Dynamic advertising with constant exogenous interference

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Abstract

We propose a model of a firm which advertises a product in a homogeneous market where an a constant exogenous interference is present. Using the framework of the Nerlove and Arrow's advertising-goodwill-demand model, we assume that the interference acts additively on the goodwill production term as a negative term. Hence we admit that the firm goodwill may become negative and associate a zero demand with negative goodwill values. For simplicity we consider a piecewise linear demand function and we are led to formulate a nonsmooth optimal control problem with infinite horizon. We obtain that an optimal advertising policy exists and takes one of two forms: either a positive and constant advertising effort, or a decreasing effort, starting at a positive level and eventually reaching the zero value at a finite exit time. In both casesa we have explicit representations of the optimal control which are obtained through the study of two auxiliary smooth optimal control problems. It is interesting to observe that the fundamental choice between staying in the market and going out of business at some time depends both on the interference size and on the initial state (goodwill).

Keywords Optimal control; Nonsmooth optimization; Advertising

1 Introduction

The use of the optimal control (and calculus of variations) to analyze the effects of advertising dates back to the seminal paper of Nerlove and Arrow [14]. This paper is the starting point for a research field which is well described in the survey papers [17], [5]. Nowadays, Nerlove and Arrow ideas are the cornerstone of advertising models used by practitioners and theorists. The *Brandaid* model of Little [12] is a discretization of Nerlove and Arrow's one, and it is one of the most important references for many discrete time models used by marketing practitioners. From a theoretical point of view the research is still moving to make the original Nerlove and Arrow's model more and more faithful to reality. Without being exhaustive, the extensions which seems to us more relevant are: the introduction of uncertainty in the model (see the very recent paper of Marinelli [13] and the references therein), the introduction of market segmentation ([3], [4]), and last but not least the consideration of several actors in the market arena. The description of advertising as a differential game is so important that it represents a field of research by itself. The most important reference in this area is the book by Jørgensen and Zaccour [10], but it is still an important and growing research subject (for some recent papers see e.g. [15], [11], [8]).

In the distribution channel problem (see [9]), the strategies of the economic agents – generally a manufacturer and a retailer – are studied assuming that advertising efforts of both the players affect the evolution of a unique goodwill (the stock summarizing the present and past advertising investments). Therefore the evolution of the goodwill G is driven by the two advertising efforts $u_1, u_2 \geq 0$ following the linear ordinary differential equation

$$\dot{G} = u_1 + u_2 - \delta G\,. \tag{1}$$

The demand of the good – distributed using this channel – is an increasing function of its goodwill. Therefore, the manufacturer and the retailer drive the goodwill evolution by advertising in order to maximize their own profits. In this setting the two advertising efforts are combined additively and the goodwill value is always positive, as in the original model.

This simple description suggests a question "What does happen when there is a negative interference between the two advertising efforts?" It is not only a theoretical point: if a retailer sells both a national and a store brand, then the advertising effort for the store brand can have a negative interference with the goodwill evolution of the national brand [1]. Here we take a slightly different point of view: we assume that two manufacturers distribute and advertise two substitutable products. We look at the situation with the eyes of the first player: the goodwill (G_1) evolution of his product is affected not only by his advertising effort ($u_1 \geq 0$), but also, and in a negative way, by that of the other player ($u_2 \geq 0$). Assuming that the advertising effort of the second player destroys part of the advertising effort of the first one we can model this situation with an ordinary differential equation similar to the previous one (1):

$$\dot{G}_1 = u_1 - u_2 - \delta G_1\,. \tag{2}$$

Under this assumption an important feature of the original Nerlove and Arrow's model is lost: the goodwill value may become negative. Two questions arise immediately: "How should we modify the demand function in order to describe scenarios with negative goodwill values?" and "Will the profit maximization problem be mathematically tractable as in the original case?" Actually, the aim of this paper is to pave the way to the analysis of a differential game where two manufacturers (the players) advertise their products and each one obtains a positive effect on its own goodwill evolution and a negative effect on the competitor's goodwill evolution.

Our model is therefore an extension of the original Nerlove and Arrow's model, where an exogenous negative interference is introduced to affect the goodwill evolution. The main assumptions are the following:

- the goodwill evolution is described using the same linear differential equation as proposed by Nerlove and Arrow, but an exogenous, constant, and negative interference is added to the advertising effect (this entails that the goodwill value may become negative);
- the demand function is linearly increasing in the goodwill value when this is positive, and zero when the goodwill has negative values (this seems to be the simplest way to interpret the negative goodwill phenomenon, and to keep the possibility of results comparison with the original model in infinite horizon, as presented in [6]);
- the advertising costs are quadratic (again a simple assumption, frequently used in the differential games framework [10], and useful for the analytical tractability);
- the objective of the decision maker is that of maximizing his discounted profit in the long run.

Therefore we study an infinite horizon optimal control problem with a nonsmooth function in the objective functional. The results of our analysis go in two directions: on one hand we suggest how a variant of the Nerlove and Arrow's model can be studied when, for some reasons, the goodwill becomes negative; on the other hand we introduce some *ad hoc* observations which allow us to solve the nonsmooth optimal control problem using the solutions of some auxiliary smooth optimal control problems.

