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Modern likelihood inference for the parameter of skewness: An application to mono-zygotic twin studies

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1 Introduction

Since Sir Francis Galton's (1876) seminal paper, twin studies have extensively been used for the quantitative ascertainment of genetic and environmental influences. Twin registries worldwide represent nowadays a valuable resource for the investigation of the similarities and dissimilarities between twins. The very large twin studies carried out during the past two decades led to much novel work, especially in genetic research van Dongen *et al.* (2012). Classical twin designs remain, nonetheless, a valuable tool in fields such as biomedicine, psychiatry and behavioral sciences, where the number of available observations is far smaller than those typical in modern twin studies.

Small sample sizes are rather common to researchers in fields such as biology, genetics, medical sciences and psychology. Inference based on the classical first order normal and χ^2 approximations may then be unreliable. The last four decades have seen the development of so-called higher order likelihood approximations, which require little more effort than is needed for their first order counterparts while providing highly accurate inferences in small samples. We refer the reader to Brazzale *et al.* (2007) for a rich collection of realistic examples and case studies, which show how to use the new theory. The aim of this paper is to encourage the use of modern likelihood-based solutions for the analysis of continuous data on mono-zygotic twins.

There are several views of how the degree of concordance between twins should be assessed Kraemer (1997); Lyons *et al.* (1997). Here, we promote the use of Azzalini's (1985) skew-normal distribution, which generalizes the standard normal distribution by allowing for asymmetry. In particular, we will use Loperfido's (2002) results, according to which the maximum, or minimum, of two random variables, whose joint distribution is bivariate exchangeable normal with correlation coefficient ρ, is skew-normally distributed with skewness parameter γ, or $-\gamma$, where $\gamma = \sqrt{(1-\rho)/(1+\rho)}$. This distribution becomes the reference model when we have censoring on the maximum (or minimum) value for each twin pair.

Estimation of the shape parameter of the skew-normal distribution can be, at times, tricky. In particular, it is not easy to compute confidence intervals. Recently, Mameli *et al.* (2012), borrowing from Loperfido's result and Fisher's z transform for ρ, obtained an asymptotic confidence interval for the skewness parameter of the distribution of the maximum/minimum under this framework. Their simulation results revealed that actual and nominal coverage of the asymptotic confidence interval are close, though its expected length increases for decreasing sample size and correlation coefficient close to -1. In this paper we explore the performance of confidence intervals for γ obtained from the small-sample solutions recently proposed in Fraser *et al.* (1999), and this in terms of both actual coverage and expected length.

The paper organizes as follows. Section 2 reviews modern likelihood-based inference. The skew-normal distribution and Loperfido's results will be introduced in Section 3. Inference on γ will be discussed in Section 4. Section 5 re-analyzes the twin data collected by Tramo *et al.* (1998) using the large- and small-sample solutions of Section 4. Their finite-sample properties will be investigated in Section 6 through simulation. Some concluding remarks are given in Section 7.

2 Likelihood-based inference

2.1 First order theory

Let $y = (y_1, \ldots, y_n)$ be a sample of size n with joint log-likelihood function $l(\theta) = l(\theta; y)$, where $\theta = (\psi, \lambda)$ is a k-dimensional parameter, ψ is the scalar parameter of interest, and λ a vector of nuisance parameters of dimension $k-1$. Under broad regularity conditions, the maximum likelihood estimate of θ, denoted by $\hat{\theta}$, may be obtained by solving the score equation $l_\theta(\hat{\theta}; y) = 0$, with $l_\theta(\theta; y) = \partial l(\theta; y)/\partial\theta$. Let $j(\theta) = \partial^2 l(\theta; y)/\partial\theta\partial\theta^\top$ represent the observed information function for θ and $j(\hat{\theta})$ the observed Fisher information. The decomposition of the parameter θ into ψ and λ leads to an analogous decomposition of the score vector $l_\theta(\theta; y)$ and of the observed information function $j(\theta)$.

The recommended likelihood pivot for making inference on ψ is the signed likelihood root

$$r(\psi) = \mathrm{sign}(\hat{\psi} - \psi)\sqrt{2(l_p(\hat{\psi}) - l_p(\psi))}. \tag{1}$$

Here $l_p(\psi) = l(\hat{\theta}_\psi)$, with $\hat{\theta}_\psi = (\psi, \hat{\lambda}_\psi)$, is the profile log-likelihood, while $\hat{\lambda}_\psi$ represents the constrained maximum likelihood estimate obtained by maximizing the log-likelihood $l(\psi, \lambda)$ with respect to λ holding ψ fixed. The signed likelihood root (1) is asymptotically standard normal up to the order $n^{-1/2}$, which leads to the first order $(1-\alpha)100\%$ confidence interval for ψ

$$\left\{\psi : |r(\psi)| \leq z_{1-\alpha/2}\right\}, \tag{2}$$

where z_p, with $p \in (0,1)$, is the pth quantile of the standard normal distribution. The standard normal approximation provides a satisfactory approximation for large sample sizes, but can be highly unreliable for small values of n. The value of ψ which satisfies equation (2) can be found numerically by calculating the function $r(\psi)$ on a grid of points ψ, which are then interpolated using a suitable smoothing function. The numerical issues, which may arise in the interpolation step, can be avoided by excluding the values of ψ close to the maximum likelihood estimate $\hat{\psi}$. The details are given in (Brazzale *et al.*, 2007, Section 9.3).

2.2 Higher order theory

A nowadays broadly known improvement to the signed likelihood root (1), which was originally introduced by Barndorff-Nielsen (1983), is the modified likelihood ratio

$$r^* = r + \frac{1}{r}\log\left(\frac{q}{r}\right), \tag{3}$$

whose finite-sample distribution may be approximated by the standard normal up to the order $n^{-\frac{3}{2}}$. Several expressions for the correction term q have been proposed, both from the frequentist and the Bayesian perspective. Here, we will focus on the developments by Fraser and Reid (1995).

To derive their formula for q, Fraser and co-author used the notion of ‘tangent exponential model’ which, at a fixed value of y, denoted y_0, approximates the true model by a local exponential model with canonical parameter $\varphi = \varphi(\theta)$, defined as

$$\varphi^\top(\theta) = l_{;V}(\theta; y_0) = \sum_{i=1}^{n} \left.\frac{\partial l(\theta; y)}{\partial y_i}\right|_{y=y_0} V_i. \tag{4}$$

Here, $l_{;V}$ indicates differentiation of the log-likelihood function in the directions given by the n columns $V_1, \ldots, V_n$ of the $n \times k$ matrix V, while $\top$ denotes matrix transposition. The matrix V can be constructed using a vector of pivotal quantities $z = \{z_1(y_1, \theta), \ldots, z_n(y_n, \theta)\}$, where each component $z_i(y_i, \theta)$ has a fixed distribution under the model. The matrix V is defined from z by

$$V = -\left(\frac{\partial z}{\partial y^\top}\right)^{-1}\left.\left(\frac{\partial z}{\partial \theta^\top}\right)\right|_{(y_0, \hat{\theta}_0)},$$

where $\hat{\theta}_0$ is the maximum likelihood estimate at y_0. The expression of the correction term q is then

$$q = \frac{|\varphi(\hat{\theta}) - \varphi(\hat{\theta}_\psi) \qquad \varphi_\lambda(\hat{\theta}_\psi)|}{|\varphi_\theta(\hat{\theta})|}\left\{\frac{|j(\hat{\theta})|}{|j_{\lambda\lambda}(\hat{\theta}_\psi)|}\right\}^{\frac{1}{2}}, \tag{5}$$

where $\varphi_\theta(\theta) = \partial\varphi(\theta)/\partial\theta^\top$ represents the matrix of partial derivatives of $\varphi(\theta)$ with respect to θ, while $\varphi_\lambda(\theta) = \partial\varphi(\theta)/\partial\lambda^\top$ identifies the $k-1$ columns of this matrix which correspond to the nuisance parameter λ. Analogously, the matrix $j_{\lambda\lambda}(\theta)$ is the $(k-1)\times(k-1)$ sub-matrix of the observed information function $j(\theta)$ with respect to the nuisance parameter λ.

The higher order $(1-\alpha)100\%$ confidence interval for ψ is given by

$$\left\{\psi : |r^*(\psi)| \le z_{1-\alpha/2}\right\}. \tag{6}$$

Again, pivot profiling (Brazzale *et al.*, 2007, Section 9.3) can be used to identify the upper and lower bounds of the confidence interval. Furthermore, the r^* pivot—like its first order counterpart r—is invariant under interest-respecting re-parametrizations, that is re-parametrizations of the form $\tau(\theta) = \tau(\psi,\lambda) = (\zeta,\eta)$ with $\zeta = \zeta(\psi)$ and $\eta = \eta(\psi,\lambda)$.

The expression of q for the case in which the nuisance parametrization is not given explicitly can be found in Fraser *et al.* (1999).

2.3 Approximations for Bayesian inference

In the Bayesian setting with a prior density $\pi(\theta)$ for θ, the analogue of the first order results of Section 2.1 is the asymptotic normality of the posterior density $\pi(\theta|y)$ for θ. The Bayesian counterpart of the correction term q in (3), which we will denote by q_B, was obtained by DiCiccio and Martin (1991) under the assumption that the nuisance parametrization is given explicitly, and results to

$$q_B = l_p'(\psi) j_p(\hat\psi)^{-\frac{1}{2}} \left\{\frac{|j_{\lambda\lambda}(\hat\theta_\psi)|}{|j_{\lambda\lambda}(\hat\theta)|}\right\}^{\frac{1}{2}} \frac{\pi(\hat\theta)}{\pi(\hat\theta_\psi)}, \tag{7}$$

where $l_p'(\psi) = dl_p(\psi)/d\psi$ is the profile score function and $j_p(\psi) = d^2 l_p(\psi)/d\psi^2$ the profile observed information function. Posterior quantiles for the parameter ψ can be found exploiting the fact that the posterior distribution function

$$\Pi(\psi_0 \mid y) = \Pr(\psi \le \psi_0 \mid y) \doteq 1 - \Phi(r_B^*)$$

may be approximated to the order $n^{-3/2}$ by the standard normal distribution function $\Phi(r_B^*)$, evaluated at

$$r_B^* = r + \frac{1}{r}\log\left(\frac{q_B}{r}\right). \tag{8}$$

Again, pivot profiling provides the upper and lower bounds of the $(1-\alpha)100\%$ credible interval for ψ given by

$$\left\{\psi : |r_B^*(\psi)| \le z_{1-\alpha/2}\right\}. \tag{9}$$

Like for q, Fraser *et al.* (1999) provide the expression of the correction term q_B for the case in which the nuisance parametrization is not given explicitly.

2.3.1 Matching priors

Given the prior $\pi(\theta)$ for θ, let $\theta^\pi_{1-\alpha}$ denote the $(1-\alpha)$th approximate posterior quantile of θ of order n^{-r}, that is, the value of θ for which

$$\Pr_{\theta|y}\left(\theta \le \theta^\pi_{1-\alpha} \mid y\right) = 1 - \alpha + O_p(n^{-r}), \tag{10}$$

with $r > 0$ and $0 < \alpha < 1$. If we also have that

$$\mathrm{Pr}_{Y|\theta}\left(\theta^{\pi}_{1-\alpha} \geq \theta \mid \theta\right) = 1 - \alpha + O_p(n^{-r}), \tag{11}$$

with $\theta^{\pi}_{1-\alpha}$ the upper bound of a frequentist one-sided $(1-\alpha)100\%$ confidence interval, the prior π is called a probability matching prior to the rth order of approximation. For such priors, Bayesian and frequentist inference for the parameter θ are in perfect agreement up to the order r.

