

Fully reconciled GDP forecasts from Income and Expenditure sides

Previsioni riconciliate del PIL dal lato del reddito e della spesa

Luisa Bisaglia, Tommaso Di Fonzo and Daniele Girolimetto

Abstract We propose a complete reconciliation procedure, resulting in a ‘one number forecast’ of the *GDP* figure, coherent with both Income and Expenditure sides’ forecasted series, and evaluate its performance on the Australian quarterly *GDP* series, as compared to the original proposal by Athanasopoulos *et al.* (2019).

Abstract *In questo lavoro viene proposta una procedura di riconciliazione delle previsioni del PIL e delle sue componenti tanto dal lato del Reddito quanto da quello della Spesa, volta a produrre previsioni coerenti rispetto ad entrambi i lati. Tale procedura, applicata alle serie trimestrali del PIL australiano, viene posta a confronto con la proposta originale di Athanasopoulos et al. (2019).*

Key words: forecast reconciliation, cross-sectional (contemporaneous) hierarchies, GDP, Income, Expenditure

1 Introduction and summary

In a recent paper, Athanasopoulos *et al.* (2019, p. 690) propose “the application of state-of-the-art forecast reconciliation methods to macroeconomic forecasting” in order to perform aligned decision making and to improve forecast accuracy. In their empirical study they consider 95 Australian Quarterly National Accounts time series, describing the Gross Domestic Product (*GDP*) at current prices from Income and Expenditure sides, interpreted as two distinct hierarchical structures. In the former case (Income), *GDP* is on the top of 15 lower level aggregates (figure 1), while in the latter (Expenditure), *GDP* is the top level aggregate of a hierarchy of 79 time series (see figures 21.5-21.7 in Athanasopoulos *et al.*, 2019, pp. 703-705).

In this paper we re-consider the results of Athanasopoulos *et al.* (2019), where the forecasts of the Australian quarterly *GDP* aggregates are separately reconciled from Income (\widetilde{GDP}^I) and Expenditure (\widetilde{GDP}^E) sides. This means that \widetilde{GDP}^I and

L. Bisaglia

Dept. Statistical Sciences, University of Padova, e-mail: luisa.bisaglia@unipd.it

T. Di Fonzo

Dept. Statistical Sciences, University of Padova, e-mail: tommaso.difonzo@unipd.it

D. Girolimetto

Dept. Statistical Sciences, University of Padova, e-mail: daniele.girolimetto@studenti.unipd.it

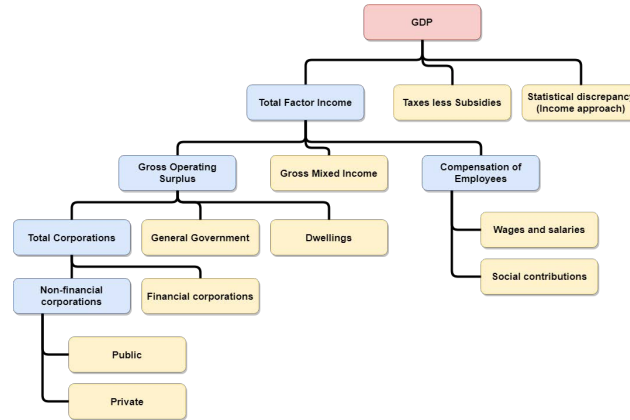


Fig. 1 Hierarchical structure of the income approach for Australian GDP. The pink cell contains the most aggregate series. The blue cells contain intermediate-level series and the yellow cells correspond to the most disaggregate bottom-level series. Source: Athanasopoulos *et al.*, 2019, p. 702.

\widehat{GDP}^E are each coherent within its own pertaining side with the other forecasted values, but in general $\widehat{GDP}^I \neq \widehat{GDP}^E$ at any forecast horizon. This circumstance could confuse and annoy the user, mostly when the discrepancy is not negligible (see Figure 2), and calls for a complete reconciliation strategy, able to produce a ‘one number forecast’ of the GDP figure, which is the main target of the paper.

