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Keywords: bootstrap, periodogram, long-memory

Contents

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1 Introduction

Bootstrap methods are resampling techniques, introduced firstly by Efron (1979), designed to assigne measures of accuracy to sample estimates. Theese techniques allow the estimation of the sample distribution of almost any statistics using only very simple methods. Independence and identical distribution of the data is the main assumptions, so the use of bootstrap methods in time series must be very judicious since otherwise the time series structure may be lost. Among the numerous bootstrap methods for time series existing in literature, we can mention the model-based resampling, block resampling, phase scrambling (see Davison and Hinkley, 1997, for a review on this argument), sieve (Kreiss, 1992) and local bootstrap (Paparoditis and Politis, 1999). Although all this methods work well with short-memory time series (for example, with time series generated by ARMA processes), do not seem adapt to time series generated by long-memory processes (like, for example, ARFIMA processes).

In this work, we propose a new method based on the empirical autocovariance function and the Durbin-Levinson algorithm that seems to give satisfactory performances especially with Gaussian long-memory processes. Then we apply this new method to two estimators of the long memory parameter d: the GPH estimator proposed by Geweke and Porter-Hudak (1983) and the local Whittle estimator proposed by Robinson (1995a).

The results of an extensive Monte Carlo experiment show that with our approach we are able to improve the performances of the considered estimators in terms of standard deviation and MSE. Moreover, our approach is computationally fast and simple to implement.

The plan of the paper is the following. In Section 2 we briefly recall long memory processes and some common methods to estimate the long-memory parameter d. Section 3 is dedicated to reviewing different existing bootstrap methods for time series. Sections 4 and 5 present the new bootstrap method and its application to long-memory estimators. Results and conclusions are reported in the last section.

2 Long memory processes

There exist different definitions of long memory processes. In particular, long memory can be expressed either in the time domain or in the frequency domain. In the time domain, a stationary discrete time series is said to be long memory if its autocorrelation function decays to zero like a power function. This definition implies that the dependence between successive observations decays slowly as the number of lags tends to infinity. On the other hand, in the frequency domain, a stationary discrete time series is said to be long memory if its spectral density behaves like $|\omega|^{-d}$ near the 0 frequency, that is the spectral density is unbounded at low frequencies.

In this paper we consider one of the most popular long memory processes that takes into account this particular behaviour of the autocorrelation and of the spectral density function, i.e. the Autoregressive Fractionally Integrated Moving Average process, ARFIMA(p, d, q) in the following, independently introduced by Granger and Joyeux (1980) and Hosking (1981). This process simply generalizes the usual ARIMA(p, d, q) process by assuming d to be fractional.

Let ε_t be a white noise process having $E[\varepsilon_t^2] = \sigma^2$. The process $\{X_t,\ t \in \mathbf{Z}\}$ is said to be an ARFIMA(p, d, q) process with $d \in (-1/2, 1/2)$, if it is stationary and satisfies the difference equation

$$\Phi(B)\,\Delta(B)\,(X_t - \mu) = \Theta(B)\,\varepsilon_t, \tag{1}$$

where $\Phi(\cdot)$ and $\Theta(\cdot)$ are polynomials in the backward shift operator B of degree p and q, respectively, $\Delta(B) = (1-B)^d = \sum_{j=0}^{\infty} \pi_j B^j$ with $\pi_j = \Gamma(j-d)/[\Gamma(j+1)\Gamma(-d)]$, and $\Gamma(\cdot)$ is the gamma function.

If $p = q = 0$ the process $\{X_t,\ t \in \mathbf{Z}\}$ is called Fractionally Integrated Noise and denoted by $I(d)$. When $d \in (0, 1/2)$ the ARFIMA(p, d, q) process is stationary and the autocorrelation function decays to zero hyperbolically at a rate $O(k^{2d-1})$, where k denotes the lag. In this case we say that the process has a long-memory behaviour. When $d \in (-1/2, 0)$ the ARFIMA(p, d, q) process is a stationary process with intermediate memory. In the following we will concentrate on $I(d)$ processes with $d \in (0, 1/2)$: for this range of values the process is stationary, invertible and possesses long-range dependence. Moreover, we will assume for convenience and without loss of generality that $\sigma^2 = 1$ and $\mu = 0$.