The outline of paper is as follows. In Section 2 we introduce the model and translate in a mathematical language the assumptions just presented. In Section 3 we prove the existence of an optimal solution and study its basic features. In Section 4 we characterize the optimal solutions under the assumption that the goodwill remains always positive. Under such assumption the decision maker sees a positive demand at all times. Symmetrically to this case, in Section 5 we characterize the optimal solutions under the assumption that the goodwill becomes zero in a finite time. We prove that, under this hypothesis, the optimal decision is to go out of business. In Section 6 we show that only one of the previous strategies can be optimal and we characterize the choices of the parameters that lead to an optimal solution with positive goodwill path and to an optimal solution where the goodwill becomes zero. In Section 7 we describe the results obtained and we suggest some future research directions.

2 The model: advertising and interference

Let $G(t)$ represent the stock of goodwill of the product/service at time t. As motivated in the previous Section, we refer to the definition of *goodwill* given by

Nerlove and Arrow [14] to describe the variable which summarises the effects of present and past advertising on the demand; the goodwill needs an advertising effort to increase, while it is subject to a spontaneous decay. The goodwill value $G(t)$ is the joint result of an advertising process, which is decided by the manufacturer, and of a known exogenous and deterministic interference.

The manufacturer's action is the advertising intensity $u(t) \geq 0$ (the activation level of an advertising medium) and we assume that the goodwill level evolve in time according to the differential equation

$$\dot{G}(t) \;=\; \gamma u(t) - \zeta - \delta\, G(t)\;, \tag{3}$$

where

- $\delta > 0$ represents the goodwill depreciation rate;
- $\gamma u > 0$ represents the effect of activating the advertising medium at level $u \in [0, u_{\max}]$;
- $\zeta > 0$ is the constant exogenous *interference*.

Moreover, we assume we know the goodwill level at the initial time,

$$G(0) \;=\; \alpha \;\geq\; 0\,. \tag{4}$$

The dynamics of the goodwill given by the linear equation (3) is essentially the same as the one proposed by [14]. In contrast with the more usual situation while using the Nerlove-Arrow's dynamics, the effective advertising intensity in equation (3), (i.e. the term $u(t) - \zeta$) may be as well negative as positive (e.g. when $u \equiv 0$). This fact may lead to negative goodwill values, which we consider admissible here, but with 0 demand associated with them. Therefore, we assume that the product demand rate is a piecewise linear function of its goodwill,

$$D(G) \;=\; \beta \cdot \max\{0, G\} \;=\; \beta \cdot [G]^{+}\;, \tag{5}$$

where $\beta > 0$ is the marginal demand with respect to the goodwill when it is positive. The assumption (5) of a demand proportional to goodwill, as far as the latter is positive, is the most elementary model of goodwill–demand relationship and in fact it is rather common in the marketing literature (see e.g. [6] and the references therein). What is new here is the definition of the demand also for negative goodwill values: the equation (5) seems to us a rather obvious choice.

Moreover, we assume that $\kappa u^2/2$, $\kappa > 0$, is the cost intensity associated with the advertising intensity (i.e. advertising activation level) u, and that $\pi > 0$ is the marginal profit of the manufacturer, gross of advertising costs. Then the discounted profit of the manufacturer over the infinite horizon $[0, +\infty)$ is

$$\Pi(u) \;=\; \int_0^{+\infty} e^{-\rho t}\left[\pi D(G(t)) - \frac{\kappa}{2} u^2(t)\right] dt\,, \tag{6}$$

where $\rho > 0$ is the discount parameter.

The manufacturer wants to maximize the profit functional, which depends on his advertising policy, $u \geq 0$.

As a consequence of the definition (5), we have that the current profit rate of the manufacturer,

$$\pi\beta \cdot [G]^{+} - \frac{\kappa}{2}u^{2}\,, \tag{7}$$

is a nonsmooth and convex function of G. Such features entail some special difficulties while searching for optimal manufacturer's behaviors.

3 Basic features of optimal solutions

Under the previous assumptions we are dealing with the following nonsmooth optimal control problem:

$$\begin{array}{ll} \max_{u(t)\in[0,u_{\max}]} & \Pi\left(u\right) = \displaystyle\int_{0}^{+\infty} e^{-\rho t}\left[\pi\beta\left[G\left(t\right)\right]^{+} - \frac{\kappa u^{2}\left(t\right)}{2}\right]dt\,, \\ & \dot{G}\left(t\right) = -\delta G\left(t\right) + \gamma u\left(t\right) - \zeta\,, \\ & G\left(0\right) = \alpha\,. \end{array} \tag{8}$$

Remark 1 We observe that there exists an optimal policy for the nonsmooth optimal control problem (8). In fact, the set of the reachable goodwill values is compact, because the motion equation is linear and the set $[0, u_{\max}]$ is compact. Therefore, we can verify that the assumption of an existence theorem (see e.g. [16, p.237, Theorem 15]) holds.

