

# Exponential stability of fast driven systems, with an application to celestial mechanics



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## ABSTRACT

We construct a normal form suited to *fast driven systems*. We call so systems including actions  $I$ , angles  $\psi$ , and one fast coordinate  $y$ , moving under the action of a vector-field  $N$  depending only on  $I$  and  $y$  and with vanishing  $I$ -components. In the absence of the coordinate  $y$ , such systems have been extensively investigated and it is known that, after a small perturbing term is switched on, the normalised actions  $I$  turn to have exponentially small variations compared to the size of the perturbation. We obtain the same result of the classical situation, with the additional benefit that no trapping argument is needed, as no small denominator arises. We use the result to prove that, in the three-body problem, the level sets of a certain function called *Euler integral* have exponentially small variations in a short time, closely to collisions.

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## 1. Description of the results

We consider a  $(n+1+m)$ -dimensional vector-field  $N$  which, expressed in local coordinates  $(I, y, \psi) \in \mathbb{P} = \mathbb{I} \times \mathbb{Y} \times \mathbb{T}^m$  (where  $\mathbb{I} \subset \mathbb{R}^n$ ,  $\mathbb{Y} \subset \mathbb{R}$  are open and connected;  $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$  is the standard torus), has the form

$$N(I, y) = v(I, y)\partial_y + \omega(I, y)\partial_\psi. \quad (1)$$

The motion equations of  $N$

$$\begin{cases} \dot{I} = 0 \\ \dot{y} = v(I, y) \\ \dot{\psi} = \omega(I, y) \end{cases}$$

can be integrated in cascade:

$$\begin{cases} I(t) = I_0 \\ y(t) = \eta(I_0, t) \\ \psi(t) = \psi_0 + \int_{t_0}^t \omega(I_0, \eta(I_0, t')) dt' \end{cases} \quad (2)$$

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with  $\eta(I_0, \cdot)$  being the general solution of the one-dimensional equation  $\dot{y}(t) = v(I_0, y)$ . This formula shows that along the solutions of  $N$  the coordinates  $I$  (“actions”) remain constant, while the motion of the coordinates  $\psi$  (“angles”) is coupled with the motion of the “driving” coordinate  $y$ . We assume that  $v$  is suitably far from vanishing (for the problem considered in the paper  $|v|$  has a positive lower bound). It is to be noted that, without further assumptions on the function  $v$  (like, for example, of being “small”, or having a stationary point) nothing prevents to the  $y$  coordinate to move *fast*. For this reason – with slight abuse due to the fact that fastness may nowise occur – we refer to the solutions in (2) as *fast driven system*. The main risk of such kind of system is that the solution  $q(t) = (I(t), y(t), \psi(t))$  of  $N$  in (2) leaves the domain  $\mathbb{P}$  at a finite time. It is then convenient to define the *exit time from  $\mathbb{P}$  under  $N$* , or, more in general, the *exit time from a given  $W \subseteq \mathbb{P}$  under the vector-field  $X$* , and denote it as  $t_{\text{ex}}^{X,W}$ , the (possibly infinite) first time that  $q(t)$  leaves  $W$ .

Let us now replace the vector-field  $N(I, y)$  with a new vector-field of the form

$$X(I, y, \psi) = N(I, y) + P(I, y, \psi) \tag{3}$$

where the “perturbation”

$$P = P_1(I, y, \psi)dI + P_2(I, y, \psi)dy + P_3(I, y, \psi)d\psi$$

is, in some sense, “small” (see the next section for precise statements). Let  $t_{\text{ex}}^{X,W}$  be the exit time from  $W$  under  $X$ , and let  $\epsilon$  be a uniform upper bound for the absolute value of  $P_1$  on  $W$ . Then, one has a linear-in-time *a-priori* bound for the variations of  $I$ , as follows

$$|I(t) - I(0)| \leq \epsilon t \quad \forall t : |t| < t_{\text{ex}}^{X,W} \quad W \subseteq \mathbb{P}. \tag{4}$$

We are interested in improving the bound (4). To the readers who are familiar with Kolmogorov–Arnold–Moser (KAM) or Nekhoroshev theories, this kind of problems is well known: see [3,24,38,44], or [11,22,25,29,47] for applications to realistic models. Those are theories originally formulated for Hamiltonian vector-fields (next extended to more general ODEs), hence, in particular, with  $n = m$  and the coordinate  $y$  absent. In those cases the unperturbed motions of the coordinates  $(I, \psi)$  are

$$I(t) = I_0, \quad \psi(t) = \psi_0 + \omega(I_0)t \tag{5}$$

and the properties of the motions after the perturbing term is switched on depend on the arithmetic properties of the frequency vector  $\omega(I_0)$ . Under suitable non-commensurability assumptions of  $\omega(I_0)$  (referred to as “Diophantine conditions”), KAM theory ensures the possibility of continuing the unperturbed motions (5) for all times. Conversely, if  $\omega(I)$  satisfies, on an open set, an analytic property known as “steepness” (which is satisfied, e.g., if  $\omega$  does not vanish and moreover if it is the gradient of a convex function), Nekhoroshev theory allows to infer – for *all* orbits – a bound as in (4), with  $e^{-C/\epsilon^a}$  replacing  $\epsilon$  and  $t_{\text{ex}}^{X,W} = e^{C/\epsilon^b}$ , with suitable  $a, b, C > 0$ . It is to be remarked that in the Nekhoroshev regime the exponential scale of  $t_{\text{ex}}^{X,W}$  is an intrinsic consequence of steepness, responsible of a process known as “capture in resonance”. In the case considered in the paper such phenomenon does not seem to exist and hence the exit time  $t_{\text{ex}}^{X,W}$  has no reason to be long. Nevertheless, motivated by an application to celestial mechanics described below, we are interested with replacing  $\epsilon$  in (4) with a smaller number. We shall prove the following result (note that steepness conditions are not needed here).

**Theorem A.** *Let  $X = N + P$  be real-analytic, where  $N$  is as in (1), with  $v \not\equiv 0$ . Under suitable “smallness” assumptions involving  $\omega, \partial\omega, \partial v$  and  $P$ , the bound in (4) holds with  $e^{-C/\epsilon^a}$  replacing  $\epsilon$ , with a suitable  $a, C > 0$ .*

A quantitative statement of [Theorem A](#) is given in [Theorem 2.1](#). In addition, in view of our application, we also discuss a version to the case when analyticity in  $\psi$  fails; this is [Theorem 2.2](#).

To describe how we shall use [Theorem A](#) (more precisely, [Theorem 2.2](#)), we make a digression on the three-body problem and the *renormalizable integrability* of the simply averaged Newtonian potential [40]. The Hamiltonian governing the motions of a three-body problem in the plane where the masses are 1,  $\mu$  and  $\kappa$ , is (see, e.g., [18])

$$H_{3b} = \left(1 + \frac{1}{\kappa}\right) \frac{\|\mathbf{y}\|^2}{2} + \left(1 + \frac{1}{\mu}\right) \frac{\|\mathbf{y}'\|^2}{2} - \frac{\kappa}{\|\mathbf{x}\|} - \frac{\mu}{\|\mathbf{x}'\|} - \frac{\kappa\mu}{\|\mathbf{x} - \mathbf{x}'\|} + \mathbf{y} \cdot \mathbf{y}'$$

where  $\mathbf{y}, \mathbf{y}' \in \mathbb{R}^2$ ;  $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ , with  $\mathbf{x} \neq 0 \neq \mathbf{x}'$  and  $\mathbf{x} \neq \mathbf{x}'$ , are impulse-position coordinates;  $\|\cdot\|$  denotes the Euclidean norm and the gravity constant has been chosen equal to 1, by a proper choice of the units system. We rescale

$$(\mathbf{y}', \mathbf{y}) \rightarrow \frac{\kappa^2}{1 + \kappa}(\mathbf{y}', \mathbf{y}), \quad (\mathbf{x}', \mathbf{x}) \rightarrow \frac{1 + \kappa}{\kappa^2}(\mathbf{x}', \mathbf{x})$$

multiply the Hamiltonian by  $\frac{1+\kappa}{\kappa^3}$  and obtain

$$H_{3b}(\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x}) = \frac{\|\mathbf{y}\|^2}{2} - \frac{1}{\|\mathbf{x}\|} + \delta \left( \frac{\|\mathbf{y}'\|^2}{2} - \frac{\alpha}{\|\mathbf{x} - \mathbf{x}'\|} - \frac{\beta}{\|\mathbf{x}'\|} \right) + \gamma \mathbf{y} \cdot \mathbf{y}' \tag{6}$$

with

$$\alpha := \frac{\mu^2(1 + \kappa)}{\kappa(1 + \mu)}, \quad \beta := \frac{\mu^2(1 + \kappa)}{\kappa^2(1 + \mu)}, \quad \gamma := \frac{\kappa}{1 + \kappa}, \quad \delta := \frac{\kappa(1 + \mu)}{\mu(1 + \kappa)}.$$

In order to simplify the analysis a little bit, we introduce a main assumption. The Hamiltonian  $H_{3b}$  in (6) includes the Keplerian term

$$\frac{\|\mathbf{y}\|^2}{2} - \frac{1}{\|\mathbf{x}\|} = -\frac{1}{2A^2}. \tag{7}$$

We assume that this term is “leading” in the Hamiltonian. By averaging theory, this assumption allows us to replace (at the cost of a small error)  $H_{3b}$  by its  $\ell$ -average

$$\bar{H} = -\frac{1}{2A^2} + \delta H \tag{8}$$

where  $\ell$  is the mean anomaly associated to (7), and<sup>1</sup>

$$H := \frac{\|\mathbf{y}'\|^2}{2} - \alpha U - \frac{\beta}{\|\mathbf{x}'\|} \tag{9}$$

with

$$U := \frac{1}{2\pi} \int_0^{2\pi} \frac{d\ell}{\|\mathbf{x}' - \mathbf{x}(\ell)\|}$$

being the “simply<sup>2</sup> averaged Newtonian potential”. We recall that the mean anomaly  $\ell$  is defined as the area spanned by  $\mathbf{x}$  on the Keplerian ellipse generated by (7) relatively to the perihelion  $\mathbf{P}$  of the ellipse, in  $2\pi$  units. From now on we focus on the motions of the averaged Hamiltonian (9), bypassing any quantitative statement concerning the averaging procedure, as this would lead much beyond the purposes of the paper.<sup>3</sup>

<sup>1</sup> Remark that  $\mathbf{y}(\ell)$  has vanishing  $\ell$ -average so that the last term in (6) does not survive.

<sup>2</sup> Here, “simply” is used as opposed to the more familiar “doubly” averaged Newtonian potential, most often encountered in the literature; e.g. [14,15,18,27,39].

<sup>3</sup> As we consider a region in phase space close where  $\mathbf{x}'$  is very close to the instantaneous Keplerian orbit of  $\mathbf{x}$ , quantifying the values of the mass parameters and the distance which allow for the averaging procedure is a delicate (even though crucial) question, which, by its nature, demands careful use of regularisations. Due to the non-trivial underlying analysis, we choose to limit ourselves to point out that the renormalizable integrability of the Newtonian potential has a nontrivial dynamical impact on the simply averaged three-body problem, which explain the existence of the motions herewith discussed, which would not be justified otherwise.

Neglecting the first term in (8), which is an inessential additive constant for  $\bar{H}$  and reabsorbing the constant  $\delta$  with a time change, we are led to look at the Hamiltonian  $H$  in (9). We denote as  $\mathbb{E}$  the Keplerian ellipse generated by Hamiltonian (7), for negative values of the energy. Without loss of generality, assume  $\mathbb{E}$  is not a circle and<sup>4</sup>  $A = 1$ . Remark that, as the mean anomaly  $\ell$  is averaged out, we lose any information concerning the position of  $\mathbf{x}$  on  $\mathbb{E}$ , so we shall only need two couples of coordinates for determining the shape of  $\mathbb{E}$  and the vectors  $\mathbf{y}'$ ,  $\mathbf{x}'$ . These are:

- the “Delaunay couple”  $(G, g)$ , where  $G$  is the Euclidean length of  $\mathbf{x} \times \mathbf{y}$  and  $g$  detects the perihelion. We remark that  $g$  is measured with respect to  $\mathbf{x}'$  (instead of with respect to a fixed direction), as the  $SO(2)$  reduction we use a rotating frame which moves with  $\mathbf{x}'$  (compare the formulae in (66) below);
- the “radial–polar couple”  $(R, r)$ , where  $r := \|\mathbf{x}'\|$  and  $R := \frac{\mathbf{y}' \cdot \mathbf{x}'}{\|\mathbf{x}'\|}$ .

Using the coordinates above, the Hamiltonian in (9) becomes

$$H(R, G, r, g) = \frac{R^2}{2} + \frac{(C - G)^2}{2r^2} - \alpha U(r, G, g) - \frac{\beta}{r} \tag{10}$$

where  $C = \|\mathbf{x} \times \mathbf{y} + \mathbf{x}' \times \mathbf{y}'\|$  is the total angular momentum of the system, and we have assumed  $\mathbf{x} \times \mathbf{y} \parallel \mathbf{x}' \times \mathbf{y}'$ , so that  $\|\mathbf{x}' \times \mathbf{y}'\| = C - \|\mathbf{x} \times \mathbf{y}\| = C - G$ .

The Hamiltonian (10) is now wearing 2 degrees-of-freedom. As the energy is conserved, its motions evolve on the 3-dimensional manifolds  $\mathcal{M}_c = \{H = c\}$ . On each of such manifolds the evolution is associated to a 3-dimensional vector-field  $X_c$ , given by the velocity field of some triple of coordinates on  $\mathcal{M}_c$ . As an example, one can take the triple  $(r, G, g)$ , even though a more convenient choice will be done below. To describe the motions we are looking for, we need to recall a remarkable property of the function  $U$ , pointed out in [40]. First of all, one has to note that  $U$  is integrable, as it is a function of  $(r, G, g)$  only. But the main point is that there exists a function  $F$  of two arguments such that

$$U(r, G, g) = F(E(r, G, g), r) \tag{11}$$

where

$$E(r, G, g) = G^2 + r\sqrt{1 - G^2} \cos g. \tag{12}$$

The function  $E$  is referred to as *Euler integral*, and we express (11) by saying that  $U$  is *renormalizable integrability via the Euler integral*. Such circumstance implies that the level sets of  $E$ , namely the curves

$$G^2 + r\sqrt{1 - G^2} \cos g = \mathcal{E} \tag{13}$$

are also level sets of  $U$ . On the other hand, the phase portrait of (13) keeping  $r$  fixed is completely explicit and has been studied in [41]. We recall it now. Let us fix (by periodicity of  $g$ ) the strip  $[-\pi, \pi] \times [-1, 1]$ . For  $0 < r < 1$  or  $1 < r < 2$  it includes two minima  $(\pm\pi, 0)$  on the  $g$ -axis; two symmetric maxima on the  $G$ -axis and one saddle point at  $(0, 0)$ . When  $r > 2$  the saddle point disappears and  $(0, 0)$  turns to be a maximum. The phase portrait includes two separatrices when  $0 < r < 1$  or  $1 < r < 2$ ; one separatrix if  $r > 2$ . These are the level sets

$$\begin{cases} \mathcal{S}_0(r) = \{\mathcal{E} = r\}, & 0 < r < 1, 1 < r < 2 \\ \mathcal{S}_1(r) = \{\mathcal{E} = 1\}, & 0 < r < 1, 1 < r < 2, r > 2 \end{cases}$$

with  $\mathcal{S}_0(r)$  being the separatrix through the saddle;  $\mathcal{S}_1(r)$  the level set through circular orbits. Rotational motions in between  $\mathcal{S}_0(r)$  and  $\mathcal{S}_1(r)$ , do exist only for  $0 < r < 1$ . The minima and the maxima are surrounded by librational motions and different motions (librations about different equilibria or rotations) are separated by  $\mathcal{S}_0(r)$  and  $\mathcal{S}_1(r)$ . All of this is represented in Fig. 1.

<sup>4</sup> We can do this as the Hamiltonian  $H_{3b}$ , rescale by a factor  $\beta^{-2}$  as  $(\mathbf{y}', \mathbf{y}) \rightarrow \beta^{-1}(\mathbf{y}', \mathbf{y})$  and  $(\mathbf{x}', \mathbf{x}) \rightarrow \beta^2(\mathbf{x}', \mathbf{x})$ .

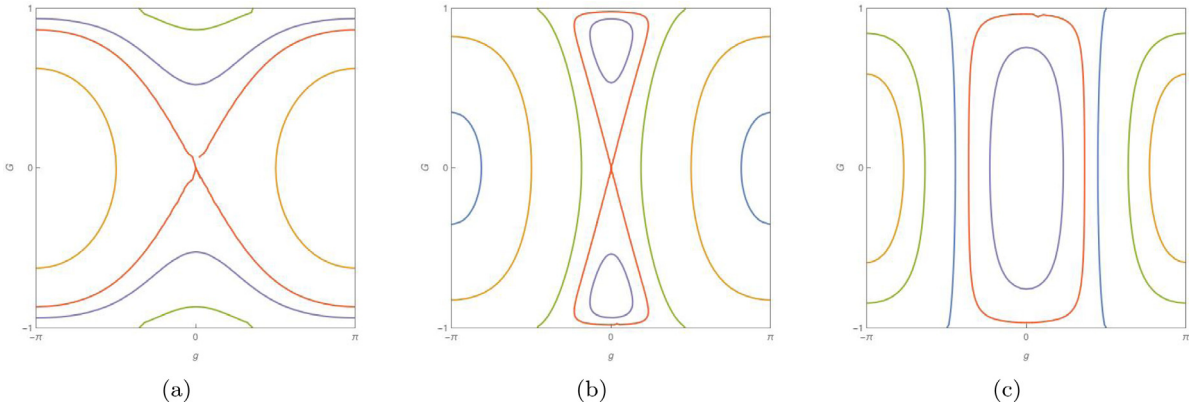


Fig. 1. Sections, at r fixed, of the level surfaces of E. (a)  $0 < r < 1$ ; (b)  $1 < r < 2$ ; (c)  $r > 2$ .

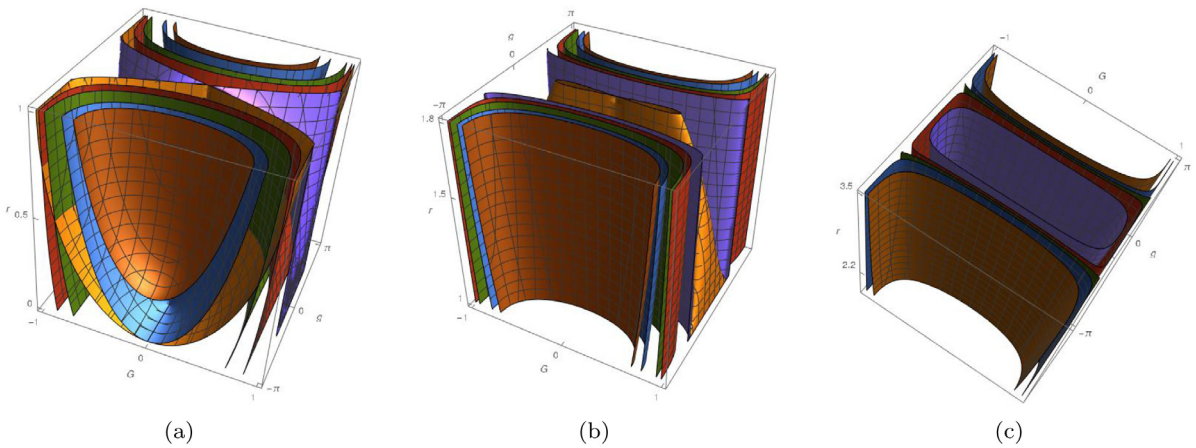


Fig. 2. Logs of the level surfaces of E in the space  $(g, G, r)$ . (a)  $0 < r < 1$ ; (b)  $1 < r < 2$ ; (c)  $r > 2$ .

In Fig. 2 the same level sets are drawn in the 3-dimensional space  $(r, G, g)$ . The spatial visualisation turns out to be useful for the purposes of the paper, as the coordinate  $r$ , which stays fixed under  $E$ , is instead moving under  $H$ , due to its dependence on  $R$ ; see (10). We denote as  $\mathcal{S}_0$  the union of all the  $\mathcal{S}_0(r)$  with  $0 \leq r \leq 2$ . It is to be noted that, while  $E$  is perfectly defined along  $\mathcal{S}_0$ ,  $U$  is not so. Indeed, as

$$\mathcal{S}_0(r) = \left\{ (G, g) : G^2 + r\sqrt{1 - G^2} \cos g = r, \quad -1 \leq G \leq 1, \quad g \in \mathbb{T} \right\} \quad 0 \leq r < 2 \quad (14)$$

we have<sup>5</sup>  $U(r, G, g) = \infty$  for  $(G, g) \in \mathcal{S}_0(r)$ , for all  $0 \leq r \leq 2$ .

The natural question now raises whether any of the  $\mathcal{E}$ -levels in Fig. 2 is an “approximate” invariant manifold for the Hamiltonian  $H$  in (10). In [42] and [16] a positive answer has been given for case  $r > 2$ , corresponding to panels (c). In this paper, we want to focus on motions close to  $\mathcal{S}_0$  with  $r$  in a left neighbourhood of 2 (panels (b)). Such portion of phase space is denoted as  $\mathcal{C}$ . By the discussion above, motions in  $\mathcal{C}$  are to be understood as “quasi-collisional”.

To state our result, we denote as  $r_s(A)$  the value of  $r$  such that the area encircled by  $\mathcal{S}_0(r_s(A))$  is  $A$ . Then the set  $\{\exists A : r = r_s(A)\}$  corresponds to  $\mathcal{S}_0$ . We prove:

<sup>5</sup> Rewriting (14) as

$$r = \frac{G^2}{1 - \sqrt{1 - G^2} \cos g}$$

tells us that  $(G, g) \in \mathcal{S}_0(r)$  if and only if  $\mathbf{x}'$  occupies in the ellipse  $\mathbb{E}$  the position with true anomaly  $\nu = \pi - g$ .

**Theorem B.** *Inside the region  $\mathcal{C}$  there exists an open set  $W$  such that along any motion with initial datum in  $W$ , for all  $t$  with  $|t| \leq t_{\text{ex}}^{X,W}$ , the ratio between the absolute variations of the Euler integral  $E$  from time 0 to time  $t$ , for all  $|t| \leq t_{\text{ex}}^{X,W}$ , and the a-priori bound  $\epsilon t$  (where  $\epsilon := |P_1|_\infty$ , with  $P_1$  being the action component of the vector-field) does not exceed  $Ce^{-L^3/C}$ , provided that the initial value of  $r$  is  $e^{-L}$  away from  $r_s(A)$ , with  $L > 0$  sufficiently large.*

The proof of [Theorem B](#), fully given in the next section, relies on a careful choice of coordinates  $(A, y, \psi)$  on  $\mathcal{M}_c$ , where  $y$  is diffeomorphic to  $r$ , while  $(A, \psi)$  are the action–angle coordinates of  $E(r, \cdot, \cdot)$ , such that the associated vector-field has the form in [\(3\)](#) with  $n = m = 1$ . The diffeomorphism  $r \rightarrow y$  allows  $X_c$  to keep its regularity upon  $\mathcal{S}_0$ .

