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Portfolio optimization and option pricing under defaultable Lévy driven models

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To Carolina and to my family

"An old man, who had spent his life looking for a winning formula (martingale), spent the last days of his life putting it into practice, and his last pennies to see it fail.

The martingale is as elusive as the soul."

Alexandre Dumas
Mille et un fantômes, 1849

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Abstract

In this thesis we study some portfolio optimization and option pricing problems in market models where the dynamics of one or more risky assets are driven by Lévy processes, and it is divided in four independent parts.

In the first part we study the portfolio optimization problem, for the logarithmic terminal utility and the logarithmic consumption utility, in a multi-defaultable Lévy driven model.

In the second part we introduce a novel technique to price European defaultable claims when the pre-defaultable dynamics of the underlying asset follows an exponential Lévy process.

In the third part we develop a novel methodology to obtain analytical expansions for the prices of European derivatives, under stochastic and/or local volatility models driven by Lévy processes, by analytically expanding the integro-differential operator associated to the pricing problem.

In the fourth part we present an extension of the latter technique which allows for obtaining analytical expansion in option pricing when dealing with path-dependent Asian-style derivatives.

KEYWORDS: portfolio optimization, stochastic control, dynamic programming, HJB equation, jump-diffusion, multi-default, direct contagion, information-induced contagion, Lévy, exponential, default, equity-credit, default intensity, change of measure, Girsanov theorem, Esscher transform, characteristic function, abstract Cauchy problem, eigenvectors expansion, Fourier inversion, local volatility, analytical approximation, partial integro-differential equation, Fourier methods, local-stochastic volatility, asymptotic expansion, pseudo-differential calculus, implied volatility, CEV, Heston, SABR, Asian options, arithmetic average process, hypoelliptic operators, ultra-parabolic operators, Black and Scholes, option pricing, Greeks.

Riassunto

In questa tesi studiamo alcuni problemi di *portfolio optimization* e di *option pricing* in modelli di mercato dove le dinamiche di uno o più titoli rischiosi sono guidate da processi di Lévy. La tesi è divisa in quattro parti indipendenti.

Nella prima parte studiamo il problema di ottimizzare un portafoglio, inteso come massimizzazione di un'utilità logaritmica della ricchezza finale e di un'utilità logaritmica del consumo, in un modello guidato da processi di Lévy e in presenza di fallimenti simultanei.

Nella seconda parte introduciamo una nuova tecnica per il prezzaggio di opzioni europee soggette a fallimento, i cui titoli sottostanti seguono dinamiche che prima del fallimento sono rappresentate da processi di Lévy esponenziali.

Nella terza parte sviluppiamo un nuovo metodo per ottenere espansioni analitiche per i prezzi di derivati europei, sotto modelli a volatilità stocastica e locale guidati da processi di Lévy, espandendo analiticamente l'operatore integro-differenziale associato al problema di prezzaggio.

Nella quarta, e ultima parte, presentiamo un estensione della tecnica precedente che consente di ottenere espansioni analitiche per i prezzi di opzioni asiatiche, ovvero particolari tipi di opzioni il cui *payoff* dipende da tutta la traiettoria del titolo sottostante.

KEYWORDS: portfolio optimization, stochastic control, dynamic programming, HJB equation, jump-diffusion, multi-default, direct contagion, information-induced contagion, Lévy, exponential, default, equity-credit, default intensity, change of measure, Girsanov theorem, Esscher transform, characteristic function, abstract Cauchy problem, eigenvectors expansion, Fourier inversion, local volatility, analytical approximation, partial integro-differential equation, Fourier methods, local-stochastic volatility, asymptotic expansion, pseudo-differential calculus, implied volatility, CEV, Heston, SABR, Asian options, arithmetic average process, hypoelliptic operators, ultra-parabolic operators, Black and Scholes, option pricing, Greeks.

Table of contents

Introduction

Lévy processes have always played a key role in mathematical finance. Already in the very first pioneering attempt to model the price fluctuations of financial markets, made by Bachelier [13] in 1900, the price of an asset traded at the Paris Bourse coincided with the most popular among all the Lévy processes: the Brownian motion. Indeed, the class of the Lévy processes includes the Brownian motion as a particular case, being the latter the "continuous part" of any Lévy process.

In the well known multiplicative version of Bachelier's model, the Black-Scholes model [33, (1973)], the dynamics of the assets are supposed to follow a geometric Brownian motion. For decades, after this ground-breaking contribution, most of the literature dealing with financial modeling only considered diffusion-based dynamics for the price of the assets, i.e. stochastic differential equations driven by one or n-dimensional Brownian motions.

Nevertheless, several empirical evidences suggest that diffusion-based models miss to reproduce some of the statistical and pathwise properties related to price fluctuations deriving from real market datas. For a complete overview of such empirical facts we refer to the monograph by Cont and Tankov [61], whereas here we limit ourself to briefly list the more evident ones.

In the first place, the time-scale invariance of log-returns predicted by Brownian-based diffusion models seems to be violated when comparing the sample paths predicted by the model with those of real markets. In particular, the presence of jumps in the latter ones irremediably contradicts scale invariance, especially when considering short-time scales. Furthermore, the presence of jumps itself in the observed trajectories is a feature that must be somehow captured by a realistic model, whereas the continuity of the Brownian motion implies the continuity of any Brownian diffusion-based dynamics. Plus, due to the Gaussian increments of Brownian motion, the stationary distribution of prices can hardly exhibit "fat tails" phenomena, which are instead sometimes observed in real market datas.

All these empirical features, which can hardly be predicted by diffusion-based models, can be naturally included when extending the class of the driving processes from the Brownian motion to the general Lévy processes. One might argue that all the properties mentioned above in the stationary distribution, as well as in the sample paths, of the observed prices can be reproduced by Gaussian-based models with local and/or stochastic volatility. Indeed, as pointed out by Bibby and Sorensen [30], an appropriate choice of the nonlinear diffusion coefficients can generate non-Gaussian diffusion processes, thus allowing for arbitrarily heavy tails. As for the jumps in the trajectories, sudden large changes in the prices can be obtained as well by fine-tuning the nonconstant coefficients of the model (see [61]), thus resulting in fluctuations in the prices that are equivalent to having actual jumps, at any given time scale.

Nevertheless, reproducing such phenomena via diffusion-based models leads to highly varying (nonstationary) diffusion coefficients in local volatility models or to unrealistically high values of "vol vol" in stochastic volatility models. By contrast, the jump part of a Lévy process includes jumps in its paths by definition, and may naturally exhibit fat-tails phenomena in its distribution. Therefore, these features naturally arise in jump-diffusion models, i.e models driven by discontinuous Lévy processes, with no need to fine-tune the parameters of the model to extreme and unrealistic values. For this reason, an increasing part of the literature in mathematical finance has been dealing, in the last fifteen years, with discontinuous stochastic models.

In this thesis we study some portfolio optimization and option pricing problems in market models where the dynamics of one or more risky assets are driven by Lévy processes, and it is divided in four independent parts. In the first part we study the portfolio optimization problem, for the logarithmic terminal utility and the logarithmic consumption utility, in a multi-defaultable Lévy driven model. In the second part we introduce a novel technique to price European defaultable claims when the pre-defaultable dynamics of the underlying asset follows an exponential Lévy process. In the third part we develop a novel methodology to obtain analytical expansions for the prices of European derivatives, under stochastic and/or local volatility models driven by Lévy processes, by analytically expanding the integro-differential operator associated to the pricing problem. In the fourth part we present an extension of the latter technique which allows for obtaining analytical expansion in option pricing when dealing with path-dependent Asian-style derivatives.

▷ **Overview of part I**

This part is based on a joint work ([186]) with Tiziano Vargiolu.

We analyze a market model given by n risky assets S^i and one riskless asset B, where any risky asset process is supposed to be the stochastic exponential driven by an n-dimensional additive process with regime-switching coefficients, as for example in [8, 44, 45, 206]. This market model naturally allows for defaults events, by assuming that the i-th driving process can jump with amplitude equal to -1. We here study the case when the regimes correspond to the default indicators of the risky assets.

This model has been chosen as a compromise between analytical tractability and flexibility in modeling various situations where risky assets are allowed to default, and can have pre-default dynamics driven by diffusion and/or jump processes, possibly with infinite random activity. Furthermore, through the dependence of the parameters on the current regime (i.e. the current default configuration), we are able to incorporate both simultaneous and information-induced contagion phenomena.

Our goal is to obtain the optimal consumption and the portfolio strategy for an investor who wants to maximize a logarithmic utility function of both his/her consumption and terminal wealth. To do this, we characterize a domain for the portfolio strategies in order for the wealth process to remain strictly positive. We then solve the utility maximization problem by means of the dynamic programming method, succeeding in proving a verification theorem based on the Hamilton-Jacobi-Bellman (HJB) equation and in exhibiting an explicit smooth solution to the HJB equation. The main conclusion of this is that the optimal consumption is an explicit linear function of the current wealth, while the optimal portfolio strategy turns out to be the maximizer of a suitable deterministic function depending only on time and on the current default indicators, but not on the current asset prices or wealth level. This represents a generalization of the model and the findings in [189], where the authors did not consider the case of one or multiple defaults, nor of regime switching and intermediate consumption.

After having characterized the optimal strategies in the general case, we present several examples with one, two or several defaultable assets, where most of the times we succeed in getting optimal strategies in closed form. Our results also allow to study with little effort the so-called *growth optimal portfolio* (GOP), and we exhibit an example where the GOP is a proper martingale or a strict local martingale depending on some boundary conditions.

▷ Overview of part II

This part is based on a joint work ([46]) with Prof. Agostino Capponi and Prof. Tiziano Vargiolu.

The research that we propose here belongs to the stream of literature focusing on pricing of defaultable bonds and vulnerable options within a joint equity-credit framework. Similarly to [156] and to Chapter 1, we used a reduced form model of default, and assumed the state dependent default intensity $(h_t)_{t\geq 0}$ to be a negative power of the stock price, i.e. $h_t \equiv h(S_t) = S_t^{-p}$. This is empirically relevant, especially in light of events occurred during the recent financial crisis. We also allowed for the possibility that the stock exhibits exogenous jumps of finite or infinite activity.

The dependence of the hazard intensity on the stock level makes the payoff of the vulnerable claim path dependent. To this purpose, we first develop a change of measure to reduce the problem to pricing an European style claim written on a free-default stock. The pricing problem then is reduced to characterizing the law of a process written as the solution of

$$dV_t = (2(1 + a)V_t + 1)dt + 2V_t dL_t,$$

with $(L_t)_{t\geq 0}$ being a Lévy process. An application of the Itô formula shows V_t to be an integral functional of e^{L_t}. In the continuous case, i.e. when L is a standard Brownian motion, the distribution of V_t has been widely studied by many authors; for instance, [77] and [213] independently derive the general expressions for the moments. A spectral representation of the transition density has been found by Linetsky, by inverting the Laplace transform in time, first obtained in [73] as the solution of an ordinary differential equation in the space variable.

Our contribution to this literature is a novel representation of the characteristic function of $\log V_t$ via a new methodology, which naturally allows for the process L_t to be a generic Lèvy process. More specifically, we proved that the characteristic function of $\log V_t$ can be characterized as the solution of a complex-valued infinite dimensional linear system of first order ordinary differential equations. After a reformulation as an abstract Cauchy problem in a suitably chosen Banach space, we obtain explicit expressions for the eigenvalues and eigenvectors of the matrix operator, and recovered an explicit eigenfunction expansion of the characteristic function. We then use this explicit representation to price defaultable bonds and options, demonstrating the accuracy and efficiency of the method.

▷ Overview of part III

This part is devoted to the development of some analytical approximations for PIDE's, arising in the pricing of European claims, under Lévy driven, possibly defaultable, models with local and/or stochastic volatilities. All the material collected in this part is based on the papers [171], [169], [183], [184] and [185], written in collaboration with Prof. Paolo Foschi, Dr. Matthew Lorig, Prof. Andrea Pascucci and M.Sc. Candia Riga.

In general, analytical approaches based on perturbation theory and asymptotic expansions have several advantages with respect to standard numerical methods: first of all, analytical approximations give closed-form solutions that exhibit an explicit dependency of the results on the underlying parameters. Moreover, analytical approaches produce much better and much faster sensitivities than numerical methods, although often accurate error estimates are not trivial to obtain.

All the chapters of this part are provided with numerical tests as to testify the accuracy and the efficiency of the proposed methodology. The Mathematica notebooks containing the general formulae and the experiments here reported are available in the authors' web-site: [167].

In Chapter 3 we consider a one-dimensional *local Lévy model* where the log-price X solves the SDE

$$dX_t = \mu(t, X_{t-})dt + \sigma(t, X_{t-})dW_t + dJ_t.$$

Here, W is a standard real Brownian motion and J is a pure-jump Lévy process, independent of W. Our main result in this chapter is a fourth order approximation formula for the characteristic function of X_t. In some particular cases, we also obtain an explicit approximation of the transition density of X_t and for the prices of European options.

Such local Lévy models have attracted an increasing interest in the theory of volatility modeling (see, for instance, [4], [48] and [60]); however, closed form pricing formulae are currently available only in few cases. Our approximation formulas provide a way to efficiently and accurately compute option prices and sensitivities by using standard and well-known Fourier methods (see, for instance, Heston [121], Carr and Madan [50], Raible [193] and Lipton [159]).

We derive the approximation formulas by introducing an "adjoint" expansion method; this is worked out in the Fourier space by considering the adjoint formulation of the pricing problem. Generally speaking, our approach makes use of Fourier analysis and PDE techniques. In the purely continuous case we also prove explicit error bounds for the expansion that generalize in a new and nontrivial way some classical estimates. Finally, we present some numerical tests under the Merton and Variance-Gamma models and show the effectiveness of the analytical approximations compared with Monte Carlo simulation.

In Chapter 5 we extend the "adjoint" expansion method in order to incorporate an exogenous stochastic-volatility given by a square-root diffusion process (Heston-type volatility). Our main result is a first-order approximation formula for the two-dimensional characteristic function of (S_t, v_t). Again, the pricing of European derivatives is done via some standard Fourier-inversion techniques.

A characterization of the two-dimensional law of the process is also important in the study of volatility derivatives, such as options on quadratic variation that have recently become a very popular instrument in financial markets. Also, our result can be used for volatility calibration purposes by Markovian projection methods via Gyöngy's lemma [113] (see, for instance, [190] and [105]). This will be object of a future investigation.

In this Chapter 5 we extend the technique introduced in Chapter 3 by admitting the possibility of default for the underlying asset, throughout the addition of a state-dependent default intensity, and by including state dependency for the Lévy measure. A state-dependent Lévy measure is an important feature because it allows for incorporating local dependence into infinite activity Lévy models that have no diffusion component, such as Variance Gamma (see [172]) and CGMY/Kobol (see [36, 47]).

Using techniques from pseudo-differential calculus, we provide explicit expansions for the Fourier transform of the transition density and of option prices. In the case of state dependent Gaussian jumps the respective inverse Fourier transforms can be explicitly computed, thus providing closed form approximations for densities and prices. Additionally, when considering defaultable bonds, approximate prices are computed as a finite sum; no numerical integration is required even in the general case.

For models with Gaussian-type jumps, we also provide pointwise error estimates for transition densities. Thus, we extend the previous results where we only consider the purely diffusive case. Additionally, our error estimates allow for jumps with locally dependent mean, variance and intensity. These results are comparable with the ones in [25], where only the case of a constant Lévy measure is considered.

Finally, in Chapter 6 we adapt, in the purely diffusion case, the methodology used in Chapter 5 by including stochastic volatility, given by an additional generic diffusion process. We derive a family of closed-form asymptotic expansions for transition densities, option prices and implied volatilities. In particular, the prices can be written as a differential operator acting on a Black-Scholes price (Gaussian density), whereas the implied volatility expansion is explicit, i.e. no numerical integration nor special functions are required. We also establish global error bounds for our asymptotic price and density expansions.

▷ **Overview of part IV**

This part is based on a joint work ([95]) with Prof. Andrea Pascucci and Prof. Paolo Foschi.

We aim to adapt the technique described in Chapter 3 to the case of hypoelliptic (fully degenerate-parabolic) two-dimensional operators, in order to apply it to the problem of pricing Asian-style claims under a general local volatility model.

Asian options are path dependent derivatives whose payoff depends on some form of averaging prices of the underlying asset. From the theoretical point of view, arithmetically-averaged Asian options have attracted an increasing interest in the last decades due to the awkward nature of the related mathematical problems. Indeed, even in the standard Black & Scholes (BS) model, when the underlying asset is a geometric Brownian motion, the distribution of the arithmetic average is not lognormal and it is quite complex to characterize it analytically. An integral representation was obtained in the pioneering work by Yor [213, 214], but with limited practical use in the valuation of Asian options.

Other asymptotic methods for Asian options with explicit error bounds were studied by Kunitomo and Takahashi [147], Shiraya and Takahashi [202], Shiraya, Takahashi and Toda [203] by Malliavin calculus techniques. Also Gobet and Miri [111] recently used Malliavin calculus to get analytical approximations and explicit error bounds; their approach is similar to ours as it is based on a Taylor expansion of the coefficients, but on the basis of preliminary numerical comparisons the resulting formulas seem to be different.

Here we consider the pricing problem for arithmetic Asian options under a local volatility, possibly time-dependent, model. Our idea is to use the natural differential geometric structure of the pricing operator regarded as a hypoelliptic (not uniformly parabolic) PDE of Kolmogorov type in $\mathbb{R}^3$. Our main results are explicit, BS-type approximation formulae not only for the option price, but also for the the terminal distribution of the asset and the average; furtherly we also get explicit approximation formulae for the Greeks that appear to be new also in the standard log-normal case. Although we do not address the theoretical problem of the convergence to get explicit error estimates, experimental results show that under the BS dynamics our formulae are extremely accurate if compared with other results in the literature. Under a general local volatility model, in comparison with Monte Carlo simulations the results are effectively exact under standard parameter regimes. The Mathematica notebook containing the general formulae and the experiments reported in this part is available in the web-site of the authors: [167].

We also mention that our method is very general and can also be applied to other path-dependent models driven by hypoelliptic degenerate PDEs; for instance, the models proposed by Hobson and Rogers [122] and Foschi and Pascucci [96].

Part I

Stochastic optimization in multi-defaultable market models

Chapter 1

Portfolio optimization in a defaultable Lèvy-driven market model

Based on a joint work ([186]) with Prof. Tiziano Vargiolu

Abstract: we analyse a market where the risky assets follow defaultable exponential additive processes, with coefficients depending on the default state of the assets. In this market we show that, when an investor wants to maximize a utility function which is logarithmic on both his/her consumption and terminal wealth, his/her optimal portfolio strategy consists in keeping proportions of wealth in the risky assets which only depend on time and on the default state of the risky assets, but not on their price nor on current wealth level; this generalizes analogous results of [189] in non-defaultable markets without intermediate consumption. While the non-defaultable case has been extensively treated in one (see e.g. [24, 28, 27, 100, 161, 132, 181]) and in more dimensions [43, 138, 145, 189], to the authors' knowledge this is the first time that such results are obtained for defaultable markets in this generality, whereas partial results (typically with only one defaultable asset) can be found in [31, 34, 41, 42, 44, 45]. We then present several examples of market where one, two or several assets can default, with the possibility of both direct and information-induced contagion, obtaining explicit optimal investment strategies in several cases. Finally, we study the growth-optimal portfolio in our framework and show an example with necessary and sufficient conditions for it to be a proper martingale or a strict local martingale.

Keywords: portfolio optimization, stochastic control, dynamic programming, HJB equation, jump-diffusion, multi-default, direct contagion, information-induced contagion.

1.1 Introduction

In the last years, mainly after the 2008 financial crisis and its aftermath, growing attention has been paid to financial models where the possibility of defaults is explicitly taken into account. However, in the literature one can only find models for financial markets with partial results (see [31, 34, 41, 42, 44, 45]) where typically only one asset can default, or models for several defaultable entities (typically CDS or CDO tranches) where the pricing problem is studied only for very specific derivatives (usually CDS or CDO, see [14, 64]).

In order to fill this gap in literature, we here present a model for a financial market where all the risky assets can possibly default, and their dynamics can depend on their default state, i.e. on which assets are already defaulted. In particular, we analyze a market model given by n risky assets S^i and one riskless asset B, where any risky asset process is supposed to be the stochastic exponential

$$\begin{cases} \mathrm{d}S_t^i = S_{t-}^i \,\mathrm{d}R_t^i, \\ S_0^i = s^i > 0, \end{cases} \qquad i = 1, \cdots, n, \tag{1.1}$$

with $R = (R^1, \cdots, R^n)$ being an n-dimensional additive process with regime-switching coefficients, as for example in [8, 44, 45, 206]. This market model naturally allows for defaults events, by assuming that the i-th driving process R^i can jump with amplitude equal to $\Delta R^i = -1$. We here study the case when the regimes correspond to the default indicators of the risky assets. Under suitable conditions on the jump measure (where jumps can be related to the default or to the risky assets' dynamics), we obtain the optimal consumption and portfolio strategy for an investor who wants to maximize a logarithmic utility function of both his/her consumption and terminal wealth. The optimal consumption turns out to be proportional to the current level of the agent's wealth, while the optimal portfolio strategy turns out to depend only on the default configuration process, i.e. it does not depend on the current value of the risky assets S^i but only on which assets are still not defaulted. This represents a generalization of the model and the findings in [189], where the authors did not consider the case of one or multiple defaults nor of intermediate consumption. On the other hand, the optimal consumption/investment strategies still depend on time as in [189] (this is also due to the non stationarity of the increments of the driving process R). After having characterized the optimal strategies in the general case, we present several examples with one, two or several defaultable assets, where most of the times we succeed in getting optimal strategies in closed form. Our results also allow to study with little effort the so-called *growth optimal portfolio* (GOP), and we exhibit an example where the GOP is a proper martingale or a strict local martingale depending on some boundary conditions.

The model in Equation (1.1) has been chosen as a compromise between analytical tractability and flexibility in modeling various situations where risky assets are allowed to default, and can have pre-default dynamics driven by diffusion and/or jump processes, possibly with infinite random activity. The naive way to model this would have been to take the dynamics in Equation (1.1), which generalizes several models where one [24, 28, 27, 100, 161, 132, 181] or several [43, 138, 145, 189] assets can exhibit jumps in their dynamics and which was already present in [189], with R still being a n-dimensional additive process, and allow for it to jump with multiplicative increments $\Delta R^i = -1$. This allows for direct contagion, as for suitable choices of the jump measure (see for example Section 6.5) simultaneous defaults are possible, but not for information-induced contagion, i.e.

where the knowledge that previous defaults had occurred does modify the dynamics of the undefaulted assets, as well as their default probabilities. This naive model is sketched out in Section 2. However, in order to take into account also information-induced effects in both the dynamics and the default probabilities, in Section 3 we show how to incorporate dependencies on past defaults in the risky assets' dynamics. This is done via a probabilistic construction, in the spirit of the one proposed in [22], where a process R with independent increments in each time interval between two consecutive defaults is built; thus, in this model R can be considered a regime-switching additive process, with regimes corresponding to the default indicators of the risky assets.

For the model in Equation (1.1), in Section 4 we study the problem of maximizing a logarithmic utility function; to do this, we characterize a domain for the portfolio strategies in order for the wealth process to remain strictly positive. We then solve the utility maximization problem by means of the dynamic programming method (in Section 5), succeeding in proving a verification theorem based on the Hamilton-Jacobi-Bellman (HJB) equation and in exhibiting an explicit smooth solution to the HJB equation. The main conclusion of this is that the optimal consumption is an explicit linear function of the current wealth, while the optimal portfolio strategy turns out to be the maximizer of a suitable deterministic function depending only on time and on the current default indicators, but not on the current asset prices or wealth level. This allows us to present several examples in Section 6, where one, two or several defaults can occur, possibly simultaneously. Particularly, several models already present in literature [14, 31, 41, 64] can be obtained as specific cases of this general framework or as starting points for the models presented here. In most of the examples, we obtain optimal investment strategies in closed form and discuss them.

In Section 7, we turn our attention to the characterization of the GOP, here defined as the portfolio which maximizes a logarithmic utility (for equivalent definitions of the GOP see for instance [56, 91]). In mathematical finance, the existence and the properties of the GOP have been widely studied by many authors, due to its relation with the numéraire portfolio. In particular, in [56] it has been shown in a quite flexible semi-martingale model that the GOP is such that all the other portfolios, evaluated with the GOP as numéraire, are supermartingales; this is called the numéraire property, which can be exploited in order to develop non-classical approaches in pricing derivative securities. For instance, in the benchmark approach by Platen [191] this can be done even in models where an Equivalen Martingale Measure (EMM) is absent. In [56] the authors proved that, even when a classical risk neutral measure does not exist, the existence of the GOP implies the existence of a numéraire under which an EMM exists. Nevertheless, GOP denominated prices might fail to be martingale and being instead strict supermartingales. Examples of this phenomenon are also given in [22, 38, 66, 146]. In this regard, we will show that in our model the inverse GOP process is either a martingale or a strict supermartingale depending on whether the growth optimal strategy is an internal or a boundary solution with respect to the domain of the admissible strategies.

We now give a brief outline of this chapter: in Section 2 a naive model, where the risky assets' prices are defaultable exponential additive models, is presented. In Section 3 we build a more general model where asset prices are driven by regime-switching additive models, with regimes corresponding to default indicators. In Section 4 we frame the portfolio optimization problem and characterize the portfolio strategies such that the portfolio wealth stays strictly positive. In Section 5 the portfolio optimization problem is solved with the dynamic programming method by using the HJB equation. In Section 6 we present several examples, with many optimal portfolios written in closed form and commented. In Section 7 we study the GOP and show in an example under which conditions the GOP is a strictly local martingale or a true martingale.

1.2 A simplified model

Before rigorously defining our framework in its full generality, we aim in this section to give a heuristic description of a simplified version of it. This approach allows the reader to get a quite intuitive idea of the dynamics involved in our model. After this brief introduction, the definition of the general setting in Section 3 will seem a natural extension of this simple one.

We consider a portfolio composed of a locally riskless asset B and n risky assets S^i, $i = 1, \cdots, n$. By considering discounted prices, we can assume without loss of generality that $B \equiv 1$. For the risky assets we assume the dynamics in Equation (1.1) where in this section $R = (R^1, \cdots, R^n)$ is an n-dimensional additive process, i.e. a process with independent increments [61], that can exhibit jumps with size -1 in any of its components, possibly simultaneously. We notice that we can rewrite Equation (1.1) in the vectorial form

$$\mathrm{d}S_t = \mathrm{diag}(S_{t-})\ \mathrm{d}R_t,$$

where $\mathrm{diag}(v)$ is the diagonal matrix in $\mathbb{R}^{n\times n}$ with principal diagonal containing the elements of v. This allows to explicitly represent the n-dimensional additive process R in Equation (1.1) via the Levy-Ito representation as

$$\begin{cases} \mathrm{d}R_t = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W_t + \int_{\mathbb{R}^n} x(N(\mathrm{d}t, \mathrm{d}x) - \nu_t(\mathrm{d}x)\mathrm{d}t), \\ R_0^i = 0, \quad i = 1, \cdots, n, \end{cases}$$

with $\mu = (\mu_1, \cdots, \mu_n) : [0, T] \to \mathbb{R}^n$, $\sigma = (\sigma_{ij})_{ij} : [0, T] \to \mathbb{R}^{n\times k}$ deterministic measurable functions, $W = (W^1, \cdots, W^k)$ a k-dimensional Brownian motion and $N(\mathrm{d}t, \mathrm{d}x)$ a jump random measure on $\mathbb{R}^+ \times \mathbb{R}^n$ with compensating measure $\nu_t(\mathrm{d}x)$. The solution of Equation (1.1) is

$$S_t^i = s^i e^{R_t^i - \frac{1}{2}\int_0^t \|\sigma(u)\|^2 du} \prod_{0<u\le t} (1 + \Delta R_u^i) e^{-\Delta R_u^i}, \quad i = 1, \cdots, n, \tag{2.2}$$

(see [192], Theorem II.37), where $\Delta R_u^i := R_u^i - R_{u-}^i$ represents the jump of R^i at the time u.

Equation (2.2) shows first that we shall impose $\Delta R_u^i \ge -1$ in order for S^i to stay non-negative, and furthermore that the process S^i reaches the *cemetery value* 0 as soon as the process R^i jumps with amplitude $\Delta R^i = -1$. Therefore, we define the *default time* τ^i as time at which S^i jumps to 0, i.e.

$$\tau^i := \min\{t > 0 | \Delta R_t^i = -1\}. \tag{2.3}$$

Now, for any $i = 1, \cdots, n$, we introduce the *default indicator process*

$$D_t^i = \mathbb{1}_{[\![\tau^i, \infty)}(t).$$

Note that D^i admits the differential representation

$$\mathrm{d}D_t^i = \int_{X^n} (1 - D_{t-}^i) \mathbb{1}_{\{x^i=-1\}}(x) N(\mathrm{d}t, \mathrm{d}x). \tag{2.4}$$

According to this setting, two or more S^i may simultaneously jump to 0 with positive probability. Indeed, this scenario is verified any time two or more R^i jump simultaneously with size -1. Nevertheless, the driving process R has independent increments, and thus, D^i_t is independent of $(D^j_u)_{u<t}$, for any $j \neq i$. To sum up, simultaneous defaults are allowed, but defaults occurred in the past can not change the probabilities of future ones. Financially speaking, within this framework we are able to capture *instantaneous contagion* but not *information-induced* one.

In order to overcome this shortcoming, a natural extension seems to let the jump measure $N(\mathrm{d}t, \mathrm{d}x)$ depend on the current default configuration D_t: this will be done in the next section.

1.3 The general setting

In this section we generalize the construction of Section 2 by introducing different regimes for the jump measure, the drift and the diffusion of the driving process $R = (R^1, \cdots, R^n)$, regimes consisting in the default indicators' vector $D = (D^1, \cdots, D^n)$. This construction is analogous to the one in [22].

Let $n, k \in \mathbb{N}$, $T > 0$, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space rich enough to support:

- a k-dimensional Brownian motion $W = (W^1, \cdots, W^k)$;
- a family $(N^d, \nu^d)_{d \in \{0,1\}^n}$, where $N^d \equiv N(d, \mathrm{d}t, \mathrm{d}x)$ are independent Poisson measures on $[0,T] \times \mathbb{R}^n$, and $\nu^d_t \equiv \nu_t(d, \mathrm{d}x)$ are the respective compensating measures.

Next, in analogy with (2.3)-(2.4), we define the default time of the i-th asset τ^i by means of the respective default indicator process D^i.

Definition 3.1. *Let the $\{0,1\}^n$-valued process $D \equiv (D_t)_{0 \le t \le T} = (D^1_t, \cdots, D^n_t)_{0 \le t \le T}$ be the unique strong solution of the n-dimensional system*

$$\mathrm{d}D^i_t = (1 - D^i_{t-}) \int_{\mathbb{R}^n} \mathbb{1}_{\{x^i = -1\}}(x) N(D_{t-}, \mathrm{d}t, \mathrm{d}x), \quad i = 1, \cdots, n, \tag{3.5}$$

with the initial conditions

$$D^i_0 = 0, \quad i = 1, \cdots, n. \tag{3.6}$$

Then, denoting with $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$ the filtration generated by $(D_t)_{t \ge 0}$, we define the $\mathbb{F}$-stopping times

$$\tau^i := \min\{t > 0 | D^i_t = 1\}, \quad i = 1, \cdots, n.$$

Remark 3.2. *(Construction) The process D can be constructed pointwise in the following way. We first consider two families of random variables $(t_k)_{0 \le k \le n}$, $t_k \in \mathbb{R}^+ \cup \{\infty\}$, and $(\zeta_k)_{0 \le k \le n}$, $\zeta_k \in \{0,1\}^n$, recursively defined as*

$$t_0 = 0, \quad \zeta_0 = 0,$$

and

$$\begin{aligned} t^i_{k+1} &:= \inf_{t > t_k} \left\{ t | \exists x \in \mathbb{R} \text{ s.t. } \mathbb{1}_{\{x_i = -1\}}(x) N(\zeta_k, \{t\}, \{x\}) = 1 \right\} \mathbb{1}_{\{\zeta^i_k = 0\}} + \infty \mathbb{1}_{\{\zeta^i_k = 1\}}, \\ t_{k+1} &:= \min_{1 \le i \le n} t^i_{k+1}, \end{aligned}$$

$$\zeta^i_{k+1} := \zeta^i_k + (1-\zeta^i_k)\mathbb{1}_{\{t_{k+1}\}}(t^i_{k+1}), \quad 1 \le i \le n.$$

Note that, in the above construction, the random variable t_k represents the k-th default time in chronological order, whereas the random vector ζ_{k+1} represents the default indicators' configuration when the k-th default event occurs. We prefer to remark one more time that index k in t_k is referred to the chronological order, and that t_k is in general different from the time τ^k, when the k-th risky asset defaults.

Then, for any $t \in [0,T]$ we define

$$D_t := \sum_{k=0}^{n} \zeta_k \mathbb{1}_{\{[t_k,t_{k+1}[\}}(t).$$

It is easy to verify that the $\{0,1\}^n$-valued process $D \equiv (D_t)_{0\le t\le T}$ is a solution for (3.5)-(3.6)*.*

We now define the driving process $R \equiv (R_t)_{0\le t\le T} = (R^1_t, \cdots, R^n_t)_{0\le t\le T}$ in (1.1) as the unique strong solution of the n-dimensional system

$$\begin{cases} \mathrm{d}R_t = \mu(t,D_t)\mathrm{d}t + \sigma(t,D_t)\mathrm{d}W_t + \int_{\mathbb{R}^n} x\big(N(D_{t-},\mathrm{d}t,\mathrm{d}x) - \nu_t(D_{t-},\mathrm{d}x)\mathrm{d}t\big), \\ R^i_0 = 0, \quad i = 1,\cdots,n, \end{cases} \tag{3.7}$$

where $\mu(\cdot,d) : [0,T] \to \mathbb{R}^n$, $\sigma(\cdot,d) : [0,T] \to \mathbb{R}^{n\times d}$ are deterministic measurable functions for any $d \in \{0,1\}^n$. Afterwards we will denote with $\sigma_i(t,d)$ the i-th row of $\sigma(t,d)$. In order for Equation (3.7) and the further computations to make sense we need the following assumptions to be satisfied.

Assumption A.1. *(Finite variance) For any $d \in \{0,1\}^n$,*

$$\int_0^T \left(\|\mu(t,d)\| + \|\sigma(t,d)\|^2 + \int_{\mathbb{R}^n} \|x\|^2 \, \nu_t(d,\mathrm{d}x) \right) \mathrm{d}t < +\infty,$$

where the $\|\cdot\|$ represent the Euclidean norms on $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$.

Assumption A.2. *(Non negativity of prices) For any $d \in \{0,1\}^n$ and $t \in [0,T]$,*

$$\mathrm{supp}(\nu^d_t) \subset X^n := \{x \in \mathbb{R}^n | x_i \ge -1 \ \forall i = 1,\cdots,n\}. \tag{3.8}$$

Assumption A.3. *(Continuity in time of the compensator) For any $d \in \{0,1\}^n$ and for any Borel set $B \subset \mathbb{R}$, $\nu^d_t(B)$ is continuous in t.*

As seen in the previous section, Assumption A.3. is equivalent to say that the risky assets prices stay a.s. non-negative for each $t \in [0,T]$. Indeed, the solution of the SDE (1.1) is still

$$S^i_t = s^i e^{R^i_t - \frac{1}{2}[R^i,R^i]^c_t} \prod_{0<s\le t} (1+\Delta R^i_s) e^{-\Delta R^i_s}, \tag{3.9}$$

(see [192], Theorem II.37) with $[R^i,R^i]^c$ being the continuous part of the quadratic variation process of R, end (3.8) implies $1+\Delta R^i_t \ge 0$ for any $i = 1,\cdots,n$ and $t \in [0,T]$. On the other hand, Equation (3.9) shows that the process S^i jumps to 0 as soon as the process R^i jumps with amplitude $\Delta R^i = -1$, and stays there at any future time. Eventually, by Definition 3.1 combined with Equation (3.7) we get

$$\tau^i = \min\{t > 0 | \Delta R^i_t = -1\} = \min\{t > 0 | S^i_t = 0\},$$

and thus, as in the previous section, the default of i-th asset coincides with its value jumping to 0.

1.4 The portfolio optimization problem

Let now $\pi_t = (\pi_t^1, \cdots, \pi_t^n)$ be a trading strategy representing the quantities of the risky assets $(S_t^1, \cdots, S_t^n)$ held in a self-financing portfolio, whose value at time t is given by

$$V_t^\pi = \pi_t^0 + \langle \pi_t, S_t \rangle = \pi_t^0 + \sum_{i=0}^{n} \pi_t^i S_t^i,$$

where $\langle \cdot, \cdot \rangle$ represents the scalar product in $\mathbb{R}^n$. In the case when $V_t^\pi > 0$, we can represent the portfolio in terms of its proportions invested in each risky asset, defining the vector $\mathfrak{h}_t := (\mathfrak{h}_t^1, \cdots, \mathfrak{h}_t^n)$ componentwise as

$$\mathfrak{h}_t^i := \frac{\pi_t^i S_t^i}{V_t^\pi}, \quad i = 1, \cdots, n. \tag{4.10}$$

Furthermore, we consider a strictly positive process $\mathfrak{c}_t$ denoting the instantaneous consumption at time t. By the self-financing property we have

$$\mathrm{d}V_t^{\mathfrak{h},\mathfrak{c}} = \sum_{i=1}^{n} \pi_{t-}^i \mathrm{d}S_t^i - \mathfrak{c}_t \mathrm{d}t = \sum_{i=1}^{n} \pi_{t-}^i S_{t-}^i \mathrm{d}R_t^i - \mathfrak{c}_t \mathrm{d}t$$

(by (4.11))

$$= V_{t-}^{\mathfrak{h},\mathfrak{c}} \langle \mathfrak{h}_{t-}, \mathrm{d}R_t \rangle - \mathfrak{c}_t \mathrm{d}t$$

where we denoted by $V^{\mathfrak{h},\mathfrak{c}}$ the portfolio value to remark that it is expressed as a function of its proportions $\mathfrak{h}$ and the consumption $\mathfrak{c}$. Here we used

$$\pi_{t-}^i S_{t-}^i = V_{t-}^\pi \mathfrak{h}_{t-}^i, \quad t \in [0, T], \tag{4.11}$$

which is still true also for $t > \tau^i$ because, by (4.10), we have

$$\mathfrak{h}_t^i = 0, \quad \forall\, t \in [\tau^i, T], \quad i = 1, \cdots, n. \tag{4.12}$$

Nevertheless, if $\mathfrak{h}$ is a generic $\mathbb{F}$-predictable process, the solution of

$$\mathrm{d}V_t^{\mathfrak{h},\mathfrak{c}} = V_{t-}^{\mathfrak{h},\mathfrak{c}} \langle \mathfrak{h}_{t-}, \mathrm{d}R_t \rangle - \mathfrak{c}_t \mathrm{d}t, \quad t \in [0, T], \tag{4.13}$$

still depends on $\mathfrak{h}^i$ even after the time τ^i. Thus, we should impose the condition (4.12) on the control variable $\mathfrak{h}$ when using Equation (4.13) for optimization purposes, but this would lead to a problem with very non standard control constraints.

In alternative, we prefer to define $V^{\mathfrak{h},\mathfrak{c}}$ as the solution of the SDE

$$\mathrm{d}V_t = V_{t-} \langle \mathrm{diag}(\underline{1} - D_{t-}) \mathfrak{h}_{t-}, \mathrm{d}R_t \rangle - \mathfrak{c}_t \mathrm{d}t, \quad t \in [0, T], \tag{4.14}$$

where $\underline{1} = (1, \cdots, 1) \in \mathbb{R}^n$, as in [42]. In this way, we need no additional conditions on $\mathfrak{h}$, as the process $V^{\mathfrak{h},\mathfrak{c}}$ is independent of $\mathfrak{h}^i$ after τ^i. In order to shorten notation we introduce the following definition.

Definition 4.1. *For any $d \in \{0,1\}^n$ and $x \in \mathbb{R}^n$, we define the vector $x^d \in \mathbb{R}^n$ as*

$$x^d = x \cdot \mathrm{diag}(\underline{1} - d).$$

In other words, x_i^d is equal to x_i if $d_i = 0$, i.e. the i-th risky asset is still alive, whereas $x_i^d = 0$ if $d_i = 1$, i.e. the i-th risky asset already defaulted. A necessary condition for $V^{\mathfrak{h},\mathfrak{c}}$ to stay $\mathbb{P}$-a.s. positive for any $t \in [0,T]$ is that

$$\langle \mathfrak{h}_{t-}^{D_{t-}}, \Delta R_t \rangle > -1 \quad \mathbb{P}\text{−a.s.} \quad \forall t \in [0,T]. \tag{4.15}$$

Indeed, by (4.14) we have

$$V_t^{\mathfrak{h},\mathfrak{c}} = V_0^{\mathfrak{h},\mathfrak{c}} - \int_0^t \mathfrak{c}_s \mathrm{d}s + \int_0^t V_{s-}^{\mathfrak{h},\mathfrak{c}} \langle \mathfrak{h}_{s-}^{D_{s-}}, \mathrm{d}R_s \rangle,$$

and therefore, as long as $V_s^{\mathfrak{h},\mathfrak{c}}$ is positive for $s \in [0,t]$, $V_t^{\mathfrak{h},\mathfrak{c}}$ jumps with size less or equal $-V_{t-}^{\mathfrak{h},\mathfrak{c}}$ if $\langle \mathfrak{h}_{t-}^{D_{t-}}, \Delta R_t \rangle \leq -1$. A sufficient condition for (4.15) to hold is

$$\mathfrak{h}_t \in H_t := \left\{ h \in \mathbb{R}^n | \langle h, x \rangle > -1 \ \ \nu_t^d(\mathrm{d}x) - a.s. \ \forall d \in \{0,1\}^n \right\}, \quad \forall t \in [0,T]. \tag{4.16}$$

Example 4.2. *If the jumps of the process R are unbounded from above, i.e.* $\mathrm{supp}(\nu_t^d) \equiv X^n$ *for any $d \in \{0,1\}^n$, with X^n as in (3.8), then H_t is the n-dimensional unit simplex in $\mathbb{R}^n$, i.e. $H_t \equiv \{h \in \mathbb{R}^n | \ h_i \geq 0, \ \sum_{i=1}^n h_i < 1\}$.*

We now define the set of admissible strategies.

Definition 4.3. *An $\mathbb{R}^{n+1}$-valued $\mathbb{F}$-predictable process $(\mathfrak{h}, \mathfrak{c}) \equiv (\mathfrak{h}_u, \mathfrak{c}_u)_{t \leq u \leq T}$ is said to be an admissible strategy if*

a) $\mathfrak{h}_u \in H$ $\mathbb{P}$-a.s. for any $u \in [t,T]$, where H is a compact convex set $H \subset \mathbb{R}^n$ such that $H \subset \mathrm{int}(\cap_{u \in [t,T]} H_u)$;

b) $\mathfrak{c}_u > 0$ $\mathbb{P}$-a.s. for any $u \in [t,T]$.

c) For any initial condition $V_t = v > 0$ and $D_t = d \in \{0,1\}^n$, the $(n+1)$-dimensional system (3.5)-(4.14) has a unique strong solution $(V,D)^{\mathfrak{h},\mathfrak{c};t,v,d} = (V_s^{\mathfrak{h},\mathfrak{c};t,v,d}, D_s^{t,d})_{s \in [t,T]}$ such that $V_s > 0$ for any $s \in [t,T]$.

We denote by $\mathcal{A}[t,T]$ the set of all admissible strategies.

Sometimes in the sequel, in order to shorten the notation, we will suppress the explicit dependence on t, v, d in $(V,D)^{\mathfrak{h},\mathfrak{c};t,v,d}$.

We aim to find the optimal control process $(\bar{\mathfrak{h}}, \bar{\mathfrak{c}}) \in \mathcal{A}[t,T]$ such that

$$\mathbb{E}[U(V_T^{\bar{\mathfrak{h}},\bar{\mathfrak{c}};t,v,d})] = \max_{(\mathfrak{h},\mathfrak{c}) \in \mathcal{A}[t,T]} \mathbb{E}\left[U(V_T^{\mathfrak{h},\mathfrak{c};t,v,d}) + \int_0^T u(t, \mathfrak{c}_t) \mathrm{d}t \right], \tag{4.17}$$

where U, u are logarithmic utility functions

$$U(x) = A \log x, \quad u(t,c) = B e^{-\delta(T-t)} \log c, \tag{4.18}$$

with $A, B, \delta \geq 0$ such that $A + B > 0$.

1.5 Dynamic programming solution

Here we use dynamic programming in order to solve the optimal control problem (4.17). For any $(\mathfrak{h}, \mathfrak{c}) \in \mathcal{A}[t,T]$, $t \in [0,T]$, $v \in \mathbb{R}^+$ and $d \in \{0,1\}^n$ we define the function

$$J^{\mathfrak{h},\mathfrak{c}}(t,v,d) := \mathbb{E}\left[U(V_T^{\mathfrak{h},\mathfrak{c};t,v,d}) + \int_0^T u(t,\mathfrak{c}_t)\mathrm{d}t\right] \tag{5.19}$$

Moreover we define the value function $J : [0,T] \times \mathbb{R}^+ \times \{0,1\}^n \to \mathbb{R}^+$ as

$$J(t,v,d) := \sup_{(\mathfrak{h},\mathfrak{c}) \in \mathcal{A}[t,T]} J^{\mathfrak{h},\mathfrak{c}}(t,v,d). \tag{5.20}$$

Following the approach in [90], by formal arguments we obtain that J solves the so-called HJB (Hamilton-Jacobi-Bellman) equation

$$-J_t(t,v,d) = \sup_{h \in H,\, c>0} \left(A^{h,c}J(t,v,d) + u(c)\right) \tag{5.21}$$

where, for any $h \in H$ and $c > 0$, $A^{h,c}$ is the infinitesimal generator of the process $(V,D)^{h,c}$, i.e.:

$$\begin{aligned} A^{h,c}J(t,v,d) &= \Big(\langle \mu(t,d), h^d\rangle v - c\Big) J_v(t,v,d) + \frac{1}{2}\langle h^d, \Sigma(t,d)h^d\rangle v^2 J_{vv}(t,v,d) \\ &+ \int_{X^n} \Big(J\big(t, v(1+\langle x, h^d\rangle), d + \chi(d,x)\big) - J(t,v,d) - \langle h^d, x\rangle v J_v(t,v,d)\Big)\nu_t^d(\mathrm{d}x), \end{aligned} \tag{5.22}$$

with

$$\Sigma(t,d) = \sigma\sigma^*(t,d),$$

and where the function $\chi : \{0,1\}^n \times R^n \to \{0,1\}^n$ is defined as

$$\chi_i(d,x) := (\underline{1} - d_i)\mathbb{1}_{\{x_i=-1\}}(x), \quad i = 1, \cdots, n. \tag{5.23}$$

Moreover, by (5.19)-(5.20) we directly obtain the terminal condition

$$J(T,v,d) = U(v) \quad v \in \mathbb{R}^+, d \in \{0,1\}^n. \tag{5.24}$$

The next theorem, which is a particular case of [90, Theorem III.8.1], rigorously connects the optimal control problem (4.17) with the HJB equation and gives us a useful characterization of the optimal control process $(\bar{\mathfrak{h}}, \bar{\mathfrak{c}})$ when it exists. Before stating the verification theorem, we formally define the domain of the operator $A^{h,c}$.

Definition 5.1. *We denote with $\mathcal{D}$ the set of the functions $f \in C^{1,2}([0,T] \times \mathbb{R}^+ \times \{0,1\}^n)$ such that, for any $(t,v,d) \in [0,T] \times \mathbb{R}^+ \times \{0,1\}^n$ and for any $(\mathfrak{h},\mathfrak{c}) \in \mathcal{A}[t,T]$ the so-called Dynkyn formula holds, i.e.*

$$\mathbb{E}[f(T,(V,D)_T^{\mathfrak{h},\mathfrak{c}})] - \mathbb{E}[f(t,(V,D)_t^{\mathfrak{h},\mathfrak{c}})] = \mathbb{E}\left[\int_t^T A^{\mathfrak{h}_u,\mathfrak{c}_u} f(u,(V,D)_u^{\mathfrak{h},\mathfrak{c}})\mathrm{d}u\right].$$

We can now state the following

Theorem 5.2. *Let $K \in \mathcal{D}$ be a classical solution of* (5.21) *with terminal condition* (5.24)*. Then, for any $(t, v, d) \in [0, T] \times \mathbb{R}^+ \times \{0, 1\}^n$ we have*

a) $K(t, v, d) \geq J^{\mathfrak{h},\mathfrak{c}}(t, v, d)$ for any admissible control $(\mathfrak{h}, \mathfrak{c}) \in \mathcal{A}[t, T]$;

b) if there exists an admissible control $(\bar{\mathfrak{h}}, \bar{\mathfrak{c}}) \in \mathcal{A}[t, T]$ such that

$$(\bar{\mathfrak{h}}_s, \bar{\mathfrak{c}}_s) \in \underset{(h,c)\in H\times\mathbb{R}^+}{\arg\max} \Big(A^{h,c} K(s, (V, D)_s^{\bar{\mathfrak{h}},\bar{\mathfrak{c}};t,v,d}) + u(s, c) \Big) \quad \mathbb{P}\text{−a.s. } \forall s \in [t, T], \tag{5.25}$$

then $K(t, v, d) = J^{\bar{\mathfrak{h}},\bar{\mathfrak{c}}}(t, v, d) = J(t, v, d)$.

Now we use Theorem 5.2 in order to solve the optimization problem (4.17). Analogously to [189] it turns out that the optimal control $(\bar{\mathfrak{h}}, \bar{\mathfrak{c}})$ in (5.25) is a Markov control policy. In particular we are going to find out that

$$\bar{\mathfrak{h}}(s) = \bar{h}(s, D_s), \quad \bar{\mathfrak{c}}(s) = \bar{c}(s) V_s^{\bar{\mathfrak{h}},\bar{\mathfrak{c}}},$$

where $\bar{h} : [0, T] \times \{0, 1\}^n \to H$ and $\bar{c} : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ are deterministic functions such that

$$(\bar{h}(t, d), \bar{c}(t, v)) \in \underset{(h,c)\in H\times\mathbb{R}^+}{\arg\max} \Big(A^{h,c} K(t, v, d) + u(t, c) \Big),$$

for any $(t, v, d) \in [0, T] \times \mathbb{R}^+ \times \{0, 1\}^n$. We are now in the position to characterize the value function J and the optimal strategy $(\bar{\mathfrak{h}}, \bar{\mathfrak{c}})$. Before to state our main result we introduce the following

Definition 5.3. *For any $d \in \{0, 1\}^n$, let $F^d : [0, T] \times H \to \mathbb{R}$ be the function*

$$F^d(t, h) := \langle \mu(t, d), h^d \rangle - \frac{1}{2} \langle h^d, \Sigma(t, d) h^d \rangle + \int_{X^n} \log\left(1 + \langle x, h^d \rangle\right) - \langle x, h^d \rangle \ \nu_t^d(\mathrm{d}x), \tag{5.26}$$

where h^d is defined as in Definition 4.1.

Theorem 5.4. *Let $U(v)$ and $u(t, c)$ be the logarithmic functions defined as in* (4.18)*. Then:*

a) Equation (5.21) *with terminal condition* (5.24) *has a classical solution K given by*

$$K(t, v, d) = \begin{cases} \left(A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)\right) \log v + \Phi^d(t) & \text{if } \delta > 0, \\ (A + B(T - t)) \log v + \Phi^d(t) & \text{if } \delta = 0, \end{cases} \tag{5.27}$$

for any $(t, v, d) \in [0, T] \times \mathbb{R}^+ \times \{0, 1\}^n$, where $(\Phi^d)_{d\in\{0,1\}^n}$ is a family of suitable C^1 deterministic functions such that $\Phi^d(T) = 0$.

b) K belongs to $\mathcal{D}$.

c) $K = J$ and an optimal control process $(\bar{\mathfrak{h}}, \bar{\mathfrak{c}})$ is given by

$$(\bar{\mathfrak{h}}(t), \bar{\mathfrak{c}}(t)) := \left(\bar{h}(t, D_{t-}), \bar{c}(t) V_{t-}^{\bar{\mathfrak{h}},\bar{\mathfrak{c}}}\right), \tag{5.28}$$

where $\bar{h} : [0, T] \times \{0, 1\}^n \to H$ is a function such that

$$\bar{h}(t, d) \in \underset{h\in H}{\arg\max}\, F^d(t, h) \tag{5.29}$$

with $F^d(t,h)$ *as in* (5.26),

$$\bar{c}(t) := \begin{cases} \frac{Be^{-\delta(T-t)}}{A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)} & \text{if } \delta > 0, \\ \frac{B}{A+B(T-t)} & \text{if } \delta = 0, \end{cases} \tag{5.30}$$

and where $V^{\bar{\mathfrak{h}},\bar{\mathfrak{c}}}$ *is the unique positive solution of*

$$\frac{\mathrm{d}V_t^{\bar{\mathfrak{h}},\bar{\mathfrak{c}}}}{V_t^{\bar{\mathfrak{h}},\bar{\mathfrak{c}}}} = \langle \mathrm{diag}(\underline{1} - D_{t-})\bar{h}(t, D_{t-}), \mathrm{d}R_t\rangle - \bar{c}(t)\mathrm{d}t, \quad t \in [0, T],$$

Before proving Theorem 5.4 we explicitly remark what follows.

Remark 5.5. *A function* $\bar{h}$ *such that* (5.29) *holds exists and, for any* $d \in \{0,1\}^n$ *is unique in its components* $\bar{h}^i$ *such that* $d_i = 0$*. In fact,* F^d *in* (5.26) *does not depend on the* i*-th components of* h *if* $d_i = 1$*, and on the other hand,* F^d *is a strictly concave function and* H *is a compact convex subset of* $\mathbb{R}^n$*. Roughly speaking, the* i*-th component of the optimal strategy* $\bar{h}$ *is not relevant after the risky asset* S^i *defaults, which is consistent with Equation* (4.14).

Remark 5.6. *In analogy with [189, Remark 3.2], we point out that the optimal Markov policy* $\bar{h}$ *does not depend on the variable* v*. Thus, the optimal strategy only depends on* t *and* D_{t-} *through* $\mu(t, D_{t-})$, $\sigma(t, D_{t-})$ *and* $\nu_t(D_{t-}, \mathrm{d}x)$*, but not on the current level of wealth* V_t*. The dependence on the risky asset prices* S_t^i, $i = 1, \cdots, n$*, is just when the process* S^i *jumps to zero, otherwise the optimal strategy is a completely deterministic function as in [189]. In the time-homogeneous case, i.e.* $\mu(t,d) \equiv \mu(d)$, $\sigma(t,d) \equiv \sigma(d)$ *and* $\nu_t^d \equiv \nu$, $\bar{\mathfrak{h}}$ *is piecewise constant in time, jumping only at the default times* τ^i, $i = 1, \cdots, n$.

Remark 5.7. *By contrast, for any* $t \in [0,T]$, $\bar{\mathfrak{c}}_t$ *is a linear function of* V_t *that only depends on the parameters* A, B, δ *of the utility functions* U *and* u*. Therefore, the optimal consumption* $\bar{\mathfrak{c}}_t$ *does not depend explicitly on default configuration* D_t*, nor on the model parameters* $\mu(t,d), \sigma(t,d), \nu_t(d,dx)$*. Furthermore, consistently with the financial intuition, the optimal consumption* $\bar{\mathfrak{c}}_t$ *is constantly equal to* 0 *when consumption the utility function* $u(t,c)$ *is constantly null, i.e.* $B = 0$.

In order to prove Theorem 5.4, we need to introduce the following notation.

Definition 5.8. *Given* $d \in \{0,1\}^n$*, we call the length of* d *the positive integer defined as*

$$l(d) := n - \sum_{i=0}^{n} d_i.$$

Moreover we establish on $\{0,1\}^n$ *the following (partial) order relation:*

$$d \leq d' \quad \text{if} \qquad d_i \geq d'_i \quad \forall i = 1, \cdots, n.$$

Note that, given $D_t = d$ *for a certain* $t \leq 0$*, the states* $d' < d$ *are the only states accessible for* D *after the time* t.

Roughly speaking, the length of d is equal to the number of risky asset that are still alive. In particular, when every risky asset is already defaulted we have $l(d) = 0$, while when every risky asset is still alive we have $l(d) = n$. We also explicitly observe that

$$d + \chi(d, x) \leq d \quad \forall d \in \{0, 1\}^n, x \in X^n.$$

Hence by (5.22), given a state $d \in \{0,1\}^n$, $A^h J(t, x, d)$ depends only on the states $d' \leq d$, i.e. the states whose alive assets are a subset of the alive ones in d; in other words, $A^{h,c} J(t, x, d)$ does not depend on the assets already defaulted.

We also need the following

Lemma 5.9. *Consider the function*

$$\psi(t, v, c) = \begin{cases} Be^{-\delta(T-t)} \log c - c\frac{A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)}{v}, & \delta > 0 \\ B \log c - c\frac{A + B(T-t)}{v}, & \delta = 0 \end{cases} \tag{5.31}$$

with $A, B \geq 0$, $A + B > 0$. Then, for any $t \in [0, T]$ and $v > 0$ we have

$$\bar{c}(t)v = \arg\max_{c>0} \psi(t, v, c), \tag{5.32}$$

where $\bar{c}$ is defined as in (5.30)*. Moreover,*

$$\begin{aligned} \max_{c>0} \psi(t, v, c) = \psi(t, v, \bar{c}(t)v) =& Be^{-\delta(T-t)} \left(\log \left(\frac{Be^{-\delta(T-t)}}{A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)} \right) - 1 \right) \\ &+ Be^{-\delta(T-t)} \log v \end{aligned} \tag{5.33}$$

if $\delta > 0$, whereas

$$\max_{c>0} \psi(t, v, c) = \psi(t, v, \bar{c}(t)v) = B \left(\log \left(\frac{B}{A + B(T-t)} \right) - 1 \right) + B \log v$$

if $\delta = 0$.