In order to characterize an optimal control of problem (8), we should use the suitable nonsmooth necessary conditions (see e.g. [2]), but the results described in that paper are not easy to apply to our problem, because we cannot find the optimal control function by a direct integration, in view of the relevant adjoint differential inclusion. Hence, we tackle the problem from a different point of view: using an observation on possibly optimal goodwill paths which reach the 0 value, we transform the analysis of problem (8) into that of two smooth optimal control problems, one with infinite horizon, the other with variable final time.

As a first step, we prove that no "cycle" may exist around an equilibrium point: an optimal state function cannot oscillate around 0. This result is quite similar to the one obtained by Hartl in [7], but here we cannot use directly the results described in that paper because we do not have *a priori* the uniqueness of the solution.

It is useful to define here some notations we will use during the whole paper. We denote by $G\left(t;u\right)$ the value at the time t of the state function associated with the control function u (i.e. $G\left(\cdot;u\right)$ is the unique solution of (3), (4) when the control function is u). Moreover, we denote by $u|_{[0,\tau]}$ the restriction of the control function u (defined in the whole programming interval) to the subinterval $[0,\tau]$. Both this notations are standard in control theory.

Finally, given a function u which is defined either on an interval $[0, \tau]$, or on $[0, +\infty)$, we denote by $\bar{u}_\tau$ the function of $[0, +\infty)$ into $\mathbb{R}$, defined as

$$\bar{u}_\tau(t) = \begin{cases} u(t) , & t \in [0, \tau) , \\ 0 , & t \in [\tau, +\infty) . \end{cases}$$

We can use this definition in two different ways mainly: if the original control u is defined in $[0, +\infty)$, then $\bar{u}_\tau$ is an admissible control which coincides with u in $[0, \tau)$, but is 0 in $[\tau, +\infty)$; on the other hand, if the control u is defined only in $[0, \tau]$, then $\bar{u}_\tau$ allows us to extend (with 0 value) the original control to the whole programming interval.

Lemma 1 *Let u be an admissible control such that for some $\tau, \vartheta > 0$*

$$\begin{array}{ll} G(t; u) > 0 , & t \in [0, \tau) , \\ G(t; u) < 0 , & t \in (\tau, \tau + \vartheta) , \\ G(\tau; u) = G(\tau + \vartheta; u) = 0 , & \end{array} \tag{9}$$

then u cannot be optimal for problem (8).

Proof. Let us assume that u is an optimal control. We notice that

$$\Pi(u) \ = \ A + B + C , \tag{10}$$

where

$$A \ = \ \int_0^\tau e^{-\rho t} \left[\pi\beta G(t; u) - \frac{\kappa}{2} u^2(t) \right] dt , \tag{11}$$

$$B \ = \ -\frac{\kappa}{2} \int_\tau^{\tau+\vartheta} u^2(t) e^{-\rho t} dt \ < \ 0 , \tag{12}$$

$$C \ = \ \int_{\tau+\vartheta}^{+\infty} e^{-\rho t} \left[\pi\beta \left[G(t; u) \right]^+ - \frac{\kappa}{2} u^2(t) \right] \ \geq \ 0 . \tag{13}$$

The inequality $B < 0$ in (12) must hold, because the equality $B = 0$ would imply $G(\tau + \vartheta; u) < 0$. Moreover, the inequality $C \geq 0$ in (13) must hold, otherwise the control $\bar{u}_{\tau+\vartheta}$ would be better than u, as

$$\Pi(\bar{u}_{\tau+\vartheta}) \ = \ A + B \ > \ A + B + C \ = \ \Pi(u) , \tag{14}$$

and u would not be optimal. Finally, $C < +\infty$ because the control function is bounded, the motion equation is (3), and therefore the state function is bounded too. Now, let us define the control

$$u^*(t) = \begin{cases} u(t) , & t \in [0, \tau) , \\ u(t + \vartheta) , & t \in [\tau, +\infty) , \end{cases} \tag{15}$$

whose associate state function is

$$G(t; u^*) = \begin{cases} G(t; u) , & t \in [0, \tau) , \\ G(t + \vartheta; u) , & t \in [\tau, +\infty) . \end{cases} \tag{16}$$

We observe that u^* is an admissible control and

$$\Pi(u) - \Pi(u^*) \; = \; B + (1 - e^{\rho\vartheta})C \; < \; 0\,. \tag{17}$$

Therefore u cannot be optimal. □

Actually, this Lemma has an interesting interpretation from a practical point of view: if the goodwill (and therefore the demand) reaches the 0 value at a time $\tau > 0$, then it will be negative (the demand will remain 0) for all $t > \tau$. Using this result we can relate the original problem to a pair of smooth optimal control problems. In particular, at the end of this Section, we will observe that, if the goodwill reaches the 0 value at a time $\tau > 0$, then the optimal control is positive and decreasing until it reaches the 0 value at the time τ, and hence it remains 0.

