$ for all $\bar{x} \in W$.

As in the proof of [11, Thm. 7.4], if $\tilde{\alpha} \in]3/2, \alpha[$ is fixed then one can easily check that conditions (3.1.15) are satisfied with $k_\epsilon = \epsilon^{2\tilde{\alpha}/3}$. By 1.4.31 and the discussion above, it follows that the Gaffney inequality (1.4.1) holds with a constant C independent of ϵ, provided ϵ is sufficiently small. To complete the proof it suffices to apply Theorem 3.2.30. □

3.4 Critical case

It is clear that condition (3.3.5) is satisfied also in the case $\alpha = 3/2$. Thus the uniform Gaffney inequality (3.2.1) holds also in the case $\alpha = 3/2$ in Theorem 3.3.6. However, in this case the convergence of Ω_ϵ to Ω in the sense of (3.1.15) is not guaranteed hence we cannot directly deduce that we have spectral stability. Thus, another method has to be used in the analysis of the stability problem for $\alpha = 3/2$. For example, in the case of non-constant periodic functions b one could use the unfolding method as in [30], adopted also in [10, 11, 53, 54]: in those papers, polyharmonic operators with various boundary conditions are addressed, and for $\alpha = 3/2$ we have spectral instability in the sense that the limiting problem differs from the given problem in Ω by a strange term appearing in the boundary conditions (as often happens in homogenization problems).

In this last section we plan to do that for the *curlcurl* operator, and as one can see from Theorems 3.4.57 and 3.4.109 a preliminary analysis would indicate that no strange limit appears in the limiting problem for $\alpha = 3/2$. On the other hand, at the moment we are not able to formulate any conjecture for the case $\alpha < 3/2$ although, on the base of the results of [30] concerning the Navier-Stokes system, a degeneration phenomenon (to Dirichlet boundary conditions) could not be excluded.

We will follow Section 8 of [11]. The variables in $\mathbb{R}^3$ will be denoted in the following way: $x = (x_1, x_2, x_3) = (\bar{x}, x_3) \in \mathbb{R}^3$ where $\bar{x} = (x_1, x_2) \in \mathbb{R}^2$ denotes the projection in the first two variables. To simplify the exposition, we restrict ourselves to particular family of domains with oscillating boundaries; in particular we assume the domain $\Omega \subset \mathbb{R}^3$ to be of the form $\Omega = W\times]-1,0[$, where $W = \left\{\bar{x} \in \mathbb{R}^2 : a_1 < x_1 < b_1, a_2 < x_2 < b_2\right\}$ is a rectangle in $\mathbb{R}^2$. We denote by g, g_ϵ the upper boundary profile functions for Ω, Ω_ϵ respectively, that is

$$\Omega_\epsilon = \left\{(\bar{x}, x_3) \in W\times]-1,0[: -1 < x_3 < g_\epsilon(\bar{x})\right\}.$$

Obviously $g \equiv 0$ in W, while we set

$$g_\epsilon(\bar{x}) := \epsilon^{3/2} b(\bar{x}/\epsilon) \tag{3.4.1}$$

for all $\bar{x} \in W$, where $b \in C^2(\mathbb{R}^2)$ is a Y-periodic non-negative fixed function. Note that since $b \geq 0$ then $\Omega \subset \Omega_\epsilon$ for any $\epsilon > 0$. We denote by Γ and Γ_ϵ the upper boundaries of Ω and Ω_ϵ respectively, namely

$$\Gamma := W \times \{0\} \quad \text{and} \quad \Gamma_\epsilon := \left\{(\bar{x}, x_3) : \bar{x} \in W, x_3 = g_\epsilon(\bar{x})\right\}.$$

For the sake of clarity, we will use ν_ϵ to denote the outer unit normal to Ω_ϵ in Γ_ϵ. We define $Y := (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ to be the basic cell in $\mathbb{R}^2$, and we consider the cell $C_\epsilon^k := \epsilon k + \epsilon Y$ for any $k \in \mathbb{Z}^2$ and any $\epsilon > 0$. We set

$$\widehat{W}_\epsilon := \bigcup_{k \in I_{W,\epsilon}} C_\epsilon^k \subset W,$$

where $I_{W,\epsilon} := \left\{k \in \mathbb{Z}^2 : C_\epsilon^k \subset W\right\}$. Observe that $\widehat{W}_\epsilon \subset W$. Finally, for any $0 < \epsilon \leq 1$ we set

$$K_\epsilon := W \times]-1, -\epsilon[\subset \Omega \quad \text{and} \quad Q_\epsilon := \widehat{W}_\epsilon \times]-\epsilon, 0[\subset \Omega. \tag{3.4.2}$$

In order to prove our results, we will make use of the *unfolding operator* introduced in [35]. We recall some of its properties in what follows.

Definition 3.4.3. Let u be a real-valued function defined on Ω. For any $\epsilon > 0$ sufficiently small the *unfolding* $\widehat{u}$ of u is the real-valued function defined on $\widehat{W}_\epsilon \times Y \times]-1/\epsilon, 0[$ by

$$\widehat{u}(\bar{x}, \bar{y}, y_3) := u\left(\epsilon \left\lfloor \frac{\bar{x}}{\epsilon} \right\rfloor + \epsilon \bar{y}, \epsilon y_3\right), \tag{3.4.4}$$

for all $(\bar{x}, \bar{y}, y_3) \in \widehat{W}_\epsilon \times Y \times]-1/\epsilon, 0[$, where $\left\lfloor \frac{\bar{x}}{\epsilon} \right\rfloor$ denotes the integer part of the vector $\frac{\bar{x}}{\epsilon}$ with respect to the unit cell Y, meaning that $\left\lfloor \frac{\bar{x}}{\epsilon} \right\rfloor = k \in \mathbb{Z}^2$ if and only if $\bar{x} \in C_\epsilon^k$.

We recall a useful lemma that will be used later in computations.

Lemma 3.4.5 (Lemma 8.7 of [11])**.** *Let $t \in [-1, 0[$ be fixed. Then*

$$\int_{\widehat{W}_\epsilon \times]t,0[} u(x)\, dx = \epsilon \int_{\widehat{W}_\epsilon \times Y \times]t/\epsilon, 0[} \widehat{u}(\bar{x}, y)\, d\bar{x} dy \tag{3.4.6}$$

for all $u \in L^1(\Omega)$ and $\epsilon > 0$ sufficiently small.

We also introduce some function spaces that will be useful in what follows and later on. We say that a function u belongs to the space $W^{1,2}_{Per_Y,loc}(Y \times (-\infty,0))$ if $u \in W^{1,2}_{loc}(\mathbb{R}^2 \times (-\infty,0))$ and moreover u is Y-periodic in the first two variables $\bar{y} \in \mathbb{R}^2$. There will be no notational difference when writing a function in $W^{1,2}_{Per_Y,loc}(Y \times (-\infty,0))$ or its restriction to $Y \times (-\infty,0)$. We also introduce the set

$$w^{1,2}_{Per_Y,loc}(Y \times (-\infty,0)) :=\big\{u \in W^{1,2}_{Per_Y,loc}(Y \times (-\infty,0)) : \\ \|D^\gamma u\|_{L^2(Y\times(-\infty,0))} < \infty \text{ for all } |\gamma| = 1\big\}.$$

For the sake of simplicity, we will always omit the vectorial notation for these spaces, and whether we are dealing with a scalar function or a vector field will be clear from the context.

Similarly, we say that a vector field u belongs to the space $X_{Per_Y,loc}(Y\times(-\infty,0))$ if $u \in H_{loc}(\mathrm{curl},\mathbb{R}^2\times(-\infty,0))\cap H_{loc}(\mathrm{div},\mathbb{R}^2\times(-\infty,0))$ and moreover u is Y-periodic in the first two variables $\bar{y} \in \mathbb{R}^2$. Again, there will be no notational difference when writing a function in $X_{Per_Y,loc}(Y \times (-\infty,0))$ or its restriction to $Y \times (-\infty,0)$. Finally we set

$$x_{Per_Y}(Y \times (-\infty,0)) :=\big\{u \in X_{Per_Y,loc}(Y \times (-\infty,0)) : \\ \|\mathrm{curl}\, u\|_{L^2(Y\times(-\infty,0))^3}, \|\mathrm{div}\, u\|_{L^2(Y\times(-\infty,0))} < \infty\big\}.$$

The following lemma is directly inspired by [11, Lemma 8.9].

Lemma 3.4.7. *Let $v_\epsilon \in H^1(\Omega)^3$ with $\|v_\epsilon\|_{H^1(\Omega)^3} \le M$ for all $\epsilon > 0$, for some $M > 0$. Let V_ϵ be defined by*

$$V_\epsilon(\bar{x}, y) := \hat{v}_\epsilon(\bar{x}, y) - \int_Y \hat{v}_\epsilon(\bar{x}, \bar{y}, 0)d\bar{y} \tag{3.4.8}$$

for $(\bar{x}, y) \in \widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0)$. Then there exists $\hat{v} \in L^2(W, w^{1,2}_{Per_Y}(Y \times (-\infty,0)))$ such that

$$\frac{V_\epsilon}{\epsilon^{1/2}} \underset{\epsilon\to0}{\rightharpoonup} \hat{v} \text{ and } \frac{D_y V_\epsilon}{\epsilon^{1/2}} \underset{\epsilon\to0}{\rightharpoonup} D_y\hat{v} \text{ in } L^2(W \times Y \times (d,0)) \text{ for any } d < 0. \tag{3.4.9}$$

In addition, if $\mathrm{div}\, v_\epsilon = 0$ in Ω, then

$$\mathrm{div}_y\, \hat{v} = 0 \quad \text{in } W \times Y \times (-\infty,0). \tag{3.4.10}$$

Moreover, for any $v \in H^1(\Omega)^3$

$$\widehat{(v \circ \Phi_\epsilon)|_\Omega} \underset{\epsilon\to0}{\longrightarrow} v(\bar{x}, 0) \text{ in } L^2(W \times Y \times (-1,0)).$$

Proof. Observe that $D_y V_\epsilon = D_y \hat{v}_\epsilon$. By Lemma 3.4.5 and the chain rule we have that

$$\int_{\widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0)} \left| \frac{D_y V_\epsilon}{\epsilon^{1/2}} \right|^2 d\bar{x} dy = \int_{\widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0)} \epsilon \left| \widehat{Dv_\epsilon} \right|^2 d\bar{x} dy$$
$$= \int_{\widehat{W}_\epsilon \times (-1,0)} |Dv_\epsilon|^2 \, dx \leq \int_\Omega |Dv_\epsilon|^2 \, dx \leq M^2$$

for all $\epsilon > 0$, hence $\left\| \frac{D_y V_\epsilon}{\epsilon^{1/2}} \right\|_{L^2(W \times Y \times (-\infty, 0))}$ is uniformly bounded with respect to ϵ. Moreover, we can apply the Poincaré-Wirtinger inequality to V_ϵ, so that we obtain that for any $d < 0$ there exists a constant $C_d > 0$ such that

$$\left\| \frac{V_\epsilon}{\epsilon^{1/2}} \right\|_{L^2(W \times Y \times (d,0))} \leq C_d \left\| \frac{D_y V_\epsilon}{\epsilon^{1/2}} \right\|_{L^2(W \times Y \times (d,0))} \leq C_d M$$

for all $\epsilon > 0$. Then we can conclude that there exists a function $\hat{v}$ defined on $W \times Y \times (d, 0)$, admitting weak first order derivatives in the variable y, with $\hat{v} \in L^2(W \times Y \times (d, 0))$ for any $d < 0$, $D_y \hat{v} \in L^2(W \times Y \times (-\infty, 0))$, such that (3.4.9) holds. Moreover, since $\operatorname{div} v_\epsilon = 0$ in Ω for all $\epsilon > 0$, by (3.4.4) we deduce that $\operatorname{div}_y \hat{v}_\epsilon(\bar{x}, \bar{y}, y_3) = \epsilon \, \widehat{\operatorname{div} v_\epsilon}(\bar{x}, \bar{y}, y_3) = 0$, which together with (3.4.9) proves that

$$\operatorname{div}_y \hat{v} = 0 \quad \text{in } W \times Y \times (-\infty, 0).$$

For the proof of the Y-periodicity of $\hat{v}$ with respect to the variables $\bar{y}$ and the last statement of the Lemma 3.4.7, we refer to Lemma 4.3 of [30] (in particular the argument in Step 3). □

Lemma 3.4.11 ([30, Lemma 4.3])**.** *Let v_ϵ be a bounded sequence in $H^1(\Omega_\epsilon)^3$ such that $v_\epsilon \cdot \nu_\epsilon = 0$ on Γ_ϵ and suppose that their restrictions to Ω weakly converge in $H^1(\Omega)^3$ to some $v \in H^1(\Omega)^3$ as $\epsilon \to 0$. Then the third component v_3 of v vanishes on Γ.*

Moreover, let $\hat{v} : W \times Y \times (-\infty, 0) \to \mathbb{R}^3$ be the weak limit in $L^2(W \times Y \times (d, 0))$ for any $d < 0$, of the sequence of functions

$$\frac{\hat{v}_\epsilon(\bar{x}, y) - \int_Y \hat{v}_\epsilon(\bar{x}, \bar{y}, 0) \, d\bar{y}}{\epsilon^{1/2}},$$

introduced in Lemma 3.4.7. Then

$$\hat{v}_3(\bar{x}, \bar{y}, 0) = \nabla b(\bar{y}) \cdot \big(v_1(\bar{x}, 0), v_2(\bar{x}, 0)\big) \quad \textit{for a.e. } (\bar{x}, \bar{y}) \in W \times Y. \tag{3.4.12}$$

Remark 3.4.13. Observe that the sign in formula (3.4.12) is different from the one in formula (4.11) of [30] since in the latter the authors consider supergraphs, while we consider subgraphs.

We define the transformation Φ_ϵ from Ω_ϵ to Ω with $\Phi_\epsilon(\bar{x}, x_3) := (\bar{x}, x_3 - h_\epsilon(\bar{x}, x_3))$ for all $(\bar{x}, x_3) \in \overline{\Omega}_\epsilon$, where

$$h_\epsilon(\bar{x}, x_3) = \begin{cases} 0, & \text{if } -1 < x_3 \leq -\epsilon, \\ g_\epsilon(\bar{x}) \left(\frac{x_3+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^3, & \text{if } -\epsilon < x_3 \leq g_\epsilon(\bar{x}). \end{cases}$$

Observe that Φ_ϵ coincides with the identity in K_ϵ, where K_ϵ was defined in (3.4.2). Moreover

$$D\Phi_\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\partial h_\epsilon}{\partial x_1} & -\frac{\partial h_\epsilon}{\partial x_2} & 1 - \frac{\partial h_\epsilon}{\partial x_3} \end{pmatrix}. \tag{3.4.14}$$

By standard calculus one can easily deduce the following

Lemma 3.4.15. *The map Φ_ϵ is a diffeomorphism of class C^2 and for all $\epsilon > 0$ sufficiently small*

$$|h_\epsilon| \leq c\epsilon^{3/2}, \quad \left|\frac{\partial h_\epsilon}{\partial x_i}\right| \leq c\epsilon^{1/2}, \quad \left|\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}\right| \leq c\epsilon^{-1/2},$$

where $c > 0$ is a constant independent of ϵ.

We also recall the following technical lemma from [11].

Lemma 3.4.16 ([11, Lemma 8.27])**.** *For all $y \in Y \times (-1, 0)$ and $i, j = 1, 2, 3$, the functions $\hat{h}_\epsilon(\bar{x}, y)$, $\widehat{\frac{\partial h_\epsilon}{\partial x_i}}(\bar{x}, y)$ and $\widehat{\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}}(\bar{x}, y)$ are independent of $\bar{x} \in W$. Moreover, $\hat{h}_\epsilon(\bar{x}, y) = O(\epsilon^{3/2})$, $\widehat{\frac{\partial h_\epsilon}{\partial x_i}}(\bar{x}, y) = O(\epsilon^{1/2})$ as $\epsilon \to 0$ and*

$$\epsilon^{1/2} \widehat{\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}}(\bar{x}, y) \xrightarrow[\epsilon \to 0]{} \frac{\partial^2 \left(b(\bar{y})(y_3 + 1)^3\right)}{\partial y_i \partial y_j}, \tag{3.4.17}$$

for all $i, j = 1, 2, 3$, uniformly in $y \in Y \times (-1, 0)$.

3.4.1 Electric case

From now on until the end of this section we will assume that there holds a uniform Gaffney inequality, that is $X_{\mathrm{N}}(\Omega_\epsilon) \subset H^1(\Omega_\epsilon)^3$ for all $\epsilon > 0$ and

$$\|u\|_{H^1(\Omega_\epsilon)^3} \leq C\|u\|_{X_{\mathrm{N}}(\Omega_\epsilon)} \quad \text{for all } u \in X_{\mathrm{N}}(\Omega_\epsilon),$$

where $C > 0$ is independent of $\epsilon > 0$.

Recall the *controvariant Piola pull-back transform* associated with the diffeomorphism Φ_ϵ, that is for any function $\varphi \in X_{\mathrm{N}}(\Omega)$

$$\mathcal{P}_\epsilon\varphi\colon = (\varphi \circ \Phi_\epsilon) D\Phi_\epsilon. \tag{3.4.18}$$

This defines a map

$$\mathcal{P}_\epsilon : X_{\mathrm{N}}(\Omega) \to X_{\mathrm{N}}(\Omega_\epsilon).$$

For all $\epsilon > 0$ let v_ϵ be the unique weak solution to

$$\begin{cases} \operatorname{curl}\operatorname{curl} v_\epsilon - \nabla \operatorname{div} v_\epsilon + v_\epsilon = f_\epsilon, & \text{in } \Omega\epsilon, \\ \operatorname{div} v_\epsilon = 0, & \text{in } \partial\Omega_\epsilon, \\ \nu \times v_\epsilon = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{3.4.19}$$

that is $v_\epsilon \in X_{\mathrm{N}}(\Omega_\epsilon)$ and

$$\int_{\Omega_\epsilon} v_\epsilon \cdot \psi \, dx + \int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \psi \, dx + \int_{\Omega_\epsilon} \operatorname{div}(v_\epsilon) \operatorname{div}(\psi) \, dx = \int_{\Omega_\epsilon} f_\epsilon \cdot \psi \, dx \tag{3.4.20}$$

for all $\psi \in X_{\mathrm{N}}(\Omega_\epsilon)$. Note that contrary to (3.2.6), here we have set $\tau = 1$ for the sake of simplicity.

Again, under the assumption that $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \le C$ for every $\epsilon > 0$, we have that the norms in $X_{\mathrm{N}}(\Omega_\epsilon)$ of the functions v_ϵ are bounded (cf. (3.2.8)). Moreover, since by hypothesis there holds a uniform Gaffney inequality, also the $H^1(\Omega_\epsilon)^3$-norms are bounded. Thus in particular the restrictions of the functions v_ϵ to Ω (which is a subset of Ω_ϵ for any $\epsilon > 0$) are bounded in $H^1(\Omega)^3$. Denote by $v \in H^1(\Omega)^3$ their weak limit in $H^1(\Omega)^3$.

Let now $\varphi \in X_{\mathrm{N}}(\Omega)$ be a fixed test function. Note that since Ω is convex (it is a parallelepiped in $\mathbb{R}^3$), there holds the Gaffney inequality and thus in particular $\varphi \in H^1(\Omega)^3$ (see, e.g., [107]). Since $\mathcal{P}_\epsilon\varphi \in X_{\mathrm{N}}(\Omega_\epsilon)$, then we have that

$$\int_{\Omega_\epsilon} v_\epsilon \cdot \mathcal{P}_\epsilon\varphi \, dx + \int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathcal{P}_\epsilon\varphi \, dx + \int_{\Omega_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon\varphi \, dx = \int_{\Omega_\epsilon} f_\epsilon \cdot \mathcal{P}_\epsilon\varphi \, dx. \tag{3.4.21}$$

Since the first two integrals of the left-hand side of (3.4.21) only involve first order derivatives of the function h_ϵ (cf. (3.1.2)), using Lemma 3.4.15 it is easy to see that

$$\int_{\Omega_\epsilon} v_\epsilon \cdot \mathcal{P}_\epsilon\varphi \, dx \xrightarrow[\epsilon\to 0]{} \int_\Omega v \cdot \varphi \, dx \quad \text{and} \quad \int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathcal{P}_\epsilon\varphi \, dx \xrightarrow[\epsilon\to 0]{} \int_\Omega \operatorname{curl} v \cdot \operatorname{curl} \varphi \, dx.$$

We now focus on the last term in the left hand-side of (3.4.21). We can write it in the following way

$$\begin{aligned} \int_{\Omega_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon\varphi \, dx = \int_{\Omega_\epsilon \setminus \Omega} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon\varphi \, dx + \int_{\Omega \setminus K_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon\varphi \, dx \\ + \int_{K_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon\varphi \, dx. \end{aligned} \tag{3.4.22}$$

As in (3.1.38), we compute

$$\begin{aligned} \operatorname{div} \mathcal{P}_\epsilon \varphi &= \sum_{m,n,i=1}^{3} \left(\frac{\partial \varphi_m}{\partial x_n}(\Phi_\epsilon) \frac{\partial (\Phi_\epsilon)_n}{\partial x_i} \frac{\partial (\Phi_\epsilon)_m}{\partial x_i} \right) + \sum_{m,i=1}^{3} \varphi_m(\Phi_\epsilon) \frac{\partial^2 (\Phi_\epsilon)_m}{\partial x_i^2} \\ &= \sum_{m,n,i=1}^{3} \underbrace{\left(\frac{\partial \varphi_m}{\partial x_n}(\Phi_\epsilon) \frac{\partial (\Phi_\epsilon)_n}{\partial x_i} \frac{\partial (\Phi_\epsilon)_m}{\partial x_i} \right)}_{\text{type A}} - \underbrace{\varphi_3(\Phi_\epsilon)\Delta h_\epsilon}_{\text{type B}}. \end{aligned} \tag{3.4.23}$$

We call the first addends of type A, the last addends of type B. We will first show that

$$\int_{\Omega_\epsilon \setminus \Omega} \operatorname{div} v_\epsilon \, \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx \xrightarrow[\epsilon \to 0]{} 0. \tag{3.4.24}$$

Indeed

$$\begin{aligned} \left(\int_{\Omega_\epsilon \setminus \Omega} \operatorname{div} v_\epsilon \, \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx \right)^2 &\le C \int_{\Omega_\epsilon \setminus \Omega} \left| \operatorname{div} \mathcal{P}_\epsilon \varphi \right|^2 \, dx \\ &\le C \sum_{m,n,i=1}^{3} \int_{\Omega_\epsilon \setminus \Omega} \left| \frac{\partial \varphi_m}{\partial x_n}(\Phi_\epsilon) \frac{\partial (\Phi_\epsilon)_n}{\partial x_i} \frac{\partial (\Phi_\epsilon)_m}{\partial x_i} \right|^2 + \int_{\Omega_\epsilon \setminus \Omega} \left| \varphi_3(\Phi_\epsilon)\Delta h_\epsilon \right|^2 dx \\ &\le C \sum_{m,n=1}^{3} \int_{\Omega_\epsilon \setminus \Omega} \left| \frac{\partial \varphi_m}{\partial x_n}(\Phi_\epsilon) \right|^2 dx + C\epsilon^{-1} \int_{\Omega_\epsilon \setminus \Omega} \left| \varphi_3(\Phi_\epsilon) \right|^2 dx \\ &\le C \sum_{m,n=1}^{3} \int_{\Phi_\epsilon(\Omega_\epsilon \setminus \Omega)} \left| \frac{\partial \varphi_m}{\partial x_n} \right|^2 dx + C\epsilon^{-1} \int_{\Phi_\epsilon(\Omega_\epsilon \setminus \Omega)} \left| \varphi_3 \right|^2 dx. \end{aligned}$$

Observe that

$$\begin{aligned} \Phi_\epsilon(\Omega_\epsilon \setminus \Omega) &= \left\{ (\bar{x}, x_3 - h_\epsilon(\bar{x}, x_3)) : \bar{x} \in W, 0 < x_3 < g_\epsilon(\bar{x}) \right\} \\ &\subset \left\{ (\bar{x}, z_3) : \bar{x} \in W, -g_\epsilon(\bar{x}) < z_3 < 0 \right\} \\ &\subset \left\{ (\bar{x}, z_3) : \bar{x} \in W, -\epsilon^{3/2} b_0 < z_3 < 0 \right\}, \end{aligned} \tag{3.4.25}$$

where $b_0 = \left\| b(\cdot) \right\|_{L^\infty(W)}$. Thus we have that $\left| \Phi_\epsilon(\Omega_\epsilon \setminus \Omega) \right| \to 0$ as ϵ goes to 0, hence

$$\int_{\Phi_\epsilon(\Omega_\epsilon \setminus \Omega)} \left| \frac{\partial \varphi_m}{\partial x_n} \right|^2 dx \xrightarrow[\epsilon \to 0]{} 0. \tag{3.4.26}$$

Moreover, by the one dimensional embedding estimate for Sobolev functions, we have that

$$\left\| \varphi(\bar{x}, \cdot) \right\|^2_{L^\infty(-1,0)} \le C \left\| \varphi(\bar{x}, \cdot) \right\|^2_{H^1(-1,0)},$$

for almost every $\bar{x} \in W$. Thus

$$\epsilon^{-1} \int_{\Phi_\epsilon(\Omega_\epsilon \setminus \Omega)} |\varphi_3|^2 \, dx \leq \epsilon^{-1} \epsilon^{3/2} b_0 \int_W \big\| \varphi(\bar{x}, \cdot) \big\|^2_{L^\infty(-1,0)} \, d\bar{x}$$
$$\leq C\epsilon^{1/2} \int_W \big\| \varphi(\bar{x}, \cdot) \big\|^2_{H^1(-1,0)} \, d\bar{x} \leq C\epsilon^{1/2} \|\varphi\|^2_{H^1(\Omega)^3} \xrightarrow[\epsilon \to 0]{} 0.$$

We now study the second term in the right-hand side of (3.4.22). To begin, recall that $Q_\epsilon = \widehat{W}_\epsilon \times (-\epsilon, 0)$ and write

$$\int_{\Omega \setminus K_\epsilon} \operatorname{div} v_\epsilon \, \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx = \int_{Q_\epsilon} \operatorname{div} v_\epsilon \, \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx + \int_{\Omega \setminus (K_\epsilon \cup Q_\epsilon)} \operatorname{div} v_\epsilon \, \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx. \tag{3.4.27}$$

Using the same argument as before, we can show that

$$\left(\int_{\Omega \setminus (K_\epsilon \cup Q_\epsilon)} \operatorname{div} v_\epsilon \, \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx \right)^2$$
$$\leq C \sum_{m,n=1}^{3} \int_{\Phi_\epsilon(\Omega \setminus (K_\epsilon \cup Q_\epsilon))} \left| \frac{\partial \varphi_m}{\partial x_n} \right|^2 dx + C\epsilon^{-1} \int_{\Phi_\epsilon(\Omega \setminus (K_\epsilon \cup Q_\epsilon))} |\varphi_3|^2 \, dx.$$

Since $\Phi_\epsilon(\Omega \setminus (K_\epsilon \cup Q_\epsilon)) \subset \Phi_\epsilon(\Omega \setminus K_\epsilon) \subset \Omega \setminus K_\epsilon$, it is clear that

$$\int_{\Phi_\epsilon(\Omega \setminus (K_\epsilon \cup Q_\epsilon))} \left| \frac{\partial \varphi_m}{\partial x_n} \right|^2 dx \xrightarrow[\epsilon \to 0]{} 0.$$

Again using the one dimensional embedding for Sobolev functions and observing that, in this case, the diameter of the set $\Phi_\epsilon(\Omega \setminus (K_\epsilon \cup Q_\epsilon))$ in the direction x_3 is $O(\epsilon)$, we deduce that

$$\epsilon^{-1} \int_{\Phi_\epsilon(\Omega \setminus (K_\epsilon \cup Q_\epsilon))} |\varphi_3|^2 \, dx \leq C \|\varphi\|^2_{H^1(\Omega \setminus (\widehat{W}_\epsilon \times (-1,0))} \xrightarrow[\epsilon \to 0]{} 0,$$

where the last limit is a consequence of the fact that $\left| \Omega \setminus (\widehat{W}_\epsilon \times (-1,0)) \right| \to 0$ as ϵ goes to 0. Thus we proved that

$$\int_{\Omega \setminus (K_\epsilon \cup Q_\epsilon)} \operatorname{div} v_\epsilon \, \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx \xrightarrow[\epsilon \to 0]{} 0. \tag{3.4.28}$$

We now study the first term in the right hand side of (3.4.27).

Lemma 3.4.29. *Let $\hat{v} \in L^2(W, w^{1,2}_{Per_Y}(Y \times (-\infty, 0)))$ be as in Lemma 3.4.7. Then*

$$\int_{Q_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx \xrightarrow[\epsilon \to 0]{} - \int_W \int_{Y \times (-1,0)} \operatorname{div}_y \hat{v} \; \Delta_y \left(b(\bar{y})(y_3 + 1)^3 \right) \, dy \; \varphi_3(\bar{x}, 0) \, d\bar{x}. \tag{3.4.30}$$

Proof. From (3.4.23) we write

$$\int_{Q_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx$$
$$= \sum_{m,n,i=1}^{3} \int_{Q_\epsilon} \operatorname{div} v_\epsilon \left(\frac{\partial \varphi_m}{\partial x_n} (\Phi_\epsilon) \frac{\partial (\Phi_\epsilon)_n}{\partial x_i} \frac{\partial (\Phi_\epsilon)_m}{\partial x_i} \right) dx - \int_{Q_\epsilon} \operatorname{div} v_\epsilon \, \varphi_3(\Phi_\epsilon) \, \Delta h_\epsilon \, dx.$$

The first order derivatives of Φ_ϵ are bounded in L^∞-norm, as one can see using Lemma 3.4.15. Moreover, observing that $\Phi_\epsilon(Q_\epsilon) \subset \Omega \setminus K_\epsilon = W \times (-\epsilon, 0)$, it is easy to see that

$$\left(\int_{Q_\epsilon} \operatorname{div} v_\epsilon \left(\frac{\partial \varphi_m}{\partial x_n} (\Phi_\epsilon) \frac{\partial (\Phi_\epsilon)_n}{\partial x_i} \frac{\partial (\Phi_\epsilon)_m}{\partial x_i} \right) dx \right)^2 \leq C \int_{\Phi_\epsilon(Q_\epsilon)} \left| \frac{\partial \varphi_m}{\partial x_n} \right|^2 dx \xrightarrow[\epsilon \to 0]{} 0.$$

Using Lemma 3.4.5 we get

$$\begin{aligned} - \int_{Q_\epsilon} \operatorname{div} v_\epsilon \; \varphi_3(\Phi_\epsilon) \; \Delta h_\epsilon \, dx = & -\epsilon \int_{W_\epsilon} \int_{Y \times (-1,0)} \widehat{\operatorname{div} v_\epsilon} \; \varphi_3(\widehat{\Phi_\epsilon(x)}) \; \widehat{\Delta h_\epsilon} \, dy dx \\ & - \epsilon \int_{W_\epsilon} \int_{Y \times (-1,0)} \frac{\operatorname{div}_y \widehat{v_\epsilon}}{\epsilon} \; \varphi_3(\widehat{\Phi_\epsilon(x)}) \; \widehat{\Delta h_\epsilon} \, dy dx \\ & - \int_{W_\epsilon} \int_{Y \times (-1,0)} \frac{\operatorname{div}_y \widehat{v_\epsilon}}{\epsilon^{\frac{1}{2}}} \; \varphi_3(\widehat{\Phi_\epsilon(x)}) \; \epsilon^{\frac{1}{2}} \widehat{\Delta h_\epsilon} \, dy dx. \end{aligned} \tag{3.4.31}$$

Applying Lemma 3.4.7 we have that there exists $\hat{v} \in L^2 \left(W, w^{1,2}_{Per_Y}(Y \times (-\infty, 0)\,) \right)$ such that in particular $\dfrac{\operatorname{div}_y \widehat{v_\epsilon}}{\epsilon^{\frac{1}{2}}} \rightharpoonup \operatorname{div}_y \hat{v}$ in $L^2(W \times Y \times (-1, 0))$ as ϵ goes to 0. Moreover, from (3.4.17) we have that $\epsilon^{\frac{1}{2}} \widehat{\Delta h_\epsilon}(\bar{x}, y) \xrightarrow[\epsilon \to 0]{} \Delta_y \left(b(\bar{y})(y_3 + 1)^3 \right)$ uniformly in $y \in Y \times (-1, 0)$. Again using Lemma 3.4.7 , we get

$$\varphi_3(\widehat{\Phi_\epsilon(x)}) \xrightarrow[\epsilon \to 0]{} \varphi_3(\bar{x}, 0)$$

in $L^2(W \times Y \times (-1, 0))$. Then it follows that

$$\int_{Q_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \varphi \, dx \xrightarrow[\epsilon \to 0]{} - \int_W \int_{Y \times (-1,0)} \operatorname{div}_y \hat{v} \; \Delta_y \left(b(\bar{y})(y_3 + 1)^3 \right) \, dy \; \varphi_3(\bar{x}, 0) \, d\bar{x}.$$

$\square$

We have thus proved the following theorem.

Theorem 3.4.32. *Let $f_\epsilon \in L^2(\Omega_\epsilon)^3, \epsilon > 0$ be a family of functions and let $f \in L^2(\Omega)^3$ be their weak limit in $L^2(\Omega)$. Let $v_\epsilon \in X_{\mathrm{N}}(\Omega_\epsilon) \cap H^1(\Omega_\epsilon)^3$ be the (unique) solution of (3.4.20) corrisponding to the Poisson datum f_ϵ for every $\epsilon > 0$. Then, possibly passing to a subsequence, there exist $v \in H^1(\Omega)^3$ and $\hat{v} \in L^2(W, w^{1,2}_{Per_Y}(Y \times (-\infty, 0)))$ such that $v_\epsilon|_\Omega \rightharpoonup v$ weakly in $H^1(\Omega)^3$, $v_\epsilon|_\Omega \to v$ strongly in $L^2(\Omega)^3$ and*

$$\int_\Omega (v \cdot \varphi + \operatorname{curl} v \cdot \operatorname{curl} \varphi + \operatorname{div} v \operatorname{div} \varphi)\, dx \\ - \int_W \int_{Y \times (-1,0)} \operatorname{div}_y \hat{v}\, \Delta_y \left(b(\bar{y})(y_3 + 1)^3 \right) dy \varphi_3(\bar{x}, 0)\, d\bar{x} = \int_\Omega f \cdot \varphi\, dx \tag{3.4.33}$$

for all $\varphi \in X_{\mathrm{N}}(\Omega)$.

We now want to characterize the function $\hat{v}$ via a weak microscopic problem. To do so we proceed as in Section 8.4 of [11].

Take $\eta \in \left(C^\infty(\overline{W} \times \overline{Y} \times (-\infty, 0]) \right)^3$ a function periodic in Y with support contained in $C \times \overline{Y} \times [d, 0]$ for some compact set $C \subset W$ and $d \in (-\infty, 0)$, and such that $\eta_1(\bar{x}, \bar{y}, 0) = \eta_2(\bar{x}, \bar{y}, 0) = 0$ for all $(\bar{x}, \bar{y}) \in W \times Y$. Set

$$\eta_\epsilon(x) := \epsilon^{1/2} \eta \left(\bar{x}, \frac{\bar{x}}{\epsilon}, \frac{x_3}{\epsilon} \right)$$

for all $\epsilon > 0$ and $x \in W \times (-\infty, 0]$. Observe that η_ϵ is admissible for sufficiently small ϵ, and its (controvariant) Piola transform can thus be used as test function in the weak formulation (3.4.20) of the problem under consideration. Thus, we write

$$\int_{\Omega_\epsilon} (v_\epsilon \cdot \mathcal{P}_\epsilon \eta_\epsilon + \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathcal{P}_\epsilon \eta_\epsilon + \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \eta_\epsilon)\, dx = \int_{\Omega_\epsilon} f_\epsilon \cdot \mathcal{P}_\epsilon \eta_\epsilon\, dx. \tag{3.4.34}$$

One can see that

$$\int_{\Omega_\epsilon} v_\epsilon \cdot \mathcal{P}_\epsilon \eta_\epsilon\, dx \xrightarrow[\epsilon \to 0]{} 0, \int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathcal{P}_\epsilon \eta_\epsilon\, dx \xrightarrow[\epsilon \to 0]{} 0 \text{ and } \int_{\Omega_\epsilon} f_\epsilon \cdot \mathcal{P}_\epsilon \eta_\epsilon\, dx \xrightarrow[\epsilon \to 0]{} 0. \tag{3.4.35}$$

We now consider the term with the divergence and write

$$\int_{\Omega_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \eta_\epsilon\, dx = \int_{\Omega_\epsilon \setminus \Omega} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \eta_\epsilon\, dx + \int_\Omega \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \eta_\epsilon\, dx. \tag{3.4.36}$$

It is easy to see that

$$\int_{\Omega_\epsilon\setminus\Omega} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \eta_\epsilon \, dx \xrightarrow[\epsilon\to 0]{} 0. \tag{3.4.37}$$

Additionally, the next lemma gives us the limit of the second term in the right-hand side of (3.4.36).