Proof. We only prove the case $\delta > 0$. For any $t \in [0, T], v > 0$ we have

$$\psi_c(t, v, c) = \frac{Be^{-\delta(T-t)}}{c} - \frac{A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)}{v} = 0$$

if and only if $c = \bar{c}(t)v$. Thus, $\bar{c}(t)v$ is the only stationary point for $\psi(t, v, \cdot)$, and since $\lim_{c \to 0} \psi(t, v, c) = \lim_{c \to \infty} \psi(t, v, c) = -\infty$, we obtain (5.32). Eventually, (5.33) follows from a direct computation. □

We now prove Theorem 5.4.

Proof of Theorem 5.4. We only prove the theorem for $\delta > 0$, as the case $\delta = 0$ is totally analogous.

Part a). By induction on $k = l(d)$. We start proving the statement when $k = 0$. In this case we clearly have $d = \mathbf{1} := (1, \cdots, 1)$, i.e. all the risky assets are defaulted. If we search for a solution of the kind $K(t, v, d)$ as in (5.27), we clearly obtain

$$A^{h,c} K(t, v, \mathbf{1}) = A^h K(t, v, 1, \cdots, 1) = -c\frac{A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)}{v},$$

so that the HJB equation becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{\mathbf{1}}(t) =& Be^{-\delta(T-t)}\log v - \sup_{c>0}\left(Be^{-\delta(T-t)}\log c - c\frac{A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)}{v}\right)\\
=& Be^{-\delta(T-t)}\log v - \sup_{c>0}\psi(t,v,c),
\end{aligned}
$$

with $\psi(t,v,c)$ as in (5.31). Thus by Lemma 5.9 we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{\mathbf{1}}(t) =& Be^{-\delta(T-t)}\log v - \psi(t,v,\bar{c}(t)v)\\
=& -Be^{-\delta(T-t)}\left(\log\left(\frac{Be^{-\delta(T-t)}}{A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)}\right)-1\right)
\end{aligned}
\tag{5.34}
$$

Therefore, we define $\Phi^{\mathbf{1}}$ as the unique solution of (5.34) provided with the terminal condition $\Phi^{\mathbf{1}}(T)=0$, so that $K(t,v,\mathbf{1})$ solves Equation (5.21) with the terminal condition (5.24).

We now assume the statement to be true for any $d' \in \{0,1\}^n$ such that $l(d') \le k-1$, and we prove it to be true for any d such that $l(d)=k$. We set

$$
K(t,v,d) = \left(A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)\right)\log v + \Phi^d(t), \tag{5.35}
$$

where Φ^d is a C^1 deterministic function such that $\Phi^d(T)=0$. Then we have

$$
\begin{aligned}
\frac{\partial K}{\partial t}(t,v,d) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi^d(t) - Be^{-\delta(T-t)},\\
v\frac{\partial K}{\partial v}(t,v,d) &= -v^2\frac{\partial^2 K}{\partial v^2}(t,v,d) = A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\frac{\mathcal{A}^{h,c}K(t,v,d)}{A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)} =& \langle\mu(t,d),h^d\rangle - \frac{c}{v} - \frac{1}{2}\langle h^d\Sigma(t,d),h^d\rangle\\
&+\int_{X^n}\left(\frac{K\big(t,v(1+\langle x,h^d\rangle),d+\chi(d,x)\big)-\Phi^d(t)}{A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)} - \log v - \langle x,h^d\rangle\right)\nu_t^d(\mathrm{d}x)
\end{aligned}
$$

(by (5.23))

$$
= \langle\mu(t),h^d\rangle - \frac{c}{v} - \frac{1}{2}\langle h^d\Sigma(t),h^d\rangle + I_1 + I_2,
$$

where

$$
I_1 = \int_{X^n\setminus\Theta^d}\left(\frac{K\big(t,v(1+\langle x,h^d\rangle),d\big)-\Phi^d(t)}{A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)} - \log v - \langle x,h^d\rangle\right)\nu_t^d(\mathrm{d}x),
$$

(by (5.35))

$$
= \int_{X^n\setminus\Theta^d}\Big(\log\left(1+\langle x,h^d\rangle\right) - \langle x,h^d\rangle\Big)\nu_t^d(\mathrm{d}x),
$$

with

$$\Theta^d = X^n \bigcap \left(\bigcup_{\substack{1 \le i \le n, \\ d_i = 0}} \{x_i = -1\} \right),$$

and where

$$I_2 = \int_{\Theta^d} \left(\frac{K\big(t, v(1 + \langle x^d, h^d \rangle), d + \chi(d,x)\big) - \Phi^d(t)}{A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)} - \log v - \langle x^d, h^d \rangle \right) \nu_t^d(\mathrm{d}x)$$

(by induction hypothesis)

$$= \int_{\Theta^d} \left(\frac{\Phi^{d+\chi(d,x)}(t) - \Phi^d(t)}{A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)} + \log\left(1 + \langle x^d, h^d \rangle\right) - \langle x^d, h^d \rangle \right) \nu_t^d(\mathrm{d}x)$$

(by (5.23))

$$= \frac{\phi^d(t) - \nu_t\left(\Theta^d\right)\Phi^d(t)}{A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)} + \int_{\Theta^d} \Big(\log\left(1 + \langle x^d, h^d \rangle\right) - \langle x^d, h^d \rangle \Big) \nu_t^d(\mathrm{d}x),$$

where ϕ^d is the continuous deterministic function

$$\phi^d(t) = \sum_{d' < d} \nu_t^d \left(\Lambda^{d'}\right) \Phi^{d'}(t),$$

with

$$\Lambda^{d'} = X^n \bigcap \left(\bigcap_{\substack{1 \le i \le n, \\ d'_i = 0}} \{x_i \ne -1\} \right) \bigcap \left(\bigcap_{\substack{1 \le i \le n, \\ d'_i = 1}} \{x_i = -1\} \right).$$

Thus we obtain

$$A^{h,c} K(t,v,d) = \phi^d(t) - \nu_t^d\left(\Theta^d\right)\Phi^d(t) + \left(A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)\right)\left(F^d(t,h) - \frac{c}{v}\right) \quad (5.36)$$

with F^d as in (5.26), and the HJB equation becomes

$$\begin{aligned}
\frac{d}{\mathrm{d}t}\Phi^d(t) &= Be^{-\delta(T-t)} \log v - \sup_{h \in H,\, c>0} \left(A^{h,c} K(t,v,d) + Be^{-\delta(T-t)} \log c \right) \\
&= Be^{-\delta(T-t)} \log v + \nu_t^d\left(\Theta^d\right)\Phi^d(t) - \phi^d(t) \\
&\quad - \left(A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)\right) \sup_{h \in H} F^d(t,h) \\
&\quad - \sup_{c>0} \left(Be^{-\delta(T-t)} \log c - c \frac{A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)}{v} \right) \\
&= \nu_t^d\left(\Theta^d\right)\Phi^d(t) - \phi^d(t) - \left(A + \frac{B}{\delta}\left(1 - e^{-\delta(T-t)}\right)\right) \sup_{h \in H} F^d(t,h) \\
&\quad + Be^{-\delta(T-t)} \log v - \sup_{c>0} \psi(t,v,c),
\end{aligned} \quad (5.37)$$

with $\psi(t,v,c)$ as in (5.31). Let us observe that $\arg\max_{h\in H} F^d(t,h)$ is not empty for any $t\in[0,T]$ because H is a compact subset of $\mathbb{R}^n$ and F^d is continuous, and thus

$$\sup_{h\in H} F^d(t,h) = F^d(t,\bar{h}(t,d)), \tag{5.38}$$

with F^d as in (5.31). Therefore, plugging (5.38)-(5.32) into (5.37) we get

$$\begin{aligned}\frac{d}{\mathrm{d}t}\Phi^d(t) =&\nu_t^d\left(\Theta^d\right)\Phi^d(t)-\phi^d(t)-\left(A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)\right)F^d(t,\bar{h}(t,d))\\&+Be^{-\delta(T-t)}\log v-\psi(t,v,\bar{c}(t)v)\end{aligned}$$

(by (5.33))

$$\begin{aligned}=&\nu_t^d\left(\Theta^d\right)\Phi^d(t)-\phi^d(t)-\left(A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)\right)F^d(t,\bar{h}(t,d))\\&-Be^{-\delta(T-t)}\left(\log\left(\frac{Be^{-\delta(T-t)}}{A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)}\right)-1\right).\end{aligned} \tag{5.39}$$

Furthermore, note that $\phi^d(t)$ and $\nu_t^d\left(\Theta^d\right)$ are continuous in t by Assumption A.4. Thus, setting $\Phi^d(\cdot)$ as the unique solution of the ODE (5.39) with terminal condition $\Phi^d(T)=0$, we have that $K(t,v,d)$ solves Equation (5.21) with terminal condition (5.24), and Part a) is proved.

Part b). In order to prove $K\in\mathcal{D}$ it is sufficient to prove that, for any $\bar{t}\in[0,T]$ and $(\mathfrak{h},\mathfrak{c})\in\mathcal{A}[\bar{t},T]$,

$$K(t,(V,D)_t^{\mathfrak{h},\mathfrak{c};\bar{t},v,d})-\int_{\bar{t}}^t A^{\mathfrak{h}_u,\mathfrak{c}_u}K(u,(V,D)_u^{\mathfrak{h},\mathfrak{c};\bar{t},v,d})\mathrm{d}u$$

is a martingale. Now, by applying the Itôformula, we obtain

$$\mathrm{d}K(t,(V,D)_t^{\mathfrak{h},\mathfrak{c}})=A^{\mathfrak{h}_t,\mathfrak{c}_t}K(t,(V,D)_t^{\mathfrak{h},\mathfrak{c}})\mathrm{d}t+\mathrm{d}M_t,$$

where

$$\begin{aligned}\mathrm{d}M_t =&a(t)\mathfrak{h}_t^{D_t}\sigma(t)\mathrm{d}W_t+\int_{\mathbb{R}^n}\Big(a(t)\log\left(1+\langle x,\mathfrak{h}_{t-}^{D_{t-}}\rangle\right)\\&+\Phi^{D_{t-}+\chi(D_{t-},x)}(t)-\Phi^{D_{t-}(t)}\Big)\big(N(D_{t-},\mathrm{d}x,\mathrm{d}t)-\nu_t(D_{t-},\mathrm{d}x)\mathrm{d}t\big),\end{aligned}$$

and where we have set $a(t):=A+\frac{B}{\delta}\left(1-e^{-\delta(T-t)}\right)$. Therefore, in order to prove the theorem is sufficient to check that M_t is a martingale. Since

$$\begin{aligned}\mathbb{E}\left[\int_{\bar{t}}^T|a(t)\mathfrak{h}_t^{D_t}\sigma(u)|^2\mathrm{d}t\right]&\le a^2(0)\mathbb{E}\left[\int_{\bar{t}}^T\|\mathfrak{h}_t\|^2\|\sigma(t)\|^2\mathrm{d}t\right]\\&\le a^2(0)\sup_{h\in H}\|h\|^2\int_{\bar{t}}^T\|\sigma(t)\|^2\mathrm{d}u<+\infty,\end{aligned}$$

the continuous part is a martingale. We observe now that, since $\mathfrak{h}$ takes values in the compact set $H\subset\mathrm{int}(\cap_{t\in[\bar{t},T]}H_t)$, there exists a constant $\delta>0$ such that $1+\langle\mathfrak{h}_t,x\rangle\ge\delta$

$\nu^d(\mathrm{d}x)$-a.s. for any $d \in \{0,1\}^n$, $t \in [\bar{t}, T]$. Thus, the function $x \to \log(1 + \langle \mathfrak{h}_t, x\rangle)$ is bounded from below and with linear growth, and so there is a constant $C > 0$ such that

$$|\log(1 + \langle \mathfrak{h}_t, x\rangle)| \leq C \sup_{h\in H} \|h\| \, \|x\| \quad \nu^d(\mathrm{d}x)\text{−a.s.} \tag{5.40}$$

for any $d \in \{0,1\}^n$, $t \in [\bar{t}, T]$. According now to the notation used in the proof of part a), we define

$$\Theta := \Theta^{(0,\cdots,0)} = \bigcup_{i=1}^{n} \{x_i = -1\}, \tag{5.41}$$

and finally, in order to verify the pure jump stochastic integral to be a martingale, we only need to check that, for any $d \in \{0,1\}^n$

$$\mathbb{E}\left[\int_{\bar{t}}^{T} \int_{\mathbb{R}^n} \left| a(t) \log\left(1 + \langle x, \mathfrak{h}_{t-}^{D_{t-}}\rangle\right) + \Phi^{D_{t-}+\chi(D_{t-},x)}(t) - \Phi^{D_{t-}(t)}\right|^2 \nu_t^d(\mathrm{d}x)\mathrm{d}t\right]$$

(by (5.23) and (5.41))

$$= \mathbb{E}\left[\int_{\bar{t}}^{T} \int_{\mathbb{R}^n} \left| a(t) \log\left(1 + \langle x, \mathfrak{h}_{t-}^{D_{t-}}\rangle\right) + \mathbb{1}_\Theta(x)\left(\Phi^{D_{t-}+\chi(D_{t-},x)}(t) - \Phi^{D_{t-}(t)}\right)\right|^2 \nu_t^d(\mathrm{d}x)\mathrm{d}t\right]$$

(by the triangular inequality)

$$\leq 2a^2(0)\,\mathbb{E}\left[\int_{\bar{t}}^{T} \int_{\mathbb{R}^n} \left|\log\left(1 + \langle x, \mathfrak{h}_{t-}^{D_{t-}}\rangle\right)\right|^2 \nu_t^d(\mathrm{d}x)\mathrm{d}t\right]$$
$$+ 2\mathbb{E}\left[\int_{\bar{t}}^{T} \int_{\Theta} \left|\Phi^{D_{t-}+\chi(D_{t-},x)}(t) - \Phi^{D_{t-}}(t)\right|^2 \nu_t^d(\mathrm{d}x)\mathrm{d}t\right]$$

(by (5.40) and Assumption A.1.)

$$\leq 2a^2(0) \int_{\bar{t}}^{T} C^2 \sup_{h\in H} \|h\|^2 \int_{\mathbb{R}^n} \|x\|^2 \nu_t^d(\mathrm{d}x) + 2\left(\max_{d'\in\{0,1\}^n} \Phi^{d'}(t)\right)^2 \nu_t^d(\Theta)\mathrm{d}t < +\infty.$$

Part c). By (5.36), (5.38) and Lemma 5.9 we have

$$(\bar{h}(t,d), \bar{c}(t)v) \in \arg\max_{h\in H,\, c>0} A^{h,c}K(t,v,d) \quad \forall v \in \mathbb{R}^+$$

for any $t \in [0,T]$ and $d \in \{0,1\}^n$. Therefore, the process $(\bar{\mathfrak{h}}_t, \bar{\mathfrak{c}}_t)$ defined in (5.28) satisfies (5.25) and the statement follows by Theorem 5.2. □

1.6 Examples

In this section we present several examples of market models with one, two or several defaultable assets. In particular, in Section 6.1 we present a general model with one nondefaultable stock, one defaultable stock and one defaultable bond, where the vulnerable assets default simultaneously, with the bond possibly recovering part of its notional. Section 6.3 and 6.4 are particular cases of this general example, where the agent cannot trade in the defaultable stock or in the defaultable bond, respectively. These two cases have already been dealt in literature in [31, 41] respectively. Section 6.4 presents a market model, inspired by [14, 64], with several defaultable bonds that cannot default simultaneously; as a consequence, the optimal portfolio proportion of each bond depends only on its dynamics and not on that of the other ones. Instead in Section 6.5 we study the same market model, with only two defaultable bonds, where we introduce the possibility of a simultaneous default; as a consequence, the optimal portfolio proportion of each bond prior to any default turns out to depend also on the dynamics of the other bond.

In the light of Remark 5.7, in the following examples we only focus on the optimal investment strategy $\bar{\mathfrak{h}}_t = \bar{h}(t, D_t)$, as the optimal consumption $\bar{\mathfrak{c}}_t$ does not depend on the choice of the model.

1.6.1 Diffusion dynamics with default

In this section we present an example of market model with three risky assets, namely one default-free stock, one defaultable stock and one defaultable bond, where we assume that the two latter assets are issued by the same entity. This model generalizes two models in [31, 41], which can be obtained by imposing a null strategy in the defaultable stock or in the defaultable bond respectively.

The risky assets' dynamics (1.1)-(3.7) takes now the form

$$\begin{aligned} \mathrm{d}S_t^i &= S_{t-}^i \,\mathrm{d}R_t^i, \quad i = 1, 2, P \\ \mathrm{d}R_t^1 &= \mu_1(t, D_t)\mathrm{d}t + \sigma_1(t, D_t)\mathrm{d}W_t, \\ \mathrm{d}R_t^2 &= \mu_2(t, D_t)\mathrm{d}t + \sigma_2(t, D_t)\mathrm{d}W_t - (\mathrm{d}N_t - \lambda(t)\mathrm{d}t), \\ \mathrm{d}R_t^P &= \mu_P(t, D_t)\mathrm{d}t - \xi(1 - D_{t-})(\mathrm{d}N_t - \lambda(t)\mathrm{d}t), \end{aligned}$$

where N_t is a 1-dimensional Poisson processes with intensity λ, acting on S^2 and S^P, and where (following [31])

$$\mu_P(t, D_t) := \xi(1 - D_{t-})\lambda(t)\left(\frac{1}{\Delta(t)} - 1\right)$$

In other words, both the stocks S^1 and S^2 follow a standard Black-Scholes dynamics, with the only admissible jump of the process (S^1, S^2, S^P) having amplitude equal to $(0, -S^2, -\xi S^P)$, and causing the default of both the stock S^2 and of the bond S^P. In this case the stock loses all its value, while the bond loses a fixed fraction $\xi \in [0, 1]$ of its value, thus allowing for a partial recovery. Notice that the drift of the defaultable bond μ_P is proportional to the difference between the intensity $\frac{\lambda}{\Delta}$ of N under an equivalent martingale measure and the intensity λ of N under the real world probability measure, under which the utility is maximized. In [31], the quantity $\frac{1}{\Delta}$ is called default event risk premium.

The compensating measure ν_t^0 is now equal to $\lambda(t) > 0$ times the Dirac delta distribution concentrated in $\{x_1 = 0, x_2 = -1, x_3 = -\xi\} \in \mathbb{R}^3$, i.e.

$$\begin{aligned} \nu_t^0\big(\{x_1 = 0, x_2 = -1, x_3 = -\xi\}\big) &= \lambda(t), \\ \nu_t^0\left(\mathbb{R}^2 \backslash \{x_1 = 0, x_2 = -1, x_3 = -\xi\}\right) &= 0, \quad \forall t \in [0, T]. \end{aligned}$$

Note that, by contrast, the post-default compensating measure ν^1 can be actually set identically equal to 0 without loss of generality, as none of the jumps of the process R^2 (thus also of R^3) occurring after the default time τ^2 have any impact on the price S_t^2, nor on the price S_t^R. Indeed, the former process has already jumped to the absorbing state 0, whereas the latter is constant because $\mu_P(t,1) \equiv 0$, and thus its dynamics is identically equal to the riskless asset.

Under this particular choice of ν_t^d, $d = 0, 1$, the subset $H_t \subset \mathbb{R}^2$ defined in (4.16) takes the form

$$H_t \equiv \{(h_1, h_2, h_P) \mid h_2 + \xi h_P < 1\}$$

For sake of simplicity we can assume, without losing generality, the convex compact subset $H \subset \mathbb{R}^3$ of Definition 4.3-a expressed in the form $H = H_1 \times H_2$, where H_1 and H_2 are convex compact subset of $\mathbb{R}$ and of the half-plane $\{(h_2, h_P) \mid h_2 + \xi h_P < 1\}$ respectively. Now, Equation (5.26) can be written, in extended form, as

$$\begin{aligned}
F^1(t,h) &= \mu_1(t,1)h_1 - \frac{1}{2}\|\sigma_1(t,1)\|^2 h_1^2, \\
F^0(t,h) &= \langle \mu(t,0), h\rangle - \frac{1}{2}\langle (h_1,h_2)\Sigma(t,0), (h_1,h_2)\rangle + \lambda(t)\big(\log(1 - h_2 - \xi h_P) + h_2 + \xi h_P\big).
\end{aligned}$$

where we denote

$$\Sigma(t,0) = \sigma\sigma^*(t,0) := \begin{pmatrix} \|\sigma_1(t,0)\|^2 & \langle \sigma_1(t,0), \sigma_2(t,0)\rangle \\ \langle \sigma_1(t,0), \sigma_2(t,0)\rangle & \|\sigma_1(t,0)\|^2 \end{pmatrix}$$

as the diffusion component of the risky bond S^P is null.

Now, as F^1 is strictly concave in h_1, the maximization problem with respect to h_1 over H_1 has a unique solution that can be either internal or on the boundary. A necessary and sufficient condition under which the maximum over H_1 is internal is that the solution of the first order condition

$$\mu_1(t,1) = \|\sigma_1(t,1)\|^2 h_1,$$

given by $h_1(t) = \frac{\mu_1(t,1)}{\|\sigma_1(t,1)\|^2}$, belongs to $\mathrm{int}(H_1)$. Thus, under this condition, the first component of $\bar{h}(t,1)$ in (5.26) is univocally determined by

$$\bar{h}_1(t,1) = \frac{\mu_1(t,1)}{\|\sigma_1(t,1)\|^2}.$$

Analogously, assuming the matrix $\mathrm{rank}\Sigma(t,0) = 2$, $F^0(t,h)$ is a strictly concave function and so the maximization problem over H has a unique solution. Moreover, we have the following

Proposition 6.1. *For any $t \in [0,T]$, the unique maximum of $F^0(t,h)$ over H is an internal point if and only if $h_1^*(t) \in H_1$, where $h_1^*(t)$ is the first component of*

$$(h_1, h_2)(t) := \Sigma^{-1} \begin{pmatrix} \mu_1(t,0) \\ \mu_2(t,0) - \lambda(t)\left(\frac{1}{\Delta(t)} - 1\right) \end{pmatrix}$$

and $(h_2^*(t), h_P^*(t)) \in H_2$, where

$$h_P^*(t) = \frac{1}{\xi}\left(1 - \Delta(t) - h_2(t)\right) \tag{6.42}$$

Under these assumptions, the unique maximizer of $F^0(t,h)$ is

$$\bar{h}(t,0) = \big(h_1^*(t), h_2^*(t), h_P^*(t)\big).$$

Proof. Being $F^0(t,h)$ strictly concave on H with respect to h, the unique maximum over H is an internal point if and only if it is the solution of the first order condition

$$\nabla_h F^0(t,h) = 0.$$

Condition (6.46) is explicitly given by

$$\begin{cases} \mu_1(t,0) &= \Sigma_{11}(t,0)h_1 + \Sigma_{21}(t,0)h_2, \\ \mu_2(t,0) &= \Sigma_{21}(t,0)h_1 + \Sigma_{22}(t,0)h_2 + \ell(t)\left(\frac{1}{1-h_2-\xi h_P} - 1\right), \\ \mu_P(t,0) &= \ell(t)\xi\left(\frac{1}{1-h_2-\xi h_P} - 1\right). \end{cases} \tag{6.43}$$

(recall that $\Sigma_{ij} = \langle\sigma_i,\sigma_j\rangle$). Now, by substituting the third equation into the second, the first two equations in (6.50) become

$$\begin{cases} \mu_1(t,0) &= \Sigma_{11}(t,0)h_1 + \Sigma_{21}(t,0)h_2, \\ \mu_2(t,0) - \lambda(t)\left(\frac{1}{\Delta(t)} - 1\right) &= \Sigma_{21}(t,0)h_1 + \Sigma_{22}(t,0)h_2, \end{cases}$$

which results in a modified Merton problem on the stocks, whose solution is given by Equation (6.1). Once we have h_2, we can easily obtain h_P from the third equation of (6.50), resulting in Equation (6.49). It is also very easy to assess that $h_2 + \xi h_P < 1$, so the triple $(h_1, h_2, h_P) \in H$. Thus, the conclusion follows. □

Corollary 6.2. *Let* $(h_1^M(t), h_2^M(t)) := \Sigma^{-1}(t,0)(\mu_1(t,0),\mu_2(t,0))$ *be the Merton optimal strategy for the undefaultable log-normal dynamics of the risky assets. Then, by calling* $\rho := \frac{\langle\sigma_1,\sigma_2\rangle}{\|\sigma_1\|\|\sigma_2\|}$ *the correlation between* S_1 *and* S_2*, under the assumptions of the previous proposition we have that*

$$\begin{pmatrix} \bar{h}_1(t) \\ \bar{h}_2(t) \end{pmatrix} = \begin{pmatrix} h_1^M(t) \\ h_2^M(t) \end{pmatrix} + \frac{\xi\lambda(t)(\frac{1}{\Delta} - 1)}{1-\rho^2} \begin{pmatrix} \frac{\rho}{\|\sigma_1\|\|\sigma_2\|} \\ \\ -\frac{1}{\|\sigma_2\|^2} \end{pmatrix}$$

In particular,

$$\bar{h}(t,0) \to h^M(t) \quad \text{as } \lambda(t) \to 0$$

Proof. A direct computation shows that

$$\begin{pmatrix} \bar{h}_1(t) \\ \bar{h}_2(t) \end{pmatrix} = \begin{pmatrix} h_1^M(t) \\ h_2^M(t) \end{pmatrix} - \Sigma^{-1} \begin{pmatrix} 0 \\ \lambda(t)(\frac{1}{\Delta} - 1) \end{pmatrix}$$

By inverting Σ, the conclusions follow. □

Remark 6.3. *If* $\rho = 0$*, i.e. when the default-free asset is independent of the defaultable part of the portfolio (bond and stock), then the optimal portfolio in the default-free asset is exactly equal to the Merton portfolio, as in [31].*

1.6.2 One default-free stock and one defaultable bond

As already said, if we impose the portfolio constraint $h_2 \equiv 0$, i.e. we do not allow our agent to invest in the defaultable stock, we obtain exactly the market model treated in [31]. In this case, the set of all admissible strategies becomes

$$H_t \equiv \{(h_1, 0, h_P) \mid h_P < 1/\xi\},$$

and again we can assume without losing generality the convex compact subset $H \subset \mathbb{R}^3$ of Definition 4.3-a expressed in the form $H = H_1 \times \{0\} \times H_2$, where H_1 and H_2 are convex compact subset of $\mathbb{R}$ and of the half-line $(-\infty, 1/\xi)$ respectively. Now, Equation (5.26) can be written, in extended form (by omitting the variable $h_2 \equiv 0$), as

$$F^1(t,h) = \mu_1(t,1)h_1 - \frac{1}{2}\left\|\sigma_1(t,1)\right\|^2 h_1^2,$$
$$F^0(t,h) = \langle \mu(t,0), h\rangle - \frac{1}{2}\left\|\sigma_1(t,1)\right\|^2 h_1^2 + \lambda(t)\big(\log(1-\xi h_P) + \xi h_P\big).$$

Now, as F^1 is again strictly concave in h_1, the maximization problem with respect to h_1 over H_1 has a unique solution that can be internal or on the boundary of H_1, leading to the exact same conclusion as in the general case in Subsection 6.2. We also notice that $F^0(t,h)$ is a strictly concave function and so the maximization problem over H has a unique solution.

Proposition 6.4. *For any $t \in [0,T]$, the unique maximum of $F^0(t,h)$ over H is an internal point if and only if*

$$h_1^*(t) := \frac{\mu_1(t,0)}{\|\sigma_1(t,0)\|^2} \in H_1$$

and

$$h_P^*(t) = \frac{1}{\xi}\left(1 - \Delta(t)\right) \in H_2.$$

Under these assumptions, the unique maximizer of $F^0(t,h)$ is

$$\bar{h}(t,0) = \big(h_1^*(t), h_2^*(t), h_P^*(t)\big).$$

Proof. Being $F^0(t,h)$ strictly concave on $(-\infty,1) \times \mathbb{R}$ with respect to h, the unique maximum over H is an internal point if and only if it is the solution of the first order condition

$$\nabla_h F^0(t,h) = 0,$$

which now reads as

$$\begin{cases} \mu_1(t,0) &= \Sigma_{11}(t,0)h_1, \\ \mu_P(t,0) &= \ell(t)\xi\left(\frac{1}{1-\xi h_P} - 1\right). \end{cases}$$

Thus the conclusion follows easily. ☐

Remark 6.5. *We obtain the same conclusion as in [31] (notice that there the utility function is $U(x) = x^\gamma/\gamma$, so mathematically speaking we obtain the same conclusions in the limiting case $\gamma \to 0$). In particular, the investment in the riskless stock is independent of the default possibility of the risky bond. Plus, due to the log-utility function, the optimal strategy of the risky bond is myopic, i.e. it does not depend on the residual investment horizon $T-t$.*

1.6.3 Two stocks, one of which defaultable

We now impose the portfolio constraint $h_P \equiv 0$, i.e. we allow our agent to invest only in the default-free and in the defaultable stocks; thus, we obtain the same market model treated in [41]. In this case, the set of admissible strategies becomes

$$H_t \equiv \{(h_1, h_2, 0) \mid h_2 < 1\},$$

and again we can assume without losing generality the convex compact subset $H \subset \mathbb{R}^3$ of Definition 4.3-a expressed in the form $H = H_1 \times H_2 \times \{0\}$, where H_1 and H_2 are convex compact subset of $\mathbb{R}$ and of the half-line $(-\infty, 1)$ respectively. Now, Equation (5.26) can be written, in extended form (by omitting the variable $h_P \equiv 0$), as

$$\begin{aligned}
F^1(t,h) &= \mu_1(t,1)h_1 - \frac{1}{2}\left\|\sigma_1(t,1)\right\|^2 h_1^2, \\
F^0(t,h) &= \langle \mu(t,0), h\rangle - \frac{1}{2}\langle (h_1,h_2)\Sigma(t,0), (h_1,h_2)\rangle + \lambda(t)\big(\log(1-h_2) + h_2\big).
\end{aligned}$$

Now, as F^1 is again strictly concave in h_1, the maximization problem with respect to h_1 over H_1 has a unique solution that can either be internal or on the boundary of H_1, leading to the exact same conclusion as in the general case in Subsection 6.2. We also notice that, if we again assume that $\text{rank}\Sigma = 2$, then $F^0(t,h)$ is a strictly concave function and so the maximization problem over H has a unique solution.

Proposition 6.6. *For any $t \in [0,T]$, the unique maximum of $F^0(t,h)$ over H is an internal point if and only if*

$$\begin{aligned}
\Delta(t) = &\left(\det\Sigma(t,0) - \Sigma_{11}(t,0)\mu_2(t,0) + \Sigma_{12}(t,0)\mu_1(t,0)\right)^2 \\
&+2\lambda(t)\Sigma_{11}(t,0)\left(\det\Sigma(t,0) + \Sigma_{11}(t,0)\mu_2(t,0) - \Sigma_{12}(t,0)\mu_1(t,0)\right) + \lambda^2(t)\Sigma_{11}^2(t,0) \geq 0,
\end{aligned} \tag{6.44}$$

and

$$\begin{aligned}
h_1^*(t) &= \frac{\mu_1(t,0) - \Sigma_{12}(t,0)h_2^*(t)}{\Sigma_{11}(t,0)} \\
h_2^*(t) &= \frac{\det\Sigma(t,0) + \Sigma_{11}(t,0)\mu_2(t,0) - \Sigma_{12}(t,0)\mu_1(t,0) + \Sigma_{11}(t,0)\lambda(t) - \sqrt{\Delta(t)}}{2\det\Sigma(t,0)},
\end{aligned} \tag{6.45}$$

belong to H_1 and H_2 respectively. Under this condition, the function $\bar{h}(t,0)$ in (5.26) *is univocally determined by*

$$\bar{h}(t,0) = \big(h_1^*(t), h_2^*(t)\big).$$

Proof. Being $F^0(t,h)$ strictly concave on $(-\infty,1)\times\mathbb{R}$ with respect to h, the unique maximum over H is an internal point if and only if it is the solution of the first order condition

$$\nabla_h F^0(t,h) = 0. \tag{6.46}$$

Condition (6.46) is explicitly given by

$$\begin{cases}
\mu_1(t,0) = \Sigma_{11}(t,0)h_1 + \Sigma_{12}(t,0)h_2 \\
\mu_2(t,0) + \ell(t) = \Sigma_{12}(t,0)h_1 + \Sigma_{22}(t,0)h_2 + \frac{\ell(t)}{1-h_2}
\end{cases} \tag{6.47}$$

Now, by substitution and by multiplying for $(1-h_1)\Sigma_{11}(t,0)$ the first equation, (6.47) becomes

$$\begin{cases} h_1 = \dfrac{\mu_1(t,0)-\Sigma_{12}(t,0)h_2}{\Sigma_{11}(t,0)} \\ a(t)h_2^2 + b(t)h_2 + c(t) = 0 \end{cases}$$

where

$$\begin{aligned} a(t) &= \det\Sigma(t,0), \\ b(t) &= -(\det\Sigma(t,0)+\Sigma_{11}(t,0)\mu_2(t,0)-\Sigma_{12}(t,0)\mu_1(t,0)) - \Sigma_{11}(t,0)\lambda(t), \\ c(t) &= \Sigma_{11}(t,0)\mu_2(t,0)-\Sigma_{12}(t,0)\mu_1(t,0). \end{aligned}$$

Thus, System (6.47) may have two solutions: $h^* = (h_1^*, h_2^*)$ as in (6.45), and $g^* = (g_1^*, g_2^*)$ given by

$$\begin{cases} g_1^*(t) &= \frac{\mu_1(t,0)-\Sigma_{12}(t,0)h_2^2(t)}{\Sigma_{11}(t,0)} \\ g_2^*(t) &= \frac{\det\Sigma(t,0)+\Sigma_{11}(t,0)\mu_2(t,0)-\Sigma_{12}(t,0)\mu_1(t,0)+\Sigma_{11}(t,0)\lambda(t)+\sqrt{\Delta(t)}}{2\det\Sigma(t,0)} \end{cases}$$

where $\Delta(t)$ is defined as in (6.44). In order to conclude it is enough to observe that g^* cannot belong to $H \subset \mathbb{R}\times(-\infty,1)$. Indeed, let us assume that $g_2^* < 1$. Then $h_2^* < g_2^*$ implies $h_2^* < 1$ and so $F^0(t,h)$ has two stationary points on $\mathbb{R}\times(-\infty,1)$, which is impossible because it is strictly concave with respect to h. □

Corollary 6.7. *Let $h^M(t) := \Sigma^{-1}(t,0)\mu(t,0)$ be the Merton optimal strategy for the undefaultable log-normal dynamics. Then*

$$\bar{h}(t,0) \to h^M(t) \quad \text{as } \lambda(t)\to 0 \tag{6.48}$$

if and only if $h^M(t)\in H$. In particular, if

$$h_1^M(t) = \frac{\Sigma_{22}(t,0)\mu_1(t,0)-\Sigma_{12}(t,0)\mu_2(t,0)}{\det\Sigma(t,0)} < 1,$$

we can always find a compact $H\subset(-\infty,1)\times\mathbb{R}$ such that (6.48) holds. In this case, we have

$$\begin{pmatrix} \bar{h}_1(t) \\ \bar{h}_2(t) \end{pmatrix} = \begin{pmatrix} h_1^M(t) \\ h_2^M(t) \end{pmatrix} + \lambda(t)A(t)\begin{pmatrix} -\Sigma_{12} \\ \Sigma_{11} \end{pmatrix} + o(\lambda(t))$$

with

$$A(t) := \frac{1}{2\det\Sigma}\left(1 - \frac{\det\Sigma(t,0)+\Sigma_{11}(t,0)\mu_2(t,0)-\Sigma_{12}(t,0)\mu_1(t,0)}{\det\Sigma(t,0)-\Sigma_{11}(t,0)\mu_2(t,0)+\Sigma_{12}(t,0)\mu_1(t,0)}\right)$$

Proof. A direct computation shows that $h^*(t) = h^M(t)$ when $\lambda(t)=0$. Then the limit follows by continuity of $h^*(t)$. For the first-order asymptotics, we have that

$$\begin{aligned} \sqrt{\Delta(t)} &= (\det\Sigma(t,0)-\Sigma_{11}(t,0)\mu_2(t,0)+\Sigma_{12}(t,0)\mu_1(t,0))\times \\ &\times\left(1+\lambda(t)\Sigma_{11}(t,0)\frac{\det\Sigma(t,0)+\Sigma_{11}(t,0)\mu_2(t,0)-\Sigma_{12}(t,0)\mu_1(t,0)}{(\det\Sigma(t,0)-\Sigma_{11}(t,0)\mu_2(t,0)+\Sigma_{12}(t,0)\mu_1(t,0))^2}+o(\lambda(t))\right). \end{aligned}$$

Hence Equation (6.7) follows. □

Remark 6.8. *In this case, if the two assets are independent, then $\Sigma_{12}\equiv 0$, and the same conclusion of the previous sections follows "at first order"; in fact, by Equation (6.7), one has that $\bar{h}_1 = h_1^M + o(\lambda(t))$, i.e. the deviations from Merton's portfolio of the non-defaultable asset are of higher order with respect to $\lambda(t)$.*

1.6.4 Several defaultable bonds

In this section we present an example of market model with several defaultable bonds, with dynamics analogous to Sections 6.2, 6.3; namely:

$$\begin{aligned} dS_t^i &= S_{t-}^i \, dR_t^i, \qquad i = 1, \dots, n, \\ dR_t^i &= \mu_i(t, D_{t-})\mathrm{d}t - \xi_i(1 - D_{t-}^i)(\mathrm{d}N_t^i - \lambda_i(t, D_{t-})\mathrm{d}t), \end{aligned}$$

where

$$\mu_i(t, D_{t-}) := \xi_i(1 - D_{t-}^i)\lambda_i(t, D_{t-}) \left(\frac{1}{\Delta_i(t)} - 1 \right)$$

and where now, the intensities of the Poisson processes N^i (both under the real world probability measure and the risk-neutral one) can possibly depend on the default state D_{t-} of the other bonds. This model is inspired by [14, 64]. Precisely, we can distinguish two relevant cases: the case when simultaneous defaults cannot occur (as in [14, 64]), and the case when they can occur. In the first case we only have information-induced contagion among bonds, whereas in the second one it is also possible to model direct contagion.

While in the next example we will focus on the case when simultaneous defaults can occur, here we focus on the case when they cannot. This is obtained by imposing that the N^i, $i = 1, \dots, n$, are independent Poisson processes conditional to the default state D. The compensating measure ν_t is then equal to

$$\nu_t(D_{t-}, \mathrm{d}x) = \sum_{i=1}^{n} (1 - D_{t-}^i)\lambda_i(t, D_{t-})\delta_{-e_i}(\mathrm{d}x)$$

where e_i is the i-th coordinate-vector in $\mathbb{R}^n$, with 1 in the i-th component and 0 in the other ones.

Under this choice of $\nu_t(d, \cdot)$, the subset $H_t \subset \mathbb{R}^n$ defined in (4.16) takes the form

$$H_t \equiv \left\{ h \mid h_i < \frac{1}{\xi_i} \quad \forall i = 1, \dots, n \right\}.$$

Again, for sake of simplicity we can assume the convex compact subset $H \subset \mathbb{R}^n$ of Definition 4.3-a expressed in the form $H = \prod_{i=1}^n H_i$, where H_i are convex compact subsets of the interval $(-\infty, \frac{1}{\xi_i})$. Now, Equation (5.26) can be written as

$$\begin{aligned} F^d(t, h) = \langle \mu(t, d), h^d \rangle + & \\ + \sum_{i=1}^{n} (1 - d^i)\lambda_i(t, d)(\log(1 - \xi_i h_i(1 - d^i)) + \xi_i h_i(1 - d^i)), & \end{aligned}$$

for all $d \in \{0, 1\}^n$. Now, as each F^d is strictly concave in all the non-null components of h^d, the maximization problem with respect to these variables over H has a unique solution that can be internal or on the boundary. In particular, we have the following

Proposition 6.9. *For any $t \in [0, T]$ and $d \in \{0, 1\}^n$, a unique maximum of $F^d(t, h)$ over H is an internal point if and only if*

$$h_i^*(t) = \frac{1}{\xi_i} (1 - \Delta_i(t)) \in H_i \tag{6.49}$$

for all $i = 1, \dots, n$ such that $d_i = 0$. Under these assumptions, $h^*(t)$ is a maximizer of $F^d(t, h)$.

Proof. Being $F^d(t,h)$ strictly concave on H with respect to the non-null variables of h^d, the unique maximum over H is an internal point if and only if it is the solution of the first order condition

$$F^d_{h_i}(t,h) = 0 \qquad \forall i \text{ such that } d_i = 0,$$

which now reads as

$$\xi_i \lambda_i(t,d) \left(\frac{1}{\Delta_i(t)} - 1 \right) = \lambda_i(t,d) \xi_i \left(\frac{1}{1 - \xi_i h_P} - 1 \right). \tag{6.50}$$

Thus the conclusion follows easily. □

Remark 6.10. *In this particular example, where there is not direct contagion, it turns out that the optimal portfolio on the i-th bond (if still alive) is uniquely determined by its coefficients, with no dependence on the coefficients of the other defaultable bonds.*

Corollary 6.11. *If $\xi_i \equiv \xi$ and $\Delta_i \equiv \Delta$, then the optimal portfolio for all the defaultable bonds is*

$$h_i^*(t) \equiv \frac{1}{\xi}(1 - \Delta(t)).$$

Remark 6.12. *The assumptions of the corollary above are qualitatively known as "name homogeneity" [14], and hold when default risks of the bonds are exchangeable, for example when bonds are of the same credit rating and/or of firms from the same industrial sector. Notice that for this conclusion it is* not *necessary to assume that $\lambda_i \equiv \lambda$.*

1.6.5 Two defaultable bonds with direct contagion

In this section we specialize the previous example to $n = 2$ but add the possibility of simultaneous default, by modifying the dynamics as

$$\begin{aligned} dS_t^i &= S_{t-}^i \, dR_t^i, \qquad i = 1, \dots, 2, \\ dR_t^i &= \mu_i(t, D_{t-}) \mathrm{d}t - \xi_i (1 - D_{t-}^i)(\mathrm{d}N_t^i - \lambda_i(t, D_{t-}) \mathrm{d}t) \\ &\quad - \xi_i (1 - D_{t-}^1)(1 - D_{t-}^2)(\mathrm{d}N_t - \lambda(t) \mathrm{d}t), \end{aligned}$$

where N^1, N^2 and N are independent Poisson processes and this time

$$\begin{aligned} \mu_i(t, D_{t-}) &:= \xi_i (1 - D_{t-}^i) \lambda_i(t, D_{t-}) \left(\frac{1}{\Delta_i(t)} - 1 \right) \\ &\quad + \xi_i (1 - D_{t-}^1)(1 - D_{t-}^2) \lambda(t, D_{t-}) \left(\frac{1}{\Delta(t)} - 1 \right), \end{aligned}$$

and where now the intensities of the Poisson processes N^i (both under the real world probability measure and the risk-neutral one) can possibly depend on the default state D_{t-} of the other bond, while the Poisson process N, with intensity λ, acts on both the defaultable bonds when they are still non-defaulted.

The compensating measure ν_t is now equal to

$$\nu_t(D_{t-}, \mathrm{d}x) = \sum_{i=1}^{n} (1 - D_{t-}^i) \lambda_i(t, D_{t-}) \delta_{-e_i}(\mathrm{d}x) + (1 - D_{t-}^1)(1 - D_{t-}^2) \lambda(t, D_{t-}) \delta_{(-1,-1)}(\mathrm{d}x),$$

where again e_i, $i = 1, 2$, is the i-th coordinate vector in $\mathbb{R}^2$, and we also have the possibility of a simultaneous jump to $(-1, -1)$ with intensity λ.

Under this choice of $\nu_t(d, \cdot)$, the subset $H_t \subset \mathbb{R}^n$ defined in (4.16) takes the form

$$H_t \equiv \left\{ h \mid h_i < \frac{1}{\xi_i} \quad \forall i = 1, 2, \quad \xi_1 h_1 + \xi_2 h_2 < 1 \right\}.$$

Also in this example, for sake of simplicity we can assume the convex compact subset $H \subset \mathbb{R}^n$ of Definition 4.3-a expressed as $H = H_1 \times H_2$, where H_i are convex compact subsets of the interval $(-\infty, \frac{1}{\xi_i})$.

Now, Equation (5.26) can be written, in extended form, as

$$\begin{aligned} F^{(0,0)}(t,h) &= \langle \mu(t,(0,0)), h \rangle + \sum_{i=1}^{2} \lambda_i(t,(0,0))(\log(1 - \xi_i h_i) + \xi_i h_i) \\ &\quad + \lambda(t,(0,0))(\log(1 - \xi_1 h_1 - \xi_2 h_2) + \xi_1 h_1 + \xi_2 h_2), \\ F^{(0,1)}(t,h) &= \mu_1(t,(0,1))h_1 + \lambda_1(t,(0,1))(\log(1 - \xi_1 h_1) + \xi_1 h_1), \\ F^{(1,0)}(t,h) &= \mu_2(t,(1,0))h_2 + \lambda_2(t,(1,0))(\log(1 - \xi_2 h_2) + \xi_2 h_2). \end{aligned} \tag{6.51}$$

Now, as each F^d is strictly concave in all the non-null components of h^d, the maximization problem with these variables over H has a unique solution that can be internal or on the boundary. More in details, we have the following

Proposition 6.13. *For any $t \in [0, T]$, for $i = 1, 2$, if*

$$h_i^*(t) = \frac{1}{\xi_i}(1 - \Delta_i(t)) \in H_i$$

then $h_i^*(t)$ is the optimal portfolio proportion of the i-th bond after the other one is defaulted. For the case $d = (0,0)$ (i.e. prior to any default), if the unique solution $(h_1^*, h_2^*) \in H_t$ of the system

$$\begin{aligned} \frac{\lambda_1}{\Delta_1} &= \frac{\lambda_1}{1 - \xi_1 h_1} + \frac{\lambda}{1 - \xi_1 h_1 - \xi_2 h_2}, \\ \frac{\lambda_2}{\Delta_2} &= \frac{\lambda_2}{1 - \xi_2 h_2} + \frac{\lambda}{1 - \xi_1 h_1 - \xi_2 h_2}, \end{aligned}$$

also belongs to $H_1 \times H_2$, then it is the optimal pre-default portfolio.

Proof. The situation when the i-th bond is already defaulted is analogous to the previous example, with exactly the same results.

Let us now pass to the case $d = (0,0)$. Since in this case $F^d(t,h)$ is strictly concave on H, the unique maximum over H is an internal point if and only if it is the solution of the first order condition

$$F^d_{h_i}(t,h) = 0 \qquad \forall i = 1, 2,$$

corresponding to Equations (6.52–6.53). Thus the conclusion follows. □

Remark 6.14. *In this example with a direct contagion, it turns out that the optimal portfolio in the i-th bond prior to any default depends (via a non-linear relation) on its coefficients and also on the coefficient of the other bond. Thus, the possibility of simultaneous defaults introduces a (non-linear) dependence among the defaultable bonds, which is somewhat analogous to the correlation effect arising in diffusion models.*

Note that, solving the system (6.52)-(6.52) requires solving a 3rd order algebraic equation. When λ tends to 0 we have the following continuity property.

Remark 6.15. *Let us denote by* $\left(h_1^{*,\lambda}(t), h_2^{*,\lambda}(t)\right)$ *the optimal strategy when both the bonds are still alive, i.e.* $d = (0,0)$*. Then we have*

$$\lim_{\lambda \to 0} \left(h_1^{*,\lambda}(t), h_2^{*,\lambda}(t)\right) = \left(h_1^{*,0}(t), h_2^{*,0}(t)\right).$$

Indeed, $F^{(0,0)}(h_1, h_2; \lambda)$ *in* (6.51) *is continuous, and thus uniformly continuous on the compact* $H_1 \times H_2 \times [0, \bar{\lambda}]$*, for any* $\bar{\lambda} > 0$*. Thus,*

$$\left(h_1^{*,\lambda}(t), h_2^{*,\lambda}(t)\right) = \arg \max_{(h_1,h_2) \in H_1 \times H_2} F^{(0,0)}(h_1, h_2; \lambda)$$

tends to

$$\arg \max_{(h_1,h_2) \in H_1 \times H_2} F^{(0,0)}(h_1, h_2; 0) = \left(h_1^{*,0}(t), h_2^{*,0}(t)\right)$$

as λ *tends to* 0*. In particular, for* $i = 1, 2$*, by Proposition* (6.9) *we have*

$$\lim_{\lambda \to 0} h_i^{*,\lambda}(t) = \frac{1}{\xi_i} \left(1 - \Delta_i(t)\right)$$

if $\frac{1}{\xi_i} \left(1 - \Delta_i(t)\right) \in H_i$*.*

1.7 GOP and GOP-denominated prices

Throughout this whole section we will consider a null utility function $u(t, c) \equiv 0$ for the consumption, i.e. $B = 0$ in (4.18). In the light of Theorem 5.4, this is equivalent to consider the optimization problem with terminal utility function $U(v)$, with null consumption rate $\mathfrak{c}_t \equiv 0$. Furthermore, we will enlarge the set of the admissible strategies $\mathcal{A}[t, T]$. In particular we drop part a) of Definition 4.3 and we only assume that $\mathfrak{h}_t$ belongs to H_t defined as in (4.16). Under this more general assumption, the optimal strategy $(\bar{\mathfrak{h}}_t)_{0 \leq t \leq T}$ that solves the logarithmic maximization problem (4.17)-(4.18) with $A = 1$ is called, when it exists, the *growth optimal* strategy. The related wealth process $V_t^{\bar{\mathfrak{h}}}$ is called *Growth Optimal Portfolio* (GOP).

As already said in the Introduction, the GOP has the so-called numéraire property [56], in the sense that all the other portfolios measured in terms of the GOP are supermartingales. The numéraire property can be used for example in the benchmark approach [191] to price contingent claims even in models where an Equivalen Martingale Measure (EMM) is absent. GOP denominated prices might however fail to be martingale and being instead strict supermartingales [22, 38, 66, 146]. We will now show that in our model the inverse GOP process is either a martingale or a strict supermartingale depending on whether the growth optimal strategy is an internal or a boundary solution with respect to the domain of the admissible strategies.

Hereafter assume that a growth optimal strategy $\bar{\mathfrak{h}}$ exists, and it is characterized as

$$\bar{\mathfrak{h}}(t) = \bar{h}(t, D_{t-}),$$

where $\bar{h}(t,\cdot) : \{0,1\}^n \to H_t$ is a deterministic function such that

$$\bar{h}(t,d) \in \arg\max_{h \in H_t} F^d(t,h), \tag{7.54}$$

for any $t \in [0,T]$, with $F^d(t,h)$ as in (5.26). For sake of simplicity, we can always assume without any loss of generality that $h_i(\bar{t},d) = 0$ if $d_i = 1$. Then, by the Itô's formula, the dynamics of the inverse GOP process $I_t := \frac{1}{V_t^{\mathfrak{h}}}$ is

$$\begin{aligned}\frac{\mathrm{d}I_t}{I_{t-}} = &-\left\langle \bar{h}(t,D_{t-}), \nabla_h F^{D_{t-}}(t,\bar{h}(t,D_{t-}))\right\rangle \mathrm{d}t - \bar{h}(t,D_{t-})\sigma(t,D_t)\mathrm{d}W_t \\ &+ \int_{X^n} \left(\frac{1}{1+\left\langle \bar{h}(t,D_{t-}),x\right\rangle} - 1 \right) \big(N(D_{t-},\mathrm{d}t,\mathrm{d}x) - \nu_t(D_{t-},\mathrm{d}x)\mathrm{d}t\big),\end{aligned}$$

Now, observe that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \left(\int_{X^n} \left(\frac{1}{1+\left\langle \bar{h}(t,D_{t-}),x\right\rangle} - 1 \right)^2 \nu_t(D_{t-},\mathrm{d}x) + \left|\bar{h}(t,D_{t-})\sigma(t,D_t)\right|^2 \right) \mathrm{d}t \right] \\ &= \int_0^T \left(\max_{d \in \{0,1\}^n} \int_{X^n} \left(\frac{1}{1+\left\langle \bar{h}(t,d),x\right\rangle} - 1 \right)^2 \nu_t(d,\mathrm{d}x) + \left|\bar{h}(t,d)\sigma(t,d)\right|^2 \right) \mathrm{d}t < +\infty,\end{aligned}$$

From this we get that $\mathbb{E}[\sup_{0\le t\le T} |I_t|^2] < +\infty$ (see [192, V.Theorem 67]) and that

$$\begin{aligned}&- I_{t-}\bar{h}(t,D_{t-})\sigma(t,D_t)\mathrm{d}W_t \\ &+ I_{t-} \int_{X^n} \left(\frac{1}{1+\left\langle \bar{h}(t,D_{t-}),x\right\rangle} - 1 \right) \big(N(D_{t-},\mathrm{d}t,\mathrm{d}x) - \nu_t(D_{t-},\mathrm{d}x)\mathrm{d}t\big)\end{aligned}$$

is the stochastic differential of a martingale. Therefore, I_t is a martingale if and only if

$$\left\langle \bar{h}(t,D_{t-}), \nabla_h F^{D_{t-}}(t,\bar{h}(t,D_{t-}))\right\rangle = 0, \quad \forall\, t \in [0,T].$$

Of course, by (7.54), $\bar{h}(t,d) \in \operatorname{int}(H_t)$ implies $\nabla_h F^d(t,\bar{h}(t,d)) = 0$, and thus, in order for I_t to be a strict supermartingale the optimal strategy $\bar{h}(t,D_t)$ has to be a boundary solution. The latter case is only possible when $\operatorname{supp}(\nu_t^d(dx))$ are not compact subsets of $\mathbb{R}^n$, and in the next subsection we will provide an example where this fact is evident.

1.7.1 Strict supermartingale inverse-GOP

We consider a market model with only one risky asset, that is $n = 1$. Therefore, we can refer to $\mu(t,0)$, $\sigma(t,0)$ and $\nu^0(t,dx)$ as $\mu(t)$, $\sigma(t)$ and $\nu(t,dx)$ respectively. Furthermore the pre-default growth optimal policy $\bar{h}(t) \equiv \bar{h}(t,0)$ is the value that maximize over $H_t \subset \mathbb{R}$ the function

$$F(t,h) \equiv F^0(t,h) = \mu(t)h - \frac{1}{2}\sigma^2(t)h^2 + \int_{-1}^{\infty} \Big(\log(1+xh) - xh \Big)\nu_t(\mathrm{d}x).$$

Now, let us set $0 \le m_t \le 1$ and $M_t \in [0,+\infty]$ such that $\operatorname{supp}(\nu_t) \subseteq [-m_t, M_t]$ for any $t \in [0,T]$, and consider the following scenarios.

a) $\nu_t(\{-m_t\}) > 0$, $\nu_t(\{M_t\}) > 0$. According with (4.16) we have $H_t = (-\frac{1}{M_t}, \frac{1}{m_t})$ for any $t \in [0, T]$. In this case the growth optimal policy exists and it is an internal point. Indeed, the function $F(t, h)$ is concave in h and

$$\lim_{h \to -\frac{1}{M_t}^+} F(t,h) = \lim_{h \to \frac{1}{m_t}^-} F(t,h) = -\infty.$$

In this case the inverse GOP I_t is a martingale.

b) $\nu_t(\{-m_t\}) = 0 < \nu_t(\{M_t\})$. We have $H_t = (-\frac{1}{M_t}, \frac{1}{m_t}]$ and the growth optimal policy still exists because $F(t, h)$ is concave in h and

$$\lim_{h \to -\frac{1}{M_t}^+} F(t,h) = -\infty.$$

In this case, differently from a), $\arg\max_{h \in H_t} F(t, h)$ might correspond to the boundary point $\frac{1}{m_t}$. A necessary and sufficient condition for this is

$$\lim_{h \to \frac{1}{m_t}^-} \partial_h F(t,h) = \mu - \sigma(t)h + \int_{-m_t}^{M_t} \left(\frac{x}{1+hx} - x \right) \nu_t(\mathrm{d}x) \geq 0, \tag{7.55}$$

that of course implies

$$\int_{-m_t}^{-m_t+\epsilon} \frac{x}{1+\frac{x}{m_t}} \nu_t(\mathrm{d}x) > -\infty.$$

Therefore, if (7.55) holds the inverse GOP I_t is a strict supermartingale.

c) $\nu_t(\{M_t\}) = 0 < \nu_t(\{-m_t\})$. In analogy with b), we have $H_t = [-\frac{1}{M_t}, \frac{1}{m_t})$ and the condition

$$\lim_{h \to -\frac{1}{M_t}^+} \partial_h F(t,h) = \mu - \sigma(t)h + \int_{-m_t}^{M_t} \left(\frac{x}{1+hx} - x \right) \nu_t(\mathrm{d}x) \leq 0,$$

in order for $\bar{h}(t)$ to coincide with boundary point $-\frac{1}{M_t}$.

d) $\nu_t(\{-m_t\}) = \nu_t(\{M_t\}) = 0$. Totally analogous to b) and c), with $\bar{h}(t)$ possibly corresponding to either $-\frac{1}{M_t}$ or $\frac{1}{m_t}$.

The above example shows that a boundary solution $1/m_t$ (or $-1/M_t$) is impossible when the compensator ν_t puts mass on the boundary m_t (or M_t) of the respective support. Indeed, as the proportion invested in the risky asset gets closer to the the boundary, the log-portfolio value gets arbitrarily close to $-\infty$ with positive probability. Note that this phenomena is actually independent of m_t being or not equal to one, and so independent of the risky asset being or not defaultable.

Part II

Pricing european claims in defaultable exponential Lévy models

Chapter 2

Pricing vulnerable claims in a Lévy driven model

Based on a joint work with ([46]) Prof. A. Capponi and Prof. T. Vargiolu.

Abstract: we obtain explicit expressions for prices of vulnerable claims written on a stock whose predefault dynamics follows a Lévy-driven SDE. The stock defaults to zero with a hazard rate intensity being a negative power of the stock price. We recover the characteristic function of the terminal log price as the solution of a complex valued infinite dimensional system of first order ordinary differential equations. We provide an explicit eigenfunction expansion representation of the characteristic function in a suitably chosen Banach space, and use it to price defaultable bonds and stock options. We present numerical results to demonstrate the accuracy and efficiency of the method.

Keywords: Lévy, exponential, default, equity-credit, reduced approach, default intensity, change of measure, Girsanov theorem, Esscher transform, characteristic function, abstract Cauchy problem, eigenvectors expansion, Fourier inversion.