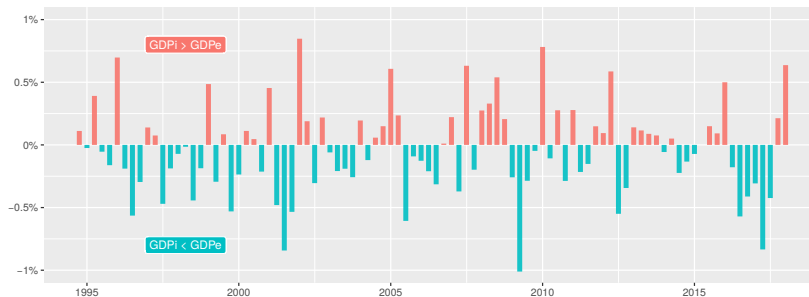


Fig. 2 Discrepancies in the reconciled 1-step-ahead GDP forecasts from Income and Expenditure sides. ARIMA base forecasts reconciled according to MinT-shr procedure (see section 3). Source data: Athanasopoulos *et al.* (2019).

We show that fully reconciled forecasts of GDP , coherent with all the reconciled forecasts from both Expenditure and Income sides, can be obtained through the classical least squares adjustment procedure proposed by Stone *et al.* (1942). It should be noted that the proposed solution has been considered by van Erven and Cugliari (2015) and Wickramasuriya *et al.* (2019) as an alternative formulation, equivalent to the regression approach by Hyndman *et al.* (2011). As far as we know, however, it has never been applied so far to distinct hierarchies sharing only the top level series. The procedure can be seen as a forecast combination (Bates and Granger, 1969) -

working on different series rather than on the output of multiple models - which makes additional use of external constraints valid for the series and their forecasts.

2 From single side to complete aggregation constraints

Denoting with x_t the actual *GDP* at time t , the relationships linking the series of, respectively, the Income and Expenditure sides hierarchies can be expressed as

$$\mathbf{y}_t^I = \mathbf{S}^I \mathbf{b}_t^I, \quad \mathbf{y}_t^E = \mathbf{S}^E \mathbf{b}_t^E, \quad t = 1, \dots, T, \quad (1)$$

where $\mathbf{y}_t^I = [x_t \quad \mathbf{a}_t^{I'} \quad \mathbf{b}_t^{I'}]'$, $\mathbf{y}_t^E = [x_t \quad \mathbf{a}_t^{E'} \quad \mathbf{b}_t^{E'}]'$, \mathbf{b}_t^I and \mathbf{b}_t^E are (10×1) and (53×1) , respectively, vectors of bottom level (disaggregated) series, \mathbf{a}_t^I and \mathbf{a}_t^E are (5×1) and (26×1) , respectively, vectors of higher levels (aggregated) series, and

$$\mathbf{S}^I = \begin{bmatrix} \mathbf{1}'_{10} \\ \mathbf{C}^I \\ \mathbf{I}_{10} \end{bmatrix}, \quad \mathbf{S}^E = \begin{bmatrix} \mathbf{1}'_{53} \\ \mathbf{C}^E \\ \mathbf{I}_{53} \end{bmatrix}$$

are contemporaneous (cross-sectional) summing matrix mapping the bottom level series into the higher-levels variables in each hierarchy, where $\mathbf{1}_k$ denotes a $(k \times 1)$ vector of ones, \mathbf{I}_k denotes the identity matrix of order k , and \mathbf{C}^I and \mathbf{C}^E are the (5×10) and (26×53) , respectively, matrices of 0's and 1's describing the aggregation relationships between the bottom level series and the higher level series (apart *GDP*) for Income (\mathbf{C}^I) and Expenditure (\mathbf{C}^E) sides. The relationships (1) can be equivalently written as

$$\mathbf{U}^I \mathbf{y}_t^I = \mathbf{0}, \quad \mathbf{U}^E \mathbf{y}_t^E = \mathbf{0}, \quad t = 1, \dots, T, \quad (2)$$

where $\mathbf{U}^I = \begin{bmatrix} \mathbf{I}_6 \\ -\mathbf{1}_{10} - \mathbf{C}^{I'} \end{bmatrix}$, and $\mathbf{U}^E = \begin{bmatrix} \mathbf{I}_{27} \\ -\mathbf{1}_{53} - \mathbf{C}^{E'} \end{bmatrix}$ are (16×6) and (80×27) matrices, respectively. The only variable subject to linear constraints on both the Income and Expenditure sides in expressions (1) and (2) being x_t (i.e., *GDP*), we can express the aggregation relationships linking the 95 'unique' variables as