If $r = 1$, $\pi(\theta)$ is called a first order probability matching prior, while for $r = 3/2$ we have a second order probability matching prior. Welch and Peers (1963) showed that the unique first order probability matching prior, when no nuisance parameter is present, is Jeffrey's prior.

The same result does not necessarily hold when θ includes a nuisance component λ. For an orthogonal parametrization, Staicu and Reid (2008) proposed to use the following prior for θ in (7),

$$\pi(\psi, \lambda) \propto i^{1/2}_{\psi\psi}(\psi, \lambda), \tag{12}$$

where $i_{\psi\psi}(\psi, \lambda)$ represents the value of the expected Fisher information function corresponding to ψ. The authors call this prior the "unique prior", as it leads to an approximation of the marginal posterior distribution of ψ accurate to the order $n^{-3/2}$. When the parametrization $\theta = (\psi, \lambda)$ is not orthogonal, their suggestion is to find an orthogonal parametrization (ψ, η) of the original model for which the prior can be expressed as (12), and then to re-express the prior in the original parametrization (ψ, λ), leading to

$$\pi(\psi, \lambda) \propto i^{1/2}_{\psi\psi.\lambda}(\psi, \lambda)\left|\frac{\partial\eta}{\partial\lambda}\right|, \tag{13}$$

with $i_{\psi\psi.\lambda}(\psi, \lambda) = i_{\psi\psi}(\psi, \lambda) - i_{\psi\lambda}(\psi, \lambda)i^{-1}_{\lambda\lambda}(\psi, \lambda)i_{\lambda\psi}(\psi, \lambda)$, where the indices ψ and λ indicate which sub-blocks of the expected Fisher information function to take. Furthermore, $\left|\partial\eta/\partial\lambda\right|$ represents the Jacobian of the transformation from (ψ, η) to (ψ, λ).

3 The skew-normal model

The skew-normal distribution was introduced by Azzalini (1985) to define a class of asymmetric parametric models which includes the standard normal as a special case. We say that a continuous random variable $Z \sim SN(\gamma)$, distributes as a skew-normal indexed by the real parameter γ, if it has density function

$$p(z; \gamma) = 2\phi(z)\Phi(\gamma z) \quad \text{with} \quad z \in \mathbb{R}.$$

Here $\phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the density and the distribution functions of the standard normal distribution. The class of skew-normal distributions can be widened by including a location parameter $\mu \in \mathbb{R}$ and a scale parameter $\sigma > 0$. Thus, if $X \sim SN(\gamma)$, then $Y = \mu + \sigma X$ is a skew normal random variable with parameters μ, σ, γ, or, $Y \sim SN(\mu, \sigma, \gamma)$ for short. Making inference on the skewness parameter is quite challenging, as the expected Fisher information becomes singular as $\gamma \to 0$. Functions for manipulating the skew-normal probability distribution and for fitting it to data are given in the R package `sn` Azzalini (2013). We refer the reader to Genton (2004) for a general treatment of the skew-normal distribution and its extensions.

In this paper we focus on the distribution of the maximum (or minimum) of an exchangeable bivariate normal random vector. Loperfido (2002) showed that a linear combination of the maximum and the minimum of a bivariate exchangeable normal random vector is skew-normally distributed with parameters specified by the following theorem.

Theorem 3.1 *Let X_1 and X_2 be two random variables whose joint distribution is bivariate normal with common mean $\mu \in \mathbb{R}$, common variance $\sigma^2 > 0$ and correlation coefficient $\rho \in (-1, 1)$. Then for any two real constants h and $k \neq -h$, the distribution of $h \min(X_1, X_2) + k \max(X_1, X_2)$ is*

$$SN\left(\mu(h+k),\ \sigma\sqrt{h^2+k^2+2\rho hk},\ \gamma = \frac{k-h}{|k+h|}\sqrt{\frac{1-\rho}{1+\rho}}\right).$$

Theorem 3.1 was subsequently generalized by Loperfido (2008) to the case where X_1 and X_2 are exchangeable, elliptical and continuous random variables. It follows that the distribution of $\max(X_1, X_2)$ is $SN(\mu, \sigma, \gamma)$ with $\gamma = \sqrt{(1-\rho)/(1+\rho)} \geq 0$, whereas the distribution of $\min(X_1, X_2)$ is $SN(\mu, \sigma, \gamma)$ with $\gamma = -\sqrt{(1-\rho)/(1+\rho)} \leq 0$. The special case of $\rho = 0$ translates into $\gamma = 1$ and $\gamma = -1$, respectively. Figure 1 shows how the shape of the distributions of $\max(X_1, X_2)$ (bold) and $\min(X_1, X_2)$ (solid) changes when ρ varies from -0.9 to $+0.9$.

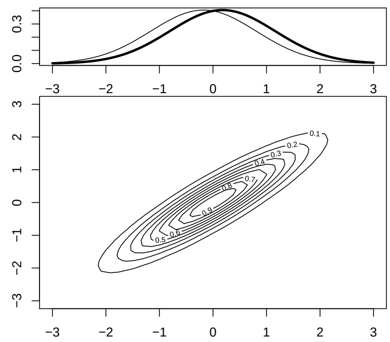
Theorem 3.1 provides the reference models for mono-zygotic twin studies for which information on the pair (X_1, X_2) is missing, and only their maximum (or minimum) value is recorded. This may, for instance, happen because of practical reasons; see Roberts (1966) for a rather early treatment. As pointed out there, because healthy mono-zygotic twins share an identical genetic mark-up, time of onset for a particular event in the first twin—such as getting a cold or developing leukaemia—is likely to closely follow in the second twin, so that only the smaller or larger record may be kept. Furthermore, working with the maximum (or minimum) of two correlated measurements can be, at times, more reliable than the study of the original values, especially if the measurements of the smaller (or larger) values are less accurate.

4 Inference on the skewness parameter $\gamma = \sqrt{\frac{1-\rho}{1+\rho}}$

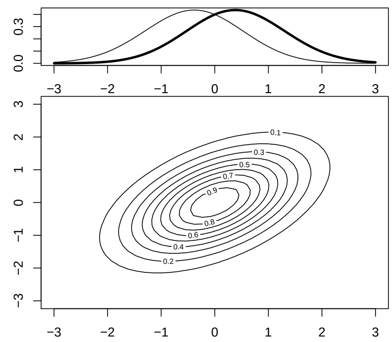
4.1 Background results

4.1.1 Exact confidence interval

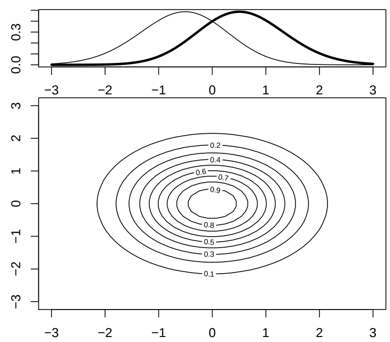
Let $Y = (X_1, X_2)$ be a bivariate normal vector with common mean 0, common variance 1 and correlation coefficient $\rho \in (-1, 1)$. Given an i.i.d. sample $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ of size n from Y, Haddad and Provost (2011) proposed a range-based exact confidence interval for ρ. The construction of the confidence interval makes use of the two random variables $D_+ = \sum_{i=1}^n (X_{1i} + X_{2i})^2$ and $D_- = \sum_{i=1}^n (X_{1i} - X_{2i})^2$. Taking advantage of the independence of $X_{1i} + X_{2i}$ and $X_{1i} - X_{2i}$ along with the fact that $X_{1i} + X_{2i} \sim$



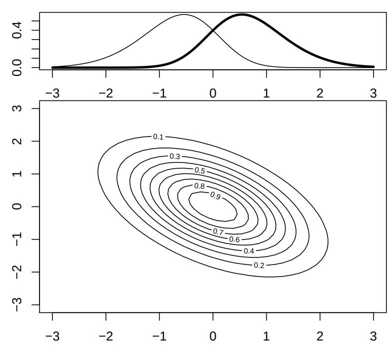
(a) $\rho = 0.9$



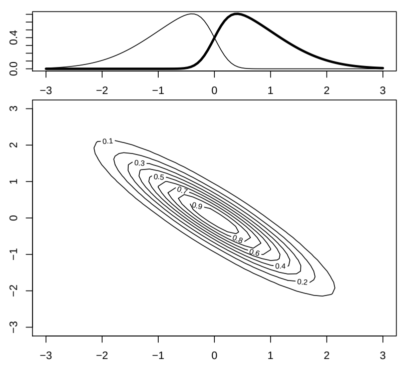
(b) $\rho = 0.5$



(c) $\rho = 0$



(d) $\rho = -0.5$



(e) $\rho = -0.9$

Figure 1: Contour plots of the bivariate standard normal distributions with correlation coefficient ρ, and corresponding distribution of the maximum (bold) and minimum (solid), for $\rho \in \{-0.9, -0.5, 0, 0.5, 0.9\}$.

$N(0, 2(1+\rho))$ and $X_{1i} - X_{2i} \sim N(0, 2(1-\rho))$, the authors derive the following pivotal quantity

$$\frac{D_+}{D_-}\frac{(1-\rho)}{(1+\rho)} \sim F_{n,n}, \tag{14}$$

where $F_{n,n}$ is Fisher's F distribution with (n, n) degrees of freedom. This gives an exact $(1-\alpha)100\%$ confidence interval for the parameter γ of the form

$$\left\{\gamma \in [0,\infty) : \sqrt{\frac{D_-}{D_+}F_{n,n}(1-\alpha/2)} < \gamma < \sqrt{\frac{D_-}{D_+}F_{n,n}(\alpha/2)}\right\}, \tag{15}$$

where $F_{n,n}(p)$, with $p \in (0, 1)$, represents the pth quantile of Fisher's F distribution with (n, n) degrees of freedom.