Lemma 3.4.38. *There holds the following limit*

$$\int_{\Omega} \operatorname{div} v_\epsilon \operatorname{div} \mathcal{P}_\epsilon \eta_\epsilon \, dx \xrightarrow[\epsilon\to 0]{} \int_{W\times Y\times(-\infty,0)} \operatorname{div}_y \hat{v}(\bar{x},y) \operatorname{div}_y \eta(\bar{x},y)\, d\bar{x}dy. \tag{3.4.39}$$

Proof. Due to the Y-periodicity of η we have that

$$\widehat{\mathcal{P}_\epsilon \eta_\epsilon}(\bar{x},y) = \epsilon^{1/2}\eta\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \bar{y}, y_3 - \epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right)\widehat{D\Phi_\epsilon}(\bar{x},y)$$

Note that since

$$h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right) = \begin{cases} \frac{\epsilon^{3/2}b(\bar{y})(y_3+1)^3}{(\epsilon^{1/2}b(\bar{y})+1)^3}, & \text{if } -1 \le y_3 < 0, \\ 0, & \text{if } -1/\epsilon < y_3 < -1, \end{cases}$$

we have that

$$D_y^\beta\left(\epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right) \xrightarrow[\epsilon\to 0]{} 0 \text{ uniformly on } W\times Y\times(-\infty,0] \tag{3.4.40}$$

for all $|\beta| \le 2$. Observe that

$$\widehat{D\Phi_\epsilon}(\bar{x},y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\epsilon^{-1}\frac{\partial\widehat{h_\epsilon}}{\partial y_1}(\bar{x},y) & -\epsilon^{-1}\frac{\partial\widehat{h_\epsilon}}{\partial y_2}(\bar{x},y) & 1-\epsilon^{-1}\frac{\partial\widehat{h_\epsilon}}{\partial y_3}(\bar{x},y) \end{pmatrix}.$$

Hence, from (3.4.40) we deduce that

$$\widehat{D\Phi_\epsilon}(\bar{x},y) \xrightarrow[\epsilon\to 0]{} \mathbb{I}_3 \text{ and } \frac{\partial}{\partial y_i}\widehat{D\Phi_\epsilon}(\bar{x},y) \xrightarrow[\epsilon\to 0]{} \mathbb{O}_3 \tag{3.4.41}$$

uniformly in $W\times Y\times(-\infty,0]$, for all $i = 1,2,3$. Since η is smooth and compactly supported, it is Lipschitz continuous together with its derivatives, and thus

$$\left\|(D^\gamma\eta)\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \bar{y}, y_3 - \epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right) - D^\gamma\eta(\bar{x},y)\right\|_{L^2(\widehat{W_\epsilon}\times Y\times(-\infty,0))} \xrightarrow[\epsilon\to 0]{} 0 \tag{3.4.42}$$

for any $|\gamma| \leq 1$ (see also (8.51) in [11]). Consider

$$\begin{aligned}
&\frac{\partial}{\partial y_i}\widehat{\mathcal{P}_\epsilon \eta_\epsilon}(\bar{x}, y) \\
&\quad = \epsilon^{1/2}\frac{\partial}{\partial y_i}\left(\eta\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \bar{y}, y_3 - \epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right)\right)\widehat{D\Phi_\epsilon}(\bar{x}, y) \\
&\qquad + \epsilon^{1/2}\eta\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \bar{y}, y_3 - \epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right)\frac{\partial}{\partial y_i}\widehat{D\Phi_\epsilon}(\bar{x}, y).
\end{aligned}$$

Then, using (3.4.41) and (3.4.42) and the chain rule, we can see that for all $i = 1, 2, 3$

$$\epsilon^{-1/2}\frac{\partial}{\partial y_i}\widehat{\mathcal{P}_\epsilon \eta_\epsilon}(\bar{x}, y) \underset{\epsilon\to 0}{\longrightarrow} \frac{\partial \eta}{\partial y_i}(\bar{x}, y) \tag{3.4.43}$$

in $L^2(W \times Y \times (-\infty, 0))$. At this point, using Lemma 3.4.5, Lemma 3.4.7 and (3.4.43) we conclude that

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{\widehat{W}_\epsilon\times(-1,0)} \operatorname{div} v_\epsilon \, \operatorname{div} \mathcal{P}_\epsilon\eta_\epsilon \, dx &= \lim_{\epsilon\to 0}\int_{\widehat{W}_\epsilon\times Y\times(-1/\epsilon,0)} \frac{\operatorname{div}_y \hat{v}_\epsilon}{\epsilon^{1/2}}\,\frac{\operatorname{div}_y \widehat{\mathcal{P}_\epsilon\eta_\epsilon}}{\epsilon^{1/2}}\, d\bar{x}dy \\
&= \int_{W\times Y\times(-\infty,0)} \operatorname{div}_y \hat{v}(\bar{x}, y)\, \operatorname{div}_y \eta(\bar{x}, y)\, d\bar{x}dy.
\end{aligned}$$

Finally, we observe that the function η_ϵ is zero outside $\widehat{W}_\epsilon \times (-1, 0))$ for ϵ small enough, by the definition of $\widehat{W}_\epsilon$ and the fact that $\eta_\epsilon(\bar{x}, x_3) = \epsilon^{1/2}\eta\left(\bar{x}, \frac{\bar{x}}{\epsilon}, \frac{x_3}{\epsilon}\right) = 0$ if $\bar{x} \notin C$, C compact subset of W. Hence trivially

$$\int_{\Omega\setminus(\widehat{W}_\epsilon\times(-1,0))} \operatorname{div} v_\epsilon \, \operatorname{div}(\mathcal{P}_\epsilon\eta_\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0.$$

□

Theorem 3.4.44. *Let $\hat{v} \in L^2(W, w^{1,2}_{Per_Y}(Y \times (-\infty, 0)))$ be the function from Theorem 3.4.32. Then*

$$\int_{W\times Y\times(-\infty,0)} \operatorname{div}_y \hat{v}(\bar{x}, y)\;\; \operatorname{div}_y \eta(\bar{x}, y) = 0, \tag{3.4.45}$$

for all $\eta \in L^2(W, x_{Per_Y}(Y \times (-\infty, 0)))$ such that $\eta_1(\bar{x}, \bar{y}, 0) = \eta_2(\bar{x}, \bar{y}, 0) = 0$ for all $(\bar{x}, \bar{y}) \in W \times Y$. Moreover, we have that

$$\nu \times \big(\hat{v}(\bar{x}, \bar{y}, 0) \times \nu\big) = \big(-\nabla b(\bar{y})\, v_3(\bar{x}, 0), 0\big) \tag{3.4.46}$$

for a.e. $(\bar{x}, \bar{y}) \in W \times Y$.

Proof. In the case the test function η is smooth, one can take the limit in equation (3.4.34) and combine (3.4.34)-(3.4.39). Then reasoning with an approximation argument one can recover the general case for a generic function in $L^2(W, x_{Per_Y}(Y \times (-\infty, 0)))$ satisfying the same boundary conditions.

Now we turn to prove formula (3.4.46). We have that $\nu_\epsilon \times v_\epsilon = 0$ on Γ_ϵ for all $\epsilon > 0$. Writing the cross product explicitly we get (for the sake of simplicity in the next formulas we put the index denoting the vector component at the top)

$$
\begin{aligned}
&\bullet\quad \nu_\epsilon^2 v_\epsilon^3 - \nu_\epsilon^3 v_\epsilon^2 = 0 \iff (0, v_\epsilon^3, -v_\epsilon^2) \perp \nu_\epsilon;\\
&\bullet\quad \nu_\epsilon^3 v_\epsilon^1 - \nu_\epsilon^1 v_\epsilon^3 = 0 \iff (-v_\epsilon^3, 0, v_\epsilon^1) \perp \nu_\epsilon;\\
&\bullet\quad \nu_\epsilon^1 v_\epsilon^2 - \nu_\epsilon^2 v_\epsilon^1 = 0 \iff (v_\epsilon^2, -v_\epsilon^1, 0) \perp \nu_\epsilon.
\end{aligned}
\tag{3.4.47}
$$

Note that this analysis is justified since by hypothesis for every $\epsilon > 0$ the vector field v_ϵ belongs to $H^1(\Omega_\epsilon)^3$ and thus it can be traced as a function in $L^2(\partial\Omega_\epsilon)^3$ defined on the boundary. Let $v \in H^1(\Omega)^3$ be the function of Theorem 3.4.32, which is the weak limit in $H^1(\Omega)^3$ of the functions v_ϵ. Thus, in particular, we have that $(v_\epsilon)_i|_\Omega \rightharpoonup v_i$ in $H^1(\Omega)$ for $i = 1, 2, 3$, and so we can apply Lemma 3.4.11 to each of the three vector fields in (3.4.47) to get that

$$
v_1(\bar{x}, 0) = v_2(\bar{x}, 0) = 0 \quad \text{a.e. in } W, \tag{3.4.48}
$$

thus $\nu \times v = 0$ on Γ, and

- $\hat{v}_2(\bar{x}, \bar{y}, 0) = -\nabla b(\bar{y}) \cdot (0, v_3)(\bar{x}, 0) = -\dfrac{\partial b}{\partial y_2}(\bar{y}) v_3(\bar{x}, 0),$
- $\hat{v}_1(\bar{x}, \bar{y}, 0) = \nabla b(\bar{y}) \cdot (-v_3, 0)(\bar{x}, 0) = -\dfrac{\partial b}{\partial y_1}(\bar{y}) v_3(\bar{x}, 0),$

for a.e. $(\bar{x}, \bar{y}) \in W \times Y$. Therefore

$$
\big(\hat{v}_1(\bar{x}, \bar{y}, 0), \hat{v}_2(\bar{x}, \bar{y}, 0)\big) = -\nabla b(\bar{y})\, v_3(\bar{x}, 0) \quad \text{a.e. on } W \times Y.
$$

Observing that $\nu = (0, 0, 1)$ on Γ, we recover that

$$
\nu \times \big(\hat{v}(\bar{x}, \bar{y}, 0) \times \nu\big) = \big(-\nabla b(\bar{y})\, v_3(\bar{x}, 0), 0\big) \quad \text{a.e. on } W \times Y.
$$

□

We also have the following

Lemma 3.4.49. *There exists a solution $\tilde v \in L^2(W, x_{Per_Y}(Y \times (-\infty, 0)))$ to the following problem:*

$$\int_{W\times Y\times(-\infty,0)} \operatorname{div}_y \tilde v(\bar x, y) \ \operatorname{div}_y \eta(\bar x, y) = 0, \tag{3.4.50}$$

for all $\eta \in L^2(W, x_{Per_Y}(Y \times (-\infty, 0)))$ with $\eta_1(\bar x, \bar y, 0) = \eta_2(\bar x, \bar y, 0) = 0$ for all $(\bar x, \bar y) \in W \times Y$, such that

$$\nu \times \big(\tilde v(\bar x, \bar y, 0) \times \nu\big) = \big(-\nabla b(\bar y)\, v_3(\bar x, 0), 0\big)$$

on $W \times Y$.

Proof. Consider a function $u \in w^{1,2}_{Per_Y}(Y \times (-\infty, 0))$ such that $u(\bar y, 0) = -b(\bar y)$ on Y and

$$\int_{Y\times(-\infty,0)} \nabla u \cdot \nabla \varphi = 0 \tag{3.4.51}$$

for all $\varphi \in w^{1,2}_{Per_Y}(Y \times (-\infty, 0))$ such that $\varphi(\bar y, 0) = 0$ for all $\bar y \in Y$. Its existence can be proved using the same argument in [11, Lemma 8.65]. In particular, testing against functions $\varphi \in C^\infty_c(Y \times (-\infty, 0))$ one can see that u is harmonic. Then the gradient ∇u belongs to $X_{Per_Y,loc}(Y \times (-\infty, 0))$ and both its curl and divergence vanish. Considering

$$\tilde v(\bar x, y) := \nabla u(y)\, v_3(\bar x, 0),$$

we observe that $\tilde v \in L^2(W, x_{Per_Y}(Y \times (-\infty, 0)))$ is such that

$$\operatorname{div}_y \tilde v(\bar x, y) = \Delta u(y)\, v_3(\bar x, 0) = 0$$

in $Y\times(-\infty, 0)$. Hence $\tilde v$ satisfies equation (3.4.50) trivially. Moreover, the boundary condition is also trivially satisfied. □

Remark 3.4.52. Let us now consider the difference $w := \hat v - \tilde v$, with $\hat v$ being the function from Theorem 3.4.32 and $\tilde v$ the function from Lemma 3.4.49. It is obvious that $w \in L^2(W, x_{Per_Y}(Y \times (-\infty, 0)))$ satisfies (3.4.50), and moreover it can be tested against itself since it has null tangential trace on $W \times Y \times \{0\}$. We immediately deduce that $\operatorname{div}_y w = 0$ in $W \times Y \times (-\infty, 0)$. Thus

$$\hat v(\bar x, y) = \nabla u(y)\, v_3(\bar x, 0) + \operatorname{curl}_y g(\bar x, y)$$

for some function g, whose regularity is such that it guarantees the above sum to be in $L^2(W, w^{1,2}_{Per_Y}(Y \times (-\infty, 0)))$, and with $u \in w^{1,2}_{Per_Y}(Y \times (-\infty, 0))$ the function satisfying (3.4.51).

Lemma 3.4.53. *Let* $u \in w^{1,2}_{Per_Y}(Y \times (-\infty, 0))$ *be the solution of* (3.4.51)*, and* $v, \hat{v}$ *be as in Theorem 3.4.32. Then* $\hat{v}(\bar{x}, y) = \nabla u(y)\, v_3(\bar{x}, 0) + \mathrm{curl}_y\, g(\bar{x}, y)$ *for almost all* $(\bar{x}, y) \in W \times Y \times (-\infty, 0)$*, with* g *such that* $\mathrm{curl}_y\, g \in L^2(W, x_{Per_Y}(Y \times (-\infty, 0)))$*.*

Moreover, the second integral in the left-hand side of (3.4.33) *vanishes, that is*

$$-\int_W \int_{Y\times(-1,0)} \mathrm{div}_y\, \hat{v}\, \Delta_y \left(b(\bar{y})(y_3+1)^3\right) dy\, \varphi_3(\bar{x}, 0)\, d\bar{x} = 0. \tag{3.4.54}$$

Proof. The explicit form of $\hat{v}$ is explained by remark (3.4.52), and thus the first part of the theorem is proved.

Using the fact that $\hat{v}(\bar{x}, y) = -\nabla u(y)\, v_3(\bar{x}, 0) + \mathrm{curl}_y\, g(\bar{x}, y)$ one can see that

$$\begin{aligned} &-\int_W \int_{Y\times]-1,0[} \mathrm{div}_y\, \hat{v}(\bar{x}, y)\ \Delta_y \left(b(\bar{y})(y_3+1)^3\right) dy\, \varphi_3(\bar{x}, 0)\, d\bar{x} \\ &= \int_W \int_{Y\times]-1,0[} \Delta u(y)\ \Delta_y \left(b(\bar{y})(y_3+1)^3\right) dy\, v_3(\bar{x}, 0)\, \varphi_3(\bar{x}, 0) d\bar{x} = 0, \end{aligned} \tag{3.4.55}$$

since u is harmonic. □

Before stating the main theorem, we make the following remark.

Remark 3.4.56. Formula (3.4.48) effectively state that the limit function $v \in H^1(\Omega)^3$ of Theorem 3.4.32 is such that $\nu \times v = 0$ on the upper profile Γ of Ω.

Theorem 3.4.57. *Let* $f_\epsilon \in L^2(\Omega_\epsilon)^3, \epsilon > 0$ *be a family of functions and let* $f \in L^2(\Omega)^3$ *be their weak limit in* $L^2(\Omega)$*. Let* $v_\epsilon \in X_{\mathrm{N}}(\Omega_\epsilon) \cap H^1(\Omega_\epsilon)^3$ *be the (unique) solution of* (3.4.20) *corrisponding to the Poisson datum* f_ϵ *for every* $\epsilon > 0$*. Then, possibly passing to a subsequence, there exist* $v \in H^1(\Omega)^3$ *with* $\nu \times v = 0$ *on* Γ *such that* $v_\epsilon|_\Omega \rightharpoonup v$ *weakly in* $H^1(\Omega)^3$*,* $v_\epsilon|_\Omega \to v$ *strongly in* $L^2(\Omega)^3$ *and*

$$\int_\Omega (v \cdot \varphi + \mathrm{curl}\, v \cdot \mathrm{curl}\, \varphi + \mathrm{div}\, v\ \mathrm{div}\, \varphi)\, dx = \int_\Omega f \cdot \varphi\, dx \tag{3.4.58}$$

for all $\varphi \in X_{\mathrm{N}}(\Omega)$*.*

Proof. Use Theorem 3.4.32 and Lemma 3.4.53. □

3.4.2 Magnetic case

In this final section of the chapter, for the sake of completeness, we study the magnetic case for the threshold critical case of $\alpha = 3/2$, which has an interest on its own.

As we will see in Theorem 3.4.109, even the magnetic case seems to be stable at the threshold of $\alpha = 3/2$, analogously to Theorem 3.4.57.

We will suppose the validity uniform Gaffney (magnetic) inequality, that is $X_{\mathrm{T}}(\Omega_\epsilon) \subset H^1(\Omega_\epsilon)^3$ for all $\epsilon > 0$, and there exists $C > 0$ (independent of ϵ) such that

$$\|u\|_{H^1(\Omega_\epsilon)^3} \leq C\|u\|_{X_{\mathrm{T}}(\Omega_\epsilon)} \quad \text{for all } u \in X_{\mathrm{T}}(\Omega_\epsilon). \tag{3.4.59}$$

Let $\{f_\epsilon\}_{\epsilon>0} \in L^2(\Omega_\epsilon)^3$ be a family of data such that $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \leq C$ for all $\epsilon > 0$, for some $C > 0$, and let $f \in L^2(\Omega)^3$ its weak limit in $L^2(\Omega)^3$, namely $f_\epsilon|_\Omega \rightharpoonup f$ weakly in $L^2(\Omega)^3$ as $\epsilon \to 0$. The *magnetic Maxwell Poisson problem* reads

$$\begin{cases} \operatorname{curl}\operatorname{curl} v_\epsilon = f_\epsilon, & \text{in } \Omega\epsilon, \\ \operatorname{div} v_\epsilon = 0, & \text{in } \Omega_\epsilon, \\ \nu_\epsilon \cdot v_\epsilon = 0, & \text{on } \partial\Omega_\epsilon, \\ \nu_\epsilon \times \operatorname{curl} v_\epsilon = 0, & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{3.4.60}$$

Its weak formulation is

$$\int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl}\psi \, dx = \int_{\Omega_\epsilon} f_\epsilon \cdot \psi \, dx \quad \text{for all } \psi \in X_{\mathrm{T}}(\operatorname{div} 0, \Omega_\epsilon) \tag{3.4.61}$$

in the unknown $v_\epsilon \in X_{\mathrm{T}}(\operatorname{div} 0, \Omega_\epsilon)$.

We will not study problem (3.4.61) but instead the following:

$$\begin{aligned} \int_{\Omega_\epsilon} v_\epsilon \cdot \psi \, dx + \int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl}\psi \, dx + \int_{\Omega_\epsilon} \operatorname{div} v_\epsilon \operatorname{div}\psi \, dx& \\ = \int_{\Omega_\epsilon} f_\epsilon \cdot \psi \, dx \quad \text{for all } \psi \in X_{\mathrm{T}}(\Omega_\epsilon),& \end{aligned} \tag{3.4.62}$$

in the unknown $v_\epsilon \in X_{\mathrm{T}}(\Omega_\epsilon)$.

In the magnetic case the appropriate push-back for the vector fields is not the controvariant Piola transform (cf. (3.4.18)), as it does not preserve the magnetic boundary condition of being tangent to the boundary. Instead, we consider the so-called *covariant Piola transform* $\mathscr{P}_\epsilon : X_{\mathrm{T}}(\Omega) \to X_{\mathrm{T}}(\Omega_\epsilon)$, associated with the diffeomorphism Φ_ϵ, defined by

$$\mathscr{P}_\epsilon\varphi := (\varphi \circ \Phi_\epsilon)(D\Phi_\epsilon)^{-T} \det D\Phi_\epsilon = (\varphi \circ \Phi_\epsilon)\mathrm{cof}_{D\Phi_\epsilon},$$

where $\mathrm{cof}_{D\Phi_\epsilon}$ denotes the cofactor matrix of $D\Phi_\epsilon$.

We now fix a generic vector field $\varphi \in X_{\mathrm{T}}(\Omega)$. Observe that by the Gaffney inequality, which holds also in Ω since it is convex (see e.g. [107]), we have that $\varphi \in H^1(\Omega)^3$. Since $\mathscr{P}_\epsilon\varphi \in X_{\mathrm{T}}(\Omega_\epsilon)$ we have that

$$\begin{aligned} \int_{\Omega_\epsilon} v_\epsilon \cdot \mathscr{P}_\epsilon\varphi \, dx + \int_{\Omega_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi \, dx + \int_{\Omega_\epsilon} \operatorname{div} v_\epsilon \operatorname{div} \mathscr{P}_\epsilon\varphi \, dx& \\ = \int_{\Omega_\epsilon} f_\epsilon \cdot \mathscr{P}_\epsilon\varphi \, dx \quad \text{for all } \psi \in X_{\mathrm{T}}(\Omega_\epsilon).& \end{aligned}$$

Note that in the following until the end of the chapter, vectors will always be understood to be row vectors. If $\mathbf{v}$ is a vector in $\mathbb{R}^3$ we denote its coordinate entries by $\mathbf{v}^{(i)}$, thus writing $\mathbf{v} = (\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)})$. In the case the vector is made of a more complex expression we will use the subscript notation together with the use of round brackets, e.g. $(\mathbf{v} \times \mathbf{w})_i$ with $\mathbf{v}, \mathbf{w}$ vectors in $\mathbb{R}^3$ or $(\mathbf{v}\, M)_i$ with M a 3×3 matrix.

Using the Levi-Civita symbols (and the Einstein notation for sums), if $F = (F^{(1)}, F^{(2)}, F^{(3)})$ is a vector field then $(\operatorname{curl} F)_k = \frac{\partial F^{(j)}}{\partial x_i}\, \xi_{ijk}$, and that if M is a 3×3 matrix then

$$(\operatorname{cof}_M)_{sj} = \frac{1}{2}\, \xi_{ss_2s_3}\, \xi_{jj_2j_3}\, M_{s_2j_2} M_{s_3j_3}.$$

Using the chain rule for the derivatives, we can write the curl of $\mathscr{P}_\epsilon\varphi$ explicitly in the following way

$$\begin{aligned}(\operatorname{curl} \mathscr{P}_\epsilon\varphi)^{(k)} =& \frac{1}{2}\xi_{ijk}\xi_{ss_2s_3}\xi_{jj_2j_3} \frac{\partial \varphi^{(s)}}{\partial x_h} \circ \Phi_\epsilon \frac{\partial \Phi_\epsilon^{(h)}}{\partial x_i} \frac{\partial \Phi_\epsilon^{(s_2)}}{\partial x_{j_2}} \frac{\partial \Phi_\epsilon^{(s_3)}}{\partial x_{j_3}} \\ &+ \xi_{ijk}\xi_{ss_2s_3}\xi_{jj_2j_3}\, \varphi^{(s)} \circ \Phi_\epsilon \frac{\partial^2 \Phi_\epsilon^{(s_2)}}{\partial x_i \partial x_{j_2}} \frac{\partial \Phi_\epsilon^{(s_3)}}{\partial x_{j_3}}.\end{aligned} \tag{3.4.63}$$

As in the electric case, since $\|f_\epsilon\|_{L^2(\Omega_\epsilon)^3} \leq C$ for all $\epsilon > 0$, using a similar argument to the one used to prove (3.2.8) and the validity of the uniform inequality (3.4.59), we can see that the norms in $H^1(\Omega_\epsilon)^3$ of the functions v_ϵ are bounded for every $\epsilon > 0$. Hence, let $v \in H^1(\Omega)^3$ be the weak limit in $H^1(\Omega)^3$ of the functions $v_\epsilon|_\Omega$. Observe that thanks to Lemma 3.4.11 the function v is such that $v \cdot \nu = 0$ on the upper profile Γ.

Lemma 3.4.64. *We have the following limits:*

$$\int_{\Omega_\epsilon} f_\epsilon \cdot \mathscr{P}_\epsilon\varphi\, dx \underset{\epsilon\to 0}{\longrightarrow} \int_\Omega f \cdot \varphi\, dx; \tag{3.4.65}$$

$$\int_{\Omega_\epsilon} \operatorname{div} v_\epsilon \cdot \operatorname{div} \mathscr{P}_\epsilon\varphi\, dx \underset{\epsilon\to 0}{\longrightarrow} \int_\Omega \operatorname{div} v \cdot \operatorname{div} \varphi\, dx; \tag{3.4.66}$$

$$\int_{K_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi\, dx \underset{\epsilon\to 0}{\longrightarrow} \int_\Omega \operatorname{curl} v \cdot \operatorname{curl} \varphi\, dx, \tag{3.4.67}$$

$$\int_{\Omega_\epsilon\setminus\Omega} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi\, dx \underset{\epsilon\to 0}{\longrightarrow} 0. \tag{3.4.68}$$

Proof. The proof of limits (3.4.65) and (3.4.67) is easy and left to the reader. To see limit (3.4.66) one needs to observe that the divergence is transformed via the covariant Piola transform in the following way

$$\operatorname{div} \mathscr{P}_\epsilon\varphi = (\operatorname{div} \varphi) \circ \Phi_\epsilon\, \det D\Phi_\epsilon,$$

which is an expression that involves only first order derivatives of Φ_ϵ, and use Lemma 3.4.15. We will only prove limit (3.4.68) in details. Proceeding similarly to the electric case, by Hölder's inequality, formula (3.4.63) and Lemma 3.4.15 we have that

$$\begin{aligned}
&\left(\int_{\Omega_\epsilon\setminus\Omega} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi \, dx\right)^2 \le \|\operatorname{curl} v_\epsilon\|^2_{L^2(\Omega_\epsilon\setminus\Omega)^3} \int_{\Omega_\epsilon\setminus\Omega} |\operatorname{curl} \mathscr{P}_\epsilon\varphi|^2 \, dx \\
&\le C \int_{\Omega_\epsilon\setminus\Omega} \left| \frac{1}{2}\xi_{ijk}\xi_{ss_2s_3}\xi_{jj_2j_3} \frac{\partial\varphi^{(s)}}{\partial x_h} \circ \Phi_\epsilon \frac{\partial\Phi_\epsilon^{(h)}}{\partial x_i} \frac{\partial\Phi_\epsilon^{(s_2)}}{\partial x_{j_2}} \frac{\partial\Phi_\epsilon^{(s_3)}}{\partial x_{j_3}} \right|^2 dx \\
&\quad + C \int_{\Omega_\epsilon\setminus\Omega} \left| \xi_{ijk}\xi_{ss_2s_3}\xi_{jj_2j_3} \, \varphi^{(s)} \circ \Phi_\epsilon \frac{\partial^2\Phi_\epsilon^{(s_2)}}{\partial x_i \partial x_{j_2}} \frac{\partial\Phi_\epsilon^{(s_3)}}{\partial x_{j_3}} \right|^2 dx \\
&\le C \int_{\Omega_\epsilon\setminus\Omega} |D\varphi \circ \Phi_\epsilon|^2 \, dx + C\epsilon^{-1} \int_{\Omega_\epsilon\setminus\Omega} |\varphi \circ \Phi_\epsilon|^2 \, dx \\
&\le C \int_{\Phi_\epsilon(\Omega_\epsilon\setminus\Omega)} |D\varphi|^2 \, dx + C\epsilon^{-1} \int_{\Phi_\epsilon(\Omega_\epsilon\setminus\Omega)} |\varphi|^2 \, dx.
\end{aligned} \tag{3.4.69}$$

Observe that

$$\begin{aligned}
\Phi_\epsilon(\Omega_\epsilon \setminus \Omega) &= \left\{(\bar{x}, x_3 - h_\epsilon(\bar{x}, x_3)) : \bar{x} \in W, 0 < x_3 < g_\epsilon(\bar{x})\right\} \\
&\subset \left\{(\bar{x}, z_3) : \bar{x} \in W, -g_\epsilon(\bar{x}) < z_3 < 0\right\} \\
&\subset \left\{(\bar{x}, z_3) : \bar{x} \in W, -\epsilon^{3/2} b_0 < z_3 < 0\right\},
\end{aligned}$$

where $b_0 = \big\|b(\cdot)\big\|_{L^\infty(W)}$. Thus we have that $\big|\Phi_\epsilon(\Omega_\epsilon \setminus \Omega)\big| \to 0$ as ϵ goes to 0, hence

$$\int_{\Phi_\epsilon(\Omega_\epsilon\setminus\Omega)} |D\varphi|^2 \, dx \xrightarrow[\epsilon\to 0]{} 0. \tag{3.4.70}$$

Moreover, by the one dimensional embedding estimate for Sobolev functions, we have that

$$\big\|\varphi(\bar{x}, \cdot)\big\|^2_{L^\infty(-1,0)} \le C \big\|\varphi(\bar{x}, \cdot)\big\|^2_{H^1(-1,0)},$$

for almost every $\bar{x} \in W$. Thus

$$\begin{aligned}
C\epsilon^{-1} \int_{\Phi_\epsilon(\Omega_\epsilon\setminus\Omega)} |\varphi|^2 \, dx &\le C\epsilon^{-1}\epsilon^{3/2} b_0 \int_W \big\|\varphi(\bar{x}, \cdot)\big\|^2_{L^\infty(-1,0)} \, d\bar{x} \\
&\le C\epsilon^{1/2} \int_W \big\|\varphi(\bar{x}, \cdot)\big\|^2_{H^1(-1,0)} \, d\bar{x} \le C\epsilon^{1/2} \|\varphi\|^2_{H^1(\Omega)^3} \xrightarrow[\epsilon\to 0]{} 0,
\end{aligned}$$

and the proof is concluded. □

Recall that $Q_\epsilon := \widehat{W}_\epsilon \times (-\epsilon, 0)$. We study the following integrals

$$\int_{\Omega\setminus K_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi\, dx = \int_{Q_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi\, dx + \int_{\Omega\setminus(K_\epsilon\cup Q_\epsilon)} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi\, dx. \tag{3.4.71}$$

By the same argument used in (3.4.69), we get

$$\left(\int_{\Omega\setminus(K_\epsilon\cup Q_\epsilon)} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi\, dx\right)^2 \leq C \int_{\Phi_\epsilon(\Omega\setminus(K_\epsilon\cup Q_\epsilon))} |D\varphi|^2\, dx + C\epsilon^{-1} \int_{\Phi_\epsilon(\Omega\setminus(K_\epsilon\cup Q_\epsilon))} |\varphi|^2\, dx.$$

Since $\Phi_\epsilon(\Omega \setminus (K_\epsilon \cup Q_\epsilon)) \subset \Phi_\epsilon(\Omega \setminus K_\epsilon) \subset \Omega \setminus K_\epsilon$, it is clear that

$$\int_{\Phi_\epsilon(\Omega\setminus(K_\epsilon\cup Q_\epsilon))} |D\varphi|^2\, dx \xrightarrow[\epsilon\to 0]{} 0.$$

Again using the one dimensional embedding for Sobolev functions and observing that in this case the diameter of the set $\Phi_\epsilon(\Omega \setminus (K_\epsilon \cup Q_\epsilon))$ in the direction x_3 is less or equal than ϵ, we have that

$$\epsilon^{-1} \int_{\Phi_\epsilon(\Omega\setminus(K_\epsilon\cup Q_\epsilon))} |\varphi|^2\, dx \leq C\|\varphi\|^2_{H^1(\Omega\setminus(\widehat{W}_\epsilon\times(-1,0))} \xrightarrow[\epsilon\to 0]{} 0,$$

where the last limit is a consequence of the fact that $\left|\Omega \setminus (\widehat{W}_\epsilon \times (-1, 0))\right| \to 0$ as ϵ goes to 0. Thus we proved that

$$\int_{\Omega\setminus(K_\epsilon\cup Q_\epsilon)} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi\, dx \xrightarrow[\epsilon\to 0]{} 0. \tag{3.4.72}$$

We now study

$$\int_{Q_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon\varphi\, dx.$$

The terms of the first type in the sum of the right-hand side of (3.4.63) go to 0 as ϵ goes to 0, since they involve only derivatives of Φ_ϵ of the first order. Thus we focus on the terms of the second type. Before doing so, we present the following lemma, which sums up Lemma 3.4.7 and Lemma 3.4.11 applied to the magnetic case. Recall also that we are under the assumption that the uniform inequality (3.4.59) is valid.