The estimation of the long-memory parameter d has been of interest to many authors. Many estimators are well described in Beran (1994). In this paper three of the most common will be considered. We will try to improve the semiparametric estimators local Whittle and GPH, while the parametric Whittle is used as benchmark.

2.1 The Whittle estimator

Several theoretical and practical advantages are possessed by the frequency domain approximate maximum likelihood method proposed by Fox and Taqqu (1986), also called Whittle estimator. This estimator extends the results of Hannan (1973), who applied Whittle's method to the estimation of the parameters of ARMA models. Fox and Taqqu's result, later generalized by Dahlhaus (1989) to the exact maximum likelihood estimator, is the basis of one of the most used methods for estimating the long (and short, if both are present) memory parameters in Gaussian time series. Giraitis and Surgailis (1990) generalized the result of Fox and Taqqu in order to prove the asymptotic normality of Whittle's estimator without the Gaussianity assumption.

The exact maximum likelihood estimator has the drawback of implying a large computational burden and it might also cause computational problems when calculating the autocovariances needed to evaluate the likelihood function (Sowell, 1992). These difficulties do not occur when using the Whittle estimator, which has the further advantage of not requiring the estimation of the mean of the series (generally unknown in practice). Besides, under some regularity assumptions (Fox and Taqqu, 1986; Dahlhaus, 1989) fulfilled by ARFIMA(p, d, q) processes, it is possible to prove that the Whittle estimator has the same asymptotic distribution as the exact maximum likelihood estimator and it converges to the true values of the parameters at the usual rate of $n^{-1/2}$, where n is the length of the series. Eventually, for Gaussian processes the Whittle estimator is asymptotically efficient in the sense of Fisher.

If the Whittle approximation to the log-likelihood function is used, the parameter vector $\boldsymbol{\theta} = (d, \phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)$ is estimated by minimizing with respect to $\boldsymbol{\theta}$ the estimated variance of the underlying white noise process:

$$\hat{\sigma}^2(\boldsymbol{\theta}) = \frac{1}{2\pi} \sum_{j=1}^{n'} \frac{I_n(\omega_j)}{f(\omega_j, \boldsymbol{\theta})},$$

where n' is the integer part of $(n-1)/2$, $I_n(\omega_j)$ denotes the periodogram of the series, defined at the Fourier frequencies $\omega_j = 2\pi j/n$ $(j = 1, \ldots, n')$, and $f(\omega_j, \boldsymbol{\theta})$ indicates the spectral density of the ARFIMA process at the Fourier frequency ω_j.

The drawback of this estimator is that it is necessary to assume the parametric form of the spectral density to be known *a priori*. If the specified spectral density function is not the correct one (as it is often the case) the estimated parameters may be dramatically biased.

2.2 The GPH estimator

This is one of the best known methods to estimate in a semi-parametric way the fractional parameters d of long-range dependence behaviour. The advantage of this method is that the specification of the model is not really necessary because the only information we need is the behaviour of the spectral density near the origin. Furthermore, the long-memory parameter can be estimated alone.

This method was first introduced by Geweke and Porter-Hudak (1983) for the Gaussian case when d belongs to $(-1/2, 0)$ and then it was developed by Robinson (1995b).

Assume that the process $\{X_t\}$, $t = 1, 2, \ldots, n$, is an ARFIMA(p, d, q) model as defined in equation (1), then we can observe that the spectral density of this model is proportional to $(4\sin^2(\omega/2))^{-d}$ near the origin, i.e.

$$f(\omega) \sim c_f (4\sin^2(\omega/2))^{-d}, \tag{2}$$

when ω tends to 0. Since the periodogram $I(\omega)$ is an asymptotically unbiased estimate of the spectral density, that is:

$$\lim_{\omega \to 0} E[I(\omega)] = f(\omega)$$

it is possible to estimate d applying the least squares method to the following equation

$$\log(I(\omega_{j,N})) = \log\{\sigma^2 f_\varepsilon(0) 2\pi\} - d\log(4\sin^2(\omega_j/2)) + u_j \tag{3}$$

where u_j, $j = 1, 2, \ldots, n^*$ are i.i.d. error terms, $\omega_{j,N} = (2\pi j/n)$, $j = 1, 2, \ldots, n^*$ and n^* is the integer part of $(n-1)/2$.