4.1.2 Large-sample confidence intervals

Haddad and Provost (2011) considered also the construction of a confidence interval for ρ when the means and variances of the bivariate random vector are unknown. In this case, the solution is no longer exact. Let (X_1, X_2) be a bivariate normal random vector with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$ where $(\mu_1, \sigma_1) \in \mathbb{R} \times \mathbb{R}_+$ and $(\mu_2, \sigma_2) \in \mathbb{R} \times \mathbb{R}_+$ are, respectively, the means and variances of X_1 and X_2, and $\rho \in (-1, 1)$ their correlation. The first step is to standardize the two components X_{1i} and X_{2i}; let X^*_{1i} and X^*_{2i} be the standardized variables. An approximate confidence interval for the parameter ρ is obtained, likewise above, by using the fact that $X^*_{1i} - X^*_{2i}$ and $X^*_{1i} + X^*_{2i}$ are nearly independent, $X^*_{1i} + X^*_{2i} \sim N(0, 2(1+\rho))$ and $X^*_{1i} - X^*_{2i} \sim N(0, 2(1-\rho))$. The pivot

$$\frac{D^*_+}{D^*_-}\frac{(1-\rho)}{(1+\rho)} = \left(\frac{1+R}{1-R}\right)\frac{(1-\rho)}{(1+\rho)}, \tag{16}$$

where $D^*_+ = \sum_{i=1}^n (X^*_{1i} + X^*_{2i})^2$ and $D^*_- = \sum_{i=1}^n (X^*_{1i} - X^*_{2i})^2$ and R is the sample correlation coefficient defined as

$$R = \frac{\sum_{i=1}^n \left(X_{1i} - \bar{X}_1\right)\left(X_{2i} - \bar{X}_2\right)}{\sqrt{\sum_{i=1}^n \left(X_{1i} - \bar{X}_1\right)^2}\sqrt{\sum_{i=1}^n \left(X_{2i} - \bar{X}_2\right)^2}},$$

follows approximately an $F_{n-1,n-1}$ distribution. The corresponding $(1-\alpha)100\%$ confidence interval for γ is

$$\left\{\gamma \in [0,\infty) : \sqrt{\left(\frac{1-R}{1+R}\right)F_{n-1,n-1}(1-\alpha/2)} < \gamma < \sqrt{\left(\frac{1-R}{1+R}\right)F_{n-1,n-1}(\alpha/2)}\right\}. \tag{17}$$

A second approximate solution to the inferential problem we are interested in can be found in Mameli *et al.* (2012). Because of the difficulties of obtaining the finite sample distribution of R, inference for ρ is commonly based on the monotonic transformation $\frac{1}{2}\ln\left((1+R)/(1-R)\right)$, called Fisher's z-transform. In particular, the distribution of

$$Z = \frac{\frac{1}{2}\ln\left(\frac{1+R}{1-R}\right) - \frac{1}{2}\ln\left(\frac{1+\rho}{1-\rho}\right)}{\frac{1}{\sqrt{n-3}}} \tag{18}$$

for $n > 50$ is approximately standard normal.This turns into an $(1-\alpha)100\%$ confidence interval for γ of the form

$$\left\{\gamma \in [0,\infty) : \exp\left(\frac{-z_{\frac{\alpha}{2}}}{\sqrt{n-3}}\right)\sqrt{\left(\frac{1-R}{1+R}\right)} < \gamma < \exp\left(\frac{z_{\frac{\alpha}{2}}}{\sqrt{n-3}}\right)\sqrt{\left(\frac{1-R}{1+R}\right)}\right\}. \tag{19}$$

Note that the upper and lower bounds of both, the confidence interval (19) proposed by Mameli *et al.* (2012) and solution (17) derived by Haddad and Provost (2011), include the multiplying factor $\sqrt{(1-R)/(1+R)}$.

4.2 Small-sample confidence intervals

4.2.1 No nuisance parameter

Reid (2003) provides the expression of the higher-order pivot r^* when interest relies on $\theta = \rho$, the correlation coefficient of a bivariate normal vector (X_1, X_2) with common means 0 and variances 1. The reference model in this case is a $(2,1)$ curved exponential family. A key quantity for the determination of the canonical parameter (4) of the approximating tangent full exponential model is the vector $V = (1-\hat{\theta})^{-1}(t-\hat{\theta}s, s-\hat{\theta}t)^\top$, obtained from the two independent pivots $Z_1 = (T+S)/(1+\theta)$ and $Z_2 = (T-S)/(1-\theta)$, with $S = n^{-1}\sum_{i=1}^n X_{1i}X_{2i}$ and $T = (2n)^{-1}\sum_{i=1}^n (X_{1i}^2 + X_{2i}^2)$, whose distribution is χ^2_n/n . The canonical parameter takes the form $\varphi(\theta) = n\{(1-\theta^2)(1-\hat{\theta}^2)\}^{-1}\{\theta(t-\hat{\theta}s)-(s-\hat{\theta}t)\}$. Later, Reid and Fraser (2010) proposed an alternative formulation, $\bar{\varphi}(\theta) = n\theta/(1-\theta^2)$, of the canonical parameter. As shown there, both formulations lead to almost the same numerical results as far as the approximation of tail areas is concerned.

Turning to the Bayesian world, we may adopt Jeffreys' prior for ρ, given by

$$\pi(\rho) \propto \frac{\sqrt{(1+\rho^2)}}{(1-\rho^2)}, \tag{20}$$

which, as stated in Section 2.3.1, provides a first order probability matching prior for a scalar parameter in the absence of nuisance parameters.

Confidence intervals for the parameter γ can be derived from the $r(\rho)$, $r^*(\rho)$ and $r^*_B(\rho)$ pivots due to their invariance under interest-respecting re-parametrizations.

4.2.2 Nuisance parameters

Let $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ be a sample from a bivariate normal distribution with real means μ_1, μ_2, variances $\sigma_1^2 > 0$, $\sigma_2^2 > 0$ and correlation $\rho \in (-1,1)$. The log-likelihood function

$$\begin{aligned} l(\theta) = & -n\left(\log(\sigma_1\sigma_2) + \frac{1}{2}\log(1-\rho^2) + \frac{\mu_1^2}{2(1-\rho^2)\sigma_1^2} + \frac{\mu_2^2}{2(1-\rho^2)\sigma_2^2} - \frac{\mu_1\mu_2\rho}{(1-\rho^2)\sigma_1\sigma_2}\right) + \\ & - \frac{1}{2(1-\rho^2)\sigma_1^2}\sum_{i=1}^n x_{1i}^2 - \frac{1}{2(1-\rho^2)\sigma_2^2}\sum_{i=1}^n x_{2i}^2 + \frac{\mu_1\sigma_2 - \mu_2\sigma_1\rho}{(1-\rho^2)\sigma_1^2\sigma_2}\sum_{i=1}^n x_{1i} + \\ & + \frac{\mu_2\sigma_1 - \mu_1\sigma_2\rho}{(1-\rho^2)\sigma_1\sigma_2^2}\sum_{i=1}^n x_{2i} + \frac{\rho}{(1-\rho^2)\sigma_1\sigma_2}\sum_{i=1}^n x_{1i}x_{2i}, \end{aligned}$$

Table 1: Measurements of the *corpus callosum* surface area for ten mono-zygotic twins Tramo *et al.* (1998). Bivariate Shapiro-Wilk test for normality: $W = 0.97$, p-value = 0.86.

1st twin	6.08	6.22	7.99	7.44	6.48	8.76	6.32	7.62	6.03	7.67
2nd twin	5.73	5.80	8.42	6.84	6.43	7.99	6.32	7.60	6.59	7.52

with $\theta = (\rho, \mu_1, \mu_2, \sigma_1, \sigma_2)$, characterizes an exponential family with canonical parameter

$$\varphi(\theta) = \left(-\frac{1}{(1-\rho^2)\sigma_1^2}, -\frac{1}{(1-\rho^2)\sigma_2^2}, \frac{\mu_1\sigma_2 - \mu_2\sigma_1\rho}{(1-\rho^2)\sigma_1^2\sigma_2}, \frac{\mu_2\sigma_1 - \mu_1\sigma_2\rho}{(1-\rho^2)\sigma_1\sigma_2^2}, \frac{\rho}{(1-\rho^2)\sigma_1\sigma_2} \right).$$

Setting $\psi = \rho$ and $\lambda = (\mu_1, \mu_2, \sigma_1, \sigma_2)$, q and r^* can readily be obtained from equations (5) and (3), respectively.

The computation of the Bayesian credible interval of Section 2.3 requires that we specify a prior for the parameter θ. The "unique prior" defined by Staicu and Reid (2008) may be calculate by referring to the orthogonal re-parametrization

$$p(x_1, x_2 \mid \psi, \eta) \propto \frac{1}{\eta_4} \exp\left\{ -\frac{1}{2(1-\psi^2)^{1/2}} \left[\frac{(x_1-\eta_1)^2}{\eta_3} + \eta_3(x_2-\eta_2)^2 - 2\psi(x_1-\eta_1)(x_2-\eta_2) \right] \right\}, \tag{21}$$

with $\psi = \rho$ and $\eta_1 = \mu_1$, $\eta_2 = \mu_2$, $\eta_3 = \sigma_1/\sigma_2$ and $\eta_4 = \sigma_1\sigma_2(1-\rho^2)^{1/2}$. Note that Gosh *et al.* (2009, 2010) used the same parametrization but with $\mu_1 = \mu_2 = 0$, so that Fisher's expected information function only includes the parameters ψ, η_3 and η_4.

Confidence and credible intervals for the parameter γ can be readily derived from the $r(\rho)$, $r^*(\rho)$ and $r^*_B(\rho)$ pivots thanks to their invariance under interest-respecting re-parametrizations.

5 A real-data example

We consider the data collected by Tramo *et al.* (1998), as available on `StatLib`. The data set submitted by the authors includes different measurements on the brains of ten pairs of mono-zygotic twins. Five twin pairs are male and the remaining five are female. Here we focus on the variable *corpus callosum* surface area; see Table 1. To assure that all conditions of Theorem 3.1 hold, we first standardize the pairs of observations $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ as in paragraph 4.1.2. The bivariate Shapiro-Wilk normality test ($W = 0.97$, p-value $= 0.86$) supports the hypothesis of bivariate normality of the standardized data. The maximum likelihood estimate of γ is $\hat{\gamma} = 0.324$. The five 95% confidence intervals for γ, computed using the methods outlined in Section 4, are given in Table 2. The interval based on the third order Bayesian solution r^*_B is wider than the confidence intervals obtained from the first order pivot r, the higher order frequentist pivot r^*, the large sample (HP) confidence interval by Haddad and Provost (2011) and the (ACI) confidence interval by Mameli *et al.* (2012).

Figure 2 shows how to compute the lower and upper bounds numerically. The intervals based on r ($1st$), r^* ($3rd$), r^*_B ($Bayes$) and ACI can be read off from the intersections of the corresponding pivots with the horizontal black lines, which represent the 2.5% and the 97.5% quantiles of the standard normal distribution. The lower and upper bounds of the HP confidence interval are computed similarly, but this time by referring to the horizontal

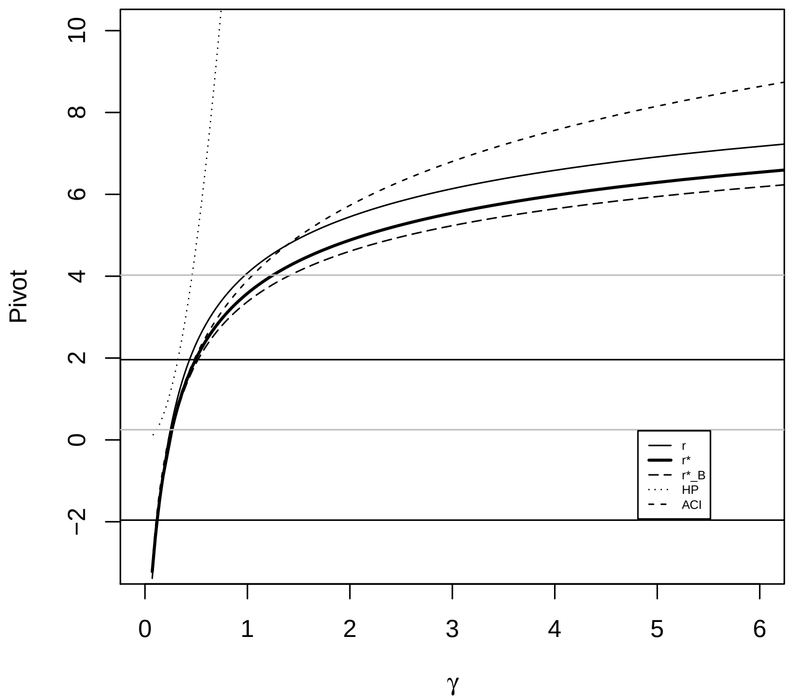


Figure 2: Tramo *et al.* (1998) *corpus callosum* surface area data. Pivot functions for the parameter γ obtained from: likelihood root r (solid), modified likelihood root r^* (bold); Bayesian modified likelihood root r^*_B (long-dashes); expression (16) by Haddad and Provost (2011) (dotted); approximate pivot used in (18) by Mameli *et al.* (2012) (dashed). Black horizontal lines: 2.5% and 97.5% normal quantiles; grey horizontal lines: 2.5% and 97.5% quantiles of the $F(9,9)$ distribution.

grey lines, which represent the 2.5%, and the 97.5% quantiles of the F distribution with $(9,9)$ degrees of freedom.