Before switching to proofs, we recall how the theme of collisions in  $N$ -body problems (with  $N \geq 3$ ) has been treated so far. As the literature in the field is countless, by no means we claim completeness. In the late 1890s H. Poincaré [\[43\]](#) conjectured the existence of special solutions in a model of the three-body problem usually referred to as planar, circular, restricted three-body problem (PCRTBP). According to Poincaré’s conjecture, when one of the primaries has a small mass  $\mu$ , the orbit of an infinitesimal body approaching a close encounter with the small primary consists of two Keplerian arcs glueing so as to form a cusp. These solutions were named by him *second species solutions*, and their existence has been next proved in [\[4–8,26,30\]](#). In the early 1900s, J. Chazy classified all the possible final motions of the three-body problem, including the possibility of collisions [\[12\]](#). The study was reconsidered in [\[1,2\]](#). After the advent of KAM theory, the existence of almost-collisional quasi-periodic orbits was proven [\[13,17,48\]](#). The papers [\[19,20,31–34,45,46\]](#) deal with rare occurrence of collisions or the existence of chaos in the proximity of collisions. In [\[23\]](#) it is proved that for PCRTBP there exists an open set in phase space of fixed measure, where the set of initial points which lead to collision is  $O(\mu^\alpha)$  dense with some  $0 < \alpha < 1$ . In [\[28\]](#) it is proved that, after collision regularisation, PCRTBP is integrable in a neighbourhood of collisions. In [\[9,10\]](#) the result has been recently extended to the spatial version, often denoted SCRTBP.

## 2. A Normal Form Theorem for fast driven systems

In [Sections 2.1–2.4](#) we state and prove a Normal Form Theorem (NFT) for real-analytic systems. For the purpose of the paper, in [Section 2.5](#) we generalise the result, allowing the dependence on the angular coordinate  $\psi$  to be just  $C^{\ell_*}$  ( $\ell_* \in \mathbb{N}$ ), rather than holomorphic. In all cases, we limit to the case  $n = m = 1$ . Generalisations to  $n, m \geq 1$  are straightforward.

### 2.1. Weighted norms

Let us consider a 3-dimensional vector-field

$$(I, y, \psi) \in \mathbb{P}_{r,\sigma,s} := \mathbb{I}_r \times \mathbb{Y}_\sigma \times \mathbb{T}_s \rightarrow X = (X_1, X_2, X_3) \in \mathbb{C}^3$$

where  $\mathbb{I} \subset \mathbb{R}$ ,  $\mathbb{Y} \subset \mathbb{R}$  are open and connected;  $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ , which has the form [\(3\)](#). As usual, if  $A \subset \mathbb{R}$  and  $r, s > 0$ , the symbols  $A_r, \mathbb{T}_s$  denote the complex  $r, s$ -neighbourhoods of  $A, \mathbb{T}$ :

$$A_r := \bigcup_{x \in A} B_r(x), \quad \mathbb{T}_s := \{ \psi = \psi_1 + i\psi_2 : \psi_1 \in \mathbb{T}, \psi_2 \in \mathbb{R}, |\psi_2| < s \},$$

with  $B_r(x)$  being the complex ball centred at  $x$  with radius  $r$ . We assume each  $X_i$  to be holomorphic in  $\mathbb{P}_{r,\sigma,s}$ , meaning the it has a finite weighted norm defined below. If this holds, we simply write  $X \in \mathcal{O}_{r,\sigma,s}^3$ .

For functions  $f : (I, y, \psi) \in \mathbb{I}_r \times \mathbb{Y}_\sigma \times \mathbb{T}_s \rightarrow \mathbb{C}$ , we write  $f \in \mathcal{O}_{r,\sigma,s}$  if  $f$  is holomorphic in  $\mathbb{P}_{r,\sigma,s}$ . We let

$$\|f\|_u := \sum_{k \in \mathbb{Z}} \sup_{\mathbb{I}_r \times \mathbb{Y}_\sigma} |f_\kappa(I, y)| e^{|k|s} \quad u = (r, \sigma, s) \tag{15}$$

where

$$f = \sum_{k \in \mathbb{Z}} f_k(\mathbf{I}, y) e^{ik\psi}$$

is the Fourier series associated to  $f$  relatively to the  $\psi$ -coordinate. For  $\psi$ -independent functions or vector-fields we simply write  $\|\cdot\|_{r,\sigma}$ .

For vector-fields  $X : (\mathbf{I}, y, \psi) \in \mathbb{I}_r \times \mathbb{Y}_\sigma \times \mathbb{T}_s \rightarrow X = (X_1, X_2, X_3) \in \mathbb{C}^3$ , we write  $X \in \mathcal{O}_{r,\sigma,s}^3$  if  $X_i \in \mathcal{O}_{r,\sigma,s}$  for  $i = 1, 2, 3$ . We define the *weighted norms*

$$\|X\|_u^w := \sum_i w_i^{-1} \|X_i\|_u$$

where  $w = (w_1, w_2, w_3) \in \mathbb{R}_+^3$  are the *weights*. The weighted norm affords the following properties.

- Monotonicity:

$$\|X\|_u^w \leq \|X\|_{u'}^w, \quad \|X\|_u^{w'} \leq \|X\|_u^w \quad \forall u \leq u', w \leq w' \tag{16}$$

where  $u \leq u'$  means  $u_i \leq u'_i$  for  $i = 1, 2, 3$ .

- Homogeneity:

$$\|X\|_u^{\alpha w} = \alpha^{-1} \|X\|_u^w \quad \forall \alpha > 0. \tag{17}$$

### 2.2. The Normal Form Theorem

We now state the main result of this section. Observe that the nature of the system does not give rise to any non-resonance condition or ultraviolet cut-off. We name Normal Form Theorem the following

**Theorem 2.1** (NFT). *Let  $u = (r, \sigma, s)$ ;  $X = N + P \in \mathcal{O}_u^3$  and let  $w = (\rho, \tau, t) \in \mathbb{R}_+^3$ . Put*

$$Q := 3 \operatorname{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma}$$

and<sup>6</sup> assume that for some  $p \in \mathbb{N}$ ,  $s_2 \in \mathbb{R}_+$ , the following inequalities are satisfied:

$$0 < \rho < \frac{r}{8}, \quad 0 < \tau < e^{-s_2} \frac{\sigma}{8}, \quad 0 < t < \frac{s}{10} \tag{18}$$

and

$$\chi := \frac{\operatorname{diam}(\mathbb{Y}_\sigma)}{s_2} \left\| \frac{\partial_y v}{v} \right\|_{r,\sigma} \leq 1 \tag{19}$$

$$\theta_1 := 2 e^{s_2} \operatorname{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \frac{\tau}{t} \leq 1 \tag{20}$$

$$\theta_2 := 4 \operatorname{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_{\Gamma} v}{v} \right\|_{r,\sigma} \frac{\rho}{\tau} \leq 1$$

$$\theta_3 := 8 \operatorname{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_{\Gamma} \omega}{v} \right\|_{r,\sigma} \frac{\rho}{t} \leq 1 \tag{21}$$

$$\eta^2 := \max \left\{ \frac{\operatorname{diam}(\mathbb{Y}_\sigma)}{t} \left\| \frac{\omega}{v} \right\|_{r,\sigma}, 2^7 e^{2s_2} Q^2 (\|P\|_u^w)^2 \right\} < \frac{1}{p}. \tag{22}$$

Then, with

$$u_* = (r_*, \sigma_*, s_*), \quad r_* := r - 8\rho, \quad \sigma_* = \sigma - 8e^{s_2}\tau, \quad s_* = s - 10t$$

there exists a real-analytic change of coordinates  $\Phi_*$  such that  $X_* := \Phi_* X \in \mathcal{O}_{u_*}^3$  and  $X_* = N + P_*$ , with

$$\|P_*\|_{u_*}^w < 2^{-(p+1)} \|P\|_u^w.$$

<sup>6</sup>  $\operatorname{diam}(\mathcal{A})$  denotes diameter of the set  $\mathcal{A}$ .



**Remark 2.1** (Proof of Theorem A). Theorem 2.1 immediately implies Theorem A, with  $C = \min\{2^{-7}Q^{-2}e^{-2s_2}\varrho^2 \log 2, t/\text{diam}\Upsilon_\sigma\}$ ,  $a = 2$ , provided that  $\varrho := \frac{\epsilon^2}{(\|P\|_u^w)^2}$  is of “order one” with respect to  $\epsilon$ . The mentioned “smallness assumptions” correspond to conditions (18)-(21) and  $\|\frac{\omega}{v}\|_{r,\sigma} \ll (\|P\|_u^w)^2$ .

2.3. The step lemma

We denote as

$$e^{\mathcal{L}_Y} = \sum_{k \geq 0} \frac{\mathcal{L}_Y^k}{k!} \tag{23}$$

the formal Lie series associated to  $Y$ , where

$$[Y, X] = J_X Y - J_Y X, \quad (J_Z)_{ij} := \partial_j Z_i$$

denotes Lie brackets of two vector-fields, with

$$\mathcal{L}_Y := [Y, \cdot]$$

being the Lie operator.

**Lemma 2.1.** Let  $X = N + P \in \mathcal{O}_u^3$ , with  $u = (r, \sigma, s)$ ,  $N$  as in (36),  $s_1, s_2 > 0$ . Assume

$$\frac{\text{diam}(\Upsilon_\sigma)}{s_1} \left\| \frac{\omega}{v} \right\|_{r,\sigma} \leq 1, \quad \frac{\text{diam}(\Upsilon_\sigma)}{s_2} \left\| \frac{\partial_y v}{v} \right\|_{r,\sigma} \leq 1 \tag{24}$$

and that  $P$  is so small that

$$Q \|P\|_u^w < 1 \quad Q := 3 \text{diam}(\Upsilon_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma}, \quad w = (\rho, \tau, t) \tag{25}$$

Let  $\rho_*, \tau_*, t_*$  be defined via

$$\begin{aligned} \frac{1}{\rho_*} &= \frac{1}{\rho} - \text{diam}(\Upsilon_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_{r,\sigma} \left( \frac{1}{\tau} - e^{s_2} \text{diam}(\Upsilon_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \frac{1}{t} \right) \\ &\quad - \text{diam}(\Upsilon_\sigma) \left( \left\| \frac{\partial_1 \omega}{v} \right\|_{r,\sigma} + e^{s_2} \text{diam}(\Upsilon_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_{r,\sigma} \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \right) \frac{1}{t} \\ \frac{1}{\tau_*} &= \frac{e^{-s_2}}{\tau} - \text{diam}(\Upsilon_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \frac{1}{t} \\ t_* &= t \end{aligned} \tag{26}$$

and assume

$$w_* = (\rho_*, \tau_*, t_*) \in \mathbb{R}_+^3, \quad u_* = (r - 2\rho_*, \sigma - 2\tau_*, s - 3s_1 - 2t_*) \in \mathbb{R}_+^3. \tag{27}$$

Then there exists  $Y \in \mathcal{O}_{u_*+w_*}^3$  such that  $X_+ := e^{\mathcal{L}_Y} X \in \mathcal{O}_{u_*}^3$  and  $X_+ = N + P_+$ , with

$$\|P_+\|_{u_*}^{w_*} \leq \frac{2Q (\|P\|_u^w)^2}{1 - Q \|P\|_u^w} \tag{28}$$

In the next section, we shall use Lemma 2.1 in the following “simplified” form.



**Lemma 2.2** (Step Lemma). *If (24), (25) and (27) are replaced with*

$$\begin{aligned} 2 e^{s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \frac{\tau}{t} &\leq 1 \\ 4 \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_I v}{v} \right\|_{r,\sigma} \frac{\rho}{\tau} &\leq 1 \\ 8 \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_I \omega}{v} \right\|_{r,\sigma} \frac{\rho}{t} &\leq 1 \end{aligned} \tag{29}$$

$$\frac{\text{diam}(\mathbb{Y}_\sigma)}{t} \left\| \frac{\omega}{v} \right\|_{r,\sigma} \leq 1, \quad \frac{\text{diam}(\mathbb{Y}_\sigma)}{s_2} \left\| \frac{\partial_y v}{v} \right\|_{r,\sigma} \leq 1 \tag{30}$$

$$0 < \rho < \frac{r}{4}, \quad 0 < \tau < \frac{\sigma}{4} e^{-s_2}, \quad 0 < t < \frac{s}{5} \tag{31}$$

$$2Q \lll P \lll_u^w < 1 \tag{32}$$

then  $X_+ = N + P_+ \in \mathcal{O}_{u_+}^3$  and

$$\lll P_+ \lll_{u_+}^w \leq 8e^{s_2} Q (\lll P \lll_u^w)^2. \tag{33}$$

with

$$u_+ := (r - 4\rho, \sigma - 4\tau e^{s_2}, s - 5t).$$

**Proof.** The inequality in (30) guarantees that one can take  $s_1 = t$ , while the inequalities in (29) and (31) imply

$$\frac{1}{\rho_*} \geq \frac{1}{2\rho}, \quad \frac{1}{\tau_*} \geq \frac{e^{-s_2}}{2\tau}$$

whence, as  $t_* = t$ ,

$$w_* < 2e^{s_2} w, \quad u_* \geq u_+ > 0.$$

Then (33) is implied by (28), monotonicity and homogeneity (16)–(17), and the inequality in (32).  $\square$

To prove Lemma 2.1, we look for a change of coordinates which conjugates the vector-field  $X = N + P$  to a new vector-field  $X_+ = N_+ + P_+$ , where  $P_+$  depends on the coordinates  $I$  at higher orders. The procedure we follow is reminiscent of classical techniques of normal form theory, where one chooses the transformation so that  $X_+ = e^{\mathcal{L}Y} X$ , with the operator  $e^{\mathcal{L}Y}$  being defined as in (23). As in the classical case,  $Y$  will be chosen as the solution of a certain ‘‘homological equation’’ which allows to eliminate the first order terms depending on  $\psi$  of  $P$ . However, as stated in Lemma 2.1, differently from the classical situation, one can take  $N = N_+$ , which is another way of saying that it is possible to choose  $Y$  such in a way to solve

$$\mathcal{L}_N[Y] = P \tag{34}$$

regardless  $P$  has vanishing average or not — or, in other words, that *also the resonant terms* of the perturbing term will be killed. Note also that no ‘‘ultraviolet cut-off’’ is used. Eq. (34) is precisely what is discussed in Lemma 2.3 and Proposition 2.1.

Fix  $y_0 \in \mathbb{Y}$ ;  $v, \omega : \mathbb{I} \times \mathbb{Y} \rightarrow \mathbb{R}$ , with  $v \not\equiv 0$ . We define, formally, the operators  $\mathcal{F}_{v,\omega}$  and  $\mathcal{G}_{v,\omega}$  as acting on functions  $g : \mathbb{I} \times \mathbb{Y} \times \mathbb{T} \rightarrow \mathbb{R}$  as

$$\begin{aligned} \mathcal{F}_{v,\omega}[g](I, y, \psi) &:= \int_{y_0}^y \frac{g\left(I, \eta, \psi + \int_y^\eta \frac{\omega(I, \eta')}{v(I, \eta')} d\eta'\right)}{v(I, \eta)} d\eta \\ \mathcal{G}_{v,\omega}[g](I, y, \psi) &:= \int_{y_0}^y \frac{g\left(I, \eta, \psi + \int_y^\eta \frac{\omega(I, \eta')}{v(I, \eta')} d\eta'\right)}{v(I, \eta)} e^{-\int_y^\eta \frac{\partial_y v(I, \eta')}{v(I, \eta')} d\eta'} d\eta \end{aligned} \tag{35}$$

Observe that, when existing,  $\mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega}$  send zero-average functions to zero-average functions. The existence  $\mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega}$  is established by the following

**Lemma 2.3.** *If inequalities (24) hold, then*

$$\mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega} : \mathcal{O}_{r,\sigma,s} \rightarrow \mathcal{O}_{r,\sigma,s-s_1}$$

and

$$\|\mathcal{F}_{v,\omega}[g]\|_{r,\sigma,s-s_1} \leq \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{g}{v} \right\|_{r,\sigma,s}, \quad \|\mathcal{G}_{v,\omega}[g]\|_{r,\sigma,s-s_1} \leq e^{s^2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{g}{v} \right\|_{r,\sigma,s}$$

The proof of Lemma 2.3 is obvious from the definitions (35).

**Proposition 2.1.** *Let*

$$N = (0, v(\mathbf{I}, y), \omega(\mathbf{I}, y)), \quad Z = (Z_1(\mathbf{I}, y, \psi), Z_2(\mathbf{I}, y, \psi), Z_3(\mathbf{I}, y, \psi)) \tag{36}$$

belong to  $\mathcal{O}_{r,\sigma,s}^3$  and assume (24). Then the ‘‘homological equation’’

$$\mathcal{L}_N[Y] = Z \tag{37}$$

has a solution  $Y \in \mathcal{O}_{r,\sigma,s-3s_1}$  verifying

$$\|Y\|_{r,\sigma,s-3s_1}^{\rho_*, \tau_*, t_*} \leq \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \quad \|Z\|_{r,\sigma}^{\rho, \tau, t} \tag{38}$$

with  $\rho_*, \tau_*, t_*$  as in (26).

**Proof.** We expand  $Y_j$  and  $Z_j$  along the Fourier basis

$$Y_j(\mathbf{I}, y, \psi) = \sum_{k \in \mathbb{Z}} Y_{j,k}(\mathbf{I}, y) e^{ik\psi}, \quad Z_j(\mathbf{I}, y, \psi) = \sum_{k \in \mathbb{Z}} Z_{j,k}(\mathbf{I}, y) e^{ik\psi}, \quad j = 1, 2, 3$$

Using

$$\mathcal{L}_N[Y] = [N, Y] = J_Y N - J_N Y$$

where  $(J_Z)_{ij} = \partial_j Z_i$  are the Jacobian matrices, we rewrite (37) as

$$\begin{aligned} Z_{1,k}(\mathbf{I}, y) &= v(\mathbf{I}, y) \partial_y Y_{1,k} + ik\omega(\mathbf{I}, y) Y_{1,k} \\ Z_{2,k}(\mathbf{I}, y) &= v(\mathbf{I}, y) \partial_y Y_{2,k} + (ik\omega(\mathbf{I}, y) - \partial_y v(\mathbf{I}, y)) Y_{2,k} - \partial_{\mathbf{I}} v(\mathbf{I}, y) Y_{1,k} \\ Z_{3,k}(\mathbf{I}, y) &= v(\mathbf{I}, y) \partial_y Y_{3,k} + ik\omega(\mathbf{I}, y) Y_{3,k} - \partial_{\mathbf{I}} \omega(\mathbf{I}, y) Y_{1,k} - \partial_y \omega(\mathbf{I}, y) Y_{2,k}. \end{aligned} \tag{39}$$

Regarding (39) as equations for  $Y_{j,k}$ , we find the solutions

$$\begin{aligned} Y_{1,k} &= \int_{y_0}^y \frac{Z_{1,k}(\mathbf{I}, \eta)}{v(\mathbf{I}, \eta)} e^{ik \int_y^\eta \frac{\omega(\mathbf{I}, \eta')}{v(\mathbf{I}, \eta')} d\eta'} d\eta \\ Y_{2,k} &= \int_{y_0}^y \frac{Z_{2,k}(\mathbf{I}, \eta) + \partial_{\mathbf{I}} v Y_{1,k}}{v(\mathbf{I}, \eta)} e^{\int_y^\eta \frac{ik\omega(\mathbf{I}, \eta') - \partial_y v(\mathbf{I}, \eta')}{v(\mathbf{I}, \eta')} d\eta'} d\eta \\ Y_{3,k} &= \int_{y_0}^y \frac{Z_{3,k}(\mathbf{I}, \eta) + \partial_{\mathbf{I}} \omega(\mathbf{I}, \eta) Y_{1,k} + \partial_y \omega(\mathbf{I}, \eta) Y_{2,k}}{v(\mathbf{I}, \eta)} e^{ik \int_y^\eta \frac{\omega(\mathbf{I}, \eta')}{v(\mathbf{I}, \eta')} d\eta'} d\eta \end{aligned}$$

multiplying by  $e^{ik\psi}$  and summing over  $k \in \mathbb{Z}$  we find

$$Y_1 = \mathcal{F}_{v,\omega}[Z_1]$$

$$\begin{aligned} Y_2 &= \mathcal{G}_{v,\omega}[Z_2] + \mathcal{G}_{v,\omega}[\partial_1 v Y_1], \\ Y_3 &= \mathcal{F}_{v,\omega}[Z_3] + \mathcal{F}_{v,\omega}[\partial_1 \omega Y_1] + \mathcal{F}_{v,\omega}[\partial_y \omega Y_2]. \end{aligned} \tag{40}$$

Then, by Lemma 2.3,

$$\begin{aligned} \|Y_1\|_{r,\sigma,s-s_1} &\leq \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \|Z_1\|_{r,\sigma,s} \\ \|Y_2\|_{r,\sigma,s-2s_1} &\leq e^{s^2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \|Z_2\|_{r,\sigma,s-2s_1} + e^{s^2} \text{diam}(\mathbb{Y}_\sigma)^2 \left\| \frac{1}{v} \right\|_{r,\sigma} \left\| \frac{\partial_1 v}{v} \right\|_{r,\sigma} \|Z_1\|_{r,\sigma,s} \\ \|Y_3\|_{r,\sigma,s-3s_1} &\leq \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \|Z_3\|_{r,\sigma,s-2s_1} + e^{s^2} \text{diam}(\mathbb{Y}_\sigma)^2 \left\| \frac{1}{v} \right\|_{r,\sigma} \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \|Z_2\|_{r,\sigma,s-2s_1} \\ &\quad + \text{diam}(\mathbb{Y}_\sigma)^2 \left\| \frac{1}{v} \right\|_{r,\sigma} \left( \left\| \frac{\partial_1 \omega}{v} \right\|_{r,\sigma} + e^{s^2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_{r,\sigma} \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \right) \|Z_1\|_{r,\sigma,s} \end{aligned}$$

Multiplying the inequalities above by  $\rho_*^{-1}, \tau_*^{-1}, t_*^{-1}$  respectively and taking the sum, we find (38), with

$$\begin{aligned} \frac{1}{\rho} &= \frac{1}{\rho_*} + e^{s^2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_{r,\sigma} \frac{1}{\tau_*} + \text{diam}(\mathbb{Y}_\sigma) \left( \left\| \frac{\partial_1 \omega}{v} \right\|_{r,\sigma} + e^{s^2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_{r,\sigma} \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \right) \frac{1}{t_*} \\ \frac{1}{\tau} &= \frac{e^{s^2}}{\tau_*} + e^{s^2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \frac{1}{t_*} \\ \frac{1}{t} &= \frac{1}{t_*}. \end{aligned}$$

We recognise that, under conditions (27),  $\rho_*, \tau_*, t_*$  in (26) solve the equations above.  $\square$

**Lemma 2.4.** *Let  $w < u \leq u_0; Y \in \mathcal{O}_{u_0}^3, W \in \mathcal{O}_u^3$ . Then*

$$\| \mathcal{L}_Y[W] \|_{u-w}^{u_0-u+w} \leq \| Y \|_{u-w}^w \| W \|_u^{u_0-u+w} + \| W \|_{u-w}^{u_0-u+w} \| Y \|_{u_0}^{u_0-u+w}.$$

**Proof.** One has

$$\begin{aligned} \| \mathcal{L}_Y[W] \|_{u-w}^{u_0-u+w} &= \| J_W Y - J_Y W \|_{u-w}^{u_0-u+w} \\ &\leq \| J_W Y \|_{u-w}^{u_0-u+w} + \| J_Y W \|_{u-w}^{u_0-u+w} \end{aligned}$$