Equation (2) is an asymptotic relation that holds only in a neighbourhood of the origin, thus if we use this relation from all periodogram ordinates $(-\pi < \omega < \pi)$ the estimator of d can be highly biased. Geweke and Porter-Hudak (1983) proposed to consider only the first $\sqrt{n}$ frequencies for the estimate since d is the memory parameter and influences mostly the lower frequencies. The higher frequencies are influenced by the short memory ARMA part.

An interesting advantage with respect to the Whittle is that the GPH estimator can be easily applied without bothering about the ARMA part of the process. The main drawback of this estimator is its high standard deviation. Moreover Agiakloglou et al. (1993) showed that it is biased in presence of ARMA parameter near the non-stationary area.

2.3 The local Whittle estimator

The local Whittle estimator is another semiparametric estimator of the memory parameter d developed by Robinson (1995a) following a suggestion of Künsch (1987). Robinson (1995a) demonstrated that the local Whittle estimator is asymptotically more efficient than the GPH in the stationary case, although it is not defined in closed form and numerical optimization methods are needed to calculate it.

It can be found minimizing the following expression:

$$R(d) = \log\left[\frac{1}{m}\sum_{j=1}^{m}\omega_j^d I_j\right] - d\frac{1}{m}\sum_{j=1}^{m}\log\omega_j^d, \tag{4}$$

where $I_j = I(\omega_j)$ is the periodogram at the Fourier frequencies and m is an integer less than $n/2$.

Under slight conditions Robinson (1995a) showed that this estimator is weakly consistent. Moreover, under stronger conditions, he proved the asymptotic normality even if the convergence rate is slower than in the Whittle case. The rate depends on $m^{1/2}$, the number of frequencies considered in the estimate. Usually it is considered a value of $m = \lfloor\sqrt{n}\rfloor$. Thus, the local Whittle estimate is much less efficient than parametric estimates, like, for example, the Whittle one, when they happen to be based on a correct model, but it is asymptotically more efficient than the GPH estimate.

3 Bootstrap for time series

Bootstrap methods were introduced firstly by Efron (1979) and it has become a popular statistical tool caused by its easiness of use combined to the advent of strong calculators. For a review of the bootstrap methodology, see Hinkley (1988); monographs on the topic include Efron and Tibshirani (1993) and Davison and Hinkley (1997).

Special care is needed when applying bootstrap techniques to time series analysis, since the correlation structure among the variables is possibly complicated and simple methods designed for independent and identically distributed variables are not appropriate. Li and Maddala (1996) discussed the difficulties found in the use of bootstrap for time series models, and gave some guidelines. More recently, Bühlmann (2002) and Politis (2003) review and compare some bootstrap methods for time series illuminating some theoretical aspects of the procedure as well as their performances on finite-sample data. In spite of the great number of papers on bootstrap techniques for time series, the problem is still open since these techniques are not always satisfactory especially if the time series exhibits long range dependence.

In this section we define the bootstrap methods used when data present long memory behaviour. We do not consider parametric bootstrap since it needs to know the correct model to work well.

3.1 Sieve bootstrap

The sieve bootstrap was first introduced by Kreiss (1992) and then developed by Bühlmann (1997). This method is based on the idea of sieve approximation: it approximates a general linear, invertible process by a finite autoregressive model with order increasing with the series length, and resampling from the approximated autoregressions. By viewing such autoregressive approximations as a sieve for the underlying infinite-order process, the bootstrap procedure may still be regarded as a non parametric one. Moreover, this method is computationally simple and yields a (conditionally) stationary bootstrap sample that does not exhibit artefacts in the dependence structure. In a very recent paper Kapetanios and Psaradakis (2006) study the properties of the sieve bootstrap for a class of linear processes with long range dependence. The authors established the first order asymptotic validity of the sieve bootstrap in the case of the sample mean and sample autocovariances, but the results of a Monte Carlo experiment are disappointing.