A further natural notation choice is useful in the following. Let w be a control function defined on $[0, +\infty)$ and let $\tau > 0$ be a given time, then we will write the symbol (τ, w), instead of $\left(\tau,\, w|_{[0,\tau]}\right)$, to denote the time-control pair with the restriction of w to the interval $[0, \tau]$ as the control component.

Lemma 2 *The time-control pair (τ, u) is an optimal solution of the variable final time problem*

$$\begin{array}{rcl} \max_{T,v} \quad \Psi(T, v) & = & \displaystyle\int_0^T e^{-\rho t}\left[\pi\beta G(t; v) - \frac{\kappa v^2(t)}{2}\right] dt\,, \\ \dot{G}(t; v) & = & -\delta G(t; v) + \gamma v(t) - \zeta\,, \\ G(0; v) & = & \alpha \; > \; 0\,, \\ G(t; v) & \geq & 0\,, \end{array} \tag{18}$$

if and only if the control $\bar{u}_\tau$, the extension of u to the interval $[\tau, +\infty)$ with 0 value, is an optimal solution of the problem (8) with $G(\tau; \bar{u}_\tau) = 0$.

Proof. ($\Rightarrow$) Let the time-control pair (τ, u) be an optimal solution of the variable final time problem (18). First of all, if $G(\tau; u) > 0$, then for sufficiently small $\varepsilon > 0$ we have that $G\left(\tau + \varepsilon;\, \bar{u}_\tau|_{[0,\tau+\varepsilon]}\right) \geq 0$. But this implies that $\Psi(\tau, u) < \Psi\left(\tau + \varepsilon,\, \bar{u}_\tau|_{[0,\tau+\varepsilon]}\right)$ and this contradicts the optimality of (u, τ). Therefore we must have $G(\tau; u) = 0$. We have proved that $G(\tau; \bar{u}_\tau) = G(\tau; u) = 0$. We notice that the control $\bar{u}_\tau$ is an admissible solution for problem (8) and

$$\Psi(\tau, u) \; = \; \Pi(\bar{u}_\tau)\;. \tag{19}$$

Let us assume that $\bar{u}_\tau$ is not optimal for problem (8), and let w be an optimal control of problem (8), which exists because of the Remark 3, then we have that $\Pi(\bar{u}_\tau) < \Pi(w)$, and therefore

$$\Psi(\tau, u) \; = \; \Pi(\bar{u}_\tau) \; < \; \Pi(w)\;. \tag{20}$$

Now, either case may occur:

a) there exists a time $\vartheta > 0$ such that $G(\vartheta; w) = 0$;

b) $G(t; w) > 0$ for all $t \in [0, +\infty)$.

Case a) implies that (τ, u) is not optimal for the problem (18), because from the inequality in (20) and Lemma 1 we obtain that $\Psi(\tau, u) < \Psi(\vartheta, w) = \Pi(w)$: a contradiction.

Then case b) must occur, so that we have

$$\lim_{T \to +\infty} \Psi(T, w) = \\ = \lim_{T \to +\infty} \int_0^T e^{-\rho t} \left[\pi \beta G(t; w) - \frac{\kappa}{2} w^2(t) \right] dt = \Pi(w) . \quad (21)$$

From this limit and the inequality (20) we obtain that there exists some T such that

$$\Psi(\tau, u) < \Psi(T, w) , \quad (22)$$

which contradicts the optimality of (τ, u) for the problem (18).

($\Leftarrow$) Let u be an optimal solution of problem (8) with $G(\tau; u) = 0$ and let us assume that (τ, u) is not optimal for the variable final time problem (18). Therefore there exists a control w and a time $\theta \in (0, +\infty]$ such that

$$\Psi(\tau, u) < \Psi(\vartheta, w) . \quad (23)$$

We notice that the following inequality holds

$$\Psi(\vartheta, w) \leq \Pi(\bar{w}_\vartheta) . \quad (24)$$

Moreover, by Lemma 1 we have that

$$\Psi(\tau, u) = \Pi(u) , \quad (25)$$

because, after the time τ, the state variable remains always negative (hence the optimal control u after τ must be identically equal to 0). Finally, from (23), (24), and (25) we obtain

$$\Pi(u) = \Psi(\tau, u) < \Psi(\vartheta, w) \leq \Pi(\bar{w}_\vartheta) , \quad (26)$$

so that u cannot be optimal for problem (8). □

The Lemma 2 allows us to conclude that, given an optimal solution of problem (8), either it has a positive goodwill at all times, or a restriction of it solves optimally a special variable final time smooth optimal control problem. In the following Sections we obtain the explicit forms of the solution in the two alternative cases.

From an economic point of view the results of this section may be summarized as follows: the decision maker, while using an optimal advertising policy, either stays in the market in the long run (the goodwill path is always positive), or goes out of the market (the goodwill becomes zero at a finite time). In the latter case the original problem coincides (in the precise sense of Lemma 2) with a free final time optimal control problem, which represents the optimal strategy to go out of business.

4 Optimal positive goodwill path

From the analysis of Section 3 we have obtained that one of the strategies candidate to be optimal could keep the goodwill value always positive. Now, assuming that such positive optimal goodwill condition holds, we want to find the advertising policy which determine it.