Now,  $(J_W Y)_i = \partial_1 W_i Y_1 + \partial_y W_i Y_2 + \partial_\psi W_i Y_3$ , so, using Cauchy inequalities,

$$\begin{aligned} \| (J_W Y)_i \|_{u-w} &\leq \| \partial_1 W_i \|_{u-w} \| Y_1 \|_{u-w} + \| \partial_y W_i \|_{u-w} \| Y_2 \|_{u-w} + \| \partial_\psi W_i \|_{u-w} \| Y_3 \|_{u-w} \\ &\leq w_1^{-1} \| W_i \|_u \| Y_1 \|_{u-w} + w_2^{-1} \| W_i \|_u \| Y_2 \|_{u-w} + w_3^{-1} \| W_i \|_u \| Y_3 \|_{u-w} \\ &= \| Y \|_{u-w}^w \| W_i \|_u \end{aligned}$$

Similarly,

$$\|(J_Y W)_i\|_{u-w} \leq \|W\|_{u-w}^{u_0-u+w} \|Y_i\|_{u_0}.$$

Taking the  $u_0 - u + w$ -weighted norms, the thesis follows.  $\square$

**Lemma 2.5.** *Let  $0 < w < u \in \mathbb{R}^3$ ,  $Y \in \mathcal{O}_{u+w}^3$ ,  $W \in \mathcal{O}_u^3$ . Then*

$$\|\mathcal{L}_Y^k[W]\|_{u-w}^w \leq 3^k k! (\|Y\|_{u+w}^w)^k \|W\|_{u-w}^w.$$

**Proof.** We apply Lemma 2.4 with  $W$  replaced by  $\mathcal{L}_Y^{i-1}[W]$ ,  $u$  replaced by  $u - (i - 1)w/k$ ,  $w$  replaced by  $w/k$  and, finally,  $u_0 = u + w$ . With  $\|\cdot\|_i^w = \|\cdot\|_{u-i\frac{w}{k}}^w$ ,  $0 \leq i \leq k$ , so that  $\|\cdot\|_0^w = \|\cdot\|_u^w$  and  $\|\cdot\|_k^w = \|\cdot\|_{u-w}^w$ ,

$$\begin{aligned} \|\mathcal{L}_Y^i[W]\|_i^{w+w/k} &= \left\| [Y, \mathcal{L}_Y^{i-1}[W]] \right\|_i^{w+w/k} \\ &\leq \|Y\|_i^{w/k} \|\mathcal{L}_Y^{i-1}[W]\|_{i-1}^{w+w/k} + \|Y\|_{u+w}^{w+w/k} \|\mathcal{L}_Y^{i-1}[W]\|_i^{w+w/k}. \end{aligned}$$

Hence, de-homogenizing,

$$\begin{aligned} \frac{k}{k+1} \|\mathcal{L}_Y^i[W]\|_i^w &\leq k \frac{k}{k+1} \|Y\|_i^w \|\mathcal{L}_Y^{i-1}[W]\|_{i-1}^w + \frac{k^2}{(k+1)^2} \|Y\|_{u+w}^w \|\mathcal{L}_Y^{i-1}[W]\|_i^w \\ &\leq \frac{k^2}{k+1} \left(1 + \frac{1}{k+1}\right) \|Y\|_{u+w}^w \|\mathcal{L}_Y^{i-1}[W]\|_{i-1}^w \end{aligned}$$

Eliminating the common factor  $\frac{k}{k+1}$  and iterating  $k$  times from  $i = k$ , by Stirling, we get

$$\begin{aligned} \|\mathcal{L}_Y^k[W]\|_{u-w}^w &\leq k^k \left(1 + \frac{1}{k}\right)^k (\|Y\|_{u+w}^w)^k \|W\|_{u-w}^w \\ &\leq e^k k! (\|Y\|_{u+w}^w)^k \|W\|_{u-w}^w \\ &< 3^k k! (\|Y\|_{u+w}^w)^k \|W\|_{u-w}^w \end{aligned}$$

as claimed.  $\square$

**Proposition 2.2.** *Let  $0 < w < u$ ,  $Y \in \mathcal{O}_{u+w}^3$ ,*

$$q := 3\|Y\|_{u+w}^w < 1.$$

*Then the Lie series  $e^{\mathcal{L}_Y}$  defines an operator*

$$e^{\mathcal{L}_Y} : \mathcal{O}_u^3 \rightarrow \mathcal{O}_{u-w}^3$$

*and its tails*

$$e_m^{\mathcal{L}_Y} = \sum_{k \geq m} \frac{\mathcal{L}_Y^k}{k!}$$

*verify*

$$\|e_m^{\mathcal{L}_Y} W\|_{u-w}^w \leq \frac{q^m}{1-q} \|W\|_u^w \quad \forall W \in \mathcal{O}_u^3.$$

**Proof of Lemma 2.1.** We look for  $Y$  such that  $X_+ := e^{\mathcal{L}_Y} X$  has the desired properties.

$$\begin{aligned} e^{\mathcal{L}_Y} X &= e^{\mathcal{L}_Y} (N + P) = N + P + \mathcal{L}_Y N + e_2^{\mathcal{L}_Y} N + e_1^{\mathcal{L}_Y} P \\ &= N + P - \mathcal{L}_N Y + P_+ \end{aligned}$$

with  $P_+ = e_2^{\mathcal{L}^Y} N + e_1^{\mathcal{L}^Y} P$ . We choose  $Y$  so that the homological equation

$$\mathcal{L}_N Y = P$$

is satisfied. By Proposition 2.1, this equation has a solution  $Y \in \mathcal{O}_{r,\sigma,s-3s_1}^3$  verifying

$$q := 3 \|Y\|_{r,\sigma,s-3s_1}^{w_*} \leq 3 \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \|P\|_u^w = Q \|P\|_u^w < 1.$$

By Proposition 2.2, the Lie series  $e^{\mathcal{L}^Y}$  defines an operator

$$e^{\mathcal{L}^Y} : W \in \mathcal{O}_{u_*+w_*} \rightarrow \mathcal{O}_{u_*}$$

and its tails  $e_m^{\mathcal{L}^Y}$  verify

$$\begin{aligned} \|e_m^{\mathcal{L}^Y} W\|_{u_*}^{w_*} &\leq \frac{q^m}{1-q} \|W\|_{u_*+w_*}^{w_*} \\ &\leq \frac{(Q \|P\|_u^w)^m}{1-Q \|P\|_u^w} \|W\|_{u_*+w_*}^{w_*} \end{aligned}$$

for all  $W \in \mathcal{O}_{u_*+w_*}^3$ . In particular,  $e^{\mathcal{L}^Y}$  is well defined on  $\mathcal{O}_u^3 \subset \mathcal{O}_{u_*+w_*}^3$ , hence  $P_+ \in \mathcal{O}_{u_*}^3$ . The bounds on  $P_+$  are obtained as follows. Using the homological equation, one finds

$$\begin{aligned} \|e_2^{\mathcal{L}^Y} N\|_{u_*}^{w_*} &= \left\| \sum_{k=1}^{\infty} \frac{\mathcal{L}_Y^{k+1} N}{(k+1)!} \right\|_{u_*}^{w_*} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \|\mathcal{L}_Y^{k+1} N\|_{u_*}^{w_*} \\ &= \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \|\mathcal{L}_Y^k P\|_{u_*}^{w_*} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k!} \|\mathcal{L}_Y^k P\|_{u_*}^{w_*} \\ &\leq \frac{Q (\|P\|_u^w)^2}{1-Q \|P\|_u^w} \end{aligned} \tag{41}$$

The bound

$$\|e_1^{\mathcal{L}^Y} P\|_{u_*}^{w_*} \leq \frac{Q (\|P\|_u^w)^2}{1-Q \|P\|_u^w} \tag{42}$$

is even more straightforward.  $\square$

#### 2.4. Proof of the Normal Form Theorem

The proof of NFT is obtained – following [44] – via iterate applications of the Step Lemma. At the base step, we let<sup>7</sup>

$$X = X_0 := N + P_0, \quad w = w_0 := (\rho, \tau, t), \quad u = u_0 := (r, \sigma, s)$$

with  $X_0 = N + P_0 \in \mathcal{O}_{u_0}^3$ . We let

$$Q_0 := 3 \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma}$$

<sup>7</sup> With slight abuse of notations, here and during the proof of Theorem 2.2, the sub-fix  $j$  will denote the value of a given quantity at the  $j$ th step of the iteration.

Conditions (29)–(32) are implied by the assumptions (20)–(22). We then conjugate  $X_0$  to  $X_1 = N + P_1 \in \mathcal{O}_{u_1}^3$ , where

$$u_1 = (r - 4\rho, \sigma - 4\tau e^{s_2}, s - 5t) =: (r_1, \sigma_1, s_1).$$

Then we have

$$\|P_1\|_{u_1}^{w_0} \leq 8e^{s_2} Q_0 \left( \|P_0\|_{u_0}^{w_0} \right)^2 \leq \frac{1}{2} \|P_0\|_{u_0}^{w_0}. \tag{43}$$

We assume, inductively, that, for some  $1 \leq j \leq p$ , we have

$$X_j = N + P_j \in \mathcal{O}_{u_j}^3, \quad \|P_j\|_{u_j}^{w_0} < 2^{-(j-1)} \|P_1\|_{u_1}^{w_0} \tag{44}$$

where

$$u_j = (r_j, \sigma_j, s_j) \tag{45}$$

with

$$r_j := r_1 - 4(j-1)\frac{\rho}{p}, \quad \sigma_j := \sigma_1 - 4e^{s_2}(j-1)\frac{\tau}{p}, \quad s_j := s_1 - 5(j-1)\frac{t}{p}.$$

The case  $j = 1$  trivially reduces to the identity  $\|P_1\|_{u_1}^{w_0} = \|P_1\|_{u_1}^{w_0}$ . We aim to apply Lemma 2.2 with  $u = u_j$  as in (45) and

$$w = w_1 := \frac{w_0}{p}, \quad \forall 1 \leq j \leq p.$$

Conditions (29)–(31) are easily seen to be implied by (20), (19), (18) and the first condition in (22) combined with the inequality  $p\eta^2 < 1$ , implied by the choice of  $p$ . We check condition (32). By homogeneity,

$$\|P_j\|_{u_j}^{w_1} = p \|P_j\|_{u_j}^{w_0} \leq p \|P_1\|_{u_1}^{w_0} \leq 8pe^{s_2} Q_0 \left( \|P_0\|_{u_0}^{w_0} \right)^2$$

whence, using

$$Q_j = 3 \operatorname{diam}(\mathbb{Y}_{\sigma_j}) \left\| \frac{1}{v} \right\|_{r_j, \sigma_j} \leq Q_0$$

we see that condition (32) is met:

$$2Q_j \|P_j\|_{u_j}^{w_1} \leq 16pe^{s_2} Q_0^2 \left( \|P_0\|_{u_0}^{w_0} \right)^2 < 1.$$

Then the Iterative Lemma can be applied and we get  $X_{j+1} = N + P_{j+1} \in \mathcal{O}_{u_{j+1}}^3$ , with

$$\|P_{j+1}\|_{u_{j+1}}^{w_1} \leq 8e^{s_2} Q_j \left( \|P_j\|_{u_j}^{w_1} \right)^2 \leq 8e^{s_2} Q_0 \left( \|P_j\|_{u_j}^{w_1} \right)^2.$$

Using homogeneity again to the extreme sides of this inequality and combining it with (44), (43) and (22), we get

$$\begin{aligned} \|P_{j+1}\|_{u_{j+1}}^{w_0} &\leq 8pe^{s_2} Q_0 \left( \|P_j\|_{u_j}^{w_0} \right)^2 \leq 8pe^{s_2} Q_0 \|P_1\|_{u_1}^{w_0} \|P_j\|_{u_j}^{w_0} \\ &\leq 64pe^{2s_2} Q_0^2 \left( \|P_0\|_{u_0}^{w_0} \right)^2 \|P_j\|_{u_j}^{w_0} \leq \frac{1}{2} \|P_j\|_{u_j}^{w_0} \\ &< 2^{-j} \|P_1\|_{u_1}^{w_0}. \end{aligned}$$

After  $p$  iterations,

$$\|P_{p+1}\|_{u_{p+1}}^{w_0} < 2^{-p} \|P_1\|_{u_1}^{w_0} < 2^{-(p+1)} \|P_0\|_{u_0}^{w_0}$$

so we can take  $X_\star = X_{p+1}$ ,  $P_\star = P_{p+1}$ ,  $u_\star = u_{p+1}$ .  $\square$

2.5. A generalisation when the dependence on  $\psi$  is smooth

**Definition 2.1.** We denote  $\mathcal{C}_{u,\ell_*}^3$ , with  $u = (r, \sigma)$ , the class of vector-fields

$$(\mathbf{I}, y, \psi) : \mathbb{P}_u := \mathbb{I}_r \times \mathbb{Y}_\sigma \times \mathbb{T} \rightarrow X = (X_1, X_2, X_3) \in \mathbb{C}^3 \quad u = (r, \sigma)$$

where each  $X_i \in \mathcal{C}_{u,\ell_*}$ , meaning that  $X_i$  is  $C^{\ell_*}$  in  $\mathbb{P} := \mathbb{I} \times \mathbb{Y} \times \mathbb{T}$ ,  $X_i(\cdot, \cdot, \psi)$  is holomorphic in  $\mathbb{I}_r \times \mathbb{Y}_\sigma$  for each fixed  $\psi$  in  $\mathbb{T}$ .

In this section we generalise [Theorem 2.1](#) to the case that  $X \in \mathcal{C}_{u,\ell_*}^3$ . We use techniques going back to J. Nash and J. Moser [\[35–37\]](#).

First of all, we need a different definition of norms<sup>8</sup> and, especially, *smoothing* operators.

1. *Generalised weighted norms.* We let

$$\|X\|_{u,\ell}^w := \sum_i w_i^{-1} \|X_i\|_{u,\ell}, \quad 0 \leq \ell \leq \ell_* \tag{46}$$

where  $w = (w_1, w_2, w_3) \in \mathbb{R}_+^3$  where, if  $f : \mathbb{P}_{r,\sigma} := \mathbb{I}_r \times \mathbb{Y}_\sigma \times \mathbb{T} \rightarrow \mathbb{C}$ , then

$$\|f\|_u := \sup_{\mathbb{I}_r \times \mathbb{Y}_\sigma \times \mathbb{T}} |f|, \quad \|f\|_{u,\ell} := \max_{0 \leq j \leq \ell} \{ \|\partial_\varphi^j f\|_u \} \quad u = (r, \sigma). \tag{47}$$

Clearly, the class  $\mathcal{O}_{r,\sigma,s}^3$  defined in [Section 2.1](#) is a proper subset of  $\mathcal{C}_{u,\ell_*}^3$

Observe that the norms [\(46\)](#) still verify monotonicity and homogeneity in [\(16\)](#) and [\(17\)](#).

2. *Smoothing.* We call *smoothing* a family of operators

$$T_K : \quad f \in \mathcal{C}_{u,\ell_*} \rightarrow T_K f \in \mathcal{C}_{u,\ell_*}, \quad K \in \mathbb{N}$$

verifying the following. Let  $R_K := I - T_K$ . There exist  $c_0 > 0$ ,  $\delta \geq 0$  such that for all  $f \in \mathcal{C}_{u,\ell_*}$ , for all  $K$ ,  $0 \leq j \leq \ell \leq \ell_*$ ,

- $\|T_K f\|_{u,\ell} \leq c_0 K^{(\ell-j+\delta)} \|f\|_{u,j} \quad \forall 0 \leq \ell \leq \ell_*$
- $\|R_K f\|_{u,j} \leq c_0 K^{-(\ell-j-\delta)} \|f\|_{u,\ell} \quad \forall 0 \leq \ell \leq \ell_*$

As an example, as suggested in [\[3\]](#), one can take

$$T_K f(\mathbf{I}, y, \psi) := \sum_{k \in \mathbb{Z}, |k|_1 \leq K} f_k(\mathbf{I}, y) e^{ik\psi}$$

which, with the definitions [\(46\)–\(47\)](#), verifies the inequalities above with  $\delta = 2$ .

We name Generalised Normal Form Theorem (GNFT) the following

**Theorem 2.2** (GNFT). *Let  $u = (r, \sigma)$ ;  $X = N + P \in \mathcal{C}_{u,\ell_*}^3$ ,  $p, \ell, K \in \mathbb{h}$  and let  $w_K = \left(\rho, \tau, \frac{1}{c_0 K^{1+\delta}}\right) \in \mathbb{R}_+^3$  and assume that for some  $s_1, s_2 \in \mathbb{R}_+$ , the following inequalities are satisfied. Put*

$$Q := 3 e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma} \tag{48}$$

then assume:

$$0 < \rho < \frac{r}{8}, \quad 0 < \tau < e^{-s_2} \frac{\sigma}{8} \tag{49}$$

<sup>8</sup> The series in [\(15\)](#) is in general diverging when  $f \in \mathcal{C}_{u,\ell_*}$ .



and

$$\chi := \max \left\{ \frac{\text{diam}(\mathbb{Y}_\sigma)}{s_1} \left\| \frac{\omega}{v} \right\|_{r,\sigma}, \frac{\text{diam}(\mathbb{Y}_\sigma)}{s_2} \left\| \frac{\partial_y v}{v} \right\|_{r,\sigma} \right\} \leq 1 \tag{50}$$

$$\begin{aligned} \theta_1 &:= 2 e^{s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} c_0 K^{1+\delta} \tau \leq 1 \\ \theta_2 &:= 4 e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_{\Gamma} v}{v} \right\|_{r,\sigma} \frac{\rho}{\tau} \leq 1 \\ \theta_3 &:= 8 e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_{\Gamma} \omega}{v} \right\|_{r,\sigma} c_0 K^{1+\delta} \rho \leq 1 \end{aligned} \tag{51}$$

$$\eta := 2^4 e^{s_2} Q \|P\|_u^{w_K} < \frac{1}{\sqrt{p}}. \tag{52}$$

Then, with

$$u_* = (r_*, \sigma_*), \quad r_* := r - 8\rho, \quad \sigma_* = \sigma - 8e^{s_2}\tau$$

there exists a real-analytic change of coordinates  $\Phi_*$  such that  $X_* := \Phi_* X \in \mathcal{C}_{u_*, \ell_*}^3$  and  $X_* = N + P_*$ , with

$$\|P_*\|_{u_*}^{w_K} \leq \max \left\{ 2^{-(p+1)} \|P\|_u^{w_K}, 2c_0 K^{-\ell+\delta} \|P\|_{u, \ell}^{w_K} \right\} \quad \forall 0 \leq \ell \leq \ell_*.$$

The result generalising Lemma 2.1 is

**Lemma 2.6.** *Let  $X = N + P \in \mathcal{C}_{u, \ell_*}^3$ , with  $u = (r, \sigma)$ ,  $N$  as in (36),  $\ell, K \in \mathbb{N}$ . Assume (24) and that  $P$  is so small that*

$$Q \|P\|_u^{w_K} < 1 \quad Q := 3e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{r,\sigma}, \quad w_K = \left( \rho, \tau, \frac{1}{c_0 K^{1+\delta}} \right) \tag{53}$$

Let  $\rho_*, \tau_*$  be defined via

$$\begin{aligned} \frac{1}{\rho_*} &= \frac{1}{\rho} - \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_{\Gamma} v}{v} \right\|_{r,\sigma} \left( \frac{e^{s_1}}{\tau} - e^{2s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} c_0 K^{1+\delta} \right) \\ &\quad - \text{diam}(\mathbb{Y}_\sigma) \left( e^{s_1} \left\| \frac{\partial_{\Gamma} \omega}{v} \right\|_{r,\sigma} + e^{2s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_{\Gamma} v}{v} \right\|_{r,\sigma} \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} \right) c_0 K^{1+\delta} \\ \frac{1}{\tau_*} &= \frac{e^{-s_2}}{\tau} - e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} c_0 K^{1+\delta} \end{aligned} \tag{54}$$

assume

$$\hat{w}_* = (\rho_*, \tau_*) \in \mathbb{R}_+^2, \quad u_* = (r - 2\rho_*, \sigma - 2\tau_*) \in \mathbb{R}_+^2$$

and put

$$w_{*,K} := \left( \hat{w}_*, \frac{1}{c_0 K^{1+\delta}} \right).$$

Then there exists  $Y \in T_K \mathcal{C}_{u_* + \hat{w}_*, \ell_*}^3$  such that  $X_+ := e^{\mathcal{L}Y} X \in \mathcal{C}_{u_*, \ell_*}^3$  and  $X_+ = N + P_+$ , with

$$\|P_+\|_{u_*}^{w_{*,K}} \leq \frac{2Q (\|P\|_u^{w_K})^2}{1 - Q \|P\|_u^{w_K}} + c K^{-\ell+\delta} \|P\|_{u, \ell}^{w_K} \quad \forall 0 \leq \ell \leq \ell_* \tag{55}$$

The simplified form of Lemma 2.6, corresponding to Lemma 2.2, is

**Lemma 2.7** (Generalised Step Lemma). Assume (24) and replace (53) and (54) with

$$\begin{aligned}
 &2 e^{s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_{r,\sigma} c_0 K^{1+\delta} \tau \leq 1 \\
 &4 e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_{r,\sigma} \frac{\rho}{\tau} \leq 1 \\
 &8 e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 \omega}{v} \right\|_{r,\sigma} c_0 K^{1+\delta} \rho \leq 1
 \end{aligned} \tag{56}$$

$$0 < \rho < \frac{r}{4}, \quad 0 < \tau < \frac{\sigma}{4} e^{-s_2} \tag{57}$$

$$2Q \lll P \lll_u^{w_K} < 1 \tag{58}$$

then  $X_+ = N + P_+ \in \mathcal{C}_{u_+, \ell_*}^3$  and

$$\lll P_+ \lll_{u_+}^{w_K} \leq 8e^{s_2} Q (\lll P \lll_u^{w_K})^2 + c K^{-\ell+\delta} \lll P \lll_{u, \ell}^{w_K} \tag{59}$$

with

$$u_+ := (r - 4\rho, \sigma - 4\tau e^{s_2}).$$

**Proof.** The inequalities in (56) guarantee

$$\frac{1}{\rho_*} \geq \frac{1}{2\rho}, \quad \frac{1}{\tau_*} \geq \frac{e^{-s_2}}{2\tau}$$

whence

$$w_{*,K} < 2e^{s_2} w_K, \quad u_* \geq u_+ > 0.$$

Then (59) is implied by (55), monotonicity and homogeneity and the inequality in (58).  $\square$

Let now  $\mathcal{F}_{v,\omega}$  and  $\mathcal{G}_{v,\omega}$  be as in (35). First of all, observe that  $\mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega}$  take  $T_K \mathcal{C}_{u, \ell_*}$  to itself. Moreover, generalising Lemma 2.3,

**Lemma 2.8.** If inequalities (24) hold, then

$$\mathcal{F}_{v,\omega}, \mathcal{G}_{v,\omega} : \mathcal{C}_{u, \ell_*} \rightarrow \mathcal{C}_{u, \ell_*}$$

and

$$\|\mathcal{F}_{v,\omega}[g]\|_{r,\sigma} \leq e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{g}{v} \right\|_{r,\sigma}, \quad \|\mathcal{G}_{v,\omega}[g]\|_{r,\sigma} \leq e^{s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{g}{v} \right\|_{r,\sigma}.$$

**Proposition 2.3.** Let

$$N = (0, v(\mathbb{I}, y), \omega(\mathbb{I}, y)), \quad Z = (Z_1(\mathbb{I}, y, \psi), Z_2(\mathbb{I}, y, \psi), Z_3(\mathbb{I}, y, \psi))$$

belong to  $\mathcal{C}_{u, \ell_*}^3$  and assume (24). Then the ‘‘homological equation’’

$$\mathcal{L}_N[Y] = Z$$

has a solution  $Y \in \mathcal{C}_{u, \ell_*}$  verifying

$$\lll Y \lll_u^{\rho_*, \tau_*, t_*} \leq e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_u \lll Z \lll_u^{\rho, \tau, t} \quad u = (r, \sigma) \tag{60}$$

with  $\rho_*, \tau_*, t_*$  defined via

$$\begin{aligned} \frac{1}{\rho_*} &= \frac{1}{\rho} - \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_u \left( \frac{e^{s_1}}{\tau} - e^{2s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_u \frac{1}{t} \right) \\ &\quad - \text{diam}(\mathbb{Y}_\sigma) \left( e^{s_1} \left\| \frac{\partial_1 \omega}{v} \right\|_u + e^{2s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_u \left\| \frac{\partial_y \omega}{v} \right\|_u \right) \frac{1}{t} \\ \frac{1}{\tau_*} &= \frac{e^{-s_2}}{\tau} - e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_u \frac{1}{t} \\ t_* &= t \end{aligned} \tag{61}$$

and provided that

$$(\rho_*, \tau_*) \in \mathbb{R}_+^2. \tag{62}$$

In particular, if  $Z \in T_K \mathcal{C}_{u, \ell_*}^3$  for some  $K \in \mathbb{N}$ , then also  $Y \in T_K \mathcal{C}_{u, \ell_*}^3$ .