Lemma 3.4.73. *For all $\epsilon > 0$ let $v_\epsilon \in X_{\mathrm{T}}(\Omega_\epsilon)$ be the (unique) solution of (3.4.62), and suppose that $v_\epsilon|_\Omega \underset{\epsilon\to 0}{\rightharpoonup} v$ weakly in $H^1(\Omega)^3$. Then the third component $v^{(3)}$ of v vanishes on Γ. Moreover, if by V_ϵ we denote the following vector field*

$$V_\epsilon(\bar{x}, y) = \hat{v}_\epsilon(\bar{x}, y) - \int_Y \hat{v}_\epsilon(\bar{x}, \bar{y}, 0) d\bar{y} \quad \text{for } (\bar{x}, y) \in \widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0),$$

then there exists $\hat{v} \in L^2(W, w^{1,2}_{Per_Y}(Y \times (-\infty, 0)))$ such that

$$\frac{V_\epsilon}{\epsilon^{1/2}} \underset{\epsilon\to 0}{\rightharpoonup} \hat{v} \text{ and } \frac{D_y V_\epsilon}{\epsilon^{1/2}} = \frac{D_y \hat{v}_\epsilon}{\epsilon^{1/2}} \underset{\epsilon\to 0}{\rightharpoonup} D_y \hat{v} \text{ in } L^2(W \times Y \times (d, 0)) \text{ for any } d < 0,$$

and the following relation holds

$$\hat{v}^{\,(3)}(\bar{x}, \bar{y}, 0) = -\frac{\partial b}{\partial y_1}(\bar{y}) v^{(1)}(\bar{x}, 0) - \frac{\partial b}{\partial y_2}(\bar{y}) v^{(2)}(\bar{x}, 0).$$

Moreover, for any $u \in H^1(\Omega)^3$

$$\widehat{(u \circ \Phi_\epsilon)|_\Omega} \underset{\epsilon\to 0}{\longrightarrow} u(\bar{x}, 0) \text{ in } L^2(W \times Y \times (-1, 0)).$$

By definition (3.4.4), Lemma 3.4.5, Lemma 3.4.16, Lemma 3.4.73, and observing that by formula (3.4.14)

$$\xi_{ss_2s_3} \frac{\partial^2 \Phi_\epsilon^{(s_2)}}{\partial x_i \partial x_{j_2}} \frac{\partial \Phi_\epsilon^{(s_3)}}{\partial x_{j_3}} = -\xi_{ss_2s_3}\, \delta_{s_23} \delta_{s_3j_3} \frac{\partial^2 h_\epsilon}{\partial x_i \partial x_{j_2}},$$

we get that

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_{Q_\epsilon} \frac{\partial v_\epsilon^{(b)}}{\partial x_a} \xi_{abk} \xi_{ijk} \xi_{ss_2s_3} \xi_{jj_2j_3}\ \varphi^{(s)} \circ \Phi_\epsilon\ \frac{\partial^2 \Phi_\epsilon^{(s_2)}}{\partial x_i \partial x_{j_2}} \frac{\partial \Phi_\epsilon^{(s_3)}}{\partial x_{j_3}}\, dx \\
&= -\lim_{\epsilon\to 0} \int_{\widehat{W}_\epsilon} \int_{Y\times(-1,0)} \Bigg(\epsilon^{-1/2} \frac{\partial\, \hat{v}_\epsilon^{(b)}}{\partial y_a}(\bar{x}, y)\, \xi_{abk} \xi_{ijk} \xi_{s3s_3} \xi_{jj_2s_3}\, \widehat{\varphi^{(s)} \circ \Phi_\epsilon}(\bar{x}, y) \\
&\qquad\qquad \cdot \epsilon^{1/2} \widehat{\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_{j_2}}}(\bar{x}, y) \Bigg) d\bar{x} dy \\
&= -\int_W \int_{Y\times(-1,0)} \frac{\partial\, \hat{v}^{\,(b)}}{\partial y_a}(\bar{x}, y)\ \varphi^{(s)}(\bar{x}, 0)\ \frac{\partial^2 [b(\bar{y})(y_3 + 1)^3]}{\partial y_i \partial y_{j_2}}\ \xi_{abk} \xi_{ijk} \xi_{s3s_3} \xi_{jj_2s_3}\, d\bar{x} dy.
\end{aligned}$$

Furthermore, since $\sum_{j=1}^{3} \xi_{ijk}\xi_{jj_2s_3} = \delta_{is_3}\delta_{kj_2} - \delta_{ij_2}\delta_{ks_3}$, we have that

$$
\begin{aligned}
&- \frac{\partial\, \hat{v}^{(b)}}{\partial y_a}(\bar{x}, y)\ \varphi^{(s)}(\bar{x}, 0)\ \frac{\partial^2[b(\bar{y})(y_3+1)^3]}{\partial y_i \partial y_{j_2}}\ \xi_{abk}\xi_{ijk}\xi_{s3s_3}\xi_{jj_2s_3} \\
&= \frac{\partial\, \hat{v}^{(b)}}{\partial y_a}(\bar{x}, y)\, \xi_{abk}\, \varphi^{(s)}(\bar{x}, 0)\, \xi_{ks3}\ \frac{\partial^2[b(\bar{y})(y_3+1)^3]}{\partial y_i \partial y_i} \\
&\quad - \frac{\partial\, \hat{v}^{(b)}}{\partial y_a}(\bar{x}, y)\, \xi_{abk}\ \frac{\partial^2[b(\bar{y})(y_3+1)^3]}{\partial y_i \partial y_k}\ \varphi^{(s)}(\bar{x}, 0)\, \xi_{is3} \\
&= \big(\operatorname{curl}_y \hat{v}(\bar{x}, y) \times \varphi(\bar{x}, 0)\big)_3\, \Delta_y[b(\bar{y})(y_3+1)^3] \\
&\quad - \Big(\operatorname{curl}_y \hat{v}(\bar{x}, y)\ D^2_y[b(\bar{y})(y_3+1)^3] \times \varphi(\bar{x}, 0)\Big)_3,
\end{aligned}
$$

where D^2_y denotes the Hessian matrix with respect to the variable $y = (\bar{y}, y_3)$. We have thus proved the following

Lemma 3.4.74. *Let $\hat{v} \in L^2(W, w^{1,2}_{Per_Y}(Y \times (-\infty, 0)))$ be as in Lemma 3.4.73, and set*

$$
M : Y \times [-1, 0] \to \mathrm{Sym}_3(\mathbb{R}), \quad M(y) := \Delta[b(\bar{y})(y_3+1)^3]\, \mathbb{I}_3 - D^2[b(\bar{y})(y_3+1)^3]. \tag{3.4.75}
$$

Then

$$
\lim_{\epsilon \to 0} \int_{Q_\epsilon} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon \varphi\, dx = \int_W \int_{Y \times (-1,0)} \big(\operatorname{curl}_y \hat{v}(\bar{x}, y)\, M(y) \times \varphi(\bar{x}, 0)\big)_3\, d\bar{x} dy. \tag{3.4.76}
$$

The next theorem is a consequence of (3.4.72) and Lemmas 3.4.64 and 3.4.74.

Theorem 3.4.77. *Let $f_\epsilon \in L^2(\Omega_\epsilon)^3, \epsilon > 0$ be a family of functions weakly converging in $L^2(\Omega)^3$ to $f \in L^2(\Omega)^3$. Let $v_\epsilon \in X_{\mathrm{T}}(\Omega) \cap H^1(\Omega)^3$ be the solution of (3.4.62) corresponding to the Poisson datum f_ϵ for every $\epsilon > 0$. Then, possibly passing to a subsequence, there exists $v \in H^1(\Omega)^3$ with $v \cdot \nu = 0$ on Γ, and $\hat{v} \in L^2(W, w^{1,2}_{Per_Y}(Y \times (-\infty, 0)))$ such that $v_\epsilon \rightharpoonup v$ weakly in $H^1(\Omega)^3$, $v_\epsilon \to v$ strongly in $L^2(\Omega)^3$ and*

$$
\begin{aligned}
&\int_\Omega (v \cdot \varphi + \operatorname{div} v\ \operatorname{div} \varphi + \operatorname{curl} v \cdot \operatorname{curl} \varphi)\, dx \\
&\qquad + \int_W \int_{Y \times (-1,0)} \big(\operatorname{curl}_y \hat{v}(\bar{x}, y)\, M(y) \times \varphi(\bar{x}, 0)\big)_3\, d\bar{x} dy = \int_\Omega f \cdot \varphi\, dx
\end{aligned} \tag{3.4.78}
$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$, with $M \in C(Y \times [-1, 0], \mathrm{Sym}_3(\mathbb{R}))$ being the matrix-valued function defined in (3.4.75).

We now want to characterize the function $\hat{v}$ introduced in Lemma 3.4.73. Proceeding as in [11, Section 8.4], we take $\eta \in C^\infty\left(\overline{W}\times\overline{Y}\times]-\infty,0]\right)^3$ periodic in Y with compact support contained in $C\times\overline{Y}\times[d,0]$, for some compact $C\subset W$ and $d<0$, and such that $\eta^{(3)}(\bar{x},\bar{y},0)=0$ in $\overline{W}\times\overline{Y}$. Set

$$\eta_\epsilon(x)=\epsilon^{1/2}\eta\left(\bar{x},\frac{\bar{x}}{\epsilon},\frac{x_3}{\epsilon}\right).$$

Observe that for $\epsilon>0$ small enough the function η_ϵ is admissible, and thus we can consider $\mathscr{P}_\epsilon\eta_\epsilon\in X_{\mathrm{T}}(\Omega_\epsilon)$ as a test function in (3.4.62), yielding

$$\int_{\Omega_\epsilon} v_\epsilon\cdot\mathscr{P}_\epsilon\eta_\epsilon\,dx+\int_{\Omega_\epsilon}\operatorname{curl} v_\epsilon\cdot\operatorname{curl}\mathscr{P}_\epsilon\eta_\epsilon\,dx+\int_{\Omega_\epsilon}\operatorname{div} v_\epsilon\,\operatorname{div}\mathscr{P}_\epsilon\eta_\epsilon\,dx=\int_{\Omega_\epsilon} f_\epsilon\cdot\mathscr{P}_\epsilon\eta_\epsilon\,dx. \tag{3.4.79}$$

Due to the presence of the factor $\epsilon^{1/2}$ it is not difficult to see that

$$\int_{\Omega_\epsilon} v_\epsilon\cdot\mathscr{P}_\epsilon\eta_\epsilon\,dx\xrightarrow[\epsilon\to0]{}0,\ \int_{\Omega_\epsilon}\operatorname{div} v_\epsilon\,\operatorname{div}\mathscr{P}_\epsilon\eta_\epsilon\,dx\xrightarrow[\epsilon\to0]{}0,\ \int_{\Omega_\epsilon} f_\epsilon\cdot\mathscr{P}_\epsilon\eta_\epsilon\,dx\xrightarrow[\epsilon\to0]{}0. \tag{3.4.80}$$

Moreover, there holds the following

Lemma 3.4.81. *We have that*

$$\int_{\Omega_\epsilon\setminus\Omega}\operatorname{curl} v_\epsilon\cdot\operatorname{curl}\mathscr{P}_\epsilon\eta_\epsilon\,dx\xrightarrow[\epsilon\to0]{}0 \tag{3.4.82}$$

and

$$\int_{\Omega}\operatorname{curl} v_\epsilon\cdot\operatorname{curl}\mathscr{P}_\epsilon\eta_\epsilon\,dx\xrightarrow[\epsilon\to0]{}\int_{W\times Y\times(-\infty,0)}\operatorname{curl}_y\hat{v}(\bar{x},y)\cdot\operatorname{curl}_y\eta(\bar{x},y)\,d\bar{x}dy. \tag{3.4.83}$$

Proof. Recalling formula (3.4.63) and using Lemma 3.4.15 we get

$$\begin{aligned}
\left(\int_{\Omega_\epsilon\setminus\Omega}\operatorname{curl} v_\epsilon\cdot\operatorname{curl}\mathscr{P}_\epsilon\eta_\epsilon\,dx\right)^2 &\le C\int_{\Omega_\epsilon\setminus\Omega}|\operatorname{curl}\mathscr{P}_\epsilon\eta_\epsilon|^2\,dx\\
&\le C\int_{\Omega_\epsilon\setminus\Omega}\left|\frac{1}{2}\xi_{ijk}\xi_{ss_2s_3}\xi_{jj_2j_3}\frac{\partial\eta_\epsilon^{(s)}}{\partial x_h}\circ\Phi_\epsilon\frac{\partial\Phi_\epsilon^{(h)}}{\partial x_i}\frac{\partial\Phi_\epsilon^{(s_2)}}{\partial x_{j_2}}\frac{\partial\Phi_\epsilon^{(s_3)}}{\partial x_{j_3}}\right|^2dx\\
&\quad+C\int_{\Omega_\epsilon\setminus\Omega}\left|\xi_{ijk}\xi_{ss_2s_3}\xi_{jj_2j_3}\,\eta_\epsilon^{(s)}\circ\Phi_\epsilon\frac{\partial^2\Phi_\epsilon^{(s_2)}}{\partial x_i\partial x_{j_2}}\frac{\partial\Phi_\epsilon^{(s_3)}}{\partial x_{j_3}}\right|^2dx\\
&\le C\int_{\Omega_\epsilon\setminus\Omega}|D\eta_\epsilon\circ\Phi_\epsilon|^2\,dx+C\epsilon^{-1}\int_{\Omega_\epsilon\setminus\Omega}|\eta_\epsilon\circ\Phi_\epsilon|^2\,dx\\
&\le C\int_{\Phi_\epsilon(\Omega_\epsilon\setminus\Omega)}|D\eta_\epsilon|^2\,dx+C\epsilon^{-1}\int_{\Phi_\epsilon(\Omega_\epsilon\setminus\Omega)}|\eta_\epsilon|^2\,dx.
\end{aligned} \tag{3.4.84}$$

The rate of convergence for the domain of integration is $\left|\Phi_\epsilon(\Omega_\epsilon \setminus \Omega)\right| = O(\epsilon^{3/2})$ (cf. (3.4.25)). Since $\|\eta_\epsilon\|_{L^\infty(\Omega)} \le C\epsilon^{1/2}$ and $\|D\eta_\epsilon\|_{L^\infty(\Omega)} \le C\epsilon^{-1/2}$, both integrals in the above sum go to 0 as ϵ goes to 0. Thus we have proved (3.4.82).

We now prove (3.4.83). Due to the Y-periodicity of η we have that

$$\widehat{\mathscr{P}_\epsilon\eta_\epsilon}(\bar{x},y) = \epsilon^{1/2}\eta\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \bar{y}, y_3 - \epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right)\widehat{\operatorname{cof}_{D\Phi_\epsilon}}(\bar{x},y)$$

From (3.4.41) we get that

$$\widehat{\operatorname{cof}_{D\Phi_\epsilon}} \xrightarrow[\epsilon\to 0]{} \mathbb{I}_3 \text{ and } \frac{\partial}{\partial y_i}\widehat{\operatorname{cof}_{D\Phi_\epsilon}} \xrightarrow[\epsilon\to 0]{} \mathbb{O}_3, \tag{3.4.85}$$

uniformly in $W \times Y \times (-\infty, 0]$, for all $i = 1,2,3$. Since η is smooth and compactly supported, it is Lipschitz continuous together with its derivatives, and thus

$$\left\|(D^\gamma\eta)\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \bar{y}, y_3 - \epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right) - D^\gamma\eta(\bar{x},y)\right\|_{L^2(\widehat{W}_\epsilon\times Y\times(-\infty,0))} \xrightarrow[\epsilon\to 0]{} 0 \tag{3.4.86}$$

for any $|\gamma| \le 1$ (cf. formula (8.51) of [11]). Consider

$$\begin{aligned}
&\frac{\partial}{\partial y_i}\widehat{\mathscr{P}_\epsilon\eta_\epsilon}(\bar{x},y)\\
&\quad = \epsilon^{1/2}\frac{\partial}{\partial y_i}\left(\eta\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \bar{y}, y_3 - \epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right)\right)\widehat{\operatorname{cof}_{D\Phi_\epsilon}}(\bar{x},y)\\
&\qquad + \epsilon^{1/2}\eta\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \bar{y}, y_3 - \epsilon^{-1}h_\epsilon\left(\epsilon\left\lfloor\frac{\bar{x}}{\epsilon}\right\rfloor + \epsilon\bar{y}, \epsilon y_3\right)\right)\frac{\partial}{\partial y_i}\widehat{\operatorname{cof}_{D\Phi_\epsilon}}(\bar{x},y).
\end{aligned}$$

From (3.4.85), (3.4.86) and using the chain rule we have that for all $i = 1,2,3$

$$\epsilon^{-1/2}\frac{\partial}{\partial y_i}\widehat{\mathscr{P}_\epsilon\eta_\epsilon}(\bar{x},y) \xrightarrow[\epsilon\to 0]{} \frac{\partial\eta}{\partial y_i}(\bar{x},y) \tag{3.4.87}$$

in $L^2(W \times Y \times (-\infty,0))$. By Lemma 3.4.5, Lemma 3.4.73 and (3.4.87) we get

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{\widehat{W}_\epsilon\times(-1,0)} \operatorname{curl} v_\epsilon \cdot \operatorname{curl}\mathscr{P}_\epsilon\eta_\epsilon\, dx &= \lim_{\epsilon\to 0}\int_{\widehat{W}_\epsilon\times Y\times(-1/\epsilon,0)} \frac{\operatorname{curl}_y \hat{v}_\epsilon}{\epsilon^{1/2}} \cdot \frac{\operatorname{curl}_y \widehat{\mathscr{P}_\epsilon\eta_\epsilon}}{\epsilon^{1/2}}\, d\bar{x}dy\\
&= \int_{W\times Y\times(-\infty,0)} \operatorname{curl}_y \hat{v}(\bar{x},y)\cdot\operatorname{curl}_y \eta(\bar{x},y)\, d\bar{x}dy.
\end{aligned}$$

Finally, by the definition of $\widehat{W}_\epsilon$ and the fact that η has compact support in the first two variables, it is not difficult to see that

$$\int_{\Omega\setminus(\widehat{W}_\epsilon\times(-1,0))} \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon \eta_\epsilon \, dx \underset{\epsilon\to 0}{\longrightarrow} 0,$$

which concludes the proof.

$\square$

Recall that a vector field u belongs to the space $X_{Per_Y,loc}(Y\times(-\infty,0))$ if $u\in H_{loc}(\operatorname{curl},\mathbb{R}^2\times(-\infty,0))\cap H_{loc}(\operatorname{div},\mathbb{R}^2\times(-\infty,0))$ and moreover u is Y-periodic in the first two variables $\bar{y}\in\mathbb{R}^2$. There will be no notational difference when writing a function in $X_{Per_Y,loc}(Y\times(-\infty,0))$ or its restriction to $Y\times(-\infty,0)$. Then

$$x_{Per_Y}(Y\times(-\infty,0)) = \big\{u\in X_{Per_Y,loc}(Y\times(-\infty,0)) : \\ \|\operatorname{curl} u\|_{L^2(Y\times(-\infty,0))^3}, \|\operatorname{div} u\|_{L^2(Y\times(-\infty,0))} < \infty\big\}.$$

Theorem 3.4.88. *Let $\hat{v}\in L^2(W, w^{1,2}_{Per_Y}(Y\times(-\infty,0)))$ be the function introduced in Lemma 3.4.73. Then*

$$\int_{W\times Y\times(-\infty,0)} \operatorname{curl}_y \hat{v}(\bar{x},y)\cdot\operatorname{curl}_y \eta(\bar{x},y)\, d\bar{x}dy = 0 \tag{3.4.89}$$

for all $\eta\in L^2(W, x^{1,2}_{Per_Y}(Y\times(-\infty,0)))$ such that $\eta^{(3)}(\bar{x},\bar{y},0)=0$ in $\overline{W}\times\overline{Y}$. Moreover

$$\hat{v}^{(3)}(\bar{x},\bar{y},0) = -\frac{\partial b}{\partial y_1}(\bar{y})v^{(1)}(\bar{x},0) - \frac{\partial b}{\partial y_2}(\bar{y})v^{(2)}(\bar{x},0), \tag{3.4.90}$$

where v is the function introduced in Theorem 3.4.77.

Proof. To see (3.4.89), in the case of smooth test functions, one simply takes the limit in equation (3.4.79) and using limits (3.4.80)-(3.4.83). To get the general case, one can reason via an approximation argument.

The boundary condition (3.4.90) is a direct consequence of Lemma 3.4.11. $\square$

We now want to describe the function $\hat{v}$ in a more explicit way, separating the variables $\bar{x}$ and y. The natural microscopic problem related to (3.4.89) is described in the following lemma.

Lemma 3.4.91. *Assume that the function b introduced in (3.4.1) is of class C^3. Then, for $i=1,2$ there exist vector fields $V_i(y)\in x_{Per_Y}(Y\times(-\infty,0))$ satisfying*

$$\int_{Y\times(-\infty,0)} \operatorname{curl} V_i\cdot\operatorname{curl}\eta\, dy = 0 \tag{3.4.92}$$

for all $\eta \in x_{Per_Y}(Y \times (-\infty, 0)))$ such that $\eta^{(3)}(\bar{y}, 0) = 0$ on Y, together with the boundary condition

$$V_i^{(3)}(\bar{y}, 0) = -\frac{\partial b}{\partial y_i}(\bar{y}) \quad \textit{for all } \bar{y} \in Y. \tag{3.4.93}$$

Moreover, if there are two solutions V_i and $\tilde{V}_i$ of (3.4.92), (3.4.93) then $V_i - \tilde{V}_i = \nabla q$ for some $q \in W^{1,2}_{Per_Y, loc}(Y \times (-\infty, 0))$.

Finally, if V_i is of class $W^{2,2}_{Per_Y}(Y \times (d, 0))$ for some $d < 0$, then $\operatorname{curl} \operatorname{curl} V_i = 0$ in $(Y \times (d, 0))$ and it satisfies the boundary condition

$$(\nu \times \operatorname{curl} V_i)(\bar{y}, 0) = 0 \quad \textit{on } Y.$$

Proof. We provide an explicit solution of problem (3.4.92)–(3.4.93), assuming b to be of class C^3. Consider the function defined by

$$u_i(y) := \begin{cases} -\frac{\partial b}{\partial y_i}(\bar{y})(y_3 + 1)^3, & -1 \leq y_3 \leq 0, \\ 0, & -\infty < y_3 \leq -1, \end{cases}$$

and take its gradient $V_i(y) := \nabla u_i(y)$. Clearly $\operatorname{curl} V_i = 0$ and thus it trivially solves (3.4.92), and its third component on $Y \times \{0\}$ coincides with $-\frac{\partial b}{\partial y_i}(\bar{y})$. Moreover its divergence is well defined and it is equal to

$$u_i(y) := \begin{cases} -\Delta\left(\frac{\partial b}{\partial y_i}(\bar{y})(y_3 + 1)^3\right), & -1 \leq y_3 \leq 0, \\ 0, & -\infty < y_3 \leq -1, \end{cases}$$

which is clearly square summable over the whole semi-infinite strip, since it is equal to zero if $y_3 < -1$. Thus V_i belongs to $x_{Per_Y}(Y \times (-\infty, 0)))$.

To see uniqueness, suppose $V \in x_{Per_Y}(Y \times (-\infty, 0)))$ satisfies (3.4.92) and the boundary condition $V^{(3)}(\bar{y}, 0) = 0$. Testing V against itself in (3.4.92) yields $\operatorname{curl} V = 0$. Hence by the Poincaré's Lemma we can conclude that $V = \nabla q$ for some scalar potential function $q \in W^{1,2}_{Per_Y, loc}(Y \times (-\infty, 0))$. □

Remark 3.4.94. Consider the vector field $-\frac{\partial b}{\partial y_1}(\bar{y})\, \hat{e}_3$. Then obviously it satisfies condition (3.4.93) for $i = 1$. Let $r < 0$ and introduce a smooth cut-off function $\rho_r :]-\infty, 0] \to \mathbb{R}$ such that $0 \leq \rho_r \leq 1$ and

$$\begin{cases} \rho_r(z) \equiv 1, & \text{for } \frac{r}{2} \leq z \leq 0, \\ \rho_r(z) \equiv 0, & \text{for } z \leq r. \end{cases}$$

Consider now the modified vector field $-\rho_r(y_3)\frac{\partial b}{\partial y_1}(\bar{y})\,\hat{e}_3$. It belongs to $x_{Per_Y}(Y\times(-\infty,0))$ and again it satisfies (3.4.93) for $i=1$, with curl equal to

$$\begin{aligned}\operatorname{curl}\left(-\rho_r(y_3)\frac{\partial b}{\partial y_1}(\bar{y})\,\hat{e}_3\right) &= -\nabla\left(\rho_r(y_3)\frac{\partial b}{\partial y_1}(\bar{y})\right)\times\hat{e}_3 \\ &= \rho_r(y_3)\left(-\frac{\partial^2 b}{\partial y_2\partial y_1}(\bar{y})\hat{e}_1+\frac{\partial^2 b}{\partial y_1\partial y_1}(\bar{y})\hat{e}_2\right).\end{aligned} \tag{3.4.95}$$

Thus we can use $\eta(y)=V_1-\rho_r(y_3)\frac{\partial b}{\partial y_1}(\bar{y})\,\hat{e}_3$ as a test function in (3.4.92). Hence

$$\begin{aligned}&\int_{Y\times(-\infty,0)}\operatorname{curl}V_1\cdot\operatorname{curl}V_2\,dy \\ &\qquad=\int_{Y\times(r,0)}\rho_r(y_3)\,\operatorname{curl}V_1(y)\cdot\left(-\frac{\partial^2 b}{\partial y_2\partial y_1}(\bar{y})\hat{e}_1+\frac{\partial^2 b}{\partial y_1\partial y_1}(\bar{y})\hat{e}_2\right)dy,\end{aligned}$$

and

$$\begin{aligned}&\int_{Y\times(-\infty,0)}|\operatorname{curl}V_2|^2dy \\ &\qquad=\int_{Y\times(r,0)}\rho_r(y_3)\,\operatorname{curl}V_2(y)\cdot\left(-\frac{\partial^2 b}{\partial y_2\partial y_1}(\bar{y})\hat{e}_1+\frac{\partial^2 b}{\partial y_1\partial y_1}(\bar{y})\hat{e}_2\right)dy.\end{aligned}$$

Analogously, using $V_2-\rho_r(y_3)\frac{\partial b}{\partial y_2}(\bar{y})\,\hat{e}_3$ we find out that

$$\begin{aligned}&\int_{Y\times(-\infty,0)}\operatorname{curl}V_1\cdot\operatorname{curl}V_2\,dy \\ &\qquad=\int_{Y\times(r,0)}\rho_r(y_3)\,\operatorname{curl}V_2(y)\cdot\left(-\frac{\partial^2 b}{\partial y_2\partial y_2}(\bar{y})\hat{e}_1+\frac{\partial^2 b}{\partial y_1\partial y_2}(\bar{y})\hat{e}_2\right)dy,\end{aligned}$$

and

$$\begin{aligned}&\int_{Y\times(-\infty,0)}|\operatorname{curl}V_1|^2dy \\ &\qquad=\int_{Y\times(r,0)}\rho_r(y_3)\,\operatorname{curl}V_1(y)\cdot\left(-\frac{\partial^2 b}{\partial y_2\partial y_2}(\bar{y})\hat{e}_1+\frac{\partial^2 b}{\partial y_1\partial y_2}(\bar{y})\hat{e}_2\right)dy.\end{aligned}$$

Passing to the limits for $r\to 0^-$ in the above equalities we find out that all the expressions on the left are null.

Remark 3.4.96. If $p_i \in W^{1,2}_{Per_Y}(Y \times (-\infty, 0))$ is a solution to the following Neumann problem

$$\begin{cases} \Delta p_i = 0, & \text{in } Y \times (-\infty, 0), \\ \frac{\partial p_i}{\partial \nu} = -\frac{\partial b}{\partial y_i}, & \text{on } Y \times \{0\}, \end{cases} \tag{3.4.97}$$

then $V_i := \nabla p_i \in x_{Per_Y}(Y \times (-\infty, 0)))$ is a solution of

$$\begin{cases} \operatorname{curl} V_i = 0, & \text{in } Y \times (-\infty, 0), \\ \operatorname{div} V_i = 0, & \text{in } Y \times (-\infty, 0), \\ V_i \cdot \nu = -\frac{\partial b}{\partial y_i}, & \text{on } Y \times \{0\}, \end{cases} \tag{3.4.98}$$

hence it trivially solves (3.4.92), (3.4.93).

Lemma 3.4.99. *Let V_1, V_2 be as in Lemma 3.4.91. Then*

$$\int_{Y \times (-1,0)} \big(\operatorname{curl} V_1(y) M(y)\big)_2 \, dy = \int_{Y \times (-\infty,0)} \big|\operatorname{curl} V_1(y)\big|^2 \, dy, \tag{3.4.100}$$

$$\int_{Y \times (-1,0)} \big(\operatorname{curl} V_2(y) M(y)\big)_1 \, dy = -\int_{Y \times (-\infty,0)} \big|\operatorname{curl} V_2(y)\big|^2 \, dy, \tag{3.4.101}$$

and

$$\begin{aligned} -\int_{Y \times (-1,0)} \big(\operatorname{curl} V_1(y) M(y)\big)_1 \, dy &= \int_{Y \times (-1,0)} \big(\operatorname{curl} V_2(y) M(y)\big)_2 \, dy \\ &= \int_{Y \times (-\infty,0)} \operatorname{curl} V_1(y) \cdot \operatorname{curl} V_2(y) dy. \end{aligned} \tag{3.4.102}$$

Moreover for both $i = 1, 2$

$$\int_{Y \times (-\infty,0)} \big(\operatorname{curl} V_i(y)\, M(y)\big)_3 \, dy = 0. \tag{3.4.103}$$

Proof. Let $\hat{e}_1 = (1, 0, 0), \hat{e}_2 = (0, 1, 0), \hat{e}_3 = (0, 0, 1)$ be the standard unit coordinate vectors in $\mathbb{R}^3$. Define

$$\eta_2(y) = \begin{cases} \nabla[b(\bar{y})(y_3 + 1)^3] \times \hat{e}_1, & \text{if } (\bar{y}, y_3) \in Y \times [-1, 0), \\ 0, & \text{if } (\bar{y}, y_3) \in Y \times] - \infty, -1), \end{cases}$$

$$\eta_1(y) = \begin{cases} \hat{e}_2 \times \nabla[b(\bar{y})(y_3 + 1)^3], & \text{if } (\bar{y}, y_3) \in Y \times [-1, 0), \\ 0, & \text{if } (\bar{y}, y_3) \in Y \times] - \infty, -1), \end{cases}$$

and

$$\eta_3(y) = \begin{cases} \pm\hat{e}_3 \times \nabla[b(\bar{y})(y_3+1)^3], & \text{if } (\bar{y}, y_3) \in Y \times [-1,0), \\ 0, & \text{if } (\bar{y}, y_3) \in Y \times]-\infty,-1). \end{cases}$$

The vector fields $\eta_j \in C^1_{Per_Y}(\overline{Y} \times (-\infty, 0])^3$ are such that $\eta_j^{(3)}(\bar{y}, 0) = -\frac{\partial b}{\partial y_i}(\bar{y})$ for $j = 1, 2$ and $\eta_3^{(3)} \equiv 0$. Moreover, computing their curl we find out that

$$\operatorname{curl} \eta_2(y) = \begin{cases} -\hat{e}_1 M(y), & \text{if } (\bar{y}, y_3) \in Y \times [-1,0), \\ 0, & \text{if } (\bar{y}, y_3) \in Y \times]-\infty,-1), \end{cases}$$

$$\operatorname{curl} \eta_1(y) = \begin{cases} \hat{e}_2 M(y), & \text{if } (\bar{y}, y_3) \in Y \times [-1,0), \\ 0, & \text{if } (\bar{y}, y_3) \in Y \times]-\infty,-1), \end{cases}$$

and

$$\operatorname{curl} \eta_3(y) = \begin{cases} \pm\hat{e}_3 M(y), & \text{if } (\bar{y}, y_3) \in Y \times [-1,0), \\ 0, & \text{if } (\bar{y}, y_3) \in Y \times]-\infty,-1) \end{cases}$$

Setting $\tilde{\eta}_j = V_j - \eta_j$ for $j = 1, 2$ and $\tilde{\eta}_3 = \eta_3$, we have that $\tilde{\eta}_j^{(3)}(\bar{y}, 0) = 0$ for all $j = 1, 2, 3$, hence we can test them in (3.4.92). Recalling that $M(y)$ is symmetric for all $y \in Y \times [-1, 0]$, we obtain the desired equalities. $\square$

Remark 3.4.104. The last two equalities (3.4.103) are superfluous for our purpose.

We now have all the ingredients to prove the next theorem.

Lemma 3.4.105. *Let V_1, V_2 be as in Lemma 3.4.91. Let $v, \hat{v}$ be the functions defined in Lemma 3.4.73. Then there exist a scalar function q harmonic in y such that*

$$\hat{v}(\bar{x}, y) = V_1(y) v^{(1)}(\bar{x}, 0) + V_2(y) v^{(2)}(\bar{x}, 0) + \nabla_y q(\bar{x}, y) \tag{3.4.106}$$

for almost all $(\bar{x}, y) \in W \times Y \times (-\infty, 0)$.

Moreover, the second integral in the left-hand side of (3.4.78) is equal to

$$\int_W \int_{Y\times(-1,0)} \left(\operatorname{curl}_y \hat{v}(\bar{x}, y)\, M(y) \times \varphi(\bar{x}, 0)\right)_3 \, d\bar{x} dy = -\int_W v(\bar{x}, 0)\, \mathfrak{M} \cdot \varphi(\bar{x}, 0)\, d\bar{x}, \tag{3.4.107}$$

where $\mathfrak{M}$ is a constant symmetric matrix defined as follows

$$\mathfrak{M} = \begin{pmatrix} \int_{Y\times(-1,0)} |\operatorname{curl} V_1|^2\, dy & \int_{Y\times(-1,0)} \operatorname{curl} V_1 \cdot \operatorname{curl} V_2\, dy & 0 \\ \int_{Y\times(-1,0)} \operatorname{curl} V_1 \cdot \operatorname{curl} V_2 & \int_{Y\times(-1,0)} |\operatorname{curl} V_2|^2\, dy & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The function $\hat{v}$ satisfies problem (3.4.89) together with the boundary condition (3.4.90). Arguing as in the proof of Lemma 3.4.91, and observing that the vector field $V_1(y)v^{(1)}(\bar{x},0)+V_2(y)v^{(2)}(\bar{x},0)$ also satisfies the two conditions (3.4.89),(3.4.90), we immediately deduce that the curl in y of the difference vanish in $W\times Y\times(\infty,0)$, and using Poincaré's Lemma we have that the function $\hat{v}$ is of the type described in (3.4.106). Hence $\operatorname{curl}_y \hat{v}(\bar{x},y) = \operatorname{curl} V_1(y)v^{(1)}(\bar{x},0) + \operatorname{curl} V_2(y)v^{(2)}(\bar{x},0)$ and so

$$
\begin{aligned}
&\int_W \int_{Y\times(-1,0)} \big(\operatorname{curl}_y \hat{v}(\bar{x},y)\, M(y) \times \varphi(\bar{x},0)\big)_3 \, d\bar{x}dy \\
&= \left(\int_{Y\times(-1,0)} \operatorname{curl} V_1(y) M(y)\, dy \times \int_W v^{(1)}(\bar{x},0)\varphi(\bar{x},0)\, d\bar{x}\right)_3 \\
&+ \left(\int_{Y\times(-1,0)} \operatorname{curl} V_2(y) M(y)\, dy \times \int_W v^{(2)}(\bar{x},0)\varphi(\bar{x},0)\, d\bar{x}\right)_3 \\
&= \int_{Y\times(-1,0)} \big(\operatorname{curl} V_1(y) M(y)\big)_1 \, dy \int_W v^{(1)}(\bar{x},0)\varphi^{(2)}(\bar{x},0)\, d\bar{x} \\
&\qquad - \int_{Y\times(-1,0)} \big(\operatorname{curl} V_1(y) M(y)\big)_2 \, dy \int_W v^{(1)}(\bar{x},0)\varphi^{(1)}(\bar{x},0)\, d\bar{x} \\
&+ \int_{Y\times(-1,0)} \big(\operatorname{curl} V_2(y) M(y)\big)_1 \, dy \int_W v^{(2)}(\bar{x},0)\varphi^{(2)}(\bar{x},0)\, d\bar{x} \\
&\qquad - \int_{Y\times(-1,0)} \big(\operatorname{curl} V_2(y) M(y)\big)_2 \, dy \int_W v^{(2)}(\bar{x},0)\varphi^{(1)}(\bar{x},0)\, d\bar{x} \\
&= -\int_{Y\times(-1,0)} |\operatorname{curl} V_1|^2 \, dy \int_W v^{(1)}(\bar{x},0)\varphi^{(1)}(\bar{x},0)\, d\bar{x} \\
&\quad - \int_{Y\times(-1,0)} |\operatorname{curl} V_2|^2 \, dy \int_W v^{(2)}(\bar{x},0)\varphi^{(2)}(\bar{x},0)\, d\bar{x} \\
&\quad - \int_{Y\times(-1,0)} \operatorname{curl} V_1 \cdot \operatorname{curl} V_2 \, dy \int_W \Big(v^{(1)}(\bar{x},0)\varphi^{(2)}(\bar{x},0) + v^{(2)}(\bar{x},0)\varphi^{(1)}(\bar{x},0)\Big)\, d\bar{x},
\end{aligned}
$$

where in the last equality we used (3.4.100), (3.4.101) and (3.4.102). Computing $v(\bar{x},0)\,\mathfrak{M}\cdot\varphi$ and integrating in $\bar{x}$ over W we conclude. □

We have thus proved that

$$
\begin{array}{ccc}
\int_{\Omega_\epsilon} (v_\epsilon \cdot \varphi\, dx + \operatorname{curl} v_\epsilon \cdot \operatorname{curl} \mathscr{P}_\epsilon \varphi\, dx + \operatorname{div} v_\epsilon \ \operatorname{div} \varphi)\, dx & = & \int_\Omega f_\epsilon \cdot \mathscr{P}_\epsilon \varphi\, dx \\
\Big\downarrow{\scriptstyle \epsilon\to 0} & & \Big\downarrow{\scriptstyle \epsilon\to 0} \\
\int_\Omega (v\cdot\varphi + \operatorname{curl} v\cdot \operatorname{curl}\varphi + \operatorname{div} v \ \operatorname{div}\varphi)\, dx - \int_\Gamma v\,\mathfrak{M}\cdot\varphi\, d\sigma & = & \int_\Omega f\cdot\varphi\, dx.
\end{array}
$$

This means that the problem that the limit function v of Theorem 3.4.77 satisfies is

$$\begin{cases} \operatorname{curl}\operatorname{curl} v = f, & \text{in } \Omega, \\ \operatorname{div} v = 0, & \text{in } \Omega, \\ \nu \cdot v = 0, & \text{on } \partial\Omega, \\ \nu \times \operatorname{curl} v + v\,\mathfrak{M} = 0, & \text{on } \Gamma. \end{cases} \tag{3.4.108}$$

Then we can state the following theorem.