6 Numerical assessment

We designed two simulations studies to assess and compare the finite-sample properties of the methods discussed in this paper. The summary statistics used are: empirical coverage (CP), upper error probability (UE), that is, the percentage of the true parameter values falling above the upper bound, lower error probability (LE), that is, the percentage of the true parameter values falling below the lower bound, and average length (AL) of the five confidence intervals considered in Section 4. All simulations were run using the numerical computing environment R R Core Team (2013).

Table 2: Tramo *et al.* (1998) *corpus callosum* surface area data. Lower (LB) and upper (UB) bounds of 95% confidence intervals for the parameter γ. Pivots used, with corresponding confidence intervals: $1st$ – likelihood root r (2); $3rd$ – modified likelihood root r^* (6); $Bayes$ – Bayesian modified likelihood root r_B^* (9); HP – Haddad and Provost (2011) (17); ACI – Mameli *et al.* (2012) (19)

Method	LB	UB	Length
$1st$	0.121	0.435	0.314
$3rd$	0.119	0.493	0.374
$Bayes$	0.114	0.518	0.404
HP	0.114	0.460	0.346
ACI	0.109	0.481	0.372

6.1 Considered scenarios

Simulation 1 considers a bivariate normal model with zero means, unit variances and unknown correlation ρ, which takes values from -0.9 to $+0.9$, with step size 0.1. The purpose is to compare the behavior of the higher order pivot r^* with its first order counterpart r, the Bayesian small-sample solution r_B^* and the exact method (14), which apply when no nuisance parameter is present, and this for small sample sizes.

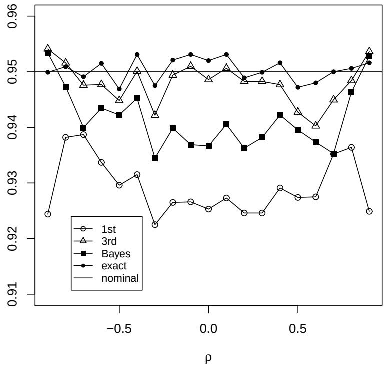
Simulation 2 wants to investigate the finite-sample performance of the confidence intervals obtained when nuisance parameters are present, again while emphasizing small sample sizes. The pivots used are the higher order solution r^*, its first order counterpart r, the Bayesian competitor r_B^*, and the large-sample solutions (17) and (19). We used $\mu_1 = \mu_2 = 7$, $\sigma_1 = \sigma_2 = 0.9$, while again $\rho \in \{-0.9, 0.8, \ldots, 0.8, 0.9\}$. Note that the simulation set-up borrows from the twin data example of Section 5, for which the maximum likelihood estimate is $\hat{\theta} = (0.900, 7.061, 6.924, 0.905, 0.872)$.

In both simulations, $10,000$ replicates are generated for the four sample sizes $n = 5, 10, 15, 20$. The simulation error amounts to ± 0.004.

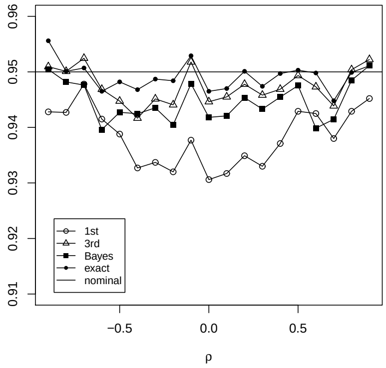
6.2 Discussion

Figure 3 shows the actual coverage of the nominal 95% confidence intervals for γ derived from equation (15) ($exact$), and by using the pivots r ($1st$), r^* ($3rd$), and r_B^* ($Bayes$) for the no-nuisance parameter case. The higher order likelihood pivots r^* and, to a somewhat lesser extent, r_B^* outperform their first order counterpart r, even for the very limited sample sizes considered in the four scenarios of Simulation 1. The differences among the four pivots fade out as the sample size increases.

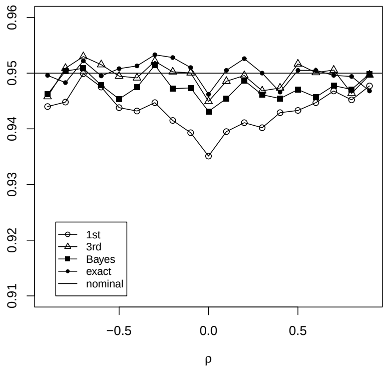
Tables 3 – 6 summarize the performance of the nominal 95% confidence intervals for γ derived from the four pivots considered in Simulation 1. Being there no nuisance parameter present, the corresponding true coverage probabilities are very close to the nominal level, as we may have expected. Inspection of the upper and lower error probabilities reveals that the r^* and r_B^* pivots also improve in terms of symmetry over r. The exact method produces confidence intervals for γ which are, on average, wider than the confidence intervals obtained from the first order solution r and the higher order pivots r^* and r_B^*. The average



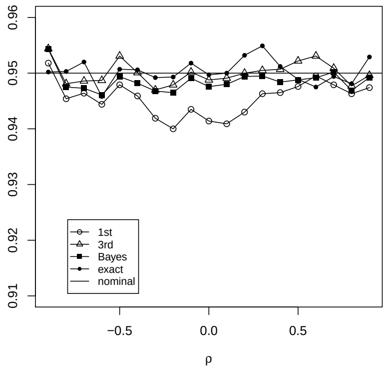
(a) $n = 5$



(b) $n = 10$



(c) $n = 15$



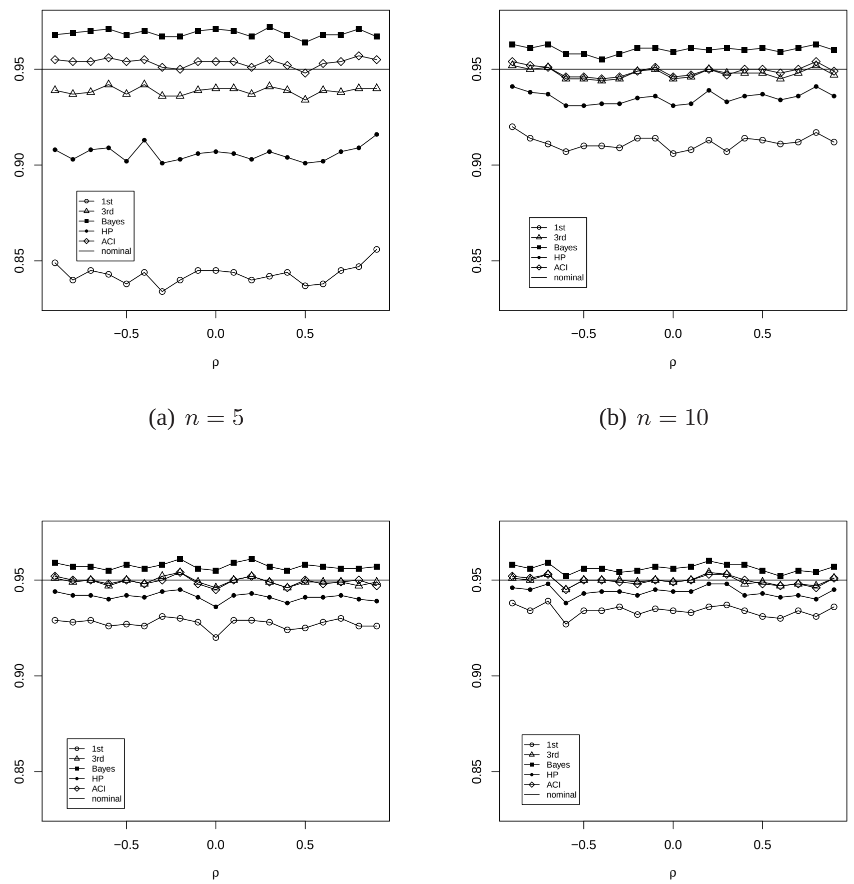
(d) $n = 20$

Figure 3: Simulation 1: bivariate normal with means 0 and variances 1. Empirical coverage of nominal two-sided 95% confidence intervals for γ for varying values of ρ and sample sizes $n = 5, 10, 15, 20$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r^*_B ($Bayes$); expression (14) by Haddad and Provost (2011) ($exact$). Based on 10,000 replicates.

length of all four confidence intervals is larger for negative values of ρ, and increases when the correlation tends to -1.

Figure 4 reports the actual coverage of the nominal 95% confidence intervals for γ obtained from expressions (16) (HP) and (18) (ACI), and by using the pivots r ($1st$), r^* ($3rd$) and r^*_B ($Bayes$), as in Simulation 2. In terms of real coverage, r^* again outperforms its first order counterpart r. It also outperforms the large-sample (HP) proposal by Haddad and Provost (2011) and, surprisingly, the Bayesian solution r^*_B. The most accurate method is the large-sample confidence interval developed by Mameli *et al.* (2012), although the differences fade out for increasing sample size.

Tables 7 – 10 summarize the performance of the nominal 95% confidence intervals for γ derived from the five methods considered in Simulation 2. The results reveal that r^*



(a) $n = 5$ (b) $n = 10$

(c) $n = 15$ (d) $n = 20$

Figure 4: Simulation 2: bivariate normal with means 7 and variances 0.9. Empirical coverage of nominal equi-taled 95% confidence intervals for γ for varying values of ρ and sample sizes $n = 5, 10, 15, 20$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r_B^* ($Bayes$); expression (16) by Haddad and Provost (2011) (HP) and expression (18) by Mameli *et al.* (2012) (ACI). Based on 10,000 replicates.

is more accurate than r, especially when the sample size is small, because of both an, on average, larger width and its capability of correctly centering the confidence intervals. The r_B^* pivot consistently over-estimates the real coverage, while guaranteeing symmetry on the tails, because of the, on average, longer confidence intervals it produces. The ACI and HP methods lead to confidence intervals for γ which are remarkably asymmetric. Their better performance with respect to, respectively, r^* and r may be explained by the, on average, larger widths of the corresponding confidence intervals. For all five methods considered, the expected length becomes larger for negative values of ρ, especially when ρ is close to -1. This is in agreement with Mameli *et al.* (2012), who noted the same behavior for their ACI confidence interval.