Before starting this analysis, we want to notice that the assumption on the upper bound for the control function is essential in the issues of Section 3 (under this hypothesis the state function is always bounded and the objective functional always converges). On the other hand, from an economic point of view, it represents an exogenous threshold which is not justified. In order to consider the real trade off between advertising cost and profit we assume, in this Section, that the upper bound $u_{\max}$ is high enough to admit certain constant controls, and precisely

$$u_{\max} \; > \; \max\left\{ \frac{\pi\beta\gamma}{(\delta+\rho)\kappa}, \frac{\zeta}{\gamma} \right\} . \tag{27}$$

With this further assumption, we can have both the useful technical condition $u(t) \in [0, u_{\max}]$ and the economic decision space wide as desired.

Now we can present the main result of this Section.

Theorem 1 *Let u be an optimal control for problem (8), and let $G(t;u) > 0$ for all $t > 0$, then*

$$u\,(t) \; \equiv \; u^* \; = \; \frac{\pi\beta\gamma}{(\delta+\rho)\kappa} \, , \tag{28}$$

and

$$\gamma u^* \; > \; \zeta \, . \tag{29}$$

Proof. The hypothesis implies that the value of the objective functional at the control u coincides with the value of the functional

$$\Phi(u) \; = \; \int_0^{+\infty} e^{-\rho t} \left[\pi\beta G\,(t;u) - \frac{\kappa}{2} u^2(t) \right] dt \, , \tag{30}$$

which has a smooth integrand function. Then the control-state pair $(u, G\,(\cdot;u))$ must satisfy the optimality necessary conditions for the problem of maximizing the objective functional (30), subject to the motion equation (3) and the initial condition (4). Moreover, using the substitution $x = G + \zeta/\delta$, we obtain the equivalent formulation

$$\begin{aligned} &\max_u \quad \int_0^T e^{-\rho t} \left[\pi\beta x\,(t) - \frac{\kappa u^2\,(t)}{2} \right] dt \; - \; \frac{\pi\beta\zeta}{\delta\rho} \, , \\ &\dot{x}\,(t) = -\delta x\,(t) + \gamma u\,(t) - \zeta \, , \\ &x\,(0) = \alpha + \zeta/\delta \, . \end{aligned} \tag{31}$$

As a consequence of the variable substitution made, the positivity condition $G(t;u) > 0$ is represented as

$$x(t) \; > \; \frac{\zeta}{\delta} \, . \tag{32}$$

Now, we can observe that (up to an additive constant in the objective functional) this problem is precisely the one studied in [6]. If $u_{\max} > \pi\beta\gamma/(\delta+\rho)\kappa$, then from these results we can conclude that (28) is the unique optimal control under the goodwill positivity condition.

The constant control u^* is compatible with the positive goodwill path hypothesis if and only if $G(t;u^*) > 0$ for all $t \in [0,+\infty)$, and this is equivalent to $\gamma u^* \geq \zeta$. On the other hand, the constant control $u(t) \equiv \zeta/\gamma$ cannot be optimal. In fact we have that

$$G\left(t;\frac{\zeta}{\gamma}\right) \; = \; G^0 e^{-\delta t} \; > \; 0 \tag{33}$$

and the profit intensity is

$$\pi\beta G^0 e^{-\delta t} - \frac{\kappa}{2}\frac{\zeta^2}{\gamma^2} \, . \tag{34}$$

This is a strictly decreasing function and it is negative for $t \in (\tau,+\infty)$, where $\tau = \delta^{-1}\ln\left(2\pi\beta\gamma G^0/\kappa\zeta\right)$. Now, the control $\bar{u}_\tau$, which is obtained by switching off the advertising process $u(t) \equiv \zeta/\gamma$ at the time τ, has the same profit intensity as u in the interval $[0,\tau]$, and a 0, i.e. a strictly greater profit intensity in $(\tau,+\infty)$. Hence the control $u(t) \equiv \zeta/\gamma$ is not optimal. □

Remark 2 Even where the special constant control (28) satisfies the inequality (29), it may not be optimal, as one can see, at a first level, by comparing the profit associated with it,

$$\Pi(u^*) \; = \; \Phi(u^*) \; = \; \frac{\pi\beta\alpha}{\delta+\rho} + \frac{\pi\beta}{(\delta+\rho)\rho}\left(\frac{\gamma u^*}{2} - \zeta\right) , \tag{35}$$

and the profit associated with the 0 control. In fact, $\Pi(u^*)$ may be negative in some cases, whereas $\Pi(0) > 0$. Furthermore, at a second level, we observe that, if $\dfrac{\delta+2\rho}{2(\delta+\rho)}\gamma u^* < \zeta \leq \gamma u^*$, then the current value of the profit intensity $\pi\beta G(t;u^*) - \kappa u^*/2$ is eventually negative and therefore u^* cannot be optimal. In fact, if τ is the first time at which the profit intensity is not positive, then the control $(u^*)_\tau$ induces a greater profit that u^*. The analysis developed so far does not provide any sufficient condition for the optimality of the constant control (28).