Theorem 3.4.109. *Let $f_\epsilon \in L^2(\Omega_\epsilon)^3, \epsilon > 0$ be a family of functions weakly converging in $L^2(\Omega)^3$ to $f \in L^2(\Omega)^3$. Let $v_\epsilon \in X_{\mathrm{T}}(\Omega) \cap H^1(\Omega)^3$ be the (unique) solution of (3.4.62) corresponding to the Poisson datum f_ϵ for every $\epsilon > 0$. Then, possibly passing to a subsequence, there exists $v \in H^1(\Omega)^3$ with $v \cdot \nu = 0$ on Γ, such that $v_\epsilon \rightharpoonup v$ weakly in $H^1(\Omega)^3$, $v_\epsilon \to v$ strongly in $L^2(\Omega)^3$ and*

$$\int_\Omega (v \cdot \varphi + \operatorname{div} v \operatorname{div} \varphi + \operatorname{curl} v \cdot \operatorname{curl} \varphi)\, dx = \int_\Omega f \cdot \varphi \, dx$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

Proof. One just needs to observe that the matrix $\mathfrak{M}$ is null, since the microscopic solutions V_i, $i = 1, 2$ in the proof of Lemma 3.4.91, since they are gradients, are irrotational (see also Remark 3.4.94 and Remark 3.4.96). □

Chapter 4

Permittivity perturbation

In this chapter we study the dependence of the eigenvalues of the electric Maxwell problem (4.1.1) upon the variation of the permittivity parameter. Note that the permeability μ has been normalized. We show the local Lipschitz continuity of single eigenvalues (see Theorem 4.2.11), as well as their continuity with respect to the weak* topology (see Theorem 4.3.6) . Then, in Section 4.4, we prove an analyticity result for the elementary symmetric functions of the eigenvalues and provide formulas for the derivatives. We use such formulas to prove a Rellich-Nagy-type result (see Theorem 4.4.11), and to study a constrained permittivity optimization problem (see Theorem 4.5.1). Finally, in Section 4.6 we prove the generic simplicity of the spectrum of problem (4.1.1).

4.1 Some preliminaries

For the sake of simplicity and to ease the notation, in the present chapter we consider column vectors. In particular, given a 3×3 matrix M and a vector $\mathbf{v} \in \mathbb{R}^3$ the matrix-vector product is indicated as $M\mathbf{v}$.

Let Ω be a bounded domain of $\mathbb{R}^3$. In this chapter we study the following eigenvalue problem

$$\begin{cases} \operatorname{curl}\operatorname{curl} u = \lambda\,\varepsilon\, u & \text{in } \Omega, \\ \operatorname{div}(\varepsilon u) = 0 & \text{in } \Omega, \\ \nu \times u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

upon variation of the permittivity parameter ε.

We denote by $L^\infty(\Omega)^{3\times 3}$ and $W^{1,\infty}(\Omega)^{3\times 3}$ the spaces of real matrix-valued functions $M = \big(M_{ij}\big)_{1\le i,j\le 3} : \Omega \to \mathbb{R}^{3\times 3}$ whose components are in $L^\infty(\Omega)$ and

$W^{1,\infty}(\Omega)$, respectively. We endow these spaces with the norms

$$\|M\|_{L^\infty(\Omega)^{3\times 3}} := \max_{1\le i,j\le 3} \|M_{ij}\|_{L^\infty(\Omega)}$$

and

$$\|M\|_{W^{1,\infty}(\Omega)^{3\times 3}} := \max_{1\le i,j\le 3} \|M_{ij}\|_{W^{1,\infty}(\Omega)}\,. \tag{4.1.2}$$

Moreover, for the sake of simplicity and to ease the notation, we will often write $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ respectively instead of $L^\infty(\Omega)^{3\times 3}$ and $W^{1,\infty}(\Omega)^{3\times 3}$, whenever it is clear from the context if we are referring to scalar or matrix-valued functions.

Let $M \in L^\infty(\Omega)^{3\times 3}$. In this sequel we will exploit the following trivial inequalities:

$$|M\zeta\cdot\zeta| \le 3\|M\|_{L^\infty(\Omega)}|\zeta|^2$$

and

$$|M\zeta| \le 3\|M\|_{L^\infty(\Omega)}\,|\zeta|$$

for all $\zeta\in\mathbb{R}^3$ and a.e. in Ω. Here $|\zeta|$ denotes the standard Euclidean vector norm in $\mathbb{R}^3$, namely $|\zeta|^2 = \sum_{i=1}^3 \zeta_i^2$. Throughout this chapter, unless otherwise specified, we will assume the following:

$$\Omega \text{ is a bounded domain of } \mathbb{R}^3 \text{ of class } C^{1,1}. \tag{4.1.3}$$

In order to study the eigenvalue problem (4.1.1), we first need to specify where we take the parameter ε. Therefore we introduce the following set of admissible permittivities

$$\mathcal{E} := \Big\{\varepsilon\in W^{1,\infty}(\Omega)\cap \operatorname{Sym}_3(\Omega) : \\ \exists\, c>0 \text{ s.t. } \varepsilon(x)\,\zeta\cdot\zeta \ge c\,|\zeta|^2 \text{ for a.a. } x\in\Omega, \text{ for all } \zeta\in\mathbb{R}^3\Big\}, \tag{4.1.4}$$

endowed with the norm defined in (4.1.2). Here $\operatorname{Sym}_3(\Omega)$ denotes the set of (3×3)-symmetric matrix-valued functions in Ω. Given $\varepsilon\in\mathcal{E}$, we denote by $m_\varepsilon>0$ the greatest positive constant that guarantees the coercivity condition in the above definition, that is

$$m_\varepsilon := \max\Big\{c>0 : \varepsilon(x)\,\zeta\cdot\zeta\ge c\,|\zeta|^2 \text{ for a.a. } x\in\Omega, \text{ for all } \zeta\in\mathbb{R}^3\Big\}. \tag{4.1.5}$$

Observe that the set $\mathcal{E}$ is open in $W^{1,\infty}(\Omega)\cap\operatorname{Sym}_3(\Omega)$. This is implied by the continuity of the map

$$\big(\mathcal{E}, \|\cdot\|_{L^\infty(\Omega)}\big)\to\mathbb{R}_+, \qquad \varepsilon\mapsto m_\varepsilon.$$

Indeed let $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$. Since $\left|(\varepsilon_2 - \varepsilon_1)\,\zeta \cdot \zeta\right| \leq 3\|\varepsilon_2 - \varepsilon_1\|_{L^\infty(\Omega)}|\zeta|^2$ a.e. in Ω, then

$$\varepsilon_2\,\zeta \cdot \zeta = \varepsilon_1\,\zeta \cdot \zeta + (\varepsilon_2 - \varepsilon_1)\,\zeta \cdot \zeta \geq \Big(m_{\varepsilon_1} - 3\|\varepsilon_2 - \varepsilon_1\|_{L^\infty(\Omega)}\Big)|\zeta|^2 .$$

Hence $m_{\varepsilon_2} \geq m_{\varepsilon_1} - 3\|\varepsilon_2 - \varepsilon_1\|_{L^\infty(\Omega)}$. Exchanging the role of ε_1 and ε_2 one can show that $m_{\varepsilon_2} \leq m_{\varepsilon_1} + 3\|\varepsilon_2 - \varepsilon_1\|_{L^\infty(\Omega)}$, thus

$$|m_{\varepsilon_2} - m_{\varepsilon_1}| \leq 3\|\varepsilon_2 - \varepsilon_1\|_{L^\infty(\Omega)}\,, \tag{4.1.6}$$

which ensures the (Lipschitz) continuity of the map defined above.

Let $\varepsilon \in \mathcal{E}$. We denote by $L^2_\varepsilon(\Omega)$ the space $L^2(\Omega)^3$ endowed with the inner product

$$\langle u, v\rangle_\varepsilon = J_\varepsilon[u][v] := \int_\Omega \varepsilon u \cdot v\,dx \qquad \forall u, v \in L^2(\Omega)^3. \tag{4.1.7}$$

Note that the above inner product induces a norm equivalent to the standard L^2-norm since

$$m_\varepsilon \int_\Omega |u|^2\,dx \leq \int_\Omega \varepsilon u \cdot u\,dx \leq 3\|\varepsilon\|_{L^\infty(\Omega)} \int_\Omega |u|^2\,dx \qquad \forall u \in L^2(\Omega)^3.$$

Recall from Chapter 1 the space

$$X^\varepsilon_{\mathrm{N}}(\Omega) = H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}\,\varepsilon, \Omega)$$

equipped with inner product

$$\langle u, v\rangle_{X^\varepsilon_{\mathrm{N}}(\Omega)} := \int_\Omega \varepsilon u \cdot v\,dx + \int_\Omega \mathrm{curl}\,u \cdot \mathrm{curl}\,v\,dx + \int_\Omega \mathrm{div}(\varepsilon u)\ \mathrm{div}(\varepsilon v)\,dx$$

for all $u, v \in X^\varepsilon_{\mathrm{N}}(\Omega)$. Recall from Chapter 1 the following closed linear subspace of $X^\varepsilon_{\mathrm{N}}(\Omega)$

$$\begin{aligned} X^\varepsilon_{\mathrm{N}}(\mathrm{div}\,\varepsilon 0, \Omega) &= \big\{u \in X^\varepsilon_{\mathrm{N}}(\Omega) : \mathrm{div}\,(\varepsilon u) = 0\big\} \\ &= \big\{u \in L^2(\Omega)^3 : \mathrm{curl}\,u \in L^2(\Omega)^3, \mathrm{div}(\varepsilon u) = 0, \nu \times u|_{\partial\Omega} = 0\big\}\,. \end{aligned}$$

If $\varepsilon \in \mathcal{E}$ and assumption (4.1.3) is valid, the space $X^\varepsilon_{\mathrm{N}}(\Omega)$ is continuously embedded into $H^1(\Omega)^3$ and there holds the following Gaffney inequality

$$\|u\|^2_{H^1(\Omega)^3} \leq C_\varepsilon\left(\|u\|_{L^2_\varepsilon(\Omega)} + \|\mathrm{curl}\,u\|^2_{L^2(\Omega)^3} + \|\,\mathrm{div}\,\varepsilon u\|^2_{L^2(\Omega)}\right) = C_\varepsilon\|u\|_{X^\varepsilon_{\mathrm{N}}(\Omega)} \tag{4.1.8}$$

for all $u \in X^\varepsilon_{\mathrm{N}}(\Omega)$, where $C_\varepsilon > 0$ is a constant independent of $u \in X_{\mathrm{N}}(\Omega)$ but possibly depending on ε. In Section 1.4 we showed a complete proof in the particular case $\varepsilon = 1$. For this more general case involving a non-unitary permittivity ε we refer to Prokhorov and Filonov [102, Thm. 1.1].

It is easy to realize that the weak formulation of problem (4.1.1) is

$$\int_\Omega \operatorname{curl} u \cdot \operatorname{curl} v \, dx = \lambda \int_\Omega \varepsilon u \cdot v \, dx \qquad \forall v \in X_{\mathrm{N}}^\varepsilon(\operatorname{div} \varepsilon 0, \Omega), \tag{4.1.9}$$

in the unknowns $\lambda \in \mathbb{R}$ (the eigenvalues) and $u \in X_{\mathrm{N}}^\varepsilon(\operatorname{div} \varepsilon 0, \Omega)$ (the eigenvectors). The eigenvalues of problem (4.1.9) are non-negative, as one can easily see by testing the eigenfunction u against itself.

Analogously to Chapter 2, for our purposes it is convenient to work in the space $X_{\mathrm{N}}^\varepsilon(\Omega)$ rather than $X_{\mathrm{N}}^\varepsilon(\operatorname{div} \varepsilon 0, \Omega)$. Hence, again, following Costabel [39] and Costabel and Dauge [40], we consider the following eigenvalue problem which presents an additional penalty term:

$$\int_\Omega \operatorname{curl} u \cdot \operatorname{curl} v \, dx + \tau \int_\Omega \operatorname{div}(\varepsilon u) \operatorname{div}(\varepsilon v) \, dx = \sigma \int_\Omega \varepsilon u \cdot v \, dx \quad \forall v \in X_{\mathrm{N}}^\varepsilon(\Omega), \tag{4.1.10}$$

in the unknowns $u \in X_{\mathrm{N}}^\varepsilon(\Omega)$ and $\sigma \in \mathbb{R}$. Here $\tau > 0$ is any fixed positive real number. Solutions of problem (4.1.9) will then corresponds to solutions u of (4.1.10) with $\operatorname{div} \varepsilon u = 0$ in Ω (see also Theorem 4.1.15 below). Observe that also the eigenvalues σ of problem (4.1.10) are non-negative. We will only consider positive eigenvalues (cf. also Definition 4.1.19) and forget about the zero eigenvalue. In any case, to be thorough, the zero eigenspace $K_{\mathrm{N}}^\varepsilon(\Omega)$ of problem (4.1.10) (and of problem (4.1.9)) is composed of those curl-free vector fields which are normal to the boundary and such that $\operatorname{div} \varepsilon u = 0$ in Ω, namely

$$K_{\mathrm{N}}^\varepsilon(\Omega) = \left\{ u \in L^2(\Omega)^3 : \operatorname{curl} u = 0 \text{ in } \Omega,\ \operatorname{div} \varepsilon u = 0 \text{ in } \Omega,\ \nu \times u = 0 \text{ on } \partial\Omega \right\}.$$

Slightly modifying the proof of [7, Prop. 3.18], we can deduce that if $m \in \mathbb{N}$ is the number of connected components of the boundary of Ω, then $\dim_{\mathbb{R}} K_{\mathrm{N}}^\varepsilon(\Omega) = m - 1$. In particular, if $\partial\Omega$ has only one connected component, $K_{\mathrm{N}}^\varepsilon(\Omega) = \{0\}$.

Via a standard procedure, we can convert problem (4.1.10) into an eigenvalue problem for a compact self-adjoint operator. Recall the map J_ε defined in (4.1.7), which is nothing but the bilinear form corresponding to the inner product of $L^2_\varepsilon(\Omega)$. Obviously J_ε can be thought as an operator acting from $L^2_\varepsilon(\Omega)$ to $(X_{\mathrm{N}}^\varepsilon(\Omega))'$. We define the operator T_ε from $X_{\mathrm{N}}^\varepsilon(\Omega)$ to $(X_{\mathrm{N}}^\varepsilon(\Omega))'$ by

$$T_\varepsilon[u][v] := \int_\Omega \varepsilon u \cdot v \, dx + \int_\Omega \operatorname{curl} u \cdot \operatorname{curl} v \, dx + \tau \int_\Omega \operatorname{div}(\varepsilon u) \operatorname{div}(\varepsilon v) \, dx \quad \forall u, v \in X_{\mathrm{N}}^\varepsilon(\Omega). \tag{4.1.11}$$

Observe that by the Riesz Theorem, T_ε is a homeomorphism from $X^\varepsilon_{\mathrm{N}}(\Omega)$ to its dual and can thus be inverted. We can therefore define the operator S_ε, acting from $L^2_\varepsilon(\Omega)$ to itself, by setting

$$S_\varepsilon := \iota_\varepsilon \circ T_\varepsilon^{-1} \circ J_\varepsilon : L^2_\varepsilon(\Omega) \to L^2_\varepsilon(\Omega), \tag{4.1.12}$$

where ι_ε denotes the embedding of $X^\varepsilon_{\mathrm{N}}(\Omega)$ into $L^2_\varepsilon(\Omega)$. We then have the following

Lemma 4.1.13. *Let $\varepsilon \in \mathcal{E}$. Then the operator S_ε is a self-adjoint operator from $L^2_\varepsilon(\Omega)$ to itself. Moreover, σ is an eigenvalue of problem (4.1.10) if and only if $\mu = (\sigma+1)^{-1}$ is an eigenvalue of the operator S_ε, the eigenvectors being the same.*

The proof is completely analogous to the one of Lemma 2.1.11, thus we skip it. If the space $X^\varepsilon_{\mathrm{N}}(\Omega)$ is compactly embedded into $L^2(\Omega)^3$, which is true under our assumptions on ε and Ω, the operator S_ε is compact and its spectrum consists of $\{0\} \cup \{\mu_n\}_{n\in\mathbb{N}}$ with μ_n being a decreasing sequence composed of positive eigenvalues of S_ε of finite multiplicity converging to zero. Accordingly, by Lemma 4.1.13, the spectrum of problem (4.1.10) is composed possibly by zero and by ε-dependent positive eigenvalues of finite multiplicity which can be arranged in an increasing sequence

$$0 < \sigma_1[\varepsilon] \le \sigma_2[\varepsilon] \le \cdots \le \sigma_n[\varepsilon] \le \cdots \nearrow +\infty.$$

Here each eigenvalue is repeated in accordance with its multiplicity. Note that the zero eigenvalue (whose multiplicity is fixed and depending only on the geometry of Ω) is present only if the boundary $\partial\Omega$ has more than one connected component.

By the min-max formula every eigenvalue can be variationally characterized as follows

$$\sigma_j[\varepsilon] = \min_{\substack{V_j \subset X^\varepsilon_{\mathrm{N}}(\Omega),\\ \dim V_j = j}} \max_{\substack{u\in V_j,\\ u\neq 0}} \frac{\int_\Omega |\operatorname{curl} u|^2\, dx + \tau \int_\Omega |\operatorname{div}(\varepsilon u)|^2\, dx}{\int_\Omega \varepsilon u \cdot u\, dx}. \tag{4.1.14}$$

Moreover, we have the following lemma whose proof, which we present here for the sake of completeness, is a minor adaptation of the one of [40, Thm 1.1].

Lemma 4.1.15. *Let Ω be as in (4.1.3). Let $\varepsilon \in \mathcal{E}$. Then the eigenpairs $(\sigma, u) \in \mathbb{R} \times X^\varepsilon_{\mathrm{N}}(\Omega)$ of problem (4.1.10) are spanned by the following two disjoint families:*

i) the pairs $(\lambda, u) \in \mathbb{R} \times X^\varepsilon_{\mathrm{N}}(\operatorname{div}\varepsilon 0, \Omega)$ solutions of problem (4.1.9);

ii) the pairs $(\tau\rho, \nabla f)$ where $(\rho, f) \in \mathbb{R} \times H^1_0(\Omega)$ is an eigenpair of the problem

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla f) = \rho f & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1.16}$$

In particular, the set of eigenvalues of problem (4.1.10) are given by the union of the set of eigenvalues of problem (4.1.9) and the set of eigenvalues of the operator $\operatorname{div}(\varepsilon\nabla\cdot)$ *with Dirichlet boundary conditions in* Ω *multiplied by* τ*.*

Proof. It is easily seen that if $(\lambda, u) \in \mathbb{R} \times X^{\varepsilon}_{\mathrm{N}}(\operatorname{div}\varepsilon 0, \Omega)$ is an eigenpair of problem (4.1.9), then it is an eigenpair of problem (4.1.10). Moreover, if $u = \nabla f$, where $f \in H^1_0(\Omega)$ is a solution of problem (4.1.16), then u is in $X^{\varepsilon}_{\mathrm{N}}(\Omega)$ and solves (4.1.10) with $\sigma = \tau\rho$.

Conversely, suppose that $(\sigma, u) \in \mathbb{R} \times X^{\varepsilon}_{\mathrm{N}}(\Omega)$ is an eigenpair of problem (4.1.10). If

$$p := \operatorname{div}(\varepsilon u) = 0,$$

then clearly $u \in X^{\varepsilon}_{\mathrm{N}}(\operatorname{div}\varepsilon 0, \Omega)$ and solves (4.1.9). Suppose now that $p \neq 0$. We set

$$H^1_0(\Omega, \operatorname{div}(\varepsilon\nabla\cdot)) := \{u \in H^1_0(\Omega) : \operatorname{div}(\varepsilon\nabla u) \in L^2(\Omega)\}.$$

Then for all $\psi \in H^1_0(\Omega, \operatorname{div}(\varepsilon\nabla\cdot))$, by taking $\nabla\psi$ as test functions in (4.1.10) we get

$$\int_\Omega \tau\, p \operatorname{div}(\varepsilon\nabla\psi)\, dx = \sigma \int_\Omega \varepsilon u \cdot \nabla\psi\, dx = -\sigma \int_\Omega p\, \psi\, dx,$$

thus

$$\int_\Omega p \left(\tau \operatorname{div}(\varepsilon\nabla\psi) + \sigma\psi\right)\, dx = 0. \tag{4.1.17}$$

Necessarily σ/τ belongs to the spectrum of the operator $-\operatorname{div}(\varepsilon\nabla\cdot)$ with Dirichlet boundary conditions, because if not we could find a $\hat{\psi}$ such that $\operatorname{div}(\varepsilon\nabla\hat{\psi}) + \frac{\sigma}{\tau}\hat{\psi} = p$, hence from (4.1.17) we would get $p = 0$, which is a contradiction. From the Fredholm alternative we deduce that p belongs to the associated eigenspace, thus $p \in H^1_0(\Omega, \operatorname{div}(\varepsilon\nabla\cdot))$ and

$$\operatorname{div}(\varepsilon\nabla p) + \frac{\sigma}{\tau}\, p = 0. \tag{4.1.18}$$

Now, we define the field

$$w := u + \frac{\tau}{\sigma}\, \nabla p \in X^{\varepsilon}_{\mathrm{N}}(\Omega).$$

If $w = 0$ then $u = -\frac{\tau}{\sigma}\, \nabla p$, and recalling (4.1.18) one deduces that (σ, u) is of the form in ii). Therefore, suppose that $w \neq 0$. Observe that w satisfies

$$\operatorname{div}(\varepsilon w) = p + \frac{\tau}{\sigma} \operatorname{div}(\varepsilon\nabla p) = 0 \quad \text{and} \quad \operatorname{curl} w = \operatorname{curl} u.$$

Hence for any $v \in X^{\varepsilon}_{\mathrm{N}}(\Omega)$

$$\begin{aligned}\int_\Omega \operatorname{curl} w \cdot \operatorname{curl} v\, dx &= \int_\Omega \left(\sigma\, \varepsilon u \cdot v - \tau\, p \operatorname{div}(\varepsilon v)\right) dx = \int_\Omega (\sigma\, \varepsilon u + \tau\, \varepsilon\nabla p) \cdot v\, dx \\ &= \sigma \int_\Omega \varepsilon w \cdot v\, dx.\end{aligned}$$

Thus the pair (σ, w) belongs to the family in i) and σ is a multiple eigenvalue of (4.1.10). In this case we can split the eigenspace corresponding to σ according to the two families in i) and ii). □

In view of the previous lemma, we introduce the following definition valid for the present chapter, which is analogous to Definition 2.1.6.

Definition 4.1.19. Let Ω be as in (4.1.3). Let $\varepsilon \in \mathcal{E}$. An eigenvalue σ of problem (4.1.10) is said to be a *Maxwell eigenvalue* if $\sigma \neq 0$ and there exists $u \in X_{\mathrm{N}}^{\varepsilon}(\mathrm{div}\varepsilon 0, \Omega)$, $u \neq 0$, such that (σ, u) is an eigenpair of problem (4.1.9). In this case, we say that u is a *Maxwell eigenvector*. We denote the set of Maxwell eigenvalues by:

$$0 < \lambda_1[\varepsilon] \leq \lambda_2[\varepsilon] \leq \cdots \leq \lambda_n[\varepsilon] \leq \cdots \nearrow +\infty,$$

where we repeat the eigenvalues in accordance with their (Maxwell) multiplicity, i.e. the dimension of the space generated by the corresponding Maxwell eigenvectors.

We stress that the introduction of problem (4.1.10) is of technical nature, but in this chapter we are mostly interested in the behavior of Maxwell eigenvalues. Accordingly, we will focus more on the behaviour of $\{\lambda_j[\varepsilon]\}_{j\in\mathbb{N}} \subset \{\sigma[\varepsilon]\}_{j\in\mathbb{N}}$ than on the behaviour of all $\{\sigma[\varepsilon]\}_{j\in\mathbb{N}}$. Note also that the Maxwell eigenvalues $\{\lambda_j[\varepsilon]\}_{j\in\mathbb{N}}$ do not depend upon the choice of the parameter $\tau > 0$ introduced with the penalty term in (4.1.10), meaning that different values of τ provide exactly the same Maxwell spectrum.

4.2 Locally Lipschitz continuity of the eigenvalues

We first focus on the continuity of the eigenvalues $\sigma_j[\varepsilon]$ of problem (4.1.10), which in particular implies the continuity of the Maxwell eigenvalues $\lambda_j[\varepsilon]$. For the sake of simplicity, in this section we will fix $\tau = 1$. Note that the results presented below remain valid independently of the value of $\tau > 0$.

We find it convenient to recall the space

$$H_{\mathrm{N}}^1(\Omega) = \left\{u \in H^1(\Omega)^3 : \nu \times u = 0 \text{ on } \partial\Omega\right\} \tag{4.2.1}$$

endowed with the usual H^1-norm, first introduced in Section 1.4.3. Note that in view of the Gaffney inequality (4.1.8), valid under our assumptions (4.1.3), for every $\varepsilon \in \mathcal{E}$ the spaces $X_{\mathrm{N}}^{\varepsilon}(\Omega)$ and $H_{\mathrm{N}}^1(\Omega)$ coincide as sets, and their respective norms are equivalent. Hence one can use the space $H_{\mathrm{N}}^1(\Omega)$ for the variational characterization of the eigenvalues: the benefit lies in the fact that in this way we do not have to deal with Hilbert spaces that may depend on the permittivity parameter ε, allowing us to compare Rayleigh quotients relative to different permittivities. The trick of identifying the spaces $X^{\varepsilon}_{\mathrm{N}}(\Omega)$ and $H^1_{\mathrm{N}}(\Omega)$ will be used several times throughout this chapter.

Hence, instead of formula (4.1.14), we will make use of the following min-max formula

$$\sigma_j[\varepsilon] = \min_{\substack{V_j \subset H^1_{\mathrm{N}}(\Omega),\\ \dim V_j = j}} \max_{\substack{u \in V_j,\\ u \neq 0}} \frac{\int_\Omega |\operatorname{curl} u|^2\, dx + \int_\Omega |\operatorname{div}(\varepsilon u)|^2\, dx}{\int_\Omega \varepsilon u \cdot u\, dx} \tag{4.2.2}$$

in order to prove our continuity result. Before doing so, we first prove a locally uniform Gaffney inequality, that can be obtained exploiting the standard inequality (4.1.8) for a fixed permittivity.

Proposition 4.2.3. *Let Ω be as in (4.1.3). Let $\tilde{\varepsilon} \in \mathcal{E}$. Then there exist two constants $\delta, C_{\mathcal{G}} > 0$ such that*

$$\|u\|^2_{H^1(\Omega)^3} \le C_{\mathcal{G}} \left(\langle \varepsilon u, u \rangle_{L^2(\Omega)^3} + \|\operatorname{curl} u\|^2_{L^2(\Omega)^3} + \|\operatorname{div} \varepsilon u\|^2_{L^2(\Omega)} \right) \tag{4.2.4}$$

for all $u \in H^1_{\mathrm{N}}(\Omega)$ and for all $\varepsilon \in \mathcal{E}$ with $\|\varepsilon - \tilde{\varepsilon}\|_{W^{1,\infty}(\Omega)} < \delta$.

Proof. First of all, we observe that if $\hat{\varepsilon} \in \mathcal{E}$ then by Lemma 4.7.1 we have that

$$\operatorname{div}(\hat{\varepsilon} u) = \operatorname{tr}(\hat{\varepsilon} D u) + (\operatorname{div} \hat{\varepsilon}) \cdot u, \tag{4.2.5}$$

where $\operatorname{tr}(\cdot)$ denotes the trace operator, and if $\hat{\varepsilon} = \left(\hat{\varepsilon}^{(1)} | \, \hat{\varepsilon}^{(2)} | \, \hat{\varepsilon}^{(3)} \right)$ with $\hat{\varepsilon}^{(k)}$ denoting the k-th column, then $\operatorname{div} \hat{\varepsilon}$ is the vector field defined by

$$\operatorname{div} \hat{\varepsilon} = \left(\operatorname{div} \hat{\varepsilon}^{(1)}, \operatorname{div} \hat{\varepsilon}^{(2)}, \operatorname{div} \hat{\varepsilon}^{(3)} \right).$$

Moreover, if M is a 3×3 matrix then the following inequalities

$$|\operatorname{tr}(\hat{\varepsilon}(x) M)| \le 9 \|\hat{\varepsilon}\|_{L^\infty(\Omega)} |M|, \tag{4.2.6}$$

$$|\operatorname{div} \hat{\varepsilon}(x)| \le 3\sqrt{3} \|\hat{\varepsilon}\|_{W^{1,\infty}(\Omega)} \tag{4.2.7}$$

hold for a.e. $x \in \Omega$, where $|M|$ denotes the matrix norm $|M| := \max_{i,j} |M_{ij}|$.

Now we fix $u \in H^1_{\mathrm{N}}(\Omega)$ and $\varepsilon \in \mathcal{E}$. From (4.1.8) we know that the Gaffney inequality holds for $\tilde{\varepsilon}$, that is there exists a constant $C_{\tilde{\varepsilon}} > 0$ independent of u such that

$$\|u\|^2_{H^1(\Omega)^3} \le C_{\tilde{\varepsilon}} \left(\langle \tilde{\varepsilon} u, u \rangle_{L^2(\Omega)^3} + \|\operatorname{curl} u\|^2_{L^2(\Omega)^3} + \|\operatorname{div} \tilde{\varepsilon} u\|^2_{L^2(\Omega)} \right). \tag{4.2.8}$$

Moreover

$$\begin{aligned} \left| \operatorname{tr}(\tilde{\varepsilon} D u)^2 - \operatorname{tr}(\varepsilon D u)^2 \right| &= \left| \operatorname{tr}\left((\tilde{\varepsilon} + \varepsilon) D u \right) \operatorname{tr}\left((\tilde{\varepsilon} - \varepsilon) D u \right) \right| \\ &\le 9^2 \, \|\tilde{\varepsilon} + \varepsilon\|_{L^\infty(\Omega)} \|\tilde{\varepsilon} - \varepsilon\|_{L^\infty(\Omega)} \, |Du|^2 \end{aligned}$$

and

$$\begin{aligned}\Big|(\operatorname{div}\tilde\varepsilon\cdot u)^2-(\operatorname{div}\varepsilon\cdot u)^2\Big| &\le \Big|\big(\operatorname{div}(\tilde\varepsilon-\varepsilon)\cdot u\big)\,\big(\operatorname{div}(\tilde\varepsilon+\varepsilon)\cdot u\big)\Big|\\ &\le (3\sqrt{3})^2\,\|\tilde\varepsilon+\varepsilon\|_{W^{1,\infty}(\Omega)}\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\,|u|^2.\end{aligned}$$

We also have that

$$\begin{aligned}&2\Big|\operatorname{tr}(\tilde\varepsilon Du)\ \operatorname{div}\tilde\varepsilon\cdot u-\operatorname{tr}(\varepsilon Du)\ \operatorname{div}\varepsilon\cdot u\Big|\\ &\quad\le 2\Big|\operatorname{tr}(\tilde\varepsilon Du)\ \operatorname{div}(\tilde\varepsilon-\varepsilon)\cdot u\Big|+2\Big|\operatorname{tr}\big((\tilde\varepsilon-\varepsilon)Du\big)\ \operatorname{div}\varepsilon\cdot u\Big|\\ &\quad\le 2\cdot 9\cdot 3\sqrt{3}\,\Big(\|\tilde\varepsilon\|_{W^{1,\infty}(\Omega)}+\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\Big)\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\,2\,|u|\,|Du|\\ &\quad\le 54\sqrt{3}\,\Big(\|\tilde\varepsilon\|_{W^{1,\infty}(\Omega)}+\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\Big)\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\,(\,|u|^2+|Du|^2).\end{aligned}$$

Thus

$$\begin{aligned}&\Big|\|\operatorname{div}(\tilde\varepsilon u)\|^2_{L^2(\Omega)}-\|\operatorname{div}(\varepsilon u)\|^2_{L^2(\Omega)}\Big|\\ &\quad\le 54\sqrt{3}\,\Big(\|\tilde\varepsilon\|_{W^{1,\infty}(\Omega)}+\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}+\|\tilde\varepsilon+\varepsilon\|_{W^{1,\infty}(\Omega)}\Big)\\ &\qquad\cdot\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\left(\int_\Omega|u|^2dx+\int_\Omega|Du|^2dx\right).\end{aligned}\tag{4.2.9}$$

Furthermore, we have that

$$\Big|\langle\tilde\varepsilon u,u\rangle_{L^2(\Omega)^3}-\langle\varepsilon u,u\rangle_{L^2(\Omega)^3}\Big|\le 3\|\tilde\varepsilon-\varepsilon\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)^3}\,.\tag{4.2.10}$$

Therefore, making use of (4.2.9) and (4.2.10) in (4.2.8) we obtain that

$$\begin{aligned}&\|u\|^2_{H^1(\Omega)^3}\\ &\le C_{\tilde\varepsilon}\Big(\langle\varepsilon u,u\rangle_{L^2(\Omega)^3}+\|\operatorname{curl}u\|^2_{L^2(\Omega)^3}+\|\operatorname{div}\varepsilon u\|^2_{L^2(\Omega)}\Big)\\ &\quad+\Big(C_{\tilde\varepsilon}\,54\sqrt{3}\,\Big(\|\tilde\varepsilon\|_{W^{1,\infty}(\Omega)}+\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}+\|\tilde\varepsilon+\varepsilon\|_{W^{1,\infty}(\Omega)}\Big)+3\Big)\\ &\qquad\cdot\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\|u\|^2_{H^1(\Omega)^3}\\ &\le C_{\tilde\varepsilon}\Big(\langle\varepsilon u,u\rangle_{L^2(\Omega)^3}+\|\operatorname{curl}u\|^2_{L^2(\Omega)^3}+\|\operatorname{div}\varepsilon u\|^2_{L^2(\Omega)}\Big)\\ &\quad+\Big(C_{\tilde\varepsilon}\,2\cdot 54\sqrt{3}\,\Big(\|\tilde\varepsilon\|_{W^{1,\infty}(\Omega)}+\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\Big)+3\Big)\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\|u\|^2_{H^1(\Omega)^3}\,.\end{aligned}$$

Hence, taking $\delta>0$ small enough such that for all $\varepsilon\in\mathcal{E}$ with $\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}<\delta$ we have that

$$1-\Big(C_{\tilde\varepsilon}\,108\sqrt{3}\,\Big(\|\tilde\varepsilon\|_{W^{1,\infty}(\Omega)}+\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}\Big)+3\Big)\|\tilde\varepsilon-\varepsilon\|_{W^{1,\infty}(\Omega)}>0.$$

Hence we get that formula (4.2.4) holds with

$$C_{\mathcal{G}} := \frac{C_{\tilde{\varepsilon}}}{1 - \delta\left(C_{\tilde{\varepsilon}}\, 108\sqrt{3}\,\left(\|\tilde{\varepsilon}\|_{W^{1,\infty}(\Omega)} + \delta\right) + 3\right)}.$$

□

We are now ready to show that the eigenvalues $\sigma_j[\varepsilon]$ of problem (4.1.10) are locally Lipschitz continuous in ε.

Theorem 4.2.11. *Let Ω be as in (4.1.3). Let $j \in \mathbb{N}$ and $\varepsilon_1 \in \mathcal{E}$. Then there exist two constants $\delta, \tilde{C} > 0$ such that*

$$\left|\sigma_j[\varepsilon_1] - \sigma_j[\varepsilon_2]\right| \leq \tilde{C}\|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)} \tag{4.2.12}$$

for all $\varepsilon_2 \in \mathcal{E}$ such that $\|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)} < \delta$.