7 Concluding remarks

In this paper we investigate the behavior of likelihood-based small-sample procedures to compute confidence intervals for the parameter of skewness which characterizes the distribution of the maximum/minimum of a bivariate normal exchangeable random vector. This distribution represents the reference model for assessing the degree of concordance of a continuos mono-zygotic twin trait when interest focuses on the pairwise maximum or minimum, as in Section 5. Extensive numerical investigation revealed that the higher order frequentist pivot r^* is highly accurate, especially for the rather small sample sizes which may be encountered, and for the challenging situation where ρ is close to -1. This is in agreement with the findings by Sun and Wong (2007), though their contribution focuses on ρ and does not consider the custom-tailored statistics of Section 4. When no nuisance parameter is present, r^* yields confidence intervals which, for practical purposes, may be considered exact. Among the four alternatives available in the presence of nuisance parameters, the only real competitor to r^*, in terms of both real coverage and required computational efforts, is the ACI confidence intervals, though it leads to, on average, longer confidence intervals which counterbalance the lack of symmetry on the tails. The potential applicability of the ACI method to studies on twins was already put forward in Mameli *et al.* (2012).

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Table 3: Summary statistics for Simulation 1: bivariate normal with means 0 and variances 1. Empirical coverage (CP), upper (UE) and lower (LE) error probability and average length (AL) of nominal two-sided 95% confidence intervals for γ, for varying values of ρ and sample size $n = 5$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r^*_B ($Bayes$); expression (15) by Haddad and Provost (2011) ($exact$). Based on 10,000 replicates; simulation error: ± 0.004.

(a)

ρ	$Method$	CP	UE	LE	AL
−0.9	$1st$	0.924	0.039	0.037	6.714
	$3rd$	0.954	0.024	0.022	6.556
	$Bayes$	0.953	0.024	0.022	6.604
	$exact$	0.950	0.024	0.026	11.329
−0.8	$1st$	0.938	0.038	0.024	4.753
	$3rd$	0.952	0.023	0.026	4.600
	$Bayes$	0.947	0.024	0.029	4.587
	$exact$	0.951	0.023	0.026	7.803
−0.7	$1st$	0.939	0.038	0.023	3.861
	$3rd$	0.948	0.024	0.029	3.702
	$Bayes$	0.940	0.024	0.036	3.648
	$exact$	0.949	0.025	0.026	6.144
−0.6	$1st$	0.934	0.040	0.026	3.345
	$3rd$	0.948	0.024	0.028	3.198
	$Bayes$	0.943	0.023	0.034	3.130
	$exact$	0.952	0.025	0.024	5.244
−0.5	$1st$	0.930	0.038	0.033	2.946
	$3rd$	0.945	0.024	0.031	2.807
	$Bayes$	0.942	0.025	0.033	2.727
	$exact$	0.947	0.025	0.028	4.483
−0.4	$1st$	0.932	0.037	0.031	2.659
	$3rd$	0.950	0.022	0.027	2.531
	$Bayes$	0.945	0.024	0.030	2.441
	$exact$	0.953	0.024	0.023	3.948
−0.3	$1st$	0.922	0.042	0.035	2.426
	$3rd$	0.942	0.027	0.031	2.314
	$Bayes$	0.934	0.032	0.033	2.234
	$exact$	0.948	0.026	0.027	3.552
−0.2	$1st$	0.926	0.039	0.034	2.241
	$3rd$	0.949	0.026	0.025	2.141
	$Bayes$	0.940	0.030	0.030	2.055
	$exact$	0.952	0.025	0.023	3.189
−0.1	$1st$	0.927	0.037	0.036	2.087
	$3rd$	0.951	0.023	0.026	1.995
	$Bayes$	0.937	0.031	0.032	1.911
	$exact$	0.953	0.023	0.024	2.854
0	$1st$	0.925	0.037	0.038	1.927
	$3rd$	0.949	0.024	0.027	1.842

(b)

ρ	$Method$	CP	UE	LE	AL
	$Bayes$	0.937	0.031	0.032	1.758
	$exact$	0.952	0.023	0.025	2.579
0.1	$1st$	0.927	0.038	0.035	1.806
	$3rd$	0.951	0.025	0.024	1.732
	$Bayes$	0.941	0.031	0.029	1.647
	$exact$	0.953	0.024	0.022	2.346
0.2	$1st$	0.925	0.033	0.042	1.695
	$3rd$	0.948	0.025	0.026	1.636
	$Bayes$	0.936	0.030	0.034	1.550
	$exact$	0.949	0.024	0.028	2.110
0.3	$1st$	0.925	0.036	0.040	1.577
	$3rd$	0.948	0.027	0.024	1.526
	$Bayes$	0.938	0.031	0.030	1.439
	$exact$	0.950	0.026	0.024	1.903
0.4	$1st$	0.929	0.032	0.039	1.460
	$3rd$	0.948	0.028	0.024	1.421
	$Bayes$	0.942	0.032	0.026	1.347
	$exact$	0.952	0.025	0.024	1.701
0.5	$1st$	0.927	0.032	0.040	1.350
	$3rd$	0.943	0.031	0.026	1.324
	$Bayes$	0.940	0.033	0.027	1.257
	$exact$	0.947	0.026	0.026	1.499
0.6	$1st$	0.928	0.031	0.041	1.236
	$3rd$	0.940	0.035	0.025	1.209
	$Bayes$	0.937	0.037	0.025	1.157
	$exact$	0.948	0.027	0.025	1.313
0.7	$1st$	0.935	0.023	0.042	1.061
	$3rd$	0.945	0.029	0.026	1.054
	$Bayes$	0.935	0.038	0.026	1.039
	$exact$	0.950	0.026	0.024	1.097
0.8	$1st$	0.936	0.025	0.038	0.865
	$3rd$	0.948	0.027	0.024	0.864
	$Bayes$	0.946	0.029	0.024	0.905
	$exact$	0.951	0.026	0.023	0.874
0.9	$1st$	0.925	0.035	0.040	0.554
	$3rd$	0.954	0.022	0.024	0.563
	$Bayes$	0.953	0.023	0.024	0.652
	$exact$	0.952	0.026	0.023	0.598

Table 4: Summary statistics for Simulation 1: bivariate normal with means 0 and variances 1. Empirical coverage (CP), upper (UE) and lower (LE) error probability and average length (AL) of nominal two-sided 95% confidence intervals for γ, for varying values of ρ and sample size $n = 10$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r^*_B ($Bayes$); expression (15) by Haddad and Provost (2011) ($exact$). Based on 10,000 replicates; simulation error: ± 0.004.

(a)

ρ	$Method$	CP	UE	LE	AL
−0.9	$1st$	0.942	0.037	0.021	4.907
	$3rd$	0.951	0.024	0.025	4.826
	$Bayes$	0.951	0.024	0.025	4.834
	$exact$	0.952	0.022	0.025	6.496
−0.8	$1st$	0.945	0.034	0.020	3.174
	$3rd$	0.950	0.025	0.025	3.144
	$Bayes$	0.949	0.025	0.026	3.154
	$exact$	0.949	0.024	0.027	4.484
−0.7	$1st$	0.946	0.035	0.019	2.588
	$3rd$	0.953	0.025	0.023	2.556
	$Bayes$	0.949	0.025	0.026	2.551
	$exact$	0.953	0.025	0.022	3.545
−0.6	$1st$	0.943	0.033	0.024	2.242
	$3rd$	0.948	0.023	0.029	2.205
	$Bayes$	0.944	0.023	0.034	2.185
	$exact$	0.952	0.024	0.024	2.968
−0.5	$1st$	0.944	0.034	0.022	1.999
	$3rd$	0.951	0.023	0.026	1.961
	$Bayes$	0.947	0.025	0.028	1.931
	$exact$	0.954	0.025	0.021	2.583
−0.4	$1st$	0.937	0.036	0.027	1.801
	$3rd$	0.946	0.026	0.028	1.766
	$Bayes$	0.943	0.028	0.030	1.730
	$exact$	0.949	0.025	0.026	2.275
−0.3	$1st$	0.937	0.034	0.029	1.649
	$3rd$	0.947	0.025	0.028	1.615
	$Bayes$	0.944	0.027	0.029	1.577
	$exact$	0.952	0.023	0.024	2.034
−0.2	$1st$	0.933	0.036	0.031	1.503
	$3rd$	0.945	0.027	0.028	1.472
	$Bayes$	0.940	0.029	0.031	1.433
	$exact$	0.947	0.027	0.026	1.825
−0.1	$1st$	0.936	0.034	0.030	1.395
	$3rd$	0.948	0.027	0.025	1.368
	$Bayes$	0.947	0.028	0.025	1.328
	$exact$	0.951	0.026	0.024	1.648
0	$1st$	0.937	0.032	0.031	1.290
	$3rd$	0.949	0.026	0.025	1.268

(b)

ρ	$Method$	CP	UE	LE	AL
0	$Bayes$	0.945	0.029	0.026	1.227
	$exact$	0.951	0.025	0.024	1.485
0.1	$1st$	0.936	0.031	0.034	1.192
	$3rd$	0.949	0.026	0.025	1.177
	$Bayes$	0.945	0.027	0.028	1.138
	$exact$	0.951	0.024	0.025	1.345
0.2	$1st$	0.935	0.033	0.032	1.106
	$3rd$	0.948	0.029	0.023	1.098
	$Bayes$	0.946	0.030	0.024	1.060
	$exact$	0.952	0.027	0.021	1.223
0.3	$1st$	0.937	0.028	0.036	1.011
	$3rd$	0.947	0.026	0.027	1.008
	$Bayes$	0.945	0.026	0.029	0.974
	$exact$	0.949	0.024	0.027	1.091
0.4	$1st$	0.938	0.027	0.035	0.920
	$3rd$	0.948	0.028	0.025	0.922
	$Bayes$	0.948	0.027	0.025	0.896
	$exact$	0.951	0.024	0.025	0.974
0.5	$1st$	0.941	0.023	0.035	0.817
	$3rd$	0.947	0.026	0.027	0.827
	$Bayes$	0.945	0.027	0.027	0.807
	$exact$	0.950	0.024	0.026	0.860
0.6	$1st$	0.942	0.022	0.036	0.704
	$3rd$	0.948	0.027	0.025	0.719
	$Bayes$	0.943	0.031	0.026	0.716
	$exact$	0.954	0.023	0.023	0.743
0.7	$1st$	0.945	0.020	0.036	0.574
	$3rd$	0.950	0.023	0.026	0.594
	$Bayes$	0.946	0.028	0.026	0.606
	$exact$	0.949	0.025	0.026	0.627
0.8	$1st$	0.943	0.020	0.037	0.415
	$3rd$	0.954	0.022	0.024	0.438
	$Bayes$	0.952	0.023	0.025	0.459
	$exact$	0.952	0.023	0.026	0.496
0.9	$1st$	0.943	0.022	0.034	0.259
	$3rd$	0.949	0.029	0.023	0.275
	$Bayes$	0.949	0.028	0.023	0.283
	$exact$	0.949	0.026	0.025	0.343

Table 5: Summary statistics for Simulation 1: bivariate normal with means 0 and variances 1. Empirical coverage (CP), upper (UE) and lower (LE) error probability and average length (AL) of nominal two-sided 95% confidence intervals for γ, for varying values of ρ and sample size $n = 15$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r_B^* ($Bayes$); expression (15) by Haddad and Provost (2011) ($exact$). Based on 10,000 replicates; simulation error: ± 0.004.