2.1 Introduction

The development of pricing frameworks, which allow for consistent valuation of equity and credit derivatives, has been subject of considerable investigation. The seminal paper [176] dates back to 1974, and establishes a linkage between equity option and bond markets via a structural model, where bondholders have absolute priority on the equity owners of the firm at default. Further studies have considered reduced form models, with default modeled as the first jump of a Poisson process with stochastic intensity, the so-called Cox process (see [148]). Such models require the specification of the conditional mean arrival rate of default, which depends on the underlying fundamentals and usually takes a parametric form to facilitate calibration to bonds spreads, credit default swaps, or equity options.

The approaches proposed in the literature have modeled the dependence of the default intensity on (1) the value of the underlying stock, (2) the stock volatility, and (3) both stock value and its volatility. In [156] the author develops a joint equity-credit framework, where the default intensity is a negative power of the underlying stock price, under Black-Scholes predefault dynamics. He provides closed-form pricing formulas for corporate bonds and stocks via a spectral expansion of the diffusion infinitesimal generator. In [49] the authors construct a model, where the default rate is an affine function of the variance of the underlying stock. Using the theory of Bessel processes, they are able to provide closed form expressions for security prices. In[52] they consider a similar model, but allow for jumps of infinite activity in the underlying stock dynamics. In [175] they build a unified credit-equity framework by time changing a diffusion to a tractable jump diffusion process with stochastic volatility. In [171] the author considered a general state-dependent Lévy-based model for the pre-default dynamics of the underlying, and, for a generic state-dependent default intensity function, they carried out a family of approximations for European options prices and for the equivalent Black&Scholes implied volatilities. In [53] they consider prices of spread options, under the assumption that stocks stay above an upper barrier before default, and drop below a lower barrier after that. In [51] they develop a local volatility formula under the joint credit-equity model in [156], and demonstrate how it can be simultaneously calibrated to CDS and equity option prices.

This chapter belongs to the stream of literature focusing on pricing of defaultable bonds and vulnerable options within a joint equity-credit framework. Similarly to [156], we use a reduced form model of default, and assume the state dependent default intensity h to be a negative power of the stock price, i.e.

$$h(S_t) = S_t^{-p}, \quad p > 0.$$

This choice was considered in [5], [12], [68], [76, p. 216], [179] and [144]. In particular, in [179] the author empirically estimated, via finite-difference or lattice methods, the value of the power parameter p to be in the range between 1.2 and 2 for Japanese bonds rated BB+ and below. Even though the main focus of these references is on pricing convertible bonds, in [5] they showed that this class of models generate implied volatility skews in stock option prices, with the parameters of the hazard rate specification controlling the slope of the skew. This establishes a link between implied volatility skews and credit spreads. The negative power intensity choice is also empirically relevant in light of events occurred during the recent financial crisis. Indeed, during the first half of 2008, Lehman stock lost 73% of its value as the credit market continued to tighten, with credit spreads increasing from 150 to about 700 basis points, see also [6].

Differently from [156], we also allow for the possibility that the stock exhibits exogenous jumps of finite or infinite activity.

The dependence of the hazard intensity on the stock level makes the payoff of the vulnerable claim path dependent. To this purpose, we firstly develop a change of measure which reduces the problem to pricing an European style claim written on the predefault log-price. Our technique generalizes [156], as the presence of jumps in the dynamics prevents a direct application of the Brownian version of the Girsanov theorem and requires the use of the Esscher transfom. We then provide a representation of the characteristic function of the predefault log-price process via a novel methodology, which departs significantly from existing literature in option pricing. More specifically, we prove that the characteristic function of the log-price process can be characterized as the solution of a complex-valued infinite dimensional linear system of first order ordinary differential equations. After reformulation as an abstract Cauchy problem in a suitably chosen Banach space, we obtain explicit expressions for the eigenvalues and eigenvectors of the matrix operator, and recover an explicit eigenfunction expansion of the characteristic function. We then use it to price defaultable bonds and options, demonstrating the accuracy and efficiency of the method.

The proposed approach transcends the specific financial application, and advances existing literature in probability theory dealing with time integrals of a geometric Brownian motion. Besides mathematical finance, such processes have wide applications, see for instance [78] for an application to insurance, and have been subject of detailed investigation. In particular, the solution of the SDE

$$dV_t = (2(1+a)V_t + 1)dt + 2V_t dL_t, \tag{1.1}$$

with $(L_t)_{t\geq 0}$ being a standard Brownian motion, turns out to be a time-integral functional of a geometric Brownian motion ([140], pp. 360-361). The distribution of V_t has been widely studied by many authors. In [77] and [213] they independently derived the general expressions for the moments. The transition density has been first derived in [210], pp. 271, Equation 38, for $a < 0$ (see also [59], [155]). In the general case, a spectral representation of the transition density has been found in [156], by inverting the Laplace transform in time, first obtained in [73] as the solution of an ordinary differential equation in the space variable. Our contribution to this literature is a novel representation of the characteristic function of $\log V_t$. Moreover, this characterization is naturally carried out for the more general case when L_t in (1.1) is a Lévy process.

Another interesting property is the connection with arithmetic Asian options. Indeed, a well known identity (see [77]) states that V_t in (1.1) is equal in law to the time integral of an exponential Brownian motion. This connection is well explained in [156], Appendix B, and has been exploited by different authors (see [73] and [155]) to price Asian style derivatives. In this context, our method might provide a novel approach for pricing arithmetic Asian options, which expands the array of currently available methods (see [111], [69], [207], and Chapter 7 of the current thesis based on [95]).

The rest of the chapter is organized as follows. Section 2.2 introduces the intensity based model of default, and provides an European type characterization of the price of the vulnerable claim. Section 2.3 develops a fully explicit representation for the characteristic function. Section 2.4 provides a numerical study to assess accuracy and computational efficiency of the proposed representation. Section 2.5 contains the conclusions. Some of the proofs are delegated to the Appendix, whereas all the figures and tables are reported in Section 2.7.

2.2 Pricing of a defaultable claim

Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space, where $\mathbb{Q}$ is a suitable pricing measure, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual hypotheses of completeness and right continuity and it is rich enough to support:

- a standard Brownian motion W;
- a pure-jump Lévy process Z, independent of W, with characteristic triplet $(0, 0, \nu)$, i.e.
$$Z_t = \int_0^t \int_{|y|>1} z\ N(ds, dz) + \int_0^t \int_{|z|<1} y\ \widetilde{N}(ds, dz),$$
where N and ν are, respectively, the jump measure and the Lévy measure of Z, whereas
$$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$$
is the compensated jump measure;
- a unit mean exponentially distributed and $\mathcal{G}$-measurable random variable ζ independent of the filtration $\mathbb{F}$.

In order for further computations to make sense, we need the Lévy measure ν to satisfy the following integrability condition:

$$\int_{|z|>1} e^z \nu(dz) < \infty.$$

We use a standard construction of the default time τ, based on doubly stochastic point processes, see [32], Section 9.2.1. More specifically, given a nonnegative $\mathbb{F}$-adapted intensity process $h = (h_t)_{0\leq t\leq T}$ we define the default time τ as

$$\tau = \inf\left\{ t \geq 0 : \int_0^t h_s ds \geq \zeta \right\}.$$

The market filtration $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T}$, which describes the information available to investors, is given by
$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \sigma(\tau \wedge u).$$
after completion and regularization on the right, see [23]. It is a well-known result (see e.g. [32], Section 6.5 for details) that the process

$$M_t = \mathbb{1}_{\{\tau\leq t\}} - \int_0^{t\wedge\tau} h_s ds \tag{2.2}$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}$. From now on, we let the default intensity depend on time through the underlying stock S_t, i.e. $h_t = h(S_t)$, so that the predefault dynamics of the underlying stock admits the form

$$\frac{dS_t}{S_{t-}} = (r - q + h(S_t))dt + \sigma dW_t + \int_{\mathbb{R}} (e^z - 1)\, \widetilde{N}\,(dt, dz)\,, \tag{2.3}$$

where $r, q \geq 0$ and $\sigma > 0$ are, respectively, the risk-free interest rate, dividend rate and volatility, and $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a nonnegative function satisfying the following standing assumption.

Assumption A. *h is a strictly decreasing function that belongs to $C^1(]0,\infty[)$, such that*

$$\lim_{s\to 0} h(s) = \infty, \quad \lim_{s\to\infty} h(s) = 0. \tag{2.4}$$

and that the SDE (2.3) admits a unique strong strictly positive solution defined on $[0,t)$ for any time $t > 0$.

Roughly speaking, the lower the asset value, the higher the default intensity; conversely, the higher the stock price, the lower the default risk.

In this way, the pure jump Lévy process Z precisely describes the jumps of the log-return. In alternative, we could have let Z describe the multiplicative jumps of S, but in that case we would have to impose that $\nu((-\infty,-1]) = 0$, see [189] for a more detailed discussion.

The defaultable stock price process is defined as $\widetilde{S}_t = \mathbb{1}_{\{\tau>t\}} S_t$. In other words, it follows the predefault dynamics (2.3) up to $\tau-$, and jumps to 0 at time τ, where it remains forever afterwards. Indeed we have

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t-}} = (r-q)dt + \sigma dW_t + \int_{\mathbb{R}} (e^z - 1)\,\widetilde{N}(dt,dz) - dM_t,$$

where M_t has been defined in (2.2). It is easy to check that $e^{-(r-q)t}\widetilde{S}_t$ is a local martingale under $\mathbb{Q}$.
We are interested in pricing a contingent claim that, at maturity t, pays a payoff φ if default did not occur and a constant recovery R otherwise. The no-arbitrage price of such a claim at time 0 is then given by

$$e^{-rt}\mathbb{E}_{\mathbb{Q}}\left[\varphi(\widetilde{S}_t)\mathbb{1}_{\{\tau>t\}} + R\mathbb{1}_{\{\tau\leq t\}}\right].$$

Using the so called Key Lemma (see [32], Lemma 5.1.2 pp.143), we have

$$\mathbb{E}_{\mathbb{Q}}\left[\varphi(\widetilde{S}_t)\mathbb{1}_{\{\tau>t\}}\right] = \mathbb{E}_{\mathbb{Q}}\left[\varphi(S_t)\mathbb{1}_{\{\tau>t\}}\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t h(S_u)du}\varphi(S_t)\right],$$
$$\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau\leq t\}}\right] = 1 - \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>t\}}\right] = 1 - \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t h(S_u)du}\right].$$

Therefore, assuming $S_0 = s > 0$, the price of the claim, denoted by $C_{\varphi,R}(t,s)$, is given by

$$C_{\varphi,R}(t,s) = e^{-rt}\left(R + \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t h(S_u)du}(\varphi(S_t) - R)\right]\right). \tag{2.5}$$

Remark 2.1. *If $R = 0$ and $\varphi(s) = (s-K)^+$, then $C_{\varphi,R}$ in (2.5) reduces to the price of a vulnerable call option with zero recovery. Similarly, if $\varphi(s) = 1$ and $R = 0$, it reduces to the price of a defaultable zero-coupon bond with zero recovery.*

Note that the expectation in (2.5) contains a path-dependent term, namely the negative exponential of the cumulative default intensity. Using a change of measure technique, which generalizes the one adopted in [156] for purely diffusive dynamics, we express $C_{\varphi,R}(t,s)$ as the expectation of a payoff function only depending on the terminal value S_t. This is key for the characteristic function approach developed in the next section. Differently from [156], who does not consider jumps in the stock dynamics, here the presence of the Lévy process $(Z_t)_{t\geq 0}$ in the dynamics of S requires to combine the use of the Girsanov theorem with the Esscher transform.

It is convenient to work with the predefault log-return process $X_t := \log S_t$. By Itô's formula, we obtain

$$dX_t = \left(r - q + h\left(e^{X_t}\right) - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1 - z\mathbb{1}_{\{|z|<1\}})\nu(dz)\right) dt \\ + \sigma dW_t + dZ_t. \tag{2.6}$$

Therefore, in terms of the initial log-price $X_0 = x$, the price of the European claim in (2.5) becomes

$$C_{\varphi,R}(t,x) = e^{-rt}\left(R + \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^t h\left(e^{X_u}\right)du}\left(\varphi\left(e^{X_t}\right) - R\right)\right]\right). \tag{2.7}$$

Lemma 2.2. *For any $t > 0$ and $x \in \mathbb{R}$ we have*

$$C_{\varphi,R}(t,x) = e^{-rt}R + e^{-qt}e^x\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{-X_t}\left(\varphi\left(e^{X_t}\right) - R\right)\right], \tag{2.8}$$

where $\widetilde{\mathbb{Q}}$ is a measure on $(\Omega, \mathcal{G})$, equivalent to $\mathbb{Q}$, defined as

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t - t\int_{\mathbb{R}} (e^z - 1 - z\mathbb{1}_{\{|z|<1\}})\nu(dz) + Z_t\right). \tag{2.9}$$

The dynamics of the process $(X_s)_{0\leq s\leq t}$ under $\widetilde{\mathbb{Q}}$ becomes

$$dX_s = \left(\bar{r} + \frac{\sigma^2}{2} + h\left(e^{X_s}\right)\right) ds + \sigma d\widetilde{W}_s + d\widetilde{Z}_s, \tag{2.10}$$

where

$$\bar{r} = r - q - \int_{\mathbb{R}} \left(e^z - 1 - ze^z\mathbb{1}_{\{|z|<1\}}\right)\nu(dz). \tag{2.11}$$

The processes $(\widetilde{W}_s)_{0\leq s\leq t}$ and $(\widetilde{Z}_s)_{0\leq s\leq t}$ are, respectively, a standard Brownian motion and a Lévy process with characteristic triplet $(0, 0, e^x \cdot \nu)$ under $\widetilde{\mathbb{Q}}$. Moreover, $\widetilde{W}$ and $\widetilde{Z}$ are independent.

Proof. By (2.6) we obtain

$$e^{-\int_0^t h\left(e^{X_u}\right)du} = e^{x+(r-q)t}e^{-X_t}\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}$$

where $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}$ is defined as in (2.9), and plugging it into (2.7) we obtain (2.8). We now prove the second part. An equivalent definition of $\widetilde{\mathbb{Q}}$ is given by

$$\frac{d\widetilde{\mathbb{Q}}}{d\overline{\mathbb{Q}}} = \exp\left(-\frac{\sigma^2}{2}t + \sigma dW_t\right),$$

where $\overline{\mathbb{Q}}$ is a probability measure, equivalent to $\mathbb{Q}$, defined as

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left(-t\int_{\mathbb{R}} (e^z - 1 - z\mathbb{1}_{\{|z|<1\}})\nu(dz) + Z_t\right).$$

We first observe that

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} = \frac{e^{Z_t}}{\mathbb{E}_{\mathbb{Q}}\left[e^{Z_t}\right]},$$

By a known result on the Esscher transform applied to Lévy processes (see for instance [188], Theorem 13.67) we obtain

$$Z_s = s\int_{\mathbb{R}} \left(e^z - 1\right) z\mathbb{1}_{\{|z|<1\}}\nu(dz) + \widetilde{Z}_s, \quad s \le t, \tag{2.12}$$

where $(\widetilde{Z})_{0\le s\le t}$ is a Lévy process with characteristic triplet $(0, 0, e^x \cdot \nu)$ under $\overline{\mathbb{Q}}$. Moreover, as W is independent of $\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}}$, W preserves the law and is independent of $\widetilde{Z}$ under $\overline{\mathbb{Q}}$. Therefore, the dynamics of $(X_s)_{0\le s\le t}$ under $\overline{\mathbb{Q}}$ is given by

$$dX_s = \left(\bar{r} - \frac{\sigma^2}{2} + h\left(e^{X_s}\right)\right)ds + \sigma dW_s + d\widetilde{Z}_s. \tag{2.13}$$

We observe now that $\left(\mathbb{E}_{\overline{\mathbb{Q}}}\left[\frac{d\widetilde{\mathbb{Q}}}{d\overline{\mathbb{Q}}}\middle| \mathcal{F}_s\right]\right)_{0\le s\le t}$ is an exponential martingale, and applying the Girsanov theorem we obtain

$$dW_s = \sigma ds + d\widetilde{W}_s, \quad s \le t, \tag{2.14}$$

where $(\widetilde{W}_s)_{0\le s\le t}$ is a standard Brownian motion under $\widetilde{\mathbb{Q}}$. Moreover, as $\widetilde{Z}$ is independent of $\frac{d\widetilde{\mathbb{Q}}}{d\overline{\mathbb{Q}}}$, $\widetilde{Z}$ preserves the law and is independent of $\widetilde{W}$ under $\widetilde{\mathbb{Q}}$. Hence, plugging (2.14) into (2.13), the dynamics of $(X_s)_{0\le s\le t}$ under $\widetilde{\mathbb{Q}}$ is given by (2.10), and this completes the proof. □

2.3 A characteristic function approach in the negative power intensity case

In this section we focus on the particular choice of h given by

$$h(s) = \beta s^{-p}, \quad p > 0, \quad \beta > 0. \tag{3.15}$$

First, we need to check that the above choice of h satisfies Assumption A. This is done in the following lemma.

Lemma 3.1. *The function h defined in Equation* (3.15) *satisfies Assumption A.*

Proof. It is straightforward to check that the function h defined in Eq. (3.15) belongs to $C^1(]0,\infty[)$ and admits the limits in Eq. (2.4). It remains to verify the second part of Assumption A, namely the global strong uniqueness and positivity of the solution of Equation (2.3). To see this, first of all let S be a strong solution of Equation (2.3) on $[\![0,\tau[\![$, where τ is an explosion time for S in the sense of [192, Theorem V.38]. This also ensures that S is the unique strong solution of Equation (2.3) on $[\![0,\tau[\![$. Define then $V_t := S_t^p = p\beta e^{Y_t}$ for all $t \in [\![0,\tau[\![$. Then V is the solution of

$$dV_t = (aV_t + b)dt + cV_t dW_t + V_{t-}\int_{\mathbb{R}}(e^{pz} - 1)\widetilde{N}(dt, dz) \tag{3.16}$$

on $\llbracket 0,\tau\llbracket$ with suitably chosen $a,b,c\in\mathbb{R}$. By [192, Theorem V.7], Equation (3.16) admits a unique global strong solution $\widetilde{V}$, which thus coincides with V on $\llbracket 0,\tau\llbracket$. Now, by letting $\widetilde{S}_t := \widetilde{V}_t^{1/p}$ for all $t\geq 0$, we have that $\widetilde{S}$ satisfies Equation (2.3): this implies that $\tau=+\infty$ a.s., and that $\widetilde{S}\equiv S$ is the unique global strong solution of Equation (2.3). Finally, it is positive by [192, Theorem V.72]. □

Thus, Eq. (2.10) becomes

$$dX_t = \left(\bar{r} + \frac{\sigma^2}{2} + \beta e^{-pX_t}\right) dt + \sigma d\widetilde{W}_t + d\widetilde{Z}_t.$$

Now, instead of working directly on the process X, we consider the process $Y_t := -\log(p\beta) + pX_t$. By Itô's formula we obtain

$$dY_t = p\left(\bar{r} + \frac{\sigma^2}{2}\right) dt + e^{-Y_t}dt + p\sigma d\widetilde{W}_t + p\ d\widetilde{Z}_t. \tag{3.17}$$

Clearly, the price in (2.8) becomes

$$C_{\varphi,R}(t,x) = e^{-rt}R + e^{-qt+x}\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\bar{\varphi}(Y_t,R)\right], \tag{3.18}$$

where

$$\bar{\varphi}(Y,R) = \left(p\beta e^{Y}\right)^{-\frac{1}{p}}\left(\varphi\left(\left(p\beta e^{Y}\right)^{\frac{1}{p}}\right) - R\right). \tag{3.19}$$

We explicitly observe that, by definition, $Y_0 = px - \log(p\beta)$.

For such a choice of the default intensity function h, the pricing problem has been studied in [156], under the assumption that the underlying asset follows a purely diffusive log-normal dynamics. In particular, the author computed the Laplace transform in time of the positive valued process e^{Y_t}, and after inverting it, he was able to provide a spectral representation for the transition density. On the contrary, our approach is based on the study of the characteristic function of the real valued process Y. This difference represents a key point for two main reasons. Firstly, from the mathematical point of view we develop a completely novel methodology. Moreover, the proposed approach naturally handles Lévy jumps in the dynamics (2.3) without any additional effort. Indeed, as we shall see shortly, the characteristic function of Y_t can be viewed as the solution of an infinite dimensional system of ODE's with respect to the time variable. Here, differently from Linetsky's approach based on the transition density, the space variable becomes a mere parameter, and the addition of Lévy jumps does not impact the structure of the differential operator.

Once the characteristic function of Y_t is known, the expectation in (3.18) can be computed using the most popular inverse Fourier techniques, as we shall see in Section 2.4.

The next two subsections are devoted to an analytical characterization of the characteristic function of Y_t. Before, the next remark shows how the methodology that we propose in this setting might be also applied to the pricing of arithmetic Asian options.

Remark 3.2. *Consider an underlying stock whose price is given by a geometric Brownian motion $S_t = e^{\sigma B_t + (r-\frac{\sigma^2}{2})t}$, where we assumed for simplicity $S_0 = 1$. Define the arithmetic average process*

$$A_t = \int_0^t S_u du = \int_0^t e^{\sigma B_u + (r-\frac{\sigma^2}{2})u} du,$$

and consider a European-type payoff function $\varphi(A_t)$. *Some typical choices of the payoff might be* $\varphi(A_t) = (\frac{A_t}{t} - K)^+$ *and* $\varphi(A_t) = (K - \frac{A_t}{t})^+$, *referring to an Asian call and put option respectively, with a positive fixed strike* K.

Now, by invariance to time reversal of Brownian motion (see [77] for more details), for any $t > 0$, A_t *is equal in law to*

$$L_t = \int_0^t e^{\sigma(B_t - B_u) + (r - \frac{\sigma^2}{2})(t-u)} du,$$

where L_t *satisfies*

$$dL_t = (rL_t + 1)dt + \sigma L_t dB_t, \qquad L_0 = 0.$$

Thus, in order to price the Asian option $\varphi(A_t)$ *it is enough to characterize the law of* L_t. *We could now consider the perturbed version of* L_t *given by*

$$dL_t^\varepsilon = (rL_t^\varepsilon + 1)dt + \sigma L_t^\varepsilon dB_t, \qquad L_0^\varepsilon = \varepsilon, \qquad 0 < \varepsilon << 1,$$

and then use our approach to determine the characteristic function of $\log(L_t^\varepsilon)$, *whose dynamics is of the form* (3.17).

2.3.1 A useful characterization of the characteristic function

We define the characteristic function of Y_t, given $Y_0 = y \in \mathbb{R}$, as

$$\phi(y; t, \xi) := \mathbb{E}\left[e^{i\xi Y_t}\right], \tag{3.20}$$

for $t > 0$ and $\xi \in \mathbb{C}$ such that the expression above is well defined. As the starting point y is fixed once and for all, in the sequel we will systematically use the shorthand notation $\phi(t, \xi)$ instead of $\phi(y; t, \xi)$. From now on we assume the Lévy measure to satisfy the following:
Assumption B. *For any* $\alpha \leq 1$,

$$\int_{|z|>1} e^{\alpha z} \nu(dz) < \infty. \tag{3.21}$$

From a financial point of view, Assumption B may appear restrictive. Admittedly, many Lévy driven models used in finance do not satisfy the integral condition (3.21). However, this assumption may be dropped so to include more general Lévy measures in our analysis if, in cases when ν does not satisfy Assumption B, we consider a limit representation for $\phi(t, \xi)$ in terms of a sequence of characteristic functions. Concretely, for any $I > 1$, denote by Y^I the solution of the SDE

$$dY^I = p\left(\bar{r} + \frac{\sigma^2}{2}\right) dt + e^{-Y_t^I} dt + p\sigma d\widetilde{W}_t + p\, d\widetilde{Z}_t^I,$$

where $\widetilde{Z}^I$ is the Lévy process defined as follows. Recall from Eq. (2.12) that $\widetilde{Z}$ is a pure jump Lévy process with characteristic triplet $(0, 0, \widetilde{\nu})$ and thus, for any $t \geq 0$, it can be represented as

$$\widetilde{Z}_t = \int_0^t \int_{|z|\leq 1} z\widetilde{J}(ds, dz) + \int_0^t \int_{|z|>1} zJ(ds, dz),$$

with $\widetilde{J}(ds,dz) = J(ds,dz) - \widetilde{\nu}(dz)ds$, and where $J(ds,dz)$ represents the jump measure of $\widetilde{Z}$. Next we define

$$\widetilde{Z}_t^I := \int_0^t \int_{|z|\le 1} z\widetilde{J}(ds,dz) + \int_0^t \int_{1<|z|\le I} zJ(ds,dz).$$

In other words, $\widetilde{Z}^I$ has all the jumps of $\widetilde{Z}$ with magnitude not greater than I. Hence, $\widetilde{Z}^I$ is a Lévy process with characteristic triplet $(0,0,\widetilde{\nu}^I)$, where

$$\widetilde{\nu}^I(dx) = \mathbb{1}_{[-I,I]}(x)\widetilde{\nu}(dx) = \mathbb{1}_{[-I,I]}(x)e^x\nu(dx).$$

In the following Lemma 3.3, we prove that for any $t > 0$, Y^I converges pointwise to Y_t as I goes to ∞. Using dominated convergence, this immediately leads to

$$\phi(t,\xi) = \lim_{I\to\infty} \phi^I(t,\xi). \tag{3.22}$$

Here, $\phi^I(t,\cdot)$ denotes the characteristic function of Y_t^I, whose associated Lévy measure $\widetilde{\nu}^I$ clearly satisfies Assumption B.

Lemma 3.3. *For all $t > 0$, $\lim_{I\to+\infty} Y_t^I = Y_t$ almost surely.*

Proof. For a fixed $I > 0$, define $V_t := S_t^p = p\beta e^{Y_t}$ and $V_t^I := p\beta e^{Y_t^I}$. Then V is the unique strong solution of Eq. (3.16) with $V_0 = s^p$, while V^I is the unique strong solution of the same equation with $V_0^I = s^p$ and $\widetilde{J}^I$ in place of $\widetilde{J}$, where $\widetilde{J}^I$ is the compensated jump measure of $\widetilde{Z}^I$. Then $U_t := V_t - V_t^I$ is the unique solution of

$$\begin{cases} dU_t &= aU_tdt + cU_tdW_t + U_{t-}\int_{\mathbb{R}}(e^{pz}-1)\widetilde{J}(dt,dz) + \\ & +V_{t-}^I\int_{\mathbb{R}}(e^{pz}-1)(\widetilde{J}-\widetilde{J}^I)(dt,dz), \quad t>0, \\ U_0 &= 0 \end{cases} \tag{3.23}$$

But $\widetilde{J} - \widetilde{J}^I \equiv 0$ a.s. as a random measure on $[\![0,\tau^I[\![\times\mathbb{R}$, with

$$\tau^I := \inf\{t > 0 : |\Delta\widetilde{Z}_t| > I\}.$$

Then, we have that $\lim_{I\to+\infty}\tau^I = +\infty$, so that for large enough $\bar{I}(=\bar{I}(\omega))$ and fixed time t, we have that $\tau^I > t$ for all $I > \bar{I}$. This immediately implies that $\int_{\mathbb{R}}(e^{pz}-1)(\widetilde{J}-\widetilde{J}^I)(\cdot,dz) \equiv 0$ on $[0,t]$, so $U \equiv 0$ is the unique solution of Equation (3.23) on $[0,t]$, i.e. $V \equiv V^I$ on $[0,t]$. □

The next lemma allows us to characterize $\phi(t,\xi)$ as the solution of an infinite ODE's system with respect to the time variable.

Lemma 3.4. *For any $\xi \in \mathbb{C}$ such that $\mathrm{Im}(\xi) \ge 0$, define $\phi(t,\xi) = \phi(y;t,\xi)$ as in (3.20). Then, we have*

$$|\phi(t,\xi)| = \left|\mathbb{E}\left[e^{i\xi Y_t}\right]\right| \le e^{tf(i\ \mathrm{Im}(\xi))} < \infty \tag{3.24}$$

and

$$\begin{cases} \frac{d}{dt}\phi(t,\xi) = f(\xi)\phi(t,\xi) + i\xi\phi(t,\xi+i), \quad t>0, \\ \phi(0,\xi) = e^{i\xi y}, \end{cases} \tag{3.25}$$

where

$$f(\xi) = p\left(r - q + \frac{\sigma^2}{2}\right)i\xi - \frac{1}{2}p^2\sigma^2\xi^2 + \psi(\xi), \tag{3.26}$$

with

$$\psi(\xi) = \int_{\mathbb{R}} \left(i\xi p\big(1-e^z\big) + e^z\big(e^{i\xi pz}-1\big)\right)\nu(dz). \tag{3.27}$$

Proof. Let us recall that $\widetilde{Z}$ is a Lévy process with characteristic triplet $(0,0,\widetilde{\nu})$, where $\widetilde{\nu}(dx) = e^x \nu(dx)$. Moreover, we will denote with $J(dx,dt)$ the jump measure of $\widetilde{Z}$, whereas

$$\widetilde{J}(dx,dt) = J(dx,dt) - \widetilde{\nu}(dx)dt$$

represents the compensated jump measure of $\widetilde{Z}$. From now on, we fix $\xi \in \mathbb{C}$ such that $\omega := \mathrm{Im}(\xi)$. For $\omega = 0$, (3.24) is trivial. In order to prove (3.24) for $\omega > 0$, we set $\bar{Y}_t := Y_t - \int_0^t e^{-Y_s} ds$ and obtain

$$|\phi(t,\xi)| = \left| \mathbb{E}\left[e^{i\xi Y_t} \right] \right| \le \mathbb{E}\Big[e^{-\omega Y_t} \Big] \le \mathbb{E}\left[e^{-\omega \bar{Y}_t} \right]. \tag{3.28}$$

Clearly,

$$d\bar{Y}_t = p\left(\bar{r} + \frac{\sigma^2}{2} \right) dt + p\sigma \, d\widetilde{W}_t + p \, d\widetilde{Z}_t,$$

and thus, $\bar{Y}/p$ is a Lévy process with characteristic triplet $\left(\left(\bar{r} + \frac{\sigma^2}{2} \right), \sigma, \widetilde{\nu} \right)$, and characteristic exponent

$$\bar{\psi}(\zeta) = i\left(\bar{r} + \frac{\sigma^2}{2} \right) \zeta - \frac{1}{2}\sigma^2\zeta^2 + \int_{\mathbb{R}} \left(e^{i\zeta z} - 1 - i\zeta z \cdot \mathbb{1}_{\{|z|<1\}} \right) \widetilde{\nu}(dz)$$

(by the definition of $\bar{r}$ in (2.11))

$$= \left(r - q + \frac{\sigma^2}{2} \right) i\zeta - \frac{1}{2}\sigma^2\zeta^2 + \int_{\mathbb{R}} \left(i\zeta\left(1 - e^z\right) + e^z\left(e^{i\zeta z} - 1\right) \right) \nu(dz) = f\left(\zeta/p\right),$$

with f defined as in (3.26). Note that, by Assumption B, we also have

$$\int_{|z|>1} e^{-\omega p z} \, \widetilde{\nu}(dz) < \infty, \quad \forall \omega > 0.$$

Therefore, a well known result on Lévy processes (see [197, Theorem 25.17]) gives

$$\mathbb{E}\left[e^{-\omega \bar{Y}_t} \right] = \mathbb{E}\left[e^{-p\omega \frac{\bar{Y}_t}{p}} \right] = e^{t\bar{\psi}(ip\omega)} = e^{tf(i\omega)} < \infty,$$

and this combined with (3.28) proves (3.24). In order to prove (3.25) we set $U_t(\xi) := e^{i\xi Y_t}$. Of course we have,

$$\phi(0,\xi) = \mathbb{E}\left[U_0(\xi) \right] = e^{i\xi y}, \tag{3.29}$$

and by the Itô's formula for Lévy processes (see for instance [188, Theorem 14.37]) we get

$$dU_t(\xi) = U_t(\xi)\left(\left(f(\xi) + i\xi e^{-Y_t} \right) dt + i\xi p\sigma d\widetilde{W}_t + \int_{\mathbb{R}} (e^{i\xi p z} - 1) \widetilde{J}(dt,dz) \right),$$

Note that

$$\int_0^t \mathbb{E}\left[|i\xi p\sigma U_s(\xi)|^2 \right] ds = |\xi p\sigma|^2 \int_0^t \mathbb{E}\left[e^{-2\omega Y_s} \right] ds = |\xi p\sigma|^2 \int_0^t \phi(s, 2i\omega) ds$$

(by (3.24))

$$\le |\xi p\sigma|^2 \int_0^t e^{sf(2i\omega)} ds < \infty,$$

and

$$\int_0^t \int_{\mathbb{R}} \mathbb{E}\left[\left|U_s(\xi)(e^{i\xi pz}-1)\right|^2\right] \widetilde{\nu}(dz)ds = \int_0^t \int_{\mathbb{R}} \mathbb{E}\left[e^{-2\omega Y_s}\right] \left|e^{i\xi pz}-1\right|^2 \widetilde{\nu}(dz)ds$$
$$= \int_0^t \phi(s,2i\omega)ds \int_{\mathbb{R}} \left|e^{-\omega pz}-1\right|^2 \widetilde{\nu}(dz)$$

(by (3.24))

$$\leq \int_0^t e^{sf(2i\omega)}ds \int_{\mathbb{R}} \left|e^{-\omega pz}-1\right|^2 \widetilde{\nu}(dz)$$

(by Assumption B and since $|e^{-\omega pz}-1|^2 = O(z^2)$ as $z \to 0$)

$$< \infty.$$

Therefore

$$\int_0^t i\xi p\sigma U_s(\xi)d\widetilde{W}_s + \int_0^t \int_{\mathbb{R}} U_s(\xi)(e^{i\xi pz}-1)\widetilde{J}(ds,dz)$$

is a martingale, and we obtain

$$\phi(t,\xi) = \mathbb{E}\left[U_t(\xi)\right] = \mathbb{E}\left[\int_0^t U_s(\xi)\left(f(\xi)+i\xi e^{-Y_s}\right)ds\right]$$

(by definition of $U_t(\xi)$)

$$= \mathbb{E}\left[\int_0^t \big(f(\xi)U_s(\xi)+i\xi U_s(\xi+i)\big)ds\right]$$

(again, applying (3.24))

$$= \int_0^t \big(f(\xi)\mathbb{E}\left[U_s(\xi)\right]+i\xi\mathbb{E}\left[U_s(\xi+i)\right]\big)ds = \int_0^t \big(f(\xi)\phi(s,\xi)+i\xi\phi(s,\xi+i)\big)ds.$$

Now, by differentiating on t and by (3.29) we get (3.25), and this concludes the proof. □

2.3.2 Eigenvalues expansion

Let us fix $\xi \in \mathbb{C} \setminus \{0\}$ such that $\mathrm{Im}(\xi) \geq 0$. Throughout this paragraph we provide a representation of the characteristic function of Y_t in the form

$$\phi(t,\xi) = \sum_{n=0}^{\infty} x_n e^{f_n t}, \tag{3.30}$$

where we define, for all $n \geq 0$,

$$f_n := f(\xi+ni), \tag{3.31}$$

$$x_n := \sum_{m=n}^{\infty} e^{i(\xi+im)y} b_{m,n}, \tag{3.32}$$

with $(b_{m,n})_{m\geq n\geq 0}$ defined as below in (3.46). We recall that $y \in \mathbb{R}$ represents the initial point Y_0 of the stochastic process $(Y_t)_{t\geq 0}$. In order to get (3.30)-(3.32) we proceed as follows. For all $t \geq 0$ set

$$\phi_n(t) := \phi(t, \xi + ni), \quad n \geq 0, \tag{3.33}$$

and define the complex valued sequence

$$\Phi_t := \big(\phi_n(t)\big)_{n\geq 0}. \tag{3.34}$$

We first show that Φ solves an abstract Cauchy problem of the form

$$\begin{cases} \frac{d}{dt}\Phi_t = A\Phi_t, \quad t > 0, \\ \Phi_0 = \big(e^{i(\xi+ni)y}\big)_{n\geq 0}, \end{cases} \tag{3.35}$$

where A is a linear operator acting on a suitable Banach space which is defined in the sequel. Then we provide a series expansion for Φ in terms of the eigenvalues and the eigenvectors of A. More rigorously, let us now consider the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ with the measure μ defined as

$$\mu(\{n\}) = e^{-f((n+1)i)\log(n+1)} \quad \forall n \in \mathbb{N}_0,$$

with f as in (3.26). We denote by X the Banach space $L^2(\mathbb{N}_0, \mu)$, i.e. the space of the complex valued sequences $(u(n))_{n\in\mathbb{N}_0}$ endowed with the norm

$$\|u\|^2 = \sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)}|u(n)|^2 < \infty.$$

Let $A : \mathcal{D}(A) \to X$ be the linear operator defined on its natural domain $\mathcal{D}(A) := \{u \in X | Au \in X\}$, given by

$$(Au)(n) = f_n u(n) + i(\xi + ni)u(n+1) \quad \forall n \geq 0. \tag{3.36}$$

In other words, A is the infinite dimensional bi-diagonal matrix $(a_{n,n}, a_{n,n+1})_{n\in\mathbb{N}_0}$, where

$$a_{n,n} = f_n, \quad a_{n,n+1} = i(\xi + ni). \tag{3.37}$$

Proposition 3.5. *For any $t \geq 0$, $\Phi_t \in \mathcal{D}(A)$. Moreover, Φ solves the Cauchy problem (3.35).*

Before proving the last proposition we need to state some preliminary results.

Lemma 3.6. *Assuming $\nu\big((-\infty, -\bar{z}]\big) > 0$ for some $\bar{z} > 0$, the following relations hold*[1]*:*

$$0 < f(ni) = \Omega(e^{np\bar{z}}) \quad \text{as } n \to \infty, \tag{3.38}$$
$$0 < f((n+1)i) - f(ni) = \Omega(e^{np\bar{z}}) \quad \text{as } n \to \infty, \tag{3.39}$$
$$f_n = O(f(ni)) \quad \text{as } n \to \infty, \tag{3.40}$$
$$0 < f_n = \Omega(e^{np\bar{z}}) \quad \text{as } n \to \infty. \tag{3.41}$$
$$0 < |f_{n+1}| - |f_n| = \Omega(e^{np\bar{z}}) \quad \text{as } n \to \infty. \tag{3.42}$$

The proof is postponed to the Appendix.

[1] $g(n) = \Omega(h(n))$ as $n \to \infty$ if there exist $C > 0$ and $\bar{n} > 0$ such that $|g(n)| \geq C|h(n)|$ for any $n > \bar{n}$.

Remark 3.7. *Note that the assumption on the Lévy measure required in the statement of Lemma 3.6 clearly cuts off the particular choice of a purely diffusive model, i.e.* $\nu \equiv 0$, *as well as models where jumps are only positive. Nevertheless, it is possible to give similar estimations leading to the proof of Proposition 3.10, as well as to any further result in this paragraph, where the leading asymptotic behavior* e^n *is replaced by* n^2. *For sake of simplicity, we omit the technical details for these particular cases.*

Proof. (Proposition 3.5) We first treat the case $t = 0$. Of course, by (3.33)-(3.34),

$$\Phi_0 = \left(e^{i(\xi+ni)y}\right)_{n\in\mathbb{N}_0}.$$

Moreover, by (3.38)

$$||\Phi_0||^2 = \sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)} e^{-2ny} < \infty,$$

and so $\Phi_0 \in X$. Now, by (3.36) we have

$$\begin{aligned}
||A\Phi_0||^2 &= \sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)} \left| f_n e^{i(\xi+ni)y} + i(\xi+ni)e^{i(\xi+(n+1)i)y}\right|^2 \\
&\leq 2\sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)-2ny} \left(|f_n|^2 + |\xi+ni|^2 e^{-2}\right) \\
&\leq 2\sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)-2ny} \left(e^{2|f_n|} + |\xi+ni|^2 e^{-2}\right),
\end{aligned}$$

and applying first (3.40), and then (3.39), there exists $\bar{n} > 0$ such that

$$\begin{aligned}
||A\Phi_0||^2 \leq\ & 2\sum_{n=0}^{\bar{n}} e^{-f((n+1)i)\log(n+1)-2ny} \left(e^{2|f_n|} + |\xi+ni|^2 e^{-2}\right) \\
&+ 2\sum_{n=\bar{n}+1}^{\infty} e^{-f((n+1)i)\log(n+1)-2ny} \left(e^{2Cf((n+1)i)} + |\xi+ni|^2 e^{-2}\right)
\end{aligned}$$

(by (3.38))

$$< \infty.$$

Therefore $A\Phi_0 \in X$ and so $\Phi_0 \in \mathcal{D}(A)$. Let us now fix $t > 0$. By (3.24) we have

$$||\Phi_t||^2 = \sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)} |\phi_n(t)|^2 \leq \sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)} e^{2tf(ni)}$$

(applying first (3.39), and then (3.38))

$$\leq \sum_{n=0}^{\bar{n}} e^{-f((n+1)i)\log(n+1)} e^{2tf(ni)} + \sum_{n=\bar{n}+1}^{\infty} e^{f((n+1)i)(2t-\log(n+1))} < \infty,$$

and so $\Phi_t \in X$. On the other hand, by (3.36),

$$\begin{aligned}||A\Phi_t||^2 &= \sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)} \left|f_n\phi_n(t) + i(\xi+ni)\phi_{n+1}(t)\right|^2 \\ &\le 2\sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)} \left(|f_n|^2|\phi_n(t)|^2 + |\xi+ni|^2|\phi_{n+1}(t)|^2\right) \\ &\le 2\sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)} \left(e^{2|f_n|}|\phi_n(t)|^2 + |\xi+ni|^2|\phi_{n+1}(t)|^2\right)\end{aligned}$$

(by (3.24))

$$\le 2\sum_{n=0}^{\infty} e^{-f((n+1)i)\log(n+1)} \left(e^{2|f_n|}e^{2tf(ni)} + |\xi+ni|^2 e^{2tf((n+1)i)}\right).$$

Now, applying first (3.40), and then (3.39), there exists $\bar{n} > 0$ such that

$$\begin{aligned}||A\Phi_t|| \le\ & 2\sum_{n=0}^{\bar{n}} e^{-f((n+1)i)\log(n+1)} \left(e^{2|f_n|}e^{2tf(ni)} + |\xi+ni|^2 e^{2tf((n+1)i)}\right) \\ & + 2\sum_{n=\bar{n}+1}^{\infty} e^{-f((n+1)i)\log(n+1)} \left(e^{2C(1+t)f((n+1)i)} + |\xi+ni|^2 e^{2tf((n+1)i)}\right)\end{aligned}$$

(by (3.38))

$$< \infty.$$

Thus, $A\Phi_t \in X$ too and so $\Phi_t \in \mathcal{D}(A)$. An iterative application of Lemma 3.4 leads to

$$\begin{cases} \frac{d}{dt}\phi_n(t) = f_n\phi_n(t) + i(\xi+ni)\phi_{n+1}(t), & t > 0, \\ \phi_n(0) = e^{i(\xi+ni)y}, \end{cases}$$

for any $n \in \mathbb{N}_0$, which is equivalent to (3.35) by (3.36). □

Following, we give three propositions which allow us to represent the initial data Φ_0 as a countable linear combination of eigenvectors of A.

Proposition 3.8. *For any $n \in \mathbb{N}_0$, f_n belongs to the discrete spectrum of A, and the corresponding eigenvectors are all the elements $u \in X$ such that*

$$u(m) = \begin{cases} u(0)\prod_{j=0}^{m-1} \frac{a_{n,n}-a_{j,j}}{a_{j,j+1}} & \text{if } m \le n, \\ 0 & \text{otherwise,} \end{cases} \tag{3.43}$$

with $(a_{j,j}, a_{j,j+1})_{j\in\mathbb{N}}$ as in (3.37).

Proof. Let us fix $n \in \mathbb{N}_0$ and consider f_n. Then, for any $u : \mathbb{N}_0 \to \mathbb{C}$,

$$(f_nI - A)u = 0 \iff (f_n - a_{m,m})u(m) - a_{m,m+1}u(m+1) = 0 \quad \forall m \in \mathbb{N}_0$$

$$\iff u(m+1) = \frac{(f_n - a_{m,m})u(m)}{a_{m,m+1}} \quad \forall m \in \mathbb{N}_0$$
$$\iff u(m) = u(0) \prod_{j=0}^{m-1} \frac{f_n - a_{j,j}}{a_{j,j+1}} \quad \forall m \in \mathbb{N}_0, \tag{3.44}$$

which is equivalent to (3.43), as (3.44) also implies $u(m) = 0$ for $m \geq n+1$. Such a u belongs to X because it has only n non null components. □

For any $n \in \mathbb{N}_0$, let us denote by v_n the eigenvector of f_n given by

$$v_n(m) = \begin{cases} \prod_{j=0}^{m-1} \frac{a_{n,n}-a_{j,j}}{a_{j,j+1}}, & \text{if } m \leq n, \\ 0. & \text{otherwise} \end{cases} \tag{3.45}$$

We explicitly observe that v_n is the only eigenvector of f_n such that $v_n(0) = 1$.

Corollary 3.9. *For any $n \in \mathbb{N}_0$, $\Psi_t^n := e^{f_n t} v_n$ satisfies*

$$\begin{cases} \frac{d}{dt}\Psi_t^n = A\Psi_t^n, & t > 0 \\ \Psi_0^n = v_n \end{cases}.$$

The next proposition shows explicitly how any element e_n of the canonical basis $(e_n)_{n\in\mathbb{N}_0}$ can be written as a linear combination of $v_0, \dots, v_n$.

Proposition 3.10. *For any $m \in \mathbb{N}_0$ we have*

$$e_m = \sum_{n=0}^{m} b_{m,n} v_n,$$

where

$$\begin{aligned} b_{m,n} &= \left(\prod_{l=0}^{n-1} \frac{a_{l,l+1}}{a_{n,n} - a_{l,l}}\right)\left(\prod_{l=n}^{m-1} \frac{a_{l,l+1}}{a_{n,n} - a_{l+1,l+1}}\right) \\ &= \left(\prod_{l=0}^{n-1} \frac{i(\xi + li)}{f_n - f_l}\right)\left(\prod_{l=n}^{m-1} \frac{i(\xi + li)}{f_n - f_{l+1}}\right), \quad 0 \leq n \leq m. \end{aligned} \tag{3.46}$$

In order to prove the last assertion, we first need to state the following algebraic property.

Lemma 3.11. *Let $(z_n)_{n\in\mathbb{N}_0}$ be a sequence in $\mathbb{C}$ such that $z_n \neq z_m$ for any $n \neq m$. Then, we have*

$$\sum_{n=k}^{m} \prod_{\substack{l=k \\ l\neq n}}^{m} \frac{1}{z_n - z_l} = 0, \quad \forall\, m > k \geq 0. \tag{3.47}$$

The proof of Lemma 3.11 is postponed to the Appendix.

Proof. (Proposition 3.10) We fix $m \geq 0$ and we set

$$u = \sum_{n=0}^{m} b_{m,n} v_n.$$

By (3.45)-(3.46) we easily get

$$u(k) = \begin{cases} 0, & k > m \\ \left(\prod_{l=0}^{m-1} \frac{a_{l,l+1}}{a_{m,m}-a_{l,l}}\right)\left(\prod_{l=0}^{m-1} \frac{a_{m,m}-a_{l,l}}{a_{l,l+1}}\right) = 1, & k = m. \end{cases}$$

Now, when $k < m$, again by (3.45)-(3.46) we obtain

$$\begin{aligned} u(k) &= \sum_{n=k}^{m} \left(\prod_{l=0}^{n-1} \frac{a_{l,l+1}}{a_{n,n} - a_{l,l}} \right) \left(\prod_{l=n}^{m-1} \frac{a_{l,l+1}}{a_{n,n} - a_{l+1,l+1}} \right) \left(\prod_{l=0}^{k-1} \frac{a_{n,n} - a_{l,l}}{a_{l,l+1}} \right) \\ &= \left(\prod_{l=k}^{m-1} a_{l,l+1} \right) \sum_{n=k}^{m} \left(\prod_{\substack{l=k \\ l \neq n}}^{m} \frac{1}{a_{n,n} - a_{l,l}} \right) \end{aligned}$$

(by Lemma 3.11)

$$= 0,$$

and this concludes the proof. ☐

The next proposition is crucial to write the initial data Φ_0 as a linear combination of the eigenvectors $\{v_n\}_{n \in \mathbb{N}_0}$.

Proposition 3.12. *We have*

$$\sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \left\| e^{i(\xi+im)y} b_{m,n} v_n \right\| < \infty. \tag{3.48}$$

In order to prove this proposition we use the following lemmas, whose proofs are postponed to the Appendix.

Lemma 3.13. *For any $m \in \mathbb{N}_0$,*

$$||b_{m,n} v_n|| \leq g(n) \left(\prod_{l=0}^{m-1} |\xi + li| \right) \left(\prod_{l=n+1}^{m} \frac{1}{|f_n - f_l|} \right), \tag{3.49}$$

with $g(n) = O(n^{1/2} K^n e^{-p\bar{z}n^2})$ as $n \to \infty$, for some $K > 0$ and for $\bar{z} > 0$ as in Lemma 3.6.

Lemma 3.14. *For any fixed $C > 0$ we have*

$$\sum_{j=0}^{\infty} C^j \, (j+n-1)! e^{-p\bar{z}j(j+1)/2} = O(n! 2^n), \quad \text{as } n \to \infty,$$

with $\bar{z} > 0$ as in Lemma 3.6.

Proof. (Proposition 3.12) We have

$$\sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \left\| e^{i(\xi+im)y} b_{m,n} v_n \right\| = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} e^{-my} \, ||b_{m,n} v_n||$$

(by (3.49))

$$\leq \sum_{n=0}^{\infty} g(n) \sum_{m=n}^{\infty} e^{-my} \left(\prod_{l=0}^{m-1} |\xi + li| \right) \left(\prod_{l=n+1}^{m} \frac{1}{|f_l - f_n|} \right)$$

(by rescaling the index $m = n + j$)

$$= \sum_{n=0}^{\infty} g(n) e^{-ny} \sum_{j=0}^{\infty} e^{-jy} \left(\prod_{l=0}^{j+n-1} |\xi + li| \right) \left(\prod_{l=1}^{j} \frac{1}{|f_{l+n} - f_n|} \right)$$

(for some constant $C > 0$)

$$\leq \sum_{n=0}^{\infty} g(n) \left(e^{-y}C\right)^n \sum_{j=0}^{\infty} \left(e^{-y}C\right)^j (j+n-1)! \prod_{l=1}^{j} \frac{1}{||f_{l+n}| - |f_n||}$$

(by (3.42))

$$\leq \sum_{n=0}^{\infty} g(n) \left(e^{-y}C\right)^n \sum_{j=0}^{\infty} \left(e^{-y}C\right)^j (j+n-1)! \prod_{l=1}^{j} \frac{1}{||f_l| - |f_0||}$$

(by (3.41))

$$\leq \sum_{n=0}^{\infty} g(n) \left(e^{-y}C\right)^n \sum_{j=0}^{\infty} \left(e^{-y}C\right)^j (j+n-1)! \prod_{l=1}^{j} C' e^{-p\bar{z}l}$$

$$= \sum_{n=0}^{\infty} g(n) \left(e^{-y}C\right)^n \sum_{j=0}^{\infty} \left(e^{-y}C'C\right)^j (j+n-1)! e^{-p\bar{z}j(j+1)/2}.$$

Hence, by Lemmas 3.13 and 3.14 we have

$$g(n) \sum_{j=0}^{\infty} \left(e^{-y}C'C\right)^j (j+n-1)! e^{-p\bar{z}j(j+1)/2} = O\left(n^{1/2}(2K)^n e^{-p\bar{z}n^2} n!\right),$$

as $n \to \infty$, and this yields (3.48). □

Corollary 3.15. *We have*

$$\Phi_0 = \sum_{n=0}^{\infty} x_n v_n,$$

with $(x_n)_{n\in\mathbb{N}_0}$ *defined as in* (3.32).

Proof. By Propositions 3.10 we have

$$\Phi_0 = \sum_{m=0}^{\infty} e^{i(\xi+im)y} e_m = \sum_{m=0}^{\infty} e^{i(\xi+im)y} \sum_{n=0}^{m} b_{m,n} v_n$$

(by Proposition 3.12 the above summations are unconditionally convergent)

$$= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} e^{i(\xi+im)y} b_{m,n} v_n = \sum_{n=0}^{\infty} x_n v_n.$$

□

Now, combining Corollaries 3.9 and 3.15, the solution of the Cauchy problem (3.35) can be represented as

$$\Phi_t = \sum_{n=0}^{\infty} x_n e^{f_n t} v_n.$$

Hence, recalling that, by (3.45), $v_n(0) = 1$ for any $n \in \mathbb{N}_0$, we obtain the above representation (3.30)-(3.32) for $\Phi_t(0) = \phi_0(t) = \phi(t, \xi)$.

2.4 Numerical analysis

We present a numerical study to assess accuracy and efficiency of the proposed methodology. We consider dynamics for the underlying stock which can be either diffusive or jump diffusive. For the latter, we consider two widely used models in finance, namely the compound Poisson and the variance gamma (VG) (see also [172]). This provides a comprehensive case study, since the methodology is analyzed both on finite and infinite activity jump processes. For the compound Poisson process, we follow [176], and assume that jumps occur at a constant rate λ, with Gaussian magnitude. The corresponding Lévy measure is

$$\nu(dz) = \lambda\phi(dz; \mu, \theta),$$

where ϕ denotes the univariate Gaussian distribution with mean μ and standard deviation θ. For the VG process, it is well known that it may be obtained as a drifted Brownian motion with drift κ and volatility δ, but evaluated at a random time specified by a Gamma process with mean 0 and variance v. We refer to [61, Section 4.4.3], for more details. The associated Lévy measure is

$$\nu(dz) = \frac{1}{v}\left(\frac{e^{-\lambda_1 z}}{z}\mathbb{1}_{(0,\infty)}(z) - \frac{e^{\lambda_2 z}}{z}\mathbb{1}_{(-\infty,0)}(z)\right) dz$$

where

$$\lambda_1 = \left(\sqrt{\frac{\kappa^2 v^2}{4} + \frac{\delta^2 v}{2}} + \frac{\kappa v}{2}\right)^{-1}, \qquad \lambda_2 = \left(\sqrt{\frac{\kappa^2 v^2}{4} + \frac{\delta^2 v}{2}} - \frac{\kappa v}{2}\right)^{-1}.$$

Note that the Lévy measure of a VG process satisfies (3.21) only for $\alpha \in (\lambda_2, \lambda_1)$, and thus Assumption B is not satisfied. Although the expansion (3.30)-(3.32) does not directly apply in this case, it does so if we consider the limiting representation given in Eq. (3.22), i.e. choose I sufficiently high. In our numerical simulations, we set the limiting parameter I to 50. We have experimented with higher values of I, and noticed that the characteristic function does not exhibit significant differences.

Similarly to [156], we parameterize the hazard rate as $h(s) = h^* \left(\frac{s^*}{s}\right)^p$, where s^* is the reference stock level and h^* the scale parameter. This translates into setting $\beta = h^*(s^*)^p$ in (3.15). Throughout the section, we consider zero recovery, i.e. $R = 0$. We present a comparison versus Monte-Carlo Euler scheme, to simulate the stock price process, consisting in 100,000 Monte Carlo runs on a time grid of 100 equally spaced time points.

2.4.1 Characteristic function

We analyze the sensitivity to truncation errors of the eigenfunction expansion. The expressions in (3.30) and (3.32) show that the characteristic function contains two infinite sums. For practical purposes, a truncation is needed. As we demonstrate next, our formula is able to recover an accurate estimate using only few terms in the sum. In particular, we use the following truncated series:

$$\phi(t,\xi) \approx \phi^{N,M}(t,\xi) := \sum_{n=0}^{N} \widetilde{x}_n^M e^{f_n t}, \qquad \widetilde{x}_n^M = \sum_{m=n}^{n+M} e^{i(\xi+im)y} b_{m,n}. \tag{4.50}$$

Figure 2.1 tests the stability of the approximation in the purely diffusive case. In particular, it plots the real part of $\phi_{N,N}$ as a function of N, and shows how the truncated sums in (4.50) converge after few terms.

Next, we evaluate the accuracy of our asymptotic representation of the characteristic function for a wide range of ξ values, and present comparisons with the Monte-Carlo method. We set $N = M = 7$ for this analysis. Tables 2.1, 2.2 and 2.3 show that the real part of the characteristic function always falls within the 99% Monte-Carlo confidence band when considering zero jumps, jumps with Gaussian magnitude, and VG-type jumps respectively.

Figures 2.2, 2.3, and 2.4 plot the difference between our truncated expansion and the corresponding Monte-Carlo estimate for $\phi(t,\xi)$, under the three usual choices for the Lévy measure. As it appears from the figures, $\phi^{N,M}(t,\xi)$ is in full agreement with the MC bands. Note that the width of the confidence interval, acting as an upper bound for the error, is always smaller that 0.002.

2.4.2 Fourier pricing formulas

We compute the expectation in (3.18) using a standard Fourier integral formula (cf. [121], [153] and [159]). For convenience, we briefly review the general pricing formula, which is expressed in terms of the characteristic function of the underlying log-price process. Following a classical result reported in several textbooks (see for instance [188, Theorem 15.6]), the expectation in (3.18) becomes

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\bar{\varphi}(Y_t,R)\right] = \frac{1}{\pi}\int_0^\infty \phi(t,-(\xi+i\alpha))\hat{\bar{\varphi}}(\xi+i\alpha,R)d\xi, \tag{4.51}$$

where $\phi(t,\cdot)$ is the characteristic function of the process Y_t in (3.17) starting from $Y_0 = y_0$, $\hat{\bar{\varphi}}(\cdot,R)$ is the Fourier transform of $\bar{\varphi}(\cdot,R)$ in (3.19), and $\alpha \in \mathbb{R}$ is a constant such that:

a) the damped payoff function

$$\bar{\varphi}_\alpha(Y,R) := e^{-\alpha Y}\bar{\varphi}(Y,R), \quad Y \in \mathbb{R},$$

is in $L^1(\mathbb{R})$, as well as its Fourier transform $\hat{\bar{\varphi}}_\alpha(\cdot,R)$;

b) $\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\alpha Y_t}\right]$ is finite.