Proof. For the sake of simplicity in this proof, given $\varepsilon \in \mathcal{E}$ and $u \in H^1_{\mathrm{N}}(\Omega)$ we set

$$\mathcal{R}[u] := \int_\Omega |\operatorname{curl} u|^2 dx, \qquad \mathcal{D}_\varepsilon[u] := \int_\Omega |\operatorname{div}(\varepsilon u)|^2 dx.$$

Let be $\delta > 0$ be as in Proposition 4.2.3 with $\tilde{\varepsilon} = \varepsilon_1$. Let $\varepsilon_2 \in \mathcal{E}$ be such that $\|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)} < \delta$ and recall that m_{ε_1}, m_{ε_2} denote the constants associated with the coercivity of $\varepsilon_1, \varepsilon_2$ respectively (see (4.1.5)). Fix $u \in H^1_{\mathrm{N}}(\Omega)$. Then

$$\begin{aligned}
&\left|\frac{\mathcal{R}[u] + \mathcal{D}_{\varepsilon_1}[u]}{\int_\Omega \varepsilon_1 u \cdot u\, dx} - \frac{\mathcal{R}[u] + \mathcal{D}_{\varepsilon_2}[u]}{\int_\Omega \varepsilon_2 u \cdot u\, dx}\right| \qquad (4.2.13)\\
&\leq \frac{\mathcal{R}[u]\left|\int_\Omega (\varepsilon_2 - \varepsilon_1) u \cdot u\, dx\right| + \left|\mathcal{D}_{\varepsilon_1}[u]\int_\Omega \varepsilon_2\, u \cdot u\, dx - \mathcal{D}_{\varepsilon_2}[u]\int_\Omega \varepsilon_1\, u \cdot u\, dx\right|}{(\int_\Omega \varepsilon_1 u \cdot u\, dx)(\int_\Omega \varepsilon_2 u \cdot u\, dx)}\\
&\leq \frac{3\,\|\varepsilon_2 - \varepsilon_1\|_{L^\infty(\Omega)}\mathcal{R}[u]\int_\Omega |u|^2\, dx}{(\int_\Omega \varepsilon_1 u \cdot u\, dx)(\int_\Omega \varepsilon_2 u \cdot u\, dx)}\\
&\quad + \frac{\left|\mathcal{D}_{\varepsilon_1}[u]\int_\Omega \varepsilon_2 u\cdot u\, dx - \mathcal{D}_{\varepsilon_1}[u]\int_\Omega \varepsilon_1 u\cdot u\, dx + \mathcal{D}_{\varepsilon_1}[u]\int_\Omega \varepsilon_1 u\cdot u\, dx - \mathcal{D}_{\varepsilon_2}[u]\int_\Omega \varepsilon_1 u\cdot u\, dx\right|}{(\int_\Omega \varepsilon_1 u \cdot u\, dx)(\int_\Omega \varepsilon_2 u \cdot u\, dx)}\\
&\leq \frac{3\,\|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)}}{m_{\varepsilon_2}}\frac{\mathcal{R}[u] + \mathcal{D}_{\varepsilon_1}[u]}{\int_\Omega \varepsilon_1 u \cdot u\, dx} + \frac{\left|\mathcal{D}_{\varepsilon_1}[u] - \mathcal{D}_{\varepsilon_2}[u]\right|}{\int_\Omega \varepsilon_2 u \cdot u\, dx}.
\end{aligned}$$

We now focus on the second term in the right hand side of the above inequality. Arguing in the same way as in inequality (4.2.9) we deduce that there exist a constant $C > 0$ not depending on ε_1, ε_2 and u such that

$$\left|\mathcal{D}_{\varepsilon_1}[u] - \mathcal{D}_{\varepsilon_2}[u]\right| \leq C \max_{i=1,2}\left\{\|\varepsilon_i\|_{W^{1,\infty}(\Omega)}\right\}\|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)}\left(\int_\Omega |u|^2 dx + \int_\Omega |Du|^2 dx\right). \tag{4.2.14}$$

Moreover, thanks to the locally uniform Gaffney inequality (4.2.4) there exists a constant $C_{\mathcal{G}} > 0$ such that for $i = 1, 2$

$$\int_\Omega |Du|^2 dx \le C_{\mathcal{G}} \int_\Omega \left(\varepsilon_i u \cdot u + |\operatorname{curl} u|^2 + |\operatorname{div}(\varepsilon_i u)|^2\right) dx.$$

Using the above inequality with $i = 2$ we get

$$\frac{\int_\Omega |Du|^2 dx}{\int_\Omega \varepsilon_2 u \cdot u\, dx} \le C_{\mathcal{G}} \left(1 + \frac{\mathcal{R}[u] + \mathcal{D}_{\varepsilon_2}[u]}{\int_\Omega \varepsilon_2 u \cdot u\, dx}\right),$$

which applied to (4.2.14) yields

$$\begin{aligned}\frac{\left|\mathcal{D}_{\varepsilon_1}[u] - \mathcal{D}_{\varepsilon_2}[u]\right|}{\int_\Omega \varepsilon_2 u \cdot u\, dx} \le C \max_{i=1,2} \left\{\|\varepsilon_i\|_{W^{1,\infty}(\Omega)}\right\} \|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)} \\ \cdot \left(\frac{1}{m_{\varepsilon_2}} + C_{\mathcal{G}} \left(1 + \frac{\mathcal{R}[u] + \mathcal{D}_{\varepsilon_2}[u]}{\int_\Omega \varepsilon_2 u \cdot u\, dx}\right)\right).\end{aligned} \tag{4.2.15}$$

Thus it follows from (4.2.13) and (4.2.15) that

$$\begin{aligned}&\frac{\mathcal{R}[u] + \mathcal{D}_{\varepsilon_1}[u]}{\int_\Omega \varepsilon_1 u \cdot u\, dx} \left(1 - 3\frac{\|\varepsilon_2 - \varepsilon_1\|_{W^{1,\infty}(\Omega)}}{m_{\varepsilon_2}}\right) \\ &\le \frac{\mathcal{R}[u] + \mathcal{D}_{\varepsilon_2}[u]}{\int_\Omega \varepsilon_2 u \cdot u\, dx} \left(1 + C_{\mathcal{G}}\, C \max_{i=1,2} \left\{\|\varepsilon_i\|_{W^{1,\infty}(\Omega)}\right\} \|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)}\right) \\ &\quad + C \max_{i=1,2} \left\{\|\varepsilon_i\|_{W^{1,\infty}(\Omega)}\right\} \|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)} \left(\frac{1}{m_{\varepsilon_2}} + C_{\mathcal{G}}\right).\end{aligned} \tag{4.2.16}$$

Eventually taking a smaller $\delta > 0$, and taking the appropriate supremum and infimum in (4.2.16), the min-max formula (4.2.2) yields

$$\begin{aligned}\sigma_j[\varepsilon_1] - \sigma_j[\varepsilon_2] \le \Bigg(&\frac{3}{m_{\varepsilon_2}} \sigma_j[\varepsilon_1] + C_{\mathcal{G}}\, C \max_{i=1,2} \left\{\|\varepsilon_i\|_{W^{1,\infty}(\Omega)}\right\} \sigma_j[\varepsilon_2] \\ &+ C \max_{i=1,2} \left\{\|\varepsilon_i\|_{W^{1,\infty}(\Omega)}\right\} \left(\frac{1}{m_{\varepsilon_2}} + C_{\mathcal{G}}\right)\Bigg) \|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)}.\end{aligned} \tag{4.2.17}$$

Exchanging the role of ε_1 and ε_2, we can therefore get the inequality (4.2.12) but with a constant possibly depending also on ε_2, that can be taken as follows

$$\begin{aligned}\widehat{C}(\varepsilon_2) &:= 3 \max \left\{\frac{\sigma_j[\varepsilon_1]}{m_{\varepsilon_2}}, \frac{\sigma_j[\varepsilon_2]}{m_{\varepsilon_1}}\right\} \\ &\quad + C \max_{i=1,2} \left\{\|\varepsilon_i\|_{W^{1,\infty}(\Omega)}\right\} \left(C_G \max_{i=1,2} \{\sigma_j[\varepsilon_i]\} + \max_{i=1,2} \left\{\frac{1}{m_{\varepsilon_i}}\right\} + C_{\mathcal{G}}\right).\end{aligned} \tag{4.2.18}$$

In order to finish the proof, it only remains to show that this constant can be chosen uniform in ε_2. Up to taking a smaller δ, we note that by (4.1.6) the constant m_{ε_2} is uniformly bounded away from zero in ε_2. Indeed by (4.1.6) one has that

$$m_{\varepsilon_2} \geq m_{\varepsilon_1} - 3\delta.$$

Moreover, $\sigma_j[\varepsilon_2]$ is also locally uniformly bounded in ε_2. Indeed, from (4.2.5), (4.2.6) and (4.2.7) it is not difficult to see that there exists a constant $C' > 0$ not depending on ε_2 such that for all $u \in H^1_{\mathrm{N}}(\Omega)$ one has

$$\int_\Omega \left|\operatorname{div}(\varepsilon_2 u)\right|^2 dx \leq C' \|\varepsilon_2\|^2_{W^{1,\infty}(\Omega)} \int_\Omega \left(|u|^2 + |Du|^2\right) dx.$$

Then, applying the standard Gaffney inequality (with unitary permittivity, cf. (1.4.1)), which we know it is valid under our assumptions and involves a geometric constant, we get that, possibly changing the value of $C' > 0$, for all $u \in H^1_{\mathrm{N}}(\Omega)$

$$\int_\Omega \left|\operatorname{div}(\varepsilon_2 u)\right|^2 dx \leq C' \|\varepsilon_2\|^2_{W^{1,\infty}(\Omega)} \int_\Omega \left(|u|^2 + |\operatorname{curl} u|^2 + |\operatorname{div} u|^2\right) dx.$$

Hence, using the min-max formula (4.2.2) for $\sigma_j[\varepsilon_2]$ we have that

$$\begin{aligned}
\sigma_j[\varepsilon_2] &= \min_{\substack{V_j \subset H^1_{\mathrm{N}}(\Omega),\\ \dim V_j = j}} \max_{\substack{u \in V_j,\\ u \neq 0}} \frac{\int_\Omega |\operatorname{curl} u|^2 dx + \int_\Omega |\operatorname{div}(\varepsilon_2 u)|^2 dx}{\int_\Omega \varepsilon_2 u \cdot u \, dx} \\
&\leq (C'+1) \frac{\|\varepsilon_2\|^2_{W^{1,\infty}(\Omega)}}{m_{\varepsilon_2}} \min_{\substack{V_j \subset H^1_{\mathrm{N}}(\Omega),\\ \dim V_j = j}} \max_{\substack{u \in V_j,\\ u \neq 0}} \left(\frac{\int_\Omega |\operatorname{curl} u|^2 dx + \int_\Omega |\operatorname{div} u|^2 dx}{\int_\Omega |u|^2 dx} + 1 \right) \\
&= (C'+1) \frac{\|\varepsilon_2\|^2_{W^{1,\infty}(\Omega)}}{m_{\varepsilon_2}} (\sigma_j[\mathbb{I}_3] + 1) \\
&\leq (C'+1) \frac{\left(\|\varepsilon_1\|_{W^{1,\infty}(\Omega)} + \delta\right)^2}{m_{\varepsilon_1} - 3\delta} (\sigma_j[\mathbb{I}_3] + 1),
\end{aligned}$$

where $\sigma_j[\mathbb{I}_3]$ is the j-th eigenvalue of problem (4.1.10) with unitary permittivity. Accordingly, the constant $\widehat{C}(\varepsilon_2)$ defined in (4.2.18) is bounded above by a constant independent of ε_2 for all $\varepsilon_2 \in \mathcal{E}$ such that $\|\varepsilon_1 - \varepsilon_2\|_{W^{1,\infty}(\Omega)} < \delta$. Thus the inequality (4.2.12) is proved.

□

4.3 Weak* continuity of the Maxwell eigenvalues

It turns out that the Maxwell eigenvalues $\lambda_j[\varepsilon]$ depends with continuity on ε not only with respect to the strong topology of $W^{1,\infty}(\Omega)$, but also with respect to its weak* topology. Here we prove it only for scalar permittivities, but we plan to address the more general case of matrix-valued ε in the future.

Let α, β, γ be positive real numbers and assume that $0 < \alpha \le \beta$. We introduce the class of admissible permittivities

$$\mathcal{A}_{\alpha,\beta,\gamma} := \left\{ \varepsilon \in W^{1,\infty}(\Omega) : \varepsilon \text{ is scalar}, \alpha \le \varepsilon(x) \le \beta \text{ a.e. in } \Omega, \, \|\nabla\varepsilon\|_{L^\infty(\Omega)^3} \le \gamma \right\}. \tag{4.3.1}$$

Observe that $\mathcal{A}_{\alpha,\beta,\gamma} \subset \mathcal{E}$. If $\varepsilon \in \mathcal{A}_{\alpha,\beta,\gamma}$, with a slight abuse of notation, by ε we denote both the scalar function $\varepsilon(x)$ and the matrix-valued function $\varepsilon(x)\mathbb{I}_3$. Before proceeding any further, we first prove the following lemma.

Lemma 4.3.2. *Let Ω be as in (4.1.3), and let $\varepsilon \in \mathcal{A}_{\alpha,\beta,\gamma}$. Then there exists a constant $C > 0$, depending on α, β and γ, such that*

$$\sigma_j[\varepsilon] \le C(\sigma_j[\mathbb{I}_3] + 1)$$

for each $j \in \mathbb{N}$. Here $\sigma_j[\mathbb{I}_3]$ is the j-th eigenvalue of problem (4.1.10) with unitary permittivity.

Proof. Let $u \in X^\varepsilon_{\mathrm{N}}(\Omega)$. Since the Gaffney inequality (4.1.8) holds, then $\operatorname{div} u \in L^2(\Omega)$ and by formula (4.7.6) we have that

$$\operatorname{div}(\varepsilon u) = \varepsilon \operatorname{div} u + \nabla\varepsilon \cdot u$$

a.e. in Ω. Thus it is not difficult to see that

$$\int_\Omega |\operatorname{div}(\varepsilon u)|^2 dx \le 2\|\varepsilon\|^2_{W^{1,\infty}(\Omega)} \int_\Omega \left(|u|^2 + |\operatorname{div} u|^2 \right) dx.$$

Hence, using the min-max formula (4.2.2) we get

$$\begin{aligned}\sigma_j[\varepsilon] &\le \max\{1, 2\|\varepsilon\|^2_{W^{1,\infty}(\Omega)}\} \min_{\substack{V_j \subset H^1_{\mathrm{N}}(\Omega),\\ \dim V_j = j}} \max_{\substack{u \in V_j,\\ u \neq 0}} \frac{\int_\Omega |u|^2 dx + \int_\Omega |\operatorname{curl} u|^2 \, dx + \int_\Omega |\operatorname{div} u|^2 \, dx}{\alpha \int_\Omega |u|^2 dx} \\ &\le C\left(\sigma_j[\mathbb{I}_3] + 1\right),\end{aligned}$$

with C depending on α, β and γ. □

We consider $W^{1,\infty}(\Omega)$ as a subspace of $L^\infty(\Omega)^4$. The inclusion is interpreted by identifying a function $f \in W^{1,\infty}(\Omega)$ as the quadruple $(f, \partial_1 f, \partial_2 f, \partial_3 f) = (f, \nabla f) \in L^1(\Omega) \times L^1(\Omega)^3 = L^\infty(\Omega)^4$. Then we endow $W^{1,\infty}(\Omega)$ with the topology induced by $L^\infty(\Omega)^4$, where we consider the weak* topology on $L^\infty(\Omega)^4$, the (topological) dual space of $L^1(\Omega)^4$. We call this topology the (induced) weak* topology on $W^{1,\infty}(\Omega)$.

Note that the subspace $W^{1,\infty}(\Omega)$ is sequentially weakly* closed in $L^\infty(\Omega)^4$. Indeed it is not difficult to see that if $f_k \in W^{1,\infty}(\Omega)$ converges weakly* in $L^\infty(\Omega)^4$ to some quadruple $(f, \mathbf{f}) \in L^\infty(\Omega)^4$, that is

$$\int_\Omega f_k g \, dx + \int_\Omega \nabla f_k \cdot \mathbf{h} \, dx \xrightarrow[k\to\infty]{} \int_\Omega f g \, dx + \int_\Omega \mathbf{f} \cdot \mathbf{h} \, dx$$

for all $(g, \mathbf{h}) \in L^1(\Omega)^4$, then $f \in W^{1,\infty}(\Omega)$ with $\mathbf{f}$ being the (weak) gradient of f. Sequential weak* closedness is sufficient for our purposes, since we will work with bounded sets in $W^{1,\infty}(\Omega)$ (cf. definition (4.3.1)), that is bounded sets in $L^\infty(\Omega)^4$, which are metrizable with respect to the weak* topology due to the fact that $L^1(\Omega)^4$ is separable (see e.g. [94, Thm. 2.6.23]). Hence in the sequel the notions of sequentially weakly* closed and weakly* closed coincide. We are thus motivated to give the following

Definition 4.3.3. We say that a sequence of (scalar) functions $f_k \in W^{1,\infty}(\Omega)$ weakly* converges to a function $f \in W^{1,\infty}(\Omega)$, denoting it by $f_k \rightharpoonup^* f$, if

$$\int_\Omega f_k g \, dx + \int_\Omega \nabla f_k \cdot \mathbf{h} \, dx \xrightarrow[k\to\infty]{} \int_\Omega f g \, dx + \int_\Omega \nabla f \cdot \mathbf{h} \, dx \quad \text{for all } (g, \mathbf{h}) \in L^1(\Omega)^4. \tag{4.3.4}$$

In particular (4.3.4) implies that f_k converges weakly* to f in $L^\infty(\Omega)$, and ∇f_k converges weakly* to ∇f in $L^\infty(\Omega)^3$.

Remark 4.3.5. Observe that if $\{f_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ is such that $f_k \rightharpoonup^* f$, then the sequence is bounded in $W^{1,\infty}(\Omega)$. Since $\Omega \subset \mathbb{R}^3$ is bounded, then $W^{1,\infty}(\Omega)$ is contained in $W^{1,p}(\Omega)$ for any $p \geq 1$. Taking $p > 3$ and observing that under our assumptions (4.1.3) on Ω we can use the Rellich-Kondrachov embedding theorem (see, e.g., [2, Thm. 6.3]), we have that we can also assume that $f_k \to f$ strongly in $L^\infty(\Omega)$.

We can now state the main result of this section. The following theorem is inspired by [67, Thm. 9.1.3], which deals with the eigenvalues of a Dirichlet Laplacian problem where a density function parameter models the study of non-homogeneous membranes. The proof is in the same spirit of that one of [82, Thm. 3.1], where the authors prove a similar result for general elliptic operators of arbitrary order subject to homogeneous boundary conditions, upon variation of the mass density. We also refer to the works by Cox and McLaughlin [43, 44] for further details.

Theorem 4.3.6. *Let Ω be a bounded domain of $\mathbb{R}^3$ of class $C^{1,1}$. For all $j \in \mathbb{N}$, the map $\varepsilon \mapsto \lambda_j[\varepsilon]$ from $\mathcal{A}_{\alpha,\beta,\gamma}$ to $\mathbb{R}$ is weakly* continuous for all $j \in \mathbb{N}$, where $\lambda_j[\varepsilon]$ is the j-th Maxwell eigenvalue of problem (4.1.10).*

Proof. Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_{\alpha,\beta,\gamma}$. We need to show that if $\varepsilon_k \rightharpoonup^* \varepsilon$ in $W^{1,\infty}(\Omega)$ as $k \to \infty$ (in the sense of (4.3.4)), then $\lambda_j[\varepsilon_k] \to \lambda_j[\varepsilon]$ for all $j \in \mathbb{N}$.

Observe that for any $j \in \mathbb{N}$, thanks to Lemma 4.3.2, we have that the sequence $\{\lambda_j[\varepsilon_k]\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Let $L_j := \sup_{k\in\mathbb{N}}\{\lambda_j[\varepsilon_k]\} < \infty$ so that

$$\lambda_j[\varepsilon_k] \le L_j \quad \text{for all } k \in \mathbb{N}. \tag{4.3.7}$$

Let $u_j[\varepsilon_k]$ be the Maxwell eigenfunctions associated with the the Maxwell eigenvalues $\lambda_j[\varepsilon_k]$. In particular, their ε_k-divergence is null, hence the weak formulation (4.1.10) reads as follows:

$$\int_\Omega \operatorname{curl} u_j[\varepsilon_k] \cdot \operatorname{curl} v \, dx = \lambda_j[\varepsilon_k] \int_\Omega \varepsilon_k \, u_k[\varepsilon_k] \cdot v \, dx \quad \text{for all } v \in X_{\mathrm{N}}^{\varepsilon_k}(\Omega). \tag{4.3.8}$$

Assume that the eigenfunctions $\{u_j[\varepsilon_k]\}_{j\in\mathbb{N}}$ form an $L^2_{\varepsilon_k}(\Omega)$-orthonormal sequence, that is they are such that $J_{\varepsilon_k}[u_j[\varepsilon_k]][u_l[\varepsilon_k]] = \int_\Omega \varepsilon_k \, u_j[\varepsilon_k] \cdot u_l[\varepsilon_k] \, dx = \delta_{jl}$ for all $j, l \in \mathbb{N}$. Thus

$$\int_\Omega \left|u_j[\varepsilon_k]\right|^2 \, dx \le \frac{1}{\alpha} \int_\Omega \varepsilon_k \left|u_j[\varepsilon_k]\right|^2 \, dx \le \frac{1}{\alpha} \tag{4.3.9}$$

and

$$\int_\Omega \left|\operatorname{curl} u_j[\varepsilon_k]\right|^2 dx = \lambda_j[\varepsilon_k] \int_\Omega \varepsilon_k \left|u_j[\varepsilon_k]\right|^2 dx \le L_j \tag{4.3.10}$$

for all $k \in \mathbb{N}$.

Observe that by Remark 4.7.5 for any $k \in \mathbb{N}$ the Maxwell eigenvectors $u_j[\varepsilon_k]$ also belong to $X_{\mathrm{N}}(\Omega)$ and

$$\operatorname{div} u_j[\varepsilon_k] = \frac{\operatorname{div}(\varepsilon_k u_j[\varepsilon_k]) - \nabla\varepsilon_k \cdot u_j[\varepsilon_k]}{\varepsilon_k} = -\frac{\nabla\varepsilon_k \cdot u_j[\varepsilon_k]}{\varepsilon_k}. \tag{4.3.11}$$

In particular

$$\int_\Omega \left|\operatorname{div} u_j[\varepsilon_k]\right|^2 dx \le \frac{\|\nabla\varepsilon_k\|^2_{L^\infty(\Omega)^3}}{\alpha^2} \int_\Omega \left|u_j[\varepsilon_k]\right|^2 dx \le \frac{\gamma^2}{\alpha^3} < \infty$$

for all $k \in \mathbb{N}$, hence the sequence $\{u_j[\varepsilon_k]\}_{k\in\mathbb{N}}$ is bounded in $X_{\mathrm{N}}(\Omega)$. It follows that, possibly passing to a subsequence, there exists $\bar{u}_j \in X_{\mathrm{N}}(\Omega)$ such that $u_j[\varepsilon_k]$ weakly converges to $\bar{u}_j$ as $k \to \infty$ in $X_{\mathrm{N}}(\Omega)$, and by (4.3.7) there exists $\bar{\lambda}_j \in \mathbb{R}$ such that $\lambda_j[\varepsilon_k] \to \bar{\lambda}_j$ as $k \to \infty$. Moreover, since the embedding of $X_{\mathrm{N}}(\Omega)$ into $L^2(\Omega)^3$ is compact (see e.g. [113]) we can also assume that $u_j[\varepsilon_k] \to \bar{u}_j$ strongly in $L^2(\Omega)^3$ as $k \to \infty$.

Now, observing that by Remark 4.7.5 also the test functions $v \in X_{\mathrm{N}}^{\varepsilon_k}(\Omega)$ belongs to $X_{\mathrm{N}}(\Omega)$ (cf. also (4.3.11)), passing to the limit in (4.3.8) we get that

$$\int_\Omega \operatorname{curl} \bar{u}_j \cdot \operatorname{curl} v \, dx = \bar{\lambda}_j \int_\Omega \varepsilon \, \bar{u}_j \cdot v \, dx \quad \text{for all } v \in X_{\mathrm{N}}(\Omega). \tag{4.3.12}$$

The limit in the left-hand side is obvious since $u_j[\varepsilon_k] \rightharpoonup \bar{u}_j$ weakly in $X_{\mathrm{N}}(\Omega)$ as $k \to \infty$. In order to see the limit in the right-hand side, one computes

$$\begin{aligned}\left|\int_{\Omega}(\varepsilon_k u_j[\varepsilon_k] - \varepsilon\bar{u}_j)\cdot\varphi dx\right| &\le \left|\int_{\Omega}\varepsilon_k(u_j[\varepsilon_k]-\bar{u}_j)\cdot\varphi\, dx\right| + \left|\int_{\Omega}(\varepsilon_k-\varepsilon)\,\bar{u}_j\cdot\varphi\, dx\right| \\ &\le \|\varepsilon_k\|_{L^\infty(\Omega)}\big\|u_j[\varepsilon_k]-\bar{u}_j\big\|_{L^2(\Omega)^3}\|\varphi\|_{L^2(\Omega)^3} + \left|\int_{\Omega}(\varepsilon_k-\varepsilon)\,\bar{u}_j\cdot\varphi\, dx\right|.\end{aligned}$$

The first term of the sum goes to 0 as $k \to \infty$ since $u_j[\varepsilon_k]$ converges to $\bar{u}_j$ strongly in $L^2(\Omega)^3$, while the second term goes to 0 since $\varepsilon_k \rightharpoonup^* \varepsilon$ and $\bar{u}_j \cdot \varphi \in L^1(\Omega)$.

Furthermore, we have that $\operatorname{div}(\varepsilon\bar{u}_j) = 0$. Indeed fix any $\eta \in C_c^\infty(\Omega)$. Then for all $k \in \mathbb{N}$

$$\int_{\Omega} \varepsilon_k\, u_j[\varepsilon_k]\cdot\nabla\eta\, dx = 0.$$

since $\operatorname{div}(\varepsilon_k u_j[\varepsilon_k]) = 0$. Taking the limit as $k \to \infty$ and reasoning as above, we get that

$$\int_{\Omega} \varepsilon\,\bar{u}_j\cdot\nabla\eta = 0,$$

which means exactly that $\operatorname{div}(\varepsilon\bar{u}_j) = 0$ in Ω. Therefore from (4.3.12) it follows that $\bar{u}_j$ is a Maxwell eigenfunction of problem (4.1.10) corresponding to the Maxwell eigenvalue $\bar{\lambda}_j$.

Observe that since $J_\varepsilon[\bar{u}_j][\bar{u}_l] = \delta_{jl}$ for all $j, l \in \mathbb{N}$, then $\{\bar{\lambda}_j\}_{j\in\mathbb{N}}$ is a divergent sequence.

In order to prove that $\bar{\lambda}_j = \lambda_j[\varepsilon]$ for all $j \in \mathbb{N}$, we assume by contradiction that there exists a Maxwell eigenfunction $\bar{u} \in X_{\mathrm{N}}^{\varepsilon}(\Omega)$ of problem (4.1.10), with $\operatorname{div}(\varepsilon\bar{u}) = 0$, corresponding to a Maxwell eigenvalue $\bar{\lambda}$, such that it is orthogonal in $L^2_\varepsilon(\Omega)$ to all the $\bar{u}_j$, $j \in \mathbb{N}$, namely that $\langle\bar{u}, \bar{u}_j\rangle_\varepsilon = J_\varepsilon[\bar{u}][\bar{u}_j] = 0$ for all $j \in \mathbb{N}$.

Before proceeding any further, we make a clarification. In the sequel, we fix $\tau > 0$ big enough so that the first j eigenvalues of problem (4.1.10) are all of pure Maxwell type, i.e. without resonances with eigenfunctions deriving from problem (4.1.16) (cf. also Remark 2.1.9). This is done in order to avoid complications in the argument that follows, especially in the use of Theorem 4.7.14. The aim is to get a contradiction: in particular we show that inequality (4.3.19), which does not depend on τ, holds for every $j \in \mathbb{N}$. This is achieved by fixing the value of $\tau > 0$ depending on $j \in \mathbb{N}$ on every iteration.

Assume that $\bar{u}$ is such that $\|\bar{u}\|_{L^2_\varepsilon(\Omega)} = 1/(\bar{\lambda}+1)$. By Theorem 4.7.14 we get that

$$-\frac{1}{2(\lambda_j[\varepsilon_k]+1)} \le \frac{1}{2}T_{\varepsilon_k}[u][u] - \big\|u - P_{j-1,\varepsilon_k}u\big\|_{L^2_{\varepsilon_k}(\Omega)} \tag{4.3.13}$$

for all $u \in X_{\mathrm{N}}^{\varepsilon_k}(\Omega)$ and $j, k \in \mathbb{N}$. Here T_ε is the operator defined in (4.1.11) and P_{j-1,ε_k} denotes the orthogonal projection in $L^2_{\varepsilon_k}(\Omega)$ of u onto the linear span of $u_1[\varepsilon_k], \dots, u_{j-1}[\varepsilon_k]$.

Note that by Remark 4.7.5 the limit function $\bar{u} \in X_{\mathrm{N}}(\Omega)$ also belongs to $X_{\mathrm{N}}^{\varepsilon_k}(\Omega)$ for any $k \in \mathbb{N}$. Since $\varepsilon_k \rightharpoonup^* \varepsilon$ as $k \to \infty$ then

$$\int_\Omega \varepsilon_k \bar{u} \cdot \bar{u}\, dx \xrightarrow[k\to\infty]{} \int_\Omega \varepsilon \bar{u} \cdot \bar{u}\, dx. \tag{4.3.14}$$

Moreover we also have that

$$\int_\Omega \left(\left(\operatorname{div}(\varepsilon_k \bar{u})\right)^2 - \left(\operatorname{div}(\varepsilon \bar{u})\right)^2 \right) dx = \int_\Omega \left(\operatorname{div}(\varepsilon_k \bar{u})\right)^2 dx \xrightarrow[k\to\infty]{} 0. \tag{4.3.15}$$

To see the limit above, thanks to formula (4.7.6) we just need to see that

$$\int_\Omega \left((\varepsilon_k \operatorname{div} \bar{u})^2 - (\varepsilon \operatorname{div} \bar{u})^2 \right) dx \xrightarrow[k\to\infty]{} 0, \tag{4.3.16}$$

$$\int_\Omega \left(\varepsilon_k \operatorname{div} \bar{u}\ \nabla \varepsilon_k \cdot \bar{u} - \varepsilon \operatorname{div} \bar{u}\ \nabla \varepsilon \cdot \bar{u} \right) dx \xrightarrow[k\to\infty]{} 0, \tag{4.3.17}$$

and

$$\int_\Omega \left((\nabla \varepsilon_k \cdot \bar{u})^2 - (\nabla \varepsilon \cdot \bar{u})^2 \right) dx \xrightarrow[k\to\infty]{} 0. \tag{4.3.18}$$

We will only show limit (4.3.18), since the other two limits are similar. We have that

$$\begin{aligned} \int_\Omega \left((\nabla \varepsilon_k \cdot \bar{u})^2 - (\nabla \varepsilon \cdot \bar{u})^2 \right) dx &\leq \int_\Omega \left(\nabla (\varepsilon_k + \varepsilon) \cdot \bar{u} \right) \left(\nabla \varepsilon_k \cdot \bar{u} - \nabla \varepsilon \cdot \bar{u} \right) dx \\ &\leq C \int_\Omega \nabla (\varepsilon_k - \varepsilon) \cdot \bar{u} |\bar{u}|\, dx \xrightarrow[k\to\infty]{} 0, \end{aligned}$$

since $\varepsilon \rightharpoonup^* \varepsilon$ as $k \to \infty$ and the vector field $\bar{u}|\bar{u}|$ is in $L^1(\Omega)^3$. Here $C > 0$ is a constant depending on γ.

Limits (4.3.14) and (4.3.15) show that $T_{\varepsilon_k}[\bar{u}][\bar{u}] \xrightarrow[k\to\infty]{} T_\varepsilon[\bar{u}][\bar{u}]$. Hence, putting $u = \bar{u}$ in (4.3.13) and passing to the limit as $k \to \infty$ we get that for all $j \in \mathbb{N}$

$$-\frac{1}{2(\bar{\lambda}_j + 1)} \leq \frac{1}{2} T_\varepsilon[\bar{u}][\bar{u}] - \|\bar{u}\|_{L^2_\varepsilon(\Omega)} = -\frac{1}{2(\bar{\lambda} + 1)} < 0. \tag{4.3.19}$$

Indeed from the fact that $\varepsilon_k \rightharpoonup^* \varepsilon$, $u_j[\varepsilon_k] \to \bar{u}_j$ strongly in $L^2(\Omega)^3$ as $k \to \infty$, and that $\langle \bar{u}, \bar{u}_i \rangle_\varepsilon = 0$ for all $i \in \mathbb{N}$, we get that $\langle \bar{u}, u_i[\varepsilon_k] \rangle_{\varepsilon_k} \xrightarrow[k\to\infty]{} 0$ and thus

$$\left\| P_{j-1,\varepsilon_k} \bar{u} \right\|_{L^2_{\varepsilon_k}(\Omega)} \xrightarrow[k\to\infty]{} 0.$$

Inequality (4.3.19) in turn implies that the sequence $\{\bar{\lambda}_j\}_{j\in\mathbb{N}}$ is bounded, hence the contradiction. Therefore necessarily $\bar{\lambda}_j = \lambda_j[\varepsilon]$ for every $j \in \mathbb{N}$ and the theorem is proved. □

4.4 Analiticity and the ε-derivative

In the previous section we have showed that the eigenvalues $\sigma_j[\varepsilon]$ of the modified problem (4.1.10), and then in particular the Maxwell eigenvalues $\lambda_j[\varepsilon]$, are locally Lipschitz continuous in $\mathcal{E}$. Here instead we are interested in proving higher regularity properties. More in detail we plan to show that the eigenvalues depend analytically upon ε when the permittivity varies in a specific (open) subset of the admissible set $\mathcal{E}$. We also provide an explicit formula for their ε-derivative, which will be useful later on to prove Theorem 4.5.1 and in Section 4.6 to prove the generic simplicity of the Maxwell spectrum. As already mentioned in the introduction, if we consider a multiple eigenvalue, a perturbation of the permittivity can in principle split the eigenvalue into different eigenvalues of lower multiplicity and thus the corresponding branches can have a corner at the splitting point. In this case we will not even have differentiability. Our strategy in order to bypass this problem is to consider the symmetric functions of multiple eigenvalues. This point of view has been first introduced by Lamberti and Lanza de Cristoforis in [79] and later successfully adopted in many other works (see, e.g., [21, 23, 84, 81, 87]).

Recall that

$$0 < \sigma_1[\varepsilon] \leq \sigma_2[\varepsilon] \leq \cdots \leq \sigma_n[\varepsilon] \leq \cdots \nearrow +\infty.$$

are the eigenvalues of problem (4.1.10), while instead

$$0 < \lambda_1[\varepsilon] \leq \lambda_2[\varepsilon] \leq \cdots \leq \lambda_n[\varepsilon] \leq \cdots \nearrow +\infty.$$

are the subset of Maxwell eigenvalues of problem (4.1.10) (see Definition 4.1.19). Also recall that, by Lemma 4.1.13, $\{\sigma_j[\varepsilon]\}_{j\in\mathbb{N}}$ coincide with the reciprocal minus one of the eigenvalues of the operator S_ε defined in (4.1.12). In order to obtain an explicit formula for the derivatives of the Maxwell eigenvalues with respect to the permittivity ε we need the following technical lemma.

Lemma 4.4.1. *Let Ω be as in (4.1.3). Let $\tilde{\varepsilon} \in \mathcal{E}$ and $\tilde{u}, \tilde{v} \in X_{\mathrm{N}}^{\tilde{\varepsilon}}(\operatorname{div} \tilde{\varepsilon}0, \Omega)$ be two Maxwell eigenvectors associated with a Maxwell eigenvalue $\tilde{\lambda}$ with permittivity $\tilde{\varepsilon}$. Then*

$$\langle d|_{\varepsilon=\tilde{\varepsilon}} S_\varepsilon[\eta][\tilde{u}], \tilde{v}\rangle_{\tilde{\varepsilon}} = \tilde{\lambda}(\tilde{\lambda}+1)^{-2} \int_\Omega \eta\tilde{u}\cdot\tilde{v}\,dx \tag{4.4.2}$$

for all $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$.

Proof. Under our assumptions on Ω, the space $X_{\mathrm{N}}^{\varepsilon}(\Omega)$ coincides with the space $H_{\mathrm{N}}^1(\Omega)$ introduced in (4.2.1) and their norm are equivalent thanks to the Gaffney inequality (4.1.8). Then, it is easily seen that the compact self-adjoint operator S_ε in $L^2(\Omega)^3$ is obtained by compositions and inversions of real-analytic maps in ε

(such as linear and multilinear continuous maps). As a consequence S_ε depends real analytically upon ε.