(a)

ρ	$Method$	CP	UE	LE	AL
-0.9	$1st$	0.948	0.034	0.018	3.706
	$3rd$	0.950	0.026	0.025	3.676
	$Bayes$	0.950	0.026	0.025	3.676
	$exact$	0.952	0.025	0.023	4.980
-0.8	$1st$	0.948	0.034	0.019	2.499
	$3rd$	0.951	0.026	0.023	2.489
	$Bayes$	0.951	0.026	0.024	2.492
	$exact$	0.950	0.027	0.023	3.441
-0.7	$1st$	0.948	0.034	0.018	2.065
	$3rd$	0.953	0.024	0.022	2.053
	$Bayes$	0.951	0.024	0.025	2.053
	$exact$	0.953	0.023	0.023	2.716
-0.6	$1st$	0.947	0.032	0.021	1.805
	$3rd$	0.951	0.024	0.024	1.789
	$Bayes$	0.947	0.024	0.029	1.780
	$exact$	0.950	0.024	0.025	2.286
-0.5	$1st$	0.949	0.030	0.020	1.613
	$3rd$	0.954	0.023	0.023	1.594
	$Bayes$	0.950	0.024	0.026	1.578
	$exact$	0.953	0.025	0.022	1.978
-0.4	$1st$	0.947	0.030	0.023	1.457
	$3rd$	0.954	0.022	0.024	1.438
	$Bayes$	0.951	0.023	0.026	1.418
	$exact$	0.951	0.025	0.024	1.741
-0.3	$1st$	0.939	0.033	0.028	1.331
	$3rd$	0.946	0.026	0.028	1.313
	$Bayes$	0.945	0.026	0.029	1.289
	$exact$	0.950	0.024	0.025	1.553
-0.2	$1st$	0.938	0.032	0.030	1.224
	$3rd$	0.946	0.026	0.028	1.207
	$Bayes$	0.945	0.027	0.028	1.182
	$exact$	0.947	0.026	0.027	1.398
-0.1	$1st$	0.939	0.030	0.030	1.126
	$3rd$	0.949	0.024	0.027	1.112
	$Bayes$	0.948	0.024	0.027	1.085
	$exact$	0.952	0.023	0.025	1.260
0	$1st$	0.939	0.031	0.030	1.037
	$3rd$	0.948	0.027	0.026	1.026

(b)

ρ	$Method$	CP	UE	LE	AL
0	$Bayes$	0.945	0.028	0.027	1.000
	$exact$	0.950	0.025	0.025	1.139
0.1	$1st$	0.940	0.030	0.031	0.951
	$3rd$	0.949	0.025	0.025	0.944
	$Bayes$	0.947	0.028	0.025	0.919
	$exact$	0.952	0.025	0.023	1.027
0.2	$1st$	0.942	0.027	0.031	0.879
	$3rd$	0.952	0.025	0.023	0.876
	$Bayes$	0.950	0.026	0.024	0.853
	$exact$	0.953	0.024	0.023	0.934
0.3	$1st$	0.940	0.025	0.035	0.798
	$3rd$	0.947	0.025	0.027	0.798
	$Bayes$	0.946	0.027	0.028	0.779
	$exact$	0.948	0.025	0.027	0.836
0.4	$1st$	0.940	0.027	0.034	0.716
	$3rd$	0.945	0.028	0.027	0.722
	$Bayes$	0.944	0.029	0.027	0.708
	$exact$	0.947	0.027	0.026	0.749
0.5	$1st$	0.944	0.025	0.032	0.620
	$3rd$	0.949	0.027	0.024	0.630
	$Bayes$	0.946	0.030	0.024	0.625
	$exact$	0.949	0.025	0.026	0.657
0.6	$1st$	0.949	0.020	0.031	0.524
	$3rd$	0.955	0.023	0.022	0.537
	$Bayes$	0.951	0.027	0.022	0.539
	$exact$	0.951	0.025	0.025	0.569
0.7	$1st$	0.946	0.019	0.035	0.419
	$3rd$	0.949	0.023	0.027	0.434
	$Bayes$	0.947	0.025	0.027	0.441
	$exact$	0.947	0.029	0.024	0.481
0.8	$1st$	0.952	0.019	0.029	0.303
	$3rd$	0.955	0.023	0.021	0.317
	$Bayes$	0.954	0.024	0.021	0.322
	$exact$	0.953	0.025	0.022	0.380
0.9	$1st$	0.947	0.021	0.033	0.193
	$3rd$	0.949	0.026	0.025	0.203
	$Bayes$	0.949	0.026	0.025	0.203
	$exact$	0.948	0.027	0.025	0.263

Table 6: Summary statistics for Simulation 1: bivariate normal with means 0 and variances 1. Empirical coverage (CP), upper (UE) and lower (LE) error probability and average length (AL) of nominal two-sided 95% confidence intervals for γ, for varying values of ρ and sample size $n = 20$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r^*_B ($Bayes$); expression (15) by Haddad and Provost (2011) ($exact$). Based on 10,000 replicates; simulation error: ± 0.004.

(a)

ρ	$Method$	CP	UE	LE	AL
−0.9	$1st$	0.947	0.033	0.020	3.088
	$3rd$	0.951	0.025	0.025	3.073
	$Bayes$	0.950	0.025	0.025	3.073
	$exact$	0.948	0.026	0.026	4.194
−0.8	$1st$	0.949	0.032	0.019	2.126
	$3rd$	0.950	0.027	0.023	2.121
	$Bayes$	0.950	0.027	0.023	2.121
	$exact$	0.950	0.027	0.023	2.876
−0.7	$1st$	0.951	0.030	0.019	1.761
	$3rd$	0.953	0.024	0.023	1.754
	$Bayes$	0.951	0.024	0.024	1.755
	$exact$	0.952	0.024	0.024	2.276
−0.6	$1st$	0.947	0.032	0.022	1.550
	$3rd$	0.951	0.024	0.024	1.540
	$Bayes$	0.947	0.024	0.029	1.537
	$exact$	0.948	0.026	0.027	1.916
−0.5	$1st$	0.943	0.032	0.025	1.387
	$3rd$	0.947	0.025	0.028	1.375
	$Bayes$	0.944	0.025	0.032	1.366
	$exact$	0.948	0.024	0.027	1.656
−0.4	$1st$	0.947	0.029	0.024	1.259
	$3rd$	0.951	0.022	0.027	1.246
	$Bayes$	0.947	0.023	0.030	1.232
	$exact$	0.951	0.023	0.026	1.459
−0.3	$1st$	0.944	0.031	0.025	1.149
	$3rd$	0.950	0.025	0.026	1.137
	$Bayes$	0.949	0.025	0.026	1.119
	$exact$	0.950	0.024	0.026	1.302
−0.2	$1st$	0.942	0.033	0.026	1.061
	$3rd$	0.948	0.029	0.024	1.050
	$Bayes$	0.947	0.028	0.025	1.031
	$exact$	0.951	0.027	0.022	1.173
−0.1	$1st$	0.941	0.033	0.026	0.972
	$3rd$	0.948	0.028	0.024	0.963
	$Bayes$	0.948	0.028	0.024	0.944
	$exact$	0.950	0.026	0.023	1.060
0	$1st$	0.944	0.029	0.027	0.896
	$3rd$	0.950	0.026	0.024	0.889

(b)

ρ	$Method$	CP	UE	LE	AL
0	$Bayes$	0.948	0.027	0.025	0.870
	$exact$	0.952	0.025	0.023	0.958
0.1	$1st$	0.940	0.029	0.030	0.818
	$3rd$	0.948	0.026	0.026	0.814
	$Bayes$	0.946	0.027	0.027	0.797
	$exact$	0.949	0.026	0.025	0.866
0.2	$1st$	0.944	0.028	0.028	0.745
	$3rd$	0.951	0.027	0.022	0.744
	$Bayes$	0.949	0.028	0.023	0.729
	$exact$	0.950	0.026	0.023	0.781
0.3	$1st$	0.944	0.025	0.031	0.673
	$3rd$	0.949	0.025	0.026	0.675
	$Bayes$	0.948	0.026	0.026	0.663
	$exact$	0.951	0.024	0.025	0.702
0.4	$1st$	0.941	0.027	0.032	0.598
	$3rd$	0.946	0.028	0.025	0.603
	$Bayes$	0.945	0.030	0.026	0.596
	$exact$	0.948	0.026	0.026	0.628
0.5	$1st$	0.946	0.023	0.031	0.520
	$3rd$	0.951	0.026	0.023	0.528
	$Bayes$	0.948	0.028	0.023	0.526
	$exact$	0.951	0.026	0.024	0.554
0.6	$1st$	0.946	0.023	0.031	0.433
	$3rd$	0.949	0.026	0.026	0.443
	$Bayes$	0.946	0.029	0.025	0.445
	$exact$	0.948	0.027	0.025	0.479
0.7	$1st$	0.946	0.020	0.034	0.342
	$3rd$	0.949	0.024	0.028	0.352
	$Bayes$	0.947	0.025	0.028	0.356
	$exact$	0.948	0.025	0.027	0.403
0.8	$1st$	0.947	0.019	0.035	0.250
	$3rd$	0.950	0.022	0.028	0.259
	$Bayes$	0.950	0.023	0.027	0.261
	$exact$	0.950	0.023	0.028	0.318
0.9	$1st$	0.949	0.019	0.032	0.162
	$3rd$	0.950	0.024	0.025	0.167
	$Bayes$	0.950	0.025	0.025	0.167
	$exact$	0.952	0.023	0.024	0.219

Table 7: Summary statistics for Simulation 2: bivariate normal with means 7 and variances 0.9. Empirical coverage (CP), upper (UE) and lower (LE) error probability and average length (AL) of nominal two-sided 95% confidence intervals for γ, for varying values of ρ and sample size $n = 5$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r^*_B ($Bayes$); expression (17) by Haddad and Provost (2011) (HP); expression (19) by Mameli *et al.* (2012) (ACI). Based on 10,000 replicates; simulation error: ± 0.004.