**Proof.** The solution (40) satisfies

$$\begin{aligned} \|Y_1\|_u &\leq e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_u \|Z_1\|_u \\ \|Y_2\|_u &\leq e^{s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_u \|Z_2\|_u + e^{2s_1+s_2} \text{diam}(\mathbb{Y}_\sigma)^2 \left\| \frac{1}{v} \right\|_u \left\| \frac{\partial_1 v}{v} \right\|_u \|Z_1\|_u \\ \|Y_3\|_u &\leq e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_u \|Z_3\|_u + e^{2s_1+s_2} \text{diam}(\mathbb{Y}_\sigma)^2 \left\| \frac{1}{v} \right\|_u \left\| \frac{\partial_y \omega}{v} \right\|_u \|Z_2\|_u \\ &\quad + \text{diam}(\mathbb{Y}_\sigma)^2 \left\| \frac{1}{v} \right\|_u \left( e^{2s_1} \left\| \frac{\partial_1 \omega}{v} \right\|_u + e^{3s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_u \left\| \frac{\partial_y \omega}{v} \right\|_u \right) \|Z_1\|_u \end{aligned}$$

Multiplying the inequalities above by  $\rho_*^{-1}, \tau_*^{-1}, t_*^{-1}$  respectively and taking the sum, we find (60), with

$$\begin{aligned} \frac{1}{\rho} &= \frac{1}{\rho_*} + e^{s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_u \frac{1}{\tau_*} \\ &\quad + \text{diam}(\mathbb{Y}_\sigma) \left( e^{s_1} \left\| \frac{\partial_1 \omega}{v} \right\|_u + e^{2s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_1 v}{v} \right\|_u \left\| \frac{\partial_y \omega}{v} \right\|_u \right) \frac{1}{t_*} \\ \frac{1}{\tau} &= \frac{e^{s_2}}{\tau_*} + e^{s_1+s_2} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{\partial_y \omega}{v} \right\|_u \frac{1}{t_*} \\ \frac{1}{t} &= \frac{1}{t_*}. \end{aligned}$$

We recognise that, under conditions (62),  $\rho_*, \tau_*, t_*$  in (61) solve the equations above. Observe that if  $Z \in T_K \mathcal{C}_{u, \ell_*}^3$ , then also  $Y \in T_K \mathcal{C}_{u, \ell_*}^3$ , as  $\mathcal{F}_{v, \omega}, \mathcal{G}_{v, \omega}$  do so.  $\square$

**Lemma 2.9.** Let  $u_0 \geq u > w \in \mathbb{R}_+^2 \times \{0\}$ ;  $Y \in T_K \mathcal{C}_{u_0, \ell_*}^3, W \in T_K \mathcal{C}_{u, \ell_*}^3$ . Put  $w_K := \left( w_1, w_2, \frac{1}{c_0 K^{1+\delta}} \right)$ . Then

$$\| \mathcal{L}_Y[W] \|_{u-w}^{u_0-u+w_K} \leq \| Y \|_{u-w}^{w_K} \| W \|_u^{u_0-u+w_K} + \| W \|_{u-w}^{u_0-u+w_K} \| Y \|_{u_0}^{u_0-u+w_K}.$$

**Proof.** By Cauchy inequalities, the definitions (46)–(47) and the smoothing properties,

$$\begin{aligned} \|(J_W Y)_i\|_{u-w} &\leq \|\partial_1 W_i\|_{u-w} \|Y_1\|_{u-w} + \|\partial_y W_i\|_{u-w} \|Y_2\|_{u-w} + \|\partial_\psi W_i\|_{u-w} \|Y_3\|_{u-w} \\ &\leq w_1^{-1} \|W_i\|_u \|Y_1\|_{u-w} + w_2^{-1} \|W_i\|_u \|Y_2\|_{u-w} + \|W_i\|_{u,1} \|Y_3\|_{u-w} \\ &\leq w_1^{-1} \|W_i\|_u \|Y_1\|_{u-w} + w_2^{-1} \|W_i\|_u \|Y_2\|_{u-w} + c_0 K^{1+\delta} \|W_i\|_u \|Y_3\|_{u-w} \\ &= \| Y \|_{u-w}^{w_K} \| W_i \|_u \end{aligned}$$

Similarly,

$$\|(J_Y W)_i\|_{u-w} \leq \|W\|_{u-w}^{u_0-u+w_K} \|Y_i\|_{u_0}.$$

Taking the  $u_0 - u + w_K$ -weighted norms, the thesis follows.  $\square$

**Lemma 2.10.** *Let  $0 < w < u \in \mathbb{R}_+^2 \times \{0\}$ ,  $w_K := \left(w_1, w_2, \frac{1}{c_0 K^{1+\delta}}\right)$ ;  $Y \in T_K \mathcal{C}_{u+w, \ell_*}^3$ ,  $W \in T_K \mathcal{C}_{u, \ell_*}^3$ . Then*

$$\|\mathcal{L}_Y^n [W]\|_{u-w}^{w_K} \leq 3^n n! \left(\|Y\|_{u+w}^{w_K}\right)^n \|W\|_{u-w}^{w_K}.$$

**Proof.** The proof copies the one of Lemma 2.5, up to invoke Lemma 2.9 at the place of Lemma 2.4 and hence replace the  $w$ 's "up" with  $w_K$ .  $\square$

**Proposition 2.4.** *Let  $0 < w < u \in \mathbb{R}_+^2 \times \{0\}$ ,  $w_K := \left(w_1, w_2, \frac{1}{c_0 K^{1+\delta}}\right)$ ,  $Y \in T_K \mathcal{C}_{u+w, \ell_*}^3$ ,*

$$q := 3 \|Y\|_{u+w}^{w_K} < 1.$$

*Then the Lie series  $e^{\mathcal{L}^Y}$  defines an operator*

$$e^{\mathcal{L}^Y} : T_K \mathcal{C}_{u, \ell_*}^3 \rightarrow T_K \mathcal{C}_{u-w, \ell_*}^3$$

*and its tails*

$$e_m^{\mathcal{L}^Y} = \sum_{k \geq m} \frac{\mathcal{L}_Y^k}{k!}$$

*verify*

$$\|e_m^{\mathcal{L}^Y} W\|_{u-w}^{w_K} \leq \frac{q^m}{1-q} \|W\|_u^{w_K} \quad \forall W \in T_K \mathcal{C}_{u, \ell_*}^3.$$

**Proof of Lemma 2.6.** All the remarks before Lemma 2.3 continue holding also in this case, except for the fact that, differently from Lemma 2.1 here we need a "ultraviolet cut-off" of the perturbing term. Namely, we split

$$\begin{aligned} e^{\mathcal{L}^Y} X &= e^{\mathcal{L}^Y} (N + P) = N + P + \mathcal{L}_Y N + e_2^{\mathcal{L}^Y} N + e_1^{\mathcal{L}^Y} P \\ &= N + T_K P - \mathcal{L}_N Y + P_+ \end{aligned}$$

with  $P_+ = e_2^{\mathcal{L}^Y} N + e_1^{\mathcal{L}^Y} P + R_K P$ . We choose  $Y$  so that the homological equation

$$\mathcal{L}_N Y = T_K P$$

is satisfied. By Proposition 2.3, this equation has a solution  $Y \in T_K \mathcal{C}_{u, \ell_*}^3$  verifying

$$q := 3 \|Y\|_u^{w_*} \leq 3e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_u \|P\|_u^{w_K} = Q \|P\|_u^{w_K} < 1.$$

with  $w_* = (\rho_*, \tau_*, t_*)$  as in (61). As  $t_* = t = \frac{1}{c_0 K^{1+\delta}}$ , We let

$$w_{*,K} := w_* , \quad \hat{w}_* := (\rho_*, \tau_*)$$

with  $(\rho_*, \tau_*)$  as in (54). By Proposition 2.4, the Lie series  $e^{\mathcal{L}^Y}$  defines an operator

$$e^{\mathcal{L}^Y} : W \in T_K \mathcal{C}_{u_*+\hat{w}_*, \ell_*} \rightarrow T_K \mathcal{C}_{u_*, \ell_*}$$

and its tails  $e_m^{\mathcal{L}^Y}$  verify

$$\|e_m^{\mathcal{L}^Y} W\|_{u_*}^{w_{*,K}} \leq \frac{(Q \|P\|_u^{w_K})^m}{1 - Q \|P\|_u^{w_K}} \|W\|_{u_*+\hat{w}_*}^{w_{*,K}}$$

for all  $W \in T_K \mathcal{C}_{u_* + \hat{w}_*, \ell_*}^3$ . In particular,  $e^{\mathcal{L}^Y}$  is well defined on  $T_K \mathcal{C}_{u, \ell_*}^3 \subset T_K \mathcal{C}_{u_* + \hat{w}_*, \ell_*}^3$ , hence  $P_+ \in \mathcal{C}_{u_*, \ell_*}^3$ . The bounds on  $P_+$  are obtained as follows. The terms  $\|e_2^{\mathcal{L}^Y} N\|_{u_*}^{w_*, K}$  and  $\|e_1^{\mathcal{L}^Y} P\|_{u_*}^{w_*, K}$  are treated quite similarly as (41) and (42):

$$\|e_2^{\mathcal{L}^Y} N\|_{u_*}^{w_*, K} \leq \frac{Q (\|P\|_{u_*}^{w_K})^2}{1 - Q \|P\|_{u_*}^{w_K}}, \quad \|e_1^{\mathcal{L}^Y} P\|_{u_*}^{w_*, K} \leq \frac{Q (\|P\|_{u_*}^{w_K})^2}{1 - Q \|P\|_{u_*}^{w_K}}$$

Moreover, here we have the term  $R_K P$ , which is obviously bounded as

$$\|R_K P\|_{u_*}^{w_*, K} \leq c K^{-\ell + \delta} \|P\|_{u_*, \ell}^{w_*, K} \leq c K^{-\ell + \delta} \|P\|_{u, \ell}^{w_K}. \quad \square$$

We are finally ready for the

**Proof of Theorem 2.2.** Analogously as in the proof of NFT, we proceed by iterate applications of the Generalised Step Lemma. At the base step, we let

$$X = X_0 := N + P_0, \quad w_0 := w_{0, K} := \left( \rho, \tau, \frac{1}{c_0 K^{1+\delta}} \right), \quad u_0 := (r, \sigma)$$

with  $X_0 = N + P_0 \in \mathcal{C}_{u_0, \ell_*}^3$ . We let

$$Q_0 := 3 e^{s_1} \text{diam}(\mathbb{Y}_\sigma) \left\| \frac{1}{v} \right\|_{u_0}$$

Conditions (56)–(58) are implied by the assumptions (48)–(52). We then conjugate  $X_0$  to  $X_1 = N + P_1 \in \mathcal{C}_{u_1, \ell_*}^3$ , where

$$u_1 = (r - 4\rho, \sigma - 4\tau e^{s_2}) =: (r_1, \sigma_1).$$

Then we have

$$\|P_1\|_{u_1}^{w_0} \leq 8e^{s_2} Q_0 (\|P_0\|_{u_0}^{w_0})^2 + c_0 K^{-\ell + \delta} \|P_0\|_{u_0, \ell}^{w_0}.$$

If  $8e^{s_2} Q_0 (\|P_0\|_{u_0}^{w_0})^2 \leq c_0 K^{-\ell + \delta} \|P_0\|_{u_0, \ell}^{w_0}$ , the proof finishes here. So, we assume the opposite inequality, which gives

$$\|P_1\|_{u_1}^{w_0} \leq 16e^{s_2} Q_0 (\|P_0\|_{u_0}^{w_0})^2 \leq \frac{1}{2} \|P_0\|_{u_0}^{w_0}. \tag{63}$$

We assume, inductively, that, for some  $1 \leq j \leq p$ , we have

$$X_j = N + P_j \in \mathcal{C}_{u_j, \ell_*}^3, \quad \|P_j\|_{u_j}^{w_0} < 2^{-(j-1)} \|P_1\|_{u_1}^{w_0} \tag{64}$$

where

$$u_j = (r_j, \sigma_j) \tag{65}$$

with

$$r_j := r_1 - 4(j-1)\frac{\rho}{p}, \quad \sigma_j := \sigma_1 - 4e^{s_2}(j-1)\frac{\tau}{p}.$$

The case  $j = 1$  is trivially true because it is the identity  $\|P_1\|_{u_1}^{w_0} = \|P_1\|_{u_1}^{w_0}$ . We aim to apply Lemma 2.7 with  $u = u_j$  as in (65) and

$$w = w_1 := \frac{w_0}{p}, \quad \forall 1 \leq j \leq p.$$

Conditions (56) and (57) correspond to (50)–(51), while (58) is implied by (52). We check condition (58). By homogeneity,

$$\|P_j\|_{u_j}^{w_1} = p \|P_j\|_{u_j}^{w_0} \leq p \|P_1\|_{u_1}^{w_0} \leq 16pe^{s_2} Q_0 (\|P_0\|_{u_0}^{w_0})^2$$

whence, using

$$Q_j = 3 \text{diam}(\mathbb{Y}_{\sigma_j}) \left\| \frac{1}{v} \right\|_{r_j, \sigma_j} \leq Q_0$$

we see that condition (32) is met:

$$2Q_j \|P_j\|_{u_j}^{w_1} \leq 32pe^{s_2}Q_0^2 (\|P_0\|_{u_0}^{w_0})^2 < 1.$$

Then the Iterative Lemma can be applied and we get  $X_{j+1} = N + P_{j+1} \in \mathcal{C}_{u_{j+1}, \ell_*}^3$ , with

$$\|P_{j+1}\|_{u_{j+1}}^{w_1} \leq 8e^{s_2}Q_j (\|P_j\|_{u_j}^{w_1})^2 \leq 8e^{s_2}Q_0 (\|P_j\|_{u_j}^{w_1})^2$$

Using homogeneity again to the extreme sides of this inequality and combining it with (64), (63) and (52), we get

$$\begin{aligned} \|P_{j+1}\|_{u_{j+1}}^{w_0} &\leq 8pe^{s_2}Q_0 (\|P_j\|_{u_j}^{w_0})^2 \leq 8pe^{s_2}Q_0 \|P_1\|_{u_1}^{w_0} \|P_j\|_{u_j}^{w_0} \\ &\leq 128pe^{2s_2}Q_0^2 (\|P_0\|_{u_0}^{w_0})^2 \|P_j\|_{u_j}^{w_0} \leq \frac{1}{2} \|P_j\|_{u_j}^{w_0} \\ &< 2^{-j} \|P_1\|_{u_1}^{w_0}. \end{aligned}$$

After  $p$  iterations,

$$\|P_{p+1}\|_{u_{p+1}}^{w_0} < 2^{-p} \|P_1\|_{u_1}^{w_0} < 2^{-(p+1)} \|P_0\|_{u_0}^{w_0}$$

so we can take  $X_* = X_{p+1}$ ,  $P_* = P_{p+1}$ ,  $u_* = u_{p+1}$ .  $\square$

### 3. Symplectic tools

In this section we describe various sets of canonical coordinates that are needed to our application. We remark that during the proof of Theorem B, we shall not use any of such sets completely, but rather a “mix” of action–angle and regularising coordinates, described below.

#### 3.1. Starting coordinates

We begin with the coordinates

$$\begin{cases} C = \|\mathbf{x} \times \mathbf{y} + \mathbf{x}' \times \mathbf{y}'\| \\ G = \|\mathbf{x} \times \mathbf{y}\| \\ R = \frac{\mathbf{y}' \cdot \mathbf{x}'}{\|\mathbf{x}'\|} \\ A = \sqrt{a} \end{cases} \quad \begin{cases} \gamma = \alpha_{\mathbf{k}}(\mathbf{i}, \mathbf{x}') + \frac{\pi}{2} \\ g = \alpha_{\mathbf{k}}(\mathbf{x}', \mathbf{P}) + \pi \\ r = \|\mathbf{x}'\| \\ \ell = \text{mean anomaly of } \mathbf{x} \text{ in } \mathbb{E} \end{cases} \tag{66}$$

where:

- $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  is a orthonormal frame in  $\mathbb{R}^2 \times \{\mathbf{0}\}$  and  $\mathbf{k} = \mathbf{i} \times \mathbf{j}$  (“ $\times$ ” denoting, as usual, the “skew-product”);
- after fixing a set of values of  $(\mathbf{y}, \mathbf{x})$  where the Kepler Hamiltonian (7) takes negative values,  $\mathbb{E}$  denotes the elliptic orbit with initial values  $(\mathbf{y}_0, \mathbf{x}_0)$  in such set;
- $a$  is the semi-major axis of  $\mathbb{E}$ ;
- $\mathbf{P}$ , with  $\|\mathbf{P}\| = 1$ , the direction of the perihelion of  $\mathbb{E}$ , assuming  $\mathbb{E}$  is not a circle;
- $\ell$  is the mean anomaly of  $\mathbf{x}$  on  $\mathbb{E}$ , defined, mod  $2\pi$ , as the area of the elliptic sector spanned from  $\mathbf{P}$  to  $\mathbf{x}$ , normalised to  $2\pi$ ;
- $\alpha_{\mathbf{w}}(\mathbf{u}, \mathbf{v})$  is the oriented angle from  $\mathbf{u}$  to  $\mathbf{v}$  relatively to the positive orientation established by  $\mathbf{w}$ , if  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ , with  $\mathbf{u}, \mathbf{v} \perp \mathbf{w}$ .

The canonical<sup>9</sup> character of the coordinates (66) has been discussed, in a more general setting, in [40]. The shifts  $\frac{\pi}{2}$  and  $\pi$  in (66) serve only to be consistent with the spatial coordinates of [40].

### 3.2. Energy–time coordinates

We now describe the “energy–time” change of coordinates

$$\phi_{et} : (\mathcal{R}, \mathcal{E}, r, \tau) \rightarrow (R, G, r, g) = (\mathcal{R} + \rho(\mathcal{E}, r, \tau), \tilde{G}(\mathcal{E}, r, \tau), r, \tilde{g}(\mathcal{E}, r, \tau)) \tag{67}$$

which integrates the function  $E(r, G, g)$  in (12), where  $\mathcal{E}$  (“energy”) denotes the generic level-set of  $E$ , while  $\tau$  is its conjugated (“time”) coordinate. The domain of the coordinates (67) is

$$\mathcal{R} \in \mathbb{R}, \quad 0 \leq r < 2, \quad -r < \mathcal{E} < 1 + \frac{r^2}{4}, \quad \tau \in \mathbb{R}, \quad \mathcal{E} \notin \{r, 1\}. \tag{68}$$

The extremal values of  $\mathcal{E}$  are taken to be the minimum and the maximum of the function  $E$  for  $0 \leq r < 2$ . The values  $r$  and  $1$  have been excluded because they correspond, in the  $(g, G)$ -plane, to the curves  $\mathcal{S}_0(r)$  and  $\mathcal{S}_1(r)$  in Fig. 1, where periodic motions do not exist.