Let now $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$. Since $J_{\tilde\varepsilon}[\tilde u] = (\tilde\lambda + 1)^{-1} T_{\tilde\varepsilon}[\tilde u]$, $J_\varepsilon[\tilde v] = (\tilde\lambda + 1)^{-1} T_{\tilde\varepsilon}[\tilde v]$, and $S_{\tilde\varepsilon}$ is symmetric, we have that

$$
\begin{aligned}
&\langle d|_{\varepsilon=\tilde\varepsilon} S_\varepsilon[\eta][\tilde u], \tilde v\rangle_{\tilde\varepsilon} \\
&\quad = \langle \iota_\varepsilon \circ T_{\tilde\varepsilon}^{-1} \circ d|_{\varepsilon=\tilde\varepsilon} J_\varepsilon[\eta][\tilde u], \tilde v\rangle_{\tilde\varepsilon} + \langle \iota_\varepsilon \circ d|_{\varepsilon=\tilde\varepsilon} T_\varepsilon^{-1}[\eta] \circ J_{\tilde\varepsilon}[\tilde u], \tilde v\rangle_{\tilde\varepsilon} \\
&\quad = J_{\tilde\varepsilon}[\tilde v]\left[\iota_\varepsilon \circ T_{\tilde\varepsilon}^{-1} \circ d|_{\varepsilon=\tilde\varepsilon} J_\varepsilon[\eta][\tilde u]\right] + J_{\tilde\varepsilon}[\tilde v]\left[\iota_\varepsilon \circ d|_{\varepsilon=\tilde\varepsilon} T_\varepsilon^{-1}[\eta] \circ J_{\tilde\varepsilon}[\tilde u]\right] \\
&\quad = (\tilde\lambda + 1)^{-1} T_{\tilde\varepsilon}[\tilde v]\left[T_{\tilde\varepsilon}^{-1} \circ d|_{\varepsilon=\tilde\varepsilon} J_\varepsilon[\eta][\tilde u] - T_{\tilde\varepsilon}^{-1} \circ d|_{\varepsilon=\tilde\varepsilon} T_\varepsilon[\eta] \circ T_{\tilde\varepsilon}^{-1} \circ J_{\tilde\varepsilon}[\tilde u]\right] \\
&\quad = (\tilde\lambda + 1)^{-1} T_{\tilde\varepsilon}\left[T_{\tilde\varepsilon}^{-1} \circ d|_{\varepsilon=\tilde\varepsilon} J_\varepsilon[\eta][\tilde u] - T_{\tilde\varepsilon}^{-1} \circ d|_{\varepsilon=\tilde\varepsilon} T_\varepsilon[\eta] \circ T_{\tilde\varepsilon}^{-1} \circ (\tilde\lambda + 1)^{-1} T_{\tilde\varepsilon}[\tilde u]\right][\tilde v] \\
&\quad = (\tilde\lambda + 1)^{-1}\left(d|_{\varepsilon=\tilde\varepsilon} J_\varepsilon[\eta][\tilde u][\tilde v] - (\tilde\lambda + 1)^{-1} d|_{\varepsilon=\tilde\varepsilon} T_\varepsilon[\eta][\tilde u][\tilde v]\right).
\end{aligned}
\tag{4.4.3}
$$

Moreover, by standard calculus,

$$
d|_{\varepsilon=\tilde\varepsilon} J_\varepsilon[\eta][\tilde u][\tilde v] = \int_\Omega \eta \tilde u \cdot \tilde v \, dx \tag{4.4.4}
$$

and

$$
d|_{\varepsilon=\tilde\varepsilon} T_\varepsilon[\eta][\tilde u][\tilde v] = \int_\Omega \eta \tilde u \cdot \tilde v \, dx + \int_\Omega \left(\mathrm{div}(\tilde\varepsilon \tilde u)\ \mathrm{div}(\eta \tilde v) + \mathrm{div}(\eta \tilde u)\ \mathrm{div}(\tilde\varepsilon \tilde v)\right) dx. \tag{4.4.5}
$$

Since $\mathrm{div}(\tilde\varepsilon \tilde u) = 0 = \mathrm{div}(\tilde\varepsilon \tilde v)$ in Ω, using (4.4.3), (4.4.4) and (4.4.5), we get (4.4.2). □

Following [79], given a finite set of indices $F \subset \mathbb{N}$, we consider those permittivities $\varepsilon \in \mathcal{E}$ for which Maxwell eigenvalues with indices in F do not coincide with Maxwell eigenvalues with indices outiside F. We then introduce the following sets:

$$
\mathcal{E}[F] := \left\{\varepsilon \in \mathcal{E} : \lambda_j[\varepsilon] \neq \lambda_l[\varepsilon]\ \forall j \in F, l \in \mathbb{N} \setminus F\right\} \tag{4.4.6}
$$

and

$$
\Theta[F] := \left\{\varepsilon \in \mathcal{E}[F] : \lambda_j[\varepsilon] \text{ have a common value } \lambda_F[\varepsilon] \text{ for all } j \in F\right\}.
$$

Let $\varepsilon \in \mathcal{E}[F]$. The elementary symmetric function of degree $s \in \{1, \ldots, |F|\}$ of the Maxwell eigenvalues with indexes in F is defined by

$$
\Lambda_{F,s}[\varepsilon] := \sum_{\substack{j_1,\ldots,j_s \in F \\ j_1 < \cdots < j_s}} \lambda_{j_1}[\varepsilon] \cdots \lambda_{j_s}[\varepsilon].
$$

In the following theorem we show that the maps $\varepsilon \mapsto \Lambda_{F,s}[\varepsilon]$ are real analytical on $\mathcal{E}[F]$ and we compute their Fréchet derivatives with respect to ε.

Theorem 4.4.7. *Let Ω be as in (4.1.3). Let F be a finite subset of $\mathbb{N}$ and $s \in \{1, \dots, |F|\}$. Then $\mathcal{E}[F]$ is open in $W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$ and the elementary symmetric function $\Lambda_{F,s}$ depend real analytically upon $\varepsilon \in \mathcal{E}[F]$.*

Moreover, if $\{F_1, \dots, F_n\}$ is a partition of F and $\tilde{\varepsilon} \in \bigcap_{k=1}^n \Theta[F]$ is such that for each $k = 1, \dots, n$ the Maxwell eigenvalues $\lambda_j[\tilde{\varepsilon}]$ assume the common value $\lambda_{F_k}[\tilde{\varepsilon}]$ for all $j \in F_k$, then the differential of the function $\Lambda_{F,s}$ at the point $\tilde{\varepsilon}$ are given by the formula

$$d|_{\varepsilon=\tilde{\varepsilon}}\Lambda_{F,s}[\eta] = -\sum_{k=1}^{n} c_k \sum_{l\in F_k} \int_\Omega \eta \tilde{E}^{(l)} \cdot \tilde{E}^{(l)}\, dx, \tag{4.4.8}$$

for all $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$, where

$$c_k := \sum_{\substack{0\le s_1\le |F_1| \\ \dots \\ 0\le s_n\le |F_n| \\ s_1+\dots+s_n=s}} \binom{|F_k|-1}{s_k-1} (\lambda_{F_k}[\tilde{\varepsilon}])^{s_k} \prod_{\substack{j=1\\ j\neq k}}^{n} \binom{|F_j|}{s_j} (\lambda_{F_j}[\tilde{\varepsilon}])^{s_j}, \tag{4.4.9}$$

and for each $k = 1, \dots, n$, $\{\tilde{E}^{(l)}\}_{l\in F_k}$ is an orthonormal basis in $L^2_{\tilde{\varepsilon}}(\Omega)$ of Maxwell eigenvectors for the eigenspace associated with $\lambda_{F_k}[\tilde{\varepsilon}]$.

Proof. Let $\tilde{\varepsilon} \in \mathcal{E}$. As we have already pointed out, Maxwell eigenvalues are independent on the choice of the parameter $\tau > 0$ in (4.1.10). Thus, to avoid problems of different enumeration between Maxwell eigenvalues and the eigenvalues of S_ε, we can fix τ big enough such that all the Maxwell eigenvalues $\{\lambda_j[\tilde{\varepsilon}]\}_{j\in F}$ are strictly smaller than any other eigenvalue of (4.1.10) which is not a Maxwell eigenvalue (i.e. an eigenvalue belonging to the family ii) in Theorem 4.1.15). In this way $\sigma_j[\tilde{\varepsilon}] = \lambda_j[\tilde{\varepsilon}]$ for all $j \in F$.

The eigenvalues μ_j of the operator S_ε and the eigenvalues σ_j of (4.1.10) satisfy $\mu_j = (\sigma_j + 1)^{-1}$. Then the sets $\mathcal{E}[F]$ and $\{\varepsilon \in \mathcal{E} : \mu_j[\varepsilon] \neq \mu_l[\varepsilon] \ \forall j \in F,\ l \in \mathbb{N} \setminus F\}$ coincide locally around $\tilde{\varepsilon}$. By Lemma 4.1.13, S_ε is a compact self-adjoint operator acting on $L^2_\varepsilon(\Omega)$. Furthermore, as already pointed out in the proof of Lemma 4.4.1, the operator S_ε depends real analytically on ε. In the same way one shows that also the inner product $\langle\cdot,\cdot\rangle_\varepsilon$ on $L^2(\Omega)^3$ depends real analytically on ε. Therefore, by the abstract result of Lamberti and Lanza de Cristoforis [79, Thm. 2.30], we have that the set $\{\varepsilon \in \mathcal{E} : \mu_j[\varepsilon] \neq \mu_l[\varepsilon] \ \forall j \in F, l \in \mathbb{N} \setminus F\}$ is open in $W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$ and that the function

$$M_{F,s}[\varepsilon] := \sum_{\substack{j_1,\dots,j_s\in F\\ j_1<\dots<j_s}} \mu_{j_1}[\varepsilon] \cdots \mu_{j_s}[\varepsilon]$$

depend real analytically on $\varepsilon \in \mathcal{E}[F]$. From this, to infer the real analyticity of the functions $\Lambda_{F,s}$ on $\varepsilon \in \mathcal{E}[F]$, one can just observe that if we set

$$\hat{\Lambda}_{F,s}[\varepsilon] := \sum_{\substack{j_1,\dots,j_s \in F \\ j_1 < \dots < j_s}} (\lambda_{j_1}[\varepsilon]+1)\cdots(\lambda_{j_s}[\varepsilon]+1),$$

then we have

$$\hat{\Lambda}_{F,s}[\varepsilon] = \frac{M_{F,\,|F|-s}[\varepsilon]}{M_{F,\,|F|}[\varepsilon]}$$

and by elementary combinatorics

$$\Lambda_{F,s}[\varepsilon] = \sum_{k=0}^{s} (-1)^{s-k} \binom{|F|-k}{s-k} \hat{\Lambda}_{F,k}[\varepsilon], \tag{4.4.10}$$

where we have set $\hat{\Lambda}_{F,0} = 1$. Then we can deduce that locally around $\tilde{\varepsilon}$ the maps $\Lambda_{F,s}[\varepsilon]$ are real analytic and accordingly the analyticity part of the statement follows since $\tilde{\varepsilon}$ is arbitrary.

Next, we turn to prove formula (4.4.8). We start by the case $n=1$, that is $F_1 = F$ and $\tilde{\varepsilon} \in \Theta[F]$. Let $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$. By [79, Thm. 2.30] we get that

$$d|_{\varepsilon=\tilde{\varepsilon}} M_{F,s}[\eta] = \binom{|F|-1}{s-1} (\lambda_F[\tilde{\varepsilon}]+1)^{1-s} \sum_{l\in F} \langle d|_{\varepsilon=\tilde{\varepsilon}}\, S_\varepsilon[\eta][\tilde{E}^{(l)}], \tilde{E}^{(l)}\rangle_{\tilde{\varepsilon}}.$$

Moreover, by using formula (4.4.2) of Lemma 4.4.1, we have that

$$\begin{aligned}
&d|_{\varepsilon=\tilde{\varepsilon}} \hat{\Lambda}_{F,s}[\eta] \\
&= \left(d|_{\varepsilon=\tilde{\varepsilon}} M_{F,|F|-s}[\eta] M_{F,|F|}\,[\tilde{\varepsilon}] - M_{F,|F|-s}\,[\tilde{\varepsilon}]\, d|_{\varepsilon=\tilde{\varepsilon}} M_{F,|F|}\,[\eta]\right) (\lambda_F[\tilde{\varepsilon}]+1)^{2|F|} \\
&= \left(\binom{|F|-1}{|F|-s-1} (\lambda_F[\tilde{\varepsilon}]+1)^{s+1-2|F|} - \binom{|F|}{s}\binom{|F|-1}{|F|-1} (\lambda_F[\tilde{\varepsilon}]+1)^{s+1-2|F|} \right) \\
&\qquad \cdot (\lambda_F[\tilde{\varepsilon}]+1)^{2|F|} \sum_{l\in F} \langle d|_{\varepsilon=\tilde{\varepsilon}}\, S_\varepsilon[\eta][\tilde{E}^{(l)}], \tilde{E}^{(l)}\rangle_{\tilde{\varepsilon}} \\
&= -\lambda_F[\tilde{\varepsilon}](\lambda_F[\tilde{\varepsilon}]+1)^{s-1} \binom{|F|-1}{s-1} \sum_{l\in F} \int_\Omega \eta \tilde{E}^{(l)} \cdot \tilde{E}^{(l)}\, dx.
\end{aligned}$$

Finally, recalling (4.4.10), we get

$$\begin{aligned}
&d|_{\varepsilon=\tilde\varepsilon}\Lambda_{F,s}[\eta]\\
&= -\lambda_F[\tilde\varepsilon]\sum_{k=1}^{s}(-1)^{s-k}(\lambda_F[\tilde\varepsilon]+1)^{k-1}\binom{|F|-k}{s-k}\binom{|F|-1}{k-1}\sum_{l\in F}\int_\Omega \eta\tilde E^{(l)}\cdot\tilde E^{(l)}\,dx\\
&= -\binom{|F|-1}{s-1}\lambda_F[\tilde\varepsilon]\sum_{k=0}^{s-1}\binom{s-1}{k}(\lambda_F[\tilde\varepsilon]+1)^{k}(-1)^{s-k-1}\sum_{l\in F}\int_\Omega \eta\tilde E^{(l)}\cdot\tilde E^{(l)}\,dx\\
&= -\binom{|F|-1}{s-1}(\lambda_F[\tilde\varepsilon])^{s}\sum_{l\in F}\int_\Omega \eta\tilde E^{(l)}\cdot\tilde E^{(l)}\,dx.
\end{aligned}$$

Next we consider the case $n \geq 2$. By means of a continuity argument, one can easily see that there exists an open neighborhood $\mathcal{W}$ of $\tilde\varepsilon$ in $\mathcal{E}[F]$ such that $\mathcal{W} \subset \bigcap_{k=1}^{n}\mathcal{E}[F_k]$. Thus

$$\Lambda_{F,s}[\varepsilon] = \sum_{\substack{0\leq s_1\leq |F_1|\\ \dots\\ 0\leq s_n\leq |F_n|\\ s_1+\dots+s_n=s}}\prod_{k=1}^{n}\Lambda_{F_k,s_k}[\varepsilon]$$

Differentiating the above equality at the point $\tilde\varepsilon$ and using formula (4.4.8) with $n=1$ to each function Λ_{F_k,s_k}, one can see that formula (4.4.8) holds true for any $n \in \mathbb{N}$. □

We conclude this section by studying the case of one-parametric families of permittivities. Using Lemma 4.4.1 and classical analytic perturbation theory we can recover a Rellich-Nagy-type theorem which allows us to describe all the eigenvalues splitting from a multiple eigenvalue of multiplicity m by means of m real-analytic functions. For classical results in analytic perturbation theory we refer to the seminal works of Rellich [103] and Nagy [109]. More up do date formulations can be found in Chow and Hale [33, Theorem 5.2, p. 487], Kato [73, Theorem 3.9, p. 393], Lamberti and Lanza de Cristoforis [79, Theorem 2.27], Rellich [104, Theorem 1, p. 33].

Theorem 4.4.11. *Let Ω be as in (4.1.3). Let $\tilde\varepsilon \in \mathcal{E}$ and let $\{\varepsilon_t\}_{t\in\mathbb{R}} \subset \mathcal{E}$ be a family depending real analytically on t and such that $\varepsilon_0 = \tilde\varepsilon$. Let $\tilde\lambda$ be a Maxwell eigenvalue of multiplicity $m \in \mathbb{N}$ and $\tilde E^{(1)}, \dots, \tilde E^{(m)}$ a corresponding orthonormal basis of Maxwell eigenvectors in $L^2_{\tilde\varepsilon}(\Omega)$ with $\varepsilon = \tilde\varepsilon$. Let $\tilde\lambda = \lambda_n[\tilde\varepsilon] = \dots = \lambda_{n+m-1}[\tilde\varepsilon]$ for some $n \in \mathbb{N}$. Then there exist an open interval $I \subset \mathbb{R}$ containing zero and m real-analytic functions $g_1, \dots, g_m$ from I to $\mathbb{R}$ such that*

$$\{\lambda_n[\varepsilon_t], \dots, \lambda_{n+m-1}[\varepsilon_t]\} = \{g_1(t), \dots, g_m(t)\} \qquad \forall t \in I.$$

Moreover, the derivatives $g_1'(0), \ldots, g_m'(0)$ of the functions $g_1, \ldots, g_m$ at zero coincide with the eigenvalues of the matrix

$$\left(-\tilde{\lambda} \int_\Omega \dot{\varepsilon}_0 \, \tilde{E}^{(i)} \cdot \tilde{E}^{(j)} \, dx\right)_{i,j=1,\ldots,m},$$

where $\dot{\varepsilon}_0$ denotes the derivative at $t = 0$ of the map $t \mapsto \varepsilon_t$.

Proof. Again, we can assume that τ is big enough such that $\tilde{\lambda}$ is strictly smaller than any eigenvalue of (4.1.10) which is not a Maxwell eigenvalue. By applying [79, Thm. 2.27, Cor. 2.28] to the operator S_ε defined in (4.1.12) we get that there exist an open interval I of $\mathbb{R}$ containing zero and m real-analytic functions $h_1, \ldots, h_m$ from I to $\mathbb{R}$ such that $\{(\lambda_n[\varepsilon_t] + 1)^{-1}, \ldots, (\lambda_{n+m-1}[\varepsilon_t] + 1)^{-1}\} = \{h_1(t), \ldots, h_m(t)\}$ for all $t \in I$. Furthermore, the derivatives at zero of the functions $h_i, i = 1, \ldots, m$ coincide with the eigenvalues of the matrix

$$\left(\langle d|_{\varepsilon=\tilde{\varepsilon}} S_\varepsilon[\dot{\varepsilon}_0] \tilde{E}^{(i)}, \tilde{E}^{(j)} \rangle_{\tilde{\varepsilon}}\right)_{i,j=1,\ldots,m}.$$

By continuity we have that, eventually further restricting the interval I, the functions h_i are away from zero for all $t \in I$. Then, setting

$$g_i(t) := \frac{1}{h_i(t)} - 1$$

we have that $\{\lambda_n[\varepsilon_t], \ldots, \lambda_{n+m-1}[\varepsilon_t]\} = \{g_1(t), \ldots, g_m(t)\}$. Finally, noticing that

$$\frac{d}{dt} g_i(t)|_{t=0} = -(\tilde{\lambda} + 1)^2 \frac{d}{dt} h_i(t)|_{t=0},$$

we deduce that the derivatives at zero of the functions g_i coincide with the eigenvalues of the matrix

$$-(\tilde{\lambda} + 1)^2 \left(\langle d|_{\varepsilon=\tilde{\varepsilon}} S_\varepsilon[\dot{\varepsilon}_0] \tilde{E}^{(i)}, \tilde{E}^{(j)} \rangle_{\tilde{\varepsilon}}\right)_{i,j=1,\ldots,m} = \left(-\tilde{\lambda} \int_\Omega \dot{\varepsilon}_0 \, \tilde{E}^{(i)} \cdot \tilde{E}^{(j)} \, dx\right)_{i,j=1,\ldots,m},$$

where this last equality is justified by Lemma 4.4.1. □

4.5 Non-existence of critical permittivities

In this section we consider the problem of finding those densities which maximize or either minimize the symmetric functions of Maxwell eigenvalues under a suitable constraint of the permittivities. Let Ω be as in (4.1.3) and M a real symmetric positive definite 3×3 matrix. We introduce

$$L_M := \left\{ \varepsilon \in \mathcal{E} : \int_\Omega \varepsilon \, dx = M \right\},$$

where by $\int_\Omega \varepsilon\, dx$ we mean the (3×3)-matrix defined by

$$\left(\int_\Omega \varepsilon\, dx\right)_{ij} := \int_\Omega \varepsilon_{ij}\, dx \qquad \forall i,j=1,2,3.$$

In the following Theorem 4.5.1 we show that the symmetric functions of Maxwell eigenvalues do not admit points of local extremum under the constraint $\varepsilon \in L_M$. Note that Theorem 4.5.1 is in the same spirit of the analog optimization problem for the eigenvalues of the Dirichlet Laplacian with mass constraint, that is for a vibrating membrane with a fixed total mass (see Lamberti [76]). Further results on the optimization of the eigenvalues of the Dirichlet Laplacian and of more general elliptic operators with respect to the mass density can be found in Cox [42], Cox and McLaughlin [43, 44], Henrot [67], Krein [75], and Lamberti and Provenzano [82].

Theorem 4.5.1. *Let Ω be as in (4.1.3). Let F be a finite subset of $\mathbb{N}$. Let $s \in \{1,\dots,|F|\}$. Let M be a real symmetric positive definite 3×3 matrix. Then the map from $\mathcal{E}[F]\cap L_M$ to $\mathbb{R}$ which takes ε to $\Lambda_{F,s}[\varepsilon]$ has no points of local minimum or maximum.*

Proof. Let $\mathcal{I}$ be the map from $\mathcal{E}$ to the set of symmetric 3×3 matrices defined by

$$\mathcal{I}[\varepsilon] := \int_\Omega \varepsilon\, dx \qquad \forall \varepsilon \in \mathcal{E}.$$

We proceed by contradiction. Assume that there exists a local minimum or maximum $\tilde\varepsilon \in \mathcal{E}$. Clearly, there exists $n\in\mathbb{N}$ and a partition $\{F_1,\dots,F_n\}$ of F such that $\tilde\varepsilon \in \bigcap_{k=1}^n \Theta[F_k]$. Since $\tilde\varepsilon$ is a local extremum, it is a critical point for the function $\Lambda_{F,s}[\varepsilon]$ subject to the constraint

$$\mathcal{I}[\varepsilon] = M.$$

This implies the existence of a Langrange multiplier, which means that there exists a matrix $\tilde N \in \mathbb{R}^{3\times 3}$ such that

$$d|_{\varepsilon=\tilde\varepsilon}\Lambda_{F,s} = -\mathrm{Tr}\left(d|_{\varepsilon=\tilde\varepsilon}\mathcal{I}\ \tilde N\right).$$

(see, e.g., Deimling [51, Thm. 26.1]). By formula (4.4.8) for the differential of $\Lambda_{F,s}$ we have that

$$\sum_{k=1}^n c_k \sum_{l\in F_k} \int_\Omega \eta \tilde E^{(l)}\cdot \tilde E^{(l)}\, dx = \frac{1}{|\partial\Omega|}\int_\Omega \mathrm{Tr}\left(\eta\tilde N\right)\, dx \qquad \forall \eta \in W^{1,\infty}(\Omega)\cap \mathrm{Sym}_3(\Omega),$$

where for each $k = 1, \dots, n$, $\{E^{(l)}\}_{l \in F_k} \subset X_{\mathrm{N}}^{\tilde{\varepsilon}}(\Omega)^3$ is an orthonormal basis in $L^2_{\tilde{\varepsilon}}(\Omega)$ of Maxwell eigenvectors for the eigenspace associated with $\lambda_{F_k}[\tilde{\varepsilon}]$. Moreover, by the Gaffney inequality (4.1.8), we also have that $\{E^{(l)}\}_{l \in F_k} \subset H^1(\Omega)^3$. Recall that c_k, $k = 1, \dots, n$, are the constants defined in (4.4.9). Since η is arbitrary, one can easily see that there exist three constants $a_1, a_2, a_3 \geq 0$ such that

$$\left(\sum_{k=1}^{n} c_k \sum_{l \in F_k} \big(\tilde{E}_j^{(l)}\big)^2 \right) = a_j \qquad \text{a.e in } \Omega,\ j = 1, 2, 3. \tag{4.5.2}$$

In particular, since $E^{(l)}$ are not null being eigenvectors, (4.5.2) implies that the three constants a_1, a_2, a_3 cannot all be zero. We also note that since the vectors $\tilde{E}^{(l)}$ are in $H^1(\Omega)^3$ they can be traced on the boundary as vector fields in $L^2(\partial\Omega)^3$. Furthermore, on $\partial\Omega$ they are parallel to the outer unit normal ν, hence there exist scalar functions $A^l \in L^2(\partial\Omega)$ such that

$$\tilde{E}^{(l)} = A^l \nu \qquad \text{on } \partial\Omega,$$

that is

$$\tilde{E}_j^{(l)} = A^l \nu_j \quad \text{on } \partial\Omega \text{ for } j = 1, 2, 3. \tag{4.5.3}$$

Then, by using (4.5.3) in (4.5.2) we obtain that

$$(\nu_j)^2 \left(\sum_{k=1}^{n} c_k \sum_{l \in F_k} (A^l)^2 \right) = \left(\sum_{k=1}^{n} c_k \sum_{l \in F_k} \big(\tilde{E}_j^{(l)}\big)^2 \right) = a_j \qquad \text{on } \partial\Omega$$

for $j = 1, 2, 3$. Taking the square root we deduce that

$$\left(|\nu_j(x)| \left(\sum_{k=1}^{n} c_k \sum_{l \in F_k} (A^l(x))^2 \right)^{\frac{1}{2}} \right)_{j=1,2,3} = \big(\sqrt{a_j}\big)_{j=1,2,3} \qquad \text{on } \partial\Omega. \tag{4.5.4}$$

Define now the vector

$$\begin{aligned} B(x) &:= \left(|\nu_j(x)| \left(\sum_{k=1}^{n} c_k \sum_{l \in F_k} (A^l(x))^2 \right)^{\frac{1}{2}} \right)_{j=1,2,3} \\ &= \left(\sum_{k=1}^{n} c_k \sum_{l \in F_k} (A^l(x))^2 \right)^{\frac{1}{2}} \big(|\nu_j(x)|\big)_{j=1,2,3}. \end{aligned}$$

Observe that since the three constants a_1, a_2, a_3 are not all zero, by (4.5.4) the vector B is constant and different from zero. In particular this implies that for almost every $x \in \partial\Omega$ the vector

$$\big(\mathrm{sign}\,(\nu_j(x))\nu_j(x)\big)_{j=1,2,3}$$

must point in the same direction of $\left(\sqrt{a_j}\right)_{j=1,2,3}$. Since Ω is bounded and of class $C^{1,1}$ (in particular it is C^1), it is not difficult to see that this leads to a contradiction. □

4.6 Simplicity of the spectrum for generic permittivities

The issue of understanding if the eigenvalues of a parameter dependent problem can be made all simple by an arbitrarily small perturbation of the parameter is a natural question and has been already investigated by several authors for different problems. For example, Albert [4] proved the generic simplicity of the spectrum of an elliptic operator with respect to the perturbation of the zeroth order term.

Moreover, the generic simplicity of the spectrum has been also considered with respect to the domain perturbation in various papers. We mention, e.g, Micheletti [95, 96] for the Laplacian and for a general elliptic operator and Ortega and Zuazua [100] and Chitour, Kateb and Long [32] for the Stokes system in two and three dimension, respectively. Finally, we also mention the more recent paper by Dabrowski [46] where the authors analyze the Laplacian with different boundary conditions and consider also singular perturbations of the domain.

A first step, as we will show in the next proposition, we prove that it is always possible to find a small perturbation of the permittivity that splits a multiple Maxwell eigenvalue into simple eigenvalues.

Proposition 4.6.1. *Let Ω be as in (4.1.3). Let $\tilde\varepsilon \in \mathcal{E}$, $\tilde\lambda$ a Maxwell eigenvalue of multiplicity $m \in \mathbb{N}$ and $\tilde E^{(1)}, \dots, \tilde E^{(m)}$ a corresponding orthonormal basis of Maxwell eigenvectors in $L^2_{\tilde\varepsilon}(\Omega)$ with $\varepsilon = \tilde\varepsilon$. Let $\tilde\lambda = \lambda_n[\tilde\varepsilon] = \cdots = \lambda_{n+m-1}[\tilde\varepsilon]$ for some $n \in \mathbb{N}$. Define*

$$\tilde\varepsilon_{t,\eta} := \tilde\varepsilon + t\eta \qquad \forall t \in \mathbb{R},$$

for all $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$, $\|\eta\|_{W^{1,\infty}(\Omega)} \le 1$. Then for all $T > 0$ there exist $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$ with $\|\eta\|_{W^{1,\infty}(\Omega)} \le 1$, and $t \in \,]0, T[$ such that $\tilde\varepsilon_{t,\eta} \in \mathcal{E}$ and the eigenvalues $\lambda_n[\tilde\varepsilon_{t,\eta}], \dots, \lambda_{n+m-1}[\tilde\varepsilon_{t,\eta}]$ are all simple.

Proof. We will prove that there exist $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$ with $\|\eta\|_{W^{1,\infty}(\Omega)} \le 1$ and $t > 0$ as small as desired such that the eigenvalues $\lambda_n[\tilde\varepsilon_{t,\eta}], \dots, \lambda_{n+m-1}[\tilde\varepsilon_{t,\eta}]$ are not all equal. Then, repeating the same argument for the eigenvalues that have still a multiplicity strictly greater than one, in a finite number of steps we are done. Note that by the continuity of the eigenvalues with respect to permittivity variations and by choosing t small enough we can avoid that the eigenvalues splitting from a multiple eigenvalue could overlap or switch position with other eigenvalues.

Hence, suppose by contradiction that there exists $T > 0$ such that for all $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$ with $\|\eta\|_{W^{1,\infty}(\Omega)} \leq 1$ and for all $t \in]0, T[$, all the eigenvalues $\lambda_n[\tilde{\varepsilon}_{t,\eta}], \ldots, \lambda_{n+m-1}[\tilde{\varepsilon}_{t,\eta}]$ coincide. As a consequence, all the right derivatives at $t = 0$ of the branches coincide. Then, if we fix η and use Theorem 4.4.11, we get that all the eigenvalues of the matrix

$$M := \left(-\tilde{\lambda} \int_\Omega \eta \tilde{E}^{(i)} \cdot \tilde{E}^{(j)} \, dx \right)_{i,j=1,\ldots,m} \tag{4.6.2}$$

coincide. Since the above matrix is a real symmetric matrix with only one eigenvalue, it is a scalar matrix. In other words, there exists $\mu[\eta] \in \mathbb{R}$ such that

$$M = \mu[\eta] \, \mathbb{I}_m, \tag{4.6.3}$$

where $\mathbb{I}_m$ denotes the $(m \times m)$-identity matrix. For $h = 1, 2, 3$ we set

$$\eta_h := \|\xi\|^{-1}_{W^{1,\infty}(\Omega)} \, \xi \, e_{hh}$$

with $0 \neq \xi \in C^1_c(\Omega)$ arbitrary and e_{hh} the (3×3)-matrix with (h, h)-entry equal to 1 and zeros elsewhere. By (4.6.2), (4.6.3) and using the above defined η_h we can recover that for all $\xi \in C^1_c(\Omega)$

$$\int_\Omega \xi \, E^{(i)}_h E^{(j)}_h \, dx = 0 \qquad \forall i, j \in \{1, \ldots, m\}, i \neq j, \quad \forall h = 1, 2, 3,$$

and

$$\int_\Omega \xi \left((E^{(i)}_h)^2 - (E^{(j)}_h)^2 \right) dx = 0 \qquad \forall i, j \in \{1, \ldots, m\}, \quad \forall h = 1, 2, 3.$$

By the fundamental lemma of calculus of variations we get that a.e. in Ω

$$E^{(i)}_h E^{(j)}_h = 0 \qquad \forall i, j \in \{1, \ldots, m\}, \; i \neq j, \quad \forall h = 1, 2, 3,$$

and

$$(E^{(i)}_h)^2 - (E^{(j)}_h)^2 = 0 \qquad \forall i, j \in \{1, \ldots, m\}, \quad \forall h = 1, 2, 3.$$

The above relations clearly implies that $E_i = 0$ for all $i \in \{1, \ldots, m\}$, which is a contradiction since they are not identically zero, being eigenfunctions. $\square$

Remark 4.6.4. The constraint $\|\eta\|_{W^{1,\infty}(\Omega)} \leq 1$ in the above proposition can be replaced by $\|\eta\|_{W^{1,\infty}(\Omega)} \leq \delta$ for any $\delta > 0$.

Remark 4.6.5. The argument we have used to split a multiple eigenvalue into several eigenvalues of lower multiplicity uses that η is a general symmetric matrix and not a scalar matrix. However, noticing in which way η_h is defined, one can easily realize that such an argument still works if η varies in the class of diagonal matrices. Instead, in the case that we restrict ourselves to scalar matrices, what we can recover by arguing in the same way is that

$$E^{(i)} \cdot E^{(j)} = 0 \qquad \forall i, j \in \{1, \ldots, m\}, i \neq j.$$

and

$$|E^{(i)}|^2 - |E^{(j)}|^2 = 0 \qquad \forall i, j \in \{1, \ldots, m\}.$$

This does not immediately lead to a contradiction. Thus, it would be interesting to investigate wether it is still possible to split the whole spectrum when the permittivies are scalar.

We are now ready to show that the whole Maxwell spectrum is generically simple with respect to the permittivity. We note that our proof is inspired by the methods of Albert [4]

Theorem 4.6.6. *Let Ω be as in (4.1.3). Let $\tilde{\varepsilon} \in \mathcal{E}$ and let $\delta > 0$ be small enough such that*

$$\tilde{\varepsilon} + \eta \in \mathcal{E}$$

for all $\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$ with $\|\eta\|_{W^{1,\infty}(\Omega)} \leq \delta$. Let

$$B_0 := \left\{\eta \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega) : \|\eta\|_{W^{1,\infty}(\Omega)} \leq \delta\right\}$$

and

$$B_n := \{\eta \in B_0 : \text{ the first } n \text{ Maxwell eigenvalues with } \varepsilon = \tilde{\varepsilon} + \eta \text{ are simple}\}$$

for $n \in \mathbb{N}$. Then

$$B := \bigcap_{n \in \mathbb{N}} B_n = \{\eta \in B_0 : \text{ all the Maxwell eigenvalues with } \varepsilon = \tilde{\varepsilon} + \eta \text{ are simple}\}$$

is dense in B_0.

Proof. The proof follows by applying the Baire's Lemma in the complete metric space B_0. In order to do this, we have to show that

i) B_n is open in B_0 for all $n \in \mathbb{N}$,

ii) B_{n+1} is dense in B_n for all $n \in \mathbb{N}$.

Statement i) follows from the continuity of the eigenvalues with respect to the permittivity parameter. Next we prove statement ii) by contradiction. Assume that B_{n+1} is not dense in B_n for some $n \in \mathbb{N}$. Then there exists $\eta \in B_n \setminus B_{n+1}$ and a neighborhood U of η in B_0 such that

$$U \subset B_n \setminus B_{n+1}.$$

Since $\eta \in B_n \setminus B_{n+1}$ then

- the first n Maxwell eigenvalues of $(\mathrm{P}_{\tilde{\varepsilon}+\eta})$ are simple,
- the $(n+1)$-th Maxwell eigenvalue of $(\mathrm{P}_{\tilde{\varepsilon}+\eta})$ has multiplicity k for some $k \in \mathbb{N}$, $k \geq 2$.

Moreover, we note that for all $\rho \in U \subset B_n \setminus B_{n+1}$ we have:

- the first n Maxwell eigenvalues of $(\mathrm{P}_{\tilde{\varepsilon}+\rho})$ are simple,
- the $(n+1)$-th Maxwell eigenvalue of $(\mathrm{P}_{\tilde{\varepsilon}+\rho})$ is not simple.

By Proposition 4.6.1 there exist $\hat{\rho} \in W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$ with $\|\hat{\rho}\|_{W^{1,\infty}(\Omega)} \leq 1$ and $t > 0$ arbitrarily small such that $\eta + t\hat{\rho} \in U$ and all the Maxwell eigenvalues of $(\mathrm{P}_{\tilde{\varepsilon}+\eta+t\hat{\rho}})$ with indices from $(n+1)$ to $(n+k)$ are simple, therefore we deduce that in particular $\eta + t\hat{\rho} \in B_{n+1}$. This is a contradiction since $U \subset B_n \setminus B_{n+1}$. $\square$

4.7 Useful facts

In the present section we collect some technical results that we exploited in this chapter.

Lemma 4.7.1. *Let Ω be an open Lipschitz subset of $\mathbb{R}^3$ of finite measure. Let $u \in H^1(\Omega)^3$ and let ε be a matrix-valued function in $W^{1,\infty}(\Omega) \cap \mathrm{Sym}_3(\Omega)$. Then there exists $\operatorname{div}(\varepsilon u) \in L^2(\Omega)$ and*

$$\operatorname{div}(\varepsilon u) = \operatorname{tr}(\varepsilon D u) + \operatorname{div} \varepsilon \cdot u \tag{4.7.2}$$

a.e. in Ω. Here above if $\varepsilon = \left(\varepsilon^{(1)} | \varepsilon^{(2)} | \varepsilon^{(3)}\right)$, with $\varepsilon^{(k)}$ denoting the k-th column of the matrix ε, then $\operatorname{div} \varepsilon$ is the vector defined as

$$\operatorname{div} \varepsilon = \left(\operatorname{div} \varepsilon^{(1)}, \operatorname{div} \varepsilon^{(2)}, \operatorname{div} \varepsilon^{(3)}\right).$$

Proof. By density there exists a sequence of smooth vector functions $\psi_n \in C_c^\infty(\mathbb{R}^3)^3$ such that $\psi_n \to u$ in $H^1(\Omega)^3$ as $n \to +\infty$. Moreover, although the Meyers-Serrin theorem is not valid for the space $W^{1,\infty}$, we have that for any matrix $\varepsilon \in W^{1,\infty}(\Omega)$ there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega)^{3\times 3} \cap W^{1,\infty}(\Omega)$ such that

$$\|\varepsilon_k - \varepsilon\|_{L^\infty(\Omega)} \to 0 \tag{4.7.3}$$

and

$$\|D\varepsilon_k\|_{L^\infty(\Omega)} \to \|D\varepsilon\|_{L^\infty(\Omega)}, \qquad D\varepsilon_k(x) \to D\varepsilon(x) \quad \text{for a.e. } x \in \Omega \tag{4.7.4}$$

as $k \to \infty$.