5 Optimal exit policy

In Section 3 we have seen that the problem (8) has an optimal solution and the optimal goodwill path can either be positive at all times, or reach the 0 level only once. In Section 4 we have characterized the optimal policy under the assumption that the goodwill is positive. In that senario the manufacturer stays in the market forever and sees a positive demand in $[0, +\infty)$. Here we assume that the optimal goodwill path reaches the 0 level at some time. From the results obtained in Section 3, this hypothesis implies that the manufacturer goes out of the market: we have to study the free final time problem (18) instead of the infinite horizon one (8).

In the following we want to obtain the explicit form of the optimal *exit* advertising policy and the necessary conditions for the occurrence of this scenario. To this end we exploit the relation with the the free final time problem stated in Lemma 2: we recall that if that variable final time problem has an optimal solution, this is optimal, in some sense, for the problem (8) too.

Theorem 2 *Let u^τ be an optimal control for problem (8), and let $G(\tau; u^\tau) = 0$ for some $\tau > 0$, then*

$$u^\tau(t) \;=\; u^* \cdot \left[1 - e^{(\delta+\rho)(t-\tau)}\right]^+ , \tag{36}$$

where the value u^* is as defined in (28) and τ is a solution of the exit time equation

$$\left(\frac{\zeta - \gamma u^*}{\delta}\right)\left(e^{\delta\tau} - 1\right) + \frac{\gamma u^*}{2\delta+\rho}\left(e^{\delta\tau} - e^{-(\delta+\rho)\tau}\right) \;=\; \alpha\,. \tag{37}$$

Proof. In view of Lemma 2, any optimal control u of the original problem under the present assumptions is related with that of the variable final time problem (18). From Lemma 1, we obtain that $G(t;u) \le 0$, $t \ge \tau$, and hence $u(t) = 0$, for all $t > \tau$. Let us consider the optimal control problem (18) without the state condition $G(t;u) \ge 0$. We call it the *relaxed variable final time* problem. The Hamiltonian is

$$H(G,u,p,t) \;=\; e^{-\rho t}\left[\pi\beta G - \frac{\kappa}{2}u^2\right] + p\left(\gamma u - \zeta - \delta G\right) , \tag{38}$$

which is a concave function of u and has the first partial derivative w.r.t. u

$$\frac{\partial H}{\partial u} \;=\; -\kappa u e^{-\rho t} + \gamma p\,. \tag{39}$$

Hence there exists a unique maximum point which is characterized by the equation

$$u \;=\; \frac{\gamma}{\kappa}\, e^{\rho t}\,[p]^+ \,. \tag{40}$$

As for the adjoint function $p(t)$, it must be a solution to the linear differential equation

$$\dot{p}(t) \;=\; -\pi\beta e^{-\rho t} + \delta p(t)\,, \tag{41}$$

and the transversality condition is

$$p(T) \;=\; 0\,. \tag{42}$$

The direct integration of this equation gives us the adjoint function

$$p(t) \;=\; \frac{\pi\beta}{\delta+\rho}\left(1-e^{(\delta+\rho)(t-T)}\right)e^{-\rho t}\,, \tag{43}$$

hence the candidate optimal control is

$$u\,(t) \;=\; \frac{\pi\beta\gamma}{(\delta+\rho)\kappa}\cdot\left(1-e^{(\delta+\rho)(t-T)}\right) \;=\; u^*\cdot\left(1-e^{(\delta+\rho)(t-T)}\right)\,. \tag{44}$$

The second equality holds because of the definition (28). Moreover, the optimal final time T must satisfy the condition

$$H\left(G\left(T;u\right),u\left(T\right),p\left(T\right),T\right)=0\,,$$

which implies that $G\left(T;u\right)=0$, because of the transversality condition (42) and of equation (44), which gives $u(T)=0$. In order to determine such a solution completely we have to find the suitable value of T. This can be found using the condition $G(0;u)=\alpha$, which is precisely equation (37). □

Again this result provides only necessary conditions for the optimality of the policy (36). In order to state whether such a control is meaningful, we have to determine the existence of the solution of the equation (37) in the variable τ. From an economic point of view it is interesting to notice that the advertising investments are concentrated in the first part of the programming interval. First of all because we are analizing the discounted profit; then because the decision maker knows that after the time τ his product/service is out of the market.

Now we know the two different kinds of policy which are possibly compatible with the necessary conditions, but we do not have a complete characterization of the possibles scenarios in terms of the problem parameters. In the following Section we want to find the conditions on the parameters which lead to a long run solution rather than a policy with an exit time.

6 Optimal advertising policies

We know that the original problem has an optimal solution and now we know that the there are two different analytical forms for the candidate optimal policies. We need a criterion to decide which of the two advertising policies is the best.