Given the sample $X_1, X_2, \ldots, X_n$, the scheme for the sieve bootstrap is as follows.

Fit an AR($p(n)$) model to the data choosing the optimal p using the AIC criterion. It is important to note that we fit the autoregressive process with increasing order $p(n)$ as the sample size n increases. Estimate the residuals:

$$\hat{\varepsilon}_{t,n} = \sum_{j=0}^{p(n)} \hat{\phi}_{j,n}(X_{t-j} - \bar{X}), \quad \hat{\phi}_{0,n} = 1 \quad (t = p+1, \ldots, n), \tag{5}$$

where $\bar{x}$ is the sample mean and $\hat{\phi}_{j,n}$ are the autoregressive coefficients. Before bootstrapping the residuals, they have to be centred. At last each bootstrap replicate can be calculated using the following recursion:

$$\sum_{j=0}^{p(n)} \hat{\phi}_{j,n}(X^*_{t-j} - \bar{X}) = \hat{\varepsilon}^*_t \tag{6}$$

where $\hat{\varepsilon}^*_t$ are the bootstrapped residuals.

3.2 The local bootstrap

Paparoditis and Politis (1999) have proposed the non-parametric local bootstrap for weakly dependent stationary processes. It produces surrogate versions of the periodogram $I(\omega_j)$ of the observed process $\{X_t\}$ so that it is useful when the aim is to make inference through the spectrum (e.g. confidence interval for the memory parameter d in case of long-memory).

Silva et al. (ress) apply the local bootstrap to the estimation of the long memory parameter d and, via simulations, compare its performance with that of other bootstrap approaches. The authors established the efficacy of the local bootstrap in terms of low bias, short confidence intervals and low CPU times.

Given the data $X_1, \cdots, X_n$, the local bootstrap algorithm that generates bootstrap replicates $I^*_x(\omega_j)$, $j = 0, 1, \cdots, n^*$ of the periodogram can then be described as follows.

1. Select a resampling width k_n where $k_n = k(n) \in \mathbb{N}$ and $k_n \leq [n/2]$.
2. Define i.i.d discrete random variables $J_1, J_2, \cdots, J_N$ taking values in the set $\{-k_n, -k_n + 1, \cdots, k_n\}$ with probability $p_{k_n,s}$, i.e. $P(J_i = s) = p_{k_n,s}$ for $s = 0, \pm 1, \cdots, \pm k_n$ such that $p_{k_n,s} = p_{k_n,-s}$.
3. The bootstrap periodogram is then defined by $I^*_X(\omega_j) = I_X(\omega_{J_j+j})$ for $j = 1, 2, \cdots, n/2$, $I^*_X(\omega_j) = I^*_X(-\omega_j)$ for $\omega_j \leq 0$ and for $\omega_j = 0$ we set $I^*_X(0) = 0$.

Paparoditis and Politis (1999) have showed that the local bootstrap is asymptotically valid but some care should be taken for the choice of the resampling widths k_n, in the case of a finite sample size n. Following a suggestion of Silva et al. (ress), in this paper we will consider $k_n = 1$ with uniform sample probability since the results are very similar when $k_n = 2$.

4 The new bootstrap method

The method we present is based on the following theorem (Ramsey, 1974).

Theorem 4.1 *Let X_t be a Gaussian, wide-sense stationary time series with mean μ and variance γ_0. Then the conditional distribution of X_t given $X_0, \cdots, X_{t-1}$ is Gaussian with mean and variance given by*

$$m_t = E(X_t|X_0, \cdots, X_{t-1}) = \sum_{j=1}^{t} \phi_{tj} X_{t-j},$$

$$v_t = Var(X_t|X_0, \cdots, X_{t-1}) = \gamma_0 \prod_{j=1}^{t} (1 - \phi_{jj}), \tag{7}$$

where ϕ_{jj} is the jth partial autocorrelation and ϕ_{tj} is the jth autoregressive coefficient in an autoregressive fit of order t.