The functions  $\tilde{G}(\mathcal{E}, r, \cdot)$ ,  $\tilde{g}(\mathcal{E}, r, \cdot)$  and  $\rho(\mathcal{E}, r, \cdot)$  appearing in (67) are, respectively,  $2\tau_p$  periodic,  $2\tau_p$  periodic,  $2\tau_p$  quasi-periodic, meaning that they satisfy

$$\mathcal{P}_{er} : \begin{cases} \tilde{G}(\mathcal{E}, r, \tau + 2j\tau_p) = \tilde{G}(\mathcal{E}, r, \tau) \\ \tilde{g}(\mathcal{E}, r, \tau + 2j\tau_p) = \tilde{g}(\mathcal{E}, r, \tau) \\ \rho(\mathcal{E}, r, \tau + 2j\tau_p) = \rho(\mathcal{E}, r, \tau) + 2j\rho(\mathcal{E}, r, \tau_p) \end{cases} \quad \forall \tau \in \mathbb{R}, \quad \forall j \in \mathbb{Z} \tag{69}$$

with  $\tau_p = \tau_p(\mathcal{E}, r)$  the period, defined below. Note that one can find a unique splitting

$$\rho(\mathcal{E}, r, \tau) = \mathcal{B}(\mathcal{E}, r)\tau + \tilde{\rho}(\mathcal{E}, r, \tau) \tag{70}$$

such that  $\tilde{\rho}(\mathcal{E}, r, \cdot)$  is  $2\tau_p$ -periodic. It is obtained taking

$$\mathcal{B}(\mathcal{E}, r) = \frac{\rho(\mathcal{E}, r, \tau_p(\mathcal{E}, r))}{\tau_p(\mathcal{E}, r)}, \quad \tilde{\rho}(\mathcal{E}, r, \tau) = \rho(\mathcal{E}, r, \tau) - \frac{\rho(\mathcal{E}, r, \tau_p(\mathcal{E}, r))}{\tau_p(\mathcal{E}, r)}\tau. \tag{71}$$

The transformation (67) turns to satisfy also the following “half-parity” symmetry:

$$\mathcal{P}_{1/2} : \begin{cases} \tilde{G}(\mathcal{E}, r, \tau) = \tilde{G}(\mathcal{E}, r, -\tau) \\ \tilde{g}(\mathcal{E}, r, \tau) = 2\pi - \tilde{g}(\mathcal{E}, r, -\tau) \\ \rho(\mathcal{E}, r, \tau) = -\rho(\mathcal{E}, r, -\tau) \end{cases} \quad \forall -\tau_p < \tau < \tau_p. \tag{72}$$

In addition, when  $-r < \mathcal{E} < r$ , one has the following “quarter-parity”

$$\mathcal{P}_{1/4} : \begin{cases} \tilde{G}(\mathcal{E}, r, \tau) = -G(\mathcal{E}, r, \tau_p - \tau) \\ \tilde{g}(\mathcal{E}, r, \tau) = \tilde{g}(\mathcal{E}, r, \tau_p - \tau) \\ \rho(\mathcal{E}, r, \tau) = \rho(\mathcal{E}, r, \tau_p) - \rho(\mathcal{E}, r, \tau_p - \tau) \end{cases} \quad \forall 0 \leq \tau \leq \tau_p. \tag{73}$$

The change (67) will be constructed using, as generating function, a solution of the Hamilton–Jacobi equation

$$E(r, G, \partial_G S_{et}) = G^2 + r\sqrt{1 - G^2} \cos(\partial_G S_{et}) = \mathcal{E}. \tag{74}$$

We choose the solution

$$S_{et}^+(\mathcal{R}, \mathcal{E}, r, G) = \begin{cases} \pi\sqrt{\alpha_+(\mathcal{E}, r)} - \int_G^{\sqrt{\alpha_+(\mathcal{E}, r)}} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma + \mathcal{R}r & -r \leq \mathcal{E} < 1 \\ \pi - \int_G^{\sqrt{\alpha_+(\mathcal{E}, r)}} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma + \mathcal{R}r & 1 \leq \mathcal{E} \leq 1 + \frac{r^2}{4} \end{cases}$$

<sup>9</sup> Namely, the change of coordinate (66) satisfies  $\sum_{i=1}^2 (dy_i \wedge dx_i + dy'_i \wedge dx'_i) = dC \wedge d\gamma + dG \wedge dg + dR \wedge dr + dA \wedge d\ell$ .



where we denote as

$$\alpha_{\pm}(\mathcal{E}, r) = \mathcal{E} - \frac{r^2}{2} \pm r\sqrt{1 + \frac{r^2}{4} - \mathcal{E}} \tag{75}$$

the real roots of

$$x^2 - 2\left(\mathcal{E} - \frac{r^2}{2}\right)x + \mathcal{E}^2 - r^2 = 0 \tag{76}$$

Note that the equation in (76) has always a positive real root all  $r, \mathcal{E}$  as in (68), so  $\alpha_+(\mathcal{E}, r)$  is positive.  $S_{et}^+$  generates the following equations

$$\begin{cases} g = -\cos^{-1} \frac{\mathcal{E} - G^2}{r\sqrt{1 - G^2}} \\ \tau = + \int_{\tilde{G}(\mathcal{E}, r, \tau)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \frac{d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}} \\ R = \mathcal{R} - \frac{1}{r} \int_{\tilde{G}(\mathcal{E}, r, \tau)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \frac{(\mathcal{E} - \Gamma^2)d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}} =: \mathcal{R} + \rho(\mathcal{E}, r, \tau) \\ r = r \end{cases} \tag{77}$$

The equations for  $g$  and  $r$  are immediate. We check the equation for  $\tau$ . Letting, for short,  $\sigma(\mathcal{E}, r) := \sqrt{\alpha_+(\mathcal{E}, r)}$ , we have

$$\begin{aligned} \tau &= \partial_{\mathcal{E}} S_{et}^+(\mathcal{R}, \mathcal{E}, r, G) \\ &= \begin{cases} \pi \partial_{\mathcal{E}} \sigma(\mathcal{E}, r) - \partial_{\mathcal{E}} \sigma(\mathcal{E}, r) g_+(\mathcal{E}, r) - \int_G^{\sigma(\mathcal{E}, r)} \partial_{\mathcal{E}} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma & -r \leq \mathcal{E} < 1 \\ -\partial_{\mathcal{E}} \sigma(\mathcal{E}, r) g_+(\mathcal{E}, r) - \int_G^{\sigma(\mathcal{E}, r)} \partial_{\mathcal{E}} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma & 1 \leq \mathcal{E} \leq 1 + \frac{r^2}{4} \end{cases} \\ &= - \int_G^{\sigma(\mathcal{E}, r)} \partial_{\mathcal{E}} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma \\ &= \int_{\tilde{G}(\mathcal{E}, r, \tau)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \frac{d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}} \end{aligned} \tag{78}$$

where, by (75),

$$g_+(\mathcal{E}, r) := \cos^{-1} \frac{\mathcal{E} - \sigma(\mathcal{E}, r)^2}{r\sqrt{1 - \sigma(\mathcal{E}, r)^2}} = \cos^{-1} \operatorname{sign} \left( \frac{r}{2} - \sqrt{1 + \frac{r^2}{4} - \mathcal{E}} \right) = \begin{cases} \pi & -r \leq \mathcal{E} < 1 \\ 0 & 1 \leq \mathcal{E} \leq 1 + \frac{r^2}{4} \end{cases}$$

Observe that  $(g_+, \sigma)$  are the coordinates of the point where  $E$  reaches its maximum on each level set (Fig. 1). The equation for  $R$  is analogous.

Eqs. (77) define the segment of the transformation (67) with  $0 \leq \tau \leq \tau_p$ , where

$$\tau_p(\mathcal{E}, r) := \int_{\beta(\mathcal{E}, r)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \frac{d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}} \tag{79}$$

is the half-period, with

$$\beta(\mathcal{E}, r) = \begin{cases} -\sqrt{\alpha_+(\mathcal{E}, r)} & \text{if } \alpha_-(\mathcal{E}, r) < 0 \\ \sqrt{\alpha_-(\mathcal{E}, r)} & \text{if } \alpha_-(\mathcal{E}, r) > 0. \end{cases} \tag{80}$$

The transformation is prolonged to  $-\tau_p < \tau < 0$  choosing the solution

$$S_{et}^- := -2\pi G - S_{et}^+$$

of (74). It can be checked that this choice provides the symmetry relation described in (72). Considering next the functions  $S_k^\pm = S_{et}^\pm + 2k\Sigma(\mathcal{E}, r)$ , where  $\Sigma$  solves<sup>10</sup>

$$\partial_{\mathcal{E}}\Sigma = \tau_p(\mathcal{E}, r), \quad \partial_r\Sigma = \rho(\mathcal{E}, r, \tau_p(\mathcal{E}, r))$$

one obtains the extension of the transformation to  $\tau \in \mathbb{R}$  verifying (69).

Observe that quarter period symmetry (67), holding in the case  $-r < \mathcal{E} < r$ , is an immediate consequence of the definitions (77).

The coordinates  $(\mathcal{R}, \mathcal{E}, r, \tau)$  are referred to as *energy-time coordinates*.

The regularity of the functions  $\tilde{G}(\mathcal{E}, r, \tau)$ ,  $\tilde{\rho}(\mathcal{E}, r, \tau)$ ,  $\mathcal{B}(\mathcal{E}, r)$  and  $\tau_p(\mathcal{E}, r)$ , which are relevant for the paper, is studied in detail in Section 4. Their holomorphy is not discussed.

### 3.3. Action-angle coordinates

We look at the transformation

$$\phi_{aa} : (\mathcal{R}_*, A_*, r_*, \varphi_*) \rightarrow (\mathcal{R}, \mathcal{E}, r, \tau)$$

defined by equations

$$\begin{cases} A_* = \mathcal{A}(\mathcal{E}, r) \\ \varphi_* = \pi \frac{\tau}{\tau_p(\mathcal{E}, r)} \\ r_* = r \\ \mathcal{R}_* = \mathcal{R} + \mathcal{B}(\mathcal{E}, r)\tau \end{cases} \tag{81}$$

with  $\mathcal{B}(\mathcal{E}, r)$  as in (71),  $\tau_p(\mathcal{E}, r)$  as in (79) and  $\mathcal{A}(\mathcal{E}, r)$  the ‘‘action function’’, defined as

$$\mathcal{A}(\mathcal{E}, r) := \begin{cases} \sqrt{\alpha_+(\mathcal{E}, r)} - \frac{1}{\pi} \int_{\beta(\mathcal{E}, r)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma & -r \leq \mathcal{E} \leq 1 \\ 1 - \frac{1}{\pi} \int_{\beta(\mathcal{E}, r)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \cos^{-1} \frac{\mathcal{E} - \Gamma^2}{r\sqrt{1 - \Gamma^2}} d\Gamma & 1 < \mathcal{E} \leq 1 + \frac{r^2}{4} \end{cases}$$

with  $\alpha_+(\mathcal{E}, r)$  and  $\beta(\mathcal{E}, r)$  being defined in (75), (80).

Geometrically,  $\mathcal{A}(\mathcal{E}, r)$  represents the area of the region encircled by the level curves of E in Fig. 1 in the former case, the area of its complement in the second case, divided by  $2\pi$ .

The canonical character of the transformation (81) is recognised looking at the generating function

$$S_{aa}(\mathcal{R}, \mathcal{E}, r_*, \varphi_*) = \varphi_*\mathcal{A}(\mathcal{E}, r_*) + \mathcal{R}r_* \tag{82}$$

and using the following relations (compare the formulae in (77) and (79))

$$\begin{aligned} \mathcal{A}_r(\mathcal{E}, r) &= -\frac{1}{\pi r} \int_{\beta(\mathcal{E}, r)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \frac{(\mathcal{E} - \Gamma^2)d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}} \\ &= \frac{1}{\pi} \rho(\mathcal{E}, r, \tau_p) \\ \mathcal{A}_{\mathcal{E}}(\mathcal{E}, r) &= \frac{1}{\pi} \int_{\beta(\mathcal{E}, r)}^{\sqrt{\alpha_+(\mathcal{E}, r)}} \frac{d\Gamma}{\sqrt{(\Gamma^2 - \alpha_-(\mathcal{E}, r))(\alpha_+(\mathcal{E}, r) - \Gamma^2)}} \\ &= \frac{1}{\pi} \tau_p(\mathcal{E}, r) \end{aligned} \tag{83}$$

<sup>10</sup> The existence of the function  $\Sigma(\mathcal{E}, r)$  follows from the arguments of the next section: compare the formula in (83).

which allow us to rewrite (81) as the transformation generated by (82):

$$\begin{cases} A_* = \mathcal{A}(\mathcal{E}, r) \\ \varphi_* = \frac{\tau}{\mathcal{A}_{\mathcal{E}}(\mathcal{E}, r)} \\ r_* = r \\ \mathcal{R}_* = \mathcal{R} + \frac{\mathcal{A}_r(\mathcal{E}, r)}{\mathcal{A}_{\mathcal{E}}(\mathcal{E}, r)}\tau. \end{cases} \tag{84}$$

The coordinates  $(\mathcal{R}_*, A_*, r_*, \varphi_*)$  are referred to as *action-angle coordinates*.

**Remark 3.1.** We conclude this section observing a non-negligible advantage while using *action-angle coordinates* compared to *energy-time* — besides the obvious one of dealing with a constant period. It is the law that relates  $R$  to  $\mathcal{R}_*$ , which is (see (67), (70) and (81))

$$R = \mathcal{R}_* + \rho_*(A_*, r_*, \varphi_*), \quad \text{with} \quad \rho_*(A_*, r_*, \varphi_*) := \tilde{\rho} \circ \phi_{aa}(A_*, r_*, \varphi_*) \tag{85}$$

where  $\tilde{\rho}$  is as in (70). Here  $\rho_*(A_*, r_*, \varphi_*)$  is a *periodic function* because so is the function  $\tilde{\rho}$ . This benefit is evident comparing with the corresponding formula with *energy-time* coordinates:

$$R = \mathcal{R} + \mathcal{B}(\mathcal{E}, r)\tau + \tilde{\rho}(\mathcal{E}, r, \tau)$$

which would include the uncomfortable linear term  $\mathcal{B}(\mathcal{E}, r)\tau$ . Incidentally, such term would unnecessarily complicate the computations we are going to present in Section 6.

### 3.4. Regularising coordinates

In this section we define the *regularising coordinates*. First of all we rewrite  $\mathcal{S}_0(r)$  in (14) in terms of  $(A_*, \varphi_*)$ :

$$\mathcal{S}_0(r_*) = \left\{ (A_*, \varphi_*) : A_* = \mathcal{A}_s(r_*), \varphi_* \in \mathbb{R} \right\} \quad 0 < r_* < 2$$

with  $\mathcal{A}_s(r_*)$  being the limiting value of  $\mathcal{A}(\mathcal{E}, r_*)$  when  $\mathcal{E} = r_*$ :

$$\mathcal{A}_s(r_*) = \begin{cases} \sqrt{r_*(2-r_*)} - \frac{1}{\pi} \int_0^{\sqrt{r_*(2-r_*)}} \cos^{-1} \frac{r_* - \Gamma^2}{r_*\sqrt{1-\Gamma^2}} d\Gamma & 0 < r_* < 1 \\ 1 - \frac{1}{\pi} \int_0^{\sqrt{r_*(2-r_*)}} \cos^{-1} \frac{r_* - \Gamma^2}{r_*\sqrt{1-\Gamma^2}} d\Gamma & 1 < r_* < 2 \end{cases}$$

We observe that the function  $\mathcal{A}_s(r_*)$  is continuous in  $[0, 2]$  (in particular,  $\mathcal{A}_s(1^-) = \mathcal{A}_s(1^+)$ ), with

$$\mathcal{A}_s(0) = 0, \quad \mathcal{A}_s(2) = 1$$

and increases smoothly between those two values, as it results from the analysis of its derivative. Indeed, letting, for short,  $\sigma_0(r_*) := \sqrt{r_*(2-r_*)}$  and proceeding analogously as (78), we get

$$\begin{aligned} \mathcal{A}'_s(r_*) &= -\frac{1}{\pi} \int_0^{\sigma_0(r_*)} \partial_{r_*} \cos^{-1} \frac{r_* - \Gamma^2}{r_*\sqrt{1-\Gamma^2}} d\Gamma \\ &= \frac{1}{\pi r_*} \int_0^{\sigma_0(r_*)} \frac{\Gamma d\Gamma}{\sqrt{\sigma_0(r_*)^2 - \Gamma^2}} \\ &= \frac{1}{\pi} \sqrt{\frac{2-r_*}{r_*}} \quad \forall 0 < r_* < 2 \end{aligned} \tag{86}$$

We denote as  $A_* \rightarrow r_s(A_*)$  the inverse function

$$r_s := \mathcal{A}_s^{-1} \tag{87}$$

and we define two different changes of coordinates

$$\phi_{rg}^k : (Y_k, A_k, y_k, \varphi_k) \rightarrow (\mathcal{R}_*, A_*, r_*, \varphi_*) \quad k = \pm 1$$

via the formulae

$$\begin{cases} \mathcal{R}_* = Y_k e^{ky_k} \\ A_* = A_k \\ r_* = -k e^{-ky_k} + r_s(A_k) \\ \varphi_* = \varphi_k + Y_k e^{ky_k} r'_s(A_k) \end{cases} \tag{88}$$

The transformations (88) are canonical, being generated by

$$S_{rg}^k(Y_k, A_k, r_*, \varphi_*) := -\frac{Y_k}{k} \log \left| \frac{r_s(A_k) - r_*}{k} \right| + A_k \varphi_*$$

The coordinates  $(Y_k, A_k, y_k, \varphi_k)$  with  $k = \pm 1$  are called *regularising coordinates*.

#### 4. A deeper insight into energy–time coordinates

In this section we study the functions  $\tilde{G}(\mathcal{E}, r, \tau)$ ,  $\tilde{\rho}(\mathcal{E}, r, \tau)$ ,  $\mathcal{B}(\mathcal{E}, r)$  and  $\tau_p(\mathcal{E}, r)$ , described in Section 3.2. We prove that  $\tilde{G}(\mathcal{E}, r, \tau)$ ,  $\tilde{\rho}(\mathcal{E}, r, \tau)$  are  $C^\infty$  provided that  $(\mathcal{E}, r)$  vary in a compact subset set of (68) and we study the behaviour of  $\mathcal{B}(\mathcal{E}, r)$  and  $\tau_p(\mathcal{E}, r)$  closely to  $S_0(r)$ .

It reveals to be useful to perform this study via suitable other functions  $\check{G}(\kappa, \theta)$ ,  $\check{\rho}(\kappa, \theta)$ ,  $\mathcal{A}(\kappa)$  and  $T_0(\kappa)$ , which we now define. We rewrite

$$\tilde{G}(\mathcal{E}, r, \tau) = \sigma(\mathcal{E}, r) \check{G}(\kappa(\mathcal{E}, r), \theta(\mathcal{E}, r, \tau)) \quad , \quad \tau_p(\mathcal{E}, r) = \frac{T_p(\kappa(\mathcal{E}, r))}{\sigma(\mathcal{E}, r)} \tag{89}$$

and

$$\rho(\mathcal{E}, r, \tau) = -\frac{\mathcal{E}\tau}{r} + \frac{\sigma(\mathcal{E}, r)}{r} \tilde{\rho}(\kappa(\mathcal{E}, r), \theta(\mathcal{E}, r, \tau)) \quad 0 \leq \theta \leq T_p(\kappa) \tag{90}$$

where (changing, in the integrals in (77), the integration variable  $\Gamma = \sigma\xi$ )  $\check{G}(\kappa, \theta)$  is the unique solution of

$$\int_{\check{G}(\kappa, \theta)}^1 \frac{d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}} = \theta \quad , \quad 0 \leq \theta \leq T_p(\kappa) \tag{91}$$

$$\tilde{\rho}(\kappa, \theta) = \int_{\check{G}(\kappa, \theta)}^1 \frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}} \quad 0 \leq \theta \leq T_p(\kappa) \tag{92}$$

and

$$T_p(\kappa) = \begin{cases} T_0(\kappa) & 0 < \kappa < 1 \\ 2T_0(\kappa) & \kappa < 0 \end{cases} \tag{93}$$

with

$$T_0(\kappa) := \int_{G_0(\kappa)}^1 \frac{d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}} \quad , \quad \text{where} \quad G_0(\kappa) := \begin{cases} \sqrt{\kappa} & 0 < \kappa < 1 \\ 0 & \kappa < 0 \end{cases} \tag{94}$$

The function  $\tilde{\rho}(\kappa, \theta)$  in (92) is further split as

$$\tilde{\rho}(\kappa, \theta) = \mathcal{A}(\kappa)\theta + \check{\rho}(\kappa, \theta) \tag{95}$$

where

$$\mathcal{A}(\kappa) = \frac{\tilde{\rho}(\kappa, T_p(\kappa))}{T_p(\kappa)} \quad , \quad \check{\rho}(\kappa, \theta) = \tilde{\rho}(\kappa, \theta) - \mathcal{A}(\kappa)\theta \tag{96}$$

Finally,  $\sigma(\mathcal{E}, r)$ ,  $\kappa(\mathcal{E}, r)$  and  $\theta(\mathcal{E}, r, \tau)$  are given by

$$\begin{aligned} \sigma(\mathcal{E}, r) &:= \sqrt{\alpha_+(\mathcal{E}, r)} = \sqrt{\mathcal{E} - \frac{r^2}{2} + r\sqrt{1 + \frac{r^2}{4}} - \mathcal{E}} \\ \kappa(\mathcal{E}, r) &:= \frac{\alpha_-(\mathcal{E}, r)}{\alpha_+(\mathcal{E}, r)} = \frac{\mathcal{E}^2 - r^2}{\left(\mathcal{E} - \frac{r^2}{2} + r\sqrt{1 + \frac{r^2}{4}} - \mathcal{E}\right)^2} \\ \theta(\mathcal{E}, r, \tau) &:= \tau\sqrt{\mathcal{E} - \frac{r^2}{2} + r\sqrt{1 + \frac{r^2}{4}} - \mathcal{E}}. \end{aligned} \tag{97}$$

The periodicity of  $\check{\rho}(\kappa, \cdot)$  (see Eq. (100)), the uniqueness of the splitting (70) and the formulae in (90) and (95) imply that  $\mathcal{A}(\kappa)$  and  $\check{\rho}(\kappa, \theta)$  are related to  $\mathcal{B}(\mathcal{E}, r)$  and  $\tilde{\rho}(\mathcal{E}, r, \tau)$  in (70) via

$$\mathcal{B}(\mathcal{E}, r) = -\frac{\mathcal{E}}{r} + \frac{\sigma(\mathcal{E}, r)^2}{r}\mathcal{A}(\kappa), \quad \tilde{\rho}(\mathcal{E}, r, \tau) = \frac{\sigma(\mathcal{E}, r)}{r}\check{\rho}(\kappa(\mathcal{E}, r), \theta(\mathcal{E}, r, \tau)). \tag{98}$$

In view of relations (89), (93) and (98), we focus on the functions  $\check{\mathcal{G}}(\kappa, \theta)$ ,  $\check{\rho}(\kappa, \theta)$ ,  $\mathcal{A}(\kappa)$  and  $T_0(\kappa)$ . The proofs of the following statements are postponed at the end of the section.

Let us denote  $\check{\mathcal{G}}_{ij}(\kappa, \theta) := \partial_{\kappa^i \theta^j}^{i+j} \check{\mathcal{G}}(\kappa, \theta)$ ,  $\check{\rho}_{ij}(\kappa, \theta) := \partial_{\kappa^i \theta^j}^{i+j} \check{\rho}(\kappa, \theta)$ .