Lemma 3 *If the exit control u^τ, as defined in (36), is optimal for the original problem and $\zeta<\gamma u^*$, then the constant control u^* cannot be optimal.*

Proof. Let both u^τ and u^* be optimal, therefore $\Pi\left(u^*\right)=\Pi\left(u^\tau\right)$. Moreover, the goodwill functions associated with these controls are lower bounded: $G\left(t;u^*\right)\geq\min\left\{\alpha,\left(\gamma u^*-\zeta\right)/\delta\right\}>0$ and $G\left(t;u^\tau\right)>-\zeta/\delta$.
For all $\varepsilon\in[0,1]$, let us define the control u_ε as the convex combination $u_\varepsilon=$

$(1-\varepsilon)u^* + \varepsilon u^\tau$. We notice that $u_\varepsilon(t) > 0$ for all $t \in [0, +\infty)$ if $\varepsilon \in [0, 1)$. The goodwill $G(t; u_\varepsilon)$ has the lower bound

$$\begin{aligned} G(t; u_\varepsilon) &= (1-\varepsilon)G(t; u^*) + \varepsilon G(t; u^\tau) \\ &\geq (1-\varepsilon)\min\{\alpha, (\gamma u^* - \zeta)/\delta\} - \varepsilon\zeta/\delta \\ &\geq \min\{\alpha - \varepsilon(\alpha + \zeta/\delta), (\gamma u^* - \zeta)/\delta - \varepsilon\gamma u^*/\delta\} , \end{aligned}$$

so that $G(t; u_\varepsilon) > 0$, for all $t > 0$ and all $\varepsilon \in [0, \bar{\varepsilon})$, where

$$\bar{\varepsilon} = \min\{1 - \zeta/(\delta\alpha + \zeta), 1 - \zeta/\gamma u^*\} .$$

Now, the integrand function in the objective functional of the original problem is stictly concave, as a function of (u, G), on the domain $(0, +\infty)^2$. Therefore, for all $\varepsilon \in (0, \bar{\varepsilon})$ we have that

$$\Pi(u_\varepsilon) \ > \ (1-\varepsilon)\Pi(u^*) + \varepsilon\Pi(u^\tau) \ = \ \Pi(u^\tau) \tag{45}$$

and this contradicts the optimality of u^τ. □

This simple result justifies the following algorithm to solve the original problem:

`step 1` check if there exists any solution to the *exit time equation* (37);

`step 2` if there are no solutions to the equation (37), then the optimal advertising policy is the constant u^*;

`step 3` if τ is a solution of this equation, then compare $\Pi(u^*)$ with $\Pi(u^\tau)$;

`step 4` repeat `step 3` for all the solutions of the equation (37): the control which maximizes the objective functional is the optimal one.

From a computational viewpoint, the algorithm can be implemented straightforwardly and it solves our original problem. Nevertheless, it does not provide us with an economic information satisfactory enough. We wish to characterize the optimal decision analytically, with respect to the values of the problem parameters. In most situations we are able to provide such a characterization and some economic justification.

The first result in this direction concerns the *high interference* case. It is not surprising that in this situation the optimal solution is an optimal strategy to exit from the market.

Theorem 3 *If* $\zeta \ > \ \dfrac{\delta + \rho}{2\delta + \rho}\gamma u^*$ *then the exit time equation has a unique solution* τ^**, and the control* u^{τ^*} *is optimal for the original problem.*

Proof. Let us define the function

$$\varphi(T) \ = \ \alpha - \frac{\gamma u^* - \zeta}{\delta} - \frac{\gamma u^*}{\delta}\left(\frac{\zeta}{\gamma u^*} - \frac{\delta + \rho}{2\delta + \rho}\right)e^{\delta T} + \frac{\gamma u^*}{2\delta + \rho}e^{-(\delta + \rho)T} , \tag{46}$$

so that the exit time equation (37) is equivalent to $\varphi(T) = 0$. We notice that $\varphi(0) = \alpha > 0$. Moreover, under the Theorem assumption, this function is decreasing in T and $\lim_{T\mapsto\infty} \varphi(T) = -\infty$. Therefore, equation (37) has a unique solution which we denote by τ^*. Moreover, under these conditions we can apply a sufficiency result [16, p.145, Th.13, and p.146, Note 26] for the free final time optimal control problem (actually, the function $F(T)$ introduced in the sufficient conditions is in our case $\pi\beta\varphi(T)$). From it we can obtain that u^{τ^*} is the optimal control for all problems with variable final time, constrained to any bounded interval $[0, T]$, $T > \tau^*$. Therefore the control u^{τ^*} is optimal for the variable final time problem without any constraint on the final time. Using Lemma 2 we can conclude that u^{τ^*} is optimal for the original problem (8) too. □

On the other hand it is quite surprising that, in the *low interference* case the constant control u^* is not always optimal. Actually, we can prove that, if the initial goodwill is sufficiently high, then u^* is optimal. But, when the initial goodwill is small, both kinds of solution (u^* and u^τ) may possibly be optimal, and in order to solve completely the problem we have to compare the two different values of the objective functional $\Pi(u^*)$ and $\Pi\left(u^{\tau^*}\right)$. This fact is described in the following two Theorems.