According to Remark 2.1, setting $\varphi(s) = (s-K)^+$ or $\varphi(s) = 1$ in (3.19), we can respectively price a call option or a bond. In the first case, it is enough to set $\alpha > 0$, compute $\hat{\bar{\varphi}}_\alpha(\cdot,R)$, and plug it into (4.51) to get

$$C_{\text{call},R}(t,x) = e^{-rt}R + \frac{e^{-qt+x}}{\pi}\int_0^\infty \frac{\phi(t,-(\xi+i\alpha))\left(\frac{K^p}{p\beta}\right)^{-\alpha+i\xi}}{(1+p(\alpha-i\xi))(\alpha-i\xi)}d\xi. \tag{4.52}$$

In the second case, $\bar{\varphi}_\alpha(\cdot, R)$ does not belong to $L^1(\mathbb{R})$ for any $\alpha \in \mathbb{R}$. Thus, we first need to perturb it as

$$\bar{\varphi}_C(\cdot, R) = (1-R)\left(p\beta e^Y\right)^{-\frac{1}{p}}\left(1-e^{-Y}C\right)\mathbb{1}_{[\log C,\infty)}(Y),$$

and after computing $\hat{\bar{\varphi}}_C(\cdot, R)$, by (4.51) we obtain

$$\begin{aligned} C_{\text{bond},R}(t,x) &= e^{-rt}R + e^{-qt+x}\lim_{C\to 0}\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\bar{\varphi}_C(Y_t,R)\right] \\ &= e^{-rt}R + \frac{e^{-qt+x}(1-R)(p\beta)^{-\frac{1}{p}}}{\pi}\lim_{C\to 0}C^{-\frac{1}{p}}\int_0^\infty \frac{\phi(t,-\xi)C^{i\xi}}{\left(\frac{1}{p}-i\xi\right)\left(1+\frac{1}{p}-i\xi\right)}d\xi. \end{aligned} \tag{4.53}$$

In the following numerical analysis, the pricing integrals appearing in (4.52) and (4.53) are computed numerically. For the bond price, in order to approximate the limit in (4.53), we have set the perturbation parameter C reasonably close to 0, i.e. $C = 0.01$. We have not experienced significant changes in the prices if the value is lowered.

Eventually, formulas (4.52) and (4.53) can be combined with the expansion (3.30)-(3.32)-(3.46) to efficiently compute call option and bond prices.

Remark 4.1. *We might observe that the pricing formula* (4.52) *requires the computation of the function* $\phi(t,\cdot)$ *on the complex argument* $-(\xi + i\alpha)$ *for any* $\xi \in \mathbb{R}$, *whereas the representation* (3.30)-(3.32)-(3.46) *is carried out throughout Section 2.3.1 and Section 2.3.2 only for* $\xi \in \mathbb{R}$. *Actually the whole analysis from Lemma 3.4 on can be repeated for any* $\xi \in \mathbb{C}$ *under the additional integrating condition*

$$\int_{|z|>1} e^{(1+\text{Im}(\xi))z}\nu(dz) < \infty,$$

which ensures the integral in (3.27) *is finite, leading to the same representation series for* $\phi(t,\xi)$.

Note that several other efficient Fourier inversion methods based on the knowledge of the characteristic function, can be used. Those include the standard fractional FFT algorithm, or the recently developed COS method (see [88]) which only requires evaluating the characteristic function on real arguments.

2.4.3 Option implied volatilities

We imply the volatility curves from call options priced using (4.52), and compare them against the corresponding curves obtained when Monte-Carlo price estimates are used. In order to offer a comparison with [156], we adopt choices for the diffusive parameters, similar to the ones used in his paper. In particular, throughout the section we choose

$$r = q = 0.03,\ \sigma = 0.3,\ p = 2,\ s = s^* = 50,\ h^* = 0.06.$$

for the pure diffusion model, and we set

$$\mu = -1,\ \lambda = 0.1,\ \theta = 0.2; \qquad v = 0.3,\ \delta = 0.2,\ \kappa = -1.$$

for the compound Poisson and variance gamma model.

The top graph of Figure 2.5 is in full agreement with [156], see Figure 4.6 therein. This is expected given that, in the diffusive case, both methods evaluate the price of an option on a stock with the same dynamics and credit risk structure. However, differently from [156], our framework can naturally handle jumps in the stock dynamics, thus producing much richer option implied volatility surfaces. This is clearly evidenced in the middle and bottom graph of Figure 2.5, where the presence of jumps with negative mean generates higher implied volatilities. In the specific example considered, we notice that the largest values for the option implied volatility are produced using the variance gamma model.

We also measure the accuracy and efficiency of the method in pricing options. For reasonably low values of N and M in (4.50), namely $N = 4$ and $M = 5$, we can see from Table 2.4 that the resulting prices always fall within the 99% Monte-Carlo confidence band. Hence, the method provides accurate prices, without sacrificing computational speed, and performs equally well under diffusive and jump-diffusive dynamics of the underlying.

The following tables report, for different maturities and levels of moneyness, the minimum number of terms needed to obtain an option price within the 99% Monte-Carlo confidence band. We find that when the expiration is small and the option is highly in or out of the money, a slightly higher number of terms are needed with respect to higher expirations and at-the-money levels. However, our analysis indicates that the number of additional terms is negligible (at most 3), suggesting that the computational performance of the method does not deteriorate when dealing with near expiration and not at-the-money options.

2.4.4 Bond prices

We perform a similar analysis when defaultable bonds are priced via Eq. (4.53). We start analyzing how the bond price (survival probability) behaves with respect to the initial stock price s. As expected, larger values of s result in larger bond prices, as they result in decreasing default probability. As it appears from Figure 2.6, when the initial stock price is low, larger values of p result in higher default probabilities (s^{-p} is increasing in p when s is small), hence lower bond prices. The presence of negative jumps exacerbates this effect. As we can see from the lower panels of Figure 2.6, when $p = 1.75$ and $p = 2$, the impact of the jumps is significant when the price of the underlying stock is low, and noticeably amplifies the default probability relatively to the purely diffusive case.

When s is high, larger values of p are associated with smaller default probabilities (s^{-p} is decreasing in p when s is large). Hence, the default probability will decrease with p, and consequently the bond price will be higher. When negative jumps are allowed, bond prices increase slower because the presence of downward jumps in the stock dynamics increases the default intensity. Under the chosen parameter configuration, low values of p do not result in major differences between the bond prices across the different models, suggesting that default probabilities are not significantly altered by the possibility of negative jumps when p is low.

Table 2.8 reports the bond prices for different expirations. We set the truncation parameters to $N = 7$ and $M = 6$. We notice that bond prices are smaller if jump-diffusion models are used for the underlying. This happens because the mean values of the jumps of the compound Poisson and variance gamma processes are negative. Despite the infinite activity of the jumps in the variance gamma model, the smallest prices are obtained under the compound Poisson model, due to the larger expected size of the jumps ($\mu = -1 > \kappa = -0.6$) and the high jump intensity ($\lambda = 1.5$). We also notice that the impact of the jumps on bond prices becomes noticeable as the time to maturity increases, and ceteris paribus, can decrease the bond price by a maximum 13% (from 0.725 in the purely diffusive case to 0.625 in the compound Poisson case).

Table 2.9 reports the bond prices under different values of p using the same truncation order. Under each model, bond prices appear to decrease in p. Additionally, when negative jumps are allowed, bond prices tend to be smaller, given the increasing hazard rate intensity.

As done earlier for options, we also perform an accuracy test for the minimum number of terms needed to obtain a bond price within the 99% Monte-Carlo confidence band. Our analysis confirms that, as for options, short maturity bonds require only a slightly higher number of terms to be accurately approximated, hence reinforcing our conclusions about the robustness of the proposed method.

2.5 Conclusions

We have introduced a novel methodology to recover the characteristic function of a defaultable stock in a Lévy driven model. The latter is obtained considering a stock dynamics which follows a geometric Lévy process before default and jumps to zero at default. The conditional mean arrival rate of default behaves as a negative power of the stock price. Our technique exploits the structure of the stochastic differential equation of (a linear rescaling of) the log-return, and recovers its characteristic function as the solution of an infinite dimensional linear system of complex-valued ODE's. After introducing a suitable Banach space endowed a weighted l_2 norm, we provided fully explicit expressions for the eigenvalues and the eigenvectors of the infinite dimensional differential operator. This yields an explicit eigenfunction expansion of the characteristic function.

By means of a numerical analysis, we demonstrated that our asymptotic representation is very accurate even when only few terms contribute to the expansion. We have used the proposed representation to price vulnerable options and defaultable bonds. Our study indicates that prices are accurate and always fall within the 99% confidence band estimated via Monte Carlo simulation.

Our findings indicate that the methodology is highly valuable for pricing and hedging of vulnerable claims, as well as for risk management and model calibration, due to its computation efficiency and accuracy.

The proposed methodology is general enough to accommodate exogenous stochastic volatility for the underlying stock. In a future continuation of the work, we plan to explore this avenue, so to consider a hazard rate depending on both the underlying stock value and its volatility. Furthermore, we would like to analyze the connection between the process V_t in (1.1) and the time integral of an exponential Lévy process. This would allow pricing arithmetic Asian options on a Lévy driven underlying, using the eigenfunction expansion approach developed in this paper.

2.6 Appendix

2.6.1 Proofs of Lemmas 3.6, 3.11, 3.13 and 3.14

Proof of Lemma 3.6. We only prove (3.38) and (3.39). The relations (3.40), (3.41) and (3.42) can be proved analogously by using the definition of f_n in (3.31).
We first prove (3.38). By (3.26)-(3.27) we have

$$f(ni) = -p\left(r - q + \frac{\sigma^2}{2}\right)n + \frac{1}{2}p^2\sigma^2 n^2 + \psi(ni), \tag{6.54}$$

with

$$\psi(ni) = \int_{\mathbb{R}} \left(-np\left(1 - e^z\right) + e^z\left(e^{-npz} - 1\right)\right)\nu(dz). \tag{6.55}$$

Clearly, the first part grows as n^2 as n goes to ∞. Now, for n large enough we have

$$\frac{d}{dz}\left(-np\left(1-e^z\right) + e^z\left(e^{-npz} - 1\right)\right) = (np-1)e^z\left(1 - e^{-npz}\right) \geq 0 \Longleftrightarrow z \geq 0,$$

and thus

$$-np\left(1-e^z\right) + e^z\left(e^{-npz} - 1\right),$$

has a minimum at $z = 0$. Therefore,

$$\psi(ni) \geq \left(-np\left(1-e^{-\bar{z}}\right) + e^{-\bar{z}}\left(e^{np\bar{z}} - 1\right)\right)\nu((-\infty, -\bar{z}])$$

for n large enough, and this prove (3.38). We now prove (3.39). By (6.54)-(6.55) we have

$$f((n+1)i) - f(ni) = -p\left(r - q + \frac{\sigma^2}{2}\right) + \frac{1}{2}p^2\sigma^2(2n+1) + \psi((n+1)i) - \psi(ni),$$

with

$$\psi((n+1)i) - \psi(ni) = \int_{\mathbb{R}} \left(-p\left(1-e^z\right) + e^{z(1-np)}\left(e^{-pz} - 1\right)\right)\nu(dz).$$

Also in this case, for n large enough one can see that

$$\frac{d}{dz}\left(-p\left(1-e^z\right) + e^{z(1-np)}\left(e^{-pz} - 1\right)\right) \geq 0 \Longleftrightarrow z \geq 0,$$

and thus

$$-p\left(1-e^z\right) + e^{z(1-np)}\left(e^{-pz} - 1\right) \geq 0$$

has a minimum in $y = 0$. Again,

$$\psi((n+1)i) - \psi(ni) \geq \left(-p\left(1-e^{-\bar{z}}\right) + e^{-\bar{z}(1-np)}\left(e^{p\bar{z}} - 1\right)\right)\nu((-\infty, -\bar{z}])$$

for any n suitably large. □

Proof of Lemma 3.11. Clearly, apart from to rescaling the summation indices, we can assume without loss of generality that $k = 0$. Let us fix $m > 0$ and note that (3.47) is true if and only if

$$p(z_m) = 0, \tag{6.56}$$

where

$$p(y) = 1 - \sum_{n=0}^{m-1} \prod_{\substack{l=0 \\ l\neq n}}^{m-1} \frac{y - z_l}{z_n - z_l}.$$

Indeed,

$$\begin{aligned} p(z_m) &= 1 - \sum_{n=0}^{m-1} \prod_{\substack{l=0 \\ l\neq n}}^{m-1} \frac{z_m - z_l}{z_n - z_l} = \left(\prod_{l=0}^{m-1} \frac{z_m - z_l}{z_m - z_l} \right) - \sum_{n=0}^{m-1} \prod_{\substack{l=0 \\ l\neq n}}^{m-1} \frac{z_m - z_l}{z_n - z_l} \\ &= \left(\prod_{l=0}^{m-1} \frac{z_m - z_l}{z_m - z_l} \right) + \sum_{n=0}^{m-1} \left(\frac{z_m - z_n}{z_n - z_m} \prod_{\substack{l=0 \\ l\neq n}}^{m-1} \frac{z_m - z_l}{z_n - z_l} \right) \\ &= \sum_{n=0}^{m} \left(\prod_{l=0}^{m-1} (z_m - z_l) \prod_{\substack{l=0 \\ l\neq n}}^{m} \frac{1}{z_n - z_l} \right) = \left(\prod_{l=0}^{m-1} (z_m - z_l) \right) \sum_{n=0}^{m} \prod_{\substack{l=0 \\ l\neq n}}^{m} \frac{1}{z_n - z_l}. \end{aligned}$$

Now we prove that $p(y) \equiv 0$, which implies $p(z_m) = 0$. Indeed, for any z_i, $0 \leq i \leq m-1$, we have

$$\prod_{\substack{l=0 \\ l\neq n}}^{m-1} \frac{z_i - z_l}{z_n - z_l} = \begin{cases} 1, & n = i \\ 0, & n \neq i. \end{cases}$$

Therefore, $p(z_i) = 0$, thus $z_0, \cdots, z_{m-1}$ are m distinct roots of p, which is a polynomial of degree at most $m-1$. Thus $p \equiv 0$ and this proves (6.56). ◻

Proof of Lemma 3.13. By definitions (3.45) and (3.46) we have

$$b_{m,n} v_n(j) = \begin{cases} \left(\prod_{l=j}^{n-1} \frac{i(\xi+li)}{f_n - f_l} \right) \left(\prod_{l=n}^{m-1} \frac{i(\xi+li)}{f_n - f_{l+1}} \right), & j \leq n \\ 0, & \text{otherwise} \end{cases}$$

for any $m \geq n \geq 0$. Therefore

$$\begin{aligned} \|b_{m,n} v_n\|^2 &= \sum_{j=0}^{n} e^{-f((j+1)i)\log(j+1)} |b_{m,n} v_n(j)|^2 \\ &= \left(\prod_{l=n}^{m-1} \frac{|\xi + li|^2}{|f_n - f_{l+1}|^2} \right) \sum_{j=0}^{n} e^{-f((j+1)i)\log(j+1)} \prod_{l=j}^{n-1} \frac{|\xi + li|^2}{|f_n - f_l|^2} \end{aligned}$$

(by the relations in Lemma 3.6, for a suitable constant $C > 0$)

$$\begin{aligned} &\leq \left(\prod_{l=n}^{m-1} \frac{|\xi + li|^2}{|f_n - f_{l+1}|^2} \right) nC^n \prod_{l=0}^{n-1} \frac{|\xi + li|^2}{|f_n - f_l|^2} \\ &= \left(\prod_{l=0}^{m-1} |\xi + li|^2 \right) \left(\prod_{l=n+1}^{m} \frac{1}{|f_n - f_l|^2} \right) nC^n \prod_{l=0}^{n-1} \frac{1}{|f_n - f_l|^2}. \end{aligned}$$

Now, setting

$$g(n) := n^{1/2} C^{n/2} \prod_{l=0}^{n-1} \frac{1}{|f_n - f_l|},$$

we get

$$\|b_{m,n} v_n\| \le g(n) \left(\prod_{l=0}^{m-1} |\xi + li| \right) \left(\prod_{l=n+1}^{m} \frac{1}{|f_n - f_l|} \right)$$

for any $m \ge n \ge 0$. Finally, by (3.42) we obtain

$$\prod_{l=0}^{n-1} \frac{1}{|f_n - f_l|} \le \prod_{l=0}^{n-1} \frac{1}{||f_n| - |f_l||} \le \prod_{l=0}^{n-1} \frac{1}{||f_n| - |f_{n-1}||} \le \prod_{l=0}^{n-1} e^{-p\bar{z}(n-1)}$$
$$= e^{-p\bar{z}(n^2+n)},$$

for any n greater than a suitable $\bar{n} \in \mathbb{N}_0$, and than the desired result. ☐

Proof of Lemma 3.14. From the well-known identity $\sum_{k=0}^{m} \binom{m}{k} p^k (1-p)^{m-k} = 1$, one has

$$\binom{m}{k} p^k (1-p)^{m-k} \le 1, \quad \forall k \le m \in \mathbb{N}_0, \ \forall p \in (0,1).$$

By letting $p = 1/2$, one has $\binom{m}{k} \le 2^m$, which means $m! \le 2^m k!(m-k)!$. By using this fact, we have

$$\begin{aligned} \sum_{j=0}^{\infty} C^j \, (j+n-1)! e^{-p\bar{z}j(j+1)/2} &\le \sum_{j=0}^{\infty} C^j 2^{j+n-1} n!(j-1)! e^{-p\bar{z}j(j+1)/2} \\ &= 2^{n-1} n! \sum_{j=0}^{\infty} (2C)^j (j-1)! e^{-p\bar{z}j(j+1)/2}. \end{aligned}$$

The latter summation converges and does not depend on n, hence the desired result. ☐

2.7 Figures and tables

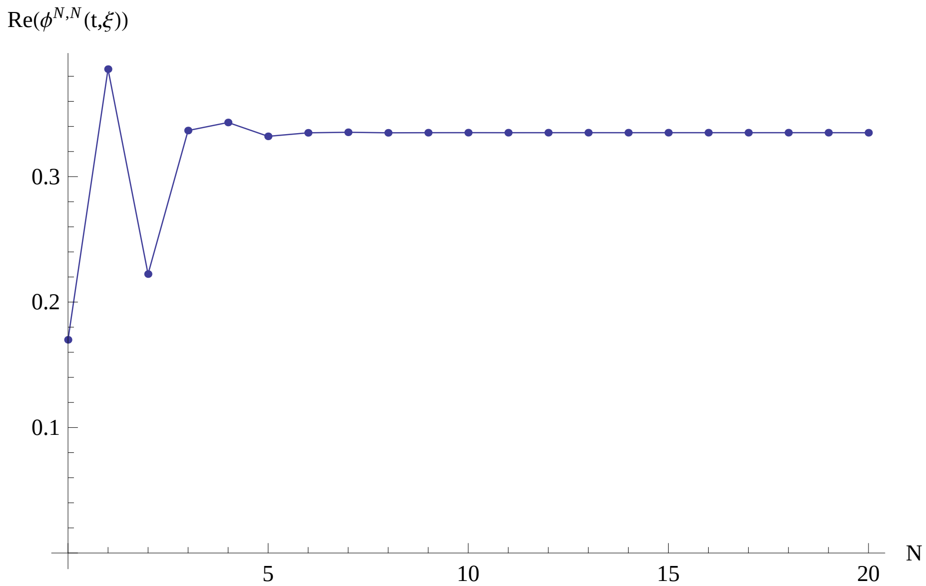

Figure 2.1: Real part of $\phi^{N,N}(t, \xi)$ as a function of N in the diffusive case. The parameters are $\sigma = 0.5$, $p = 0.5$, $t = 1$, $s = s^* = 30$, $h^* = 0.03$, $r = 0.05$, $q = 0.05$, and $\xi = 0.2$.

ξ	$Re(\phi(t, \xi))$	$Im(\phi(t, \xi))$
0.5	-0.256 (-0.256,-0.255)	0.926 (0.926,0.926)
1	-0.732 (-0.733,-0.732)	-0.438 (-0.439,-0.437)
1.5	0.507 (0.504,0.507)	-0.484 (-0.486,-0.484)
2	0.251 (0.249,0.252)	0.468 (0.467,0.470)
2.5	-0.363 (-0.364,-0.361)	0.083 (0.081,0.085)
3	0.010 (0.007,0.011)	-0.241 (-0.243,-0.240)
3.5	0.137 (0.136,0.139)	0.044 (0.041,0.045)
4	-0.043 (-0.044,-0.041)	0.067 (0.065,0.069)

Table 2.1: Characteristic function of the terminal log-price process, under diffusion dynamics. The parameters are $\sigma = 0.8$, $p = 1$, $t = 0.5$, $s = s^* = 50$, $h^* = 0.03$, $r = q = 0.03$; the truncation parameters in (4.50) are $N = M = 7$. The values in parenthesis represent the 99% Monte-Carlo confidence interval.

ξ	$Re(\phi(t,\xi))$	$Im(\phi(t,\xi))$
0.5	-0.261 (-0.262,-0.260)	0.922 (0.922,0.923)
1	-0.720 (-0.721,-0.719)	-0.445 (-0.447,-0.444)
1.5	0.510 (0.509,0.512)	-0.462 (-0.463,-0.461)
2	0.226 (0.225,0.228)	0.465 (0.464,0.467)
2.5	-0.354 (-0.355,-0.352)	0.059 (0.056,0.060)
3	0.028 (0.026,0.030)	-0.228 (-0.231,-0.228)
3.5	0.125 (0.124,0.128)	0.055 (0.053,0.056)
4	-0.049 (-0.051,-0.048)	0.057 (0.055,0.059)

Table 2.2: Characteristic function of the terminal log-price process, assuming Gaussian jumps. The parameters are $\sigma = 0.8$, $\lambda = 0.15$, $\mu = -0.7$, $\delta = 0.3$, $p = 1$, $t = 0.5$, $s = s^* = 50$, $h^* = 0.03$, $r = q = 0.03$; the truncation parameters in (4.50) are $N = M = 7$. The values in parenthesis represent the 99% Monte-Carlo confidence interval.

ξ	$Re(\phi(t,\xi))$	$Im(\phi(t,\xi))$
0.5	-0.356 (-0.356,-0.355)	0.856 (0.856,0.856)
1	-0.510 (-0.512,-0.509)	-0.544 (-0.545,-0.543)
1.5	0.512 (0.511,0.514)	-0.148 (-0.150,-0.147)
2	-0.081 (-0.084,-0.081)	0.337 (0.335,0.338)
2.5	-0.146 (-0.147,-0.144)	-0.148 (-0.150,-0.147)
3	0.114 (0.111,0.115)	-0.022 (-0.024,-0.020)
3.5	-0.024 (-0.026,-0.023)	0.055 (0.053,0.056)
4	-0.015 (-0.016,-0.013)	-0.025 (-0.028,-0.025)

Table 2.3: Characteristic function of the terminal log-price process, assuming VG-type jumps. The parameters are $\sigma = 0.8$, $v = 0.4$, $\mu = -1.2$, $\delta = 0.2$, $p = 1$, $t = 0.5$, $s = s^* = 50$, $h^* = 0.03$, $r = q = 0.03$; the truncation parameters in (4.50) are $N = M = 7$. The values in parenthesis represent the 99% Monte-Carlo confidence interval.

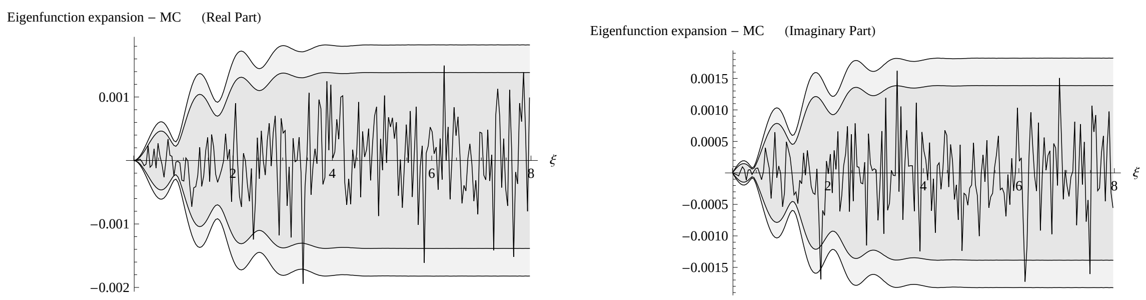

Figure 2.2: Diffusion case. Difference between our truncated expansion and the Monte Carlo values for $\phi(t,\xi)$ as a function of ξ. The dark and the light grey areas represent respectively the 95% and the 99% confidence intervals. The parameters are $\sigma = 0.5$, $p = 1$, $t = 1$, $s = s^* = 50$, $h^* = 0.03$, $r = q = 0$; the truncation parameters in (4.50) are $N = M = 7$.

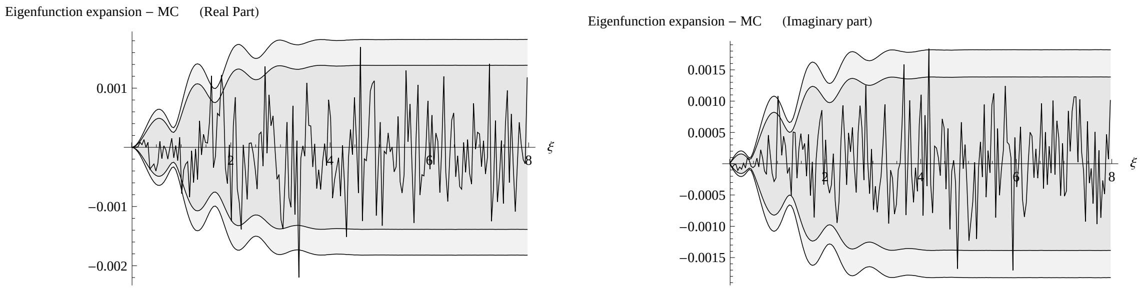

Figure 2.3: Compound Poisson. Difference between our truncated expansion and the Monte Carlo values for $\phi(t,\xi)$ as a function of ξ. The dark and the light grey areas represent respectively the 95% and the 99% confidence intervals. The parameters are $\sigma = 0.5$, $\lambda = 0.2$, $\mu = -0.5$, $\delta = 0.1$, $p = 1$, $t = 1$, $s = s^* = 50$, $h^* = 0.03$, $r = q = 0$; the truncation parameters in (4.50) are $N = M = 7$.

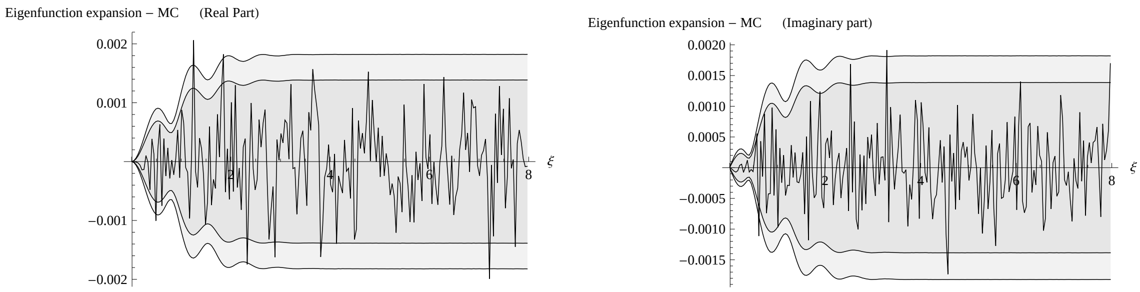

Figure 2.4: Variance Gamma. Difference between our truncated expansion and the Monte Carlo values for $\phi(t,\xi)$ as a function of ξ. The dark and the light grey areas represent respectively the 95% and the 99% confidence intervals. The parameters are $\sigma = 0.5$, $v = 0.5$, $\mu = -0.8$, $\delta = 0.3$, $p = 1$, $t = 1$, $s = s^* = 50$, $h^* = 0.03$, $r = q = 0$; the truncation parameters in (4.50) are $N = M = 7$.

Strikes	Diffusion	Compound Poisson	Variance Gamma
25	26.50 (26.42,26.52)	29.72 (29.81, 30.30)	29.1 (28.79, 29.58)
30	22.51 (22.40,22.56)	27.05 (26.91, 27.31)	26.08 (25.86,26.44)
35	18.75 (18.67,18.77)	24.42 (24.04,24.72)	23.33 (23.12, 23.61)
40	15.35 (15.26,15.45)	21.99 (21.79, 22.34)	20.85 (20.58, 21.12)
45	12.39 (12.37,12.44)	19.78 (19.67,19.89)	18.63 (18.35, 19.08)
50	9.87 (9.75,9.97)	17.76 (17.59, 17.98)	16.65 (16.23, 16.95)
55	7.80 (7.72,7.96)	15.41 (15.68, 16.15)	14.88 (14.60, 15.02)
60	6.12 (6.08,6.20)	13.86 (13.92, 14.52)	13.3 (13.05,13.55)
65	4.77 (4.73,4.88)	12.43 (12.52, 12.91)	11.89 (11.53, 12.15)
70	3.71 (3.67, 3.76)	11.09 (11.03, 11.48)	10.64 (10.39, 10.86)

Table 2.4: Option prices and corresponding 99% Monte-Carlo confidence intervals. The expiration time is $t = 2$.

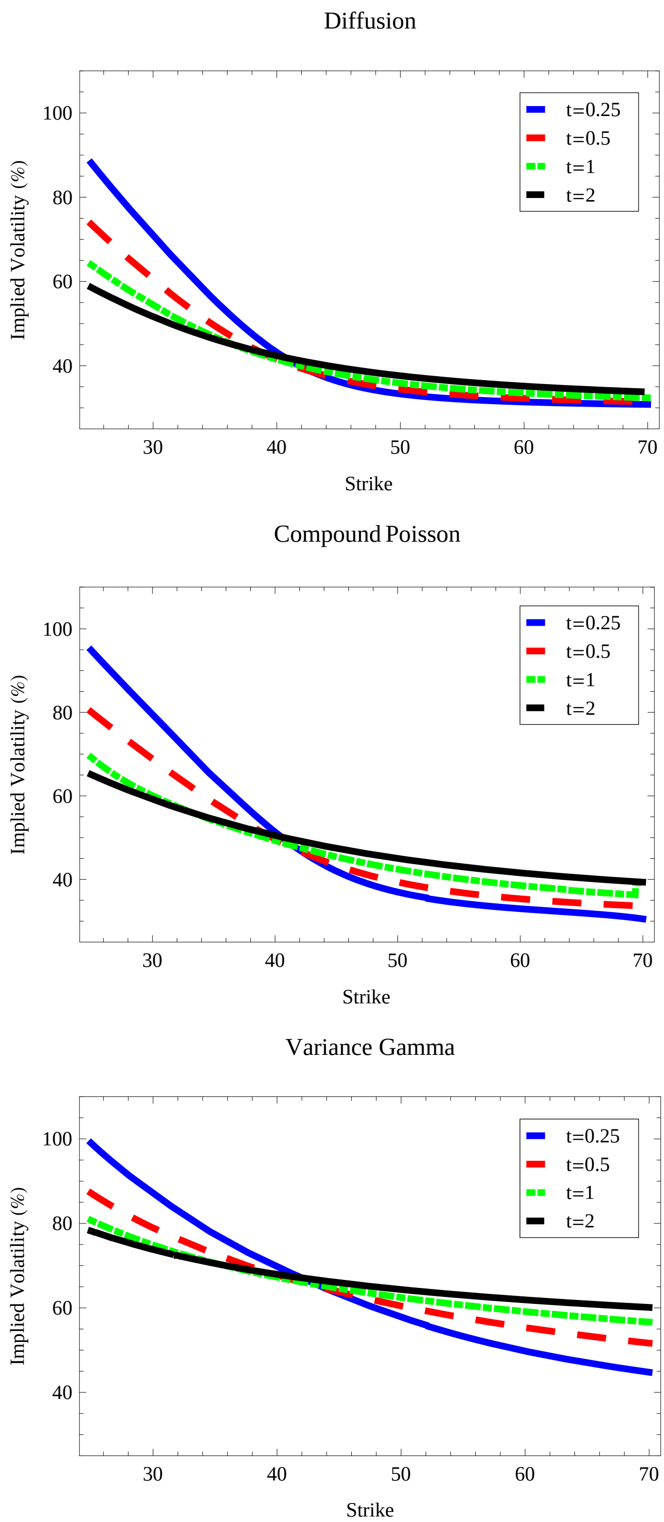

Figure 2.5: Implied volatility curves for times to expiration $t = 0.25$, $t = 0.5$, $t = 1$, and $t = 2$.

Diffusion			
Maturities	$K = 25$	$K = 50$	$K = 75$
1/52	5 (25.012)	4(0.861)	6 (8.214 $\times 10^{-6}$)
1/12	4 (25.065)	3 (1.842)	6 (3.408 $\times 10^{-6}$)
1/4	4 (25.191)	3 (3.283)	3 (0.012)
1/2	4 (25.398)	3 (4.775)	3 (0.172)
1	4 (25.768)	3 (6.908)	3 (0.924)
2	4 (26.494)	3 (9.880)	3 (2.881)

Table 2.5: Truncation parameter N needed for the option price to enter the 99% Monte-Carlo confidence band for the diffusion model. The numbers in parenthesis represents the corresponding option price obtained using the truncated characteristic function $\phi^{N,N}(t, \xi)$.

Compound Poisson			
Maturities	$K = 25$	$K = 50$	$K = 75$
1/52	3 (25.023)	3(0.888)	5 (8.260 $\times 10^{-6}$)
1/12	3 (25.114)	3 (1.966)	5 (10.6 $\times 10^{-5}$)
1/4	3 (25.335)	3 (3.655)	4 (0.016)
1/2	3 (25.654)	2 (5.452)	3 (0.239)
1	3 (26.260)	2 (8.092)	3 (1.305)
2	2 (27.333)	2 (11.768)	2 (4.072)

Table 2.6: Truncation parameter N needed for the option price to enter the 99% Monte-Carlo confidence band for the compound Poisson model. The numbers in parenthesis represents the corresponding option price obtained using the truncated characteristic function $\phi^{N,N}(t, \xi)$.

Variance Gamma			
Maturities	$K = 25$	$K = 50$	$K = 75$
1/52	2 (25.028)	2(1.204)	4 (8.852 $\times 10^{-6}$)
1/12	2 (25.132)	2 (2.995)	4 (7.35 $\times 10^{-5}$)
1/4	2 (25.458)	2 (5.731)	4 (0.164)
1/2	2 (26.024)	2 (8.385)	2 (1.355)
1	2 (27.179)	2 (11.978)	2 (4.453)
2	2 (29.155)	2 (16.657)	2 (9.53)

Table 2.7: Truncation parameter N needed for the option price to enter the 99% Monte-Carlo confidence band for the variance gamma model. The numbers in parenthesis represents the corresponding option price obtained using the truncated characteristic function $\phi^{N,N}(t, \xi)$.

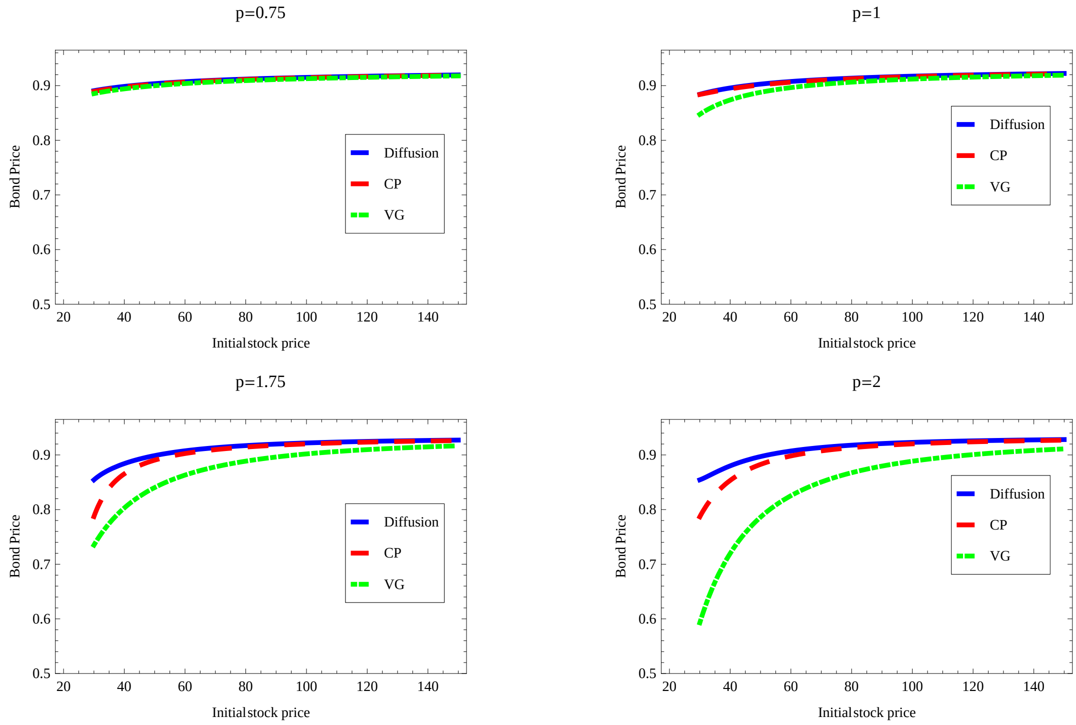

Figure 2.6: Bond price as a function of the initial stock value for different levels of p: $p = 0.75$, $p = 1$, $p = 1.75$, and $p = 2$. We set $t = 1$, $r = 0.07$, $q = 0$, $\sigma = 0.7$, $s^* = 50$, $h^* = 0.03$. For the compound Poisson model (CP in the legend), we set $\mu = -0.5$, $\lambda = 0.4$, $\theta = 0.1$. For the variance gamma model (VG in the legend), we set $\kappa = -0.7$, $\delta = 0.2$, $v = 0.7$.

t	Diffusion	Compound Poisson	Variance Gamma
1	0.94	0.928	0.936
2	0.883	0.861	0.876
3	0.829	0.791	0.82
4	0.776	0.714	0.760
5	0.725	0.625	0.711

Table 2.8: Bond prices for different expirations. We set $p = 1$, $r = q = 0.03$, $\sigma = 0.3$, $s = s^* = 20$, $h^* = 0.03$. The parameters of the compound Poisson process are $\mu = -1$, $\lambda = 1.5$, $\theta = 0.1$. The parameters of the variance gamma process are $v = 0.2$, $\delta = 0.1$, $\kappa = -0.6$.

p	Diffusion	Compound Poisson	Variance Gamma
1	0.902	0.900	0.901
1.25	0.902	0.890	0.900
1.5	0.900	0.895	0.897
1.75	0.899	0.887	0.895
2	0.896	0.885	0.888

Table 2.9: Bond prices for difference choices of p. We set $t = 1$, $r = 0.07$, $q = 0$, $\sigma = 0.5$, $s = s^* = 20$, $h^* = 0.03$. The parameters of the compound Poisson process are: $\mu = -0.5$, $\lambda = 0.5$, $\theta = 0.1$. The parameters of the variance gamma process are $v = 0.2$, $\delta = 0.1$, $\kappa = -0.6$.

Maturities	Diffusion	Compound Poisson	Variance Gamma
1/52	6 (0.999)	6 (0.998)	5 (0.999)
1/12	5 (0.995)	5 (0.995)	4 (0.995)
1/4	5 (0.984)	5 (0.985)	4 (0.984)
1/2	4 (0.969)	5 (0.968)	3 (0.968)
1	4 (0.94)	3 (0.933)	2 (0.931)
2	4 (0.883)	3 (0.858)	2 (0.852)

Table 2.10: Truncation parameter N needed for the bond prices to enter the 99% Monte-Carlo confidence band. The numbers in parenthesis represents the corresponding bond price obtained using the truncated characteristic function $\phi^{N,N}(t,\xi)$. We set $p = 1$, $r = q = 0.03$, $\sigma = 0.3$, $s = s^* = 20$, $h^* = 0.03$. The parameters of the compound Poisson process are: $\mu = -0.5$, $\lambda = 1.5$, $\theta = 0.1$. The parameters of the variance gamma process are $v = 0.2$, $\delta = 0.1$, $\kappa = -0.6$.

Part III

Analytical expansions for parabolic PIDE's and application to option pricing

Chapter 3

Adjoint expansions in local Lévy models

Based on a joint work ([185]) with Candia Riga and Prof. A. Pascucci.

Abstract: we propose a novel method for the analytical approximation in local volatility models with Lévy jumps. The main result is an expansion of the characteristic function in a local Lévy model, which is worked out in the Fourier space by considering the adjoint formulation of the pricing problem. Combined with standard Fourier methods, our result provides efficient and accurate pricing formulae. In the case of Gaussian jumps, we also derive an explicit approximation of the transition density of the underlying process by a heat kernel expansion: the approximation is obtained in two ways, using PIDE techniques and working in the Fourier space. Numerical tests confirm the effectiveness of the method.

Keywords: Lévy process, local volatility, analytical approximation, partial integro-differential equation, Fourier methods.

3.1 Introduction

Throughout this chapter we consider a one-dimensional *local Lévy model* where the log-price X solves the SDE

$$dX_t = \mu(t, X_{t-})dt + \sigma(t, X_{t-})dW_t + dJ_t. \tag{1.1}$$

In (1.1), W is a standard real Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ with the usual assumptions on the filtration and J is a pure-jump Lévy process, independent of W, with Lévy triplet $(\mu_1, 0, \nu)$. We denote by

$$T \mapsto X_T^{t,x}$$

the solution of (1.1) starting from x at time t and by

$$\phi_{X_T^{t,x}}(\xi) = E\left[e^{i\xi X_T^{t,x}}\right], \qquad \xi \in \mathbb{R},$$

the characteristic function of $X_T^{t,x}$. Our main result in this chapter is a fourth order approximation formula of $\phi_{X_T^{t,x}}$. In some particular cases, we also obtain an explicit approximation of the transition density of X.

Local Lévy models of the form (1.1) have attracted an increasing interest in the theory of volatility modeling (see, for instance, [4], [48] and [60]); however to date only in a few cases closed pricing formulae are available. Our approximation formulas provide a way to compute efficiently and accurately option prices and sensitivities by using standard and well-known Fourier methods (see, for instance, Heston [121], Carr and Madan [50], Raible [193] and Lipton [159]).

We derive the approximation formulas by introducing an "adjoint" expansion method; this is worked out in the Fourier space by considering the adjoint formulation of the pricing problem. Generally speaking, our approach makes use of Fourier analysis and PDE techniques. In Section 3.2, we present the general procedure that allows to approximate analytically the transition density (or the characteristic function), in terms of the solutions of a sequence of nested Cauchy problems. Then we also prove explicit error bounds for the expansion that generalize in a new and nontrivial way some classical estimates. In the second part of the paper (Sections 3.3 and 3.4) the previous Cauchy problems are solved explicitly by using different approaches. In Section 3.3 we focus on the special class of local Lévy models with Gaussian jumps and we provide a heat kernel expansion of the transition density of the underlying process. The same results are derived in an alternative way in Subsection 3.3.1, by working in the Fourier space.

Section 3.4 contains the main contribution of the chapter: we consider the general class of local Lévy models and provide high order approximations of the characteristic function. Since all the computations are carried out in the Fourier space, we are forced to introduce *a dual formulation* of the approximating problems, which involves the adjoint (forward) Kolmogorov operator. Even if at first sight the adjoint expansion method seems a bit odd, it turns out to be much more natural and simpler than the direct formulation. Although the interplay between perturbation methods and Fourier analysis has been previously studied in finance by other authors (cf. [152], [137] and [182]), to the best of our knowledge this is the first time that this approach is considered in relation to this setting. This approach seems to be advantageous for several reasons:

- working in the Fourier space is natural and allows to get simple and clear results;

- we can treat the entire class of Lévy processes and not only jump-diffusions or processes which can be approximated by heat kernel expansions. Potentially, we can take as leading term of the expansion every process which admits an explicit characteristic function and not necessarily a Gaussian kernel;

- our method can be extended to the case of stochastic volatility or multi-asset models;

- higher order approximations are rather easy to derive and the approximation results are generally very accurate. Potentially it is possible to derive approximation formulae for the characteristic function and plain vanilla options, at any prescribed order: for example, in Subsection 3.4.1 we provide also the 3^{rd} and 4^{th} order expansions of the characteristic function, used in the numerical tests of Section 3.5. The Mathematica notebook with the implemented formulae is freely available on the website of the authors.

For completeness, in the last part of Section 3.4, a standard pricing integral formula for European options is stated. Finally, in Section 3.5, we present some numerical tests under the Merton and Variance-Gamma models and show the effectiveness of the analytical approximations compared with Monte Carlo simulation.

Comparison with the literature. Analytical approximations and their applications to finance have been studied by several authors in the last decades because of their great importance in the calibration and risk management processes. The large body of the existing literature (see, for instance, [115], [125], [209], [106], [26], [62], [54]) is mainly devoted to purely diffusive (local and stochastic volatility) models or, as in [25] and [212], to local volatility (LV) models with Poisson jumps, which can be approximated by Gaussian kernels.

The classical result by Hagan [115] is a particular case of our expansion, in the sense that for a standard LV model with time-homogeneous coefficients our formulae reduce to Hagan's ones (see Section 3.3.1). While Hagan's results are heuristic, here we also provide explicit error estimates for time-dependent coefficients as well.

The results of Section 3.3 on the approximation of the transition density for jump-diffusions are essentially analogous to the results in [25]; however in [25], Malliavin techniques for LV models with Merton jumps are used and a first order expansion is derived. Here we use different techniques (PDE and Fourier methods) which allows to handle the more general class of local Lévy processes and to achieve higher order approximations.

Our approach is also more general than the so-called "parametrix" methods recently proposed in [62] and [54] as an approximation method in finance. The parametrix method is based on repeated application of Duhamel's principle which leads to a recursive integral representation of the fundamental solution: the main problem with the parametrix approach is that, even in the simplest case of a LV model, it is hard to compute explicitly the parametrix approximations of order greater than one. As a matter of fact, [62] and [54] only contain first order formulae. The adjoint expansion method contains the parametrix approximation *as a particular case*, that is at order zero and in the purely diffusive case. However the general construction of the adjoint expansion is substantially different and allows us to find explicit higher-order formulae for the general class of local Lévy processes.

3.2 General framework

In a local Lévy model, we assume that the risk-neutral dynamics of the underlying asset process X is given by equation (1.1). In order to guarantee the martingale property for the discounted asset price $\widetilde{S}_t := S_0 e^{X_t - rt}$, we set

$$\mu(t,x) = \bar{r} - \mu_1 - \frac{\sigma^2(t,x)}{2}, \tag{2.2}$$

where

$$\bar{r} = r - \int_{\mathbb{R}} \left(e^y - 1 - y \mathbb{1}_{\{|y|<1\}} \right) \nu(dy).$$

Provided that $X_T^{t,x}$ has density $\Gamma(t,x;T,\cdot)$, then its characteristic function is equal to

$$\phi_{X_T^{t,x}}(\xi) = \int_{\mathbb{R}} e^{i\xi y} \Gamma(t,x;T,y) dy.$$

Notice that $\Gamma(t,x;T,y)$ is the fundamental solution of the Kolmogorov operator

$$\begin{aligned} Lu(t,x) = & \frac{\sigma^2(t,x)}{2} \left(\partial_{xx} - \partial_x \right) u(t,x) + \bar{r} \partial_x u(t,x) + \partial_t u(t,x) \\ & + \int_{\mathbb{R}} \left(u(t,x+y) - u(t,x) - \partial_x u(t,x) y \mathbb{1}_{\{|y|<1\}} \right) \nu(dy). \end{aligned} \tag{2.3}$$

Example 2.1. *Let J be a compound Poisson process with Gaussian jumps, that is*

$$J_t = \sum_{n=1}^{N_t} Z_n$$

where N_t is a Poisson process with intensity ℓ and Z_n are i.i.d. random variables independent of N_t with Normal distribution $\mathcal{N}_{m,\partial^2}$. In this case, $\nu = \ell \mathcal{N}_{m,\partial^2}$ and

$$\mu_1 = \int_{|y|<1} y \nu(dy).$$

Therefore the drift condition (2.2) *reduces to*

$$\mu(t,x) = r_0 - \frac{\sigma^2(t,x)}{2},$$

where

$$r_0 = r - \int_{\mathbb{R}} \left(e^y - 1 \right) \nu(dy) = r - \lambda \left(e^{m + \frac{\delta^2}{2}} - 1 \right). \tag{2.4}$$

Moreover, the characteristic operator can be written in the equivalent form

$$\begin{aligned} Lu(t,x) = & \frac{\sigma^2(t,x)}{2} \left(\partial_{xx} - \partial_x \right) u(t,x) + r_0 \partial_x u(t,x) + \partial_t u(t,x) \\ & + \int_{\mathbb{R}} \left(u(t,x+y) - u(t,x) \right) \nu(dy). \end{aligned} \tag{2.5}$$

Example 2.2. *Let J be a Variance-Gamma process (cf. [173]) obtained by subordinating a Brownian motion with drift θ and standard deviation ϱ, by a Gamma process with variance κ and unitary mean. In this case the Lévy measure is given by*

$$\nu(dx) = \frac{e^{-\ell_1 x}}{\kappa x}\mathbb{1}_{\{x>0\}}dx + \frac{e^{\ell_2 x}}{\kappa|x|}\mathbb{1}_{\{x<0\}}dx \tag{2.6}$$

where

$$\ell_1 = \left(\sqrt{\frac{\theta^2\kappa^2}{4} + \frac{\varrho^2\kappa}{2}} + \frac{\theta\kappa}{2}\right)^{-1}, \qquad \ell_2 = \left(\sqrt{\frac{\theta^2\kappa^2}{4} + \frac{\varrho^2\kappa}{2}} - \frac{\theta\kappa}{2}\right)^{-1}.$$

The risk-neutral drift in (1.1) *is equal to*

$$\mu(t,x) = r_0 - \frac{\sigma^2(t,x)}{2}$$

where

$$r_0 = r + \frac{1}{\kappa}\log\left(1-\ell_1^{-1}\right)\left(1+\ell_2^{-1}\right) = r + \frac{1}{\kappa}\log\left(1-\kappa\left(\theta+\frac{\varrho^2}{2}\right)\right), \tag{2.7}$$

and the expression of the characteristic operator L is the same as in (2.5) *with ν and r_0 as in* (2.6) *and* (2.7) *respectively.*

Our goal is to give an accurate analytic approximation of the characteristic function and, when possible, of the transition density of X. The general idea is to consider an approximation of the volatility coefficient σ. More precisely, to shorten notations we set

$$a(t,x) = \sigma^2(t,x) \tag{2.8}$$

and we assume that a is regular enough: more precisely, for a fixed $N \in \mathbb{N}$, we make the following

Assumption $\mathbf{A}_N$. *The function $a = a(t,x)$ is continuously differentiable with respect to x up to order N. Moreover, the function a and its derivatives in x are bounded and Lipschitz continuous in x, uniformly with respect to t.*

Next, we fix a basepoint $\bar{x} \in \mathbb{R}$ and consider the N^{th}-order Taylor polynomial of $a(t,x)$ about $\bar{x}$:

$$\alpha_0(t) + 2\sum_{n=1}^{N}\alpha_n(t)(x-\bar{x})^n,$$

where $\alpha_0(t) = a(t,\bar{x})$ and

$$\alpha_n(t) = \frac{1}{2}\frac{\partial_x^n a(t,\bar{x})}{n!}, \qquad n \le N. \tag{2.9}$$

Then we introduce the n^{th}-order approximation of L:

$$L_n := L_0 + \sum_{k=1}^{n}\alpha_k(t)(x-\bar{x})^k\left(\partial_{xx} - \partial_x\right), \qquad n \le N, \tag{2.10}$$

where

$$
\begin{aligned}
L_0 u(t,x) = {} & \frac{\alpha_0(t)}{2}\left(\partial_{xx}u(t,x) - \partial_x u(t,x)\right) + \bar{r}\partial_x u(t,x) + \partial_t u(t,x) \\
& + \int_{\mathbb{R}} \left(u(t,x+y) - u(t,x) - \partial_x u(t,x) y \mathbb{1}_{\{|y|<1\}}\right) \nu(dy).
\end{aligned}
\tag{2.11}
$$

Following the perturbation method proposed in [183], and also recently used in [95] for the approximation of Asian options, the n^{th}-order approximation of the fundamental solution Γ of L is defined by

$$
\Gamma^n(t,x;T,y) := \sum_{k=0}^{n} G^k(t,x;T,y), \qquad t < T, \;\; x,y \in \mathbb{R}. \tag{2.12}
$$

The leading term G^0 of the expansion in (2.12) is the fundamental solution of L_0 and, for any $(T,y) \in \mathbb{R}_+ \times \mathbb{R}$ and $k \le N$, the functions $G^k(\cdot,\cdot;T,y)$ are defined recursively in terms of the solutions of the following sequence of Cauchy problems on the strip $]0,T[\times\mathbb{R}$:

$$
\begin{cases}
L_0 G^k(t,x;T,y) = -\sum\limits_{h=1}^{k} \left(L_h - L_{h-1}\right) G^{k-h}(t,x;T,y) \\
\qquad\qquad\qquad\;\; = -\sum\limits_{h=1}^{k} \alpha_h(t)(x-\bar{x})^h \left(\partial_{xx} - \partial_x\right) G^{k-h}(t,x;T,y), \\
G^k(T,x;T,y) = 0.
\end{cases}
\tag{2.13}
$$

In the sequel, when we want to specify explicitly the dependence of the approximation Γ^n on the basepoint $\bar{x}$, we shall use the notation

$$
\Gamma^{\bar{x},n}(t,x;T,y) \equiv \Gamma^n(t,x;T,y). \tag{2.14}
$$

In Section 3.3 we show that, in the case of a LV model with Gaussian jumps, it is possible to find *the explicit solutions* to the problems (2.13) by an iterative argument. When general Lévy jumps are considered, it is still possible to compute the explicit solution of problems (2.13) *in the Fourier space*. Indeed, in Section 3.4, we get an expansion of the characteristic function $\phi_{X_T^{t,x}}$ having as leading term the characteristic function of the process whose Kolmogorov operator is L_0 in (2.11).

We explicitly notice that, if the function σ only depends on time, then *the approximation in* (2.12) *is exact at order zero.*

We now provide global error estimates for the approximation in the purely diffusive case. The proof is postponed to the Appendix.

Theorem 2.3. *Assume the diffusion coefficient to be time-independent, i.e.* $a(t,x) \equiv a(x)$, *and satisfying the parabolicity condition*

$$
m \le \frac{a(x)}{2} \le M, \qquad x \in \mathbb{R},
$$

where m, M *are positive constants and let* $\bar{x} = x$ *or* $\bar{x} = y$ *in* (2.14). *Under Assumption* A_{N+1}, *for any* $\varepsilon > 0$ *we have*

$$
\left|\Gamma(t,x;T,y) - \Gamma^{\bar{x},N}(t,x;T,y)\right| \le g_N(T-t)\bar{\Gamma}^{M+\varepsilon}(t,x;T,y), \tag{2.15}
$$

for $x, y \in \mathbb{R}$ and $t \in [0, T[$, where $\bar{\Gamma}^M$ is the Gaussian fundamental solution of the heat operator

$$M\partial_{xx} + \partial_t,$$

and $g_N(s) = O\left(s^{\frac{N+1}{2}}\right)$ as $s \to 0^+$.

Remark 2.4. *Although the estimate* (2.15) *is stated and proved only when the diffusion coefficient is time-independent, the same result to hold also in the time-dependent case. Indeed, the same argument used in the Appendix to prove Theorem 2.3 can be repeated for a time-dependent diffusion, under the additional hypothesis: $a(\cdot, x) \in C[0, T]$ for any $x \in \mathbb{R}$. For a detailed proof in the time-dependent and multi-dimensional case we refer to the novel paper [168].*

Theorem 2.3 improves some known results in the literature. In particular in [26] asymptotic estimates for option prices in terms of $(T-t)^{\frac{N+1}{2}}$ are proved under a stronger assumption on the regularity of the coefficients, equivalent to Assumption A_{3N+2}. Here we provide error estimates for the transition density: error bounds for option prices can be easily derived from (2.15). Moreover, for small N it is not difficult to find the explicit expression of g_N.

Estimate (2.15) also justifies a time-splitting procedure which nicely adapts to our approximation operators, as shown in detail in Remark 2.7 in [183].

3.3 LV models with Gaussian jumps

In this section we consider the SDE (1.1) with J as in Example 2.1, namely J is a compound Poisson process with Gaussian jumps. Clearly, in the particular case of a constant diffusion coefficient $\sigma(t, x) \equiv \sigma$, we have the classical Merton jump-diffusion model [177]:

$$X_t^{\text{Merton}} = \left(r_0 - \frac{\sigma^2}{2}\right) t + \sigma W_t + J_t,$$

with r_0 as in (2.4). We recall that the analytical approximation of this kind of models has been recently studied by Benhamou, Gobet and Miri in [25] by Malliavin calculus techniques.

The expression of the pricing operator L was given in (2.5) and in this case the leading term of the approximation (cf. (2.11)) is equal to

$$\begin{aligned} L_0 v(t, x) = &\frac{\alpha_0(t)}{2} \left(\partial_{xx} v(t, x) - \partial_x v(t, x)\right) + r_0 \partial_x v(t, x) \\ &+ \partial_t v(t, x) + \int_{\mathbb{R}} \left(v(t, x + y) - v(t, x)\right) \nu(dy). \end{aligned} \tag{3.16}$$

The fundamental solution of L_0 is the transition density of a Merton process, that is

$$G^0(t, x; T, y) = e^{-\ell(T-t)} \sum_{n=0}^{+\infty} \frac{(\ell(T-t))^n}{n!} \Gamma_n(t, x; T, y), \tag{3.17}$$

where

$$\begin{aligned}\Gamma_n(t,x;T,y) &= \frac{1}{\sqrt{2\pi\left(A(t,T)+n\partial^2\right)}}\, e^{-\frac{\left(x-y+(T-t)r_0-\frac{1}{2}A(t,T)+nm\right)^2}{2\left(A(t,T)+n\partial^2\right)}}, \\ A(t,T) &= \int_t^T \alpha_0(s)ds.\end{aligned} \tag{3.18}$$

In order to determine the explicit solution to problems (2.13) for $k \geq 1$, we use some elementary properties of the functions $(\Gamma_n)_{n\geq 0}$. The following lemma can be proved as Lemma 2.2 in [183].