$$\mathbf{U}' \mathbf{y}_t = \mathbf{0}, \quad t = 1, \dots, T, \quad (3)$$

where $\mathbf{y}_t = [x_t \quad \mathbf{a}_t^{I'} \quad \mathbf{b}_t^{I'} \quad \mathbf{a}_t^{E'} \quad \mathbf{b}_t^{E'}]'$ is a (95×1) vector, $\mathbf{0}$ is a (33×1) null vector, and \mathbf{U}' is the following (33×95) matrix:

$$\mathbf{U}' = \begin{bmatrix} 1 & \mathbf{0}'_5 & -\mathbf{1}'_{10} & \mathbf{0}'_{26} & \mathbf{0}'_{53} \\ 1 & \mathbf{0}'_5 & \mathbf{0}'_{10} & \mathbf{0}'_{26} & -\mathbf{1}'_{53} \\ \mathbf{0}_5 & \mathbf{I}_5 & -\mathbf{C}^I & \mathbf{0}_{5 \times 26} & \mathbf{0}_{5 \times 53} \\ \mathbf{0}_{26} & \mathbf{0}_{26 \times 5} & \mathbf{0}_{26 \times 10} & \mathbf{I}_{26} & -\mathbf{C}^E \end{bmatrix}. \quad (4)$$

3 Optimal point forecast reconciliation

Forecast reconciliation is a post-forecasting process aimed at improving the quality of the *base* forecasts for a system of hierarchical/grouped, and more generally linearly constrained, time series (Hyndman *et al.*, 2011, Panagiotelis *et al.*, 2019) by exploiting the constraints that the series in the system must fulfill, whereas in general the base forecasts don't. In this framework, as base forecasts we mean the $(n \times 1)$ vector $\hat{\mathbf{y}}_{T+h} \equiv \hat{\mathbf{y}}_h$ of unbiased point forecasts, with forecast horizon $h > 0$, for the $n > 1$ variables of the system.

Following Stone *et al.* (1942), we consider the classical measurement model

$$\hat{\mathbf{y}}_h = \mathbf{y}_h + \boldsymbol{\varepsilon}_h, \quad E(\boldsymbol{\varepsilon}_h) = \mathbf{0}, \quad E(\boldsymbol{\varepsilon}_h \boldsymbol{\varepsilon}_h') = \mathbf{W}_h, \quad (5)$$

where $\hat{\mathbf{y}}_h$ is the available measurement, \mathbf{y}_h is the target forecast vector, and $\boldsymbol{\varepsilon}_h$ is a zero-mean measurement error, with covariance \mathbf{W}_h , which is a $(n \times n)$ p.d. matrix, for the moment assumed known. Given a $(n \times K)$ matrix of constant values \mathbf{U} , summarizing the K linear constraints valid for the n series of the system ($n > K$), in general it is $\mathbf{U}'\hat{\mathbf{y}}_h \neq \mathbf{0}$, and we look for reconciled forecasts $\tilde{\mathbf{y}}_h$ such that $\mathbf{U}'\tilde{\mathbf{y}}_h = \mathbf{0}$.

The reconciled forecasts $\tilde{\mathbf{y}}_h$ can be found as the solution to the linearly constrained quadratic minimization problem:

$$\tilde{\mathbf{y}}_h = \arg \min_{\mathbf{y}_h} (\hat{\mathbf{y}}_h - \mathbf{y}_h)' \mathbf{W}_h^{-1} (\hat{\mathbf{y}}_h - \mathbf{y}_h), \quad \text{s.t. } \mathbf{U}'\mathbf{y}_h = \mathbf{0},$$

which is given by

$$\tilde{\mathbf{y}}_h = \left[\mathbf{I}_n - \mathbf{W}_h \mathbf{U} (\mathbf{U}' \mathbf{W}_h \mathbf{U})^{-1} \mathbf{U}' \right] \hat{\mathbf{y}}_h. \quad (6)$$