Theorem 4 *If* $\zeta = \dfrac{\delta+\rho}{2\delta+\rho}\gamma u^*$ *then the exit time equation has a unique solution* τ^* *if and only if* $\alpha < \dfrac{\gamma u^*}{2\delta+\rho}$ *and the control* u^{τ^*} *is optimal for the original problem. Otherwise, if* $\alpha \geq \dfrac{\gamma u^*}{2\delta+\rho}$*, the exit time equation has no solutions and the optimal control of the original problem is* u^**.*

Proof. We notice that now the function (46) is

$$\varphi(T) \;=\; \alpha - \frac{\gamma u^* - \zeta}{\delta} + \frac{\gamma u^*}{2\delta+\rho} e^{-(\delta+\rho)T}. \tag{47}$$

From this representation, we verify easily that φ is strictly decreasing, $\varphi(0) = \alpha > 0$, and $\lim\limits_{T\mapsto\infty} \varphi(T) = \alpha - \dfrac{\gamma u^* - \zeta}{\delta}$. Therefore, the equation (37) has a unique solution if and only if

$$\alpha \;<\; \frac{\gamma u^* - \zeta}{\delta} \;=\; \frac{\gamma u^*}{2\delta+\rho}, \tag{48}$$

and we can apply a sufficiency result [16, p.145, Th.13] for the free final time optimal control problem. On the other hand, if (48) is not satisfied, the problem (18) has no solutions and this implies that u^* must be the optimal solution of the original control problem. □

Theorem 5 *If* $\zeta < \dfrac{\delta+\rho}{2\delta+\rho}\gamma u^*$ *then the exit time equation has solutions if and only if*

$$\alpha \;\le\; \hat{\alpha}(\zeta, u^*) \;=\; \frac{\gamma u^* - \zeta}{\delta} - \frac{\gamma u^*}{\delta}\left(1 - \frac{\zeta}{\gamma u^*}\Big/\frac{\delta+\rho}{2\delta+\rho}\right)^{\frac{\delta+\rho}{2\delta+\rho}} . \tag{49}$$

Therefore if this inequality is not satisfied, then u^* *is the optimal control for the original problem.*

Proof. We notice that the function (46) under these assumptions is convex because $\varphi''(T) > 0$ for all $T \in [0, +\infty)$. Therefore, this function reaches its minimum at the time $T_{\min}$ characterized by the equation $\varphi'(T_{\min}) = 0$. By a direct computation we obtain that

$$T_{\min} \;=\; -\frac{1}{2\delta+\rho}\ln\left(1 - \frac{\zeta}{\gamma u^*}\Big/\frac{\delta+\rho}{2\delta+\rho}\right), \tag{50}$$

hence the minimum value is

$$\varphi(T_{\min}) \;=\; \alpha - \hat{\alpha}(\zeta, u^*) . \tag{51}$$

The exit time equation has solutions if and only if $\varphi(T_{\min}) \le 0$, which is equivalent to (49). Therefore, when the inequality (49) does not hold, no control satisfies the necessary conditions for problem (18), hence the control u^* is optimal for the original problem (8). □

We observe that the function $\hat{\alpha}(\zeta, u^*)$, on the right hand side of inequality (49), takes the values

$$\hat{\alpha}(0, u^*) \;=\; 0 \qquad \text{and} \qquad \hat{\alpha}\left(\frac{\delta+\rho}{2\delta+\rho}\gamma u^*, u^*\right) \;=\; \frac{\gamma u^*}{2\delta+\rho}, \tag{52}$$

at the zero and the low level border interference points, and is strictly increasing in ζ, in fact

$$\frac{\partial \hat{\alpha}(\zeta, u^*)}{\partial \zeta} \;>\; 0 . \tag{53}$$

The monotonicity is not surprising: the higher the interference is, the higher the initial goodwill value has to be for the constant control u^* to be optimal.

The difficulties in the Theorem 3 come from the hypotheses of the sufficient conditions for the free final time optimal control problem (see [16, p.145, Th.13]). Under the assumptions of Theorem 3 we cannot apply the sufficient conditions, hence we do not know if the two controls that satisfy the necessary conditions for problem (18) are really optimal. The analysis in Theorem 3 leaves some uncertainy if the interference is small and the initial goodwill value is small too, we cannot decide which policy is optimal in general, without comparing the objective fuctional values at some special controls. On the other hand, in the proof of Theorem 3 we have seen that the exit final time equation can have at most two solutions, and we can reject the second as irrelevant when it exists. Therefore in the last step of the algorithm of the previous Section we have to compare at most two solutions, which is a very simple task from a numerical point of view.

7 Conclusion and directions for future research

This paper aims to be a first step in the consideration of advertising interference phenomena in the Nerlove and Arrow's model framework, with the assumption that interference wipes out part of a firm advertising effort, as done also in [1]. Such assumption introduces the possibility of negative goodwill values, which in turn poses interpretation questions in the original model and analytical difficulties.

Here, we propose a way to deal with this situation and we prove that, under suitable assumptions, we can characterize the optimal advertising policies. In our analysis, we consider only a firm and we see the actions of the other competitors as a constant exogenous interference. The further step in the research is to consider a competitive setting, where each player advertises to increase the goodwill of his product and to decrease the goodwill of the competitors' ones.

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