The coefficients ϕ_{tj} and ϕ_{jj} can easily be obtained by the Durbin-Levinson (see Brockwell and Davis, 1991) recursion:

$$\phi_{tt} = N_t/D_t \tag{8}$$

$$\phi_{tj} = \phi_{t-1,j} - \phi_{tt}\phi_{t-1,t-j}, \quad j = 1, \cdots, n-1, \tag{9}$$

where

$$\begin{aligned} N_0 &= 0 \\ D_0 &= 1 \\ N_t &= \rho_t - \sum_{j=1}^{t-1} \phi_{t-1,j}\rho_{t-j} \\ D_t &= D_{t-1} - N_{t-1}^2/D_{t-1}. \end{aligned}$$

and ρ_t is the autocorrelation function of X_t at lag t.

The hypotheses of Theorem 4.1 admit all processes with an MA-infinite representation, e.g. autoregressive moving average processes and ARFIMA with $0 \leq d < 1/2$. Moreover the sample autocorrelations are consistent with the theoretical autocorrelations.

Instead of using a theoretical autocovariance function, our idea is to use the empirical autocorrelation function of a time series to generate bootstrap time series through the conditional mean, the conditional variance and the Durbin-Levinson algorithm. From now on we will call shortly this new procedure ACF bootstrap. The steps to generate a bootstrap series are:

1. obtain the empirical autocorrelation function, $\hat{\rho}_t$, from the observed time series X_t;

2. generate a starting value of X_0^* from an $N(0, v_0)$ distribution where v_0 is the sample variance of X_t;

3. perform the Durbin-Levinson recursion for ϕ_{tt} and ϕ_{tj} in (8) and (9) based on the empirical autocorrelation function; calculate v_t based on equation (7) and m_t^* as follows

$$m_t^* = E(X_t|X_0, \cdots, X_{t-1}) = \sum_{j=1}^{t} \phi_{tj} X_{t-j}^*,$$

   thus m_t^* is based on the past values of the bootstrap series and the observed autocovariance function of the original one;

4. generate the bootstrap replicates of X_t from $N(m_t^*, v_t)$.

4.1 Properties

The surrogate series $X_{t,b}^*$, $b = 1, \cdots, B$, with B number of bootstrap replicates, have the following main properties:

- the autocovariance function is asymptotically unbiased:

$$E[\gamma_{k,b}^*] = E[X_t^* X_{t-k}^*] = EE[\gamma_{k,b}^*|\{X_t\}] = EE[X_t^* X_{t-k}^*|\{X_t\}] =$$

$$E\left[\left(\frac{n-k}{n}\right)\hat{\gamma}_k\right] = \left(\frac{n-k}{n}\right) E[\hat{\gamma}_k] = \left(\frac{n-k}{n}\right)^2 \gamma_k;$$

- since the spectrum transformation is a linear operator, the surrogate spectra has the same property:

$$E\left[I_b^*(\omega_j)\right] = EE\left[I_b^*(\omega_j)|\{X_t\}\right] =$$

$$E\left[\sum_{k=-(n-1)}^{n-1} \gamma_{k,b}^* \cos\omega k\right] = \sum_{k=-(n-1)}^{n-1} \left(\frac{n-k}{n}\right)^2 \gamma_k \cos\omega k;$$

- if we consider each bootstrap series $X_{t,b}^*$ as a single observation, we can notice that they are independent and identically distributed conditionally to the observed series X_t. Besides, the i.i.d. property is preserved if we consider any transformation $f(X_{t,b}^*)$ of the data.

The third property is very promising to be exploited to estimate the spectrum of a linear model. Thus we present a new estimator of the spectrum as follows

$$\bar{I}^*(\omega_j) = \frac{1}{B} \sum_{l=1}^{B} I_b^*(\omega_j). \tag{10}$$

It is easy to show that $\bar{I}^*(\omega_j)$ is asymptotically unbiased. In fact, if, without loss of generality, we will assume that the observed process X_t is zero mean, it is easy to show that the expected value of $\bar{I}^*(\omega_j)$ is:

$$E\left[\bar{I}^*(\omega_j)\right] = E\left[\frac{1}{B}\sum_{b=1}^{B} I_b^*(\omega_j)\right] = \frac{1}{B}\sum_{b=1}^{B} E\left[\sum_{-(n-1)}^{n-1} \hat{\gamma}_{k,b}^* \cos\omega_j k\right] =$$

$$\sum_{-(n-1)}^{n-1} E\left[\hat{\gamma}_{k,b}^*\right]\cos\omega_j k = \sum_{-(n-1)}^{n-1}\left(\frac{n-k}{n}\right)^2 \gamma_k \cos\omega k.$$