(a)

ρ	$Method$	CP	UE	LE	AL
−0.9	$1st$	0.842	0.118	0.041	11.313
	$3rd$	0.938	0.035	0.026	12.490
	$Bayes$	0.970	0.016	0.014	14.316
	HP	0.904	0.072	0.024	18.031
	ACI	0.956	0.034	0.010	24.343
−0.8	$1st$	0.849	0.107	0.044	8.565
	$3rd$	0.940	0.035	0.025	9.071
	$Bayes$	0.968	0.017	0.015	10.617
	HP	0.909	0.066	0.025	12.216
	ACI	0.953	0.034	0.013	16.492
−0.7	$1st$	0.847	0.102	0.051	7.001
	$3rd$	0.938	0.034	0.028	7.743
	$Bayes$	0.968	0.018	0.014	9.189
	HP	0.906	0.064	0.030	9.559
	ACI	0.955	0.031	0.014	12.906
−0.6	$1st$	0.843	0.101	0.056	5.865
	$3rd$	0.939	0.034	0.027	6.644
	$Bayes$	0.969	0.017	0.014	7.967
	HP	0.906	0.062	0.031	7.785
	ACI	0.954	0.031	0.015	10.510
−0.5	$1st$	0.839	0.100	0.061	5.078
	$3rd$	0.940	0.033	0.028	5.883
	$Bayes$	0.969	0.017	0.014	7.096
	HP	0.906	0.059	0.035	6.647
	ACI	0.954	0.029	0.016	8.974
−0.4	$1st$	0.840	0.096	0.064	4.439
	$3rd$	0.937	0.034	0.029	5.206
	$Bayes$	0.967	0.017	0.015	6.317
	HP	0.905	0.058	0.038	5.807
	ACI	0.950	0.031	0.020	7.840
−0.3	$1st$	0.845	0.089	0.067	3.892
	$3rd$	0.939	0.032	0.030	4.637
	$Bayes$	0.969	0.016	0.015	5.650
	HP	0.906	0.053	0.040	5.044
	ACI	0.954	0.027	0.020	6.810
−0.2	$1st$	0.835	0.088	0.077	3.426
	$3rd$	0.938	0.030	0.032	4.143
	$Bayes$	0.969	0.013	0.018	5.066
	HP	0.903	0.050	0.047	4.417
	ACI	0.951	0.024	0.025	5.963
−0.1	$1st$	0.842	0.082	0.076	3.055
	$3rd$	0.937	0.032	0.031	3.726
	$Bayes$	0.969	0.014	0.017	4.569
	HP	0.905	0.048	0.047	3.956
	ACI	0.951	0.025	0.024	5.341
0	$1st$	0.833	0.083	0.084	2.762
	$3rd$	0.937	0.031	0.032	3.405
	$Bayes$	0.969	0.015	0.015	4.186

(b)

ρ	$Method$	CP	UE	LE	AL
0	HP	0.901	0.049	0.050	3.561
	ACI	0.953	0.023	0.024	4.808
0.1	$1st$	0.839	0.081	0.080	2.473
	$3rd$	0.938	0.034	0.028	3.073
	$Bayes$	0.969	0.016	0.015	3.784
	HP	0.905	0.048	0.047	3.197
	ACI	0.953	0.024	0.023	4.316
0.2	$1st$	0.843	0.070	0.087	2.149
	$3rd$	0.941	0.028	0.031	2.721
	$Bayes$	0.970	0.014	0.016	3.360
	HP	0.906	0.041	0.053	2.766
	ACI	0.954	0.020	0.026	3.735
0.3	$1st$	0.843	0.068	0.089	1.898
	$3rd$	0.940	0.029	0.030	2.433
	$Bayes$	0.969	0.016	0.016	3.010
	HP	0.906	0.041	0.053	2.444
	ACI	0.953	0.021	0.026	3.300
0.4	$1st$	0.845	0.062	0.094	1.657
	$3rd$	0.944	0.027	0.029	2.156
	$Bayes$	0.972	0.014	0.014	2.674
	HP	0.908	0.037	0.055	2.133
	ACI	0.956	0.019	0.026	2.880
0.5	$1st$	0.838	0.058	0.104	1.412
	$3rd$	0.936	0.026	0.038	1.879
	$Bayes$	0.970	0.012	0.018	2.337
	HP	0.905	0.033	0.062	1.813
	ACI	0.953	0.015	0.033	2.448
0.6	$1st$	0.841	0.056	0.104	1.217
	$3rd$	0.942	0.026	0.033	1.652
	$Bayes$	0.971	0.013	0.016	2.061
	HP	0.909	0.031	0.060	1.559
	ACI	0.956	0.014	0.030	2.105
0.7	$1st$	0.850	0.052	0.098	1.012
	$3rd$	0.939	0.029	0.032	1.415
	$Bayes$	0.968	0.016	0.016	1.772
	HP	0.909	0.031	0.060	1.290
	ACI	0.955	0.015	0.030	1.741
0.8	$1st$	0.849	0.045	0.105	0.794
	$3rd$	0.939	0.027	0.034	1.160
	$Bayes$	0.968	0.014	0.018	1.459
	HP	0.911	0.026	0.063	0.995
	ACI	0.955	0.012	0.033	1.344
0.9	$1st$	0.852	0.041	0.106	0.569
	$3rd$	0.939	0.029	0.032	0.894
	$Bayes$	0.968	0.017	0.016	1.125
	HP	0.913	0.024	0.064	0.675
	ACI	0.956	0.012	0.032	0.912

Table 8: Summary statistics for Simulation 2: bivariate normal with means 7 and variances 0.9. Empirical coverage (CP), upper (UE) and lower (LE) error probability and average length (AL) of nominal two-sided 95% confidence intervals for γ, for varying values of ρ and sample size $n = 10$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r^*_B ($Bayes$); expression (17) by Haddad and Provost (2011) (HP); expression (19) by Mameli *et al.* (2012) (ACI). Based on 10,000 replicates; simulation error: ± 0.004.

(a)

ρ	$Method$	CP	UE	LE	AL
-0.9	$1st$	0.914	0.055	0.031	7.005
	$3rd$	0.948	0.026	0.026	7.427
	$Bayes$	0.960	0.020	0.020	7.903
	HP	0.938	0.041	0.021	7.477
	ACI	0.952	0.032	0.016	8.037
-0.8	$1st$	0.911	0.059	0.031	4.830
	$3rd$	0.949	0.027	0.025	5.128
	$Bayes$	0.962	0.020	0.018	5.487
	HP	0.934	0.044	0.022	5.137
	ACI	0.951	0.033	0.016	5.521
-0.7	$1st$	0.917	0.053	0.031	3.749
	$3rd$	0.948	0.029	0.023	4.015
	$Bayes$	0.959	0.022	0.019	4.304
	HP	0.939	0.040	0.021	4.049
	ACI	0.949	0.033	0.018	4.352
-0.6	$1st$	0.911	0.053	0.035	3.091
	$3rd$	0.948	0.026	0.026	3.318
	$Bayes$	0.961	0.020	0.019	3.557
	HP	0.936	0.038	0.026	3.371
	ACI	0.951	0.030	0.019	3.623
-0.5	$1st$	0.910	0.051	0.038	2.647
	$3rd$	0.949	0.026	0.026	2.846
	$Bayes$	0.963	0.018	0.019	3.050
	HP	0.934	0.039	0.027	2.903
	ACI	0.951	0.029	0.020	3.121
-0.4	$1st$	0.910	0.050	0.040	2.305
	$3rd$	0.948	0.026	0.026	2.487
	$Bayes$	0.961	0.019	0.020	2.664
	HP	0.935	0.036	0.029	2.534
	ACI	0.951	0.028	0.021	2.723
	$1st$	0.917	0.049	0.034	2.059
-0.3	$3rd$	0.952	0.026	0.022	2.227
	$Bayes$	0.962	0.020	0.018	2.386
	HP	0.939	0.037	0.024	2.266
	ACI	0.951	0.028	0.020	2.435
-0.2	$1st$	0.910	0.046	0.043	1.816
	$3rd$	0.947	0.025	0.028	1.972
	$Bayes$	0.960	0.019	0.021	2.113
	HP	0.933	0.034	0.033	2.000
	ACI	0.948	0.026	0.026	2.150
-0.1	$1st$	0.912	0.047	0.041	1.637
	$3rd$	0.946	0.027	0.026	1.785
	$Bayes$	0.958	0.021	0.021	1.913
	HP	0.934	0.035	0.031	1.804
	ACI	0.947	0.028	0.025	1.938
0	$1st$	0.914	0.044	0.042	1.466
	$3rd$	0.946	0.027	0.026	1.605
	$Bayes$	0.959	0.021	0.020	1.720

(b)

ρ	$Method$	CP	UE	LE	AL
0	HP	0.935	0.033	0.033	1.615
	ACI	0.947	0.027	0.026	1.736
0.1	$1st$	0.909	0.046	0.045	1.330
	$3rd$	0.949	0.026	0.026	1.463
	$Bayes$	0.963	0.018	0.018	1.569
	HP	0.934	0.033	0.033	1.465
	ACI	0.950	0.024	0.026	1.575
0.2	$1st$	0.912	0.040	0.048	1.178
	$3rd$	0.950	0.025	0.025	1.304
	$Bayes$	0.960	0.020	0.020	1.398
	HP	0.935	0.029	0.036	1.298
	ACI	0.950	0.023	0.027	1.395
0.3	$1st$	0.913	0.040	0.047	1.060
	$3rd$	0.951	0.024	0.025	1.180
	$Bayes$	0.962	0.019	0.019	1.266
	HP	0.936	0.029	0.036	1.168
	ACI	0.952	0.021	0.027	1.255
0.4	$1st$	0.914	0.042	0.044	0.941
	$3rd$	0.947	0.030	0.024	1.055
	$Bayes$	0.961	0.022	0.016	1.132
	HP	0.936	0.031	0.033	1.037
	ACI	0.949	0.025	0.026	1.114
0.5	$1st$	0.914	0.037	0.050	0.821
	$3rd$	0.951	0.025	0.024	0.928
	$Bayes$	0.961	0.020	0.019	0.997
	HP	0.937	0.026	0.037	0.905
	ACI	0.952	0.021	0.028	0.972
0.6	$1st$	0.915	0.032	0.053	0.703
	$3rd$	0.950	0.024	0.026	0.803
	$Bayes$	0.963	0.018	0.020	0.863
	HP	0.938	0.023	0.038	0.774
	ACI	0.952	0.018	0.031	0.832
0.7	$1st$	0.912	0.034	0.054	0.590
	$3rd$	0.948	0.027	0.025	0.682
	$Bayes$	0.962	0.019	0.018	0.734
	HP	0.933	0.025	0.042	0.649
	ACI	0.951	0.018	0.031	0.698
0.8	$1st$	0.918	0.029	0.054	0.465
	$3rd$	0.952	0.023	0.025	0.551
	$Bayes$	0.962	0.019	0.019	0.593
	HP	0.940	0.021	0.039	0.510
	ACI	0.954	0.016	0.030	0.548
0.9	$1st$	0.914	0.027	0.059	0.328
	$3rd$	0.946	0.024	0.030	0.420
	$Bayes$	0.960	0.018	0.021	0.453
	HP	0.935	0.019	0.046	0.347
	ACI	0.948	0.015	0.037	0.373

Table 9: Summary statistics for Simulation 2: bivariate normal with means 7 and variances 0.9. Empirical coverage (CP), upper (UE) and lower (LE) error probability and average length (AL) of nominal two-sided 95% confidence intervals for γ, for varying values of ρ and sample size $n = 15$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r^*_B ($Bayes$); expression (17) by Haddad and Provost (2011) (HP); expression (19) by Mameli *et al.* (2012) (ACI). Based on 10,000 replicates; simulation error: ± 0.004.