Lemma 3.1. *For any $x, y, \bar{x} \in \mathbb{R}$, $t < s < T$ and $n, k \in \mathbb{N}_0$, we have*

$$\Gamma_{n+k}(t,x;T,y) = \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)\Gamma_k(s,\eta;T,y)d\eta, \tag{3.19}$$

$$\partial_y^k \Gamma_n(t,x;T,y) = (-1)^k \partial_x^k \Gamma_n(t,x;T,y), \tag{3.20}$$

$$(y-\bar{x})^k \Gamma_n(t,x;T,y) = V_{t,T,x,n}^k \Gamma_n(t,x;T,y), \tag{3.21}$$

where $V_{t,T,x,n}$ is the operator defined by

$$V_{t,T,x,n} f(x) = \left(x - \bar{x} + (T-t)r_0 - \frac{1}{2}A(t,T) + nm\right) f(x) + \left(A(t,T)+n\partial^2\right)\partial_x f(x). \tag{3.22}$$

Our first results are the following first and second order expansions of the transition density Γ.

Theorem 3.2 (1st order expansion)**.** *The solution G^1 of the Cauchy problem* (2.13) *with $k = 1$ is given by*

$$G^1(t,x;T,y) = \sum_{n,k=0}^{+\infty} J_{n,k}^1(t,T,x)\Gamma_{n+k}(t,x;T,y). \tag{3.23}$$

where $J_{n,k}^1(t,T,x)$ is the differential operator defined by

$$J_{n,k}^1(t,T,x) = e^{-\ell(T-t)}\frac{\ell^{n+k}}{n!k!}\int_t^T \alpha_1(s)(s-t)^n(T-s)^k V_{t,s,x,n} ds\,(\partial_{xx}-\partial_x). \tag{3.24}$$

Proof. By the standard representation formula for solutions to the non-homogeneous parabolic Cauchy problem (2.13) with null final condition, we have

$$G^1(t,x;T,y) = \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta)\alpha_1(s)(\eta-\bar{x})(\partial_{\eta\eta}-\partial_\eta)G^0(s,\eta;T,y)d\eta ds =$$

(by (3.21))

$$= \sum_{n=0}^{+\infty} \frac{\ell^n}{n!}\int_t^T \alpha_1(s)e^{-\ell(s-t)}(s-t)^n.$$

$$\cdot V_{t,s,x,n} \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)(\partial_{\eta\eta} - \partial_\eta) G^0(s,\eta;T,y) d\eta ds$$

(by parts)

$$= e^{-\ell(T-t)} \sum_{n,k=0}^{+\infty} \frac{\ell^{n+k}}{n!k!} \int_t^T \alpha_1(s)(T-s)^k (s-t)^n \cdot$$
$$\cdot V_{t,s,x,n} \int_{\mathbb{R}} (\partial_{\eta\eta} + \partial_\eta)\Gamma_n(t,x;s,\eta)\Gamma_k(s,\eta;T,y) d\eta ds =$$

(by (3.20) and (3.19))

$$= e^{-\ell(T-t)} \sum_{n,k=0}^{\infty} \frac{\ell^{n+k}}{n!k!} \int_t^T \alpha_1(s)(T-s)^k (s-t)^n V_{t,s,x,n} ds \cdot$$
$$\cdot (\partial_{xx} - \partial_x)\Gamma_{n+k}(t,x;T,y)$$

and this proves (3.23)-(3.24). □

Remark 3.3. *A straightforward but tedious computation shows that the operator* $J^1_{n,k}(t,T,x)$ *can be rewritten in the more convenient form*

$$J^1_{n,k}(t,T,x) = \sum_{i=1}^{3} \sum_{j=0}^{1} f^1_{n,k,i,j}(t,T)(x-\bar{x})^j \partial_x^i, \tag{3.25}$$

for some deterministic functions $f^1_{n,k,i,j}$.

Theorem 3.4 (2nd order expansion). *The solution* G^2 *of the Cauchy problem* (2.13) *with* $k=2$ *is given by*

$$G^2(t,x;T,y) = \sum_{n,h,k=0}^{+\infty} J^{2,1}_{n,h,k}(t,T,x)\Gamma_{n+h+k}(t,x;T,y) + \sum_{n,k=0}^{\infty} J^{2,2}_{n,k}(t,T,x)\Gamma_{n+k}(t,x;T,y), \tag{3.26}$$

where

$$J^{2,1}_{n,h,k}(t,T,x) = \frac{\ell^n}{n!} \int_t^T \alpha_1(s) e^{-\ell(s-t)} (s-t)^n V_{t,s,x,n} (\partial_{xx} - \partial_x) \widetilde{J}^1_{n,h,k}(t,s,T,x) ds$$
$$J^{2,2}_{n,k}(t,T,x) = e^{-\ell(T-t)} \frac{\ell^{n+k}}{n!k!} \int_t^T \alpha_2(s)(s-t)^n (T-s)^k V^2_{t,s,x,n} ds\, (\partial_{xx} - \partial_x)$$

and $\widetilde{J}^1_{n,h,k}$ *is the "adjoint" operator of* $J^1_{h,k}$*, defined by*

$$\widetilde{J}^1_{n,h,k}(t,s,T,x) = \sum_{i=1}^{3} \sum_{j=0}^{1} f^1_{h,k,i,j}(s,T) V^j_{t,s,x,n} \partial_x^i \tag{3.27}$$

with $f^1_{h,k,i,j}$ *as in* (3.25)*. Also in this case we have the alternative representation*

$$J^{2,1}_{n,h,k}(t,T,x) = \sum_{i=1}^{6} \sum_{j=0}^{2} f^{2,1}_{n,h,k,i,j}(t,T)(x-\bar{x})^j \partial_x^i$$

$$J^{2,2}_{n,k}(t,T,x) = \sum_{i=1}^{6}\sum_{j=0}^{2} f^{2,2}_{n,k,i,j}(t,T)(x-\bar{x})^j \partial_x^i,$$

with $f^{2,1}_{n,h,k,i,j}$ *and* $f^{2,2}_{n,k,i,j}$ *deterministic functions.*

Proof. We show a preliminary result: from formulae (3.25) and (3.27) for J^1 and $\widetilde{J}^1$ respectively, it follows that

$$\int_{\mathbb{R}} \Gamma_n(t,x;s,\eta) J^1_{h,k}(s,T,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta =$$

(by (3.20) and (3.21))

$$\begin{aligned}
&= \int_{\mathbb{R}} \widetilde{J}^1_{n,h,k}(s,T,x)\Gamma_n(t,x;s,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta \\
&= \widetilde{J}^1_{n,h,k}(s,T,x)\int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta =
\end{aligned}$$

(by (3.19))

$$= \widetilde{J}^1_{n,h,k}(s,T,x)\Gamma_{n+h+k}(x,t;T,y). \tag{3.28}$$

Now we have

$$G^2(t,x;T,y) = I_1 + I_2,$$

where, proceeding as before,

$$\begin{aligned}
I_1 &= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta)\alpha_1(s)(\eta-\bar{x})(\partial_{\eta\eta}-\partial_\eta)G^1(s,\eta;T,y)d\eta ds \\
&= \sum_{n,h,k=0}^{+\infty} \frac{\ell^n}{n!}\int_t^T \alpha_1(s)e^{-\ell(s-t)}(s-t)^n \cdot \\
&\quad \cdot V_{t,s,x,n}\int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)(\partial_{\eta\eta}-\partial_\eta)J^1_{h,k}(s,T,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta ds \\
&= \sum_{n,h,k=0}^{+\infty} \frac{\ell^n}{n!}\int_t^T \alpha_1(s)e^{-\ell(s-t)}(s-t)^n \cdot \\
&\quad \cdot V_{t,s,x,n}(\partial_{xx}-\partial_x)\int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)J^1_{h,k}(s,T,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta ds =
\end{aligned}$$

(by (3.28))

$$\begin{aligned}
&= \sum_{n,h,k=0}^{+\infty} \frac{\ell^n}{n!}\int_t^T \alpha_1(s)e^{-\ell(s-t)}(s-t)^n V_{t,s,x,n}(\partial_{xx}-\partial_x)\widetilde{J}^1_{n,h,k}(s,T,x)ds\Gamma_{n+h+k}(x,t;T,y) \\
&= \sum_{n,h,k=0}^{+\infty} J^{2,1}_{n,h,k}(t,T,x)\Gamma_{n+h+k}(t,x;T,y)
\end{aligned}$$

and

$$I_2 = \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta)\alpha_2(s)(\eta-\bar{x})^2(\partial_{\eta\eta}-\partial_\eta)G^0(s,\eta;T,y)d\eta ds$$

$$
\begin{aligned}
&= e^{-\ell(T-t)} \sum_{n,k=0}^{+\infty} \frac{\ell^{n+k}}{n!k!} \int_t^T \alpha_2(s)(T-s)^k(s-t)^n \cdot \\
&\cdot V_{t,s,x,n}^2 \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)(\partial_{\eta\eta} - \partial_\eta)\Gamma_k(s,\eta;T,y) d\eta ds \\
&= e^{-\ell(T-t)} \sum_{n,k=0}^{+\infty} \frac{\ell^{n+k}}{n!k!} \int_t^T \alpha_2(s)(T-s)^k(s-t)^n \cdot \\
&\cdot V_{t,s,x,n}^2 (\partial_{xx} - \partial_x) \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)\Gamma_k(s,\eta;T,y) d\eta ds \\
&= e^{-\ell(T-t)} \sum_{n,k=0}^{+\infty} \frac{\ell^{n+k}}{n!k!} \int_t^T \alpha_2(s)(T-s)^k(s-t)^n V_{t,s,x,n}^2 \, ds \, (\partial_{xx} - \partial_x)\Gamma_{n+k}(t,x;T,y) \\
&= \sum_{n,k=0}^{+\infty} J_{n,k}^{2,2}(t,T,x)\Gamma_{n+k}(t,x;T,y).
\end{aligned}
$$

This concludes the proof. □

Remark 3.5. *For a more concise explicit representation of each term G^n in the density expansion* (2.12) *we refer to the novel paper [170].*

Remark 3.6. *Since the derivatives of a Gaussian density can be expressed in terms of Hermite polynomials, the computation of the terms of the expansion* (2.12) *is very fast. Indeed, we have*

$$
\frac{\partial_x^i \Gamma_n(t,x;T,y)}{\Gamma_n(t,x;T,y)} = \frac{(-1)^i h_{i,n}(t,T,x-y)}{\left(2\left(A(t,T) + n\partial^2\right)\right)^{\frac{i}{2}}}
$$

where

$$
h_{i,n}(t,T,z) = \mathbf{H}_i\left(\frac{z + (T-t)\mu_0 - \frac{1}{2}A(t,T) + nm}{\sqrt{2\left(A(t,T) + n\partial^2\right)}}\right)
$$

and $\mathbf{H}_i = \mathbf{H}_i(x)$ denotes the Hermite polynomial of degree i. Thus we can rewrite the terms $\left(G^k\right)_{k=1,2}$ in (3.23) *and* (3.26) *as follows:*

$$
\begin{aligned}
G^1(t,x;T,y) &= \sum_{n,k=0}^{\infty} \mathbf{G}_{n,k}^1(t,x;T,y)\Gamma_{n+k}(t,x;T,y) \\
G^2(t,x;T,y) &= \sum_{n,h,k=0}^{\infty} \mathbf{G}_{n,h,k}^{2,1}(t,x;T,y)\Gamma_{n+h+k}(t,x;T,y) \\
&\quad + \sum_{n,k=0}^{\infty} \mathbf{G}_{n,k}^{2,2}(t,x;T,y)\Gamma_{n+k}(t,x;T,y),
\end{aligned}
\tag{3.29}
$$

where

$$
\mathbf{G}_{n,k}^1(t,x;T,y) = \sum_{i=1}^{3}(-1)^i \sum_{j=0}^{1} f_{n,k,i,j}^1(t,T)(x-\bar{x})^j \frac{h_{i,n+k}(t,T,x-y)}{\left(2\left(A(t,T) + (n+k)\partial^2\right)\right)^{\frac{i}{2}}}
$$

$$\mathbf{G}^{2,1}_{n,h,k}(t,x;T,y) = \sum_{i=1}^{6}(-1)^i \sum_{j=0}^{1} f^{2,1}_{n,h,k,i,j}(t,T)(x-\bar{x})^j \frac{h_{i,n+h+k}(t,T,x-y)}{\left(2\left(A(t,T)+(n+h+k)\partial^2\right)\right)^{\frac{i}{2}}}$$

$$\mathbf{G}^{2,2}_{n,k}(t,x;T,y) = \sum_{i=1}^{6}(-1)^i \sum_{j=0}^{1} f^{2,2}_{n,k,i,j}(t,T)(x-\bar{x})^j \frac{h_{i,n+k}(t,T,x-y)}{\left(2\left(A(t,T)+(n+k)\partial^2\right)\right)^{\frac{i}{2}}}.$$

In the practical implementation, we truncate the series in (3.17) *and* (3.29) *to a finite number of terms, say* $M \in \mathbb{N} \cup \{0\}$. *Therefore we put*

$$G^0_M(t,x;T,y) = e^{-\ell(T-t)} \sum_{n=0}^{M} \frac{(\ell(T-t))^n}{n!} \Gamma_n(t,x;T,y),$$

$$G^1_M(t,x;T,y) = \sum_{n,k=0}^{M} \mathbf{G}^1_{n,k}(t,x;T,y)\Gamma_{n+k}(t,x;T,y),$$

$$G^2_M(t,x;T,y) = \sum_{n,h,k=0}^{M} \mathbf{G}^{2,1}_{n,h,k}(t,x;T,y)\Gamma_{n+h+k}(t,x;T,y) + \sum_{n,k=0}^{M} \mathbf{G}^{2,2}_{n,k}(t,x;T,y)\Gamma_{n+k}(t,x;T,y),$$

and we approximate the density Γ *by*

$$\Gamma^2_M(t,x;T,y) := G^0_M(t,x;T,y) + G^1_M(t,x;T,y) + G^2_M(t,x;T,y). \tag{3.30}$$

Next we denote by $C(t,S_t)$ the price at time $t<T$ of a European option with payoff function ϕ and maturity T; for instance, $\phi(y) = (y-K)^+$ in the case of a Call option with strike K. From the expansion of the density in (3.30), we get the following second order approximation formula.

Corollary 3.7. *We have*

$$C(t,S_t) \approx e^{-r(T-t)} u_M(t,\log S_t)$$

where

$$\begin{aligned} u_M(t,x) &= \int_{\mathbb{R}^+} \frac{1}{S}\Gamma^2_M(t,x;T,\log S)\phi(S)dS \\ &= e^{-\ell(T-t)} \sum_{n=0}^{M} \frac{(\ell(T-t))^n}{n!} \mathrm{CBS}_n(t,x) \\ &\quad + \sum_{n,k=0}^{M} \left(J^1_{n,k}(t,T,x) + J^{2,2}_{n,k}(t,T,x)\right) \mathrm{CBS}_{n+k}(t,x) \\ &\quad + \sum_{n,h,k=0}^{M} J^{2,1}_{n,h,k}(t,T,x)\mathrm{CBS}_{n+h+k}(t,x) \end{aligned}$$

and $\mathrm{CBS}_n(t,x)$ *is the BS price*[1] *under the Gaussian law* $\Gamma_n(t,x;T,\cdot)$ *in* (3.18), *namely*

$$\mathrm{CBS}_n(t,x) = \int_{\mathbb{R}^+} \frac{1}{S}\Gamma_n(t,x;T,\log S)\phi(S)dS.$$

[1] Here the BS price is expressed as a function of the time t and of the log-asset x.

3.3.1 Simplified Fourier approach for LV models

Equation (1.1) with $J = 0$ reduces to the standard SDE of a LV model. In this case we can simplify the proof of Theorems 3.2-3.4 by using Fourier analysis methods. Let us first notice that L_0 in (3.16) becomes

$$L_0 = \frac{\alpha_0(t)}{2}(\partial_{xx} - \partial_x) + r\partial_x + \partial_t, \tag{3.31}$$

and its fundamental solution is the Gaussian density

$$G^0(t,x;T,y) = \frac{1}{\sqrt{2\pi A(t,T)}} e^{-\frac{\left(x-y+(T-t)r-\frac{1}{2}A(t,T)\right)^2}{2A(t,T)}},$$

with A as in (3.18).

Corollary 3.8 (1st order expansion)**.** *In case of $\ell = 0$, the solution G^1 in (3.23) is given by*

$$G^1(t,x;T,y) = J^1(t,T,x)G^0(t,x;T,y)$$

where $J^1(t,T,x)$ is the differential operator

$$J^1(t,T,x) = \int_t^T \alpha_1(s)V_{t,s,x}ds\,(\partial_{xx} - \partial_x), \tag{3.32}$$

with $V_{t,s,x} \equiv V_{t,s,x,0}$ as in (3.22), that is

$$V_{t,T,x}f(x) = \left(x - \bar{x} + (T-t)r - \frac{1}{2}A(t,T)\right)f(x) + A(t,T)\partial_x f(x).$$

Proof. Although the result follows directly from Theorem 3.2, here we propose an alternative proof of formula (3.32). The idea is to determine the solution of the Cauchy problem (2.13) in the Fourier space, where all the computation can be carried out more easily; then, using the fact that the leading term G^0 of the expansion is a Gaussian kernel, we are able to compute explicitly the inverse Fourier transform to get back to the analytic approximation of the transition density.

Since we aim at showing the main ideas of an alternative approach, for simplicity we only consider the case of time-independent coefficients, precisely we set $\alpha_0 = 2$ and $r = 0$. In this case we have

$$L_0 = \partial_{xx} - \partial_x + \partial_t$$

and the related Gaussian fundamental solution is equal to

$$G^0(t,x;T,y) = \frac{1}{\sqrt{4\pi(T-t)}} e^{-\frac{(x-y-(T-t))^2}{4(T-t)}}.$$

Now we apply the Fourier transform (in the variable x) to the Cauchy problem (2.13) with $k = 1$ and we get

$$\begin{cases} \partial_t \hat{G}^1(t,\xi;T,y) = \left(\xi^2 - i\xi\right)\hat{G}^1(t,\xi;T,y) \\ \qquad\qquad\qquad\qquad +\alpha_1(i\partial_\xi + \bar{x})\left(-\xi^2 + i\xi\right)\hat{G}^0(t,\xi;T,y), \\ \hat{G}^1(T,\xi;T,y) = 0, \qquad \xi \in \mathbb{R}. \end{cases} \tag{3.33}$$

Notice that

$$\hat{G}^0(t,\xi;T,y) = e^{-\xi^2(T-t)+i\xi(y+(T-t))}. \tag{3.34}$$

Therefore the solution to the ordinary differential equation (3.33) is

$$\hat{G}^1(t,\xi;T,y) = -\alpha_1 \int_t^T e^{(s-t)(-\xi^2+i\xi)}(i\partial_\xi + \bar{x})\left((-\xi^2+i\xi)\hat{G}^0(s,\xi;T,y)\right) ds =$$

(using the identity $f(\xi)(i\partial_\xi + \bar{x})(g(\xi)) = (i\partial_\xi + \bar{x})(f(\xi)g(\xi)) - ig(\xi)\partial_\xi f(\xi)$)

$$\begin{aligned} &= -\alpha_1 \int_t^T (i\partial_\xi + \bar{x})\left((-\xi^2+i\xi)e^{(s-t)(-\xi^2+i\xi)}\hat{G}^0(s,\xi;T,y)\right) ds \\ &\quad + i\alpha_1 \int_t^T (-\xi^2+i\xi)\hat{G}^0(s,\xi;T,y)\partial_\xi e^{(s-t)(-\xi^2+i\xi)} ds = \end{aligned}$$

(by (3.34))

$$\begin{aligned} &= -\alpha_1 \int_t^T (i\partial_\xi + \bar{x})\left((-\xi^2+i\xi)e^{i\xi(y+(T-t))-\xi^2(T-t)}\right) ds \\ &\quad + i\alpha_1 \int_t^T (-\xi^2+i\xi)(s-t)(-2\xi+i)e^{i\xi(y+(T-t))-\xi^2(T-t)} ds = \end{aligned}$$

(again by (3.34))

$$\begin{aligned} &= -\alpha_1(T-t)(i\partial_\xi + \bar{x})\left((-\xi^2+i\xi)\hat{G}^0(t,\xi;T,y)\right) \\ &\quad + i\alpha_1 \frac{(T-t)^2}{2}(-\xi^2+i\xi)(-2\xi+i)\hat{G}^0(t,\xi;T,y). \end{aligned}$$

Thus, inverting the Fourier transform, we get

$$\begin{aligned} G^1(t,x;T,y) &= \alpha_1(T-t)(x-\bar{x})(\partial_x^2 - \partial_x)G^0(t,x;T,y) + \\ &\quad - \alpha_1 \frac{(T-t)^2}{2}(-2\partial_x^3 + 3\partial_x^2 - \partial_x)G^0(t,x;T,y) \\ &= \alpha_1 \left((T-t)^2\partial_x^3 + \left((x-\bar{x})(T-t) - \frac{3}{2}(T-t)^2\right)\partial_x^2 + \right. \\ &\quad \left. + \left(-(x-\bar{x})(T-t) + \frac{(T-t)^2}{2}\right)\partial_x\right) G^0(t,x;T,y), \end{aligned}$$

where the operator acting on $G^0(t,x;T,y)$ is exactly the same as in (3.32). □

Remark 3.9. *As in Remark 3.3, operator $J^1(t,T,x)$ can also be rewritten in the form*

$$J^1(t,T,x) = \sum_{i=1}^{3}\sum_{j=0}^{1} f_{i,j}^1(t,T)(x-\bar{x})^j\partial_x^i, \tag{3.35}$$

where $f_{i,j}^1$ are deterministic functions whose explicit expression can be easily derived.

The previous argument can be used to prove the following second order expansion.

Corollary 3.10 (2nd order expansion)**.** *In case of $\ell = 0$, the solution G^2 in* (3.26) *is given by*

$$G^2(t, x; T, y) = J^2(t, T, x)G^0(t, x; T, y)$$

where

$$J^2(t, T, x) = \int_t^T \alpha_1(s)V_{t,s,x}(\partial_{xx} - \partial_x)\widetilde{J}^1(t, s, T, x)ds + \int_t^T \alpha_2(s)V^2_{t,s,x}ds\,(\partial_{xx} - \partial_x)$$

and $\widetilde{J}^1$ is the "adjoint" operator of J^1, defined by

$$\widetilde{J}^1(t, s, T, x) = \sum_{i=1}^{3}\sum_{j=0}^{1} f^1_{i,j}(s, T)V^j_{t,s,x}\partial^i_x$$

with $f^1_{i,j}$ as in (3.35)*.*

Remark 3.11. *In a standard LV model, the leading operator of the approximation, i.e. L_0 in* (3.31)*, has a Gaussian density G^0 and this allowed us to use the inverse Fourier transform in order to get the approximated density. This approach does not work in the general case of models with jumps because typically the explicit expression of the fundamental solution of an integro-differential equation is not available. On the other hand, for several Lévy processes used in finance, the characteristic function is known explicitly even if the density is not. This suggests that the argument used in this section may be adapted to obtain an approximation of the characteristic function of the process instead of its density. This is what we are going to investigate in Section 3.4.*

3.4 Local Lévy models

In this section, we provide an expansion of the characteristic function for the local Lévy model (1.1). We denote by

$$\hat{\Gamma}(t, x; T, \xi) = \mathcal{F}\left(\Gamma(t, x; T, \cdot)\right)(\xi)$$

the Fourier transform, with respect to the second spatial variable, of the transition density $\Gamma(t, x; T, \cdot)$; clearly, $\hat{\Gamma}(t, x; T, \xi)$ is the characteristic function of $X^{t,x}_T$. Then, by applying the Fourier transform to the expansion (2.12), we find

$$\phi_{X^{t,x}_T}(\xi) \approx \sum_{k=0}^{n} \hat{G}^k(t, x; T, \xi). \tag{4.36}$$

Now we recall that $G^k(t, x; T, y)$ is defined, as a function of the variables (t, x), in terms of the sequence of Cauchy problems (2.13). Since the Fourier transform in (4.36) is performed with respect to the variable y, in order to take advantage of such a transformation it seems natural to characterize $G^k(t, x; T, y)$ as a solution of the *adjoint operator* in the dual variables (T, y).

To be more specific, we recall the definition of adjoint operator. Let L be the operator in (2.3); then its adjoint operator $\widetilde{L}$ satisfies (actually, it is defined by) the identity

$$\int_{\mathbb{R}^2} u(t, x)Lv(t, x)dxdt = \int_{\mathbb{R}^2} v(t, x)\widetilde{L}u(t, x)dxdt$$

for all $u, v \in C_0^\infty$. More explicitly, by recalling notation (2.8), we have

$$\widetilde{L}^{(T,y)}u(T,y) = \frac{a(T,y)}{2}\partial_{yy}u(T,y) + b(T,y)\partial_y u(T,y) - \partial_T u(T,y) + c(T,y)u(T,y) \\ + \int_{\mathbb{R}} \left(u(T,y+z) - u(T,y) - z\partial_y u(T,y)\mathbb{1}_{\{|z|<1\}}\right)\bar{\nu}(dz),$$

where

$$b(T,y) = \partial_y a(T,y) - \left(\bar{r} - \frac{a(T,y)}{2}\right), \qquad c(T,y) = \frac{1}{2}(\partial_{yy} + \partial_y)a(T,y),$$

and $\bar{\nu}$ is the Lévy measure with reverted jumps, i.e. $\bar{\nu}(dx) = \nu(-dx)$. Here the superscript in $\widetilde{L}^{(T,y)}$ is indicative of the fact that the operator $\widetilde{L}$ is acting in the variables (T, y).

By a classical result (cf., for instance, [104]) the fundamental solution $\Gamma(t, x; T, y)$ of L is also a solution of $\widetilde{L}$ in the dual variables, that is

$$\widetilde{L}^{(T,y)}\Gamma(t,x;T,y) = 0, \qquad t < T,\ x, y \in \mathbb{R}. \tag{4.37}$$

Going back to approximation (4.36), the idea is to consider the series of the dual Cauchy problems of (2.13) in order to solve them by Fourier-transforming in the variable y and finally get an approximation of $\phi_{X_T^{t,x}}$.

For sake of simplicity, from now on we only consider the case of time-independent coefficients: the general case can be treated in a completely analogous way. First of all, we consider the integro-differential operator L_0 in (2.11), which in this case becomes

$$L_0^{(t,x)}u(t,x) = \frac{\alpha_0}{2}(\partial_{xx} - \partial_x)u(t,x) + \bar{r}\partial_x u(t,x) + \partial_t u(t,x) \\ + \int_{\mathbb{R}} \left(u(t,x+y) - u(t,x) - y\partial_x u(t,x)\mathbb{1}_{\{|y|<1\}}\right)\nu(dy), \tag{4.38}$$

and its adjoint operator

$$\widetilde{L}_0^{(T,y)}u(T,y) = \frac{\alpha_0}{2}(\partial_{yy} + \partial_y)u(T,y) - \bar{r}\partial_y u(T,y) - \partial_T u(T,y) \\ + \int_{\mathbb{R}} \left(u(T,y+z) - u(T,y) - z\partial_y u(T,y)\mathbb{1}_{\{|z|<1\}}\right)\bar{\nu}(dz).$$

By (4.37), for any $(t, x) \in \mathbb{R}^2$, the fundamental solution $G^0(t, x; T, y)$ of L_0 solves the dual Cauchy problem

$$\begin{cases} \widetilde{L}_0^{(T,y)}G^0(t,x;T,y) = 0, & T > t,\ y \in \mathbb{R}, \\ G^0(t,x;t,\cdot) = \partial_x. \end{cases} \tag{4.39}$$

It is remarkable that a similar result holds for the higher order terms of the approximation (4.36). Indeed, let us denote by L_n the n^{th} order approximation of L in (2.10):

$$L_n = L_0 + \sum_{k=1}^{n} \alpha_k (x - \bar{x})^k (\partial_{xx} - \partial_x)$$

Then we have the following result.

Theorem 4.1. *For any $k \geq 1$ and $(t,x) \in \mathbb{R}^2$, the function $G^k(t,x;\cdot,\cdot)$ in (2.13) is the solution of the following dual Cauchy problem on $]t,+\infty[\times\mathbb{R}$*

$$\begin{cases} \widetilde{L}_0^{(T,y)} G^k(t,x;T,y) = -\sum\limits_{h=1}^{k} \left(\widetilde{L}_h^{(T,y)} - \widetilde{L}_{h-1}^{(T,y)} \right) G^{k-h}(t,x;T,y), \\ G^k(t,x;t,y) = 0, \qquad y \in \mathbb{R}, \end{cases} \tag{4.40}$$

where

$$\widetilde{L}_h^{(T,y)} - \widetilde{L}_{h-1}^{(T,y)} = \alpha_h (y-\bar{x})^{h-2} \Big((y-\bar{x})^2 \partial_{yy} + (y-\bar{x})\,(2h + (y-\bar{x}))\,\partial_y + h\,(h-1+y-\bar{x}) \Big).$$

Proof. By the standard representation formula for the solutions of the *backward* parabolic Cauchy problem (2.13), for $k \geq 1$ we have

$$G^k(t,x;T,y) = \sum_{h=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_h^{(s,\eta)} G^{k-h}(s,\eta;T,y) d\eta ds, \tag{4.41}$$

where to shorten notation we have set

$$M_h^{(t,x)} = L_h^{(t,x)} - L_{h-1}^{(t,x)}.$$

By (4.39) and since

$$\widetilde{M}_h^{(T,y)} = \widetilde{L}_h^{(T,y)} - \widetilde{L}_{h-1}^{(T,y)}.$$

the assertion is equivalent to

$$G^k(t,x;T,y) = \sum_{h=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(s,\eta;T,y) \widetilde{M}_h^{(s,\eta)} G^{k-h}(t,x;s,\eta) d\eta ds, \tag{4.42}$$

where here we have used the representation formula for the solutions of the *forward* Cauchy problem (4.40) with $k \geq 1$.

We proceed by induction and first prove (4.42) for $k = 1$. By (4.41) we have

$$\begin{aligned} G^1(t,x;T,y) &= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_1^{(s,\eta)} G^0(s,\eta;T,y) d\eta ds \\ &= \int_t^T \int_{\mathbb{R}} G^0(s,\eta;T,y) \widetilde{M}_1^{(s,\eta)} G^0(t,x;s,\eta) d\eta ds, \end{aligned}$$

and this proves (4.42) for $k = 1$.

Next we assume that (4.42) holds for a generic $k > 1$ and we prove the thesis for $k+1$. Again, by (4.41) we have

$$\begin{aligned} G^{k+1}(t,x;T,y) &= \sum_{j=1}^{k+1} \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_j^{(s,\eta)} G^{k+1-j}(s,\eta;T,y) d\eta ds \\ &= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_{k+1}^{(s,\eta)} G^0(s,\eta;T,y) d\eta ds \\ &\quad + \sum_{j=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_j^{(s,\eta)} G^{k+1-j}(s,\eta;T,y) d\eta ds \end{aligned}$$

(by the inductive hypothesis)

$$
\begin{aligned}
&= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_{k+1}^{(s,\eta)} G^0(s,\eta;T,y) d\eta ds \\
&\quad + \sum_{j=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_j^{(s,\eta)} \cdot \\
&\quad \cdot \sum_{h=1}^{k+1-j} \int_s^T \int_{\mathbb{R}} G^0(\tau,\zeta;T,y) \widetilde{M}_h^{(\tau,\zeta)} G^{k+1-j-h}(s,\eta;\tau,\zeta) d\zeta d\tau d\eta ds \\
&= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_{k+1}^{(s,\eta)} G^0(s,\eta;T,y) ds d\eta \\
&\quad + \sum_{h=1}^{k} \sum_{j=1}^{k+1-h} \int_t^T \int_t^\tau \int_{\mathbb{R}^2} G^0(t,x;s,\eta) G^0(\tau,\zeta;T,y) \cdot \\
&\quad \cdot M_j^{(s,\eta)} \widetilde{M}_h^{(\tau,\zeta)} G^{k+1-j-h}(s,\eta;\tau,\zeta) d\eta d\zeta ds d\tau \\
&= \int_t^T \int_{\mathbb{R}} G^0(s,\eta;T,y) \widetilde{M}_{k+1}^{(s,\eta)} G^0(t,x;s,\eta) ds d\eta \\
&\quad + \sum_{h=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(\tau,\zeta;T,y) \widetilde{M}_h^{(\tau,\zeta)} \cdot \\
&\quad \cdot \left(\sum_{j=1}^{k+1-h} \int_t^\tau \int_{\mathbb{R}} G^0(t,x;s,\eta) M_j^{(s,\eta)} G^{k+1-h-j}(s,\eta;\tau,\zeta) d\eta ds \right) d\zeta d\tau
\end{aligned}
$$

(again by (4.41))

$$
\begin{aligned}
&= \int_t^T \int_{\mathbb{R}} G^0(t,\eta;T,y) \widetilde{M}_{k+1}^{(s,\eta)} G^0(t,x;s,\eta) ds d\eta \\
&\quad + \sum_{h=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(\tau,\zeta;T,y) \widetilde{M}_h^{(\tau,\zeta)} G^{k+1-h}(t,x;\tau,\zeta) d\zeta d\tau \\
&= \sum_{h=1}^{k+1} \int_t^T \int_{\mathbb{R}} G^0(\tau,\zeta;T,y) \widetilde{M}_h^{(\tau,\zeta)} G^{k+1-h}(t,x;\tau,\zeta) d\zeta d\tau.
\end{aligned}
$$

□

Next we solve problems (4.39)-(4.40) by applying the Fourier transform in the variable y and using the identity

$$
\mathcal{F}_y \left(\widetilde{L}_0^{(T,y)} u(T,y) \right) (\xi) = \psi(\xi) \hat{u}(T,\xi) - \partial_T \hat{u}(T,\xi),
$$

where

$$
\psi(\xi) = -\frac{\alpha_0}{2}(\xi^2 + i\xi) + i\bar{r}\xi + \int_{\mathbb{R}} \left(e^{iz\xi} - 1 - iz\xi \mathbb{1}_{\{|z|<1\}} \right) \nu(dz).
$$

We remark explicitly that ψ is the characteristic exponent of the Lévy process

$$dX_t^0 = \left(\bar{r} - \frac{\alpha_0}{2}\right) dt + \sqrt{\alpha_0} dW_t + dJ_t,$$

whose Kolmogorov operator is L^0 in (4.38). Then:

- from (4.39) we obtain the ordinary differential equation

$$\begin{cases} \partial_T \hat{G}^0(t,x;T,\xi) = \psi(\xi)\hat{G}^0(t,x;T,\xi), & T > t, \\ \hat{G}^0(t,x;t,\xi) = e^{i\xi x}. \end{cases}$$

  with solution

$$\hat{G}^0(t,x;T,\xi) = e^{i\xi x + (T-t)\psi(\xi)} \tag{4.43}$$

  which is the 0^{th} order approximation of the characteristic function $\phi_{X_T^{t,x}}$.

- from (4.40) with $k = 1$, we have

$$\begin{cases} \partial_T \hat{G}^1(t,x;T,\xi) = \psi(\xi)\hat{G}^1(t,x;T,\xi) \\ \qquad\qquad +\alpha_1 \left((i\partial_\xi + \bar{x})(\xi^2 + i\xi) - 2i\xi + 1\right) \hat{G}^0(t,x;T,\xi) \\ \hat{G}^1(t,x;t,\xi) = 0, \end{cases}$$

  with solution

$$\hat{G}^1(t,x;T,\xi) = \int_t^T e^{\psi(\xi)(T-s)} \alpha_1 \left((i\partial_\xi + \bar{x})(\xi^2 + i\xi) - 2i\xi + 1\right) \hat{G}^0(t,x;s,\xi) ds =$$

  (by (4.43))

$$\begin{aligned} &= -e^{ix\xi + \psi(\xi)(T-t)} \alpha_1 \int_t^T (\xi^2 + i\xi) \left(x - \bar{x} - i(s-t)\psi'(\xi)\right) ds \\ &= -\hat{G}^0(t,x;T,\xi)\alpha_1(T-t)(\xi^2 + i\xi) \left(x - \bar{x} - \frac{i}{2}(T-t)\psi'(\xi)\right), \end{aligned} \tag{4.44}$$

  which is the first order term in the expansion (4.36).

- regarding (4.40) with $k = 2$, a straightforward computation based on analogous arguments shows that the second order term in the expansion (4.36) is given by

$$\hat{G}^2(t,x;T,\xi) = \hat{G}^0(t,x;T,\xi) \sum_{j=0}^{2} g_j(T-t,\xi)(x-\bar{x})^j \tag{4.45}$$

  where

$$\begin{aligned} g_0(s,\xi) &= \frac{1}{2}s^2\alpha_2\xi(i+\xi)\psi''(\xi) \\ &\quad - \frac{1}{6}s^3\xi(i+\xi)\psi''(\xi)\left(\alpha_1^2(i+2\xi) - 2\alpha_2\psi''(\xi) + \alpha_1^2\xi(i+\xi)\right) \\ &\quad - \frac{1}{8}s^4\alpha_1^2\xi^2(i+\xi)^2\psi''(\xi)^2, \\ g_1(s,\xi) &= \frac{1}{2}s^2\xi(i+\xi)\left(\alpha_1^2(1-2i\xi) + 2i\alpha_2\psi''(\xi)\right) - \frac{1}{2}s^3 i\alpha_1^2\xi^2(i+\xi)^2\psi''(\xi), \\ g_2(s,\xi) &= -\alpha_2 s\xi(i+\xi) + \frac{1}{2}s^2\alpha_1^2\xi^2(i+\xi)^2. \end{aligned}$$

Plugging (4.43)-(4.44)-(4.45) into (4.36), we finally get the second order approximation of the characteristic function of X. In Subsection 3.4.1, we also provide the expression of $\hat{G}^k(t,x;T,\xi)$ for $k=3,4$, appearing in the 4^{th} order approximation.

Remark 4.2. *The basepoint $\bar{x}$ is a parameter which can be freely chosen in order to sharpen the accuracy of the approximation. In general, the simplest choice $\bar{x}=x$ seems to be sufficient to get very accurate results, and the experiments performed in Section 3.5, as well as the ones proposed in the next Chapters, are made in accord with this choice. Nevertheless, in the works by Hagan [115] and many other authors, the proxy volatility is not computed at the spot x, but at the average between strike and spot, or along the most likely path (see [11]). In fact, the mid-point for call-put options benefit of symmetry, which induces many simplifications in formulas. Plus, incorporating this possibility in the formulas would lead likely to even more accurate results. We aim to get back to this point in a future paper, in order to show the improvement induced by different choices of the initial point $\bar{x}$, such as $\bar{x}=\frac{x+\log K}{2}$.*

Remark 4.3. *To overcome the use of the adjoint operators, it would be interesting to investigate an alternative approach to the approximation of the characteristic function based of the following remarkable symmetry relation valid for time-homogeneous diffusions*

$$m(x)\Gamma(0,x;t,y)=m(y)\Gamma(0,y;t,x) \tag{4.46}$$

where m is the so-called density of the speed measure

$$m(x)=\frac{2}{\sigma^2(x)}\exp\left(\int_1^x\left(\frac{2r}{\sigma^2(z)}-1\right)dz\right).$$

Relation (4.46) *is stated in [129] and a complete proof can be found in [83].*

For completeness, we close this section by stating an integral pricing formula for European options proved by Lewis [153]; the formula is given in terms of the characteristic function of the underlying log-price process. Formula below (and other Fourier-inversion methods such as the standard, fractional FFT algorithm or the recent COS method [88]) can be combined with the expansion (4.36) to price and hedge efficiently hybrid LV models with Lévy jumps.

We consider a risky asset $S_t=e^{X_t}$ where X is the process whose risk-neutral dynamics under a martingale measure Q is given by (1.1). We denote by $H(t,S_t)$ the price at time $t<T$, of a European option with underlying asset S, maturity T and payoff $f=f(x)$ (given as a function of the log-price): to fix ideas, for a Call option with strike K we have

$$f^{\text{Call}}(x)=(e^x-K)^+.$$

The following theorem is a classical result which can be found in several textbooks (see, for instance, [188]).

Theorem 4.4. *Let*

$$f_\gamma(x)=e^{-\gamma x}f(x)$$

and assume that there exists $\gamma\in\mathbb{R}$ such that

i) $f_\gamma,\hat{f}_\gamma\in L^1(\mathbb{R})$*;*

ii) $E^Q\left[S_T^{\gamma}\right]$ *is finite.*

Then, the following pricing formula holds:

$$H(t,S_t)=\frac{e^{-r(T-t)}}{\pi}\int_0^{\infty}\hat{f}(\xi+i\gamma)\phi_{X_T^{t,\log S_t}}(-(\xi+i\gamma))d\xi.$$

For example, f^{Call} verifies the assumptions of Theorem 4.4 for any $\gamma>1$ and we have

$$\hat{f}^{\text{Call}}(\xi+i\gamma)=\frac{K^{1-\gamma}e^{i\xi\log K}}{(i\xi-\gamma)(i\xi-\gamma+1)}.$$

Other examples of typical payoff functions and the related Greeks can be found in [188].

3.4.1 High order approximations

The analysis of Section 3.4 can be carried out to get approximations of arbitrarily high order. Below we give the more accurate (but more complicated) formulae up to the 4$^{\text{th}}$ order that we used in the numerical section. In particular we give the expression of $\hat{G}^k(t,x;T,\xi)$ in (4.36) for $k=3,4$. For simplicity, we only consider the case of time-homogeneous coefficients and $\bar{x}=x$.

We have

$$\hat{G}^3(t,x;T,\xi)=\hat{G}^0(t,x;T,\xi)\sum_{j=3}^{7}g_j(\xi)(T-t)^j$$

where

$$\begin{aligned}
g_3(\xi)&=\frac{1}{2}\alpha_3(1-i\xi)\xi\psi^{(3)}(\xi),\\
g_4(\xi)&=\frac{1}{6}i\xi(i+\xi)\Big(2\psi'(\xi)\left(\alpha_1\alpha_2-3\alpha_3\psi''(\xi)\right)+\alpha_1\alpha_2\left(3(i+2\xi)\psi''(\xi)+2\xi(i+\xi)\psi^{(3)}(\xi)\right)\Big),\\
g_5(\xi)&=\frac{1}{24}(1-i\xi)\xi\Big(-8\alpha_1\alpha_2(i+2\xi)\psi'(\xi)^2+6\alpha_3\psi'(\xi)^3\\
&\quad+\alpha_1\psi'(\xi)\left(\alpha_1^2(-1+6\xi(i+\xi))-16\alpha_2\xi(i+\xi)\psi''(\xi)\right)\\
&\quad+\alpha_1^3\xi(i+\xi)\left(3(i+2\xi)\psi''(\xi)+\xi(i+\xi)\psi^{(3)}(\xi)\right)\Big),\\
g_6(\xi)&=-\frac{1}{12}i\alpha_1\xi^2(i+\xi)^2\psi'(\xi)\Big(\alpha_1^2(i+2\xi)\psi'(\xi)-2\alpha_2\psi'(\xi)^2+\alpha_1^2\xi(i+\xi)\psi''(\xi)\Big),\\
g_7(\xi)&=-\frac{1}{48}i\left(\alpha_1\xi(i+\xi)\psi'(\xi)\right)^3.
\end{aligned}$$

Moreover, we have

$$\hat{G}^4(t,x;T,\xi)=\hat{G}^0(t,x;T,\xi)\sum_{j=3}^{9}g_j(\xi)(T-t)^j$$

where

$$g_3(\xi)=-\frac{1}{2}\alpha_4\xi(i+\xi)\psi^{(4)}(\xi),$$

$$
\begin{aligned}
g_4(\xi) = & \frac{1}{6}\xi(i+\xi)\Big(2\psi''(\xi)\left(\alpha_2^2+3\alpha_1\alpha_3-3\alpha_4\psi''(\xi)\right) \\
& +2\left(\left(\alpha_2^2+2\alpha_1\alpha_3\right)(i+2\xi)-4\alpha_4\psi'(\xi)\right)\psi^{(3)}(\xi)+\left(\alpha_2^2+2\alpha_1\alpha_3\right)\xi(i+\xi)\psi^{(4)}(\xi)\Big), \\
g_5(\xi) = & -\frac{1}{24}\xi(i+\xi)\Big(\alpha_1^2\alpha_2(-7+44\xi(i+\xi))\psi''(\xi)-\left(7\alpha_2^2+15\alpha_1\alpha_3\right)\xi(i+\xi)\psi''(\xi)^2 \\
& -2\psi'(\xi)^2\left(2\alpha_2^2+9\alpha_1\alpha_3-18\alpha_4\psi''(\xi)\right) \\
& +\psi'(\xi)\Big((i+2\xi)\left(8\alpha_1^2\alpha_2-\left(14\alpha_2^2+33\alpha_1\alpha_3\right)\psi''(\xi)\right) \\
& -\left(10\alpha_2^2+21\alpha_1\alpha_3\right)\xi(i+\xi)\psi^{(3)}(\xi)\Big) \\
& +3\alpha_1^2\alpha_2\xi(i+\xi)\left(4(i+2\xi)\psi^{(3)}(\xi)+\xi(i+\xi)\psi^{(4)}(\xi)\right)\Big), \\
g_6(\xi) = & \frac{1}{120}\xi(i+\xi)\Big(2\left(8\alpha_2^2+21\alpha_1\alpha_3\right)(i+2\xi)\psi'(\xi)^3-24\alpha_4\psi'(\xi)^4 \\
& +2\psi'(\xi)^2\left(\alpha_1^2\alpha_2(11-70\xi(i+\xi))+\left(26\alpha_2^2+57\alpha_1\alpha_3\right)\xi(i+\xi)\psi''(\xi)\right) \\
& +\alpha_1^2\psi'(\xi)\Big((i+2\xi)\left(\alpha_1^2(-1+12\xi(i+\xi))-112\alpha_2\xi(i+\xi)\psi''(\xi)\right) \\
& -38\alpha_2\xi^2(i+\xi)^2\psi^{(3)}(\xi)\Big)+\alpha_1^2\xi(i+\xi)\Big(\alpha_1^2(-7+36\xi(i+\xi))\psi''(\xi) \\
& -26\alpha_2\xi(i+\xi)\psi''(\xi)^2+\alpha_1^2\xi(i+\xi)\left(6(i+2\xi)\psi^{(3)}(\xi)+\xi(i+\xi)\psi^4(\xi)\right)\Big)\Big), \\
g_7(\xi) = & \frac{1}{144}\xi^2(i+\xi)^2\Big(-32\alpha_1^2\alpha_2(i+2\xi)\psi'(\xi)^3+2\left(4\alpha_2^2+9\alpha_1\alpha_3\right)\psi'(\xi)^4 \\
& +2\alpha_1^4\xi^2(i+\xi)^2\psi''(\xi)^2+\alpha_1^2\psi'(\xi)^2\left(\alpha_1^2(-5+26\xi(i+\xi))-47\alpha_2\xi(i+\xi)\psi''(\xi)\right) \\
& +\alpha_1^4\xi(i+\xi)\psi'(\xi)\left(13(i+2\xi)\psi''(\xi)+3\xi(i+\xi)\psi^{(3)}(\xi)\right)\Big), \\
g_8(\xi) = & \frac{1}{48}\alpha_1^2\xi^3(i+\xi)^3\psi'(\xi)^2\Big(\alpha_1^2(i+2\xi)\psi'(\xi)-2\alpha_2\psi'(\xi)^2+\alpha_1^2\xi(i+\xi)\psi''(\xi)\Big), \\
g_9(\xi) = & \frac{1}{384}\alpha_1^4\xi^4(i+\xi)^4\psi'(\xi)^4.
\end{aligned}
$$

3.5 Numerical tests

In this section our approximation formulae (4.36) are tested and compared with a standard Monte Carlo method. We consider up to the 4^{th} order expansion (i.e. $n=4$ in (4.36)) even if in most cases the 2^{nd} order seems to be sufficient to get very accurate results. We analyze the case of a constant elasticity of variance (CEV) volatility function with Lévy jumps of Gaussian or Variance-Gamma type.

Remark 5.1. *Note that, for purely diffusive local volatility models, the accuracy of our approximating technique has been tested in [183], where several comparisons with other well known approximations (cf. Hagan [115] and Henry-Labordère [118]) have been proposed.*

Thus, we consider the log-price dynamics (1.1) with

$$
\sigma(t,x)=\sigma_0 e^{(\beta-1)x}, \quad \beta\in[0,1],\ \sigma_0>0,
$$

and J as in Examples 2.1 and 2.2 respectively. In our experiments we assume that the initial stock price is $S_0=1$, the risk-free rate is $r=5\%$, the CEV volatility parameter is

$\sigma_0 = 20\%$ and the CEV exponent is $\beta = \frac{1}{2}$. Moreover we use an Euler Monte Carlo method with 200 time-steps per year and 500 000 replications.

3.5.1 Tests under CEV-Merton dynamics

In order to assess the performance of our approximations for pricing Call options in the CEV-Merton model, we consider the following set of parameters: the jump intensity is $\ell = 30\%, 50\%$, the average jump size is $m = -10\%$ and the jump volatility is $\delta = 40\%$.

Figures 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6 show the performance of the approximations against the Monte Carlo 95% and 99% confidence intervals, marked in dark and light gray respectively. In particular, Figures 3.1 ($\lambda = 30\%$) and 3.4 ($\lambda = 50\%$) show the cross-sections of absolute (left) and relative (right) errors of the 1^{st} (dotted line), 2^{nd} (dashed line), 3^{rd} (solid line) order approximations for the price of a Call with short-term maturity $T = 0.25$ and strike K ranging from 0.5 to 1.5. The relative error is defined as

$$\frac{\text{Call}^{\text{approx}} - \text{Call}^{\text{MC}}}{\text{Call}^{\text{MC}}}$$

where $\text{Call}^{\text{approx}}$ and Call^{MC} are the approximated and Monte Carlo prices respectively.

In Figures 3.2 and 3.5 we repeat the test for the medium-term maturity $T = 1$ and the strike K ranging from 0.5 to 2.5. Finally in Figures 3.3 and 3.6 we consider the long-term maturity $T = 10$ and the strike K ranging from 0.5 to 4.

Other experiments that are not reported here, show that the 2^{nd} order expansion (3.30), which is valid only in the case of Gaussian jumps, gives the same results as formula (4.36) with $n = 2$, at least if the truncation index M is suitable large, namely $M \geq 8$ under standard parameter regimes. For this reason we have only used formula (4.36) for our tests.

Tables 3.1 ($\lambda = 30\%$) and 3.2 ($\lambda = 50\%$) show a deeper analysis of the error for several combinations (K, T) of strikes and maturities, by means of the comparison with a high-precision Monte Carlo simulation. For the latter, we use a time grid consisting of $250\,T$ equally spaced points and average the results across 10^7 independent samples. For both the price and the implied volatility, we report our 4-th order approximation (PPR 4-th) and the boundaries of the 95% MC confidence interval.

3.5.2 Tests under CEV-Variance-Gamma dynamics

In this subsection we repeat the previous tests in the case of the CEV-Variance-Gamma model. Specifically, we consider the following set of parameters: the variance of the Gamma subordinator is $\kappa = 15\%$, the drift and the volatility of the Brownian motion are $\theta = -10\%$ and $\sigma = 20\%$ respectively.

The results are reported in Figures 3.7, 3.8 and 3.9. Notice that, for longer maturities and deep out-of-the-money options, the lower order approximations give good results in terms of absolute errors but only the 4^{th} order approximation lies inside the confidence regions. For a more detailed comparison, in Figures 3.8 and 3.9 we plot the 2^{nd} (dotted line), 3^{rd} (dashed line), 4^{th} (solid line) order approximations. Similar results are obtained for a wide range of parameter values.

Analogously to Table 3.1, in Table 3.3 we compare our Call price approximation with a high-precision Monte Carlo simulation for several strikes and maturities. For both the price and the implied volatility, we report our 4-th order approximation (PPR 4-th) and the boundaries of the 95% MC confidence interval.

3.6 Appendix

3.6.1 Proof of Theorem 2.3

In this appendix we prove Theorem 2.3 under Assumption A_{N+1} where $N \in \mathbb{N}$ is fixed. For simplicity we only consider the case of $r = 0$. Recalling notation (2.9), we put

$$L_0 = \frac{\alpha_0}{2}\left(\partial_{xx} - \partial_x\right) + \partial_t \tag{6.47}$$

and

$$L_n = L_0 + \sum_{k=1}^{n} \alpha_k (x - \bar{x})^k \left(\partial_{xx} - \partial_x\right), \qquad n \le N. \tag{6.48}$$

Our idea is to modify and adapt the standard characterization of the fundamental solution given by the parametrix method originally introduced by Levi [151]. The parametrix method is a constructive technique that allows to prove the existence of the fundamental solution Γ of a parabolic operator with variable coefficients of the form

$$Lu(t,x) = \frac{a(x)}{2}\left(\partial_{xx} - \partial_x\right) u(t,x) + \partial_t u(t,x).$$

In the standard parametrix method, for any fixed $\xi \in \mathbb{R}$, the fundamental solution Γ_ξ of the frozen operator

$$L_\xi u(t,x) = \frac{a(\xi)}{2}\left(\partial_{xx} - \partial_x\right) u(t,x) + \partial_t u(t,x)$$

is called a *parametrix* for L. A fundamental solution $\Gamma(t,x;T,y)$ for L can be constructed starting from $\Gamma_y(t,x;T,y)$ by means of an iterative argument and by suitably controlling the errors of the approximation.

Our main idea is to *use the* N^{th}*-order approximation* $\Gamma^N(t,x;T,y)$ *in* (2.12)-(2.13) *(related to* L_n *in* (6.47)-(6.48)*) as a parametrix.* In order to prove the error bound (2.15), we carefully generalize some Gaussian estimates: in particular, for $N = 0$ we are back into the classical framework, but in general we need accurate estimates of the solutions of the nested Cauchy problems (2.13).

By analogy with the classical approach (see, for instance, [101] or the recent and more general presentation in [71]), we have that Γ takes the form

$$\Gamma(t,x;T,y) = \Gamma^N(t,x;T,y) + \int_t^T \int_{\mathbb{R}} \Gamma^0(t,x;s,\xi)\Phi^N(s,\xi;T,y)d\xi ds$$

where Φ^N is the function in (6.49) below, which is determined by imposing the condition $L\Gamma = 0$. More precisely, we have

$$0 = L\Gamma(z;\zeta) = L\Gamma^N(z;\zeta) + \int_t^T \int_{\mathbb{R}} L\Gamma^0(z;w)\Phi^N(w;\zeta)dw - \Phi^N(z;\zeta),$$

where, to shorten notations, we have set $z = (t, x)$, $w = (s, \xi)$ and $\zeta = (T, y)$. Equivalently, we have

$$\Phi^N(z;\zeta) = L\Gamma^N(z;\zeta) + \int_t^T \int_{\mathbb{R}} L\Gamma^0(z;w)\Phi^N(w;\zeta)dw$$

and therefore by iteration

$$\Phi^N(z;\zeta) = \sum_{n=0}^{\infty} Z_n(z;\zeta) \tag{6.49}$$

where

$$\begin{aligned} Z_0^N(z;\zeta) &= L\Gamma^N(z;\zeta), \\ Z_{n+1}^N(z;\zeta) &= \int_t^T \int_{\mathbb{R}} L\Gamma^0(z;w)Z_n(w;\zeta)dw. \end{aligned}$$

The thesis is a consequence of the following lemmas.

Lemma 6.1. *For any $n \leq N$ the solution of* (2.13)*, with L_n as in* (6.47)-(6.48)*, takes the form*

$$G^n(t,x;T,y) = \sum_{\substack{i\leq n,\, j\leq n(n+3),\, k\leq \frac{n(n+5)}{2} \\ i+j-k\geq n}} c^n_{i,j,k}(x-\bar{x})^i(\sqrt{T-t})^j \partial_x^k G^0(t,x;T,y), \tag{6.50}$$

where $c^n_{i,j,k}$ are polynomial functions of $\alpha_0, \alpha_1, \dots, \alpha_n$.

Proof. We proceed by induction on n. For $n = 0$ the thesis is trivial. Next by (2.13) we have $G^{n+1}(t,x;T,y) = I_{n,2} - I_{n,1}$ where

$$I_{n,l} = \sum_{h=1}^{n+1} \alpha_h \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta)(\eta-\bar{x})^h \partial_\eta^l G^{n+1-h}(s,\eta;T,y)d\eta ds, \quad l = 1, 2.$$

We only analyze the case $l = 2$ since the other one is analogous. By the inductive hypothesis (6.50), we have that $I_{n,2}$ is a linear combination of terms of the form

$$\int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta)(\sqrt{T-s})^j(\eta-\bar{x})^{h+i-p}\partial_\eta^{k+2-p}G^0(s,\eta;T,y)d\eta ds \tag{6.51}$$

for $p = 0, 1, 2$ and $h = 1, \dots, n+1$; moreover we have

$$i + j - k \geq n + 1 - h, \tag{6.52}$$

$$i \leq n + 1 - h, \tag{6.53}$$

$$j \leq (n+1-h)(n+4-h) \leq n(n+3), \tag{6.54}$$

$$k \leq \frac{(n+1-h)(n+6-h)}{2} \leq \frac{n(n+5)}{2}. \tag{6.55}$$

Again we focus only on $p = 0$, the other cases being analogous: then by properties (3.21), (3.20) and (3.19), we have that the integral in (6.51) is equal to

$$\int_t^T (\sqrt{T-s})^j V_{t,s,x}^{h+i} ds\, \partial_x^{k+2} G^0(t,x;T,y) \tag{6.56}$$

where $V_{t,T,x} \equiv V_{t,T,x,0}$ is the operator in (3.22). Now we remark that $V^n_{t,s,x}$ is a finite sum of the form

$$V^n_{t,s,x} = \sum_{\substack{0 \le j_1, \frac{j_2}{2}, j_3 \le n \\ j_1 + j_2 - j_3 \ge n}} b^n_{j_1,j_2,j_3} (x - \bar{x})^{j_1} (\sqrt{s-t})^{j_2} \partial_x^{j_3}$$

for some constants $b^n_{j_1,j_2,j_3}$. Thus the integral in (6.56) is a linear combination of terms of the form

$$(x - \bar{x})^{j_1} (\sqrt{T-s})^{j+2+j_2} \partial_x^{k+2+j_3} G^0(t, x; T, y)$$

where

$$0 \le j_1, \frac{j_2}{2}, j_3 \le h + i, \tag{6.57}$$

$$j_1 + j_2 - j_3 \ge h + i. \tag{6.58}$$

Eventually we have

$$j_1 + j + j_2 + 2 - (k + 2 + j_3) \ge$$

(by (6.58))

$$\ge i + j - k + h \ge$$

(by (6.52))

$$\ge n + 1.$$

On the other hand, by (6.57) and (6.53) we have

$$j_1 \le h + i \le n + 1.$$

Moreover, by (6.57), (6.53) and (6.54) we have

$$j + 2 + j_2 \le j + 2 + 2(n+1) \le n(n+3) + 2 + 2(n+1) = (n+1)(n+4).$$

Finally, by (6.57), (6.53) and (6.55) we have

$$k + 2 + j_3 \le k + 2 + h + i \le k + n + 3 \le \frac{n(n+5)}{2} + n + 3 = \frac{(n+1)(n+6)}{2}.$$

This concludes the proof. □

Now we set $\bar{x} = y$ and prove the thesis only in this case: to treat the case $\bar{x} = x$, it suffices to proceed in a similar way by using the backward parametrix method introduced in [62].