5 Monte Carlo results

In this section, to assess the validity of the ACF bootstrap method with respect to the existing methods in literature, we conduct experiments with simulated data. In particular, we apply the presented method on long memory time series. We used the ACF, the sieve and the local bootstrap to estimate the long-memory parameter d by the GPH and the local Whittle methods.

We suggest the following versions of the estimators,

$$\hat{d}_{\text{GPH}} = \frac{1}{B}\sum_{b=1}^{B}\hat{d}_{b,\text{GPH}}, \tag{11}$$

for the GPH and

$$\hat{d}_{\text{lW}} = \frac{1}{B}\sum_{b=1}^{B}\hat{d}_{b,\text{lW}}, \tag{12}$$

for the local Whittle, where $\hat{d}_{b,\text{GPH}}$ and $\hat{d}_{b,\text{lW}}$ are the values observed in the $l-$th surrogate series. [1]

The functions we use are written in R language (see RDevelopmentCoreTeam, 2006) and are available upon request by the authors.

In the simulation study we consider series generated by $I(d)$ models for different values of the long-memory parameter, i.e. $d = 0.1, 0.2, 0.3, 0.4, 0.45$. The considered sample sizes are $T = 300, 500, 1000$.

The series are generated using the recursive Durbin-Levinson algorithm (see Brockwell and Davis, 1991). For each model we consider $S = 1000$ independent realizations. The number of bootstrap replications is $B = 1000$. For a given estimation

[1] At the beginning of our work we considered also a version of the GPH estimator based on the spectrum estimate given in (10). In this case the regression equation become:

$$\log\{\bar{I}^*(\omega_j)\} = \log\{\sigma^2 f_\varepsilon(0)2\pi\} - d\log\{4\sin^2(\omega_j/2)\} + u. \tag{13}$$

The estimator and its standard deviation are calculated following the simple rules of a linear regression model. It is possible to show that the two bootstrap versions of the GPH estimator are asymptotically equivalent thus we will concentrate on the first one.

method we calculate the Monte Carlo estimates, that is

$$\hat{d} = \frac{1}{S}\sum_{i=1}^{S}\hat{d}_i,$$

where $\hat{d}_i$ are the estimated values for a single realization together with the standard error (s.e.) and the mean square error (MSE). For GPH and local Whittle estimators we also calculate the bootstrap estimates

$$\hat{d}^* = \frac{1}{S}\sum_{i=1}^{S}\hat{d}_i^* \tag{14}$$

where d_i^* is calculated as in (11) and (12).

The performances of ACF bootstrap are compared with sieve and local bootstrap in terms of the s.e. and the MSE. The results are presented in tables 1-4 where also results on the Whittle method (used as a benchmark) are included. Moreover, the tables report results on $\hat{d}^*$ (mean), standard error of $\hat{d}^*$ (s.e.) and MSE of $\hat{d}^*$ (mse) for the two considered estimators and for the three bootstrap methods.

The Monte Carlo estimates are in accordance with known results (see, for example Bisaglia and Gugan, 1998). As we expected, the Whittle estimator outperforms largely all the others, since it is a parametric estimator in the best conditions (the estimates are based on the correct parametric model).

Comparing the bootstrap methods it is evident that the sieve bootstrap exhibits the worst performance and we think that it can not reproduce long-memory processes. With regard to the other two methods, the ACF is a slightly more biased but the standard deviation and the mean square error are always smaller with respect to the local bootstrap. Both of them give satisfactory results compared to the same estimators in the Monte Carlo simulations. By increasing the sample sizes also the performances of these two methods increase.