(a)

ρ	$Method$	CP	UE	LE	AL
-0.9	$1st$	0.926	0.046	0.028	5.407
	$3rd$	0.946	0.028	0.026	5.552
	$Bayes$	0.954	0.023	0.023	5.771
	HP	0.939	0.038	0.023	5.410
	ACI	0.948	0.033	0.019	5.629
-0.8	$1st$	0.924	0.047	0.029	3.562
	$3rd$	0.944	0.030	0.027	3.747
	$Bayes$	0.953	0.025	0.022	3.905
	HP	0.937	0.039	0.024	3.696
	ACI	0.946	0.034	0.019	3.846
-0.7	$1st$	0.928	0.043	0.029	2.757
	$3rd$	0.949	0.026	0.026	2.884
	$Bayes$	0.955	0.022	0.023	3.005
	HP	0.941	0.035	0.024	2.911
	ACI	0.950	0.030	0.021	3.029
-0.6	$1st$	0.925	0.045	0.030	2.314
	$3rd$	0.948	0.027	0.025	2.417
	$Bayes$	0.956	0.022	0.021	2.518
	HP	0.939	0.036	0.024	2.453
	ACI	0.949	0.031	0.020	2.553
-0.5	$1st$	0.928	0.040	0.032	1.988
	$3rd$	0.949	0.026	0.025	2.079
	$Bayes$	0.956	0.022	0.021	2.166
	HP	0.942	0.033	0.025	2.110
	ACI	0.950	0.029	0.021	2.195
-0.4	$1st$	0.927	0.042	0.032	1.753
	$3rd$	0.948	0.027	0.025	1.837
	$Bayes$	0.956	0.022	0.022	1.913
	HP	0.940	0.034	0.026	1.861
	ACI	0.948	0.029	0.022	1.936
-0.3	$1st$	0.926	0.041	0.033	1.560
	$3rd$	0.948	0.025	0.026	1.637
	$Bayes$	0.958	0.020	0.022	1.705
	HP	0.940	0.032	0.028	1.655
	ACI	0.948	0.027	0.024	1.722
-0.2	$1st$	0.925	0.040	0.035	1.393
	$3rd$	0.951	0.025	0.024	1.466
	$Bayes$	0.958	0.022	0.020	1.527
	HP	0.943	0.031	0.026	1.478
	ACI	0.952	0.027	0.022	1.538
-0.1	$1st$	0.928	0.038	0.034	1.250
	$3rd$	0.950	0.026	0.024	1.319
	$Bayes$	0.956	0.023	0.021	1.375
	HP	0.942	0.031	0.027	1.327
	ACI	0.949	0.027	0.024	1.381
0	$1st$	0.926	0.038	0.036	1.127
	$3rd$	0.949	0.025	0.026	1.192
	$Bayes$	0.958	0.021	0.022	1.242

(b)

ρ	$Method$	CP	UE	LE	AL
	HP	0.941	0.030	0.029	1.195
	ACI	0.949	0.025	0.026	1.244
0.1	$1st$	0.929	0.033	0.037	1.015
	$3rd$	0.950	0.025	0.026	1.078
	$Bayes$	0.958	0.019	0.022	1.123
	HP	0.942	0.027	0.030	1.077
	ACI	0.950	0.024	0.026	1.121
0.2	$1st$	0.924	0.035	0.041	0.910
	$3rd$	0.948	0.024	0.028	0.970
	$Bayes$	0.957	0.020	0.023	1.011
	HP	0.939	0.027	0.033	0.965
	ACI	0.949	0.022	0.029	1.004
0.3	$1st$	0.925	0.032	0.043	0.815
	$3rd$	0.948	0.024	0.028	0.873
	$Bayes$	0.956	0.019	0.024	0.910
	HP	0.938	0.025	0.037	0.865
	ACI	0.948	0.021	0.031	0.900
0.4	$1st$	0.928	0.032	0.040	0.722
	$3rd$	0.951	0.025	0.024	0.776
	$Bayes$	0.961	0.021	0.018	0.809
	HP	0.942	0.026	0.031	0.766
	ACI	0.952	0.022	0.026	0.797
0.5	$1st$	0.927	0.033	0.040	0.639
	$3rd$	0.951	0.025	0.024	0.690
	$Bayes$	0.960	0.020	0.020	0.720
	HP	0.944	0.025	0.031	0.677
	ACI	0.953	0.020	0.027	0.705
0.6	$1st$	0.929	0.028	0.043	0.548
	$3rd$	0.949	0.024	0.027	0.596
	$Bayes$	0.956	0.021	0.023	0.622
	HP	0.941	0.023	0.036	0.582
	ACI	0.950	0.020	0.031	0.605
0.7	$1st$	0.930	0.029	0.041	0.461
	$3rd$	0.950	0.026	0.024	0.505
	$Bayes$	0.958	0.023	0.020	0.527
	HP	0.943	0.024	0.033	0.490
	ACI	0.952	0.021	0.027	0.509
0.8	$1st$	0.932	0.028	0.040	0.364
	$3rd$	0.950	0.026	0.024	0.404
	$Bayes$	0.957	0.022	0.021	0.422
	HP	0.943	0.024	0.033	0.386
	ACI	0.952	0.019	0.029	0.402
0.9	$1st$	0.929	0.028	0.043	0.255
	$3rd$	0.949	0.027	0.024	0.305
	$Bayes$	0.956	0.024	0.021	0.319
	HP	0.942	0.023	0.035	0.265
	ACI	0.950	0.020	0.030	0.275

Table 10: Summary statistics for Simulation 2: bivariate normal with means 7 and variances 0.9. Empirical coverage (CP), upper (UE) and lower (LE) error probability and average length (AL) of nominal two-sided 95% confidence intervals for γ, for varying values of ρ and sample size $n = 20$. Pivots used: likelihood root r ($1st$), modified likelihood root r^* ($3rd$); Bayesian modified likelihood root r^*_B ($Bayes$); expression (17) by Haddad and Provost (2011) (HP); expression (19) by Mameli *et al.* (2012) (ACI). Based on 10,000 replicates; simulation error: ± 0.004.

(a)

ρ	$Method$	CP	UE	LE	AL
-0.9	$1st$	0.935	0.040	0.025	4.488
	$3rd$	0.949	0.025	0.025	4.718
	$Bayes$	0.956	0.023	0.021	4.852
	HP	0.947	0.033	0.020	4.414
	ACI	0.952	0.030	0.018	4.536
-0.8	$1st$	0.930	0.043	0.027	2.945
	$3rd$	0.946	0.028	0.026	3.060
	$Bayes$	0.952	0.025	0.023	3.152
	HP	0.940	0.037	0.022	3.043
	ACI	0.947	0.034	0.019	3.128
-0.7	$1st$	0.932	0.040	0.028	2.303
	$3rd$	0.950	0.024	0.026	2.376
	$Bayes$	0.954	0.022	0.024	2.446
	HP	0.942	0.034	0.024	2.404
	ACI	0.948	0.030	0.022	2.471
-0.6	$1st$	0.933	0.038	0.029	1.926
	$3rd$	0.950	0.024	0.025	1.986
	$Bayes$	0.956	0.021	0.023	2.045
	HP	0.944	0.032	0.024	2.010
	ACI	0.949	0.029	0.022	2.066
-0.5	$1st$	0.935	0.036	0.028	1.662
	$3rd$	0.951	0.025	0.024	1.716
	$Bayes$	0.957	0.021	0.021	1.766
	HP	0.946	0.031	0.023	1.734
	ACI	0.951	0.028	0.021	1.782
-0.4	$1st$	0.932	0.038	0.030	1.469
	$3rd$	0.946	0.028	0.026	1.519
	$Bayes$	0.953	0.025	0.022	1.564
	HP	0.941	0.033	0.026	1.533
	ACI	0.947	0.030	0.023	1.576
-0.3	$1st$	0.929	0.039	0.032	1.305
	$3rd$	0.946	0.028	0.026	1.351
	$Bayes$	0.952	0.025	0.023	1.391
	HP	0.941	0.033	0.027	1.361
	ACI	0.946	0.030	0.024	1.399
-0.2	$1st$	0.936	0.032	0.032	1.165
	$3rd$	0.952	0.022	0.026	1.209
	$Bayes$	0.957	0.020	0.023	1.245
	HP	0.946	0.027	0.027	1.216
	ACI	0.952	0.024	0.025	1.250
-0.1	$1st$	0.935	0.033	0.032	1.046
	$3rd$	0.951	0.025	0.024	1.088
	$Bayes$	0.957	0.022	0.022	1.120
	HP	0.946	0.028	0.026	1.091
	ACI	0.951	0.026	0.024	1.122
0	$1st$	0.932	0.035	0.033	0.952
	$3rd$	0.947	0.027	0.026	0.992
	$Bayes$	0.954	0.024	0.023	1.021

(b)

ρ	$Method$	CP	UE	LE	AL
0	HP	0.940	0.031	0.029	0.994
	ACI	0.947	0.027	0.026	1.021
0.1	$1st$	0.934	0.032	0.034	0.856
	$3rd$	0.950	0.025	0.025	0.894
	$Bayes$	0.955	0.023	0.022	0.920
	HP	0.944	0.028	0.028	0.893
	ACI	0.950	0.025	0.025	0.918
0.2	$1st$	0.936	0.030	0.034	0.769
	$3rd$	0.953	0.023	0.024	0.805
	$Bayes$	0.959	0.019	0.022	0.829
	HP	0.947	0.025	0.029	0.802
	ACI	0.953	0.022	0.025	0.824
0.3	$1st$	0.930	0.032	0.038	0.687
	$3rd$	0.947	0.024	0.029	0.722
	$Bayes$	0.954	0.021	0.025	0.744
	HP	0.940	0.026	0.035	0.717
	ACI	0.946	0.022	0.031	0.737
0.4	$1st$	0.936	0.029	0.035	0.612
	$3rd$	0.951	0.025	0.024	0.645
	$Bayes$	0.957	0.021	0.022	0.664
	HP	0.945	0.025	0.030	0.638
	ACI	0.951	0.022	0.027	0.656
0.5	$1st$	0.932	0.030	0.037	0.539
	$3rd$	0.946	0.027	0.027	0.571
	$Bayes$	0.952	0.025	0.023	0.588
	HP	0.941	0.026	0.032	0.563
	ACI	0.946	0.024	0.030	0.578
0.6	$1st$	0.932	0.032	0.036	0.467
	$3rd$	0.948	0.028	0.023	0.496
	$Bayes$	0.954	0.025	0.021	0.511
	HP	0.942	0.027	0.031	0.487
	ACI	0.950	0.023	0.027	0.501
0.7	$1st$	0.934	0.027	0.039	0.390
	$3rd$	0.950	0.025	0.026	0.417
	$Bayes$	0.955	0.022	0.023	0.430
	HP	0.943	0.023	0.034	0.408
	ACI	0.949	0.020	0.031	0.419
0.8	$1st$	0.933	0.027	0.040	0.309
	$3rd$	0.948	0.026	0.027	0.332
	$Bayes$	0.955	0.022	0.024	0.343
	HP	0.944	0.021	0.035	0.322
	ACI	0.950	0.019	0.031	0.331
0.9	$1st$	0.938	0.024	0.038	0.215
	$3rd$	0.953	0.024	0.023	0.248
	$Bayes$	0.958	0.021	0.021	0.256
	HP	0.948	0.020	0.032	0.221
	ACI	0.954	0.018	0.028	0.228

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