**Proposition 4.1.** *Let  $0 \neq \kappa < 1$  fixed. The functions  $\check{\mathcal{G}}_{ij}(\kappa, \cdot)$  and  $\check{\rho}_{ij}(\kappa, \cdot)$  are continuous for all  $\theta \in \mathbb{R}$ .*

This immediately implies

**Corollary 4.1.** *Let  $\mathbb{K} \subset \mathbb{R}$  a compact set, with  $0, 1 \notin \mathbb{K}$ . Then  $\check{\mathcal{G}}, \check{\rho}$  are  $C^\infty(\mathbb{K} \times \mathbb{T})$ .*

Concerning  $T_0(\kappa)$ , we have

**Proposition 4.2.** *Let  $0 \neq \kappa < 1$ , and let  $T_0(\kappa)$  be as in (94). Then one can find two real numbers  $C^*, \mathcal{R}^*, \mathcal{S}^*$  and two functions  $\mathcal{R}(\kappa), \mathcal{S}(\kappa)$  verifying*

$$\mathcal{R}(0) = 1 = \mathcal{S}(0), \quad 0 \leq \mathcal{R}(\kappa) \leq \mathcal{R}^*, \quad 0 \leq \mathcal{S}(\kappa) \leq \mathcal{S}^* \quad \forall \kappa \in (-1, 1)$$

such that

$$T_0'(\kappa) = -\frac{\mathcal{R}(\kappa)}{2\kappa}, \quad T_0''(\kappa) = \frac{\mathcal{S}(\kappa)}{4\kappa^2}, \quad \forall 0 \neq \kappa < 1$$

In particular,

$$|T_0(\kappa)| \leq \frac{\mathcal{R}^*}{2} \left| \log |\kappa| \right| + C^*, \quad |T_0'(\kappa)| \leq \frac{\mathcal{R}^*}{2} \left| \kappa \right|^{-1}, \quad |T_0''(\kappa)| \leq \frac{\mathcal{S}^*}{4} \left| \kappa \right|^{-2}.$$

Finally, as for  $\mathcal{A}(\kappa)$ , we have

**Proposition 4.3.** *Let  $0 \neq \kappa < 1$ , and let  $\mathcal{A}(\kappa)$  be as in (96). Then one can find  $C^* > 0$  such that*

$$|\mathcal{A}(\kappa)| \leq C^* \left| \log |\kappa| \right|^{-1}, \quad |\mathcal{A}'(\kappa)| \leq C^* \left| \kappa \right|^{-1}, \quad |\mathcal{A}''(\kappa)| \leq C^* \left| \kappa \right|^{-2}$$

**Proof of Propositions 4.1–4.3.** Relations (69), (72) and (73) provide

$$\begin{cases} \check{G}(\kappa, \theta + 2jT_p) = \check{G}(\kappa, \theta) & \forall \theta \in \mathbb{R}, j \in \mathbb{Z} \quad \forall 0 \neq \kappa < 1 \\ \check{G}(\kappa, -\theta) = \check{G}(\kappa, \theta) & \forall 0 \leq \theta \leq T_p(\kappa) \quad \forall 0 \neq \kappa < 1 \\ \check{G}(\kappa, T_p - \theta) = -\check{G}(\kappa, \theta) & \forall 0 \leq \theta \leq T_0(\kappa) \quad \forall \kappa < 0. \end{cases} \tag{99}$$

$$\begin{cases} \check{\rho}(\kappa, \theta + 2jT_p) = \check{\rho}(\kappa, \theta), & \forall \theta \in \mathbb{R}, j \in \mathbb{Z} \quad \forall 0 \neq \kappa < 1 \\ \check{\rho}(\kappa, -\theta) = -\check{\rho}(\kappa, \theta) & \forall 0 \leq \theta \leq T_p(\kappa) \quad \forall 0 \neq \kappa < 1 \\ \check{\rho}(\kappa, T_p - \theta) = -\check{\rho}(\kappa, \theta) & \forall 0 \leq \theta \leq T_0(\kappa) \quad \forall \kappa < 0. \end{cases} \tag{100}$$

The following lemmata are obvious

**Lemma 4.1.** *Let  $g(\kappa, \cdot)$  verify (99) with  $T_p(\kappa) = \pi$  for all  $\kappa$  and  $T_0$  as in (93). Then the functions  $g_{ij}(\kappa, \theta) := \partial_{\kappa^i, \theta^j}^{i+j} g(\kappa, \theta)$  are continuous on  $\mathbb{R}$  if and only if they are continuous in  $[0, T_0]$  and verify*

$$\begin{cases} \text{no further condition} & \text{if } j \in 2\mathbb{N}, \quad 0 < \kappa < 1 \\ g_{ij}(\kappa, \frac{\pi}{2}) = 0 & \text{if } j \in 2\mathbb{N}, \quad \kappa < 0 \\ g_{ij}(\kappa, 0) = 0 = g_{ij}(\kappa, \pi) & \text{if } j \in 2\mathbb{N} + 1, \quad 0 < \kappa < 1 \\ g_{ij}(\kappa, 0) = 0 & \text{if } j \in 2\mathbb{N} + 1, \quad \kappa < 0 \end{cases} \tag{101}$$

**Lemma 4.2.** *Let  $g(\kappa, \cdot)$  verify (100) with  $T_p(\kappa) = \pi$  for all  $\kappa$  and  $T_0$  as in (93). Then  $g_{ij}(\kappa, \cdot)$ , where  $g_{ij}(\kappa, \theta) := \partial_{\kappa^i, \theta^j}^{i+j} g(\kappa, \theta)$ , are continuous on  $\mathbb{R}$  if and only if they are continuous in  $[0, T_0(\kappa)]$  and verify*

$$\begin{cases} g_{ij}(\kappa, 0) = g_{ij}(\kappa, \pi) = 0 & \text{if } j \in 2\mathbb{N}, \quad 0 < \kappa < 1 \\ g_{ij}(\kappa, 0) = g_{ij}(\kappa, \frac{\pi}{2}) = 0 & \text{if } j \in 2\mathbb{N} \quad \kappa < 0 \\ \text{no further condition} & \text{if } j \in 2\mathbb{N} + 1 \end{cases} \tag{102}$$

**Proof of Proposition 4.1.** (i) The function  $\check{G}(\kappa, \cdot)$  is  $C^\infty(\mathbb{R})$  for all  $0 \neq \kappa < 1$  [21]. Then so is the function  $g(\kappa, \cdot)$ , where  $g(\kappa, \theta) := \check{G}(\kappa, \frac{T_p(\kappa)}{\pi}\theta)$ . Then (101) hold true for  $g(\kappa, \theta)$  with  $i = 0$ . Hence, the derivatives  $g_{ij}(\kappa, \theta)$ , which exist for all  $0 \neq \kappa < 1$ , also verify (101). Then  $g_{ij}(\kappa, \cdot)$  are continuous for all  $0 \neq \kappa < 1$  and so are the  $\check{G}_{ij}(\kappa, \cdot)$ .

(ii) We check conditions (102) for the function  $g(\kappa, \theta) := \check{\rho}(\kappa, \frac{T_p(\kappa)}{\pi}\theta)$ , in the case  $j = 0$ . Using (92), (91) and (96), we get, for  $0 < \kappa < 1$ ,

$$g(\kappa, 0) = \check{\rho}(\kappa, 0) = 0, \quad g(\kappa, \pi) = \check{\rho}(\kappa, T_p(\kappa)) = \widehat{\rho}(\kappa, T_p) - \frac{\widehat{\rho}(\kappa, T_p)}{T_p} T_p = 0. \tag{103}$$

while, for  $\kappa < 0$ ,

$$g(\kappa, 0) = \check{\rho}(\kappa, 0) = 0, \quad g\left(\kappa, \frac{\pi}{2}\right) = \check{\rho}(\kappa, T_0(\kappa)) = \widehat{\rho}(\kappa, T_0) - \frac{\widehat{\rho}(\kappa, T_0)}{T_0} T_0 = 0. \tag{104}$$

The identities (103) and (104) still hold replacing  $g$  with any  $g_{i0}(\kappa, \theta)$ , with  $i \in \mathbb{N}$ , therefore, any  $g_{i0}(\kappa, \theta)$  satisfies (102). Let us now consider the case  $j \neq 0$ . Again by (92), (91) and (96),

$$\check{\rho}_\theta(\kappa, \theta) = \check{G}(\kappa, \theta)^2 - \mathcal{A}(\kappa) \tag{105}$$

so, for any  $j \neq 0$ ,

$$\check{\rho}_{ij}(\kappa, \theta) = \partial_{\kappa^i, \theta^j}^{i+j-1} \left( \check{G}(\kappa, \theta)^2 \right)$$

Then the  $\check{\rho}_{ij}(\kappa, \cdot)$  with  $j \neq 0$  are continuous because so is  $\check{G}_{ij}(\kappa, \cdot)$ .  $\square$

**Proof of Proposition 4.2.** The function  $T_0(\kappa)$  in (94) is studied in detail in Appendix A. Combining Lemma A.1 and Proposition A.1 and taking the  $\kappa$ -primitive of such relations, one obtains Proposition 4.2.

**Proof of Proposition 4.3.**

$$\begin{aligned} \mathcal{A}(\kappa) &= \frac{1}{T_0(\kappa)} \int_{G_0(\kappa)}^1 \frac{\sqrt{\xi^2 - \kappa}}{\sqrt{1 - \xi^2}} d\xi + \kappa \\ \mathcal{A}'(\kappa) &= \frac{1}{2} + (\kappa - \mathcal{A}(\kappa)) \frac{T_0'(\kappa)}{T_0(\kappa)} = \frac{1}{2} - (\kappa - \mathcal{A}(\kappa)) \frac{\mathcal{R}(\kappa)}{2\kappa T_0(\kappa)} \\ &= \frac{1}{2} - \frac{\mathcal{R}(\kappa)}{2T_0(\kappa)} + \frac{\mathcal{A}(\kappa)\mathcal{R}(\kappa)}{2\kappa T_0(\kappa)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}''(\kappa) &= (1 - \mathcal{A}'(\kappa)) \frac{T_0'(\kappa)}{T_0(\kappa)} + (\kappa - \mathcal{A}(\kappa)) \left( \frac{T_0''(\kappa)}{T_0(\kappa)} - \frac{(T_0'(\kappa))^2}{(T_0(\kappa))^2} \right) \\ &= \frac{T_0'(\kappa)}{2T_0(\kappa)} - 2(\kappa - \mathcal{A}(\kappa)) \frac{(T_0'(\kappa))^2}{(T_0(\kappa))^2} + (\kappa - \mathcal{A}(\kappa)) \frac{T_0''(\kappa)}{T_0(\kappa)} \\ &= -\frac{\mathcal{R}(\kappa)}{4\kappa T_0(\kappa)} - 2(\kappa - \mathcal{A}(\kappa)) \frac{\mathcal{R}(\kappa)^2}{4\kappa^2 T_0(\kappa)^2} + (\kappa - \mathcal{A}(\kappa)) \frac{\mathcal{S}(\kappa)}{4\kappa^2 T_0(\kappa)} \quad \square \end{aligned}$$

### 5. The function $F(\mathcal{E}, r)$

In this section we study the function  $F(\mathcal{E}, r)$  in (11). Specifically, we aim to prove the following

**Proposition 5.1.**  $F(\mathcal{E}, r)$  is well defined and smooth for all  $(\mathcal{E}, r)$  with  $0 \leq r < 2$  and  $-r \leq \mathcal{E} < 1 + \frac{r^2}{4}$ ,  $\mathcal{E} \neq r$ . Moreover, there exists a number  $C > 0$  and a neighbourhood  $\mathcal{O}$  of  $0 \in \mathbb{R}$  such that, for all  $0 \leq r < 2$  and all  $-r \leq \mathcal{E} < 1 + \frac{r^2}{4}$  such that  $\mathcal{E} - r \in \mathcal{O}$ ,

$$|F(\mathcal{E}, r)| \leq C \log |\mathcal{E} - r|^{-1}, \quad |\partial_{\mathcal{E}, r} F(\mathcal{E}, r)| \leq C |\mathcal{E} - r|^{-1}, \quad |\partial_{\mathcal{E}, r}^2 F(\mathcal{E}, r)| \leq C |\mathcal{E} - r|^{-2}. \quad (106)$$

To prove Proposition 5.1 we need an analytic representation of the function  $F$ , which we proceed to provide. In terms of the coordinates (66), the function  $U$  in (10) is given by (recall we have fixed  $\Lambda = 1$ )

$$\begin{aligned} U(r, G, g) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - \sqrt{1 - G^2} \cos \xi) d\xi}{\sqrt{(1 - \sqrt{1 - G^2} \cos \xi)^2 + 2r((\cos \xi - \sqrt{1 - G^2}) \cos g - G \sin \xi \sin g) + r^2}} \end{aligned} \quad (107)$$

where  $\xi$  is the eccentric anomaly. By [40],  $U$  remains constant along the level curves, at  $r$  fixed, of the function  $E(r, \cdot, \cdot)$  in (12). Therefore, the function  $F(\mathcal{E}, r)$  which realises (11) is nothing else than the value that  $U(r, \cdot, \cdot)$  takes at a chosen fixed point  $(G_0(\mathcal{E}, r), g_0(\mathcal{E}, r))$  of the level set  $\mathcal{E}$  in Fig. 1. For the purposes<sup>11</sup> of the paper, we choose such point to be the point where the  $\mathcal{E}$ -level curve attains its maximum. It follows from the discussion in Section 3.2 that the coordinates of such point are

$$\begin{cases} G_+(\mathcal{E}, r) = \sqrt{\alpha_+(\mathcal{E}, r)} \\ g_+(\mathcal{E}, r) = \begin{cases} \pi & -r \leq \mathcal{E} < 1 \\ 0 & 1 \leq \mathcal{E} \leq 1 + \frac{r^2}{4} \end{cases} \end{cases} \quad (108)$$

<sup>11</sup> Compare (109) with the simpler formula proposed in [42], however valid only for values of  $\mathcal{E}$  in the interval  $[-r, r)$ .



where  $\alpha_+(\mathcal{E}, r)$  is as in (75). Replacing (108) into (107), we obtain

$$F(\mathcal{E}, r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |e(\mathcal{E}, r)| \cos \xi) d\xi}{\sqrt{(1 - |e(\mathcal{E}, r)| \cos \xi)^2 + 2s(\mathcal{E}, r)r(\cos \xi - |e(\mathcal{E}, r)|) + r^2}} \tag{109}$$

with

$$e(\mathcal{E}, r) = \frac{r}{2} - \sqrt{1 + \frac{r^2}{4} - \mathcal{E}}, \quad s(\mathcal{E}, r) := \text{sign}(e(\mathcal{E}, r)) = \begin{cases} -1 & -r \leq \mathcal{E} < 1 \\ +1 & 1 < \mathcal{E} \leq 1 + \frac{r^2}{4} \end{cases}$$

To study the regularity of F, it turns to be useful to rewrite the integral (109) as twice the integral on the half period  $[0, \pi]$  and next to make two subsequent changes of variable. The first time, with  $z = s(\mathcal{E}, r) \cos x$ . It gives the following formula, which will be used below.

$$F(\mathcal{E}, r) = \frac{1}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1 - z^2}} \frac{(1 - e(\mathcal{E}, r)z) dz}{\sqrt{(1 - e(\mathcal{E}, r)z)^2 + 2r(z - e(\mathcal{E}, r)) + r^2}} \tag{110}$$

We denote as

$$z_{\pm}(\mathcal{E}, r) := \frac{e(\mathcal{E}, r) - r}{e(\mathcal{E}, r)^2} \pm \frac{\sqrt{r(r - 2e(\mathcal{E}, r))(1 - e(\mathcal{E}, r)^2)}}{e(\mathcal{E}, r)^2} \tag{111}$$

the roots of the polynomial under the square root, which, as we shall see below, are real under conditions (68). As a second change, we let  $z = \frac{1 - \beta^2 t^2}{1 + \beta^2 t^2}$ . This leads to write  $F(\mathcal{E}, r)$  as

$$F(\mathcal{E}, r) = \frac{2(1 - e(\mathcal{E}, r))}{\pi |e(\mathcal{E}, r)| \sqrt{(z_-(\mathcal{E}, r) + 1)(z_+(\mathcal{E}, r) - 1)}} \left( \frac{1 + e(\mathcal{E}, r)}{1 - e(\mathcal{E}, r)} j_0(\kappa(\mathcal{E}, r)) - \frac{2e(\mathcal{E}, r)}{1 - e(\mathcal{E}, r)} j_{\beta(\mathcal{E}, r)}(\kappa(\mathcal{E}, r)) \right) \tag{112}$$

where  $j_{\beta}(\kappa)$  is the elliptic integral

$$j_{\beta}(\kappa) := \int_0^{+\infty} \frac{1}{1 + \beta t^2} \frac{dt}{\sqrt{(1 + t^2)(1 + \kappa t^2)}} \tag{113}$$

and  $\beta, \kappa$  are taken to be

$$\beta(\mathcal{E}, r) := \frac{z_-(\mathcal{E}, r) - 1}{1 + z_-(\mathcal{E}, r)}, \quad \kappa(\mathcal{E}, r) := \frac{(1 + z_+(\mathcal{E}, r))(z_-(\mathcal{E}, r) - 1)}{(1 + z_-(\mathcal{E}, r))(z_+(\mathcal{E}, r) - 1)}.$$

The elliptic integrals  $j_{\beta}(\kappa)$  in (113) are studied in Appendix A: compare Proposition A.1.

In terms of  $(e, r)$ , the inequalities in (68) become

$$r \in [0, 2], \quad e \in \left[-1, \frac{r}{2}\right] \setminus \{0, r - 1\} \subset [-1, 1] \tag{114}$$

where  $\{e = -1\}$  corresponds to the minimum level  $\{\mathcal{E} = -r\}$ ;  $\{e = r - 1\}$  corresponds to the separatrix level  $\mathcal{S}_0(r)$ ;  $\{e = 0\}$  corresponds to the separatrix level  $\mathcal{S}_1(r)$  and, finally,  $\{e = \frac{r}{2}\}$  corresponds to maximum level  $\{\mathcal{E} = 1 + \frac{r^2}{4}\}$ . It is so evident that the discriminant in (111) is not negative under conditions (114), so  $z_{\pm}(\mathcal{E}, r)$  are real under (68), as claimed. In addition, one can easily verify that, for any  $(r, e)$  as (114), it is  $e^2 + e - r \leq 0$ . This implies

$$z_+ + 1 = \frac{e(\mathcal{E}, r)^2 + e(\mathcal{E}, r) - r}{e(\mathcal{E}, r)^2} + \frac{\sqrt{r(r - 2e(\mathcal{E}, r))(1 - e(\mathcal{E}, r)^2)}}{e(\mathcal{E}, r)^2} < 0 \quad \forall e \neq r - 1.$$

Moreover, since

$$z_-(\mathcal{E}, r) < z_+(\mathcal{E}, r) \quad \forall r \neq 0, \mathcal{E} \neq 1 + \frac{r^2}{4}, \mathcal{E} \neq -r, (\mathcal{E}, r) \neq (2, 2) \tag{115}$$

we have

$$\beta(\mathcal{E}, r) > 0 \quad \forall (\mathcal{E}, r) \text{ as in (115)}$$

and

$$0 < \kappa(\mathcal{E}, r) < 1 \quad \forall (\mathcal{E}, r) \text{ as in (115)} \quad \text{and} \quad \mathcal{E} \neq r - 1.$$

Combining these informations with the formula in (112) and with Proposition A.1, we conclude that  $F(\mathcal{E}, r)$  is smooth for all  $r \neq 0$ ,  $\mathcal{E} \neq 1$ ,  $\mathcal{E} \neq 1 + \frac{r^2}{2}$ ,  $\mathcal{E} \neq \pm r$ ,  $(\mathcal{E}, r) \neq (2, 2)$  and that (106) holds. However, the representation in (110) allows to extend regularity for  $F(\mathcal{E}, r)$  to the domain  $0 \leq r < 2$ ,  $-r \leq \mathcal{E} < 1 + \frac{r^2}{4}$ ,  $\mathcal{E} \neq r$ , as claimed.  $\square$

### 6. Proof of theorem B

In this section we state and prove a more precise statement of Theorem B, which is Theorem 6.1.

The framework is as follows:

- fix an energy level  $c$ ;
- change the time via

$$\frac{dt}{dt'} = e^{-2ky} \quad k = \pm 1 \tag{116}$$

where  $t'$  is the new time and  $t$  the old one. The new time  $t'$  is soon renamed  $t$ ;

- look at the ODE

$$\partial_t q_k = X^{(k)}(q_k; c)$$

for the triple  $q_k = (A_k, y_k, \psi)$  where  $A_k, y_k$  are as in (88), while  $\psi = \varphi_*$ , with  $\varphi_*$  as in (81) in  $\mathbb{P}_k$ , where

$$\mathbb{P}_k(\varepsilon_-, \varepsilon_+, L_-, L_+, \xi) := \left\{ (A_k, y_k, \psi) : 1 - 2\varepsilon_+ < A_k \leq 1 - 2\varepsilon_- , L_- + 2\xi \leq ky_k \leq L_+ - 2\xi, \right. \\ \left. \psi \in \mathbb{T} \right\}$$

with  $\xi < (L_+ - L_-)/4$ . Observe that

- the projection of  $\mathbb{P}_+$  in the plane  $(g, G)$  in Fig. 1 is an inner region of  $\mathcal{S}_0(r)$  and  $r$  varies in a  $\varepsilon$ -left neighbourhood of 2;
- the projection of  $\mathbb{P}_-$  in the plane  $(g, G)$  in Fig. 1 is an outer region of  $\mathcal{S}_0(r)$  and  $r$  varies in a  $\varepsilon$ -left neighbourhood of 2;
- the boundary of  $\mathbb{P}_\kappa$  includes  $\mathcal{S}_0$  if  $L_+ = \infty$ ; it has a positive distance from it if  $L_+ < +\infty$ .

We shall prove

**Theorem 6.1.** *There exist a graph  $\mathcal{G}_k \subset \mathbb{P}_k(\varepsilon_-, \varepsilon_+, L_-, L_+, \xi)$  and a number  $L_* > 1$  such that for any  $L_- > L_*$  there exist  $\varepsilon_-, \varepsilon_+, L_+, \xi$ , an open neighbourhood  $W_k \supset \mathcal{G}_k$  such that along any orbit  $q_k(t)$  such that  $q_k(0) \in W_k$ ,*

$$|A(q_k(t)) - A(q_k(0))| \leq C_0 \epsilon e^{-L_*^3 t} \quad \forall t : |t| < t_{\text{ex}}$$

where  $t_{\text{ex}}$  is the first  $t$  such that  $q(t) \notin W_k$  and  $\epsilon$  is an upper bound for  $\|P_1\|_{W_k}$  (with  $P_1$  being the first component of  $P$ ).

**Proof.** For definiteness, from now on we discuss the case  $k = +1$  (outer orbits). The case  $k = -1$  (inner orbits) is pretty similar. We neglect to write the sub-fix “+1” everywhere. As the proof is long and technical, we divide it in paragraphs. We shall take

$$\mathcal{G} = \left\{ (A, y, \psi_\circ(A, y)), 1 - 2\varepsilon_+ \leq A \leq 1 - 2\varepsilon_- , L_- + 2\xi \leq y \leq L_+ - 2\xi \right\} \subset \mathbb{P}$$

with  $\varepsilon_-, \varepsilon_+, L_-, L_+, \psi_\circ$  to be chosen below.

Step 1. The vector-field  $X$ . As  $\psi$  is one of the *action-angle coordinates*, while  $A, y$  are two among the *regularising coordinates*, we need the expressions of the Hamiltonian (10) written in terms of those two sets. The Hamiltonian (10) written in *action-angle coordinates* is

$$H_{aa}(\mathcal{R}_*, A_*, r_*, \varphi_*) = \frac{(\mathcal{R}_* + \rho_*(A_*, r_*, \varphi_*))^2}{2} + \alpha F_*(A_*, r_*) + \frac{(C - G_*(A_*, r_*, \varphi_*))^2}{2r_*^2} - \frac{\beta}{r_*}$$

where

$$G_*(A_*, r_*, \varphi_*) := G \circ \phi_{aa}(A_*, r_*, \varphi_*), \quad F_*(A_*, r_*) := F \circ \phi_{aa}(A_*, r_*) \tag{117}$$

with  $\phi_{aa}$  as in (81), while  $\tilde{G}(\mathcal{E}, r, \tau)$ ,  $F(\mathcal{E}, r)$  as in (67), (11), respectively,  $\rho_*$  is as in (85). The Hamiltonian (10) written in *regularising coordinates* is

$$H_{rg}(Y, A, y, \varphi) = \frac{(Ye^y + \rho_*(A, r_o(A, y), \varphi_o(Y, A, y, \varphi)))^2}{2} + \alpha F_*(A, r_o(A, y)) + \frac{(C - G_*(A, r_o(A, y), \varphi_o(Y, A, y, \varphi)))^2}{2r_o(A, y)^2} - \frac{\beta}{r_o(A, y)}$$

where  $r_o(A, y)$ ,  $\varphi_o(Y, A, y, \varphi)$  are the right hand sides of the equations for  $r_*$ ,  $\varphi_*$  in (88), with  $k = +1$ .