Table 4 reports the gain, calculated in terms of percentage, when using ACF and local bootstrap with respect to the Monte Carlo version, for the GPH and local Whittle bootstrap estimator. The following is the formula:

$$GAIN\% = \frac{\hat{d} - \hat{d}_i^*}{\hat{d}} \times 100$$

where i = GPH, lW.

The results confirm that the gain is always bigger for the ACF bootstrap.

In conclusion we can say that our method is promising with long-memory processes. It can help to obtain better results with some semi-parametric estimators of the memory parameter d, like the GPH and the local Whittle, that usually have high standard deviation. Moreover the ACF bootstrap outperforms the other two bootstrap methods we considered. The method is open to future developments with confidence interval, hypothesis testing and theoretic results.

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		Monte Carlo			SIEVE		ACF		Local B.	
		Whittle	GPH	Local W.	GPH	Local W.	GPH	Local W.	GPH	Local W.
0.1	mean	0.085	0.105	0.114	0.018	0.068	0.071	0.124	0.098	0.122
	s.e.	0.045	0.192	0.115	0.047	0.032	0.150	0.088	0.183	0.101
	mse	0.002	0.037	0.013	0.009	0.002	0.023	0.008	0.034	0.011
0.2	mean	0.185	0.203	0.202	0.070	0.105	0.173	0.194	0.193	0.201
	s.e.	0.049	0.194	0.142	0.087	0.069	0.154	0.112	0.188	0.128
	mse	0.003	0.038	0.020	0.024	0.014	0.024	0.013	0.035	0.016
0.3	mean	0.286	0.310	0.295	0.144	0.163	0.271	0.271	0.293	0.285
	s.e.	0.049	0.197	0.156	0.124	0.109	0.156	0.130	0.187	0.145
	mse	0.003	0.039	0.024	0.039	0.030	0.025	0.018	0.035	0.021
0.4	mean	0.388	0.413	0.388	0.251	0.258	0.369	0.356	0.389	0.374
	s.e.	0.051	0.196	0.161	0.150	0.141	0.156	0.141	0.188	0.153
	mse	0.003	0.039	0.026	0.045	0.040	0.025	0.022	0.035	0.024
0.45	mean	0.440	0.453	0.427	0.299	0.304	0.405	0.392	0.430	0.412
	s.e.	0.053	0.204	0.167	0.159	0.153	0.163	0.148	0.193	0.160
	mse	0.003	0.041	0.029	0.048	0.045	0.029	0.025	0.038	0.027

Table 1: Results of the estimators analysed with parameter's values $d = 0.1, 0.2, 0.3, 0.4, 0.45$, length's series $n = 300$, bootstrap's replications $B = 1000$ and replications $S = 1000$

		Monte Carlo			SIEVE		ACF		Local B.	
		Whittle	GPH	Local W.	GPH	Local W.	GPH	Local W.	GPH	Local W.
0.1	mean	0.096	0.090	0.107	0.017	0.059	0.077	0.116	0.093	0.115
	s.e.	0.170	0.036	0.102	0.042	0.031	0.131	0.080	0.164	0.092
	mse	0.029	0.001	0.011	0.009	0.003	0.018	0.007	0.027	0.009
0.2	mean	0.211	0.190	0.198	0.073	0.101	0.179	0.189	0.203	0.196
	s.e.	0.173	0.038	0.132	0.092	0.078	0.141	0.109	0.166	0.120
	mse	0.030	0.002	0.017	0.024	0.016	0.020	0.012	0.027	0.015
0.3	mean	0.311	0.293	0.294	0.156	0.168	0.277	0.273	0.300	0.287
	s.e.	0.176	0.037	0.135	0.118	0.109	0.134	0.117	0.169	0.126
	mse	0.031	0.001	0.018	0.035	0.029	0.019	0.015	0.029	0.016
0.4	mean	0.408	0.396	0.395	0.263	0.268	0.379	0.368	0.391	0.383
	s.e.	0.172	0.037	0.142	0.146	0.144	0.139	0.129	0.168	0.139
	mse	0.030	0.001	0.020	0.040	0.038	0.020	0.018	0.028	0.020
0.45	mean	0.465	0.446	0.446	0.323	0.329	0.428	0.415	0.444	0.433
	s.e.	0.172	0.038	0.143	0.154	0.154	0.140	0.133	0.167	0.141
	mse	0.030	0.001	0.021	0.040	0.039	0.020	0.019	0.028	0.020