The key item in expression (6) is matrix \mathbf{W}_h , which is generally unknown and must be either assumed known or estimated. In agreement with Athanasopoulos *et al.* (2019), denoting with $\widehat{\mathbf{W}}_1$ the $(n \times n)$ covariance matrix of the in-sample one-step-ahead base forecasts errors of the n series in the system, we consider 3 cases:

- OLS: $\mathbf{W}_h = \sigma^2 \mathbf{I}_n$
- WLS: $\mathbf{W}_h = \widehat{\mathbf{W}}_D = \text{diag}\{\hat{w}_{11}, \dots, \hat{w}_{nn}\}$
- MinT-shr: $\mathbf{W}_h = \widehat{\mathbf{W}}_{shr} = \lambda \widehat{\mathbf{W}}_D + (1 - \lambda) \widehat{\mathbf{W}}_1$

where $\widehat{\mathbf{W}}_{shr}$ is the shrunk version of $\widehat{\mathbf{W}}_1$, with diagonal target and shrinkage intensity parameter λ proposed by Schäfer and Strimmer (2005) (more details can be found in Wickramasuriya *et al.*, 2019).

4 The accuracy of the reconciled forecasts of the Australian GDP

According to the notation of the previous section, for the complete Australian *GDP* accounts from both Income and Expenditure sides, it is $n = 95$, $K = 33$, and matrix \mathbf{U}' is given by (4). In addition, the available time series span over the period 1984:Q1 - 2018:Q4.

Base forecasts for the $n = 95$ separate time series have been obtained by Athanasopoulos *et al.* (2019) through simple univariate ARIMA models¹, selected using the `auto.arima` function of the R-package `forecast`. We did not change this first, crucial step in the forecast reconciliation workflow, since the focus is on the potential of forecast reconciliation.²

Our reconciliation proposal is applied within the same forecasting experiment designed by Athanasopoulos *et al.* (2019). They consider forecasts from $h = 1$ quarter ahead up to $h = 4$ quarters ahead using an *expanding* window, where the first training sample is set from 1984:Q4 to 1994:Q3 and forecasts are produced for 1994:Q4 to 1995:Q3. The base forecasts are reconciled using OLS, WLS and MinT-shr procedures, and the accuracy is measured by the Mean Squared Error (*MSE*).

Figure 3 shows the *skill scores* using *MSE*, that is the percentage changes in *MSE* registered by each reconciliation procedure, relative to base forecasts, computed such that positive values signal an improvement in forecasting accuracy over the base forecasts. The left and the central columns of the figure refer to the results for the Income and Expenditure sides variables separately considered, while the right column shows the results of the procedure proposed in this paper.

The results confirm also for the enlarged system the findings of Athanasopoulos *et al.* (2019, p. 709):

- reconciliation methods improve forecast accuracy relative to base forecasts;
- negative skill scores are registered only for OLS-reconciled forecasts of bottom level series ($h = 2, 3, 4$);
- MinT-shr is the best reconciliation procedure in most cases.

In addition, looking at the second row of figure 3, we see that for any forecast horizon the improvements in the unique *GDP* reconciled forecasts are always larger than those registered for \widetilde{GDP}^E . The same happens with \widetilde{GDP}^I , $h = 3, 4$, while for $h = 1, 2$ the skill scores are very close.

References

1. Athanasopoulos, G., Gamakumara, P., Panagiotelis, A., Hyndman, R.J., Affan, M. (2019), Hierarchical Forecasting, in Fuleky, P. (ed.), *Macroeconomic Forecasting in the Era of Big Data*, Cham, Springer, 689–719.
2. Bates, J.M., Granger, G.W.J. (1969), The combination of forecasts, *Operational Research Quarterly*, 20, 4, 461–468.
3. Hyndman, R.J., Ahmed, R.A., Athanasopoulos, G., Shang, H.L. (2011), Optimal combination forecasts for hierarchical time series, *Computational Statistics & Data Analysis*, 55, 9, 2579–2589.
4. Panagiotelis, A., Gamakumara, P., Athanasopoulos, G., Hyndman, R.J. (2019), *Forecast reconciliation: A geometric view with new insights on bias correction*, Department of Econometrics and Business Statistics, Monash University, Working Paper 18/19.
5. Schäfer, J.L., Strimmer, K. (2005), A Shrinkage Approach to Large-Scale Covariance Matrix Estimation and Implications for Functional Genomics, *Statistical Applications in Genetics and Molecular Biology*, 4, 1.