Taking the  $\varphi_*$ -projection of Hamilton equation of  $H_{aa}$ , and the  $(A, y)$ -projection of Hamilton equation of  $H_{rg}$ , changing the time as prescribed in (116) and reducing the energy via

$$\mathcal{R}_* + \rho_*(A, r_o(A, y), \psi) = Ye^y + \rho_*(A, r_o(A, y), \psi) = \mathcal{Y}(A, y, \psi; c)$$

with

$$\mathcal{Y}(A, y, \psi; c) := \pm \sqrt{2 \left( c - \alpha F_*(A, r_o(A, y)) - \frac{(C - G_*(A, r_o(A, y), \psi))^2}{2r_o(A, y)^2} + \frac{\beta}{r_o(A, y)} \right)} \tag{118}$$

we find that the evolution for the triple  $q = (A, y, \psi)$  during the time  $t$  is governed by the vector-field

$$\left\{ \begin{aligned} X_1(A, y, \psi; c) &= e^{-2y} \frac{C - G_*(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{*,3}(A, r_o(A, y), \psi) - e^{-2y} \rho_{*,3}(A, r_o(A, y), \psi) \\ &\quad \times \mathcal{Y}(A, y, \psi; c) \\ X_2(A, y, \psi; c) &= -e^{-y} \frac{C - G_*(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{*,3}(A, r_o(A, y), \psi) r'_s(A) \\ &\quad + e^{-y} (1 + \rho_{*,3}(A, r_o(A, y), \psi) r'_s(A)) \mathcal{Y}(A, y, \psi; c) \\ X_3(A, y, \psi; c) &= \alpha e^{-2y} F_{*,1}(A, r_o(A, y)) - e^{-2y} \frac{C - G_*(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{*,1}(A, r_o(A, y), \psi) \\ &\quad + e^{-2y} \rho_{*,1}(A, r_o(A, y), \psi) \mathcal{Y}(A, y, \psi; c) \end{aligned} \right.$$

where we have used the notation, for  $f = \rho_*, G_*, F_*$ ,

$$f_1(A, r_*, \psi) := \partial_A f(A, r_*, \psi), \quad f_3(A, r_*, \psi) := \partial_\psi f(A, r_*, \psi).$$

Step 2. *Splitting the vector-field.* We write

$$X(A, y, \psi; c) = N(A, y; c) + P(A, y, \psi; c)$$

with

$$\begin{cases} N_1(A, y; c) = 0 \\ N_2(A, y; c) = v(A, y; c) := e^{-y} \sqrt{2(c - \alpha F_*(A, r_o(A, y)))} \\ N_3(A, y; c) = \omega(A, y; c) := \alpha e^{-2y} F_{*,1}(A, r_o(A, y)) \end{cases}$$

hence,

$$\left\{ \begin{aligned} P_1 &= e^{-2y} \frac{C - G_*(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{*,3}(A, r_o(A, y), \psi) - e^{-2y} \rho_{*,3}(A, r_o(A, y), \psi) \mathcal{Y}(A, y, \psi; c) \\ P_2 &= -e^{-y} \frac{C - G_*(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{*,3}(A, r_o(A, y), \psi) r'_s(A) + e^{-y} \rho_{*,3}(A, r_o(A, y), \psi) r'_s(A) \\ &\quad \cdot \mathcal{Y}(A, y, \psi; c) + e^{-y} \left( \mathcal{Y}(A, y, \psi; c) - \sqrt{2(c - \alpha F_*(A, r_o(A, y)))} \right) \\ P_3 &= -e^{-2y} \frac{C - G_*(A, r_o(A, y), \psi)}{r_o(A, y)^2} G_{*,1}(A, r_o(A, y), \psi) + e^{-2y} \rho_{*,1}(A, r_o(A, y), \psi) \mathcal{Y}(A, y, \psi; c) \end{aligned} \right. \quad (119)$$

The application of NFT relies on the smallness of the perturbing term  $P$ . In the case in point, the “greatest” term of  $P$  is the component  $P_2$ , and precisely  $\rho_{*,3}$ . This function is not uniformly small. For this reason, we need to look at its zeros and localise around them. The localisation (described in detail below) carries the holomorphic perturbation  $P$  to a perturbation  $\tilde{P}$ , which is smaller, but *no longer holomorphic*. We shall apply GNFT to the new vector-field  $\tilde{X} = N + \tilde{P}$ .

*Step 3. Localisation about non-trivial zeros of  $\rho_{*,3}$ .* The following lemma gives an insight on the term  $\rho_{*,3}$ , appearing in (119). It will be proved in Appendix B.

**Lemma 6.1.** *For any  $\mathcal{A}_s(r_*) < A < 1$  ( $0 < A < \mathcal{A}_s(r_*)$ ) there exists  $0 < \psi_*(A, r_*) < \pi$  ( $0 < \psi_*(A, r_*) < \pi/2$ ) such that  $\rho_{*,3}(A, r_*, \psi_*(A, r_*)) \equiv 0$  (and  $\rho_{*,3}(A, r_*, \pi - \psi_*(A, r_*)) \equiv 0$ ). Moreover, there exists  $C > 0$  such that, for any  $\delta > 0$  one can find a neighbourhood  $V_*(A, r_*; \delta)$  of  $\psi_*(A, r_*)$  (and a neighbourhood  $V'(A, r_*; \delta)$  of  $\pi - \psi_*(A, r_*)$ ) such that*

$$\left( \begin{aligned} |\rho_{*,3}(A, r_*, \psi)| &\leq C \frac{\sigma_*(A, r_*)}{r_*} \delta \quad \forall \psi \in V_*(A, r_*; \delta) \\ \left( |\rho_{*,3}(A, r_*, \psi)| \leq C \frac{\sigma_*(A, r_*)}{r_*} \delta \quad \forall \psi \in V_*(A, r_*; \delta) \cup V'(A, r_*; \delta) \right) \end{aligned} \right) \quad (120)$$

We now let

$$\psi_o(A, y) := \psi_*(A, r(A, y)) , \quad V_o(A, y; \delta) := V_*(A, r(A, y); \delta) .$$

For definiteness, from now on, we focus on orbits with initial datum  $(A_0, y_0, \psi_0)$  such that  $\psi_0$  is close to  $\psi_o(A_0, y_0)$ . The symmetrical cases can be similarly treated.

Let  $W_o(A, y; \delta) \subset V_o(A, y; \delta)$  an open set and let  $g(A, y, \cdot)$  be a  $C^\infty$ ,  $2\pi$ -periodic function such that, in each period  $[\psi_o(A, y) - \pi, \psi_o(A, y) + \pi]$  satisfies

$$g(A, y, \psi; \delta) \begin{cases} \equiv 1 & \forall \psi \in W_o(A, y; \delta) \\ \equiv 0 & \forall \psi \in [\psi_o(A, y) - \pi, \psi_o(A, y) + \pi] \setminus V_o(A, y; \delta) \\ \in (0, 1) & \forall \psi \in V_o(A, y; \delta) \setminus W_o(A, y; \delta) \end{cases} \quad (121)$$

The function  $g$  is chosen so that

$$\sup_{0 \leq \ell < \ell_*} \|g\|_{u, \ell} \leq 1. \quad (122)$$

As an example, one can take  $g(A, y, \psi; \delta) = \chi(\psi - \psi_o(A, y))$ , with

$$\chi(\theta) = \begin{cases} 1 & |\theta| \leq a \\ 1 - \frac{\int_a^\theta e^{-\frac{\zeta}{(\theta-a)(b-\theta)}} d\zeta}{\int_a^b e^{-\frac{\zeta}{(\theta-a)(b-\theta)}} d\zeta} & a < \theta \leq b \\ 0 & \theta > b \\ \chi(-\theta) & \theta < -a \end{cases}$$

with  $0 < a < b$  so small that  $B_a(\psi_o(A, y)) \subset W_o(A, y; \delta)$ ,  $B_b(\psi_o(A, y)) \subset V_o(A, y; \delta)$ . If  $\zeta \in (0, 1)$  is sufficiently small (depending on  $\ell_*$ ), then (122) is met.

Let

$$\tilde{P}(A, y, \psi; \delta) := g(A, y, \psi; \delta)P(A, y, \psi). \tag{123}$$

We let

$$\tilde{X} := N + \tilde{P}$$

and

$$\mathbb{P}_{\varepsilon_-, \xi} = \mathbb{A}_{\varepsilon_-} \times \mathbb{Y}_{\xi} \times \mathbb{T}, \tag{124}$$

where  $\mathbb{A} = [1 - 2\varepsilon_+, 1 - 2\varepsilon_-]$ ,  $\mathbb{Y} = [L_- + 2\xi, L_+ - 2\xi]$  and  $\varepsilon_- < \varepsilon_+$ ,  $\xi$  are sufficiently small, and  $u = (\varepsilon_-, \xi)$ . By construction,  $\tilde{X}$  and  $\tilde{P} \in \mathcal{C}_{u, \infty}^3$ . In particular,  $\tilde{P} \in \mathcal{C}_{u, \ell_*}^3$ , for all  $\ell_* \in \mathbb{N}$ . Below, we shall fix a suitably large  $\ell_*$ .

*Step 4. Bounds.* The following uniform bounds follow rather directly from the definitions. Their proof is deferred to Appendix B, in order not to interrupt the flow.

$$\begin{aligned} \left\| \frac{1}{v} \right\|_u &\leq C \frac{e^{L_+}}{\alpha L_-^{\frac{1}{2}}}, \quad \left\| \frac{\partial_A v}{v} \right\|_u \leq C \frac{e^{L_+}}{L_- \sqrt{\varepsilon_-}}, \quad \left\| \frac{\partial_y v}{v} \right\|_u \leq 1 + C \frac{e^{L_+ - L_-}}{L_-^2} \\ \left\| \frac{\omega}{v} \right\|_u &\leq C \frac{e^{L_+ - L_-}}{L_-^{3/2}}, \quad \left\| \frac{\partial_A \omega}{v} \right\|_u \leq C \frac{e^{2L_+ - L_-}}{L_-^{3/2} \varepsilon_-^{\frac{1}{2}}}, \quad \left\| \frac{\partial_y \omega}{v} \right\|_u \leq C \frac{e^{2L_+ - 2L_-}}{L_-^{3/2}} \end{aligned} \tag{125}$$

$$\begin{aligned} \|\tilde{P}_1\|_u &\leq C e^{-2L_-} \max\left\{ |C| L_+ \sqrt{\varepsilon_+}, L_+ \varepsilon_+, \delta \sqrt{\varepsilon_+} \sqrt{\alpha L_+} \right\} \\ \|\tilde{P}_2\|_u &\leq C e^{-L_-} \max\left\{ |C| L_+ \sqrt{\frac{\varepsilon_+}{\varepsilon_-}}, L_+ \frac{\varepsilon_+}{\sqrt{\varepsilon_-}}, \sqrt{\frac{\varepsilon_+}{\varepsilon_-}} \delta \sqrt{\alpha L_+}, (\alpha L_-)^{-\frac{1}{2}} \max\{|C|^2, \varepsilon_+^2, \beta\} \right\} \\ \|\tilde{P}_3\|_u &\leq C e^{-2L_-} \max\left\{ |C| \frac{\sqrt{\varepsilon_+}}{\varepsilon_-}, \frac{\varepsilon_+}{\varepsilon_-}, \frac{\sqrt{\varepsilon_+}}{\varepsilon_-} \sqrt{\alpha L_+} \right\} \end{aligned} \tag{126}$$

Here  $C$  is a number not depending on  $L_-, L_+, \xi, \varepsilon_-, \varepsilon_+, c, |C|, \beta, \alpha$  and the norms are meant as in Section 2.5, in the domain (124). Remark that the validity of (126) is subject to condition

$$L_- \geq C \alpha^{-1} \max\{|c|, |C|^2, \varepsilon_+, \beta\}. \tag{127}$$

which will be verified below.

*Step 5. Application of GNFT and conclusion.* Fix  $s_1, s_2 > 0$ . Define

$$\rho := \frac{\varepsilon_-}{16}, \quad \tau := e^{-s_2} \frac{\xi}{16}, \quad w_K := \left( \frac{\varepsilon_-}{16}, \frac{e^{-s_2} \xi}{16}, \frac{1}{c_0 K^{1+\delta}} \right)$$

so that (49) is satisfied. With these choices, as a consequence of the bounds in (125)–(126), one has

$$\begin{aligned} \chi &\leq C(L_+ - L_-) \max\left\{ \frac{e^{L_+ - L_-}}{s_1 L_-^{3/2}}, \frac{1}{s_2} \left( 1 + C \frac{e^{L_+ - L_-}}{L_-^2} \right) \right\} \\ \theta_1 &\leq C e^{s_1} (L_+ - L_-) \xi K^{1+\delta} \frac{e^{2L_+ - 2L_-}}{L_-^{3/2}} \\ \theta_2 &\leq C e^{s_1 + s_2} (L_+ - L_-) \frac{\sqrt{\varepsilon_-} e^{L_+}}{\xi L_-} \\ \theta_3 &\leq C e^{s_1} (L_+ - L_-) K^{1+\delta} \frac{e^{2L_+ - L_-}}{L_-^{3/2}} \end{aligned}$$

$$\begin{aligned} \eta &\leq C e^{s_1+s_2}(L_+ - L_-) \frac{e^{L_+-L_-}}{\alpha L_-^{\frac{1}{2}}} \max \left\{ e^{-L-\varepsilon_-^{-1}} \max \left\{ |C|L_+ \sqrt{\varepsilon_+}, L_+ \varepsilon_+, \delta \sqrt{\varepsilon_+} \sqrt{\alpha L_+} \right\}, \right. \\ &e^{s_2} \xi^{-1} \max \left\{ |C|L_+ \sqrt{\frac{\varepsilon_+}{\varepsilon_-}}, L_+ \frac{\varepsilon_+}{\sqrt{\varepsilon_-}}, \sqrt{\frac{\varepsilon_+}{\varepsilon_-}} \delta \sqrt{\alpha L_+}, (\alpha L_-)^{-\frac{1}{2}} \max \{ |C|^2, \varepsilon_+^2, \beta \} \right\}, \\ &\left. e^{-L-} K^{1+\delta} \max \left\{ |C| \frac{\sqrt{\varepsilon_+}}{\varepsilon_-}, \frac{\varepsilon_+}{\varepsilon_-}, \frac{\sqrt{\varepsilon_+}}{\varepsilon_-} \sqrt{\alpha L_+} \right\} \right\} \end{aligned} \tag{128}$$

We now discuss inequalities (49)–(52) and (127). We choose  $s_i, L_{\pm}, \varepsilon_{\pm}$  and  $K$  to be the following functions of  $L$  and  $\xi$ , with  $0 < \xi < 1 < L$ :

$$L_- = L, \quad \varepsilon_{\pm} = c_{\pm} L^2 e^{-2L}, \quad L_+ = L + 10\xi, \quad s_1 = C_1 \xi L^{-\frac{3}{2}}, \quad s_2 = C_1 \xi, \quad K = \left[ \left( \frac{c_1}{\xi \sqrt{L}} \right)^{\frac{1}{1+\delta}} \right]$$

with  $0 < c_1 < 1 < C_1$  and  $0 < c_- < c_+ < 1$  suitably fixed, so as to have  $K > 0$ . A more stringent relation between  $\xi$  and  $L$  will be specified below. We take

$$|C| < c_1 L^2 e^{-2L}, \quad \beta < c_1 L^4 e^{-4L}, \quad \delta < c_1 L^{3/2} e^{-L}$$

In view of (128), it is immediate to check that there exist suitable numbers  $0 < c_1 < 1 < C_1$  depending only on  $c, c_+, c_-$  and  $\alpha$  such that inequalities (49)–(51) and (127) are satisfied and

$$\eta < C_2 L^{-\frac{3}{2}}$$

An application of GNFT conjugates  $\tilde{X} = N + \tilde{P}$  to a new vector-field  $\tilde{X}_* = N + \tilde{P}_*$ , with the first component of the vector  $\tilde{P}_*$  being bounded as

$$\|\tilde{P}_{*,1}\|_{u_*} \leq \varepsilon_- \|\tilde{P}_*\|_{u_*}^{w_K} \leq \varepsilon_- \max \left\{ 2^{-c_2 L^3} \|\tilde{P}\|_{u_*}^{w_K}, 2c_0 K^{-\ell+\delta} \|\tilde{P}\|_{u,\ell}^{w_K} \right\}$$

Using (122), (123), that  $\tilde{P}$  vanishes outside  $V_o$ , the chain rule and the holomorphy of  $P(A, y, \cdot)$ ,

$$\|\tilde{P}\|_{u,\ell}^{w_K} \leq 2^\ell \|P_{V_o}\|_{u,\ell}^{w_K} \leq 2^\ell \frac{\ell!}{s^\ell} \|P_{(V_o)_s}\|_{u}^{w_K} \quad \forall 0 \leq \ell \leq \ell_*$$

where  $P_{(V_o)_s}(A, y, \psi)$  denotes the restriction of  $P(A, y, \cdot)$  on  $(V_o)_s$ , while  $s$  is the analyticity radius of  $P(A, y, \cdot)$ . We take  $s$  so small that

$$\|P_{(V_o)_s}\|_{u}^{w_K} \leq 2 \|P_{V_o}\|_{u}^{w_K}$$

Then we have

$$\begin{aligned} \|\tilde{P}_{*,1}\|_{u_*} &\leq 2\varepsilon_- \max \left\{ 2^{-c_2 L^3}, c_0 2^{\ell+1} \ell! s^{-\ell} K^{-\ell+\delta} \right\} \|P_{V_o}\|_{u}^{w_K} \\ &\leq 2\varepsilon_- 2^{-c_2 L^3} \|P_{(V_o)_s}\|_{u}^{w_K} \leq 2\varepsilon_- 2^{-c_2 L^3} Q^{-1} \end{aligned}$$

where we have used the inequality

$$c_0 2^{\ell+1} \ell! s^{-\ell} K^{-\ell+\delta} \leq 2^{-c_2 L^3} \tag{129}$$

which will be discussed below. On the other hand, analogous techniques as the ones used to obtain (126) provide

$$c\varepsilon \leq \|P_{1V}\|_u \leq \varepsilon, \quad cL^{\frac{1}{2}} e^{-L} \leq Q^{-1} \leq CL^{\frac{1}{2}} e^{-L}.$$

with  $\varepsilon := CL^3 e^{-4L}$  and  $0 < c < 1$ . So,

$$\|\tilde{P}_{*,1}\|_{u_*} \leq C_3 2^{-c_3 L^3} \varepsilon$$

which is what we wanted to prove. It remains to discuss (129). By Stirling and provided that  $\ell > 2\delta$ , (129) is implied by

$$K > 1, \quad \left( \frac{4c_0\sqrt{2\pi}\ell^{\frac{3}{2}}}{es\sqrt{K}} \right)^\ell \leq 2^{-c_2L^3}$$

These inequalities are satisfied by choosing  $\ell$ ,  $\ell_*$  and  $\xi$  to be related to  $L$  such in a way that

$$\ell = \max \left\{ \left[ c_2L^3 \right] + 1, [2\delta] + 1, \left[ \left( \frac{1}{2\pi} \frac{e^2\sigma^2}{64c_0^2} \right)^{\frac{1}{3}} \right] + 1 \right\}, \quad \ell_* > \ell$$

$$K = \left[ \left( \frac{c_1}{\xi\sqrt{L}} \right)^{\frac{1}{1+\delta}} \right] > 2\pi \frac{64c_0^2}{e^2\sigma^2} \ell^3 > 1. \quad \square$$

**CRedit authorship contribution statement**

**Qinbo Chen:** Formal analysis, Writing - review & editing. **Gabriella Pinzari:** Conceptualization, Methodology, Formal analysis, Writing - original draft, Writing - review & editing, Supervision, Project administration, Funding acquisition.

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**Appendix A. The elliptic integrals  $T_0(\kappa)$  and  $j_\beta(\kappa)$**

The functions  $T_0(\kappa)$  in (94) and  $j_\beta(\kappa)$  in (113) are complete elliptic integrals. We use this appendix to store some useful material concerning such functions.

First of all, in the definition of  $T_0(\kappa)$ , we change the integration variable, letting  $\xi \rightarrow \frac{1}{\xi}$ , so as to rewrite

$$T_0(\kappa) = \int_1^{\frac{1}{G_0(\kappa)}} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa\xi^2)}} \quad 0 \neq \kappa < 1 \tag{130}$$

with  $G_0(\kappa)$  as in (94). Next, we look at the complex-valued function

$$g(\kappa) := \int_1^{+\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa\xi^2)}} \quad \kappa \in \mathbb{R} \setminus \{0, 1\} \tag{131}$$

which is easily related to  $T_0(\kappa)$  and  $j_0(\kappa)$ :

**Lemma A.1.** *Let  $0 \neq \kappa < 1$ . Then*

$$T_0(\kappa) = \begin{cases} g(\kappa) & \text{if } \kappa < 0 \\ j_0(\kappa) = \Re g(\kappa) & \text{if } 0 < \kappa < 1 \end{cases} \tag{132}$$

**Proof.** We have only to prove that  $T_0(\kappa) = j_0(\kappa)$  when  $0 < \kappa < 1$ , as the other relations are immediate, from (130) and (131). We write

$$T_0(\kappa) = \left( \int_0^{+\infty} - \int_0^1 - \int_{\frac{1}{\sqrt{\kappa}}}^{+\infty} \right) \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa\xi^2)}}. \tag{133}$$



We deform the integration path of the first integral at right hand side stretching the real path  $\xi \in [0, +\infty)$  to the purely imaginary line  $z = iy$ , with  $y \in [0, +\infty)$ , so that

$$\int_0^\infty \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa\xi^2)}} = \int_0^\infty \frac{dy}{\sqrt{(y^2 + 1)(1 + \kappa y^2)}} = j_0(\kappa) \tag{134}$$

Combining this with the observation that, for  $0 < \kappa < 1$ ,  $T_0(\kappa)$  and  $j_0(\kappa)$  are real while the two latter integrals in (133) are purely imaginary, we have  $T_0(\kappa) = j_0(\kappa)$ , as claimed.  $\square$

**Remark A.1.** It follows from the proof of Lemma A.1 (compare (133)–(134)) that, in the sense of complex integrals,

$$\left( \int_0^1 + \int_{\frac{1}{\sqrt{\kappa}}}^{+\infty} \right) \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa\xi^2)}} \equiv 0, \quad \forall 0 < \kappa < 1. \tag{135}$$

This identity can be also directly checked, using proper changes of coordinate combined with cuts of the complex plane, in order to make the square roots single-valued in a neighbourhood of the real axis.