Table 2: Results of the estimators analysed with parameter's values $d = 0.1, 0.2, 0.3, 0.4, 0.45$, length's series $n = 500$, bootstrap's replications $B = 1000$ and replications $S = 1000$

		Monte Carlo			SIEVE		ACF		Local B.	
		Whittle	GPH	Local W.	GPH	Local W.	GPH	Local W.	GPH	Local W.
0.1	mean	0,094	0,097	0,103	0,012	0,046	0,082	0,108	0,094	0,107
	s.e.	0,027	0,134	0,090	0,027	0,020	0,107	0,072	0,131	0,081
	mse	0,001	0,018	0,008	0,008	0,003	0,012	0,005	0,017	0,007
0.2	mean	0,196	0,204	0,197	0,066	0,086	0,185	0,187	0,199	0,195
	s.e.	0,025	0,138	0,107	0,073	0,063	0,112	0,092	0,134	0,100
	mse	0,001	0,019	0,012	0,023	0,017	0,013	0,009	0,018	0,010
0.3	mean	0,297	0,301	0,290	0,155	0,163	0,278	0,270	0,292	0,281
	s.e.	0,026	0,143	0,112	0,107	0,104	0,115	0,102	0,140	0,108
	mse	0,001	0,020	0,013	0,033	0,029	0,014	0,011	0,020	0,012
0.4	mean	0,397	0,406	0,396	0,266	0,273	0,384	0,372	0,397	0,388
	s.e.	0,027	0,140	0,114	0,133	0,136	0,113	0,109	0,137	0,114
	mse	0,001	0,020	0,013	0,036	0,035	0,013	0,013	0,019	0,013
0.45	mean	0,448	0,461	0,451	0,341	0,351	0,438	0,426	0,450	0,443
	s.e.	0,026	0,136	0,114	0,140	0,145	0,112	0,110	0,133	0,114
	mse	0,001	0,019	0,013	0,031	0,031	0,013	0,013	0,018	0,013

Table 3: Results of the estimators analysed with parameter's values $d = 0.1, 0.2, 0.3, 0.4, 0.45$, length's series $n = 1000$, bootstrap's replications $B = 1000$ and replications $S = 1000$

		length series 300				length series 500				length series 1000			
		GPH		Local Whittle		GPH		Local Whittle		GPH		Local Whittle	
		ACF	loc.B.	ACF	loc.B.	ACF	loc.B.	ACF	loc.B.	ACF	loc.B.	ACF	loc.B.
0.1	s.e.	21.66	4.32	23.30	11.84	22.95	3.27	21.49	9.77	19.94	2.19	19.62	9.69
	mse	36.40	8.51	37.90	19.87	38.76	6.31	36.32	16.87	34.04	4.15	34.75	17.90
0.2	s.e.	20.79	2.98	20.83	9.95	18.35	4.18	17.09	8.59	18.87	2.52	14.58	7.17
	mse	35.36	5.74	37.16	18.93	32.09	8.55	30.53	16.37	33.07	5.03	25.70	13.64
0.3	s.e.	20.91	5.25	16.24	7.08	23.97	4.17	13.07	6.58	19.33	2.22	8.42	3.07
	mse	35.37	10.33	26.44	12.85	40.68	8.54	20.65	11.88	32.63	4.05	9.77	4.11
0.4	s.e.	20.44	4.01	12.60	5.12	19.47	2.32	9.25	2.41	19.40	2.63	4.22	-0.17
	mse	34.45	7.97	16.74	7.92	33.85	4.55	12.66	3.51	33.81	5.33	2.40	-1.38
0.45	s.e.	19.82	5.22	11.70	4.26	18.45	3.25	7.47	1.70	17.79	2.09	3.34	-0.41
	mse	30.83	9.21	11.85	5.03	32.35	6.94	8.47	2.06	32.02	4.70	2.07	-1.21

Table 4: Percentage of gain comparing the Monte Carlo results of estimators GPH and local Whittle with the bootstrap results.