¹ The R scripts, the data and the results of the paper by Athanasopoulos *et al.* (2019) are available in the github repository located at <https://github.com/PuwasalaG/Hierarchical-Book-Chapter>.

² Athanasopoulos *et al.* (2019) point out that this fast and flexible approach performs well in forecasting Australian GDP aggregates, even compared to other more complex methods.

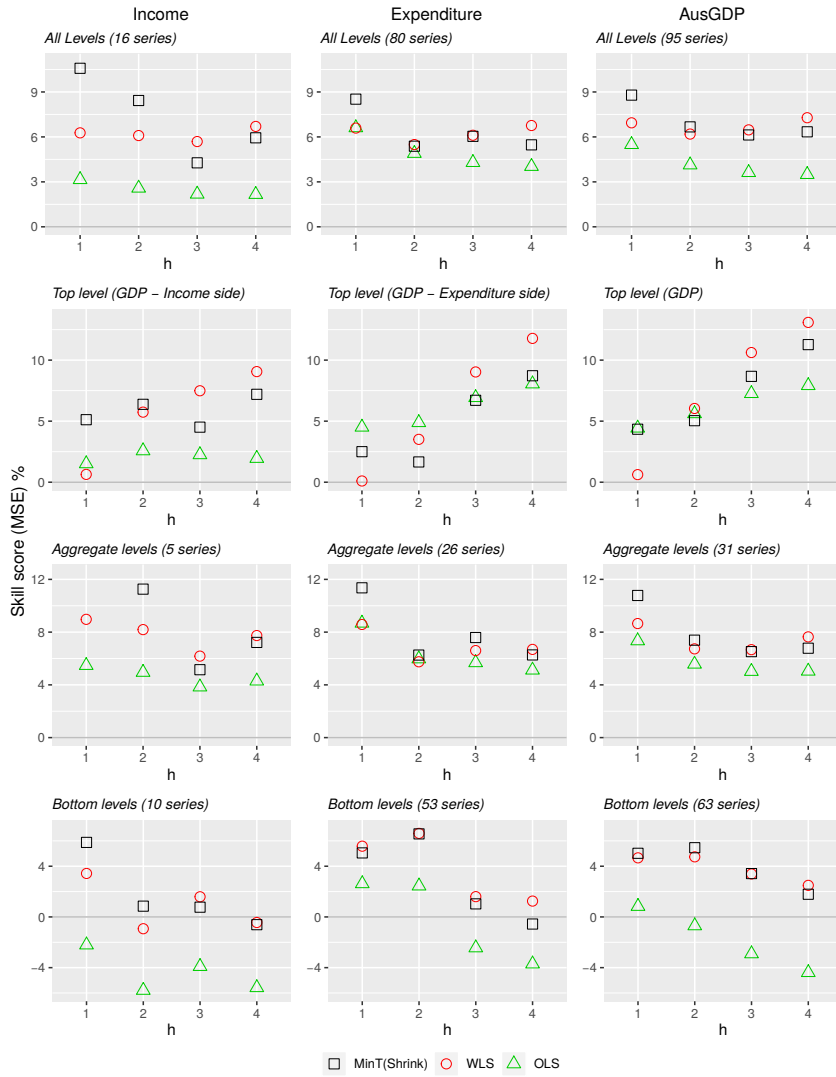


Fig. 3 Skill scores for reconciled point forecasts from alternative methods (with reference to base forecasts) using MSE.

6. Stone, R., Champenowne, D.G., Meade, J.E. (1942), The precision of national income estimates, *The Review of Economic Studies*, 9, 2, 111–125.
7. Van Erven, T., Cugliari, J. (2015), Game-theoretically Optimal Reconciliation of Contemporaneous Hierarchical Time Series Forecasts, in Antoniadis, A., Poggi, J.M., Brossat, X., (eds.) *Modeling and Stochastic Learning for Forecasting in High Dimensions*, Berlin, Springer, 297–317.
8. Wickramasuriya, S.L., Athanasopoulos, G., Hyndman, R.J. (2019), Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization, *Journal of the American Statistical Association*, 114, 526, 804–819.