Lemma 6.2. *For any $\epsilon, \tau > 0$ there exists a positive constant C, only dependent on $\varepsilon, \tau, m, M, N$ and $\max_{k \le N} \|\alpha_k\|_\infty$, such that*

$$|\partial_{xx} G^n(t, x; T, y)| \le C(T-t)^{\frac{n-2}{2}} \bar{\Gamma}^{M+\epsilon}(t, x; T, y), \tag{6.59}$$

for any $n \le N$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \le \tau$.

Proof. By Lemma 6.1 with $\bar{x} = y$, we have

$$|\partial_{xx} G^n(t,x;T,y)| \le \sum_{\substack{i\le n,\, j\le n(n+3),\, k\le \frac{n(n+5)}{2} \\ i+j-k\ge n}} |c^n_{i,j,k}| \left(\sqrt{T-t}\right)^j \left|\partial_{xx}\left((x-y)^i \partial^k_x G^0(t,x;T,y)\right)\right|.$$

Then the thesis follows from the boundedness of the coefficients α_k, $k \le N$, (cf. Assumption A_N) and the following standard Gaussian estimates (see, for instance, Lemma A.1 and A.2 in [62]):

$$\begin{aligned} \partial^k_x G^0(t,x;T,y) &\le c\left(\sqrt{T-t}\right)^{-k} \bar{\Gamma}^{M+\epsilon}(t,x;T,y), \\ \left(\frac{x-y}{\sqrt{T-t}}\right)^k G^0(t,x;T,y) &\le c\,\bar{\Gamma}^{M+\epsilon}(t,x;T,y), \end{aligned} \tag{6.60}$$

where c is a positive constant which depends on k, m, M, ε and τ. □

Lemma 6.3. *For any $\epsilon, \tau > 0$ there exists a positive constant C, only dependent on $\varepsilon, \tau, m, M, N$ and $\max_{k\le N+1} \|\alpha_k\|_\infty$, such that*

$$\left|Z^N_n(t,x;T,y)\right| \le \kappa_n (T-t)^{\frac{N+n-1}{2}} \bar{\Gamma}^{M+\epsilon}(t,x;T,y),$$

for any $n \in \mathbb{N}$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T-t \le \tau$, where

$$\kappa_n = C^n \frac{\Gamma_E\left(\frac{1+N}{2}\right)}{\Gamma_E\left(\frac{n+1+N}{2}\right)}$$

and Γ_E denotes the Euler Gamma function.

Proof. On the basis of definitions (2.12) and (2.13), by induction we can prove the following formula:

$$Z^N_0(z;\zeta) = L\Gamma^N(z;\zeta) = \sum_{n=0}^{N} (L - L_n) G^{N-n}(z;\zeta). \tag{6.61}$$

Indeed, for $N = 0$ we have

$$L\Gamma^0(z;\zeta) = (L - L_0) G^0(z;\zeta),$$

because $L_0 G^0(z;\zeta) = 0$ by definition. Then, assuming that (6.61) holds for $N \in \mathbb{N}$, for $N+1$ we have

$$L\Gamma^{N+1}(z;\zeta) = L\Gamma^N(z;\zeta) + LG^{N+1}(z;\zeta) =$$

(by inductive hypothesis and (2.13))

$$\begin{aligned} &= \sum_{n=0}^{N} (L - L_n) G^{N-n}(z;\zeta) + (L - L_0) G^{N+1}(z;\zeta) \\ &\quad - \sum_{n=1}^{N+1} (L_n - L_{n-1}) G^{N+1-n}(z;\zeta) \end{aligned}$$

$$= \sum_{n=1}^{N+1} (L - L_{n-1}) G^{N-(n-1)}(z;\zeta) + (L - L_0) G^{N+1}(z;\zeta)$$
$$- \sum_{n=1}^{N+1} (L_n - L_{n-1}) G^{N+1-n}(z;\zeta)$$
$$= (L - L_0) G^{N+1} + \sum_{n=1}^{N+1} (L - L_n) G^{N+1-n}(z;\zeta)$$

from which (6.61) follows.

Then, by (6.61) and Assumption A_{N+1} we have

$$\left| Z_0^N(z;\zeta) \right| \leq \sum_{n=0}^{N} \|\alpha_{n+1}\|_\infty |x-y|^{n+1} \left| (\partial_{xx} - \partial_x) G^{N-n}(z;\zeta) \right|$$

and for $n = 0$ the thesis follows from estimates (6.59) and (6.60). In the case $n \geq 1$, proceeding by induction, the thesis follows from the previous estimates by using the arguments in Lemma 4.3 in [71]: therefore the proof is omitted. □

3.7 Figures and tables

		Call prices		Implied volatility (%)	
T	K	PPR 4-th	MC 95% c.i.	PPR 4-th	MC 95% c.i.
0.25	0.5	0.50669	0.50648 – 0.50666	57.81	54.03 – 57.31
	0.75	0.26324	0.26304 – 0.26321	37.91	37.48 – 37.84
	1	0.05515	0.05501 – 0.05514	24.58	24.50 – 24.57
	1.25	0.00645	0.00637 – 0.00645	30.48	30.39 – 30.49
	1.5	0.00305	0.00300 – 0.00306	42.05	41.93 – 42.07
1	0.5	0.52720	0.52700 – 0.52736	38.82	38.35 – 39.20
	1	0.13114	0.13097 – 0.13125	27.06	27.01 – 27.08
	1.5	0.01840	0.01836 – 0.01852	29.04	29.03 – 29.10
	2	0.00566	0.00566 – 0.00575	34.45	34.45 – 34.55
	2.5	0.00209	0.00208 – 0.00214	37.65	37.62 – 37.77
10	0.5	0.72942	0.72920 – 0.73045	32.88	32.81 – 33.21
	1	0.52316	0.52293 – 0.52411	29.67	29.64 – 29.80
	5	0.05625	0.05604 – 0.05664	26.12	26.09 – 26.17
	10	0.01241	0.01091 – 0.01126	27.05	26.54 – 26.66
	15	0.00933	0.00369 – 0.00393	30.22	27.03 – 27.22

Table 3.1: Merton-CEV with $\lambda = 30\%$, $\delta = 40\%$ and $m = -10\%$.

T	K	Call prices PPR 4-th	Call prices MC 95% c.i.	Implied volatility (%) PPR 4-th	Implied volatility (%) MC 95% c.i.
0.25	0.5	0.50705	0.50688 – 0.5071	61.91	60.09 – 62.31
	0.75	0.26579	0.26562 – 0.26582	42.46	42.20 – 42.52
	1	0.06098	0.06086 – 0.06102	27.53	27.47 – 27.56
	1.25	0.01039	0.01030 – 0.01041	34.61	34.53 – 34.63
	1.5	0.00513	0.00506 – 0.00514	46.24	46.13 – 46.26
1	0.5	0.52935	0.52926 – 0.52969	42.98	42.83 – 43.54
	1.	0.14732	0.14719 – 0.14752	31.32	31.28 – 31.37
	1.5	0.02933	0.02922 – 0.02942	33.48	33.44 – 33.52
	2.	0.01020	0.01011 – 0.01024	38.43	38.36 – 38.47
	2.5	0.00414	0.00407 – 0.00416	41.39	41.29 – 41.43
10	0.5	0.74509	0.74480 – 0.74644	37.56	37.48 – 37.95
	1	0.56118	0.56088 – 0.56244	34.65	34.61 – 34.81
	5	0.10586	0.10530 – 0.10631	31.41	31.36 – 31.45
	10	0.03283	0.03120 – 0.03191	31.82	31.52 – 31.65
	15	0.01861	0.01362 – 0.01417	33.31	31.82 – 32.00

Table 3.2: Merton-CEV with $\lambda = 50\%$, $\delta = 40\%$ and $m = -10\%$.

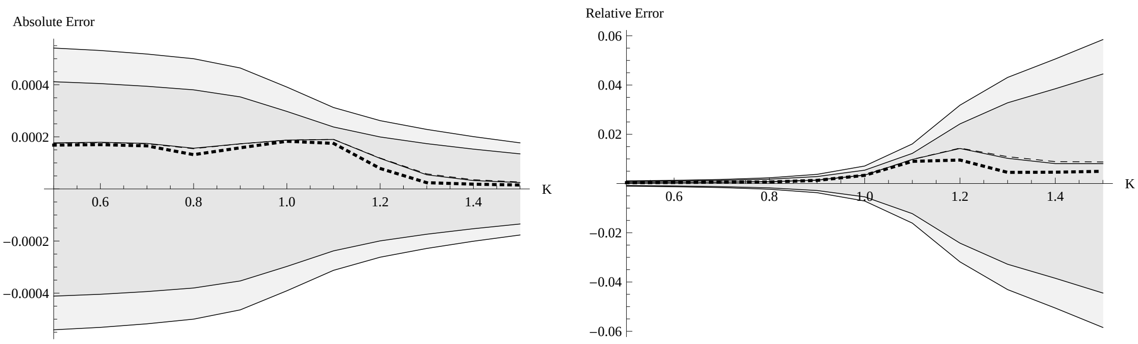

Figure 3.1: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** model ($\lambda = 30\%$) with maturity $\mathbf{T = 0.25}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{1.5}]$. The shaded bands show the 95% (dark gray) and 99% (light gray) Monte Carlo confidence regions

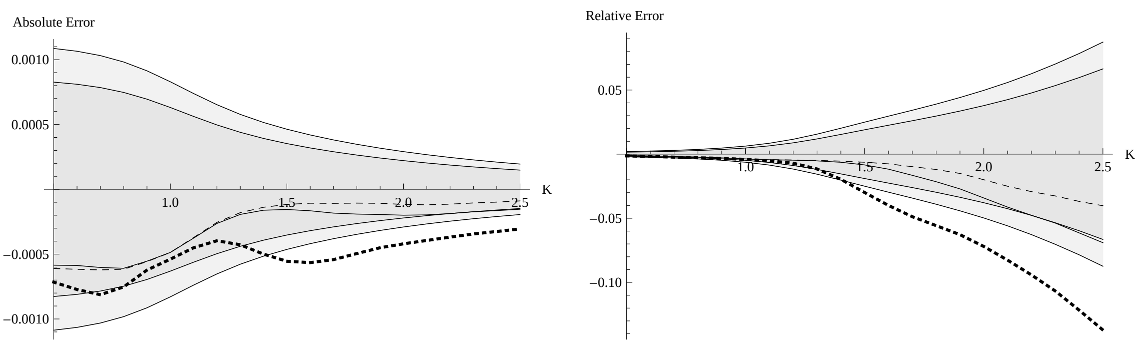

Figure 3.2: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** model ($\lambda = 30\%$) with maturity $\mathbf{T = 1}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{2.5}]$

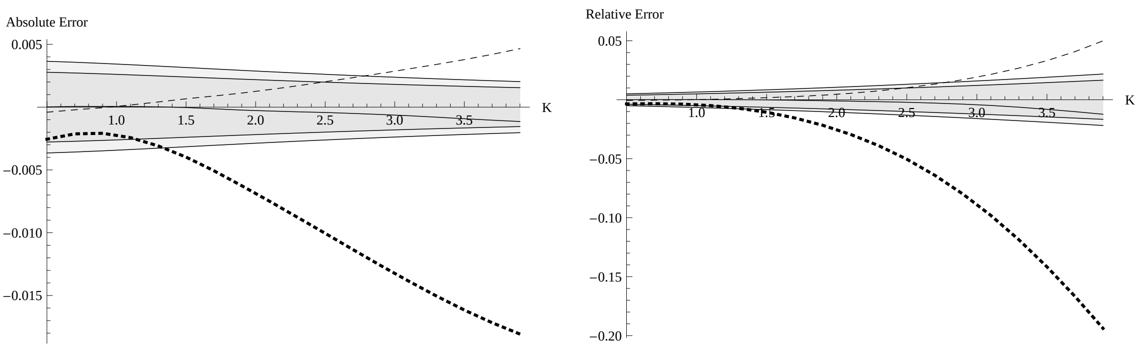

Figure 3.3: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** ($\lambda = 30\%$) model with maturity $\mathbf{T = 10}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{4}]$

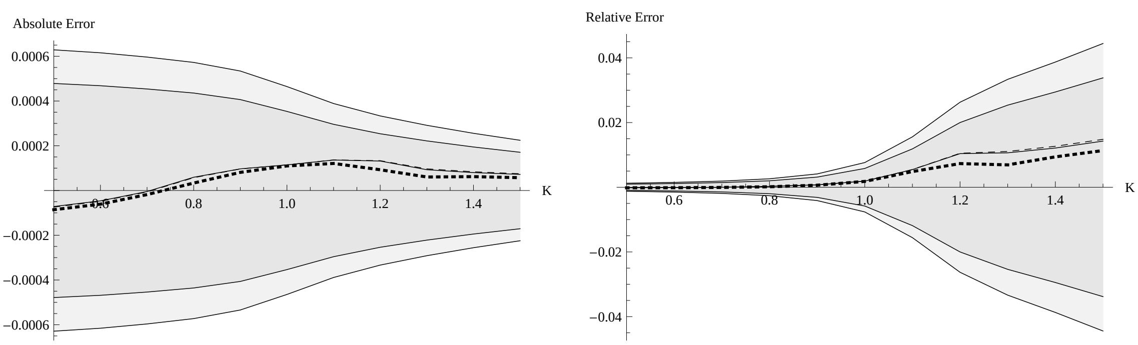

Figure 3.4: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** model ($\lambda = 50\%$) with maturity $\mathbf{T = 0.25}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{1.5}]$. The shaded bands show the 95% (dark gray) and 99% (light gray) Monte Carlo confidence regions

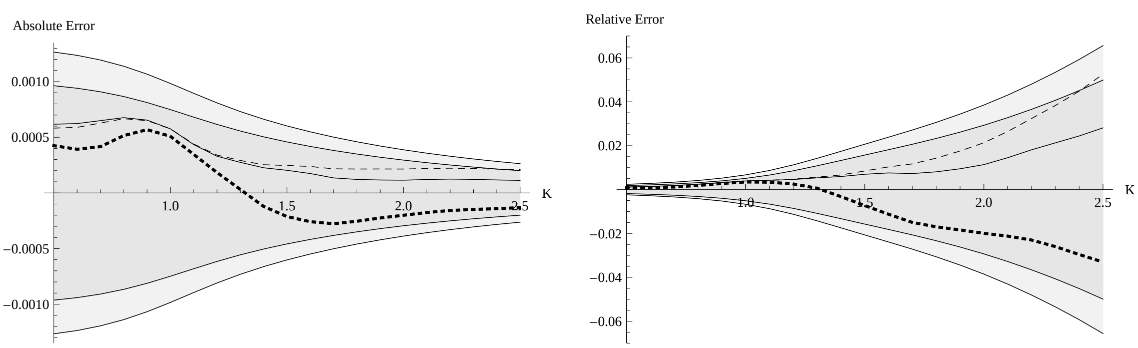

Figure 3.5: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** model ($\lambda = 50\%$) with maturity $\mathbf{T = 1}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{2.5}]$

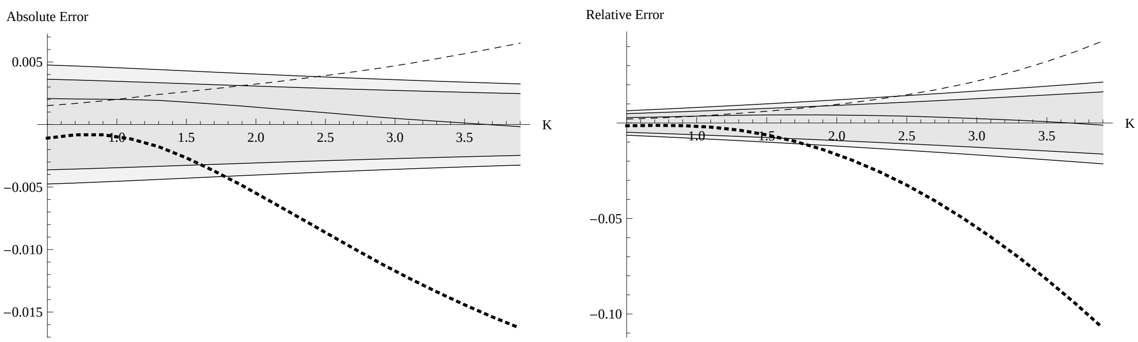

Figure 3.6: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** ($\lambda = 50\%$) model with maturity $\mathbf{T = 10}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{4}]$

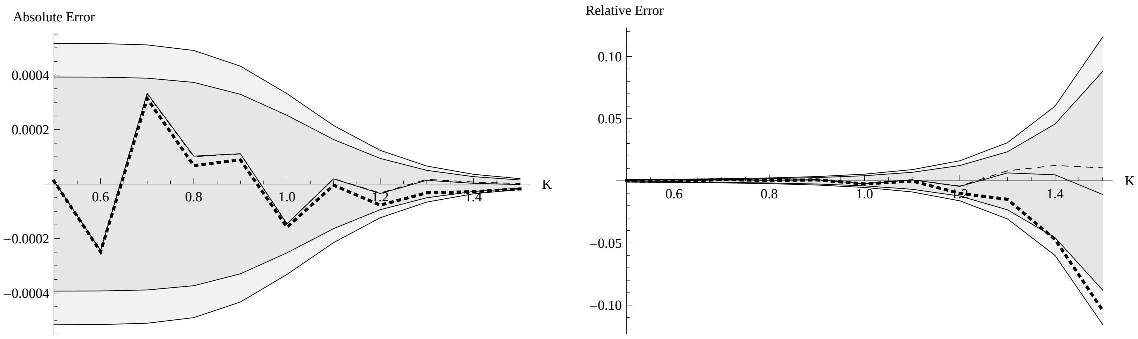

Figure 3.7: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Variance-Gamma** model with maturity $\mathbf{T = 0.25}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{1.5}]$. The shaded bands show the 95% (dark gray) and 99% (light gray) Monte Carlo confidence regions

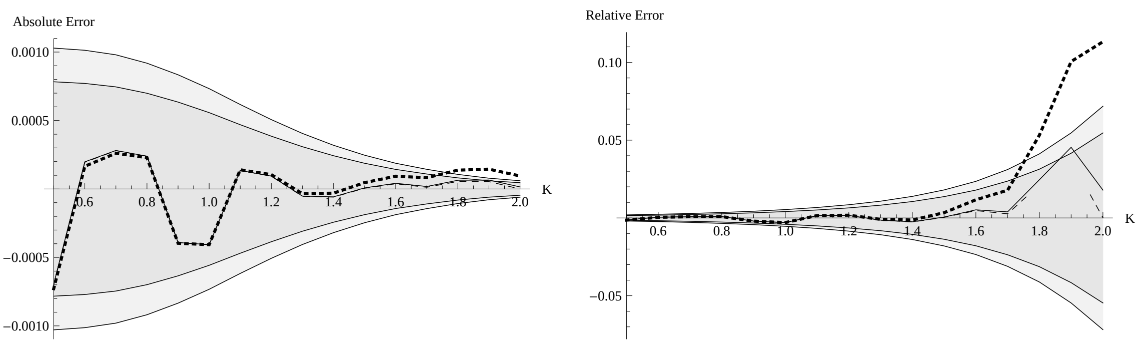

Figure 3.8: Absolute (left) and relative (right) errors of the 2nd (dotted line), 3rd (dashed line), 4th (solid line) order approximations of a Call price in the **CEV-Variance-Gamma** model with maturity $\mathbf{T = 1}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{2.5}]$

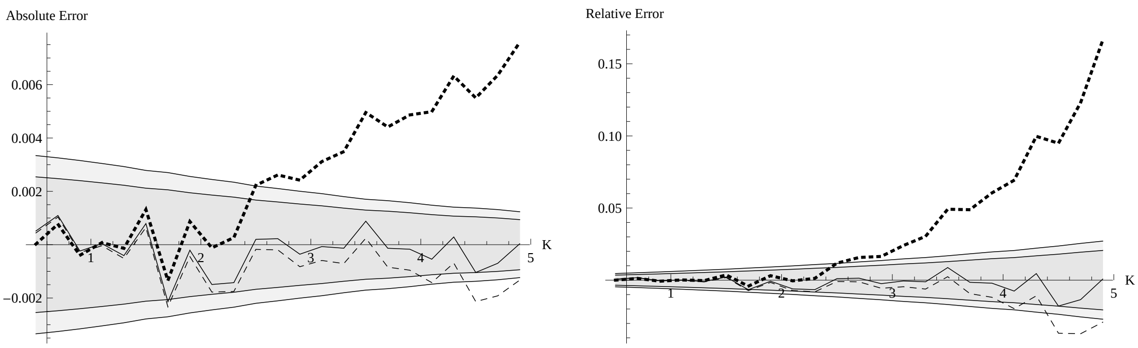

Figure 3.9: Absolute (left) and relative (right) errors of the 2nd (dotted line), 3rd (dashed line), 4th (solid line) order approximations of a Call price in the **CEV-Variance-Gamma** model with maturity $\mathbf{T = 10}$ and strike $\mathbf{K \in [0.5, 5]}$

		Call prices		Implied volatility (%)	
T	K	PPR 4-th	MC 95% c.i.	PPR 4-th	MC 95% c.i.
	0.8	0.23708	0.23704 − 0.23722	55.61	55.57 − 55.72
	0.9	0.15489	0.15482 − 0.15497	47.09	47.05 − 47.14
0.25	1	0.08413	0.08403 − 0.08415	39.29	39.24 − 39.30
	1.1	0.03436	0.03426 − 0.03433	33.27	33.22 − 33.26
	1.2	0.00968	0.00961 − 0.00965	29.28	29.21 − 29.25
	0.5	0.54643	0.54630 − 0.54679	61.02	60.91 − 61.30
	0.75	0.35456	0.35438 − 0.35479	52.35	52.28 − 52.44
1	1	0.20071	0.20049 − 0.20082	45.42	45.36 − 45.45
	1.5	0.03394	0.03374 − 0.03387	35.16	35.09 − 35.14
	2	0.00188	0.00185 − 0.00188	29.08	29.01 − 29.07
	0.5	0.80150	0.80279 − 0.80502	52.60	52.95 − 53.53
	1	0.66691	0.66775 − 0.66990	49.09	49.21 − 49.52
10	5	0.22948	0.22836 − 0.22986	42.02	41.93 − 42.05
	10	0.08664	0.08418 − 0.08518	39.21	38.93 − 39.05
	15	0.04058	0.03607 − 0.03676	37.93	37.13 − 37.26

Table 3.3: VG-CEV with $\kappa = 15\%$, $\theta = -10\%$ and $\sigma = 20\%$.

Chapter 4

Adjoint expansions in local Lévy models enhanced with Heston-type volatility

Based on a joint work ([184]) with Prof. A. Pascucci.

Abstract: we present new approximation formulas for local stochastic volatility models, possibly including Lévy jumps. Our main result is an expansion of the characteristic function, which is worked out in the Fourier space. Combined with standard Fourier methods, our result provides efficient and accurate formulas for the prices and the Greeks of plain vanilla options. We finally provide numerical results to illustrate the accuracy with real market data.

Keywords: local stochastic volatility, Lévy process, analytical approximation, characteristic function, partial integro-differential equation, Fourier methods.

4.1 Introduction

Over the last decade local stochastic volatility (LSV) models have become an industry standard for option pricing. Nowadays several financial institutions have incorporated LSV into their front office systems. Also the literature on LSV is rapidly increasing and demonstrates the ongoing interest in such models (we refer, for instance, to [133], [159], [2], [194], [120], [55], [199], [99] and [84]); an exhaustive overview on the use of LSV in foreign exchange markets can be found in [58] and [158]. More generally, LSV models including Lévy jumps (i.e. JLSV models) have also been proposed with the aim of increasing the smile in the short end: see, for instance, [160], [48] and the bibliography therein.

The drawback of LSV and JLSV models is that the more realistic dynamics comes at the cost of an additional theoretical complexity and a greater difficulty in the numerical solution of the pricing problem. For instance, a typical calibration of a JLSV model requires efficient numerical schemes for the solution of a two-dimensional partial integro-differential equation.

In Chapter 3 (see also [183] and [185]) we proposed the so-called adjoint expansion (AE) method, a general technique which yields accurate approximations of the characteristic function of a general class of processes. Basically, the AE method is an evolution of the well-known expansion formula by Hagan and Woodward in their pioneering paper [115] on the CEV model. The purpose of this chapter is to show that the AE method is sufficiently general to provide efficient approximation formulas for the price and the Greeks of plain vanilla options in a particular JLSV model. The advantages of approximation formulas are well known: under standard parameter regimes they are generally fast, accurate and keep track of the qualitative model information.

Throughout this chapter we assume the following risk-neutral dynamics for the log-price of the underlying asset:

$$\begin{cases} dX_t = \left(\bar{r} - \dfrac{\sigma^2(t, X_t)v_t}{2}\right) dt + \sigma(t, X_t)\sqrt{v_t}dW_t^1 + dZ_t, \\ dv_t = k(\theta - v_t)dt + \eta\sqrt{v_t}\left(\varrho dW_t^1 + \sqrt{1-\varrho^2}dW_t^2\right), \end{cases} \tag{1.1}$$

where $W = \left(W^1, W^2\right)$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the usual assumptions on the filtration, Z is a pure-jump Lévy process, independent of W, with Lévy triplet $(0, 0, \nu)$. We suppose that

$$\int_{|y|\geq 1} e^y \nu(dy) < \infty,$$

which is a quite reasonable integrability condition in financial applications. If the short rate and the dividend yield are denoted by r and q respectively, then $\bar{r}$ in (1.1) is determined by imposing that the discounted asset price $\widetilde{S}_t := e^{-(r-q)t+X_t}$ is a martingale: thus we have

$$\bar{r} = r - q - \int_{\mathbb{R}} \left(e^y - 1 - y\mathbb{1}_{\{|y|<1\}}\right) \nu(dy).$$

The relevant quantities in (1.1) are the local volatility *function* σ, the Lévy measure ν and the variance parameters (initial variance v_0, speed of mean reversion k, long-term variance θ, vol-of-vol η and correlation ϱ).

We remark explicitly that the AE method applies to a general SDE for the variance process

$$dv_t = a(t, v_t)dt + b\,(t, v_t)\left(\varrho dW_t^1 + \sqrt{1-\varrho^2}dW_t^2\right).$$

The general case will be treated subsequently in Chapter 6, whereas in this chapter, preferring simplicity over generality, we assume the dynamics (1.1) which seems to be sufficiently comprehensive to include several classical models. Just to quote a few, particular cases of (1.1) are *in the purely diffusive case* ($\nu \equiv 0$): the classical Heston model [121] (with σ constant), the "quadratic" LSV model by Lipton [159] (see also the so-called Tremor model [211]), the CEV-Heston type model proposed in [55]. *In the case of models with jumps:* the Merton [177] and Bates [20] models (with constant σ and a Gaussian Lévy measure ν), the local Lévy models [48] (with $k = \eta = 0$ and generic local volatility function σ and Lévy measure ν), the "Universal vol model" by Lipton [159] (with a generic local volatility function σ and a Poisson Lévy measure ν).

Another reason why we focus on the specific dynamics (1.1) is that the characteristic function of the square-root process is known explicitly and this perfectly fits the AE method, because no further approximation is introduced besides the LV's one. Furthermore, the mean-reverting dynamics is popular and widely used.

Let us explicitly remark that we are interested in computing an approximation of the two-dimensional characteristic function of (S_t, v_t), and not only of $E\left[e^{i\xi X_t}\right]$. This is important in the study of volatility derivatives, such as options on quadratic variation that have recently become a very popular instrument in financial markets. Also, our result can be used for volatility calibration purposes by Markovian projection methods via Gyöngy's lemma [113] (see, for instance, [190] and [105]). This will be object of a future investigation.

Comparison with the literature on analytical approximations. Analytical approximations and their applications to finance have been studied by several authors in the last decades. Nevertheless, to the best of our knowledge, no other analytical approximation formulae for local stochastic volatility models (possibly also incorporating Lévy jumps) is available in the literature. As a matter of fact, the large body of the existing literature (see, for instance, [115], [29], [125], [87], [209], [9], [106], [26], [62], [135]) is mainly devoted to purely diffusive (local or stochastic volatility) models or, as in [25] and [212], to local volatility models with Poisson jumps that is a very particular case because approximation is still possible by using Gaussian kernels.

Differently from other asymptotic and perturbation methods, for the AE approximation we have explicit error bounds (cf. [185]) albeit for a restricted class of models; namely, the results of Chapter 3 (see Theorem 2.3) apply to uniform parabolic pricing PDEs. Extension to degenerate equations or partial integro-differential equations related to models with jumps seems to be possible, even though further research is required.

The rest of this chapter is organized as follows: In Section 4.2 we set the notations and introduce the AE approach. The expansion of characteristic functions is presented in Section 4.3. In Subsection 4.3.1 we compute the characteristic function of the process (X_t, v_t) when the volatility function σ is constant: in this setting we give an alternative proof of some known results. Our main results are contained in Theorem 3.10 where we derive a first-order approximation formula for the characteristic function of (1.1). Next, in Subsection 4.3.3 we show how to get the second order approximation. Since the explicit approximation formula is rather long, for greater convenience we have implemented it using Mathematica; the notebook is freely available. Section 4.4 is devoted to the numerical experiments; we test the accuracy of the approximation in models with and without jumps and for different specifications of the volatility function, including the classical CEV volatility model. Finally, the Appendix contains some technical proofs, and the explicit expression of the coefficients composing our approximate expansion.

4.2 Approximation methodology: the Adjoint Expansion

In this section we briefly extend the approximation scheme proposed in Chapter 3 (see also [183], [185]) and generalize it to JLSV models. To introduce the general methodology, we first fix the notations.

Let $\Gamma = \Gamma(x, v; t, y, w)$ denote the transition density of the process $(X_t, v_t)_{t\geq 0}$ in (1.1) starting from $(x, v) \in \mathbb{R} \times \mathbb{R}_+$ at time 0. Since the starting point (x, v) is fixed once and for all, in the sequel we shall systematically use the shorthand notation $\Gamma(t, y, w)$ in place of $\Gamma(x, v; t, y, w)$.

For simplicity we assume time-independent coefficients even if our method applies to the general case without any difficult except the length of the formulas. Moreover, we assume that the vol-of-vol parameter in (1.1) is equal to one:

$$\eta = 1.$$

This assumption is not really restrictive because the change of variables

$$v_t = \eta^2 \widetilde{v}_t, \qquad \sigma(x) = \frac{\widetilde{\sigma}(x)}{\eta}, \qquad \theta = \eta^2 \widetilde{\theta},$$

transforms (1.1) into

$$\begin{aligned} dX_t &= \left(\bar{r} - \frac{\widetilde{\sigma}^2(X_t)\widetilde{v}_t}{2}\right) dt + \widetilde{\sigma}(X_t)\sqrt{\widetilde{v}_t} dW_t^1 + dZ_t, \\ d\widetilde{v}_t &= k(\widetilde{\theta} - \widetilde{v}_t)dt + \sqrt{\widetilde{v}_t}\left(\varrho dW_t^1 + \sqrt{1-\varrho^2} dW_t^2\right). \end{aligned}$$

We recall that Z is a pure-jump Lévy process, independent of W, with Lévy triplet $(0, 0, \nu)$. This means that

$$dZ_t = \int_{|y|\geq 1} yJ(dt, dy) + \int_{|y|<1} y\widetilde{J}(dt, dy),$$

where J and $\widetilde{J}$ are the jump measure and the compensated jump measure of Z respectively.

To introduce the Adjoint Expansion method, we consider the Kolmogorov PIDE of (X_t, v_t) in (1.1):

$$\begin{aligned} Lu(t, x, v) = &\frac{v}{2}\left(\sigma^2(x)\left(\partial_{xx} - \partial_x\right) + 2\varrho\sigma(x)\partial_{xv} + \partial_{vv}\right) u(t, x, v) \\ &+ \bar{r}\partial_x u(t, x, v) + k(\theta - v)\partial_v u(t, x, v) + \partial_t u(t, x, v) \\ &+ \int_{\mathbb{R}} \left(u(t, x + y, v) - u(t, x, v) - \partial_x u(t, x, v) y \mathbb{1}_{\{|y|<1\}}\right) \nu(dy). \end{aligned} \tag{2.2}$$

We fix a basepoint $\bar{x} \in \mathbb{R}$ and define the n^{th} order approximation of L:

$$
\begin{aligned}
L_n u(t,x,v) = {} & \frac{v}{2}\left(T_{n,\sigma^2}(x)\left(\partial_{xx}-\partial_x\right)+2\varrho T_{n,\sigma}(x)\partial_{xv}+\partial_{vv}\right)u(t,x,v)\\
& + \bar{r}\partial_x u(t,x,v)+k(\theta-v)\partial_v u(t,x,v)+\partial_t u(t,x,v)\\
& + \int_{\mathbb{R}}\left(u(t,x+y,v)-u(t,x,v)-\partial_x u(t,x,v)y\mathbb{1}_{\{|y|<1\}}\right)\nu(dy).
\end{aligned}
\tag{2.3}
$$

In (2.3) $T_{n,f}(x)$ denotes the n^{th}-order Taylor polynomial of the function f around $\bar{x}$. In particular the 0^{th} order approximation is given by

$$
\begin{aligned}
L_0 u(t,x,v) = {} & \frac{v}{2}\left(\bar{\sigma}_0^2\left(\partial_{xx}-\partial_x\right)+2\varrho\bar{\sigma}_0\partial_{xv}+\partial_{vv}\right)u(t,x,v)\\
& + \bar{r}\partial_x u(t,x,v)+k(\theta-v)\partial_v u(t,x,v)+\partial_t u(t,x,v)\\
& + \int_{\mathbb{R}}\left(u(t,x+y,v)-u(t,x,v)-\partial_x u(t,x,v)y\mathbb{1}_{\{|y|<1\}}\right)\nu(dy).
\end{aligned}
\tag{2.4}
$$

where $\bar{\sigma}_0 = \sigma(\bar{x})$. Then L_0 is simply obtained by freezing the volatility coefficient σ at $\bar{x}$.

Remark 2.1. *Again, the choice of $\bar{x}$ is somewhat arbitrary. However, a convenient choice that seems to work well in most applications is to choose $\bar{x}$ near X_t, i.e. the current level of X. See Remark 4.2 for further explanations.*

Following [183] and [185], the n^{th}-order expansion of the density Γ of (X_t, v_t) is given by

$$
\Gamma_n(t,y,w)=\sum_{j=0}^{n} G_j(t,y,w), \qquad (t,y,w)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+, \tag{2.5}
$$

where $G_0 \equiv \Gamma_0$ is the transition density of the process in (1.1) with $\sigma \equiv \sigma(\bar{x})$ and $G_j = G_j(t,y,w)$, for $j \geq 1$, is the solution of the Cauchy-Dirichlet problem

$$
\begin{aligned}
& \widetilde{L}_0 G_j = -\sum_{h=1}^{j}\left(\widetilde{L}_h-\widetilde{L}_{h-1}\right)G_{j-h}, && \text{in } \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+,\\
& G_j(0,y,w)=0, && (y,w)\in\mathbb{R}\times\mathbb{R}_+,\\
& G_j(t,y,0)=0, && (t,y)\in\mathbb{R}_+\times\mathbb{R},
\end{aligned}
\tag{2.6}
$$

where $\widetilde{L}_n$ is the adjoint operator of L_n in (2.3). More explicitly, we have

$$
\begin{aligned}
\widetilde{L}_0 u(t,y,w) = {} & \frac{1}{2}\left(\bar{\sigma}_0^2\left(\partial_{yy}+\partial_y\right)+2\varrho\bar{\sigma}_0\partial_{yw}+\partial_{ww}\right)\left(wu(t,y,w)\right)\\
& - \bar{r}\partial_y u(t,y,w)-\partial_w\left(k(\theta-w)u(t,y,w)\right)-\partial_t u(t,y,w)\\
& + \int_{\mathbb{R}}\left(u(t,y+z,w)-u(t,y,w)-\partial_y u(t,y,w)z\mathbb{1}_{\{|z|<1\}}\right)\bar{\nu}(dz).
\end{aligned}
\tag{2.7}
$$

where $\bar{\nu}(dz) = \nu(-dz)$ denotes the Lévy measure with reverted jumps. Moreover we have

$$
\left(\widetilde{L}_h-\widetilde{L}_{h-1}\right)u(t,y,w)=\frac{1}{h!}\left(\frac{\bar{\bar{\sigma}}_h}{2}\left(\partial_{yy}+\partial_y\right)+\varrho\bar{\sigma}_h\partial_{yw}\right)\left(w(y-\bar{x})^h u(t,y,w)\right), \tag{2.8}
$$

where

$$\bar{\sigma}_n = \frac{d^n \sigma}{dx^n}(\bar{x}), \qquad \bar{\bar{\sigma}}_n = \frac{d^n \sigma^2}{dx^n}(\bar{x}), \qquad n \in \mathbb{N} \cup \{0\}.$$

The rationale behind the previous definitions is that the recursive sequence of problems (2.6) provides an iterative approximation of the transition density where the main term is the density G_0 of the process with constant diffusion (roughly speaking, related to a Heston model with Lévy jumps); the higher order terms are corrections that can be expressed as derivatives of Γ_0.

Remark 2.2. *The question whether asymptotic error estimates similar to the ones proved in Theorem 3.2 do hold in this setting, is a challenging task that is not deeply investigated here. The additional difficulty is due to the degenerate nature of the pricing operator L, whose differential part is not uniformly parabolic. The parametrix method used in the Chapter 3 to prove Theorem 3.2, is general enough to be adapted to this degenerate framework though, but some global pointwise estimates for the leading term Γ_0 in the expansion* (2.5) *would be necessary. We aim to come back to this problem in a forthcoming paper.*

4.3 Adjoint expansion of characteristic functions

The AE scheme is well suited for Fourier methods because problems (2.6) can be explicitly solved in the Fourier space. More precisely, let us consider the characteristic function of (X_t, v_t), defined as the Fourier-Laplace transform of Γ:

$$\hat{\Gamma}(t, \xi, \omega) := E\left[e^{i\xi X_t - \omega v_t}\right] = \int_{\mathbb{R}} \int_0^{\infty} e^{i\xi y - \omega w} \Gamma(t, y, w) dy dw,$$

for $(t, \xi, \omega) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$. By applying the Fourier-Laplace transform to (2.5), we get the following expansion of the characteristic function

$$\hat{\Gamma}(t, \xi, \omega) \;\approx\; \hat{\Gamma}_n(t, \xi, \omega) = \sum_{j=0}^{n} \hat{G}_j(t, \xi, \omega). \tag{3.9}$$

The following proposition shows that the functions $\hat{G}_j$ in (3.9) are solutions of *first order* partial differential equations; this allows us to compute them explicitly as we shall do in the next subsections.

Proposition 3.1.

i) The function $\hat{G}_0 = \hat{\Gamma}_0$ solves the Cauchy problem

$$\begin{cases} -Y\hat{\Gamma}_0(t, \xi, \omega) + A(\xi, \omega)\hat{\Gamma}_0(t, \xi, \omega) = 0, & (t, \xi, \omega) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \\ \hat{\Gamma}_0(0, \xi, \omega) = e^{ix\xi - v\omega}, & (\xi, \omega) \in \mathbb{R} \times \mathbb{R}_+, \end{cases} \tag{3.10}$$

where

$$A(\xi, \omega) = -k\theta\omega + i\bar{r}\xi + \phi(\xi),$$

with

$$\phi(\xi) = \int_{\mathbb{R}} \left(e^{iz\xi} - 1 - iz\xi \mathbb{1}_{\{|z|<1\}}\right) \nu(dz) \tag{3.11}$$

and Y is the first order differential operator

$$Y = \partial_t + B(\xi,\omega)\partial_\omega,$$

with

$$B(\xi,\omega) = \alpha(\xi) - \beta(\xi)\omega + \frac{\omega^2}{2}, \qquad \alpha(\xi) = -\frac{\bar{\sigma}_0^2}{2}\xi(\xi+i), \qquad \beta(\xi) = i\xi\varrho\bar{\sigma}_0 - k; \quad (3.12)$$

ii) for $j = 1,2$ the function $\hat{G}_j$ solves the Cauchy problem

$$\begin{cases} -Y\hat{G}_j(t,\xi,\omega) + A(\xi,\omega)\hat{G}_j(t,\xi,\omega) = \hat{\Gamma}_0(t,\xi,\omega)\,\mathcal{H}_j(t,\xi,\omega), \\ \hat{G}_j(0,\xi,\omega) = 0, \end{cases} \quad (3.13)$$

where

$$\mathcal{H}_1(t,\xi,\omega) = \bar{\sigma}_1\xi\left(\bar{\sigma}_0(\xi+i) + i\varrho\omega\right)\left(i\left(\partial_\xi\psi_0\partial_\omega\psi_0 + \partial_{\xi\omega}\psi_0\right) + \bar{x}\partial_\omega\psi_0\right)(t,\xi,\omega), \quad (3.14)$$

with (see also Proposition 3.3 below for a more explicit expression of ψ_0)

$$\psi_0(t,\xi,\omega) = \log\hat{\Gamma}_0(0,\xi,\omega),$$

while, due to its longer expression, $\mathcal{H}_2$ is given in Subsection 4.3.3.

Part *i)* of Proposition 3.1 is proved in Subsection 4.3.1 where we also solve Cauchy problem (3.10) and compute $\hat{\Gamma}_0$ explicitly. Part *ii)* of Proposition 3.1 is proved in Subsections 4.3.2 and 4.3.3. In the case of local Lévy models, in [185] we derived explicit fourth-order formulas (i.e. with $n = 4$ in (3.9)) for the characteristic function. When, as in the present paper, stochastic volatility is included, the approximation formulas become definitely more involved: for this reason, in this more general setting we only compute $\hat{G}_n$ for $n = 1,2$.

4.3.1 0^{th} order approximation

The purpose of this section is to derive the 0^{th} order approximation $\hat{\Gamma}_0$ of the characteristic function of (X_t, v_t), obtained by freezing the volatility coefficient. First of all we prove the first part of Proposition 3.1.

Proof of Proposition 3.1-i). We set $U_t = e^{i\xi X_t - \omega v_t}$. By the Itôformula (see, for instance, formula (14.56) in [188]) we have

$$dU_t = \left(B(\xi,\omega)v_t + A(\xi,\omega)\right)U_t dt + dM_t \quad (3.15)$$

where

$$M_t = \int_0^t U_s\sqrt{v_s}\left(\left(i\xi\bar{\sigma}_0 - \omega\varrho\right)dW_s^1 - \omega\sqrt{1-\varrho^2}dW_s^2\right) + \int_0^t U_{s-}\int_{\mathbb{R}}\left(e^{i\xi y} - 1\right)\widetilde{J}(ds,dy)$$

is a local martingale. Actually, M is also a strict martingale because

$$E\left[\int_0^t |U_s|^2 v_s ds\right] = E\left[\int_0^t e^{-2\omega v_s} v_s ds\right] \leq E\left[\int_0^t \frac{1}{2e\omega}ds\right] = \frac{t}{2e\omega},$$

and

$$E\left[\int_0^t\int_{\mathbb{R}}\left|U_s\left(e^{i\xi y}-1\right)\right|^2\nu(dy)ds\right]\leq T\int_{\mathbb{R}}\left|e^{i\xi y}-1\right|^2\nu(dy)<\infty$$

for any $\xi\in\mathbb{R}$. Therefore, taking the expectation in (3.15), we get

$$\hat{\Gamma}_0(t,\xi,\omega)=E\left[U_t\right]=e^{ix\xi-v\omega}+E\left[\int_0^t\left(B(\xi,\omega)v_s+A(\xi,\omega)\right)U_sds\right]=$$

(changing the order of integration and using that $E\left[v_sU_s\right]=-\partial_wE\left[U_s\right]$)

$$=e^{ix\xi-v\omega}+\int_0^t\left(-B(\xi,\omega)\partial_\omega\hat{\Gamma}_0(s,\xi,\omega)+A(\xi,\omega)\hat{\Gamma}_0(s,\xi,\omega)\right)ds.$$

The thesis follows by differentiating with respect to t. □

Remark 3.2. *Let L_0 in* (2.4) *be the Kolmogorov operator of (X_t,v_t) with $\sigma\equiv\bar{\sigma}_0$. It is well-known that the transition density Γ_0 of (X_t,v_t) (with $\sigma\equiv\bar{\sigma}_0$) is a fundamental solution of L_0. As such, Γ_0 solves the following Cauchy problem with a Dirac delta as initial datum*[1]*:*

$$\begin{cases}\widetilde{L}_0\Gamma_0(t,y,w)=0, & (t,y,w)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+,\\ \Gamma_0(0,\cdot,\cdot)=\partial_{(x,v)},\end{cases}\tag{3.16}$$

where $\widetilde{L}_0$ is the adjoint operator of L_0, which acts in the dual variables (t,y,w) and whose explicit expression is given in (2.7)*.*

At first glance, it seems that problem (3.10) *could be obtained from a direct application of the Fourier-Laplace transform to* (3.16)*. However, a rigorous analysis (see [89] and [174]) reveals that in general the fundamental solution of L is not unique*[2]*. This is a rather subtle point: in general, the transform of a fundamental solution satisfies* (3.10) *with an additional non-homogeneous term coming from some integration by parts in the Fourier-Laplace transformation; on the contrary, as shown by Proposition 3.1, the characteristic function is the solution of the homogeneous problem* (3.10)*. From an analytical perspective, non-uniqueness is related to the boundary condition at $w=0$: namely, when $2k\theta<\eta^2$ a boundary condition is required because the origin is attainable. This prevents us from applying directly the Fourier-Laplace transform to problem* (3.16) *because that problem does not identify uniquely the characteristic function of the underlying stochastic process.*

Problem (3.10) can be solved by the classical method of characteristics: precisely, let us denote by $\gamma^{t,\omega,\xi}=\gamma^{t,\omega,\xi}(s)$ the integral curve of Y, solution of the Cauchy problem

$$\begin{cases}\dfrac{d}{ds}\gamma^{t,\omega,\xi}(s)=Y\left(\gamma^{t,\omega,\xi}(s)\right),\\ \gamma^{t,\omega,\xi}(t)=(t,\omega),\end{cases}$$

[1] The initial datum is attained in the sense of measures, that is

$$\lim_{(t,x,v)\to(0,\bar{x},\bar{v})}\int_{\mathbb{R}\times\mathbb{R}_+}\Gamma_0(x,v;t,y,w)f(y,w)dydw=f(\bar{x},\bar{v})$$

for any continuous and bounded function f.

[2] A fundamental solution of L is a solution of (3.16) which is integrable together with its derivatives.

or more explicitly

$$\begin{cases} \dfrac{d}{ds}\gamma_1^{t,\omega,\xi}(s) = 1, \quad \gamma_1^{t,\omega,\xi}(t) = t \\ \dfrac{d}{ds}\gamma_2^{t,\omega,\xi}(s) = B\left(\xi, \gamma_2^{t,\omega,\xi}(s)\right), \quad \gamma_2^{t,\omega,\xi}(t) = \omega. \end{cases} \tag{3.17}$$

Then we have

$$\hat{\Gamma}_0(t,\xi,\omega) = e^{\psi_0(t,\xi,\omega)} \tag{3.18}$$

where

$$\psi_0(t,\xi,\omega) = ix\xi - v\gamma_2^{t,\omega,\xi}(0) + \int_0^t A\left(\xi, \gamma_2^{t,\omega,\xi}(s)\right) ds. \tag{3.19}$$

The following proposition provides the explicit expression on the integral curve $\gamma^{t,\omega,\xi}$. First we notice that the coefficient B in (3.12) can be written in the equivalent form

$$B(\xi,\omega) = \frac{1}{2}\left(\omega - a(\xi)\right)\left(\omega - b(\xi)\right)$$

where

$$a(\xi) = \beta(\xi) + D(\xi), \quad b(\xi) = \beta(\xi) - D(\xi), \quad D(\xi) = \sqrt{\beta(\xi)^2 - 2\alpha(\xi)}. \tag{3.20}$$

Proposition 3.3. *We have*

$$\gamma^{t,\omega,\xi}(s) = \left(s, \frac{b(\xi)g(\xi,\omega)e^{-D(\xi)(t-s)} - a(\xi)}{g(\xi,\omega)e^{-D(\xi)(t-s)} - 1}\right) \tag{3.21}$$

where

$$g(\xi,\omega) = \frac{a(\xi) - \omega}{b(\xi) - \omega}.$$

Moreover the explicit expression of ψ_0 *in* (3.18) *is given by*

$$\psi_0(t,\xi,\omega) = ix\xi - v\gamma_2^{t,\omega,\xi}(0) + t\left(i\xi\bar{r} + \phi(\xi)\right) - k\theta\left(a(\xi)t - 2\log I_1(t,\xi,\omega)\right) \tag{3.22}$$

and

$$I_1(t,\xi,\omega) = \frac{g(\xi,\omega) - 1}{g(\xi,\omega)e^{-D(\xi)t} - 1}.$$

Proof. Clearly we have $\gamma_1^{t,\omega,\xi}(s) = s$. In order to find $\gamma_2^{t,\omega,\xi}$ we set

$$F(\xi,\gamma,\omega) = \int_\gamma^\omega \frac{d\tau}{B(\xi,\tau)} \tag{3.23}$$

or, more explicitly,

$$F(\xi,\gamma,\omega) = \frac{1}{D(\xi)} \log\frac{(a(\xi) - \omega)(b(\xi) - \gamma)}{(b(\xi) - \omega)(a(\xi) - \gamma)},$$

with a, b and D as in (3.20). Then, by separation of variables, the second ODE in (3.17) is equivalent to

$$F\left(\xi, \gamma_2^{t,\omega,\xi}(s), \omega\right) = t - s \tag{3.24}$$

which is equivalent to (3.21).

Finally, we compute the integral in (3.19) by the change of variable $\tau = \gamma_2^{t,\omega,\xi}(s)$ and by (3.17): we have

$$\int_0^t A\left(\xi, \gamma_2^{t,\omega,\xi}(s)\right) ds = \int_{\gamma_2^{t,\omega,\xi}(0)}^{\omega} \frac{A(\xi,\tau)}{B(\xi,\tau)} d\tau, \tag{3.25}$$

and since $\frac{A(\xi,\tau)}{B(\xi,\tau)}$ is a rational function of τ, a direct computation concludes the proof. $\square$

Remark 3.4. *For the Heston model ($\nu \equiv 0$ and constant σ), the expression of the characteristic exponent in* (3.22) *coincides with the formulas given by Bakshi and Cao and Chen [15], Duffie, Pan and Singleton [75], Gatheral [105], Lord and Kahl [163]. The standard proof of such formulas is based on the general properties of affine processes: our alternative proof is introductory to the method used in the next section to compute higher order approximations of the characteristic function when σ is non-constant.*

As explicitly noticed by Lord and Kahl [163], if we substitute the argument I_1 of the logarithm in (3.22) *with the algebraically equivalent expression*

$$I_2(t,\xi,\omega) = \frac{e^{D(\xi)t}\left(g(\xi,\omega) - 1\right)}{g(\xi,\omega) - e^{D(\xi)t}},$$

then, in the numerical implementation, we may get discontinuities caused by the fact that the complex logarithm is a multi-valued function and most software packages automatically select its principal branch.

4.3.2 First order approximations

In this section we prove the second part of Proposition 3.1 for $j = 1$ and then determine the explicit expression of $\hat{G}_1$.