The advantage of looking at  $g(\kappa)$  instead of  $T_0(\kappa)$  is that the integration path in (131) is  $\kappa$ -independent, and this turns to be useful when taking  $\kappa$ -derivatives. The main result at this respect in this section is the following

**Proposition A.1.**

- Let  $\kappa \in \mathbb{R} \setminus \{0, 1\}$  and let  $g(\kappa)$  be as in (131). There exist two positive real numbers  $\overline{\mathcal{R}}^*$ ,  $\overline{\mathcal{S}}^*$  and two complex numbers

$$\overline{\mathcal{R}}(\kappa), \overline{\mathcal{S}}(\kappa) \in \begin{cases} \mathbb{R}_+ & \text{if } \kappa < 0 \\ \mathbb{C} & \text{if } 0 < \kappa < 1 \\ i\mathbb{R}_+ & \text{if } \kappa > 1 \end{cases}$$

with

$$\Re \overline{\mathcal{R}}(0) = \Re \overline{\mathcal{S}}(0) = 1, \quad 0 \leq \Re \overline{\mathcal{R}}(\kappa) \leq \overline{\mathcal{R}}^*, \quad 0 \leq \Re \overline{\mathcal{S}}(\kappa) \leq \overline{\mathcal{S}}^* \quad \forall \kappa \in (-1, 1)$$

such that

$$g'(\kappa) = -\frac{\overline{\mathcal{R}}(\kappa)}{2\kappa} \quad g''(\kappa) = +\frac{\overline{\mathcal{S}}(\kappa)}{4\kappa^2} \quad \forall \kappa \in \mathbb{R} \setminus \{0, 1\}.$$

- Let  $\beta \geq 0$ ;  $0 < \kappa < 1$ ,  $j_\beta(\kappa)$  as in (113). There exist two positive numbers  $\mathcal{R}_\beta^*$  and  $\mathcal{S}_\beta^* \in \mathbb{R}$  and two real functions  $\mathcal{R}_\beta(\kappa)$ ,  $\mathcal{S}_\beta(\kappa)$  satisfying

$$\begin{aligned} \mathcal{R}_\beta(0) = \mathcal{S}_\beta(0) &= \begin{cases} 1 & \text{if } \beta = 0 \\ 0 & \text{if } \beta > 0 \end{cases} \\ 0 \leq \mathcal{R}_\beta(\kappa) \leq \mathcal{R}_\beta^*, \quad 0 \leq \mathcal{S}_\beta(\kappa) \leq \mathcal{S}_\beta^* & \quad \forall \beta \geq 0 \quad \forall \kappa \in (0, 1) \end{aligned} \tag{136}$$

such that

$$j'_\beta(\kappa) = -\frac{\mathcal{R}_\beta(\kappa)}{2\kappa} \quad j''_\beta(\kappa) = +\frac{\mathcal{S}_\beta(\kappa)}{4\kappa^2} \quad \forall 0 < \kappa < 1.$$

**Proof.** We prove the first statement. We distinguish two cases.

Case 1:  $\kappa < 0$  or  $\kappa > 1$ . The integral takes real values when  $\kappa < 0$ ; purely imaginary ones when  $\kappa > 1$ :

$$g(\kappa) = \begin{cases} \int_1^{+\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa\xi^2)}} & \kappa < 0 \\ -i \int_1^{+\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(\kappa\xi^2 - 1)}} & \kappa > 1 \end{cases}$$

The function under the integral is bounded above by  $\frac{1}{\min\{1, \sqrt{|\kappa|\}}\sqrt{\xi^4-1}}$  when  $\kappa < 0$ ; by  $\frac{1}{\xi^2-1}$  when  $\kappa > 1$ . Both such bounds are integrable. Then it is possible to derive under the integral, and we obtain

$$g'(\kappa) = \begin{cases} \frac{1}{2} \int_1^{+\infty} \frac{\xi^2 d\xi}{\sqrt{(\xi^2-1)(1-\kappa\xi^2)^3}} & \kappa < 0 \\ \frac{i}{2} \int_1^{+\infty} \frac{\xi^2 d\xi}{\sqrt{(\xi^2-1)(\kappa\xi^2-1)^3}} & \kappa > 1 \end{cases}$$

and

$$g''(\kappa) = \begin{cases} \frac{3}{4} \int_1^{+\infty} \frac{\xi^4 d\xi}{\sqrt{(\xi^2-1)(1-\kappa\xi^2)^5}} & \kappa < 0 \\ -\frac{3}{4}i \int_1^{+\infty} \frac{\xi^4 d\xi}{\sqrt{(\xi^2-1)(\kappa\xi^2-1)^5}} & \kappa > 1 \end{cases}$$

We change variable  $1 - \kappa\xi^2 = \eta$  when  $\kappa < 0$ ,  $\kappa\xi^2 - 1 = \eta$  when  $\kappa > 1$  and rewrite

$$g'(\kappa) \begin{cases} \frac{1}{4|\kappa|} \int_{1-\kappa}^{+\infty} \sqrt{\frac{\eta-1}{(\eta-1+\kappa)\eta^3}} d\eta & \kappa < 0 \\ \frac{i}{4|\kappa|} \int_{\kappa-1}^{+\infty} \sqrt{\frac{\eta+1}{(\eta+1-\kappa)\eta^3}} d\eta & \kappa > 1 \end{cases}$$

and

$$g''(\kappa) = \begin{cases} \frac{3}{8|\kappa|^2} \int_{1-\kappa}^{+\infty} (\eta-1) \sqrt{\frac{\eta-1}{(\eta-1+\kappa)\eta^5}} d\eta & \kappa < 0 \\ -\frac{3}{8|\kappa|^2}i \int_{\kappa-1}^{+\infty} (\eta+1) \sqrt{\frac{\eta+1}{(\eta+1-\kappa)\eta^5}} d\eta & \kappa > 1 \end{cases}$$

so we take

$$\bar{\mathcal{R}}(\kappa) = \begin{cases} \frac{1}{2} \int_{1-\kappa}^{+\infty} \sqrt{\frac{\eta-1}{(\eta-1+\kappa)\eta^3}} d\eta & \kappa < 0 \\ \frac{i}{2} \int_{\kappa-1}^{+\infty} \sqrt{\frac{\eta+1}{(\eta+1-\kappa)\eta^3}} d\eta & \kappa > 1 \end{cases}$$

and

$$\bar{\mathcal{S}}(\kappa) = \begin{cases} \frac{3}{2} \int_{1-\kappa}^{+\infty} (\eta-1) \sqrt{\frac{\eta-1}{(\eta-1+\kappa)\eta^5}} d\eta & \kappa < 0 \\ -\frac{3}{2}i \int_{\kappa-1}^{+\infty} (\eta+1) \sqrt{\frac{\eta+1}{(\eta+1-\kappa)\eta^5}} d\eta & \kappa > 1 \end{cases}$$

Observe that, if  $-1 < \kappa < 0$ ,

$$\Re \bar{\mathcal{R}}(0^-) = 1 = \Re \bar{\mathcal{S}}(0^-)$$

and

$$0 \leq \Re \bar{\mathcal{R}}(\kappa) = \frac{1}{2} \int_{1-\kappa}^{+\infty} \sqrt{\frac{\eta-1}{(\eta-1+\kappa)\eta^3}} d\eta \leq \frac{1}{2} \int_1^{+\infty} \sqrt{\frac{\eta-1}{(\eta-2)\eta^3}} d\eta$$

$$0 \leq \Re \bar{\mathcal{S}}(\kappa) \leq \frac{3}{2} \int_1^{+\infty} (\eta-1) \sqrt{\frac{\eta-1}{(\eta-2)\eta^5}} d\eta.$$

Case 2:  $0 < \kappa < 1$ . We split  $g(\kappa)$  into its real and imaginary parts. Using (132) and (135), we obtain

$$g(\kappa) = + \int_1^{\frac{1}{\sqrt{\kappa}}} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa\xi^2)}} + \int_{\frac{1}{\sqrt{\kappa}}}^{+\infty} \frac{d\xi}{\sqrt{(\xi^2 - 1)(1 - \kappa\xi^2)}} \\ = + \int_0^\infty \frac{dy}{\sqrt{(y^2 + 1)(1 + \kappa y^2)}} + i \int_0^1 \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - \kappa\xi^2)}}$$

Notice that also in this case, the functions under the integrals may be bounded by integrable functions:  $\frac{1}{\sqrt{\kappa(y^2+1)}}$  for the former;  $\frac{1}{\sqrt{1-\xi^2}} \frac{1}{\sqrt{1-\kappa}}$  in the latter. Again, we can derive under the integral, and obtain

$$g'(\kappa) = -\frac{1}{2} \int_0^\infty \frac{y^2 dy}{\sqrt{(y^2 + 1)(1 + \kappa y^2)^3}} + \frac{i}{2} \int_0^1 \frac{\xi^2 d\xi}{\sqrt{(1 - \xi^2)(1 - \kappa\xi^2)^3}}$$

and

$$g''(\kappa) = +\frac{3}{4} \int_0^\infty \frac{y^4 dy}{\sqrt{(y^2 + 1)(1 + \kappa y^2)^5}} + \frac{3}{4} i \int_0^1 \frac{\xi^4 d\xi}{\sqrt{(1 - \xi^2)(1 - \kappa\xi^2)^5}}$$

Then, letting  $1 + \kappa y^2 = \eta$  in the first respective integrals, and  $1 - \kappa\xi^2 = \eta$  in the second ones,

$$g'(\kappa) = -\frac{1}{4\kappa} \int_1^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^3}} + \frac{i}{4\kappa} \int_{1-\kappa}^1 \sqrt{\frac{1 - \eta}{(\eta - 1 + \kappa)\eta^3}} d\eta$$

and

$$g''(\kappa) = +\frac{3}{8\kappa^2} \int_1^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^5}} + \frac{3}{8\kappa^2} i \int_{1-\kappa}^1 (1 - \eta) \sqrt{\frac{1 - \eta}{(\eta - 1 + \kappa)\eta^5}} d\eta$$

and we can take

$$\overline{\mathcal{R}}(\kappa) := \frac{1}{2} \int_1^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^3}} - \frac{i}{2} \int_{1-\kappa}^1 \sqrt{\frac{1 - \eta}{(\eta - 1 + \kappa)\eta^3}} d\eta$$

and

$$\overline{\mathcal{S}}(\kappa) = \frac{3}{2} \int_1^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^5}} + \frac{3}{2} i \int_{1-\kappa}^1 (1 - \eta) \sqrt{\frac{1 - \eta}{(\eta - 1 + \kappa)\eta^5}} d\eta$$

Notice now that

$$\Re \overline{\mathcal{R}}(0^+) = 1 = \Re \overline{\mathcal{S}}(0^+)$$

and

$$0 \leq \Re \overline{\mathcal{R}}(\kappa) = \frac{1}{2} \int_1^{+\infty} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^3}} \leq \frac{1}{2} \int_1^{+\infty} \eta^{-\frac{3}{2}} = 1$$

and

$$0 \leq \Re \overline{\mathcal{S}}(\kappa) = \frac{3}{2} \int_1^{+\infty} (\eta - 1) \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^5}} \leq \frac{3}{2} \int_1^{+\infty} \eta^{-\frac{5}{2}} = 1$$

for all  $0 < \kappa < 1$ .

The proof for  $j_\beta(\kappa)$  is completely analogous to the case 2 above (with the difference that we do not have the imaginary part in that case). One finds

$$\mathcal{R}_\beta(\kappa) = \frac{1}{2} \int_1^{+\infty} \frac{1}{1 + \frac{\beta}{\kappa}(\eta - 1)} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^3}}$$

and

$$\mathcal{S}_\beta(\kappa) = \frac{3}{2} \int_1^{+\infty} \frac{\eta - 1}{1 + \frac{\beta}{\kappa}(\eta - 1)} \sqrt{\frac{\eta - 1}{(\eta - 1 + \kappa)\eta^5}}$$

which verify (136).  $\square$

**Appendix B. Technicalities**

In this section of the appendix we prove the bounds in (125), (126) and Lemma 6.1.

**Proof of (125).** We let

$$\begin{aligned}
 \mathcal{E}_*(A_*, r_*) &:= \mathcal{E} \circ \phi_{aa}(A_*, r_*), & \mathcal{E}_o(A, y) &:= \mathcal{E}_* \circ \phi_{rg}(A, y) = \mathcal{E}_*(A, r_o(A, y)) \\
 \mathcal{B}_*(A_*, r_*) &:= \mathcal{B} \circ \phi_{aa}(A_*, r_*), & \mathcal{B}_o(A, y) &:= \mathcal{B}_* \circ \phi_{rg}(A, y) = \mathcal{B}_*(A, r_o(A, y)) \\
 T_{p,*}(A_*, r_*) &:= T_p \circ \phi_{aa}(A_*, r_*), & T_{p,o}(A, y) &:= T_{p,*} \circ \phi_{rg}(A, y) = T_{p,*}(A, r_o(A, y)) \\
 F_o(A, y) &:= F_* \circ \phi_{rg}(A, y) = F_*(A, r_o(A, y)) = F(\mathcal{E}_o(A, y), r_o(A, y)) \\
 F_{*,1,o}(A, y) &:= F_{*,1} \circ \phi_{rg}(A, y) = F_{*,1}(A, r_o(A, y))
 \end{aligned} \tag{137}$$

(with  $F, T_p, \mathcal{B}$  as in (117), (93)–(94), (81)) so as to write, more rapidly,

$$v(A, y; c) = e^{-y} \sqrt{2(c - \alpha F_o(A, y))}, \quad \omega(A, y; c) = \alpha e^{-2y} F_{*,1,o}(A, y)$$

and

$$\begin{aligned}
 \frac{1}{v} &= \frac{e^y}{\sqrt{2(c - \alpha F_o(A, y))}}, & \frac{\partial_A v}{v} &= -\frac{\alpha}{2} \frac{\partial_A F_o(A, y)}{c - \alpha F_o(A, y)}, & \frac{\partial_y v}{v} &= -1 - \frac{\alpha}{2} \frac{\partial_y F_o(A, y)}{c - \alpha F_o(A, y)}, \\
 \frac{\omega}{v} &= \alpha e^{-y} \frac{F_{*,1,o}(A, y)}{\sqrt{2(c - \alpha F_o(A, y))}}, & \frac{\partial_A \omega}{v} &= \alpha e^{-y} \frac{\partial_A F_{*,1,o}(A, y)}{\sqrt{2(c - \alpha F_o(A, y))}} \\
 \frac{\partial_y \omega}{v} &= -2\alpha e^{-y} \frac{F_{*,1,o}(A, y)}{\sqrt{2(c - \alpha F_o(A, y))}} + \alpha e^{-y} \frac{\partial_y F_{*,1,o}(A, y)}{\sqrt{2(c - \alpha F_o(A, y))}}
 \end{aligned} \tag{138}$$

We evaluate the right hand sides of (138), by means of the chain rule:

$$\begin{aligned}
 F_{*,1,o} &= \frac{F_{\mathcal{E}}(\mathcal{E}_o, r_o)}{\hat{T}_{p,o}}, & \partial_A F_{*,1,o} &= \frac{\partial_{\mathcal{E}}^2 F(\mathcal{E}_o, r_o) \partial_A \mathcal{E}_o + \partial_{\mathcal{E}_r}^2 F r'_s(A)}{\hat{T}_{p,o}} - \frac{\partial_{\mathcal{E}} \hat{T}_p \partial_A \mathcal{E}_o + \partial_r \hat{T}_p r'_s(A)}{\hat{T}_{p,o}^2} F_{\mathcal{E}} \\
 \partial_y F_{*,1,o} &= \frac{\partial_{\mathcal{E}}^2 F(\mathcal{E}_o, r_o) \partial_y \mathcal{E}_o - e^{-y} \partial_{\mathcal{E}_r}^2 F}{\hat{T}_{p,o}} - \frac{\partial_{\mathcal{E}} \hat{T}_p \partial_y \mathcal{E}_o - e^{-y} \partial_r \hat{T}_p}{\hat{T}_{p,o}^2} F_{\mathcal{E}} \\
 \partial_A F_o &= F_{\mathcal{E}}(\mathcal{E}_o, r_o) \partial_A \mathcal{E}_o + F_r(\mathcal{E}_o, r_o) r'_s(A), & \partial_y F_o &= F_{\mathcal{E}}(\mathcal{E}_o, r_o) \partial_y \mathcal{E}_o - e^{-y} F_r(\mathcal{E}_o, r_o)
 \end{aligned}$$

where we have neglected to write the arguments (e.g.,  $F_{\mathcal{E}}(\mathcal{E}_o(A, y), r_o(A, y))$ , etc.) and where, again by the chain<sup>12</sup> rule,

$$\partial_A \mathcal{E}_o = \frac{1}{\hat{T}_{p,o}(A, y)} - r'_s(A) \mathcal{B}_o(A, y), \quad \partial_y \mathcal{E}_o = e^{-y} \mathcal{B}_o(A, y)$$

As a result of the discussions in Sections 4, 5 and Appendix A, the functions  $F, T_p$  and  $\mathcal{B}$  in (137) verify

$$\begin{aligned}
 C' \log |\kappa|^{-1} &\leq |F|, |T_p|, |1/\mathcal{B}| \leq C \log |\kappa|^{-1}, & C' |\kappa|^{-1} &\leq |\partial_{\mathcal{E},r} F|, |\partial_{\mathcal{E},r} T_p|, |\partial_{\mathcal{E},r} \mathcal{B}| \leq C |\kappa|^{-1} \\
 C' |\kappa|^{-2} &\leq |\partial_{\mathcal{E},r}^2 F|, |\partial_{\mathcal{E},r}^2 T_p|, |\partial_{\mathcal{E},r}^2 \mathcal{B}| \leq C |\kappa|^{-2}
 \end{aligned}$$

with  $\kappa = O(\mathcal{E} - r) = O(e^{-y})$  so that

$$\begin{aligned}
 C' L_- &\leq |F_o|, |T_{p,o}|, |\mathcal{B}_o| \leq C L_+ \\
 C' e^{L_-} &\leq |\partial_{\mathcal{E},r} F(\mathcal{E}_o, r_o)|, |\partial_{\mathcal{E},r} T_p(\mathcal{E}_o, r_o)|, |\partial_{\mathcal{E},r} \mathcal{B}(\mathcal{E}_o, r_o)| \leq C e^{L_+} \\
 C' e^{2L_-} &\leq |\partial_{\mathcal{E},r}^2 F(\mathcal{E}_o, r_o)|, |\partial_{\mathcal{E},r}^2 T_p(\mathcal{E}_o, r_o)|, |\partial_{\mathcal{E},r}^2 \mathcal{B}(\mathcal{E}_o, r_o)| \leq C e^{2L_+}
 \end{aligned} \tag{139}$$

<sup>12</sup> Use  $\partial_{A_*} \mathcal{E}_* = \frac{1}{\partial_{A_*} \mathcal{E}} \circ \phi_{aa} = \frac{1}{\hat{T}_{p,*}(A_*, r_*)}$  and  $\partial_{r_*} \mathcal{E}_* = -\frac{\partial_{A_*} \mathcal{E}}{\partial_{A_*} \mathcal{E}} \circ \phi_{aa} = -\mathcal{B}_*(A_*, r_*)$ , implied by (84).

Finally, using (86)–(87), one has

$$r'_s(A) = \frac{1}{\mathcal{A}'_s(r)} \Big|_{r=r_s(A)} = \pi \sqrt{\frac{r_s(A)}{2 - r_s(A)}}$$

whence

$$|r'_s(A)| \leq \frac{C}{\sqrt{\varepsilon_-}} \tag{140}$$

and collecting the bounds above into (138), we find (125).

**Proof of (126).** We use some results from Section 4. Taking in count (89), (90), (96) and (97) and letting

$$\begin{aligned} \sigma_*(A, r_*) &:= \sigma \circ \phi_{aa}(A, r_*), \quad \kappa_*(A, r_*) := \kappa \circ \phi_{aa}(A, r_*) \\ \hat{T}_{p,*}(A, r_*) &:= \sigma_*(A, r_*) \hat{T}_{p,*}(A, r_*) := \frac{T_p(\kappa_*(A, r_*))}{\pi}, \end{aligned}$$

we have that

$$G_*(A, r_*, \psi) = \sigma_*(A, r_*) \check{G}(\kappa_*(A, r_*), \hat{T}_{p,*}(A, r_*) \psi) \tag{141}$$

$$\rho_*(A, r_*, \psi) = \frac{\sigma_*(A, r_*)}{r_*} \check{\rho}(\kappa_*(A, r_*), \hat{T}_{p,*}(A, r_*) \psi) \tag{142}$$

By the chain rule

$$\begin{aligned} G_{*,3}(A, r_*, \psi) &= \partial_\psi G_*(A, r_*, \psi) \\ &= \sigma_*(A, r_*) \partial_\psi \check{G}(\kappa_*(A, r_*), \hat{T}_{p,*}(A, r_*) \psi) \\ &= \sigma_*(A, r_*) \hat{T}_{p,*}(A, r_*) \check{G}_3(\kappa_*(A, r_*), \hat{T}_{p,*}(A, r_*) \psi) \end{aligned} \tag{143}$$

Similarly,

$$\rho_{*,3}(A, r_*, \psi) = \frac{\sigma_*(A, r_*)}{r_*} \hat{T}_{p,*}(A, r_*) \check{\rho}_3(\kappa_*(A, r_*), \hat{T}_{p,*}(A, r_*) \psi) \tag{144}$$

By the definitions in (121)–(123), if

$$\tilde{\mathbb{P}}_{\varepsilon,\xi} := \bigcup_{(A,y) \in \mathcal{A}_{\varepsilon_-} \times \mathbb{Y}_\xi} \{A\} \times \{y\} \times V_o(A, y; \delta)$$

then

$$\|\tilde{P}_i\|_{\mathbb{P}_{\varepsilon,\xi}} \leq \|P_i(A, y, \psi)\|_{\tilde{\mathbb{P}}_{\varepsilon,\xi}}$$

so we proceed to uniformly upper bound the  $|P_i|$  in  $\tilde{\mathbb{P}}_{\varepsilon,\xi}$ .

- By Proposition 4.1,

$$|\check{G}(\kappa, \theta)|, |\check{G}_3(\kappa, \theta)| \leq C$$

- By (141), (143) and (139),

$$|G_*(A, r_o(A, y), \psi)| \leq C\sqrt{\varepsilon_+}, \quad |G_{*,3}(A, r_o(A, y), \psi)| \leq CL_+\sqrt{\varepsilon_+} \tag{145}$$

- Both the inequalities in (145) hold (with the same proof) if  $r_o(A, y)$  is replaced by a generic  $r \in \mathfrak{R}_*(A, \cdot)$ . Then,

$$|G_{*,1}(A, r_o(A, y), \psi)| \leq C \frac{\sqrt{\varepsilon_+}}{\varepsilon_-}$$

- Similarly, by (142),  $|\rho_*(A, r_*, \psi)| \leq \sqrt{\varepsilon_+}$ , hence

$$|\rho_{*,1}(A, r_o(A, y), \psi)| \leq C \frac{\sqrt{\varepsilon_+}}{\varepsilon_-}$$

- The function  $\mathcal{Y}(A, y, \psi; c)$  defined in (118) verifies

$$|\mathcal{Y}| \leq C\sqrt{\alpha L_+}$$

having used the simplifying assumption (127).

- By Lemma 6.1,

$$|\rho_{*,3}(A, r_o(A, y), \psi)| \leq C\sqrt{\varepsilon_+} \delta$$

- Recall (140).
- Using the previous bounds into (119) and writing the last term in the definition of  $P_2$  as

$$e^{-y} \frac{\frac{(C-G_*(A, r_o(A, y), \psi))^2}{2r_o(A, y)^2} - \frac{\beta}{r_o(A, y)}}{\mathcal{Y}(A, y, \psi; c) + \sqrt{2(c - \alpha F_*(A, r_o(A, y)))}}$$

we obtain, for  $\|P_i\|_{\tilde{P}_{\varepsilon, \xi}}$ , the bounds at the right hand sides of (126).

**Proof of Lemma 6.1.** Recall (144) and the expression of  $\check{\rho}_\theta(\kappa, \theta)$  in Eq. (105). Equation

$$\check{\rho}_\theta(\kappa, \theta) = \check{G}(\kappa, \theta)^2 - \mathcal{A}(\kappa) = 0 \tag{146}$$

has a unique solution

$$0 < \theta_*(\kappa) < T_0(\kappa)$$

if and only if

$$G_0(\kappa)^2 < \mathcal{A}(\kappa) < 1.$$

On the other hand, it is immediate to check that such inequality holds for all  $0 \neq \kappa < 1$ . Indeed, if  $0 < \kappa < 1$ , then  $G_0(\kappa)^2 = \kappa$  and we have

$$\kappa < \mathcal{A}(\kappa) = \frac{\int_{\sqrt{\kappa}}^1 \frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}}}{\int_{\sqrt{\kappa}}^1 \frac{d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}}} < 1.$$

If  $\kappa < 0$ , then  $G_0(\kappa)^2 = 0$  and we have

$$0 < \mathcal{A}(\kappa) = \frac{\int_0^1 \frac{\xi^2 d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}}}{\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(\xi^2-\kappa)}}} < 1$$

As a consequence of the formula (146), combined with the continuity of  $\check{G}(\kappa, \cdot)$ , we find  $V(\kappa; \delta) \subset (0, T_0(\kappa))$  (and  $V'(\kappa; \delta) \subset (0, T_0(\kappa))$  when  $\kappa < 0$ ) such that

$$|\check{\rho}_3(\kappa, \theta)| \leq \frac{C\delta}{T_p(\kappa)} \quad \forall \theta \in V(\kappa; \delta) \quad \left( \forall \theta \in V(\kappa; \delta) \cup V'(\kappa; \delta) \right)$$

which implies (120), after using (144).  $\square$

### Appendix C. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.na.2021.112306>.

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