For any $k, n \in \mathbb{N}$ the vector field $\varepsilon_k \psi_n$ belongs to $C^\infty(\Omega)^3$, hence the equality

$$\operatorname{div}(\varepsilon_k \psi_n) = \operatorname{tr}(\varepsilon_k D\psi_n) + \operatorname{div} \varepsilon_k \cdot \psi_n$$

is valid in Ω by standard calculus. Therefore if $\varphi \in C_c^\infty(\Omega)$ is any test function, we have that

$$\int_\Omega \varepsilon_k \, \psi_n \cdot \nabla\varphi \, dx = -\int_\Omega \left(\operatorname{tr}(\varepsilon_k D\psi_n) + \operatorname{div} \varepsilon_k \cdot \psi_n\right) \varphi \, dx.$$

As $k \to +\infty$, from (4.7.3) we get that the left-hand side goes to $\int_\Omega \varepsilon \, \psi_n \cdot \nabla\varphi \, dx$, while thanks to (4.7.3), (4.7.4) and the dominated convergence theorem, the right-hand side goes to $-\int_\Omega (\operatorname{tr}(\varepsilon D\psi_n) + \operatorname{div} \varepsilon \cdot \psi_n) \, \varphi \, dx$. Hence

$$\int_\Omega \varepsilon \, \psi_n \cdot \nabla\varphi \, dx = -\int_\Omega \left(\operatorname{tr}(\varepsilon D\psi_n) + \operatorname{div} \varepsilon \cdot \psi_n\right) \varphi \, dx.$$

Taking the limit for $n \to \infty$ we get

$$\int_\Omega \varepsilon \, u \cdot \nabla\varphi \, dx = -\int_\Omega \left(\operatorname{tr}(\varepsilon Du) + \operatorname{div} \varepsilon \cdot u\right) \varphi \, dx.$$

The above formula holds for all $\varphi \in C_c^\infty(\Omega)$, and by definition of weak ε-divergence formula (4.7.2) is valid. □

Remark 4.7.5. Note that in the case that the matrix-valued permittivity parameter ε is scalar, the function u can be taken in $H(\operatorname{div}, \Omega)$ and formula (4.7.2) reads as follows

$$\operatorname{div}(\varepsilon u) = \varepsilon \operatorname{div} u + \nabla\varepsilon \cdot u. \tag{4.7.6}$$

4.7.1 Auchmuty principle

The following part is an adaptation to problem (4.1.10) of arguments from [13]. It is worthy to note that for these results we would only require $X^\varepsilon_{\mathrm{N}}(\Omega)$ to be compactly embedded in $L^2_\varepsilon(\Omega)$, with possibly less assumptions on the permittivity parameter ε and on the set Ω. Nonetheless, we will assume ε to be in $\mathcal{E}$ (defined in (4.1.4)) and Ω to be a bounded domain of $\mathbb{R}^3$ of class $C^{1,1}$ (as in (4.1.3)).

Define $\hat{f} : X^\varepsilon_{\mathrm{N}}(\Omega) \to \mathbb{R}$ as follows

$$\hat{f}(u) := \frac{1}{2}T_\varepsilon[u][u] = \frac{1}{2}\int_\Omega \varepsilon\, u\cdot u\, dx + \frac{1}{2}\int_\Omega |\mathrm{curl}\, u|^2\, dx + \frac{\tau}{2}\int_\Omega \left|\mathrm{div}(\varepsilon u)\right|^2 dx, \quad (4.7.7)$$

where T_ε is the operator defined in (4.1.11), equivalent to the inner product of $X^\varepsilon_{\mathrm{N}}(\Omega)$. Let $M \geq 1$ be a natural number and suppose $\sigma_1[\varepsilon], \dots, \sigma_M[\varepsilon]$ are the first M eigenvalues of (4.1.10) (repeated according to their multiplicity) with $u_1, \dots, u_M$ the corresponding orthonormal eigenfunctions (where the orthonormality is taken in $L^2_\varepsilon(\Omega)$). Let P_M be the orthogonal projection (with respect to the inner product in $L^2_\varepsilon(\Omega)$) of $X^\varepsilon_{\mathrm{N}}(\Omega)$ onto the subspace spanned by $\{u_1, \dots, u_M\}$, that is

$$P_M u = \sum_{i=1}^{M} \langle u, u_i\rangle_\varepsilon u_i \quad \text{for all } u \in X^\varepsilon_{\mathrm{N}}(\Omega).$$

Define $f_{M+1} : X^\varepsilon_{\mathrm{N}}(\Omega) \to \mathbb{R}$ as follows

$$f_{M+1}(u) := \hat{f}(u) - \left\|(I - P_M)u\right\|_{L^2_\varepsilon(\Omega)}. \quad (4.7.8)$$

Remark 4.7.9. If $\tilde{u} \in X^\varepsilon_{\mathrm{N}}(\Omega)$ is an eigenfunction of (4.1.10) with eigenvalue $\tilde{\sigma}[\varepsilon]$, then

$$T[\tilde{u}][v] = (\tilde{\sigma}[\varepsilon] + 1)J_\varepsilon[\tilde{u}][v]$$

for all $v \in X^\varepsilon_{\mathrm{N}}(\Omega)$, where J_ε is defined in (4.1.7). Thus

$$f_{M+1}(\tilde{u}) = \frac{1 + \tilde{\sigma}[\varepsilon]}{2}\|\tilde{u}\|^2_{L^2_\varepsilon(\Omega)} - \left\|(I - P_M)\tilde{u}\right\|_{L^2_\varepsilon(\Omega)}. \quad (4.7.10)$$

Lemma 4.7.11. *The functional f_{M+1} is weakly lower semi-continuous and coercive on $X^\varepsilon_{\mathrm{N}}(\Omega)$. It is Gâteaux-differentiable on $X^\varepsilon_{\mathrm{N}}(\Omega) \setminus \{0\}$ and*

$$\langle \mathrm{D}\, f_{M+1}(u), v\rangle = T_\varepsilon[u][v] - \left\|(I - P_M)u\right\|^{-1}_{L^2_\varepsilon(\Omega)} J_\varepsilon[(I - P_M)u][(I - P_M)v] \quad (4.7.12)$$

for all $v \in X^\varepsilon_{\mathrm{N}}(\Omega)$.

Proof. Since $\|(I-P_M)u\|_{L^2_\varepsilon(\Omega)} \le \|u\|_{L^2_\varepsilon(\Omega)} \le \|u\|_{X^\varepsilon_{\mathrm{N}}(\Omega)}$, then

$$\frac{f_{M+1}(u)}{\|u\|_{X^\varepsilon_{\mathrm{N}}(\Omega)}} \ge \frac{1}{2}\frac{T_\varepsilon[u][u]}{\|u\|_{X^\varepsilon_{\mathrm{N}}(\Omega)}} - 1 \to +\infty \quad \text{as } \|u\|_{X^\varepsilon_{\mathrm{N}}(\Omega)} \to +\infty,$$

i.e. f_{M+1} is coercive.

It is not difficult to see that the functional $\hat{f} = \frac{1}{2}T_\varepsilon$ is strictly convex, that is for any $u, v \in X^\varepsilon_{\mathrm{N}}(\Omega), u \neq v$, we have that

$$\hat{f}\left(\frac{u+v}{2}\right) < \frac{\hat{f}(u)+\hat{f}(v)}{2}.$$

Moreover, since $\hat{f}$ is strongly continuous on $X^\varepsilon_{\mathrm{N}}(\Omega)$, it is also weakly lower semi-continuous. Additionally, the embedding $\iota_\varepsilon : X^\varepsilon_{\mathrm{N}}(\Omega) \to L^2_\varepsilon(\Omega)$ is compact, therefore the L^2_ε-norm is sequentially weakly continuous on $X^\varepsilon_{\mathrm{N}}(\Omega)$. Since the orthogonal projection P_M is also weakly continuous, we have that f_{M+1} is sequentially weakly lower semi-continuous, which in turn implies the weakly lower semi-continuity because f_{M+1} is coercive.

Finally, the Gâteaux derivative is just a simple computation; we omit the details. □

In the sequel we shall also need the following

Corollary 4.7.13. *Let $\hat{u} \in \langle u_1, \dots, u_M\rangle^{\perp}_{L^2_\varepsilon(\Omega)}$ be in the orthogonal space in $L^2_\varepsilon(\Omega)$ of the span of $\{u_1, \dots, u_M\}$. Suppose moreover that $\hat{u}$ is a non-zero critical point of f_{M+1} on $X^\varepsilon_{\mathrm{N}}(\Omega)$. Then $\hat{u}$ solves problem (4.1.10) with associated eigenvalue $\sigma[\varepsilon] = \|\hat{u}\|^{-1}_{L^2_\varepsilon(\Omega)} - 1$.*

Proof. If $0 \neq \hat{u} \in X^\varepsilon_{\mathrm{N}}(\Omega)$ is a critical point of f_{M+1} then $\langle \mathrm{D}\, f_{M+1}(\hat{u}), v\rangle = 0$ for all $v \in X^\varepsilon_{\mathrm{N}}(\Omega)$. By formula (4.7.12) and the fact that $\hat{u}$ is L^2_ε-orthogonal to $u_i, i = 1, \dots, M$, which in turns implies that $P_M\hat{u} = 0$ and $J_\varepsilon[\hat{u}][P_M v] = 0$ for any $v \in X^\varepsilon_{\mathrm{N}}(\Omega)$, we see that $\hat{u}$ solves problem (4.1.10) with $\sigma[\varepsilon] = \|\hat{u}\|^{-1}_{L^2_\varepsilon(\Omega)} - 1$. □

We can finally prove the following theorem. (cf. also [13, Thm. 7.4]).

Theorem 4.7.14 (Auchmuty Principle)**.** *The functional f_{M+1} defined in (4.7.8) attains its minimum $-\frac{1}{2}(\sigma_{M+1}[\varepsilon]+1)^{-1}$ on $X^\varepsilon_{\mathrm{N}}(\Omega)$, where $\sigma_{M+1}[\varepsilon]$ is the $(M+1)$-th eigenvalue of problem (4.1.10). This minimum is attained at $\pm u_{M+1}$ where u_{M+1} is a solution of (4.1.10) corresponding to the eigenvalue $\sigma_{M+1}[\varepsilon]$ and such that $\|u_{M+1}\|_{L^2_\varepsilon(\Omega)} = (\sigma_{M+1}[\varepsilon]+1)^{-1}$.*

Proof. Since $X^\varepsilon_{\mathrm{N}}(\Omega)$ is reflexive, its bounded subsets are weakly pre-compact. Hence, by Lemma 4.7.11 we have that f_{M+1} attains a finite minimum on $X^\varepsilon_{\mathrm{N}}(\Omega)$.

Observe that $f_{M+1}(0) = 0$. We complete the orthonormal set of eigenfunctions $u_1, \dots, u_M$ to a L^2_ε-orthonormal basis $\{u_i\}_{i\in\mathbb{N}} \subset X^\varepsilon_{\mathrm{N}}(\Omega)$. Now, for any $u \in X^\varepsilon_{\mathrm{N}}(\Omega)$ write $u = P_M u + w$, where

$$P_M u = \sum_{i=1}^{M} J_\varepsilon[u][u_i]\, u_i \quad \text{and} \quad w = (I - P_M)u = \sum_{i=M+1}^{+\infty} J_\varepsilon[u][u_i] u_i\,.$$

Observe that $w \in \langle u_1, \dots, u_M\rangle^\perp_{L^2_\varepsilon(\Omega)}$, thus the vector fields $P_M u$ and w are orthogonal in $L^2_\varepsilon(\Omega)$, i.e. $J_\varepsilon[P_M u][w] = 0$. Moreover, observe that for all $i = 1, \dots, M$

$$\int_\Omega \operatorname{curl} u_i \cdot \operatorname{curl} v\, dx + \int_\Omega \operatorname{div}(\varepsilon u_i) \operatorname{div}(\varepsilon v)\, dx = \sigma_i[\varepsilon] J_\varepsilon[u_i][v] \qquad \text{for all } v \in X^\varepsilon_{\mathrm{N}}(\Omega). \tag{4.7.15}$$

Since for all $i = 1, \dots, M$ we have that $J_\varepsilon[u_i][w] = 0$ then, setting $v = w$ in (4.7.15), we see that $\int_\Omega \operatorname{curl} u_i \cdot \operatorname{curl} w\, dx + \int_\Omega \operatorname{div}(\varepsilon u_i) \operatorname{div}(\varepsilon v)\, dx = 0$. Hence

$$\int_\Omega \operatorname{curl} P_M u \cdot \operatorname{curl} w\, dx + \int_\Omega \operatorname{div}(\varepsilon P_M u) \operatorname{div}(\varepsilon v)\, dx = 0.$$

Therefore $T_\varepsilon[u][u] = T_\varepsilon[P_M u][P_M u] + T_\varepsilon[w][w]$ and

$$f_{M+1}(u) = \frac{1}{2} T_\varepsilon[u][u] - \|w\|_{L^2_\varepsilon(\Omega)} \geq T_\varepsilon[w][w] - \|w\|_{L^2_\varepsilon(\Omega)} = f_{M+1}(w).$$

Thus the minimum of f_{M+1} occurs in the orthogonal space $\langle u_1, \dots, u_M\rangle^\perp_{L^2_\varepsilon(\Omega)}$. If $\hat{u} \in \langle u_1, \dots, u_M\rangle^\perp_{L^2_\varepsilon(\Omega)}$ is a non-zero critical point of f_{M+1}, by Corollary 4.7.13 one has that $\hat{u}$ is an eigenfunction of problem (4.1.10) associated to the eigenvalue $\sigma[\varepsilon] = \|\hat{u}\|^{-1}_{L^2_\varepsilon(\Omega)} - 1$. Hence, observing that $P_M \hat{u} = 0$ and using (4.7.10) we get that

$$f_{M+1}(\hat{u}) = -\frac{1}{2}\|\hat{u}\|_{L^2_\varepsilon(\Omega)} = -\frac{1}{2}(\sigma[\varepsilon] + 1)^{-1} < 0.$$

The last inequality is justified because the eigenvalues $\sigma[\varepsilon]$ are non-negative. In particular f_{M+1} is minimized when $\sigma[\varepsilon]$ is the smallest eigenvalue of problem (4.1.10) corresponding to an eigenfunction $\hat{u}$ which is orthogonal to $u_1, \dots, u_M$, that is when $\sigma[\varepsilon] = \sigma_{M+1}[\varepsilon]$, and in this case $\hat{u} = \pm\hat{u}_{M+1}$ with $\|\hat{u}_{M+1}\|_{L^2_\varepsilon(\Omega)} = (\sigma_{M+1}[\varepsilon]+1)^{-1}$. $\square$

Chapter 5

Appendix

The present chapter is divided as follows. Section 5.1 contains the detailed proof of the vector decomposition from [17]. Meanwhile, Section 5.2 contains regularity results for the Laplace operator subject to Dirichlet boundary conditions taken from the monograph [92].

We make a careful tracking of all the constants involved in Theorem 5.1.3 and Theorem 5.2.39.

5.1 Birman and Solomyak's decomposition of X_{N}

Recall that

$$H^1_{\mathrm{N}}(\Omega) = X_{\mathrm{N}}(\Omega) \cap H^1(\Omega)^3$$

and

$$E(\Omega) = \left\{ \nabla\varphi : \varphi \in H^1_0(\Omega), \Delta\varphi \in L^2(\Omega) \right\}.$$

Define also

$$\nabla H^1_0(\Omega) := \left\{ \nabla\psi : \psi \in H^1_0(\Omega) \right\},$$

and denote by $J(\Omega)$ the orthogonal space of $\nabla H^1_0(\Omega)$ in $L^2(\Omega)^3$, so that we can write

$$L^2(\Omega)^3 = \nabla H^1_0(\Omega) \oplus_{L^2} J(\Omega).$$

It is straightforward to see that $J(\Omega)$ is the set of divergence-free vector fields, namely

$$J(\Omega) = \left\{ u \in H(\mathrm{div}, \Omega) : \mathrm{div}\, u = 0 \right\}.$$

Remark 5.1.1. By Proposition 1.3.7 it follows that if Ω is a bounded Lipschitz subset of $\mathbb{R}^3$ whose boundary has only one connected component, the following decomposition

$$X_{\mathrm{N}}(\Omega) = E(\Omega) \oplus_{L^2} \left(X_{\mathrm{N}}(\Omega) \cap J(\Omega) \right).$$

holds. In other words, it is possible to split any vector field $\tilde{u} \in X_{\mathrm{N}}(\Omega)$ as follows

$$\tilde{u} = \nabla q + u,$$

where $u \in X_{\mathrm{N}}(\operatorname{div} 0, \Omega)$ and $q \in H^1_0(\Omega)$ is the (weak) solution to the following problem

$$\begin{cases} \Delta q = \operatorname{div} \tilde{u}, & \text{in } \Omega, \\ q = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.1.2}$$

Theorem 5.1.3 ([17, Thm. 4.1])**.** *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^3$. Then there exist two linear continuous operators P and Q*

$$P : X_{\mathrm{N}}(\Omega) \to \left(H^1_{\mathrm{N}}(\Omega), \, \|\cdot\|_{H^1(\Omega)^3}\right), \qquad Q : X_{\mathrm{N}}(\Omega) \to \left(E(\Omega), \, \|\cdot\|_{H(\operatorname{div},\Omega)}\right),$$

such that $u = Pu + Qu$ for any $u \in X_{\mathrm{N}}(\Omega)$. In particular there exist a constant $C_{BS} > 0$ such that

$$\|Pu\|^2_{H^1(\Omega)} + \|Qu\|^2_{L^2(\Omega)} + \|\operatorname{div} Qu\|^2_{L^2(\Omega)} \leq C_{BS} \|u\|^2_{X_{\mathrm{N}}(\Omega)} \tag{5.1.4}$$

for all $u \in X_{\mathrm{N}}(\Omega)$.

Proof. We shall prove the theorem only for the case Ω homeomorphic to a ball; the general case of a bounded Lipschitz open set in $\mathbb{R}^3$ can be then recovered by means of a partition of unity.

We will first prove it for $u \in X_{\mathrm{N}}(\Omega)$ with $\operatorname{div} u = 0$. Therefore, take any $u \in X_{\mathrm{N}}(\Omega) \cap H(\operatorname{div} 0, \Omega)$, and let $f = \operatorname{curl} u$. Then $f \in J(\Omega)$; in order to see this we just need to show that for any $\psi \in H^1_0(\Omega)$, $\int_\Omega f \cdot \nabla\psi \, dx = 0$. In particular this implies that $f \in H(\operatorname{div}, \Omega)$ and $\operatorname{div} f = 0$. We will actually show a stronger result, i.e. that the vector field f is in the orthogonal of $\nabla H^1(\Omega)$ in $L^2(\Omega)^3$. Indeed, let $\psi \in H^1(\Omega)$. Then $\nabla\psi \in H(\operatorname{curl}, \Omega)$ with $\operatorname{curl} \nabla\psi = 0$, so that

$$\int_\Omega f \cdot \nabla\psi \, dx = \int_\Omega \operatorname{curl} u \cdot \nabla\psi \, dx = -\int_\Omega u \cdot \operatorname{curl} \nabla\psi \, dx + \int_{\partial\Omega} (\nu \times u) \cdot \nabla\psi \, dx = 0, \tag{5.1.5}$$

where we recall that the boundary integral in the above formula is zero due to the electric boundary condition the vector field $u \in X_{\mathrm{N}}(\Omega)$ satisfies. Therefore, not only f is a divergence-free vector field, but also the normal trace $\gamma_\nu f$ of f is zero on the boundary $\partial\Omega$. To see this, by (5.1.5) and the fact that $\operatorname{div} f = 0$, we get that

$$\int_{\partial\Omega} (f \cdot \nu) \, \psi \, dx = \int_{\partial\Omega} (\operatorname{curl} u \cdot \nu) \psi \, d\sigma = \int_\Omega \operatorname{curl} u \cdot \nabla\psi \, dx + \int_\Omega \operatorname{div} \operatorname{curl} u \, \psi \, dx = 0$$

for any $\psi \in H^1(\Omega)$. Since $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$, we can conclude that $\gamma_\nu f = 0$. Thus the function u solves the following problem:

$$\begin{cases} \operatorname{curl} u = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u \times \nu = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.1.6}$$

The function u is uniquely characterized by (5.1.6), since the homogenous problem (i.e. with $f = 0$) has only zero as a solution, due to the fact Ω is homeomorphic to a ball. Indeed, suppose that u solves (5.1.6) with $f = 0$. Then $\operatorname{curl} u = 0$ in Ω, and so $u = \nabla\eta$ for some $\eta \in H^1(\Omega)$ (see e.g. Th.2.9 of [62]). Since $\operatorname{div} u = 0$ in Ω, then η is a harmonic function, and due to the boundary condition $u \times \nu = 0$, η must be constant on the boundary $\partial\Omega$, which has only one connected component because we assumed Ω homeomorphic to a ball. Hence η is constant in all of Ω, and consequently $u \equiv 0$.

Let now B be an open ball such that $\overline{\Omega} \subset B$, and extend f by zero to the whole B. Then we still have that $\tilde{f} \in J(B)$, where $\tilde{f}$ is the extension-by-zero of f. Indeed for any test function $\psi \in H_0^1(B)$ we have that $\psi \in H^1(\Omega)$, thus by (5.1.5)

$$\int_B \tilde{f} \cdot \nabla\psi = \int_\Omega f \cdot \nabla\psi = 0.$$

Hence $\tilde{f} \in H(\operatorname{div}, B)$ with $\operatorname{div} \tilde{f} = 0$.

Recall the Newtonian potential $\mathcal{N}g \in L^2(\mathbb{R}^3)$ of a square integrable function $g \in L^2(\mathbb{R}^3)$, namely

$$\mathcal{N}g(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{g(y)}{|x-y|}\, dy.$$

It is such that $-\Delta\mathcal{N}g = g$. We then use a particular solution in $H^1(\Omega)^3$ of the system $\operatorname{curl} w = \tilde{f}, \operatorname{div} w = 0$ as the curl of the (vector) Newtonian potential of $\tilde{f}$, that is

$$w(x) := \operatorname{curl} \mathcal{N}\tilde{f}(x) = \frac{1}{4\pi} \operatorname{curl} \int_B \frac{\tilde{f}(y)}{|x-y|}\, dy. \tag{5.1.7}$$

Since w is a curl, then it is obviously divergence free. Moreover w is a divergence-free vector field, as shown below:

$$\begin{aligned} \operatorname{div}_x \left(\int_B \frac{\tilde{f}(y)}{|x-y|} dy \right) &= \int_B \operatorname{div}_x \left(\frac{\tilde{f}(y)}{|x-y|} \right) dy = \sum_{i=1}^3 \int_B -\frac{\tilde{f}_i(y)}{|x-y|^3}(x_i - y_i) dy \\ &= \sum_{i=1}^3 \int_B \tilde{f}_i(y) \partial_{y_i} \left(\frac{1}{|x-y|} \right) dy \\ &= -\int_B (\operatorname{div}_y \tilde{f}) \frac{1}{|x-y|} dy + \int_{\partial B} (\tilde{f} \cdot \nu) \frac{1}{|x-y|} dy = 0, \end{aligned}$$

where the last passage is justified because $\operatorname{div}\tilde{f} = 0$ in B and $\tilde{f} = 0$ on ∂B. Recalling that $\Delta = -\operatorname{curl}\operatorname{curl} + \nabla \operatorname{div}$, then one can see

$$\operatorname{curl} w = -\Delta \mathcal{N}\tilde{f} = \tilde{f}.$$

Then in particular $\operatorname{curl} w = f$ in Ω and

$$\|w\|_{H^1(B)^3} \leq C_{\mathcal{N}} \|f\|_{L^2(\Omega)^3}\,, \tag{5.1.8}$$

where we can choose the constant $C_{\mathcal{N}}$ in a way that does not depend on Ω, but just on the fixed ball B containing Ω (the operator $\mathcal{N}$ is a bounded linear operator from $L^2(\mathbb{R}^3)$ to $H^2(\mathbb{R}^3)$: for these and other properties of the Newtonian potential one can see [61]). Since $\operatorname{curl} w = \tilde{f} = 0$ in $A = B \setminus \Omega$, whose fundamental group is trivial, there exists a function $\Phi_A \in H^2(A)$ such that $w = \nabla \Phi_A$ in A (see e.g. Thm. 2.9 of [62]). Moreover, we can choose Φ_A such that $\int_A \Phi_A = 0$. Since the boundary of Ω is Lipschitz, there exists a continuous linear extension operator $E : H^2(A) \to H^2(B)$. Set

$$\Phi := E\,\Phi_A. \tag{5.1.9}$$

Moreover, we have that $\gamma_\tau(w - \nabla\Phi) = 0$ on $\partial\Omega$. To see this it suffices to show that

$$\int_\Omega \operatorname{curl}(w - \nabla\Phi) \cdot z\, dx = \int_\Omega (w - \nabla\Phi) \cdot \operatorname{curl} z\, dx \tag{5.1.10}$$

for all $z \in H^1(\Omega)^3$. In order to see that, consider compact set K such that $\overline{\Omega} \subset K \subset B$ and consider a smooth function $\zeta \in C_c^\infty(B)$ such that $\operatorname{supp}\zeta \subset K$ and $\zeta \equiv 1$ in Ω. Define $\hat{z} := \zeta z$. It is obvious that $\hat{z} \equiv z$ in Ω and that $\hat{z} \in H_0^1(B)$. Thus, since both w and Φ are actually defined on B, with $w = \nabla\Phi$ in A, we proceed as follows:

$$\begin{aligned}
\int_\Omega (w - \nabla\Phi) \cdot \operatorname{curl} z\, dx &= \int_B (w - \nabla\Phi) \cdot \operatorname{curl} \hat{z}\, dx \\
&= \int_B \operatorname{curl}(w - \nabla\Phi) \cdot \hat{z}\, dx + \int_{\partial B} (w - \nabla\Phi) \cdot (\nu \times \hat{z})\, d\sigma \\
&= \int_B \operatorname{curl}(w - \nabla\Phi) \cdot \hat{z}\, dx \\
&= \int_\Omega \operatorname{curl}(w - \nabla\Phi) \cdot z\, dx.
\end{aligned}$$

Observe that (see remark after the proof)

$$\|\nabla\Phi\|_{H^1(\Omega)^3} \leq \tilde{C} \|f\|_{L^2(\Omega)^3}\,. \tag{5.1.11}$$

Now let $\varphi \in H_0^1(\Omega)$ be a function defined by the equation

$$\int_\Omega \nabla\varphi \cdot \nabla\eta \, dx = -\int_\Omega \Delta\Phi \, \eta \, dx \quad \text{for all } \eta \in H_0^1(\Omega),$$

that is, φ is the (unique) weak solution of

$$\begin{cases} \Delta\varphi = \Delta\Phi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.1.12}$$

Then $w - \nabla\Phi + \nabla\varphi$ solves (5.1.6): the electric boundary condition to have null tangential trace is verified because (5.1.10) holds and the fact that φ is null at the boundary. Therefore $w - \nabla\Phi + \nabla\varphi$ is equal to u, and we have that $u = Pu + Qu$ with $Pu = w - \nabla\Phi$ and $Qu = \nabla\varphi$. Finally, using (5.1.8) and (5.1.11) we have that (5.1.4) is valid. Indeed

$$\|Pu\|_{H^1(\Omega)} \leq \|w\|_{H^1(\Omega)^3} + \|\nabla\Phi\|_{H^1(\Omega)^3} \leq (C_{\mathcal{N}} + \tilde{C})\|f\|_{L^2(\Omega)^3} .$$

Moreover $\nabla\varphi = u - Pu$ and $\Delta\varphi = \Delta\Phi$, so that

$$\|\nabla\varphi\|_{L^2(\Omega)^3} \leq \|u\|_{L^2(\Omega)^3} + \|Pu\|_{L^2(\Omega)^3} \leq \|u\|_{L^2(\Omega)^3} + (C_{\mathcal{N}} + \tilde{C})\|f\|_{L^2(\Omega)^3}$$

and

$$\|\Delta\varphi\|_{L^2(\Omega)} = \|\Delta\Phi\|_{L^2(\Omega)} \leq \tilde{C}\|f\|_{L^2(\Omega)^3} .$$

Remebering that $f = \operatorname{curl} u$, we get (5.1.4). We have thus proved the theorem in the case of $u \in X_{\mathrm{N}}(\operatorname{div} 0, \Omega)$.

If now we take a generic $\tilde{u} \in X_{\mathrm{N}}(\Omega)$, as said in Remark 5.1.1 it is possible to write $\tilde{u} = \nabla q + u$ with $u \in X_{\mathrm{N}}(\operatorname{div} 0, \Omega)$ and $q \in H_0^1(\Omega)$ such that $\nabla q \in H(\operatorname{div}, \Omega)$, since Ω is homeomorphic to a ball and thus its boundary has only one connected component. Observe that $\operatorname{curl} \tilde{u} = \operatorname{curl} u$ and $\operatorname{div} \tilde{u} = \Delta q$. Then $\tilde{u} = Pu + Qu + \nabla q$, thus $\tilde{u} = P\tilde{u} + Q\tilde{u} = Pu + Q\tilde{u}$, where $Q\tilde{u} = Qu + \nabla q$. We now have that

$$\|Q\tilde{u}\|_{L^2(\Omega)^3} \leq \|\tilde{u}\|_{L^2(\Omega)^3} + \|Pu\|_{L^2(\Omega)^3} \leq \|\tilde{u}\|_{L^2(\Omega)^3} + (C_{\mathcal{N}} + \tilde{C})\|\operatorname{curl} \tilde{u}\|_{L^2(\Omega)^3} .$$

Moreover

$$\|\operatorname{div} Q\tilde{u}\|_{L^2(\Omega)} \leq \|\operatorname{div} Qu\|_{L^2(\Omega)} + \|\operatorname{div} \tilde{u}\|_{L^2(\Omega)} \leq \tilde{C}\|\operatorname{curl} \tilde{u}\|_{L^2(\Omega)^3} + \|\operatorname{div} \tilde{u}\|_{L^2(\Omega)} .$$

Therefore we recover (5.1.4) also in the general case of $\tilde{u} \in X_{\mathrm{N}}(\Omega)$. $\square$

Remark 5.1.13 (Regarding inequality (5.1.11))**.** Let $C_{\mathcal{N}}$ be a positive constant such that $\|\mathcal{N}f\|_{H^2(\Omega)} \leq C_{\mathcal{N}}\|f\|_{L^2(\Omega)}$ for all $f \in L^2(\Omega)$, where $\mathcal{N}$ is the Newtonian potential of f. Since the fundamental group for A is trivial and $\operatorname{curl} w = 0$ in A, there exists $\Phi_A \in H^2(A)$ such that $w = \nabla\Phi_A$ in A. Moreover we can choose Φ_A such that $\int_A \Phi_A = 0$. Hence, denoting with C_E the operator norm of the extension $E : H^2(A) \to H^2(B)$, we get

$$\begin{aligned}\|\nabla\Phi\|_{H^1(\Omega)^3} &\le \|\nabla\Phi\|_{H^1(B)^3} \le \|\Phi\|_{H^2(B)} \le C_E\|\Phi_A\|_{H^2(A)}\\ &\le C_E\, c_{\mathcal{P}}\|\nabla\Phi_A\|_{H^1(A)^3} = C_E\, c_{\mathcal{P}}\|w\|_{H^1(A)^3}\\ &\le C_E\, c_{\mathcal{P}}\, C_{\mathcal{N}}\|f\|_{L^2(\Omega)^3}\,.\end{aligned} \tag{5.1.14}$$

Here $c_{\mathcal{P}}$ is the constant of the Poincaré-Wirtinger inequality (see, e.g., [88, Thm. 13.27]), that we can apply since A is a bounded connected open set with Lipschitz boundary. Thus the constant $\tilde{C}$ in (5.1.11) is such that $\tilde{C} = C_E\, c_{\mathcal{P}}\, C_{\mathcal{N}}$.

Remark 5.1.15. Since $\partial A = \partial\Omega \cup \partial B$ is Lipschitz, the extension operator E from $H^2(A)$ to $H^2(B)$ exists, and it is just the restriction to B composed with the usual extension operator $E : H^2(A) \to H^2(\mathbb{R}^3)$. Moreover, we have that

$$\|Eg\|_{H^2(\mathbb{R}^3)} \le C_E\|g\|_{H^2(A)} \quad \text{for all } g \in H^2(\mathbb{R}^3) \tag{5.1.16}$$

where the constant C_E depends only on the Lipschitz character of the boundary of Ω. See [25] for more details.

Remark 5.1.17. As one can see from the proof in Theorem 5.1.3 and the previous remarks, the constant C_{BS} of (5.1.4) depends on the Newtonian constant $C_{\mathcal{N}}$ in (5.1.8), the constant C_E in (5.1.16), and on the Poincaré constant $c_{\mathcal{P}}$ of the set Ω. We can therefore conclude that C_{BS} depends on the Lipschitz character of $\partial\Omega$ and on the Lebesgue measure of Ω.

5.2 More details on the regularity of elliptic boundary value problems

In this section we restrict our attention on some aspects of the regularity theory for solutions of the Poisson problem for the Laplacian with Dirichlet boundary conditions. We will present in detail results taken from the monograph [92] by Maz'ya and Shaposhnikova which are useful for our purposes, specifically for Section 1.4.3 and Section 3.3.

We will work in $\mathbb{R}^N$ with $N \ge 2$. For the sake of simplicity, we will often omit $\mathbb{R}^N$ in the notation of function spaces or norms, as it will be clear from the context. In general, if S is a Banach space of functions defined on $\mathbb{R}^N$, by S_{loc} we denote the space of functions (defined on $\mathbb{R}^N$) that belong to S whenever multiplied by a smooth compactly supported function, namely

$$S_{loc} = \left\{ u : \eta u \in S \text{ for all } \eta \in C_c^\infty(\mathbb{R}^N) \right\}.$$

Let V be an open set of $\mathbb{R}^N$, $p \in [1, \infty)$ and $l > 0$ be a non-integer. Define

$$(D_{p,l}u)(x) := \left(\int_V \frac{|\nabla_{[l]}u(x) - \nabla_{[l]}u(y)|^p}{|x-y|^{N+p\{l\}}} dy, \right)^{1/p},$$

where $[l], \{l\}$ denote the integer and fractional parts of l respectively, and ∇_k stands for the gradient of order $k \in \mathbb{N} \cup \{0\}$. The *fractional Sobolev space* $W_p^l(V)$ is the space of functions in $W_p^{[l]}(V)$ with the finite norm

$$\|u\|_{W_p^l(V)} := \|u\|_{W_p^{[l]}(V)} + \left\|D_{p,l}u\right\|_{L^p(V)}.$$

By *multiplier* multiplier acting from one function space S_1 into another S_2 we mean a function which defines a bounded linear mapping of S_1 into S_2 by point-wise multiplication. In the next definition we will give the precise notion of a multiplier acting in pairs of fractional Sobolev spaces.

Definition 5.2.1. A function γ defined on V belongs to $M(W_p^m(V) \to W_p^l(V))$ if $\gamma u \in W_p^l(V)$ for all $u \in W_p^m(V)$, where $m \geq l \geq 0$ and $p \in [1, \infty)$. The norm in $M(W_p^m(V) \to W_p^l)$ is defined as the norm of the operator of multiplication

$$\|\gamma\|_{M(W_p^m(V) \to W_p^l(V))} := \sup \left\{ \|\gamma u\|_{W_p^l(V)} : \|u\|_{W_p^m(V)} \leq 1 \right\}.$$

We will use the notation $MW_p^l(V)$ instead of $M(W_p^l(V) \to W_p^l(V))$.