Proof of Proposition 3.1-ii) for $j = 1$. We first notice that, for $h = 1$, (2.8) reads

$$\left(\widetilde{L}_1 - \widetilde{L}_0\right) G_0(t,y,w) = \bar{\sigma}_1 \left(\bar{\sigma}_0 \left(\partial_{yy} + \partial_y\right) + \varrho \partial_{yw}\right) \left(w(y - \bar{x}) G_0(t,y,w)\right)$$

and therefore the Fourier-Laplace transform of

$$-\left(\widetilde{L}_1 - \widetilde{L}_0\right) G_0(t,y,w)$$

is equal to

$$\bar{\sigma}_1 \xi \left(\bar{\sigma}_0 \left(\xi + i\right) + i\varrho\omega\right) \left(i\partial_{\xi\omega} + \bar{x}\partial_\omega\right) \hat{G}_0(t,\xi,\omega) = \hat{G}_0(t,\xi,\omega)\, \mathcal{H}_1(t,\xi,\omega)$$

with $\mathcal{H}_1$ as in (3.14). By the same argument used in the proof of the *i)*-part, we have that problem (2.6) with $j = 1$ is transformed into (3.13). $\square$

The rest of the section is devoted to the computation of the term $\hat{G}_1$. First of all, the solution of problem (3.13) is given by

$$\hat{G}_1(t,\xi,\omega) = -\int_0^t e^{\int_s^t A\left(\xi, \gamma_2^{t,\omega,\xi}(\tau)\right) d\tau} \hat{G}_0\left(s, \xi, \gamma_2^{t,\omega,\xi}(s)\right) \mathcal{H}_1\left(s, \xi, \gamma_2^{t,\omega,\xi}(s)\right) ds.$$

Notice that by (3.18)-(3.19) and the identity

$$\gamma_2^{s,\gamma_2^{t,\omega,\xi}(s),\xi}(0) = \gamma_2^{t,\omega,\xi}(0), \tag{3.26}$$

we have

$$\begin{aligned}
\hat{G}_0\left(s,\xi,\gamma_2^{t,\omega,\xi}(s)\right) &= \exp\left(ix\xi - v\gamma_2^{s,\gamma_2^{t,\omega,\xi}(s),\xi}(0) + \int_0^s A\left(\xi,\gamma_2^{s,\gamma_2^{t,\omega,\xi}(s),\xi}(\tau)\right)d\tau\right) \\
&= \exp\left(ix\xi - v\gamma_2^{t,\omega,\xi}(0) + \int_0^s A\left(\xi,\gamma_2^{t,\omega,\xi}(\tau)\right)d\tau\right),
\end{aligned}$$

and therefore the expression of $\hat{G}_1$ drastically reduces to

$$\hat{G}_1(t,\xi,\omega) = -\hat{G}_0(t,\xi,\omega)\int_0^t \mathcal{H}_1\left(s,\xi,\gamma_2^{t,\omega,\xi}(s)\right)ds. \tag{3.27}$$

Now we compute explicitly the integral in (3.27). To do this, we state some preliminary results whose proof is postponed to the appendix. In the sequel we denote by $\partial_h f$ the derivative with respect to the h-th argument of the function f: in particular we have

$$\begin{aligned}
\partial_1 B(\xi,\omega) &= \partial_\xi B(\xi,\omega) = -i\varrho\bar{\sigma}_0\omega - \frac{\bar{\sigma}_0^2}{2}(2\xi+i), \\
\partial_2 B(\xi,\omega) &= \partial_\omega B(\xi,\omega) = \omega - i\xi\varrho\bar{\sigma}_0 + k.
\end{aligned}$$

Lemma 3.5. *We have*

$$\partial_\omega\gamma_2^{s,\omega,\xi}(0) = F_1\left(\xi,\gamma_2^{s,\omega,\xi}(0),\omega\right), \tag{3.28}$$

$$\partial_\xi\gamma_2^{s,\omega,\xi}(0) = F_2\left(\xi,\gamma_2^{s,\omega,\xi}(0),\omega\right), \qquad s\in[0,t], \tag{3.29}$$

where

$$F_k(\xi,\gamma,\omega) = \mathbf{c}_{k,0}(\xi,\gamma,\omega) + \mathbf{c}_{k,1}(\xi,\gamma,\omega)F(\xi,\gamma,\omega), \qquad k=1,2, \tag{3.30}$$

with F in (3.23) *and*

$$\begin{aligned}
\mathbf{c}_{1,0}(\xi,\gamma,\omega) &= \frac{B(\xi,\gamma)}{B(\xi,\omega)}, \qquad \mathbf{c}_{1,1}(\xi,\gamma,\omega) = 0, \\
\mathbf{c}_{2,0}(\xi,\gamma,\omega) &= \frac{(\omega-\gamma)\left(a'(\xi)(\omega-b(\xi))(\gamma-b(\xi)) - b'(\xi)(\omega-a(\xi))(\gamma-a(\xi))\right)}{4D(\xi)B(\xi,\omega)}, \\
\mathbf{c}_{2,1}(\xi,\gamma,\omega) &= -\frac{D'(\xi)B(\xi,\gamma)}{D(\xi)}.
\end{aligned}$$

Lemma 3.6. *We have*

$$\partial_\omega\psi_0(s,\xi,\omega) = F_3\left(\xi,\gamma_2^{s,\omega,\xi}(0),\omega\right), \tag{3.31}$$

$$\partial_\xi\psi_0(s,\xi,\omega) = F_4\left(\xi,\gamma_2^{s,\omega,\xi}(0),\omega\right), \tag{3.32}$$

$$\partial_{\xi\omega}\psi_0(s,\xi,\omega) = F_5\left(\xi,\gamma_2^{s,\omega,\xi}(0),\omega\right), \tag{3.33}$$

for $s \in [0,t]$*, where*

$$F_k(\xi,\gamma,\omega) = \mathbf{c}_{k,0}(\xi,\gamma,\omega) + \mathbf{c}_{k,1}(\xi,\gamma,\omega)F(\xi,\gamma,\omega), \qquad k = 3,4,5, \tag{3.34}$$

with F *as in* (3.23) *and*

$$\begin{aligned}
\mathbf{c}_{3,0}(\xi,\gamma,\omega) &= \frac{A(\xi,\omega) - A(\xi,\gamma) - vB(\xi,\gamma)}{B(\xi,\omega)}, \qquad \mathbf{c}_{3,1}(\xi,\gamma,\omega) = 0,\\
\mathbf{c}_{4,0}(\xi,\gamma,\omega) &= ix - v\mathbf{c}_{2,0}(\xi,\gamma,\omega) - k\theta\frac{(\omega-\gamma)\left(b'(\xi)(\omega-a(\xi)) - a'(\xi)(\omega-b(\xi))\right)}{2D(\xi)B(\xi,\omega)},\\
\mathbf{c}_{4,1}(\xi,\gamma,\omega) &= i\bar{r} + \phi(\xi) - v\mathbf{c}_{2,1}(\xi,\gamma,\omega) - k\theta\left(\frac{a(\xi)b'(\xi) - a'(\xi)b(\xi)}{a(\xi)-b(\xi)} + \gamma\frac{D'(\xi)}{D(\xi)}\right),\\
\mathbf{c}_{5,0}(\xi,\gamma,\omega) &= \frac{\mathbf{c}_{2,0}(\xi,\gamma,\omega)}{B(\xi,\omega)}\left(k\theta + \frac{v}{2}\left(a(\xi)+b(\xi)-2\gamma\right)\right)\\
&\quad - \frac{v\partial_1 B(\xi,\gamma) + \mathbf{c}_{3,0}(\xi,\gamma,\omega)\partial_1 B(\xi,\omega)}{B(\xi,\omega)},\\
\mathbf{c}_{5,1}(\xi,\gamma,\omega) &= \frac{\mathbf{c}_{2,1}(\xi,\gamma,\omega)}{B(\xi,\omega)}\left(k\theta + \frac{v}{2}\left(a(\xi)+b(\xi)-2\gamma\right)\right),
\end{aligned}$$

Lemma 3.7. *For any* $s \in [0,t]$*, we have*

$$\mathcal{H}_1(s,\xi,\omega) = F_6\left(\xi, \gamma_2^{s,\omega,\xi}(0), \omega\right),$$

where

$$F_6(\xi,\gamma,\omega) = \bar{\sigma}_1\xi\left(\bar{\sigma}_0(\xi+i) + i\varrho\omega\right)\left(\mathbf{h}_0(\xi,\gamma,\omega) + i\mathbf{h}_1(\xi,\gamma,\omega)F(\xi,\gamma,\omega)\right),$$

with F *as in* (3.23) *and*

$$\mathbf{h}_0(\xi,\gamma,\omega) = \mathbf{c}_{3,0}(\xi,\gamma,\omega)\left(\bar{x} + i\mathbf{c}_{4,0}(\xi,\gamma,\omega)\right) + i\mathbf{c}_{5,0}(\xi,\gamma,\omega), \tag{3.35}$$

$$\mathbf{h}_1(\xi,\gamma,\omega) = \mathbf{c}_{3,0}(\xi,\gamma,\omega)\mathbf{c}_{4,1}(\xi,\gamma,\omega) + \mathbf{c}_{5,1}(\xi,\gamma,\omega). \tag{3.36}$$

Lemma 3.8. *For any* $s \in [0,t]$*,we have*

$$\begin{aligned}
\mathcal{H}_1\left(s,\xi,\gamma_2^{t,\omega,\xi}(s)\right) &= \bar{\sigma}_1\xi\left(\bar{\sigma}_0(\xi+i) + i\varrho\,\gamma_2^{t,\omega,\xi}(s)\right)\\
&\cdot\left(\mathbf{h}_0\left(\xi,\gamma_2^{t,\omega,\xi}(0),\gamma_2^{t,\omega,\xi}(s)\right) + is\mathbf{h}_1\left(\xi,\gamma_2^{t,\omega,\xi}(0),\gamma_2^{t,\omega,\xi}(s)\right)\right).
\end{aligned} \tag{3.37}$$

Remark 3.9. *The coefficients* $\mathbf{h}_0$ *and* $\mathbf{h}_1$ *in* (3.35)-(3.36) *are rational functions of* ω *because they are linear combinations of* $\frac{1}{B(\xi,\omega)}$ *and* $\frac{\omega}{B(\xi,\omega)}$. *On the other hand, by* (3.21)-(3.20) *we can easily check the following identities:*

$$\begin{aligned}
\frac{1}{B\left(\xi,\gamma_2^{t,\omega,\xi}(s)\right)} &= \frac{\left(g(\xi,\omega)e^{-D(\xi)(t-s)} - 1\right)^2}{2D(\xi)^2\, g(\xi,\omega)e^{-D(\xi)(t-s)}},\\
\frac{\gamma_2^{t,\omega,\xi}(s)}{B\left(\xi,\gamma_2^{t,\omega,\xi}(s)\right)} &= \frac{\left(g(\xi,\omega)e^{-D(\xi)(t-s)} - 1\right)\left(b(\xi)g(\xi,\omega)e^{-D(\xi)(t-s)} - a(\xi)\right)}{2D(\xi)^2\, g(\xi,\omega)e^{-D(\xi)(t-s)}}.
\end{aligned} \tag{3.38}$$

Therefore it turns out that $\mathcal{H}_1\left(s,\xi,\gamma_2^{t,\omega,\xi}(s)\right)$ *is a Laurent polynomial (i.e. a polynomial with also negative powers) of* $e^{-D(\xi)(t-s)}$. *Specifically, by* (3.38) *we have*

$$\frac{\mathbf{c}_{3,0}\left(\xi,\gamma,\gamma_2^{t,\omega,\xi}(s)\right)}{g(\xi,\omega)e^{-D(\xi)(t-s)}-1}=\sum_{j=-1}^{0} d_{3,0,j}(\xi,\gamma)\left(g(\xi,\omega)e^{-D(\xi)(t-s)}\right)^j,$$
$$\mathbf{c}_{4,0}\left(\xi,\gamma,\gamma_2^{t,\omega,\xi}(s)\right)=\sum_{j=-1}^{1} d_{4,0,j}(\xi,\gamma)\left(g(\xi,\omega)e^{-D(\xi)(t-s)}\right)^j,$$
$$\frac{\mathbf{c}_{5,0}\left(\xi,\gamma,\gamma_2^{t,\omega,\xi}(s)\right)}{\left(g(\xi,\omega)e^{-D(\xi)(t-s)}-1\right)^2}=\sum_{j=-2}^{0} d_{5,0,j}(\xi,\gamma)\left(g(\xi,\omega)e^{-D(\xi)(t-s)}\right)^j,$$
$$\frac{\mathbf{c}_{5,1}\left(\xi,\gamma,\gamma_2^{t,\omega,\xi}(s)\right)}{\left(g(\xi,\omega)e^{-D(\xi)(t-s)}-1\right)^2}=d_{5,1,-1}(\xi,\gamma)\left(g(\xi,\omega)e^{-D(\xi)(t-s)}\right)^{-1},$$

for some coefficients $d_{k,h,j}$ *whose expression is given explicitly in Appendix 4.6.2.*

We also notice that $\mathbf{c}_{4,1}(\xi,\gamma,\omega)$ *is independent of* ω *and thus it will be denoted simply by* $\mathbf{c}_{4,1}(\xi,\gamma)$.

Collecting all the previous results, now the integral in (3.27) can be straightforwardly computed as the following theorem shows.

Theorem 3.10. *For any* $(t,\xi,\omega)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+$, *the solution of the Cauchy problem* (3.13) *is given by*

$$\hat{G}_1(t,\xi,\omega)=-\hat{G}_0(t,\xi,\omega)\,\bar{\sigma}_1\xi\Big(\bar{\sigma}_0(\xi+i)\,J_0\left(t,\gamma_2^{t,\omega,\xi}(0),\xi,\omega\right)+i\varrho J_1\left(t,\gamma_2^{t,\omega,\xi}(0),\xi,\omega\right)\Big),$$

where[3], *for* $j=0,1$,

$$\begin{aligned}J_j(t,\xi,\gamma,\omega)=&\sum_{k=-2}^{2}\ell_{j,k}(\xi,\gamma)g(\xi,\omega)^k\left(\frac{1-e^{-kD(\xi)t}}{kD(\xi)}\right)\\&+i\sum_{k=-1}^{1}\ell'_{j,k}(\xi,\gamma)g(\xi,\omega)^k\left(\frac{e^{-kD(\xi)t}-1+kD(\xi)t}{k^2D(\xi)^2}\right),\end{aligned}\tag{3.39}$$

and the functions $\ell_{j,k}=\ell_{j,k}(\xi,\gamma)$ *and* $\ell'_{j,k}=\ell'_{j,k}(\xi,\gamma)$ *are given in Appendix 4.6.2.*

Proof. Using (3.35)-(3.36) and Remark 3.9, tedious but straightforward computations show that for $j=0,1$

$$\begin{aligned}\left(\gamma_2^{t,\omega,\xi}(s)\right)^j\mathbf{h}_0\left(\xi,\gamma,\gamma_2^{t,\omega,\xi}(s)\right)&=\sum_{k=-2}^{2}\ell_{j,k}(\xi,\gamma)\left(g(\xi,\omega)e^{-D(\xi)(t-s)}\right)^k,\\\left(\gamma_2^{t,\omega,\xi}(s)\right)^j\mathbf{h}_1\left(\xi,\gamma,\gamma_2^{t,\omega,\xi}(s)\right)&=\sum_{k=-1}^{1}\ell'_{j,k}(\xi,\gamma)\left(g(\xi,\omega)e^{-D(\xi)(t-s)}\right)^k.\end{aligned}\tag{3.40}$$

[3] By convention, for $k=0$ we use the asymptotic values

$$\frac{1-e^{-kD(\xi)t}}{kD(\xi)}=t,\qquad\qquad\frac{e^{-kD(\xi)t}-1+kD(\xi)t}{k^2D(\xi)^2}=\frac{t^2}{2}.$$

Then the thesis follows from (3.27), (3.37) and (3.40), because

$$\int_0^t e^{-kD(\xi)(t-s)} s^j ds = (-1)^j \frac{1 - e^{-kD(\xi)t} - jkD(\xi)t}{(kD(\xi))^{j+1}},$$

for any $k \in \mathbb{Z} \setminus \{0\}$ and $j = 0, 1$. □

4.3.3 Second order approximation

In this section we show how to compute the second order approximation of the characteristic function $\hat{\Gamma}$. Although we do not find explicitly the correction term $\hat{G}_2$, still we provide an integral expression which is analogous to (3.27). We begin by completing the proof of Proposition 3.1.

Proof of Proposition 3.1-ii) for $j = 2$. As in Subsection 4.3.2 we first observe that formula (2.8) with $h = 2$ becomes

$$\left(\widetilde{L}_2 - \widetilde{L}_1\right) G_0(t, y, w) = \left(\left(\bar{\sigma}_1^2 + \bar{\sigma}_0\bar{\sigma}_2\right)(\partial_{yy} + \partial_y) + \frac{1}{2}\bar{\sigma}_2\varrho\partial_{yw}\right)\left(w(y - \bar{x})^2 G_0(t, y, w)\right).$$

Thus the Fourier-Laplace transform of

$$-\left(\widetilde{L}_2 - \widetilde{L}_1\right) G_0(t, y, w)$$

is given by

$$-\frac{\xi}{2}\left((i + \xi)\left(\bar{\sigma}_1^2 + \bar{\sigma}_0\bar{\sigma}_2\right) + i\varrho\bar{\sigma}_2\omega\right)\left(\bar{x}^2\partial_\omega + 2i\bar{x}\partial_{\omega\xi} - \partial_{\xi\xi\omega}\right)\hat{G}_0(t, \xi, \omega),$$

or more explicitly by

$$\mathcal{H}_{2,1}(t, \xi, \omega)\hat{G}_0(t, \xi, \omega)$$

where

$$\begin{aligned}\mathcal{H}_{2,1}(t, \xi, \omega) = &-\frac{\xi}{2}\left((i + \xi)\left(\bar{\sigma}_1^2 + \bar{\sigma}_0\bar{\sigma}_2\right) + i\varrho\bar{\sigma}_2\omega\right)\cdot \\ &\cdot\left(2i\left(\bar{x} + i\partial_\xi\psi_0\right)\partial_{\xi\omega}\psi_0 + \partial_\omega\psi_0\left(\left(\bar{x} + i\partial_\xi\psi_0\right)^2 - \partial_{\xi\xi}\psi_0\right) - \partial_{\xi\xi\omega}\psi_0\right)(t, \xi, \omega),\end{aligned}$$

and ψ_0 is as in (3.22). Analogously, the Fourier-Laplace transform of

$$-\left(\widetilde{L}_1 - \widetilde{L}_0\right) G_1(t, y, w)$$

is equal to

$$\bar{\sigma}_1\xi\left(\bar{\sigma}_0(\xi + i) + i\varrho\omega\right)\left(i\partial_{\xi\omega} + \bar{x}\partial_\omega\right)\hat{G}_1(t, \xi, \omega) = \mathcal{H}_{2,2}(t, \xi, \omega)\hat{G}_0(t, \xi, \omega),$$

where

$$\begin{aligned}\mathcal{H}_{2,2}(t, \xi, \omega) = &\,\bar{\sigma}_1\xi\left(\bar{\sigma}_0(\xi + i) + i\varrho\omega\right)\cdot \\ &\cdot\Big((\bar{x} + i\partial_\xi\psi_0)\partial_\omega\bar{g} + (\partial_\omega\psi_0(\bar{x} + i\partial_\xi\psi_0) + i\partial_{\xi\omega}\psi_0)\bar{g} + i(\partial_\omega\psi_0\partial_\xi\bar{g} + \partial_{\xi\omega}\bar{g})\Big)(t, \xi, \omega)\end{aligned}$$

and

$$\bar{g}(t,\xi,\omega) = -\bar{\sigma}_1\xi\Big(\bar{\sigma}_0\,(\xi+i)\,J_0\left(t,\gamma_2^{t,\omega,\xi}(0),\xi,\omega\right) + i\varrho J_1\left(t,\gamma_2^{t,\omega,\xi}(0),\xi,\omega\right)\Big),$$

with J_0, J_1 as in (3.39). Then the thesis follows with

$$\mathcal{H}_2 = \mathcal{H}_{2,1} + \mathcal{H}_{2,2}.$$

□

As for problem (3.13), finally we get

$$\hat{G}_2(t,\xi,\omega) = -\hat{G}_0\,(t,\xi,\omega)\int_0^t \mathcal{H}_2\left(s,\xi,\gamma_2^{t,\omega,\xi}(s)\right)ds,$$

where, by Remark 3.9, $\mathcal{H}_2\left(s,\xi,\gamma_2^{t,\omega,\xi}(s)\right)$ is a Laurent polynomial of $e^{-D(\xi)(t-s)}$. In particular the explicit expression of $\hat{G}_2$ can be obtained by evaluating the above integral.

4.4 Numerical experiments

In this section we test the accuracy and the efficiency of our pricing formulas for Call options. For our comparison we use an Euler-Monte Carlo simulation: specifically, we implemented the second-order discretization scheme of the SDE (1.1) proposed by Glasserman (cf. Section 3.4 in [110]). We acknowledge that several authors suggested other techniques for the exact and the semi-exact simulation of the underlying trajectories under the Heston model. However, the standard Euler discretization allows to include the local volatility feature without additional effort and can be extended to the more general class of local Lévy models. In all the experiments the Euler-Monte Carlo simulation has been executed with 10^6 sample paths and the time discretization of 300 steps per year: the 99% confidence interval is considered as a reference.

In the next subsections we examine two particular specifications of the local volatility function σ: the classical CEV volatility and a locally quadratic volatility function. We will call them the CEV-Heston and the Quad-Heston models respectively. First we consider models without jumps (i.e. $Z \equiv 0$ in (1.1)), then we also include Merton jumps in the log-return dynamics, that is we assume

$$Z_t = \sum_{n=1}^{N_t} z_n$$

where N_t is a Poisson process with intensity ℓ and z_n are i.i.d. random variables, independent of N_t, with normal distribution $\mathcal{N}_{m,\partial^2}$. In this case the characteristic operator can be written in the form

$$\begin{aligned} Lu(t,x,v) =& \frac{v}{2}\left(\sigma^2(x)\left(\partial_{xx}-\partial_x\right) + 2\varrho\sigma(x)\eta\partial_{xv} + \eta^2\partial_{vv}\right)u(t,x,v) \\ &+ r_0\partial_x u(t,x,v) + k(\theta - v)\partial_v u(t,x,v) + \partial_t u(t,x,v) \\ &+ \int_{\mathbb{R}} \left(u(t,x+y,v) - u(t,x,v)\right)\nu(dy). \end{aligned}$$

where $\nu = \ell\mathcal{N}_{m,\partial^2}$ is the Lévy measure and

$$r_0 = r - q - \int_{\mathbb{R}} (e^y - 1)\,\nu(dy) = r - \lambda\left(e^{m+\frac{\delta^2}{2}} - 1\right).$$

Moreover the function ϕ in (3.11) reduces to

$$\phi(\xi) = \int_{\mathbb{R}} \left(e^{iz\xi} - 1\right)\nu(dz) = \lambda\left(e^{im\xi - \frac{\delta^2\xi^2}{2}} - 1\right).$$

In the following experiments the initial value of the variance process will be set equal to $v_0 = 0.16$ (i.e. the initial volatility is equal to 40%) in the purely diffusive case and equal to $v_0 = 0.09$ (i.e. the initial volatility is equal to 30%) in the jump-diffusion case. The other parameters of the variance process in (1.1) are equal to

$$k = 5, \qquad \theta = (\sqrt{v_0} - 0.1)^2, \qquad \rho = -0.7, \qquad \eta = 0.9. \tag{4.41}$$

The initial price of the underlying asset, the risk-free rate and the dividend rate are set to

$$S_0 = 1, \qquad r = 0.05, \qquad q = 0, \tag{4.42}$$

respectively. The values of the parameters are chosen according to the calibration in [178] to real market data. When also jumps are considered, the parameters of the compound Poisson process Z are equal to

$$\ell = 0.3, \qquad m = -0.1, \qquad \delta = 0.4. \tag{4.43}$$

Finally, the initial point of the Taylor expansions $T_{n,\sigma}(x)$ and $T_{n,\sigma^2}(x)$ in (2.3) is $\bar{x} = \log S_0$.

All the following computations were performed using Mathematica 8 on a Intel(R) Core (TM) i7-2600 CPU@3.40GHz with 16 GB.

4.4.1 Fourier pricing formula

To compute option prices and Greeks we use a standard Fourier integral formula (cf. Heston [121], Lewis [153] and Lipton [159]). For reader's convenience we briefly review the general pricing formula which is expressed in terms of the characteristic function of the underlying log-price process and therefore can be combined with the expansion (3.9) to compute efficiently prices and sensitivities of European and American options under JLSV models.

We consider a risky asset $S_t = e^{X_t}$ where X is the process whose risk-neutral dynamics is given by (1.1). We denote by $H(S_0, T)$ the initial price of a European option with underlying asset S, maturity T and payoff $f = f(x)$ (given as a function of the log-price): to fix ideas, for a Call option with strike K we have

$$f^{\text{Call}}(x) = (e^x - K)^+.$$

The following theorem is a classical result which can be found in several textbooks (see, for instance, [188]).

Theorem 4.1. *Let*

$$f_\gamma(x) = e^{-\gamma x} f(x)$$

and assume that there exists $\gamma \in \mathbb{R}$ *such that*

i) $f_\gamma, \hat{f}_\gamma \in L^1(\mathbb{R})$*;*

ii) $E^Q\left[S_T^\gamma\right]$ *is finite.*

Then, the following pricing formula holds:

$$H(S_0, T) = \frac{e^{-rT}}{\pi} \int_0^\infty \hat{f}(\xi + i\gamma)\phi_{X_T}(-(\xi + i\gamma))d\xi, \tag{4.44}$$

where ϕ_{X_T} *is the characteristic function of* X_T*. For the Delta we have*

$$\partial_{S_0} H(S_0, T) = \frac{e^{-rT}}{S_0 \pi} \int_0^\infty (\gamma - i\xi)\hat{f}(\xi + i\gamma)\phi_{X_T}(-(\xi + i\gamma))d\xi.$$

For example, f^{Call} verifies the assumptions of Theorem 4.1 for any $\gamma > 1$ and we have

$$\hat{f}^{\text{Call}}(\xi + i\gamma) = \frac{K^{1-\gamma} e^{i\xi \log K}}{(i\xi - \gamma)(i\xi - \gamma + 1)}.$$

Other examples of typical payoff functions and the related Greeks can be found in [188].

Remark 4.2. *Expansion* (3.9) *provides an explicit approximation of the characteristic function. In fact the computation time in our experiments on Call prices is only due to the numerical inversion of the Fourier formula* (4.44)*. In our tests we used a direct integration method based on the built-in Mathematica function* `NIntegrate`*; however, several other efficient Fourier inversion methods can be used, such as the standard, fractional FFT algorithm or the recent COS method [88]. To check the accuracy we also implemented the COS method and verified that the computed prices were identical up to eight digits.*

Remark 4.3. *Assume that we want to calibrate a model to a volatility surface by means of formula* (4.44) *for Call and Put options. Then we notice that the function* ϕ_{X_T}*, which is the computationally expensive part in* (4.44)*, does not depend on* K*: thus it is sufficient to evaluate it only once to compute prices with different strikes. This allows the use of cache techniques to speed up the computation of option prices and the related Greeks (see Kilin [142]). In other terms we can give a "vector input" of strikes to obtain a vector of output values.*

4.4.2 Quad-Heston and Quad-Bates models

In the first two experiments we consider a (locally) quadratic volatility function: in terms of the asset price S, it is defined as

$$0.2 + \left(a + bS + cS^2\right) e^{-dS} \tag{4.45}$$

where a, b, c, d are real parameters. In our experiments we put

$$a = 1.2, \qquad b = 1.6, \qquad c = 0.4, \qquad d = 1.4. \tag{4.46}$$

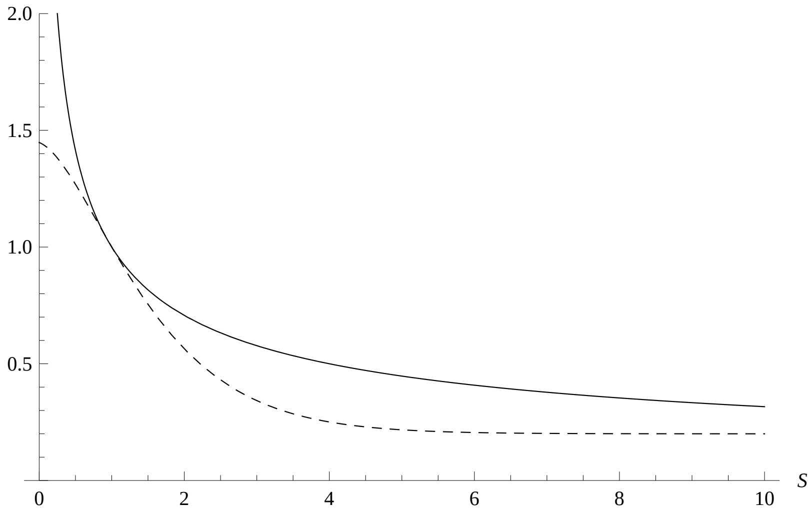

Figure 4.1: CEV (solid line) and Quad (dashed line) volatility functions with $\beta = \frac{1}{2}$ and the parameters as in (4.46)

The volatility function is bounded and tends to 0.2 as S tends to infinity. In this case, since the pricing operator L in (2.2) is uniformly parabolic, we conjecture our technique to provide a good approximation, being asymptotically exact as T approaches zero: we recall that, for uniform parabolic PDEs, convergence results and explicit error bounds for the Adjoint Expansion approximation were proved in [185]. The specification (4.45) is similar in spirit to the model proposed by Lipton in [159] where an exact solution is presented in the case of zero correlation ($\varrho = 0$).

The comparison between Monte Carlo and the first order approximation are reported in Table 4.1 for the Quad-Heston model and in Table 4.2 for the Quad-Bates model with parameters as in (4.41)-(4.42)-(4.43)-(4.46). In order to test our approximation formula for real quoted strikes, we increase the range of strike according to maturity. The numerical tests confirm that the approximation is almost exact for short maturities. More generally the quality of the approximation seems to be satisfactory for most strikes and maturities.

4.4.3 CEV-Heston and CEV-Bates models

As a second test we consider the well-known CEV local volatility function

$$S^{\beta-1}, \qquad \beta \in]0,1[.$$

A combination of the CEV and Heston dynamics was also recently proposed by Choi, Fouque and Kim [55]. Apart from its popularity, we consider the CEV volatility because it is a degenerate model: specifically, the volatility diverges as S approaches 0 and the CEV-Heston pricing operator L in (2.2) is not uniformly parabolic in both the x and v variables (see the graphs of the CEV and Quad volatility functions in Figure 4.1). Despite this, the numerical tests show that our first order approximation performs rather well. The results seem to be satisfactory especially for shorter maturities. Table 4.3 (CEV-Heston model) and Table 4.4 (CEV-Bates model) show the prices and implied volatilities of Call options for a wide range of strikes and for maturities up to 5 years.

Figure 4.2 (CEV-Heston) and Figure 4.3 (CEV-Bates) compare the relative error of the 0-th and the 1-st order approximation for the implied volatility, with respect to the 95% (dark grey area) and 99% (bright grey area) confidence interval of the Monte Carlo simulation. In order to show the consistence of the method, the latter experiments have been operated for a different choice of the parameters and for 25 different strikes for each maturity.

CEV-Heston model calibration

In what follows we aim to calibrate the parameters of the CEV-Heston model to a portion of real market implied volatility surface, by means of our 1-st order approximation formulae for the Call prices. In particular, we are considering the official implied volatilities referred to the index S&P500, registered at closure in date January 24th, 2012.

Such a calibration experiment has a dual purpose. In the first place it shows the effectiveness, and thus the consistency, of our pricing formula for a wide range of parameters. On the other hand, it provides a financial justification for the use of such models combining both local and stochastic volatility. In particular, we will globally calibrate the parameters of the model over three different maturities, namely T equal to 87, 115 and 142 days, finding out the CEV-Heston model to better fit the market implied volatility surface than the Heston model.

The calibration procedure has been operated by means of the Mathematica pre-defined optimization function *FindFit*, and taking place in two main steps. At first, we obtained a continuous parametrization of the real market data by interpolating the latter with a suitable continuous function (Figure 4.4). In regard to this, in order to obtain a better fitting, we did not considered those range of log-moneyness wherein the respective Call options are highly illiquid.

Then, we calibrated the parameters of the model with respect to the interpolated data by means of our 1-st order pricing formula, on a grid of 41 equidistant values for the log-moneyness, for each maturity. The resulting values for the optimal parameters turn out to be the following:

$$v_0 = 0.146^2, \quad k = 7.607, \quad \theta = 0.238^2, \quad \eta = 1.500, \quad \rho = -0.692, \quad \beta = 0.469, \qquad (4.47)$$

whose corresponding RMSE (over all the 3×41 sample points) is equal to 0.267%. The graphical representation of the optimal fitting is reported in Figure 4.5. Eventually, this procedure has been repeated for the Heston model giving back the following values for the optimal parameters:

$$v_0 = 0.145^2, \quad k = 8.363, \quad \theta = 0.245^2, \quad \eta = 1.778, \quad \rho = -0.734,$$

and a RMSE equal to 0.296%. The interest and the dividend rates have been set as $r = 0.02$ and $q = 0$ respectively throughout both the experiments.

4.4.4 The Greeks

Our approximation formula allows to compute also the sensitivities of plain vanilla options in a fast and accurate way. Figures 4.6 and 4.7 depict the graph of the Delta of a Call under the Heston, CEV-Heston and CEV-Bates models with parameters as in (4.41)-(4.42)-(4.43) with maturities $T = \frac{1}{12}$ and $T = 1$ respectively. Incidentally, the pictures show a significant effect of jumps on the hedging strategies for out-of-the-money options. Notice that, usual no-arbitrage bounds, like having the Delta in the interval $[0, 1]$, are not violated.

4.5 Conclusions

In Chapter 3 we introduced the adjoint expansion (AE) method which yields new analytical approximation formulas for the characteristic function and plain vanilla options in local Lévy models. Motivated by widespread use of local-stochastic volatility models and their popularity, in this chapter we extend our previous results by including a mean-reverting stochastic volatility component. In this generality, the computation of the approximations is much more demanding and leads to rather long, yet explicit, formulas: here we develop all the detailed computations to derive the first order formula and show how the second order one can be obtained. Numerical tests in models with or without jumps show that the achieved accuracy is satisfactory under standard parameter regimes: the approximation is almost exact for short maturities or low volatility. However the use of second order expansions is recommended to improve the accuracy in case the volatility function is not well approximated by affine functions. The Mathematica notebook with the code used in the experiments is freely available in the web-site of the authors.

From a theoretical perspective, further research is required to get rigorous convergence results for the general class of JSLV models: so far, only the case of uniformly parabolic PDEs has been studied in Chapter 3.

From a practical perspective, it would be interesting to obtain directly analytical approximations of the implied volatility; as for this concern, as well as higher order approximation formulas, we refer to Chapter 5 and Chapter 6 (see also [171] and [169]).

4.6 Appendix

4.6.1 Proofs of Subsection 4.3.2

Proof of Lemma 3.5. The thesis follows directly by differentiating (3.23) w.r.t. ω and ξ respectively and using the fact that

$$\partial_3 F(\xi,\gamma,\omega) = \frac{1}{B(\xi,\omega)}, \qquad \partial_2 F(\xi,\gamma,\omega) = -\frac{1}{B(\xi,\gamma)},$$

and

$$\partial_1 F\left(\xi,\gamma,\omega\right) = -\frac{D'(\xi)}{D(\xi)} F\left(\xi,\gamma,\omega\right) + \frac{a'(\xi)(\omega-\gamma)}{D(\xi)(a(\xi)-\omega)(a(\xi)-\gamma)} - \frac{b'(\xi)(\omega-\gamma)}{D(\xi)(b(\xi)-\omega)(b(\xi)-\gamma)}. \tag{6.48}$$

□

Proof of Lemma 3.6. By (3.22) and (3.25) we have

$$\partial_\omega \psi_0(s,\xi,\omega) = -v\partial_\omega \gamma_2^{s,\omega,\xi}(0) + \partial_\omega \int_{\gamma_2^{s,\omega,\xi}(0)}^{\omega} \frac{A(\xi,\tau)}{B(\xi,\tau)} d\tau$$

and (3.31) follows from (3.28) and the identity

$$\partial_\omega \int_{\gamma_2^{s,\omega,\xi}(0)}^{\omega} \frac{A(\xi,\tau)}{B(\xi,\tau)} d\tau = \frac{A(\xi,\omega)}{B(\xi,\omega)} - \frac{A\left(\xi,\gamma_2^{s,\omega,\xi}(0)\right)}{B\left(\xi,\gamma_2^{s,\omega,\xi}(0)\right)} \partial_\omega \gamma_2^{s,\omega,\xi}(0) = \frac{A(\xi,\omega) - A\left(\xi,\gamma_2^{s,\omega,\xi}(0)\right)}{B(\xi,\omega)}.$$

Moreover, we have

$$\partial_\xi \psi_0(s,\xi,\omega) = ix - v\partial_\xi \gamma_2^{s,\omega,\xi}(0) + \partial_\xi \int_0^s A\left(\xi,\gamma_2^{s,\omega,\xi}(z)\right) dz$$

and

$$\begin{aligned} &\partial_\xi \int_0^s A\left(\xi,\gamma_2^{s,\omega,\xi}(z)\right) dz \\ &= \int_0^s \left(\partial_1 A\left(\xi,\gamma_2^{s,\omega,\xi}(z)\right) + \partial_2 A\left(\xi,\gamma_2^{s,\omega,\xi}(z)\right) \partial_\xi \gamma_2^{s,\omega,\xi}(z)\right) dz = \end{aligned}$$

(recalling that $A(\xi,\omega) = -k\theta\omega + i\bar{r}\xi + \phi(\xi)$)

$$= \int_0^s \left(i\bar{r} + \phi(\xi) - k\theta\partial_\xi \gamma_2^{s,\omega,\xi}(z)\right) dz =$$

(by (3.29))

$$= \int_0^s \left(i\bar{r} + \phi(\xi) - k\theta B\left(\xi,\gamma_2^{s,\omega,\xi}(z)\right) \int_{\gamma_2^{s,\omega,\xi}(z)}^{\omega} \partial_\xi \frac{1}{B(\xi,\varsigma)} d\varsigma\right) dz =$$

(setting $\tau = \gamma_2^{s,\omega,\xi}(z)$)

$$= (i\bar{r} + \phi(\xi))\, F\left(\xi,\gamma_2^{s,\omega,\xi}(0),\omega\right) - k\theta \int_{\gamma_2^{s,\omega,\xi}(0)}^{\omega} \int_\tau^\omega \partial_\xi \frac{1}{B(\xi,\varsigma)} d\varsigma d\tau =$$

(interchanging the order of integration)

$$= (i\bar{r} + \phi(\xi))\, F\left(\xi,\gamma_2^{s,\omega,\xi}(0),\omega\right) - k\theta \int_{\gamma_2^{s,\omega,\xi}(0)}^{\omega} \left(\tau - \gamma_2^{s,\omega,\xi}(0)\right) \partial_\xi \frac{1}{B(\xi,\tau)} d\tau.$$

Now it is crucial to observe that

$$\partial_\xi \frac{\tau}{B(\xi,\tau)} = \frac{a(\xi)b'(\xi) - a'(\xi)b(\xi)}{a(\xi) - b(\xi)} \frac{1}{B(\xi,\tau)} + \frac{1}{D(\xi)} \left(\frac{a(\xi)a'(\xi)}{(\tau - a(\xi))^2} - \frac{b(\xi)b'(\xi)}{(\tau - b(\xi))^2}\right)$$

and therefore we have

$$\begin{aligned} \int_\gamma^\omega (\tau - \gamma)\, \partial_\xi \frac{1}{B(\xi,\tau)} d\tau &= \frac{a(\xi)b'(\xi) - a'(\xi)b(\xi)}{a(\xi) - b(\xi)} F(\xi,\gamma,\omega) \\ &+ \frac{\omega - \gamma}{D(\xi)} \left(\frac{a(\xi)a'(\xi)}{(a(\xi) - \omega)(a(\xi) - \gamma)} - \frac{b(\xi)b'(\xi)}{(b(\xi) - \omega)(b(\xi) - \gamma)}\right) - \gamma\partial_1 F(\xi,\gamma,\omega). \end{aligned}$$

Plugging (6.48) into the previous equation we get (3.32).

Finally, (3.33) follows by differentiating (3.31) with respect to the variable ξ: indeed we have

$$\partial_\xi \mathbf{c}_{3,0}\left(\xi, \gamma_2^{s,\omega,\xi}(0), \omega\right)$$
$$= -\frac{\partial_2 A\left(\xi, \gamma_2^{s,\omega,\xi}(0)\right)\partial_\xi \gamma_2^{s,\omega,\xi}(0) + v\partial_1 B\left(\xi, \gamma_2^{s,\omega,\xi}(0)\right) + v\partial_2 B\left(\xi, \gamma_2^{s,\omega,\xi}(0)\right)\partial_\xi \gamma_2^{s,\omega,\xi}(0)}{B(\xi,\omega)}$$
$$- \frac{\partial_1 B(\xi,\omega)\left(A(\xi,\omega) - A\left(\xi, \gamma_2^{s,\omega,\xi}(0)\right) - vB\left(\xi, \gamma_2^{s,\omega,\xi}(0)\right)\right)}{B^2(\xi,\omega)}$$

and using (3.29) we get (3.33). ☐

Proof of Lemma 3.7. It follows directly from Lemma 3.6 and formula (3.14). ☐

Proof of Lemma 3.8. The thesis follows directly from Lemma 3.7, (3.26), and the identity

$$F\left(\xi, \gamma_2^{s,\gamma_2^{t,\omega,\xi}(s),\xi}(0), \gamma_2^{t,\omega,\xi}(s)\right) = s,$$

which is a consequence of (3.24). ☐

4.6.2 Coefficients of the first order expansion

We give the explicit expression of the coefficients $\ell_{j,k}$, $\ell'_{j,k}$ and $d_{k,h,j}$ appearing in Remark 3.9 and Theorem 3.10:

$$\begin{aligned}
\ell_{0,-2} &= -i(d_{3,0,-1}d_{4,0,-1} - d_{5,0,-2}),\\
\ell_{0,-1} &= -\bar{x}d_{3,0,-1} + i(-d_{3,0,0}d_{4,0,-1} + d_{3,0,-1}(d_{4,0,-1} - d_{4,0,0}) - 2d_{5,0,-2} + d_{5,0,-1}),\\
\ell_{0,0} &= \bar{x}(d_{3,0,-1} - d_{3,0,0}) + i(d_{3,0,0}(d_{4,0,-1} - d_{4,0,0}) + d_{3,0,-1}(d_{4,0,0} - d_{4,0,1})\\
&\quad + d_{5,0,-2} - 2d_{5,0,-1} + d_{5,0,0}),\\
\ell_{0,1} &= \bar{x}d_{3,0,0} + i(d_{3,0,0}(d_{4,0,0} - d_{4,0,1}) + d_{3,0,-1}d_{4,0,1} + d_{5,0,-1} - 2d_{5,0,0}),\\
\ell_{0,2} &= i(d_{3,0,0}d_{4,0,1} + d_{5,0,0}),\\
\ell'_{0,-1} &= -\mathbf{c}_{4,1}d_{3,0,-1} + d_{5,1,-1},\\
\ell'_{0,0} &= \mathbf{c}_{4,1}(d_{3,0,-1} - d_{3,0,0}) - 2d_{5,1,-1},\\
\ell'_{0,1} &= \mathbf{c}_{4,1}d_{3,0,0} + d_{5,1,-1},\\
\ell_{1,-2} &= -ia(\xi)(d_{3,0,-1}d_{4,0,-1} - d_{5,0,-2}),\\
\ell_{1,-1} &= ib(\xi)(d_{3,0,-1}d_{4,0,-1} - d_{5,0,-2})\\
&\quad - a(\xi)(\bar{x}d_{3,0,-1} + i(d_{3,0,0}d_{4,0,-1} + d_{3,0,-1}d_{4,0,0} + d_{5,0,-2} - d_{5,0,-1})),\\
\ell_{1,0} &= b(\xi)(\bar{x}d_{3,0,-1} + i(d_{3,0,0}d_{4,0,-1} + d_{3,0,-1}d_{4,0,0} + d_{5,0,-2} - d_{5,0,-1}))\\
&\quad - a(\xi)(\bar{x}d_{3,0,0} + i(d_{3,0,0}d_{4,0,0} + d_{3,0,-1}d_{4,0,1} + d_{5,0,-1} - d_{5,0,0})),\\
\ell_{1,1} &= b(\xi)(\bar{x}d_{3,0,0} + i(d_{3,0,0}d_{4,0,0} + d_{3,0,-1}d_{4,0,1} + d_{5,0,-1} - d_{5,0,0}))\\
&\quad - ia(\xi)(d_{3,0,0}d_{4,0,1} + d_{5,0,0}),\\
\ell_{1,2} &= ib(\xi)(d_{3,0,0}d_{4,0,1} + d_{5,0,0}),\\
\ell'_{1,-1} &= a(\xi)(-\mathbf{c}_{4,1}d_{3,0,-1} + d_{5,1,-1}),
\end{aligned}$$

$$\ell'_{1,0} = \mathbf{c}_{4,1}(b(\xi)d_{3,0,-1} - a(\xi)d_{3,0,0}) - (a(\xi) + b(\xi))d_{5,1,-1},$$
$$\ell'_{1,1} = b(\xi)(\mathbf{c}_{4,1}d_{3,0,0} + d_{5,1,-1})$$

where $\mathbf{c}_{4,1} = \mathbf{c}_{4,1}(\xi, \gamma)$ are defined in (3.34) with $k = 4$, and

$$d_{3,0,-1}(\xi,\gamma) = -\frac{\left(-\gamma\eta^2 + D(\xi) + \beta(\xi)\right)(v\gamma - 2k\theta + vD(\xi) - v\beta(\xi))}{4D(\xi)^2},$$
$$d_{3,0,0}(\xi,\gamma) = \frac{(\gamma + D(\xi) - \beta(\xi))(-v\gamma + 2k\theta + v(D(\xi) + \beta(\xi)))}{4D(\xi)^2},$$
$$d_{4,0,-1}(\xi,\gamma) = \frac{(\gamma - D(\xi) - \beta(\xi))(v\gamma - 2k\theta + vD(\xi) - v\beta(\xi))(D'(\xi) + \beta'(\xi))}{4D(\xi)^2},$$
$$d_{4,0,0}(\xi,\gamma) = \frac{2D(\xi)(ixD(\xi) + (-v\gamma + k\theta + v\beta(\xi))D'(\xi))}{2D(\xi)^2} - \frac{\left(vD(\xi)^2 + (\gamma - \beta(\xi))(v\gamma - 2k\theta - v\beta(\xi))\right)\beta'(\xi)}{2D(\xi)^2},$$
$$d_{4,0,1}(\xi,\gamma) = \frac{(\gamma + D(\xi) - \beta(\xi))(-v\gamma + 2k\theta + v(D(\xi) + \beta(\xi)))(D'(\xi) - \beta'(\xi))}{4D(\xi)^2},$$
$$d_{5,0,-2}(\xi,\gamma) = \frac{(-\gamma + D(\xi) + \beta(\xi))^2(v\gamma - k\theta + vD(\xi) - v\beta(\xi))(D'(\xi) + \beta'(\xi))}{8D(\xi)^4},$$
$$d_{5,0,-1}(\xi,\gamma) = \frac{\left(D(\xi)^2 - (-\gamma + \beta(\xi))^2\right)(vD(\xi)D'(\xi) + (v\gamma - k\theta - v\beta(\xi))\beta'(\xi))}{4D(\xi)^4},$$
$$d_{5,0,0}(\xi,\gamma) = \frac{(\gamma + D(\xi) - \beta(\xi))^2(-v\gamma + k\theta + v(D(\xi) + \beta(\xi)))(D'(\xi) - \beta'(\xi))}{8D(\xi)^4},$$
$$d_{5,1,-1}(\xi,\gamma) = \frac{\mathbf{c}_{2,1}(\xi,\gamma)(-v\gamma + k\theta + v\beta(\xi))}{2D(\xi)^2},$$

with $\mathbf{c}_{2,1}$ is as in (3.30) with $k = 2$, β is as in (3.12) and D is as in (3.20).

4.7 Figures and tables

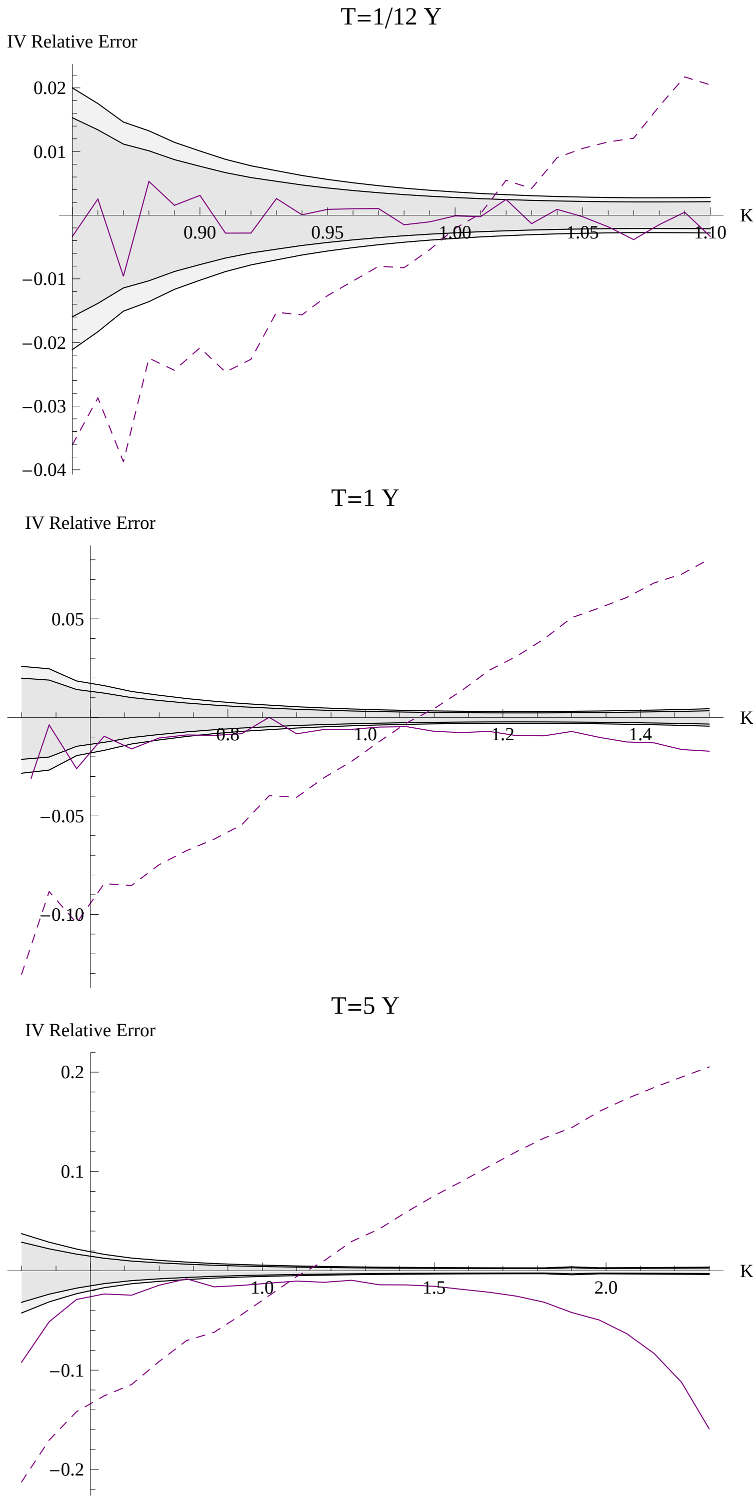

Figure 4.2: Implied volatilities in the CEV-Heston model for $T = 1/12, 1, 5$. Plot of the relative error for the 0-th order (dashed line) and the 1-st order approximation as a function of the strike, with respect to the 95% (dark grey area) and the 99% (bright grey area) confidence interval of the Monte Carlo simulation. Here, $\beta = \frac{2}{3}$, $S_0 = 1$, $\sqrt{v_0} = 0.3$, $r = 0.05$, $q = 0$, whereas the variance parameters are $k = 3$, $\sqrt{\theta} = 0.2$, $\rho = -0.5$ and $\eta = 0.8$.

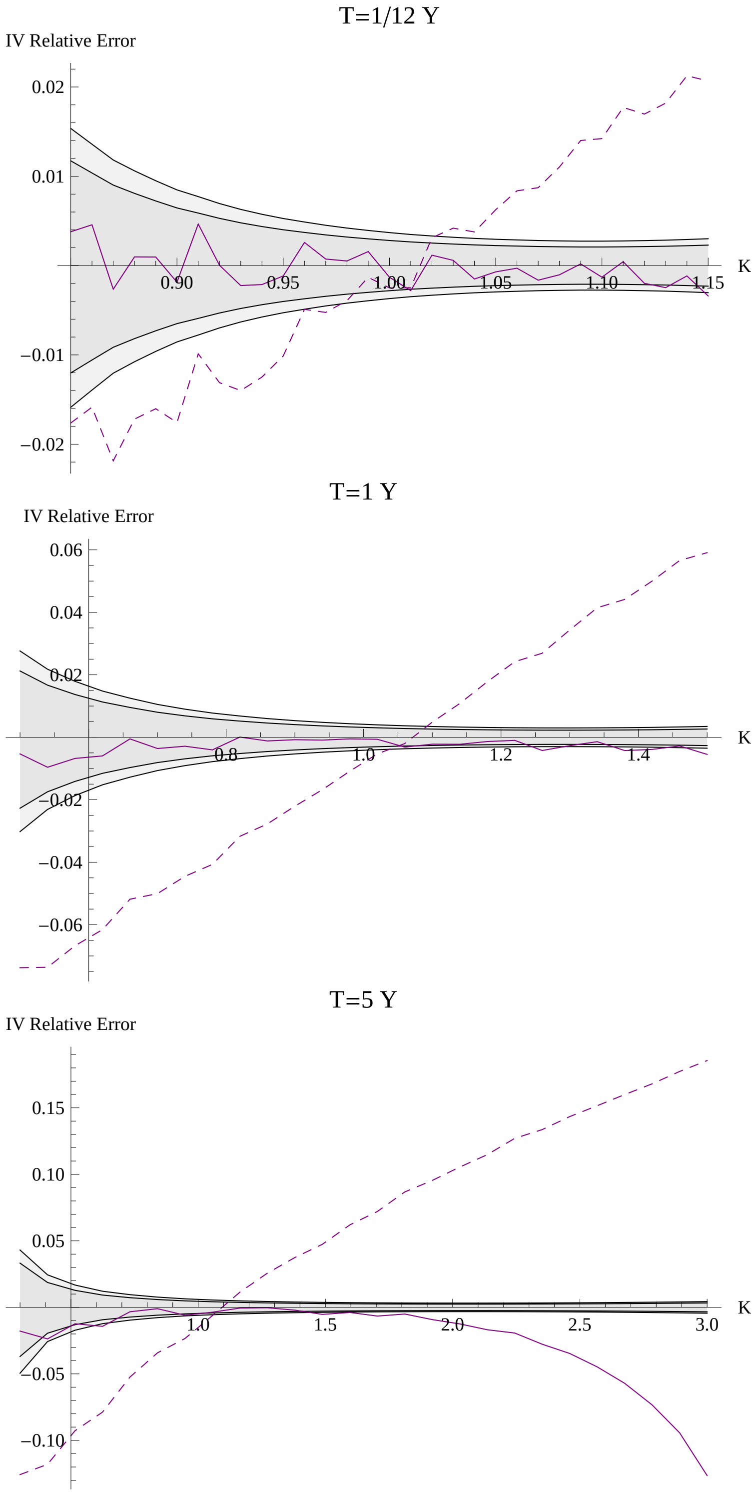

Figure 4.3: Implied volatilities in the CEV-Bates model for $T = 1/12, 1, 5$. Plot of the relative error for the 0-th order (dashed line) and the 1-st order approximation as a function of the strike, with respect to the 95% (dark grey area) and the 99% (bright grey area) confidence interval of the Monte Carlo simulation. Here, $\beta = \frac{2}{3}$, $S_0 = 1$, $\sqrt{v_0} = 0.35$, $r = 0.05$, $q = 0$, the variance parameters are $k = 3$, $\sqrt{\theta} = 0.2$, $\rho = -0.5$, $\eta = 0.8$, and the jumps parameters are $\lambda = 0.1$, $m = -0.2$ and $\delta = 0.1$.

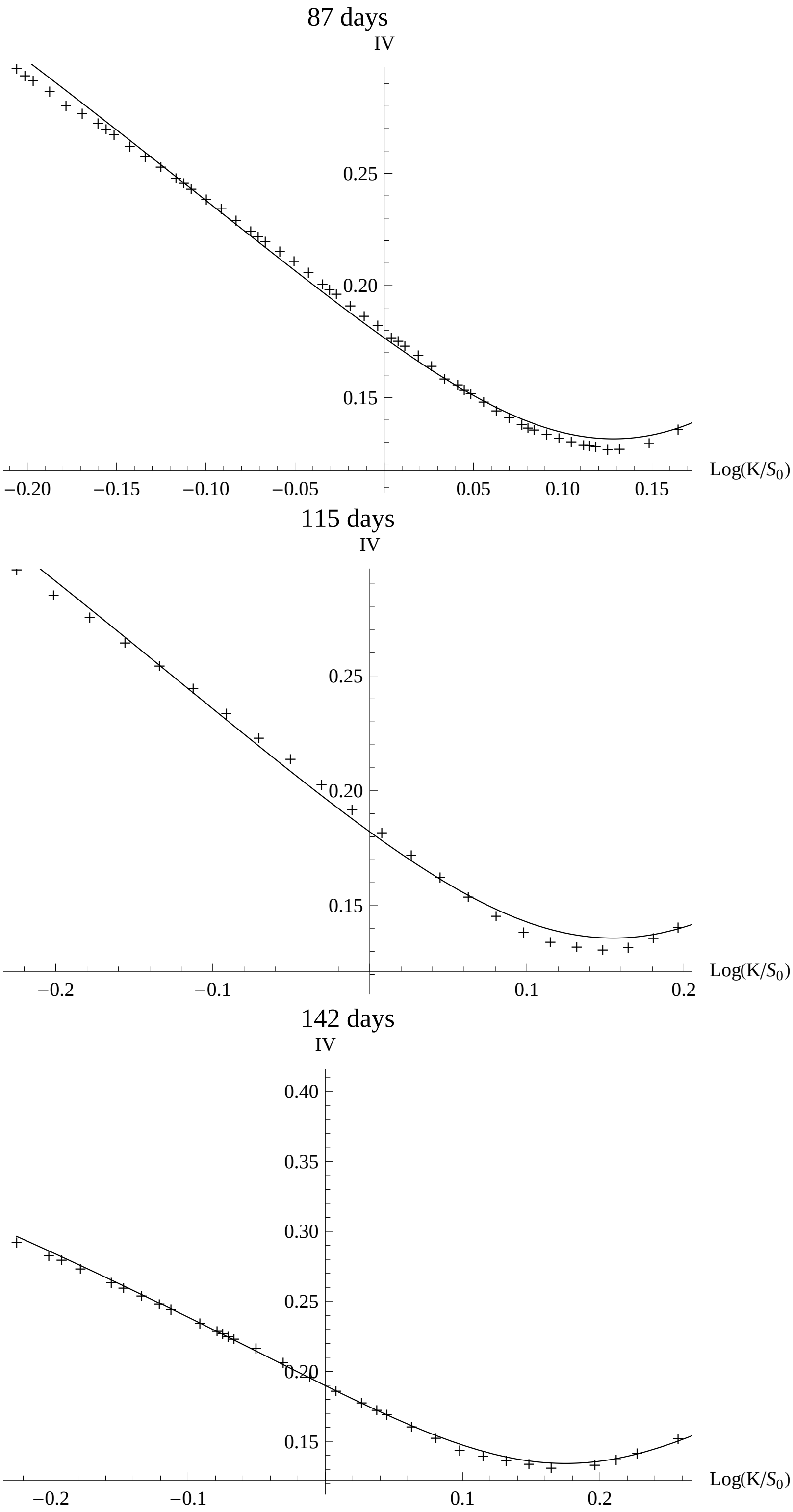

Figure 4.4: Interpolation of real market implied volatilities referred to the index S&P500 at closure in date January 24th 2012, as a function of the log-moneyness, for 87, 115 and 142-days maturities.

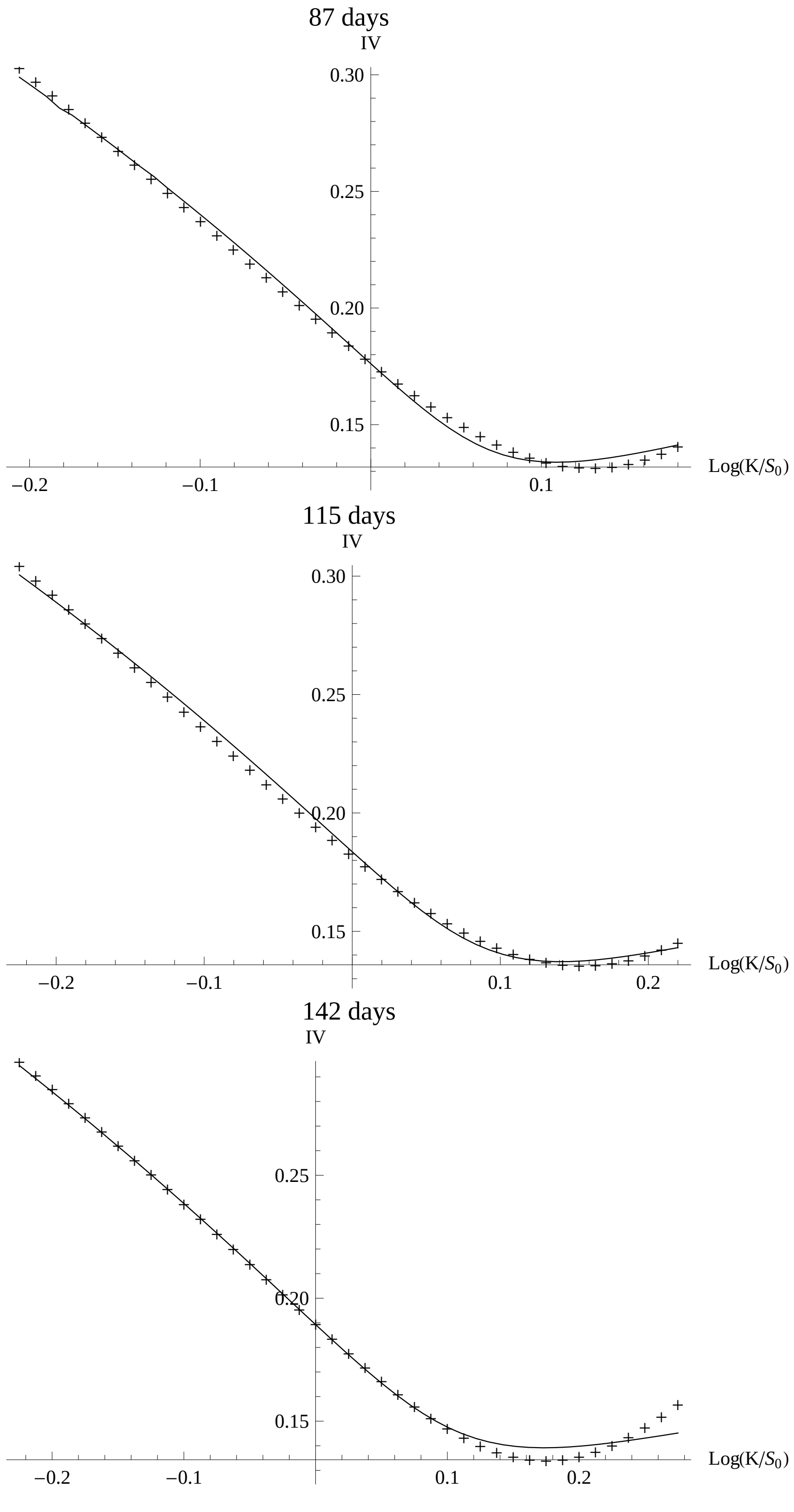

Figure 4.5: Fitting of the calibrated CEV-Heston model (optimal parameters as in (4.47)) to the interpolated real market volatilities, on a grid of 41 equidistant values for the log-moneyness, and for 87, 115 and 142-days maturities.