Remark 5.2.2. Observe that the case $m < l$ is not interesting since, under these circumstances, $M(W_p^m(V) \to W_p^l(V)) = \{0\}$. To see this, suppose $m < l$ and let $\gamma \in M(W_p^m(V) \to W_p^l(V))$. Then

$$\|\gamma\|_{M(W_p^m(V) \to W_p^l(V))} \geq \frac{\left\|\gamma\, e^{itx_1}\, \eta\right\|_{W_p^l(V)}}{\|e^{itx_1}\eta\|_{W_p^m(V)}},$$

where $t > 0$ and η is an arbitrary non-zero function in $C_c^\infty(V)$. Therefore

$$\|\gamma\|_{M(W_p^m(V) \to W_p^l(V))} \geq t^{l-m} \left(\frac{\|\gamma\eta\|_{L^p(V)}}{\|\eta\|_{L^p(V)}} + o(1) \right)$$

as $t \to +\infty$, and this is possible only if $\gamma = 0$.

We introduce the notion of *Besov space* B_p^l (cf. also Definition 5.2.54). Let $l = k + \alpha$ where $\alpha \in (0,1]$ and k is a non-negative integer. Set

$$\Delta_h u(x) := u(x+h) - u(x) \tag{5.2.3}$$

and

$$\Delta_h^{(2)} u(x) := \Delta_h \Delta_h u(x) = u(x+2h) - 2u(x+h) + u(x). \tag{5.2.4}$$

Define

$$(\mathfrak{D}_{p,l} u)(x) := \left(\int_{\mathbb{R}^N} \left| \Delta_h^{(2)} \nabla_k u(x) \right|^p |h|^{-N-p\alpha} dh \right)^{1/p}, \tag{5.2.5}$$

where ∇_k stands for the gradient of order $k \in \mathbb{N} \cup \{0\}$. Then, the space $B_p^l(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{B_p^l(\mathbb{R}^N)} := \left\| \mathfrak{D}_{p,l} u \right\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)}. \tag{5.2.6}$$

Note that for $\{l\} > 0$, the spaces $B_p^l(\mathbb{R}^N)$ and $W_p^l(\mathbb{R}^N)$ have the same elements and their norms are equivalent since

$$(2 - 2^{\{l\}}) D_{p,l} u \le \mathfrak{D}_{p,l} u \le (2 + 2^{\{l\}}) D_{p,l} u. \tag{5.2.7}$$

See also [92, Ch.4]

As in the case of fractional Sobolev spaces, we introduce the space of multipliers acting between Besov spaces.

Definition 5.2.8. A function γ defined on $\mathbb{R}^N$ belongs to $M(B_p^m \to B_p^l)$ if $\gamma u \in B_p^l$ for all $u \in B_p^m$, where $m \ge l \ge 0$ and $p \in [1, \infty)$. The norm in $M(B_p^m \to B_p^l)$ is defined as the norm of the operator of multiplication

$$\|\gamma\|_{M(B_p^m \to B_p^l)} := \sup \left\{ \|\gamma u\|_{B_p^l} : \|u\|_{B_p^m} \le 1 \right\}.$$

If $m = l$, we denote the space $M(B_p^l \to B_p^l)$ simply by MB_p^l.

A useful characterization of multipliers in pairs of Besov spaces is given by the next theorem. Note that here and in the sequel, two values are a and b are called *equivalent* ($a \sim b$) if there exist positive constants c and C, depending only on m, l, p, N (and analogous parameters), such that

$$ca \le b \le Cb.$$

Theorem 5.2.9 ([92, Thm. 4.1.1])**.** *Let $l > 0$ and $p \in (1, \infty)$. A function γ belongs to MB_p^l if and only if $\gamma \in B_{p,loc}^l$, $\mathfrak{D}_{p,l}\gamma \in M(B_p^l \to L^p)$, and $\gamma \in L^\infty$. The equivalence relation*

$$\|\gamma\|_{M(B_p^l \to B_p^l)} \sim \sup_E \frac{\left\| \mathfrak{D}_{p,l}\gamma \right\|_{L^p(E)}}{[C_{p,l}(E)]^{1/p}} + \|\gamma\|_{L^\infty} \tag{5.2.10}$$

holds, where E is an arbitrary compact set in $\mathbb{R}^N$. The equivalence relation (5.2.10) remains valid if the condition $d(E) \le 1$ is added, where $d(E)$ is the diameter of E.

For the definition of (p,l)-capacity $C_{p,l}(E)$ of a compact set $E \subset \mathbb{R}^N$ one can see section 1.2.2 of [92]. We only observe that it is equivalent to the following quantity

$$\inf\left\{\|u\|_{W_p^l}^p : u \in C_c^\infty(\mathbb{R}^N), u \geq 1 \text{ on } E\right\}.$$

5.2.1 Regularity of the Dirichlet laplacian

In the present section we will mainly follow Chapter 14 of [92]. Much of the results in [92] are more general than the ones we present here, in the sense that they deal with a more general class of operators arising from elliptic boundary value problem in a bounded domain. Here we are mainly interested in the Poisson problem for the Laplace operator subject to Dirichlet boundary conditions.

In the following we consider Ω to be a bounded domain in $\mathbb{R}^N$. The Poisson problem for the Dirichlet Laplacian with datum $f \in L^2(\Omega)$ reads as follows

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.2.11}$$

A function $u \in H_0^1(\Omega)$ is a (weak) solution of (5.2.11) if

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$

for all $v \in H_0^1(\Omega)$. It is a standard fact that if the boundary of Ω is sufficiently smooth (say of class C^2), the solution u belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ and the following a priori estimate

$$\|u\|_{H^2(\Omega)} \leq c\left(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{L^1(\Omega)}\right) \tag{5.2.12}$$

holds, where $c > 0$ is a constant depending only on Ω. We refer to Section 6.3.2 of [52] or to Section 5.3.3 of the monograph [110] for further details. Observe that since the solution to (5.2.11) is unique, then the last norm on the right-hand side of (5.2.12) can be omitted (see e.g. [52, Thm. 6, §6.2]), yielding

$$\|u\|_{H^2(\Omega)} \leq c\|\Delta u\|_{L^2(\Omega)}.$$

See also the following remark.

Remark 5.2.13 (cf. [54, Lemma 1])**.** If Ω is bounded domain in $\mathbb{R}^N$, then for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$

$$\|u\|_{L^1(\Omega)} \leq |\Omega|^{\frac{1}{2}}\|u\|_{L^2(\Omega)} \leq c_{\mathcal{P}}^2|\Omega|^{\frac{1}{2}}\|\Delta u\|_{L^2(\Omega)},$$

where $c_{\mathcal{P}}$ denotes the Poincaré constant (see, e.g., [88, Thm. 13.27]).

It turns out that the H^2-regularity for solutions of (5.2.11) and the consequent a priori estimate (5.2.12) is valid for a larger class of domains than those with boundary of class C^2 (see Theorems 5.2.16 and 5.2.39).

In what follows we will deal with parameters l, p, with $l \geq 2$ a natural number and $p \in (1, \infty)$. The parameter l stands for the order of the differential operator while p denotes the order of integrability. For problem (5.2.11) one has that $l = 2$ and $p = 2$.

Moreover, to ease the notation and to be more clear, we will denote the multiplier norm of a function γ by $\|\gamma; V\|_{MW_p^l}$ instead of $\|\gamma\|_{MW_p^l(V)}$.

The following definition can be found in Section 14.3.1 of [92]. Recall also Definition 1.4.19.

Definition 5.2.14 (Class $M_p^{l-1/p}(\delta)$)**.** Let Ω be a bounded domain in $\mathbb{R}^N$ and $\delta > 0$. If $p(l-1) \leq N$ we say that $\partial\Omega$ belongs to the class $M_p^{l-1/p}(\delta)$ if for each point $O \in \partial\Omega$ there exists a coordinate $y := x_j$ for some $j \in \{1, \ldots, n\}$, a neighbourhood U and a special Lipschitz domain of the form

$$G = \Big\{ z = (x, y) : x \in \mathbb{R}^{N-1}, y > \varphi(x) \Big\}$$

such that $U \cap \Omega = U \cap G$ and

$$\nabla\varphi \in MW_p^{l-1-1/p}(\mathbb{R}^{N-1}) \quad \text{with} \quad \Big\|\nabla\varphi; \mathbb{R}^{N-1}\Big\|_{MW_p^{l-1-1/p}} \leq \delta. \tag{5.2.15}$$

Note that the class $M_p^{l-1/p}(\delta)$ is contained in the class $C^{0,1}$ of Lipschitz domains. Moreover, observe that if $\partial\Omega \in M_p^{l-1/p}(\delta)$ for some $\delta > 0$ then $\partial\Omega \in W_p^{l-1/p}$, i.e. it can be locally described by a function in $W_p^{l-1/p}(\mathbb{R}^{N-1})$. This is because

$$MW_p^{l-1-1/p}(\mathbb{R}^{N-1}) \subset W_{p,loc}^{l-1-1/p}(\mathbb{R}^{N-1}).$$

The next theorem concerns the H^2-regularity for the solutions of (5.2.11). It is a particular case of [92, Thm. 14.5.3]. Observe that if $l = p = 2$ then $l-1-1/p = 3/2$.

Theorem 5.2.16. *Let Ω be a bounded domain in $\mathbb{R}^3$ such that $\partial\Omega \in M_2^{3/2}(\delta)$ for some $\delta > 0$, and let $f \in L^2(\Omega)$. Then the solution $u \in H_0^1(\Omega)$ of (5.2.11) belongs to $H_0^1(\Omega) \cap H^2(\Omega)$.*

Note that the index $3/2$ is somewhat sharp, as the following proposition shows. It is a particular case of [92, Theorem 14.6.3].

Proposition 5.2.17. *Let Ω be a bounded domain in $\mathbb{R}^3$ such that $\partial\Omega \in C^1$. Assume the normal to $\partial\Omega$ satisfies the Dini condition, namely*

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty,$$

where ω is the modulus of continuity of the normal. If, for a function $f \in C_c^\infty(\Omega)$, there exists a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

then $\partial\Omega \in W_2^{3/2}$.

For our purposes we are interested in a better understanding of the validity of (5.2.12). In Theorem 5.2.39 we will eventually see that a sufficient condition for the a priori estimate to hold is that the boundary $\partial\Omega$ belongs to the class $M_2^{3/2}(\delta)$ for a sufficiently small $\delta > 0$. We first need to introduce some machinery.

In the sequel we write a generic point in $\mathbb{R}^N$ as (x, y) with $x \in \mathbb{R}^{N-1}$ and $y > 0$.

Definition 5.2.18 (see section 9.4.1 of [92])**.** Let $p \in (1, \infty), l \geq 1$, and let U, V be open subsets of $\mathbb{R}^N$. A quasi-isometric mapping $\varkappa : U \to V$ is called a (p, l)-diffeomorphism if all elements of its Jacobi matrix $\partial\varkappa$ belong to the space of multipliers $MW_p^{l-1}(U)$. By $\|\partial\varkappa; U\|_{MW_p^{l-1}}$ we denote the sum of the norms of the elements of $\partial\varkappa$ in $MW_p^{l-1}(U)$.

Let $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $L > 0$. Given a function $\zeta \in C_c^\infty(\mathbb{R}^{N-1})$ such that $\zeta \geq 0$ and $\int_{\mathbb{R}^{N-1}} \zeta(t)\, dt = 1$, we introduce the operator $\mathcal{T}$ by the formula

$$(\mathcal{T}\varphi)(x, y) := \int_{\mathbb{R}^{N-1}} \zeta(t)\varphi(x + ty)\, dt, \quad y > 0. \tag{5.2.19}$$

The function $\mathcal{T}\varphi$ is an extension of φ onto $\mathbb{R}_+^N$.

Lemma 5.2.20 ([92, Thm. 8.7.2])**.** *Suppose $\nabla\varphi \in MW_p^{l-1-1/p}(\mathbb{R}^{N-1})$. Then $\nabla(\mathcal{T}\varphi) \in MW_p^{l-1}(\mathbb{R}_+^N)$ and*

$$\left\|\nabla(\mathcal{T}\varphi); \mathbb{R}_+^N\right\|_{MW_p^{l-1}} \leq c_1 \left\|\nabla\varphi; \mathbb{R}^{N-1}\right\|_{MW_p^{l-1-1/p}}, \tag{5.2.21}$$

where $c_1 > 0$ is a constant depending only on l, p and N.

Let

$$G = \Big\{(x,y) \in \mathbb{R}^N : x \in \mathbb{R}^{N-1}, y > \varphi(x)\Big\}, \tag{5.2.22}$$

and let K be a sufficiently large constant depending on L. We define the mapping

$$\lambda : \mathbb{R}^N_+ \to G, \quad \lambda(\xi,\eta) \mapsto (x,y) := (\xi, K\eta + (\mathcal{T}\varphi)(\xi,\eta)), \tag{5.2.23}$$

with $\xi \in \mathbb{R}^{N-1}$ and $\eta > 0$. The codomain of λ is G provided K is large enough. Indeed we need to check that

$$K\eta + (\mathcal{T}\varphi)(\xi,\eta) - \varphi(\xi) > 0.$$

Since φ is Lipschitz with constant L then

$$\varphi(\xi + t\eta) - \varphi(\xi) \geq -L\,\eta|t|\,.$$

Hence

$$\begin{aligned} K\eta + (\mathcal{T}\varphi)(\xi,\eta) - \varphi(\xi) &= K\eta + \int_{\mathbb{R}^{N-1}} \zeta(t)\,\big(\varphi(\xi+t\eta) - \varphi(\xi)\big)\,dt \\ &\geq \left(K - L\int_{\mathbb{R}^{N-1}} \zeta(t)|t|\,dt\right)\eta > 0 \end{aligned}$$

provided that $K > L\int_{\mathbb{R}^{N-1}} \zeta(t)|t|\,dt > 0$.

The following lemma also explains why we need the constant K to be sufficiently larger than L.

Lemma 5.2.24 ([92, Lemma 9.4.5]). *For any $\xi \in \mathbb{R}^N$ the mapping*

$$\alpha_\xi : \mathbb{R}_+ \to \mathbb{R}, \quad \alpha_\xi(\eta) \mapsto y = K\eta + (\mathcal{T}\varphi)(\xi,\eta)$$

is one to one, and the inverse is Lipschitz. Moreover the Lipschitz constant of α_ξ^{-1} is not greater than $(K-L)^{-1}$ and

$$\left|\alpha_{\xi_1}^{-1}(y) - \alpha_{\xi_2}^{-1}(y)\right| \leq cL(K-cL)^{-1}\|\xi_1 - \xi_2\|_{\mathbb{R}^{N-1}}\,, \tag{5.2.25}$$

where $c > 0$ is a constant depending on N.

With the previous lemma one can prove the following one.

Lemma 5.2.26 ([92, Lemma 9.4.6]). *Let $l \geq 2$ be an integer and $p \in (1,\infty)$. Further, let G be as in (5.2.22) with φ a Lipschitz function defined on $\mathbb{R}^{N-1}$ such that $\nabla\varphi \in MW_p^{l-1-1/p}(\mathbb{R}^{N-1})$. Then the mapping λ defined in (5.2.23) is a (p,l)-diffeomorphism of $\mathbb{R}^N_+$ onto G, whose Jacobi matrix $\partial\lambda$ is given by*

$$\partial\lambda = \begin{pmatrix} \mathbb{I}_{N-1} & 0 \\ \nabla_\xi(\mathcal{T}\varphi) & K + \frac{\partial(\mathcal{T}\varphi)}{\partial\eta} \end{pmatrix}, \tag{5.2.27}$$

where $\mathbb{I}_{N-1}$ is the identity $(N-1)\times(N-1)$ matrix.

Lemma 5.2.28 ([92, Lemma 9.4.2]). *If $\varkappa$ is a (p,l)-diffeomorphism, then $\varkappa^{-1}$ is also a (p,l)-diffeomorphism.*

From now on we set $l = p = 2$, focusing our attention to problem (5.2.11). Observe that the if Ω belongs to the class $M_2^{3/2}(\delta)$, then its profile function φ is such that $\nabla\varphi \in MW_2^{1/2}(\mathbb{R}^{N-1})$. By the Lemma 5.2.26 and Lemma 5.2.28 we have that λ and λ^{-1} are both $(2,2)$-diffeomorphisms, hence $\partial\lambda \in M_2^1(\mathbb{R}_+^N)$.

Let S be the differential operator characterized by the following relation

$$Sv := \left[\Delta(v \circ \lambda^{-1})\right] \circ \lambda, \tag{5.2.29}$$

The differential operator S is the transformation of the Laplacian Δ under the change of variables given by λ. It operates on functions v defined on $\mathbb{R}_+^N$ and gives back a function still defined on $\mathbb{R}_+^N$.

Lemma 5.2.30 (Lemma 14.4.8 of [92]). *For all $v \in (H^2 \cap H_0^1)(\mathbb{R}_+^N)$*

$$\left\|(S-\Delta)v; \mathbb{R}_+^N\right\|_{L^2} \le c_2 \left\|\mathbb{I}_N - \partial\lambda; \mathbb{R}_+^N\right\|_{MW_2^1} \left\|v; \mathbb{R}_+^N\right\|_{H^2}, \tag{5.2.31}$$

where c_2 is a continuous function of the norm of $\partial\lambda$ in $MW_2^1(\mathbb{R}_+^N)$ independent of v.

If the Lipschitz constant L of the function φ is small enough, then we can put $K = 1$ in the definition (5.2.23). Consequently the Jacobian matrix $\partial\lambda$ of λ is

$$\partial\lambda = \begin{pmatrix} & & & 0 \\ & \mathbb{I}_{N-1} & & \vdots \\ & & & 0 \\ \frac{\partial(\mathcal{T}\varphi)}{\partial\xi_1} & \dots & \frac{\partial(\mathcal{T}\varphi)}{\partial\xi_{N-1}} & 1 + \frac{\partial(\mathcal{T}\varphi)}{\partial\eta} \end{pmatrix}, \tag{5.2.32}$$

Moreover we have that

$$\left\|\mathbb{I}_N - \partial\lambda; \mathbb{R}_+^N\right\|_{MW_2^1} \sim \left\|\nabla(\mathcal{T}\varphi); \mathbb{R}_+^N\right\|_{MW_2^1},$$

and by Lemma 5.2.20 we also have that

$$\left\|\nabla(\mathcal{T}\varphi); \mathbb{R}_+^N\right\|_{MW_2^1} \le c_1 \left\|\nabla\varphi; \mathbb{R}^{N-1}\right\|_{MW_2^{1/2}}, \tag{5.2.33}$$

with $c_1 > 0$ depending only on N. Hence the following inequality

$$\left\|\mathbb{I}_N - \partial\lambda; \mathbb{R}_+^N\right\|_{MW_2^1} \le \hat{c} \left\|\nabla\varphi; \mathbb{R}^{N-1}\right\|_{MW_2^{1/2}} \tag{5.2.34}$$

holds, where $\hat{c} > 0$ is a dimensional constant.

From Lemmas 14.4.4 and 14.4.5 of [92] one can deduce the next result. See also [92, Coroll. 14.4.2].

Lemma 5.2.35. *The map $\lambda : \mathbb{R}^N_+ \to G$ is bi-Lipschitz and there exists a constant $\tilde{c} > 0$ such that*

$$\left\| u \circ \lambda; \mathbb{R}^N_+ \right\|_{L^2} \leq \tilde{c} \|u; G\|_{L^2}, \quad \left\| v \circ \lambda^{-1}; G \right\|_{L^2} \leq \tilde{c} \left\| v; \mathbb{R}^N_+ \right\|_{L^2} \tag{5.2.36}$$

for all $u \in L^2(G), v \in L^2(\mathbb{R}^N_+)$,

$$\left\| u \circ \lambda; \mathbb{R}^N_+ \right\|_{H^2} \leq \tilde{c} \|u; G\|_{H^2} \tag{5.2.37}$$

for all $u \in H^2(G) \cap H^1_0(G)$, and

$$\left\| v \circ \lambda^{-1}; G \right\|_{H^2} \leq \tilde{c} \left\| v; \mathbb{R}^N_+ \right\|_{H^2} \tag{5.2.38}$$

for all $v \in H^2(\mathbb{R}^N_+) \cap H^1_0(\mathbb{R}^N_+)$. The constant $\tilde{c}$ depends only N and on the $MW_2^{1/2}$-multiplier norm of $\nabla\varphi$ in a continuous way.

Finally, we can prove the main theorem about the a priori estimate (5.2.12).

Theorem 5.2.39 ([92, Thm. 14.5.1])**.** *If $\partial\Omega$ belongs to the class $M_2^{3/2}(\delta)$ for some δ sufficientyl small (depending only on N), then*

$$\|u\|_{H^2(\Omega)} \leq C_M \left(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{L^1(\Omega)} \right) \tag{5.2.40}$$

for all $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Here $C_M > 0$ is a constant that depends on N, δ and on a suitable atlas of Ω.

Proof. Let $O \in \partial\Omega$. By the $M_2^{3/2}(\delta)$ condition there exists a coordinate $y := x_j$ for some $j \in \{1, \ldots, n\}$, a neighbourhood U and a special Lipschitz domain $G = \left\{ z = (x, y) : x \in \mathbb{R}^{N-1}, y > \varphi(x) \right\}$ such that $U \cap \Omega = U \cap G$ and

$$\left\| \nabla\varphi; \mathbb{R}^{N-1} \right\|_{MW_2^{1/2}} \leq \delta. \tag{5.2.41}$$

In particular we have that the Lipschitz constant of the profile function φ is small (see Theorem 5.2.9). Thus we can put $K = 1$ in the definition (5.2.23) of the mapping λ, and consequently inequality (5.2.34) holds.

It is a standard result (see e.g. [3] or Section 5.3.3 of [110]) that for all $v \in H^2(\mathbb{R}^N_+)$ with support in $B_1 \cap \overline{\mathbb{R}^N_+}$, where B_1 denotes the open unit ball in $\mathbb{R}^N$, there exists a constant $c_{ADN} > 0$ depending on N such that

$$\left\| v; \mathbb{R}^N_+ \right\|_{H^2} \leq c_{ADN} \left\| \Delta v; \mathbb{R}^N_+ \right\|_{L^2}. \tag{5.2.42}$$

Choose now a smooth compactly supported function $\sigma \in \mathcal{C}_c^\infty(U)$. Let u be a function in $H^2(\Omega) \cap H_0^1(\Omega)$ and set $v := \sigma u \circ \lambda$. Note that v belongs to $H^2(\mathbb{R}^N_+) \cap H_0^1(\mathbb{R}^N_+)$ precisely because $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and λ is a $(2,2)$-diffeomorphism, thus its Jacobian matrix entries are multipliers in MW_2^1. Moreover, choosing the localizing function σ appropriately, we have that $\operatorname{supp} v \subset B_1 \cap \overline{\mathbb{R}^N_+}$. By Lemma 5.2.35 we have that

$$\|\sigma u; U \cap \Omega\|_{H^2} \leq \tilde{c} \left\| v; \mathbb{R}^N_+ \right\|_{H^2} \leq \tilde{c}\, c_{ADN} \left\| \Delta v; \mathbb{R}^N_+ \right\|_{L^2}. \tag{5.2.43}$$

Moreover, recalling that $Sv = [\Delta(v \circ \lambda^{-1})] \circ \lambda$ (see (5.2.29)) and using Lemma 5.2.30 together with inequality (5.2.34) we get

$$\begin{aligned} \left\| \Delta v; \mathbb{R}^N_+ \right\|_{L^2} &\leq \left\| Sv; \mathbb{R}^N_+ \right\|_{L^2} + \left\| (\Delta - S)v; \mathbb{R}^N_+ \right\|_{L^2} \leq \left\| Sv; \mathbb{R}^N_+ \right\|_{L^2} + c_2\, \hat{c}\, \delta \left\| v; \mathbb{R}^N_+ \right\|_{H^2} \\ &\leq \tilde{c} \left\| \Delta(\sigma u); U \cap \Omega \right\| + \tilde{c}\, c_2\, \hat{c}\, \delta \|\sigma u; U \cap \Omega\|_{H^2}, \end{aligned} \tag{5.2.44}$$

where in the last passage we again made use of Lemma 5.2.35. Combining (5.2.43) and (5.2.44) yields

$$\|\sigma u; U \cap \Omega\|_{H^2} \leq \tilde{c}^2 c_{ADN} \left(\left\| \Delta(\sigma u); U \cap \Omega \right\| + c_2\, \hat{c}\, \delta \|\sigma u; U \cap \Omega\|_{H^2} \right),$$

from which we get that

$$\|\sigma u\|_{H^2(\Omega)} \leq \frac{\tilde{c}^2 c_{ADN}}{1 - \tilde{c}^2 c_{ADN} c_2\, \hat{c}\, \delta} \left\| \Delta(\sigma u) \right\|_{L^2(\Omega)}. \tag{5.2.45}$$

The last passage is justified if δ is small enough so that $\tilde{c}^2 c_{ADN}\, c_2\, \hat{c}\, \delta < 1$.

We have thus obtained the (local) a priori inequality (5.2.45). To conclude, we consider sufficiently small neighbourhoods $U_j, j = 1, \dots, s$ that generate a (finite) open covering of $\overline{\Omega}$, and take a partition of unity $\{\sigma_j\}_{j=1,\dots,s}$ subject to that open cover. Denoting $u_j := \sigma_j u$ we have that $u = \sum_j u_j$. We have that for any $j = 1, \dots, s$ we have that (5.2.45) holds. Moreover

$$\Delta u_j = (\Delta \sigma_j) u + \nabla \sigma_j \cdot \nabla u + \sigma_j (\Delta u),$$

thus

$$\left\| \Delta u_j \right\|_{L^2(\Omega)} \leq C \left(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right) \tag{5.2.46}$$

for any $j = 1, \dots, s$, where $C = \sup_{j \in \{1,\dots,s\}} \left\| \sigma_j \right\|_{C^2(\Omega)}$. Hence

$$\begin{aligned} \|u\|_{H^2(\Omega)} &\leq \sum_{j=1}^s \left\| u_j \right\|_{H^2(\Omega)} \leq \sum_{j=1}^s \frac{\tilde{c}^2 c_{ADN}}{1 - \tilde{c}^2 c_{ADN}\, c_2\, c\, \delta} \left\| \Delta u_j \right\|_{L^2(\Omega)} \\ &\leq \frac{C\, \tilde{c}^2 c_{ADN}}{1 - \tilde{c}^2 c_{ADN}\, c_2\, \hat{c}\, \delta}\, s \left(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right). \end{aligned} \tag{5.2.47}$$

Finally, it remains to apply Proposition 5.2.50 with $\tau > 0$ such that

$$\tau < \frac{1 - \tilde{c}^2 c_{ADN}\, c_2\, \hat{c}\, \delta}{s\, C\, \tilde{c}^2 c_{ADN}}$$

to obtain

$$\|u\|_{H^2(\Omega)} \le s\, C \left(\frac{1}{\tilde{c}^2\, c_{ADN}} - c_2\, \hat{c}\, \delta - C\, \tau\, s \right)^{-1} \Big(\|\Delta u\|_{L^2(\Omega)} + c(\tau) \|u\|_{L^1(\Omega)} \Big) . \tag{5.2.48}$$

□

Remark 5.2.49. Taking a closer look at the constants appearing in the inequality (5.2.47), we have that the constant c_{ADN} and the constant $\hat{c}$ (from (5.2.34)) are dimensional. The constant C and the parameter s (the number of cuboids composing a suitable open covering of Ω) depend on the atlas associated to Ω (cf. Definition 1.0.1). Finally, the constants $\tilde{c}$ and c_2 depends on the dimension N and are continuous functions of the $MW_2^{1/2}$ multiplier norm of $\nabla\varphi$, hence they eventually depend on δ (cf. Definition 5.2.14).

The detailed analysis done until now is what ultimately motivated Definition 1.4.19 and effectively proves Theorem 1.4.22.

Proposition 5.2.50. *Let $k, N \in \mathbb{N}$ be natural numbers and $p \in [1, N)$. Let Ω be a bounded Lipschitz set in $\mathbb{R}^N$. Then for every $\tau > 0$ there exists $c(\tau) > 0$ such that*

$$\|f\|_{W^{k-1,p}(\Omega)} \le \tau \|f\|_{W^{k,p}(\Omega)} + c(\tau) \|f\|_{L^1(\Omega)}$$

for any function $f \in W^{k,p}(\Omega)$.

Proof. Let $\tau > 0$ and suppose by contradiction that there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset W^{k,p}(\Omega)$ such that

$$\big\|f_j\big\|_{W^{k-1,p}(\Omega)} \ge \tau \big\|f_j\big\|_{W^{k,p}(\Omega)} + j \big\|f_j\big\|_{L^1(\Omega)} .$$

We can directly assume that $\big\|f_j\big\|_{W^{k-1,p}(\Omega)} = 1$ for all $j \in \mathbb{N}$. Hence $\big\|f_j\big\|_{W^{k,p}(\Omega)} \le \tau^{-1}$, thus by the Rellich-Kondrachov embedding theorem (see, e.g., [2, Thm. 6.3]) we can extract a subsequence that converges to some f strongly in $W^{k-1,p}(\Omega)$. In particular $\|f\|_{W^{k-1,p}(\Omega)} = 1$. At the same time, since $\big\|f_j\big\|_{L^1(\Omega)} \le j^{-1}$ for all $j \in \mathbb{N}$, we have that $f_j \to 0$ in $L^1(\Omega)$. Thus necessarily $f = 0$, a contradiction. □

5.2.2 The condition $M_2^{3/2}(\delta)$

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. In this last section we briefly focus on a local characterization of the class $M_p^{l-1/p}(\delta)$, ultimately deducing a simpler sufficient condition for the inclusion of the boundary $\partial\Omega$ into $M_p^{l-1/p}(\delta)$ (cf. (5.2.56)). These results, presented in their general form, are taken from Section 14.6.3 of [92].

Note that by Theorem 5.2.9 and by (5.2.7), the multiplier norm of $\nabla\varphi$ in $MW_p^{l-1-1/p}(\mathbb{R}^{N-1})$ is equivalent to

$$\sup_{\substack{E\subset\mathbb{R}^{N-1}\\ d(E)\le 1}} \frac{\left\|D_{p,l-1/p}\,\varphi\right\|_{L^p(E)}}{[C_{p,l-1-1/p}(E)]^{1/p}} + \|\nabla\varphi\|_{L^\infty(\mathbb{R}^{N-1})}\,.$$

The following theorem gives a local characterization of the class $M^{l-1/p}(\delta)$.

Theorem 5.2.51 ([92, Thm. 14.6.4]). *Let $p(l-1)\le N$. The class $M_p^{l-1/p}(\delta)$ has the following equivalent description. For any point $O\in\partial\Omega$ there exists a neighbourhood U and a special Lipschitz domain $G=\left\{z=(x,y): x\in\mathbb{R}^{N-1}, y>\varphi(x)\right\}$ such that $U\cap\Omega=U\cap G$ and*

$$\lim_{\rho\to 0}\left(\sup_{E\subset B_\rho}\frac{\left\|D_{l-1/p}(\varphi;B_\rho)\right\|_{L^p(E)}}{[C_{p,l-1-1/p}(E)]^{1/p}} + \|\nabla\varphi\|_{L^\infty(B_\rho)}\right)\le c\,\delta. \tag{5.2.52}$$

Here B_ρ denotes the ball in $\mathbb{R}^{N-1}$ centred in O and of radius ρ, $c>0$ is a constant which depends only on l,p,N. The term $D_{j-1/p}(\varphi;B_\rho)$, with $j\ge 2$ a natural number, has the following definition

$$D_{j-1/p}(\varphi;B_\rho)(x) := \left(\int_{B_\rho}\frac{\left|\nabla_{j-1}\varphi(x)-\nabla_{j-1}\varphi(y)\right|^p}{|x-y|^{N+p-2}}dy\right)^{1/p}.$$

Using Theorem 5.2.51 and properties of the capacity $C_{p,l}(e)$ (see [92, §9.6.1]) we get the following result.

Corollary 5.2.53 ([92, Coroll. 14.6.1]). *The following statements hold.*

(i) *If $p(l-1)<N$ and*

$$\lim_{\rho\to 0}\left(\sup_{E\subset B_\rho}\frac{\left\|D_{l-1/p}(\varphi;B_\rho)\right\|_{L^p(E)}}{|E|^{\frac{N-p(l-1)}{(N-1)p}}} + \|\nabla\varphi\|_{L^\infty(B_\rho)}\right)< c\,\delta,$$

then $\partial\Omega\in M_p^{l-1/p}(\delta)$.

(ii) *If* $p(l-1) = N$ *and*

$$\lim_{\rho\to 0}\left(\sup_{E\subset B_\rho}\big\|D_{l-1/p}(\varphi;B_\rho)\big\|_{L^p(E)}\big|\log|E|\big|^{\frac{p-1}{p}}+\|\nabla\varphi\|_{L^\infty(B_\rho)}\right)<c\,\delta,$$

then $\partial\Omega\in M_p^{l-1/p}(\delta)$.

Here $|E|$ *denotes the* $N-1$*-dimensional Lebesgue measure of the set* E*, and* $c>0$ *is the same constant of Theorem 5.2.51.*

Recall that $l\geq 2$ is an integer and $p\in(1,\infty)$. We now want to give a final characterization of the class $M_p^{l-1/p}(\delta)$ in terms of Besov spaces $B^\mu_{q,\theta}$. Note that the spaces B^l_p introduced in (5.2.6) coincide with $B^l_{p,p}$. Recall the notion Δ_h introduced in (5.2.3) and that ∇_k denotes the gradient of order $k\in\mathbb{N}\cup\{0\}$.

Definition 5.2.54. Let $\mu>0$ be such that its fractional part $\{\mu\}>0$ is positive, and let $q,\theta\geq 1$. By $B^\mu_{q,\theta}(\mathbb{R}^N)$ we denote the space of functions in $\mathbb{R}^N$ having finite norm

$$\|u\|_{B^\mu_{q,\theta}(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N}\big\|\Delta_h\nabla_{[\mu]}u\big\|^\theta_{L^q}|h|^{-N-\theta\{\mu\}}\,dh\right)^{\frac{1}{\theta}}+\|u\|_{W_q^{[\mu]}(\mathbb{R}^N)}.$$

If $\theta=\infty$, we define the space $B^\mu_{q,\infty}(\mathbb{R}^N)$ of functions in $\mathbb{R}^N$ with the norm

$$\|u\|_{B^\mu_{q,\infty}(\mathbb{R}^N)}=\sup_{h\in\mathbb{R}^N}|h|^{-\{\mu\}}\big\|\Delta_h\nabla_{[\mu]}u\big\|_{L^q(\mathbb{R}^N)}+\|u\|_{W_q^{[\mu]}(\mathbb{R}^N)}.$$

For more details on Besov spaces (and thus fractional Sobolev spaces) we refer to [88, §17].

We say that the boundary of the bounded Lipschitz domain Ω belongs to $B^{l-1/p}_{q,p}$ if for any point of $\partial\Omega$ there exists a neighbourhood in which $\partial\Omega$ is specified in Cartesian coordinates by a functions φ satisfying (cf. Definition 5.2.54)

$$\int_{\mathbb{R}^{N-1}}\left(\int_{\mathbb{R}^{N-1}}\big|\nabla_{l-1}\varphi(x+h)-\nabla_{l-1}\varphi(x)\big|^q\,dx\right)^{p/q}|h|^{2-N-p}dh<\infty.$$

Corollary 5.2.55 ([92, Coroll. 14.6.2])**.** *Let* $p(l-1)\leq N$ *and let* Ω *be a bounded Lipschitz domain with* $\partial\Omega\in B^{l-1/p}_{q,p}$*, where*

$$q\in[p(n-1)/p(l-1)-,\infty]\quad\textit{if } p(l-1)<N$$

and

$$q\in(p,\infty]\quad\textit{if } p(l-1)=N.$$

Further, let $\partial\Omega$ *be locally defined in Cartesian coordinates by* $y=\varphi(x)$*, where* φ *is a function with a Lipschitz constant les than* $c\,\delta$*. Here* $c>0$ *is the same constant of Theorem 5.2.51. Then* $\partial\Omega\in M_p^{l-1/p}(\delta)$*.*

Setting $q = \infty$ in Corollary 5.2.55 we obtain a simple sufficient condition for the inclusion in the class $M_p^{l-1/p}(\delta)$, that is the integrability at zero of the following integral

$$\int_0 \left(\frac{\omega_{l-1}(t)}{t}\right)^p dt < \infty, \tag{5.2.56}$$

where ω_{l-1} is the modulus of continuity of $\nabla_{l-1}\varphi$. From this, it is not difficult to see that bounded Lipschitz domains of class $C^{1+\beta}$ with $\beta > 1/2$ are in the class $M_2^{3/2}(\delta)$ for any $\delta > 0$, and therefore they are such that Theorems 5.2.16 and 5.2.39 both hold.

Condition (5.2.56) is what we use in Theorem 1.4.23, and, via Corollary 1.4.30 and Corollary 1.4.31, ultimately what we need in order to justify the results in Section 3.3 regarding families of oscillating boundaries.

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