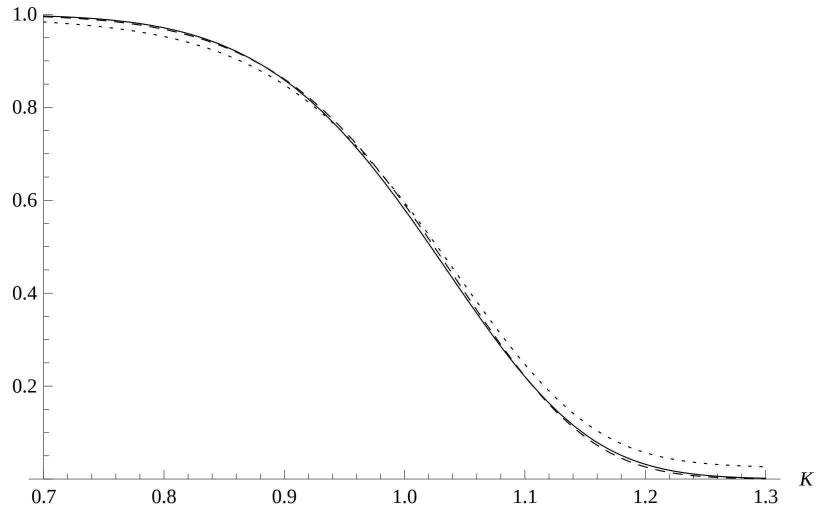

Figure 4.6: Plot of the Delta in Heston (continuous line), CEV-Heston (dashed line) and CEV-Bates (dotted line) models. The maturity is $T = \frac{1}{12}$ and the other parameters are as in (4.41)-(4.42)-(4.43)

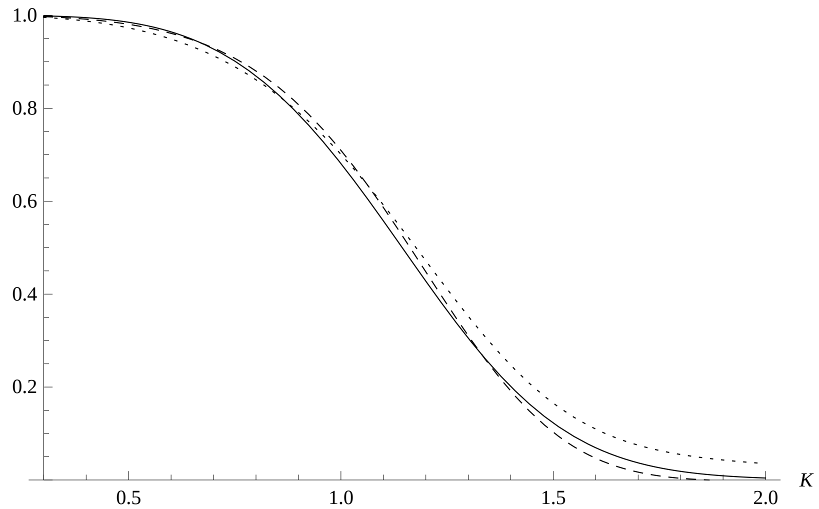

Figure 4.7: Plot of the Delta in Heston (continuous line), CEV-Heston (dashed line) and CEV-Bates (dotted line) models. The maturity is $T = 1$ and the other parameters are as in (4.41)-(4.42)-(4.43)

T	K	Call prices MC 99% c.i.	PP1	Implied volatility (%) MC 99% c.i.	PP1	Ex. time (sec.) MC	PP1
$\frac{1}{12}$	0.85	0.1590 – 0.1595	0.1593	44.64 – 45.71	45.18	59	0.30
	0.9	0.1154 – 0.1158	0.1157	42.28 – 42.90	42.74		
	0.95	0.0768 – 0.0772	0.0771	39.99 – 40.38	40.30		
	1	0.0455 – 0.0459	0.0456	37.81 – 38.08	37.89		
	1.05	0.0228 – 0.0230	0.0229	35.45 – 35.65	35.55		
	1.1	0.0092 – 0.0093	0.0092	33.25 – 33.42	33.31		
	1.15	0.0028 – 0.0028	0.0028	31.16 – 31.33	31.25		
1	0.5	0.5298 – 0.5312	0.5314	43.65 – 45.82	46.05	706	0.22
	0.7	0.3577 – 0.3590	0.3592	38.70 – 39.42	39.50		
	0.9	0.2085 – 0.2096	0.2097	33.57 – 33.90	33.94		
	1	0.1471 – 0.1480	0.1481	31.25 – 31.49	31.52		
	1.1	0.0968 – 0.0976	0.0975	29.16 – 29.34	29.33		
	1.3	0.0312 – 0.0316	0.0321	25.24 – 25.38	25.54		
	1.5	0.0060 – 0.0062	0.0062	22.15 – 22.27	22.24		
2.5	0.3	0.7383 – 0.7405	0.7397	42.66 – 46.88	45.41	1995	0.25
	0.5	0.5746 – 0.5767	0.5780	38.28 – 39.55	40.31		
	0.75	0.3934 – 0.3953	0.3964	34.22 – 34.72	35.03		
	1	0.2430 – 0.2445	0.2458	30.32 – 30.60	30.83		
	1.3	0.1139 – 0.1149	0.1166	26.51 – 26.67	26.93		
	1.7	0.0262 – 0.0267	0.0282	22.22 – 22.34	22.73		
	2	0.0052 – 0.0054	0.0044	19.50 – 19.62	18.91		
5	0.3	0.7742 – 0.7771	0.7772	39.05 – 41.75	41.89	3675	0.30
	0.5	0.6381 – 0.6408	0.6443	35.69 – 36.74	38.02		
	0.75	0.4889 – 0.4914	0.4943	32.83 – 33.34	33.93		
	1	0.3591 – 0.3613	0.3644	29.94 – 30.25	30.69		
	1.5	0.1649 – 0.1663	0.1729	25.15 – 25.32	26.05		
	2	0.0554 – 0.0562	0.0642	21.29 – 21.40	22.52		
	2.5	0.0117 – 0.0121	0.0124	18.15 – 18.25	18.36		

Table 4.1: Call options in the Quad-Heston model with initial variance $\sqrt{v_0} = 0.4$, $S_0 = 1$ and the other parameters as in (4.41)-(4.42)-(4.46): maturities (1^{st} column), strikes (2^{nd} column), 99% confidence interval of MC prices ($3^{\text{rd}} - 4^{\text{th}}$ columns), first order formula (5^{th} column), 99% confidence interval of MC implied volatilities ($6^{\text{th}} - 7^{\text{th}}$ columns), first order implied volatilities (8^{th} column), MC computation time (9^{th} column), first order formula computation time (10^{th} column)

T	K	Call prices MC 99% c.i.	Call prices PP1	Implied volatility (%) MC 99% c.i.	Implied volatility (%) PP1	Ex. time (sec.) MC	Ex. time (sec.) PP1
	0.85	0.1576 – 0.1581	0.1580	41.42 – 42.58	42.41		
	0.9	0.1121 – 0.1125	0.1123	37.50 – 38.18	37.87		
	0.95	0.0708 – 0.0712	0.0710	33.81 – 34.23	33.99		
$\frac{1}{12}$	1	0.0372 – 0.0375	0.0373	30.52 – 30.82	30.66	95	0.31
	1.05	0.0151 – 0.0153	0.0152	28.02 – 28.29	28.12		
	1.1	0.0052 – 0.0054	0.0053	27.47 – 27.84	27.72		
	1.15	0.0027 – 0.0029	0.0028	30.95 – 31.50	31.22		
	0.5	0.5289 – 0.5304	0.5295	42.18 – 44.66	43.26		
	0.7	0.3530 – 0.3545	0.3545	36.06 – 36.89	36.90		
	0.9	0.1998 – 0.2010	0.2006	30.88 – 31.27	31.12		
1	1	0.1359 – 0.1370	0.1369	28.30 – 28.60	28.56	1182	0.36
	1.1	0.0855 – 0.0865	0.0862	26.32 – 26.57	26.49		
	1.3	0.0312 – 0.0319	0.0319	25.23 – 25.48	25.45		
	1.5	0.0168 – 0.0174	0.0170	28.31 – 28.61	28.39		
	0.3	0.7372 – 0.7397	0.7378	39.86 – 45.41	41.56		
	0.5	0.5700 – 0.5724	0.5724	35.18 – 36.83	36.88		
	0.75	0.3857 – 0.3879	0.3873	32.11 – 32.71	32.55		
2.5	1	0.2352 – 0.2371	0.2367	28.93 – 29.27	29.19	3012	0.27
	1.3	0.1129 – 0.1144	0.1142	26.35 – 26.59	26.55		
	1.7	0.0430 – 0.0442	0.0434	26.13 – 26.37	26.20		
	2	0.0249 – 0.0259	0.0250	27.34 – 27.61	27.34		
	0.3	0.7704 – 0.7738	0.7729	34.43 – 38.66	37.63		
	0.5	0.6304 – 0.6337	0.6351	32.48 – 33.92	34.49		
	0.75	0.4796 – 0.4827	0.4832	30.90 – 31.56	31.65		
5	1	0.3536 – 0.3565	0.3565	29.14 – 29.56	29.56	5801	0.36
	1.5	0.1782 – 0.1806	0.1806	26.65 – 26.92	26.93		
	2	0.0880 – 0.0899	0.0894	25.63 – 25.87	25.81		
	3	0.0282 – 0.0295	0.0269	26.12 – 26.40	25.83		

Table 4.2: Call options in the Quad-Bates model with initial variance $\sqrt{v_0} = 0.3$, $S_0 = 1$ and the other parameters as in (4.41)-(4.42)-(4.43)-(4.46): maturities (1$^{\text{st}}$ column), strikes (2$^{\text{nd}}$ column), 99% confidence interval of MC prices (3$^{\text{rd}}$ $-$ 4$^{\text{th}}$ columns), first order formula (5$^{\text{th}}$ column), 99% confidence interval of MC implied volatilities (6$^{\text{th}}$ $-$ 7$^{\text{th}}$ columns), first order implied volatilities (8$^{\text{th}}$ column), MC computation time (9$^{\text{th}}$ column), first order formula computation time (10$^{\text{th}}$ column)

T	K	Call prices MC 99% c.i.	Call prices PP1	Implied volatility (%) MC 99% c.i.	Implied volatility (%) PP1	Ex. time (sec.) MC	Ex. time (sec.) PP1
	0.85	0.1589 – 0.1594	0.1592	44.51 – 45.59	45.08		
	0.9	0.1155 – 0.1159	0.1157	42.37 – 42.99	42.66		
	0.95	0.0769 – 0.0773	0.0771	40.05 – 40.44	40.26		
$\frac{1}{12}$	1	0.0455 – 0.0458	0.0456	37.76 – 38.02	37.89	27	0.30
	1.05	0.0229 – 0.0232	0.0230	35.56 – 35.76	35.57		
	1.1	0.0093 – 0.0094	0.0093	33.40 – 33.57	33.37		
	1.15	0.0029 – 0.0029	0.0028	31.40 – 31.57	31.34		
	0.5	0.5312 – 0.5327	0.5312	45.82 – 47.76	45.76		
	0.7	0.3586 – 0.3599	0.3588	39.16 – 39.80	39.30		
	0.9	0.2094 – 0.2105	0.2095	33.84 – 34.18	33.86		
1	1	0.1478 – 0.1488	0.1480	31.45 – 31.70	31.49	324	0.22
	1.1	0.0978 – 0.0985	0.0976	29.39 – 29.58	29.35		
	1.3	0.0325 – 0.0329	0.0325	25.66 – 25.80	25.65		
	1.5	0.0071 – 0.0073	0.0065	22.90 – 23.03	22.49		
	0.3	0.7415 – 0.7438	0.7395	48.48 – 51.56	45.06		
	0.5	0.5775 – 0.5797	0.5775	40.02 – 41.23	40.00		
	0.75	0.3954 – 0.3973	0.3958	34.75 – 35.26	34.86		
2.5	1	0.2458 – 0.2474	0.2456	30.83 – 31.12	30.80	880	0.26
	1.3	0.1174 – 0.1185	0.1172	27.07 – 27.24	27.03		
	1.7	0.0312 – 0.0317	0.0294	23.45 – 23.58	23.01		
	2	0.0085 – 0.0088	0.0053	21.39 – 21.52	19.59		
	0.3	0.7797 – 0.7828	0.7768	43.96 – 46.21	41.50		
	0.5	0.6437 – 0.6466	0.6434	37.82 – 38.86	37.70		
	0.7	0.5220 – 0.5247	0.5219	34.50 – 35.10	34.48		
5	1	0.3637 – 0.3660	0.3642	30.59 – 30.93	30.66	1611	0.28
	1.5	0.1750 – 0.1767	0.1744	26.29 – 26.48	26.23		
	2	0.0706 – 0.0716	0.0668	23.38 – 23.52	22.88		
	2.5	0.0235 – 0.0241	0.0152	21.21 – 21.34	19.16		

Table 4.3: Call options in the CEV-Heston model with $\beta = \frac{1}{2}$, initial variance $\sqrt{v_0} = 0.4$, $S_0 = 1$ and the other parameters as in (4.41)-(4.42): maturities (1^{st} column), strikes (2^{nd} column), 99% confidence interval of MC prices ($3^{\text{rd}} - 4^{\text{th}}$ columns), first order formula (5^{th} column), 99% confidence interval of MC implied volatilities ($6^{\text{th}} - 7^{\text{th}}$ columns), first order implied volatilities (8^{th} column), MC computation time (9^{th} column), first order formula computation time (10^{th} column)

		Call prices		Implied volatility (%)		Ex. time (sec.)	
T	K	MC 99% c.i.	PP1	MC 99% c.i.	PP1	MC	PP1
$\frac{1}{12}$	0.85	0.1577 – 0.1582	0.1580	41.72 – 42.86	42.35	63	0.19
	0.9	0.1119 – 0.1123	0.1123	37.21 – 37.89	37.82		
	0.95	0.0708 – 0.0712	0.0709	33.83 – 34.25	33.97		
	1	0.0372 – 0.0376	0.0373	30.56 – 30.85	30.66		
	1.05	0.0150 – 0.0153	0.0152	27.95 – 28.22	28.13		
	1.1	0.0052 – 0.0055	0.0053	27.60 – 27.98	27.75		
	1.15	0.0027 – 0.0029	0.0028	31.04 – 31.59	31.23		
1	0.5	0.5293 – 0.5308	0.5294	42.83 – 45.22	43.12	746	0.33
	0.7	0.3540 – 0.3555	0.3543	36.63 – 37.45	36.80		
	0.9	0.2001 – 0.2014	0.2004	30.98 – 31.37	31.08		
	1	0.1361 – 0.1372	0.1368	28.36 – 28.65	28.55		
	1.1	0.0858 – 0.0868	0.0863	26.39 – 26.64	26.51		
	1.3	0.0316 – 0.0324	0.0320	25.37 – 25.62	25.50		
	1.5	0.0169 – 0.0175	0.0170	28.35 – 28.64	28.41		
2.5	0.3	0.7377 – 0.7402	0.7378	41.29 – 46.35	41.36	1969	0.20
	0.5	0.5724 – 0.5748	0.5722	36.84 – 38.37	36.73		
	0.75	0.3865 – 0.3887	0.3870	32.32 – 32.93	32.46		
	1	0.2362 – 0.2382	0.2366	29.11 – 29.46	29.18		
	1.3	0.1148 – 0.1163	0.1145	26.64 – 26.89	26.60		
	1.7	0.0441 – 0.0452	0.0438	26.35 – 26.59	26.28		
	2	0.0257 – 0.0266	0.0252	27.55 – 27.82	27.42		
5	0.3	0.7732 – 0.7767	0.7727	38.03 – 41.44	37.41	3717	0.33
	0.5	0.6354 – 0.6388	0.6347	34.61 – 35.96	34.33		
	0.75	0.4816 – 0.4848	0.4828	31.32 – 31.99	31.57		
	1	0.3566 – 0.3596	0.3564	29.57 – 30.00	29.55		
	1.5	0.1821 – 0.1846	0.1814	27.09 – 27.37	27.01		
	2	0.0932 – 0.0952	0.0905	26.29 – 26.53	25.95		
	3	0.0311 – 0.0324	0.0278	26.73 – 27.02	26.02		

Table 4.4: Call options in the CEV-Bates model with $\beta = \frac{1}{2}$, initial variance $\sqrt{v_0} = 0.3$, $S_0 = 1$ and the other parameters as in (4.41)-(4.42)-(4.43): maturities (1$^{\text{st}}$ column), strikes (2$^{\text{nd}}$ column), 99% confidence interval of MC prices (3$^{\text{rd}} - 4^{\text{th}}$ columns), first order formula (5$^{\text{th}}$ column), 99% confidence interval of MC implied volatilities (6$^{\text{th}} - 7^{\text{th}}$ columns), first order implied volatilities (8$^{\text{th}}$ column), MC computation time (9$^{\text{th}}$ column), first order formula computation time (10$^{\text{th}}$ column)

Chapter 5

A family of density expansions for Lévy-type processes with default

Based on a joint work ([171]) with Dr. Matthew Lorig and Prof. A. Pascucci.

Abstract: we consider a defaultable asset whose risk-neutral pricing dynamics are described by an exponential Lévy-type martingale subject to default. This class of models allows for local volatility, local default intensity, and a locally dependent Lévy measure. Generalizing and extending the novel adjoint expansion technique of [185], we derive a family of asymptotic expansions for the transition density of the underlying as well as for European-style option prices and defaultable bond prices. For the density expansion, we also provide error bounds for the truncated asymptotic series. Additionally, for pure diffusion processes, we derive an asymptotic expansion for the implied volatility induced by European calls/puts. Our method is numerically efficient; approximate transition densities and European option prices are computed via Fourier transforms; approximate bond prices are computed as finite series. Additionally, as in [185], for models with Gaussian-type jumps, approximate option prices can be computed in closed form. Numerical examples confirming the effectiveness and versatility of our method are provided, as is sample Mathematica code.

Keywords: Local volatility, Lévy-type process, Asymptotic expansion, Pseudo-differential calculus, Defaultable asset.

5.1 Introduction

A local volatility model is a model in which the volatility σ_t of an asset X is a function of time t and the present level of X. That is, $\sigma_t = \sigma(t, X_t)$. Among local volatility models, perhaps the most well-known is the constant elasticity of variance (CEV) model in [65]. One advantage of local volatility models is that transition densities of the underlying – as well as European option prices – are often available in closed-form as infinite series of special functions (see [157] and references therein). Another advantage of local volatility models is that, for models whose transition density is not available in closed form, accurate density and option price approximations are readily available (see, [183], for example). Finally, in [82] the author shows that one can always find a local volatility function $\sigma(t, x)$ that fits the market's implied volatility surface exactly. Thus, local volatility models are quite flexible.

Despite the above advantages, local volatility models do suffer some shortcomings. Most notably, local volatility models do not allow for the underlying to experience jumps, the need for which is well-documented in literature (see [85] and references therein). Recently, there has been much interest in combining local volatility models and models with jumps. In [4], for example, they discuss extensions of the implied diffusion approach in [82] to asset processes with Poisson jumps (i.e., jumps with finite activity). In [25] the authors derive analytically tractable option pricing approximations for models that include local volatility and a Poisson jump process. Their approach relies on asymptotic expansions around small diffusion and small jump frequency/size limits.

We also recall that in the Chapter 3 we have already considerd general local volatility models with independent Lévy jumps (possibly with infinite activity), and we have constructed an approximated solution by expanding the local volatility function as a Taylor power series. While all of the methods just mentioned allow for local volatility and independent jumps, none of these methods allow for state-dependent jumps.

Stochastic jump-intensity was recently identified as an important feature of equity models (see [57]). A locally dependent Lévy measure allows for this possibility. Recently, two different approaches have been taken to modeling assets with locally-dependent jump measures. In [175] they time-change a local volatility model with a Lévy subordinator. In addition to admitting exact option-pricing formulas, the subordination technique results in a locally-dependent Lévy measure. In [165] the author considers another class of models that allow for state-dependent jumps. The author builds a Lévy-type processes with local volatility, local default intensity, and a local Lévy measure by considering state-dependent perturbations around a constant coefficient Lévy process. In addition to pricing formula, the author provides an exact expansion for the induced implied volatility surface.

In this chapter, we consider scalar Lévy-type processes with regular coefficients, which naturally include all the models mentioned above. Generalizing and extending the methods described in Chapter 3 (see [185]), we derive a family of asymptotic expansions for the transition densities of these processes, as well as for European-style derivative prices and defaultable bond prices. The key contributions of this chapter are as follows:

- We allow for a locally-dependent Lévy measure and local default intensity, whereas in Chapter 3 we have only considered a locally independent Lévy measure and did not allow for the possibility of default. A state-dependent Lévy measure is an important feature because it allows for incorporating local dependence into infinite activity Lévy models that have no diffusion component, such as Variance Gamma (see [172]) and

CGMY/Kobol (see [36, 47]).

- As we make no small diffusion or small jump size/intensity assumption, our formulae are valid for any Lévy type process with smooth and bounded coefficients, independent of the relative size of the coefficients.

- Whereas in Chapter 3 we have expanded the local volatility and drift functions as a Taylor series about an arbitrary point, i.e. $f(x) = \sum_n a_n(x - \bar{x})^n$, in order to achieve our approximation result, we expand the local volatility, drift, killing rate and Lévy measure in an arbitrary basis, i.e. $f(x) = \sum_n c_n B_n(x)$. This is advantageous because the Taylor series typically converges only locally, whereas other choices of the basis functions B_n may provide global convergence in suitable functional spaces.

- Using techniques from pseudo-differential calculus, we provide explicit formulae for the Fourier transform of every term in the transition density and option-pricing expansions. In the case of state dependent Gaussian jumps the respective inverse Fourier transforms can be explicitly computed, thus providing closed form approximations for densities and prices. In the general case, the density and pricing approximations can be computed quickly and easily as inverse Fourier transforms. Additionally, when considering defaultable bonds, approximate prices are computed as a finite sum; no numerical integration is required even in the general case.

- For models with Gaussian-type jumps, we provide pointwise error estimates for transition densities. Thus, we extend the previous results where we only considered the purely diffusive case. Additionally, our error estimates allow for jumps with locally dependent mean, variance and intensity. Thus, for models with Gaussian-type jumps, our results also extend the results in [25], where only the case of a constant Lévy measure is considered.

- For local volatility models with no jumps and no possibility of default, we provide an asymptotic expansion for implied volatilities corresponding to European calls/puts. As with approximate bond prices, approximate implied volatilities are computed as a finite sum of simple functions; no numerical integration is required.

The rest of this chapter proceeds as follows. In Section, 5.2 we introduce a general class of exponential Lévy-type models with locally-dependent volatility, default intensity and Lévy measure. We also describe our modeling assumptions. Next, in Section 5.3, we introduce the European option-pricing problem and derive a partial integro-differential equation (PIDE) for the price of an option. In Section 5.4 we derive a formal asymptotic expansion (in fact, a family of asymptotic expansions) for the function that solves the option pricing PIDE (Theorem 4.4). Next, in Section 5.5, we provide rigorous error estimates for our asymptotic expansion for models with Gaussian-type jumps (Theorem 5.1). In Section 5.6 we derive an implied volatility expansion for pure diffusion models (i.e., models with no default and no jumps). This expansion involves no special functions and no integrals and can therefore be computed extremely quickly. Lastly, in Section 5.7, we provide numerical examples that illustrate the effectiveness and versatility of our methods. Some concluding remarks are given in Section 5.8. Some technical proofs and some Mathematica code are provided in the Appendix.

5.2 General Lévy-type exponential martingales

For simplicity, we assume a frictionless market, no arbitrage, zero interest rates and no dividends. Our results can easily be extended to include locally dependent interest rates and dividends. We take, as given, an equivalent martingale measure $\mathbb{Q}$, chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{Q})$ satisfying the usual hypothesis of completeness and right continuity. The filtration $\mathcal{F}_t$ represents the history of the market. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{Q}$. We consider a defaultable asset S whose risk-neutral dynamics are given by

$$\begin{cases} S_t = \mathbb{I}_{\{\zeta > t\}} e^{X_t}, \\ dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{\mathbb{R}} d\overline{N}_t(t, X_{t-}, dz)z, \\ d\overline{N}_t(t, X_{t-}, dz) = dN_t(t, X_{t-}, dz) - \nu(t, X_{t-}, dz)dt, \\ \zeta = \inf\left\{ t \geq 0 : \int_0^t \gamma(s, X_s)ds \geq \mathcal{E} \right\} \end{cases} \tag{2.1}$$

Here, X is a Lévy-type process with local drift function $\mu(t, x)$, local volatility function $\sigma(t, x) \geq 0$ and state-dependent Lévy measure $\nu(t, x, dz)$. We shall denote by $\mathcal{F}_t^X$ the filtration generated by X. The random variable $\mathcal{E} \sim \text{Exp}(1)$ has an exponential distribution and is independent of X. Note that ζ, which represents the default time of S, is constructed here trough the so-called *canonical construction* (see [32]), and is the first arrival time of a doubly stochastic Poisson process with local intensity function $\gamma(t, x) \geq 0$. This way of modeling default is also considered in a local volatility setting in [49, 156], and for exponential Lévy models in [46].

We assume that the coefficients are measurable in t and suitably smooth in x to ensure the existence of a solution to (2.1) (see [181], Theorem 1.19). We also assume the following boundedness condition which is rather standard in the financial applications: there exists a Lévy measure

$$\bar{\nu}(dz) := \sup_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}} \nu(t, x, dz)$$

such that

$$\int_{\mathbb{R}} \bar{\nu}(dz) \min(1, z^2) < \infty, \qquad \int_{|z| \geq 1} \bar{\nu}(dz) e^z < \infty, \qquad \int_{|z| \geq 1} \bar{\nu}(dz)|z| < \infty. \tag{2.2}$$

Since ζ is not $\mathcal{F}_t^X$-measurable we introduce the filtration $\mathcal{F}_t^D = \sigma\left(\{\zeta \leq s\}, s \leq t\right\}$ in order to keep track of the event $\{\zeta \leq t\}$. The filtration of a market observer, then, is $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^D$. In the absence of arbitrage, S must be an $\mathcal{F}_t$-martingale. Thus, the drift $\mu(t, x)$ is fixed by $\sigma(t, x)$, $\nu(t, x, dz)$ and $\gamma(t, x)$ in order to satisfy the martingale condition[1]

$$\mu(t, x) = \gamma(t, x) - a(t, x) - \int_{\mathbb{R}} \nu(t, x, dz)(e^z - 1 - z), \qquad a(t, x) := \frac{1}{2}\sigma^2(t, x). \tag{2.3}$$

[1] We provide a derivation of the martingale condition in Section 5.3 Remark 3.2 below.

We remark that the existence of the density of X is not strictly necessary in our analysis. Indeed, since our formulae are carried out in Fourier space, we provide approximations of the characteristic function of X and all of our computations are still formally correct even when dealing with distributions that are not absolutely continuous with respect to the Lebesgue measure.

We will relax some of thee assumptions for the numerical examples provided in Section 5.7. Even without the above assumptions in force, our numerical results indicate that our approximation techniques work well.

5.3 Option pricing

We consider a European derivative expiring at time T with payoff $H(S_T)$ and we denote by V its no-arbitrage price. For convenience, we introduce

$$h(x) := H(e^x) \qquad \text{and} \qquad K := H(0).$$

Proposition 3.1. *The price V_t is given by*

$$V_t = K + \mathbb{I}_{\{\zeta>t\}}\mathbb{E}\left[e^{-\int_t^T \gamma(s,X_s)ds}\left(h(X_T)-K\right)|X_t\right], \qquad t \leq T. \tag{3.4}$$

The proof can be found in Section 2.2 of [156]. Because our notation differs from that of [156], and because a short proof is possible by using the results of [131], for the reader's convenience, we provide a derivation of Proposition 3.1 here.

Proof. Using risk-neutral pricing, the value V_t of the derivative at time t is given by the conditional expectation of the option payoff

$$\begin{aligned} V_t &= \mathbb{E}\left[H(S_T)|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[h(X_T)\mathbb{I}_{\{\zeta>T\}}|\mathcal{F}_t\right] + K\mathbb{E}\left[\mathbb{I}_{\{\zeta\leq T\}}|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[h(X_T)\mathbb{I}_{\{\zeta>T\}}|\mathcal{F}_t\right] + K - K\mathbb{E}\left[\mathbb{I}_{\{\zeta>T\}}|\mathcal{F}_t\right] \\ &= K + \mathbb{I}_{\{\zeta>t\}}\mathbb{E}\left[e^{-\int_t^T \gamma(s,X_s)ds}\left(h(X_T)-K\right)|\mathcal{F}_t^X\right] \\ &= K + \mathbb{I}_{\{\zeta>t\}}\mathbb{E}\left[e^{-\int_t^T \gamma(s,X_s)ds}\left(h(X_T)-K\right)|X_t\right], \end{aligned}$$

where we have used Corollary 7.3.4.2 from [131] to write

$$\mathbb{E}\left[(h(X_T)-K)\mathbb{I}_{\{\zeta>T\}}|\mathcal{F}_t\right] = \mathbb{I}_{\{\zeta>t\}}\mathbb{E}\left[(h(X_T)-K)e^{-\int_t^T \gamma(s,X_s)ds}|\mathcal{F}_t^X\right].$$

□

Remark 3.2. *By Proposition 3.1 with $K=0$ and $h(x)=e^x$, we have that the martingale condition $S_t = \mathbb{E}\left[S_T|\mathcal{F}_t\right]$ is equivalent to*

$$\mathbb{I}_{\{\zeta>t\}}e^{X_t} = \mathbb{I}_{\{\zeta>t\}}\mathbb{E}\left[e^{-\int_t^T \gamma(s,X_s)ds + X_T}|\mathcal{F}_t\right].$$

Therefore, we see that S is a martingale if and only if the process $\exp\left(-\int_0^t \gamma\left(s,X_s\right)ds + X_t\right)$ *is a martingale. The drift condition (2.3) follows by applying the Itô's formula to the process* $\exp\left(-\int_0^t \gamma\left(s,X_s\right)ds + X_t\right)$ *and setting the drift term to zero.*

From (3.4) one sees that, in order to compute the price of an option, we must evaluate functions of the form[2]

$$v(t,x) := \mathbb{E}\left[e^{-\int_t^T \gamma(s,X_s)ds} h(X_T)|X_t = x\right]. \tag{3.5}$$

By a direct application of the Feynman-Kac representation theorem, see for instance [188, Theorem 14.50], the classical solution of the following Cauchy problem,

$$(\partial_t + \mathcal{A}^{(t)})v = 0, \qquad\qquad v(T,x) = h(x), \tag{3.6}$$

when it exists, is equal to the function $v(t,x)$ in (3.5), where

$$\begin{aligned}\mathcal{A}^{(t)} f(x) &= \gamma(t,x)(\partial_x f(x) - f(x)) + a(t,x)(\partial_x^2 f(x) - \partial_x f(x)) \\ &\quad - \int_{\mathbb{R}} \nu(t,x,dz)(e^z - 1 - z)\partial_x f(x) + \int_{\mathbb{R}} \nu(t,x,dz)(f(x+z) - f(x) - z\partial_x f(x)),\end{aligned} \tag{3.7}$$

is the characteristic operator of the SDE (2.1). In order to shorten the notation, in the sequel we will suppress the explicit dependence on t in $\mathcal{A}^{(t)}$ by referring to it just as $\mathcal{A}$.

Sufficient conditions for the existence and uniqueness of solutions of second order elliptic integro-differential equations are given in Theorem II.3.1 of [104]. We denote by $p(t,x;T,y)$ the fundamental solution of the operator $(\partial_t + \mathcal{A})$, which is defined as the solution of (3.6) with $h = \delta_y$. Note that $p(t,x;T,y)$ represents also the transition density of $\log S$ [3]

$$p(t,x;T,y)dy = \mathbb{Q}[\log S_T \in dy | \log S_t = x], \qquad x,y \in \mathbb{R}, \qquad t < T.$$

Note also that $p(t,x;T,y)$ is not a probability density since (due to the possibility that $S_T = 0$) we have

$$\int_{\mathbb{R}} p(t,x;T,y)dy \leq 1.$$

Given the existence of the fundamental solution of $(\partial_t + \mathcal{A})$, we have that for any h that is integrable with respect to the density $p(t,x;T,\cdot)$, the Cauchy problem (3.6) has a classical solution that can be represented as

$$v(t,x) = \int_{\mathbb{R}} h(y)p(t,x;T,y)dy.$$

Remark 3.3. *If $\mathcal{G}$ is the generator of a scalar Markov process and $dom(\mathcal{G})$ contains $\mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decaying functions on $\mathbb{R}$, then $\mathcal{G}$ must have the following form:*

$$\mathcal{G}f(x) = -\gamma(x)f(x) + \mu(x)\partial_x f(x) + a(x)\partial_x^2 f(x) + \int_{\mathbb{R}} \nu(x,dz)(f(x+z) - f(x) - \mathbb{I}_{\{|z|<R\}} z\partial_x f(x)), \tag{3.8}$$

where $\gamma \geq 0$, $a \geq 0$, ν is a Lévy measure for every x and $R \in [0,\infty]$ (see [123], Proposition 2.10). If one enforces on $\mathcal{G}$ the drift and integrability conditions (2.2) *and* (2.3)*, which are needed to ensure that S is a martingale, and allow setting $R = \infty$, then the operators* (3.7) *and* (3.8) *coincide (in the time-homogeneous case). Thus, the class of models we consider in this paper encompasses all non-negative scalar Markov martingales that satisfy the regularity and boundedness conditions of Section 5.2.*

[2]Note: we can accommodate stochastic interest rates and dividends of the form $r_t = r(t,X_t)$ and $q_t = q(t,X_t)$ by simply making the change: $\gamma(t,x) \to \gamma(t,x) + r(t,x)$ and $\mu(t,x) \to \mu(t,X_t) + r(t,X_t) - q(t,X_t)$.

[3]Here with $\log S$ we denote the process $X_t \mathbb{I}_{\{\zeta > t\}} - \infty \mathbb{I}_{\{\zeta \leq t\}}$.

Remark 3.4. *In what follows we shall systematically make use of the language of pseudo-differential calculus. More precisely, let us denote by*

$$\psi_\xi(x) = \psi_x(\xi) = \frac{1}{\sqrt{2\pi}} e^{i\xi x}, \qquad x, \xi \in \mathbb{R}, \tag{3.9}$$

the so-called oscillating exponential function. Then $\mathcal{A}$ can be characterized by its action on oscillating exponential functions. Indeed, we have

$$\mathcal{A}\psi_\xi(x) = \phi(t, x, \xi)\psi_\xi(x),$$

where

$$\begin{aligned} \phi(t, x, \xi) &= \gamma(t, x)(i\xi - 1) + a(t, x)(-\xi^2 - i\xi) \\ &\quad - \int_{\mathbb{R}} \nu(t, x, dz)(e^z - 1 - z)i\xi + \int_{\mathbb{R}} \nu(t, x, dz)(e^{i\xi z} - 1 - i\xi z), \end{aligned} \tag{3.10}$$

is called the symbol of $\mathcal{A}$. Noting that

$$e^{z\partial_x} u(x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \partial_x^n u(x) = u(x + z),$$

for any analytic function $u(x)$, we have

$$\int_{\mathbb{R}} \nu(t, x, dz) \left(u(x + z) - u(x) - z\partial_x u(x)\right) = \int_{\mathbb{R}} \nu(t, x, dz) \left(e^{z\partial_x} - 1 - z\partial_x\right) u(x). \tag{3.11}$$

Then $\mathcal{A}$ can be represented as

$$\mathcal{A} = \phi(t, x, \mathcal{D}), \qquad \mathcal{D} = -i\partial_x,$$

since by (3.10) *and* (3.11)

$$\begin{aligned} \phi(t, x, \mathcal{D}) &= \gamma(t, x)(\partial_x - 1) + a(t, x)(\partial_x^2 - \partial_x) \\ &\quad - \int_{\mathbb{R}} \nu(t, x, dz)(e^z - 1 - z)\partial_x + \int_{\mathbb{R}} \nu(t, x, dz) \left(e^{z\partial_x} - 1 - z\partial_x\right). \end{aligned}$$

If coefficients $a(t), \gamma(t), \nu(t, dz)$ are independent of x, then we have the usual characterization of $\mathcal{A}$ as a multiplication by ϕ operator in the Fourier space:

$$\mathcal{A} = \mathcal{F}^{-1}\left(\phi(t, \cdot)\mathcal{F}\right), \qquad \phi(t, \cdot) := \phi(t, x, \cdot),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the (direct) Fourier and inverse Fourier transform operators respectively:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)dx, \qquad \mathcal{F}^{-1}f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} f(\xi)d\xi.$$

Moreover, if the coefficients $a, \gamma, \nu(dz)$ are independent of both t and x, then $\mathcal{A}$ is the generator of a Lévy process X and $\phi(\cdot) := \phi(t, x, \cdot)$ is the characteristic exponent of X:

$$\mathbb{E}\left[e^{i\xi X_t}\right] = e^{t\phi(\xi)}.$$

5.4 Density and option price expansions (a formal description)

Our goal is to construct an approximate solution of Cauchy problem (3.6). We assume that the symbol of $\mathcal{A}$ admits an expansion of the form

$$\phi(t,x,\xi) = \sum_{n=0}^{\infty} B_n(x)\phi_n(t,\xi), \tag{4.12}$$

where $\phi_n(t,\xi)$ is of the form

$$\begin{aligned}\phi_n(t,\xi) &= \gamma_n(t)(i\xi - 1) + a_n(t)(-\xi^2 - i\xi) \\ &\quad - \int_{\mathbb{R}} \nu_n(t,dz)(e^z - 1 - z)i\xi + \int_{\mathbb{R}} \nu_n(t,dz)(e^{iz\xi} - 1 - iz\xi).\end{aligned} \tag{4.13}$$

and $\{B_n\}_{n\geq 0}$ is some expansion basis with B_n being an analytic function for each $n \geq 0$, and $B_0 \equiv 1$ (see Examples 4.1, 4.2 and 4.3 below). Note that $\phi_n(t,\xi)$ is the symbol of an operator

$$\mathcal{A}_n := \phi_n(t,\mathcal{D}), \qquad \mathcal{D} = -i\partial_x, \tag{4.14}$$

so that

$$\mathcal{A}_n \psi_\xi(x) = \phi_n(t,\xi)\psi_\xi(x).$$

Thus, formally the generator $\mathcal{A}$ can be written as follows

$$\mathcal{A} = \sum_{n=0}^{\infty} B_n(x)\mathcal{A}_n. \tag{4.15}$$

Note that $\mathcal{A}_0$ is the generator of a time-dependent Lévy-type process $X^{(0)}$. In the time-independent case $X^{(0)}$ is a Lévy process and $\phi_0(\cdot) := \phi_0(t,\cdot)$ is its characteristic exponent.

Example 4.1 (Taylor series expansion)**.** *[185] approximate the drift and diffusion coefficients of $\mathcal{A}$ as a power series about an arbitrary point $\bar{x} \in \mathbb{R}$. In our more general setting, this corresponds to setting $B_n(x) = (x - \bar{x})^n$ and expanding the diffusion and killing coefficients $a(t,\cdot)$ and $\gamma(t,\cdot)$, as well as the Lévy measure $\nu(t,\cdot,dz)$ as follows:*

$$\begin{cases} a(t,x) = \displaystyle\sum_{n=0}^{\infty} a_n(t,\bar{x})B_n(x), & a_n(t,\bar{x}) = \dfrac{1}{n!}\partial_x^n a(t,\bar{x}), \\ \gamma(t,x) = \displaystyle\sum_{n=0}^{\infty} \gamma_n(t,\bar{x})B_n(x), & \gamma_n(t,\bar{x}) = \dfrac{1}{n!}\partial_x^n \gamma(t,\bar{x}), \\ \nu(t,x,dz) = \displaystyle\sum_{n=0}^{\infty} \nu_n(t,\bar{x},dz)B_n(x), & \nu_n(t,\bar{x},dz) = \dfrac{1}{n!}\partial_x^n \nu(t,\bar{x},dz). \end{cases} \tag{4.16}$$

In this case, (4.12) *and* (4.15) *become (respectively)*

$$\phi(t,x,\xi) = \sum_{n=0}^{\infty}(x-\bar{x})^n\phi_n(t,\xi), \qquad \mathcal{A} = \sum_{n=0}^{\infty}(x-\bar{x})^n\phi_n(t,\mathcal{D}),$$

where, for all $n \geq 0$, the symbol $\phi_n(t,\xi)$ is given by (4.13) *with coefficients given by* (4.16)*. Again, the choice of $\bar{x}$ is somewhat arbitrary. However, a convenient choice that seems to work well in most applications is to choose $\bar{x}$ near X_t, the current level of X (see also Remark 4.2 for further explanations). Hereafter, to simplify notation, when discussing implementation of the Taylor-series expansion, we suppress the $\bar{x}$-dependence: $a_n(t,\bar{x}) \to a_n(t)$, $\gamma_n(t,\bar{x}) \to \gamma_n(t)$ and $\nu_n(t,\bar{x},dz) \to \nu_n(t,dz)$.*

Example 4.2 (Two-point Taylor series expansion)**.** *Suppose f is an analytic function with domain $\mathbb{R}$ and $\bar{x}_1, \bar{x}_2 \in \mathbb{R}$. Then the two-point Taylor series of f is given by*

$$f(x) = \sum_{n=0}^{\infty} \left(c_n(\bar{x}_1,\bar{x}_2)(x-\bar{x}_1) + c_n(\bar{x}_2,\bar{x}_1)(x-\bar{x}_2)\right)(x-\bar{x}_1)^n(x-\bar{x}_2)^n, \tag{4.17}$$

where

$$c_0(\bar{x}_1,\bar{x}_2) = \frac{f(\bar{x}_2)}{\bar{x}_2-\bar{x}_1}, \quad c_n(\bar{x}_1,\bar{x}_2) = \sum_{k=0}^{n} \frac{(k+n-1)!}{k!n!(n-k)!} \frac{(-1)^k k \partial_{\bar{x}_1}^{n-k} f(\bar{x}_1) + (-1)^{n+1} n \partial_{\bar{x}_2}^{n-k} f(\bar{x}_2)}{(\bar{x}_1-\bar{x}_2)^{k+n+1}}. \tag{4.18}$$

For the derivation of this result we refer the reader to [86, 162]. Note truncating the two-point Taylor series expansion (4.17) *at $n = m$ results in an expansion which of f which is of order $\mathcal{O}(x^{2n+1})$.*

The advantage of using a two-point Taylor series is that, by considering the first n derivatives of a function f at two points $\bar{x}_1$ and $\bar{x}_2$, one can achieve a more accurate approximation of f over a wider range of values than if one were to approximate f using $2n$ derivatives at a single point (i.e., the usual Taylor series approximation).

If we associate expansion (4.17) *with an expansion of the form $f(x) = \sum_{n=0}^{\infty} f_n B_n(x)$ then $f_0 B_0(x) = c_n(\bar{x}_1,\bar{x}_2)(x-\bar{x}_1) + c_n(\bar{x}_2,\bar{x}_1)(x-\bar{x}_2)$, which is affine in x. Thus, the terms in the two-point Taylor series expansion would not be a suitable basis in* (4.12) *since $B_0(x) \neq 1$. However, one can always introduce a constant M and define a function*

$$F(x) := f(x) - M, \qquad \text{so that} \qquad f(x) = M + F(x). \tag{4.19}$$

Then, one can express f as

$$f(x) = M + \sum_{n=1}^{\infty} \left(C_{n-1}(\bar{x}_1,\bar{x}_2)(x-\bar{x}_1) + C_{n-1}(\bar{x}_2,\bar{x}_1)(x-\bar{x}_2)\right)(x-\bar{x}_1)^{n-1}(x-\bar{x}_2)^{n-1}, \tag{4.20}$$

where the C_n are as given in (4.18) *with $f \to F$. If we associate expansion* (4.20) *with an expansion of the form $f(x) = \sum_{n=0}^{\infty} f_n B_n(x)$, then we see that $f_0 B_0(x) = M$ and one can choose $B_0(x) = 1$. Thus, as written in* (4.20)*, the terms of the two-point Taylor series can be used as a suitable basis in* (4.12)*.*

Consider the following case: suppose $a(t,x)$, $\gamma(t,x)$ and $\nu(t,x,dz)$ are of the form

$$a(t,x) = f(x)A(t), \qquad \gamma(t,x) = f(x)\Gamma(t), \qquad \nu(t,x,dz) = f(x)\mathcal{N}(t,dz), \tag{4.21}$$

so that $\phi(t,x,\xi) = f(x)\Phi(t,\xi)$ with

$$\Phi(t,\xi) = \Gamma(t)(i\xi - 1) + A(t)(-\xi^2 - i\xi)$$

$$- \int_{\mathbb{R}} \mathcal{N}(t, dz)(e^z - 1 - z)i\xi + \int_{\mathbb{R}} \mathcal{N}(t, dz)(e^{i\xi z} - 1 - i\xi z).$$

It is certainly plausible that the symbol of $\mathcal{A}$ would have such a form since, from a modeling perspective, it makes sense that default intensity, volatility and jump-intensity would be proportional. Indeed, the Jump-to-default CEV model (JDCEV) of [49, 51] has a similar restriction on the form of the drift, volatility and killing coefficients.

Now, under the dynamics of (4.21)*, observe that $\phi(t, x, \xi)$ and $\mathcal{A}$ can be written as in* (4.12) *and* (4.15) *respectively with $B_0 = 1$ and*

$$B_n(x) = \left(C_{n-1}(\bar{x}_1, \bar{x}_2)(x - \bar{x}_1) + C_{n-1}(\bar{x}_2, \bar{x}_1)(x - \bar{x}_2)\right)(x - \bar{x}_1)^{n-1}(x - \bar{x}_2)^{n-1}, \qquad n \geq 1. \tag{4.22}$$

As above C_n (capital "C") are given by (4.18) *with $f \to F := f - M$ and*

$$\phi_0(t, \xi) = M\Phi(t, \xi), \qquad \phi_n(t, \xi) = \Phi(t, \xi), \qquad n \geq 1.$$

As in example 4.1, the choice of $\bar{x}_1$, $\bar{x}_2$ and M is somewhat arbitrary. But, a choice that seems to work well is to set $\bar{x}_1 = X_t - \Delta$ and $\bar{x}_2 = X_t + \Delta$ where $\Delta > 0$ is a constant and $M = f(X_t)$. It is also a good idea to check that, for a given choice of $\bar{x}_1$ and $\bar{x}_2$, the two-point Taylor series expansion provides a good approximation of f in the region of interest.

Note we assumed the form (4.21) *only for sake of simplicity. Indeed, the general case can be accommodated by suitably extending expansion* (4.12) *to the more general form*

$$\phi(t, x, \xi) = \sum_{n=0}^{\infty} \sum_{i=1}^{3} B_{i,n}(x)\phi_{i,n}(t, \xi),$$

where $\phi_{i,n}$ for $i = 1, 2, 3$ are related to the diffusion, jump and default symbols respectively. For brevity, however, we omit the details of the general case.

Example 4.3 (Non-local approximation in weighted L^2-spaces)**.** *Suppose $\{B_n\}_{n\geq 0}$ is a fixed orthonormal basis in some (possibly weighted) space $L^2(\mathbb{R}, \mathfrak{m}(x)dx)$ and that $\phi(t, \cdot, \xi) \in L^2(\mathbb{R}, \mathfrak{m}(x)dx)$ for all (t, ξ). Then we can represent $\phi(t, x, \xi)$ in the form* (4.12) *where now the $\{\phi_n\}_{n\geq 0}$ are given by*

$$\phi_n(t, \xi) = \langle B_n(\cdot), \phi(t, \cdot, \xi)\rangle_{\mathfrak{m}}, \quad n \geq 0.$$

A typical example would be to choose Hermite polynomials H_n centered at $\bar{x}$ as basis functions, which (as normalized below) are orthonormal under a Gaussian weighting

$$B_n(x) = H_n(x - \bar{x}), \qquad H_n(x) := \frac{1}{\sqrt{(2n)!!\sqrt{\pi}}} \frac{\partial_x^n \exp(-x^2)}{\exp(-x^2)}, \qquad n \geq 0. \tag{4.23}$$

In this case, we have

$$\phi_n(t, \xi) = \langle \phi(t, \cdot, \xi), B_n\rangle_{\mathfrak{m}} := \int_{\mathbb{R}} \phi(t, x, \xi)B_n(x)\mathfrak{m}(x)dx, \qquad \mathfrak{m}(x) := \exp\big(-(x - \bar{x})^2\big).$$

Once again, the choice of $\bar{x}$ is arbitrary. But, it is logical to choose $\bar{x}$ near X_t, the present level of the underlying X. In Figure 5.1 we compare the 2nd order two-point Taylor series, *the 4th order (usual) Taylor series and the 4th order Hermite polynomial expansion for an exponential function. Note that, in the case of an L^2 orthonormal basis, differentiability of the coefficients $(a(t,\cdot), \gamma(t,\cdot), \nu(t,\cdot,dz))$ is not required. This is a significant advantage over the Taylor and two-point Taylor basis functions considered in Examples 4.1 and 4.2, which do require differentiability of the coefficients.*

Now, returning to Cauchy problem (3.6), we suppose that $v = v(t,x)$ can be written as follows

$$v = \sum_{n=0}^{\infty} v_n. \tag{4.24}$$

Following [185], we insert expansions (4.15) and (4.24) into Cauchy problem (3.6) and find

$$(\partial_t + \mathcal{A}_0)v_0 = 0, \qquad v_0(T,x) = h(x), \tag{4.25}$$

$$(\partial_t + \mathcal{A}_0)v_n = -\sum_{k=1}^{n} B_k(x)\mathcal{A}_k v_{n-k}, \qquad v_n(T,x) = 0. \tag{4.26}$$

We are now in a position to find the explicit expression for $\widehat{v}_n$, the Fourier transform of v_n in (4.25)-(4.26).

Theorem 4.4. *Suppose $h \in L^1(\mathbb{R}, dx)$ and let $\widehat{h}$ denote its Fourier transform. Suppose further that v_n and its Fourier transform $\widehat{v}_n$ exist, and that both the left and right hand side of (4.25)-(4.26) belong to $L^1(\mathbb{R}, dx)$. Then $\widehat{v}_n(t,\xi)$ is given by*

$$\widehat{v}_0(t,\xi) = \exp\left(\int_t^T \phi_0(s,\xi)ds\right)\widehat{h}(\xi), \tag{4.27}$$

$$\widehat{v}_n(t,\xi) = \sum_{k=1}^{n}\int_t^T \exp\left(\int_t^s \phi_0(u,\xi)du\right) B_k(i\partial_\xi)\phi_k(s,\xi)\widehat{v}_{n-k}(s,\xi)ds, \qquad n \geq 1. \tag{4.28}$$

Note that the operator $B_k(i\partial_\xi)$ acts on everything to the right of it.

Proof. See Appendix 5.9.1. □

Remark 4.5. *Assuming $\widehat{v}_n \in L^1(\mathbb{R}, dx)$, one recovers v_n using*

$$v_n(t,x) = \int_{\mathbb{R}} d\xi \frac{1}{\sqrt{2\pi}} e^{i\xi x}\widehat{v}_n(t,\xi). \tag{4.29}$$

As previously mentioned, to obtain the FK transition densities $p(t,x;T,y)$ one simply sets $h(x) = \delta_y(x)$. In this case, $\widehat{h}(\xi)$ becomes $\psi_y(-\xi)$.

When the coefficients (a,γ,ν) are time-homogeneous, then the results of Theorem 4.4 simplify considerably, as we show in the following corollary.

Corollary 4.6 (Time-homogeneous case)**.** *Suppose that X has time-homogeneous dynamics with the local variance, default intensity and Lévy measure given by $a(x)$, $\gamma(x)$ and $\nu(x,dz)$ respectively. Then the symbol $\phi_n(t,\xi) = \phi_n(\xi)$ is independent of t. Define*

$$\tau(t) := T - t.$$

Then, for $n \leq 0$ we have

$$v_n(t,x) = u_n(\tau(t), x)$$

where

$$\begin{aligned} \widehat{u}_0(\tau,\xi) &= e^{\tau\phi_0(\xi)}\widehat{h}(\xi), \qquad (4.30)\\ \widehat{u}_n(\tau,\xi) &= \sum_{k=1}^{n}\int_0^{\tau} e^{(\tau-s)\phi_0(\xi)} B_k(i\partial_\xi)\phi_k(\xi)\widehat{u}_{n-k}(s,\xi)ds, \qquad n \geq 1. \end{aligned}$$

Proof. The proof is an algebraic computation. For brevity, we omit the details. □

Example 4.7. *Consider the Taylor density expansion of Example 4.1. That is, $B_n(x) = (x-\bar{x})^n$. Then, in the time-homogeneous case, we find that $\widehat{u}_1(t,\xi)$ and $\widehat{u}_2(t,\xi)$ are given explicitly by*

$$\widehat{u}_1(t,\xi) = e^{t\phi_0(\xi)}\left(t\widehat{h}(\xi)\bar{x}\phi_1(\xi) + it\phi_1(\xi)\widehat{h}'(\xi) + \frac{1}{2}it^2\widehat{h}(\xi)\phi_1(\xi)\phi_0'(\xi) + it\widehat{h}(\xi)\phi_1'(\xi)\right), \qquad (4.31)$$

$$\begin{aligned} \widehat{u}_2(t,\xi) = e^{t\phi_0(\xi)}\Bigg(&\frac{1}{2}t^2\widehat{h}(\xi)\bar{x}^2\phi_1^2(\xi) + t\widehat{h}(\xi)\bar{x}^2\phi_2(\xi) - it^2\bar{x}\phi_1^2(\xi)\widehat{h}'(\xi)\\ &- 2it\bar{x}\phi_2(\xi)\widehat{h}'(\xi) - \frac{1}{2}it^3\widehat{h}(\xi)\bar{x}\phi_1^2(\xi)\phi_0'(\xi) - it^2\widehat{h}(\xi)\bar{x}\phi_2(\xi)\phi_0'(\xi)\\ &- \frac{1}{2}t^3\phi_1(\xi)^2\widehat{h}'(\xi)\phi_0'(\xi) - t^2\phi_2(\xi)\widehat{h}'(\xi)\phi_0'(\xi) - \frac{1}{8}t^4\widehat{h}(\xi)\phi_1^2(\xi)(\phi_0'(\xi))^2\\ &- \frac{1}{3}t^3\widehat{h}(\xi)\phi_2(\xi)(\phi_0'(\xi))^2 - \frac{3}{2}it^2\widehat{h}(\xi)\bar{x}\phi_1(\xi)\phi_1'(\xi) - \frac{3}{2}t^2\phi_1(\xi)\widehat{h}'(\xi)\phi_1'(\xi)\\ &- \frac{2}{3}t^3\widehat{h}(\xi)\phi_1(\xi)\phi_0'(\xi)\phi_1'(\xi) - \frac{1}{2}t^2\widehat{h}(\xi)(\phi_1'(\xi))^2 - 2it\widehat{h}(\xi)\bar{x}\phi_2'(\xi) - 2t\widehat{h}'(\xi)\phi_2'(\xi)\\ &- t^2\widehat{h}(\xi)\phi_0'(\xi)\phi_2'(\xi) - \frac{1}{2}t^2\phi_1(\xi)^2\widehat{h}''(\xi) - t\phi_2(\xi)\widehat{h}''(\xi) - \frac{1}{6}t^3\widehat{h}(\xi)\phi_1^2(\xi)\phi_0''(\xi)\\ &- \frac{1}{2}t^2\widehat{h}(\xi)\phi_2(\xi)\phi_0''(\xi) - \frac{1}{2}t^2\widehat{h}(\xi)\phi_1(\xi)\phi_1''(\xi) - t\widehat{h}(\xi)\phi_2''(\xi)\Bigg). \qquad (4.32) \end{aligned}$$

Higher order terms are quite long. However, they can be computed quickly and explicitly using the Mathematica code provided in Appendix 5.9.2. The code in the Appendix can be easily modified for use with other basis functions.

Remark 4.8. *As in [185], when considering models with Gaussian-type jumps, i.e., models with a state-dependent Lévy measure $\nu(t,x,dz)$ of the form* (5.35) *below, all terms in the expansion for the transition density become explicit. Likewise, for models with Gaussian-type jumps, all terms in the expansion for the price of an option are also explicit, assuming the payoff is integrable against Gaussian functions. Indeed, as we shall see in Appendix 5.9.3, the main term $p^{(0)}(t,x;T,y)$ is given by a series of Gaussian densities; further approximations $p^{(N)}(t,x;T,y)$ can be represented as a suitable differential operator applied to $p^{(0)}(t,x;T,y)$ acting on the variable x (see for instance Proposition 9.8 for the first order). For a concise and explicit representation of the approximation $p^{(n)}$ in the case of Gaussian-type jumps, we refer to the novel paper [170].*

Remark 4.9. *Many common payoff functions (e.g. calls and puts) are not integrable: $h \notin L^1(\mathbb{R}, dx)$. Such payoffs may sometimes be accommodated using generalized Fourier* *transforms. Assume*

$$\widehat{h}(\xi) := \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\xi x} h(x) < \infty, \qquad \textit{for some } \xi = \xi_r + i\xi_i \textit{ with } \xi_r, \xi_i \in \mathbb{R}.$$

Assume also that $\phi(t,x,\xi_r+i\xi_i)$ is analytic as a function of ξ_r. Then the formulas appearing in Theorem 4.4 and Corollary 4.6 are valid and integration in (4.29) is with respect to ξ_r (i.e., $d\xi \to d\xi_r$). For example, the payoff of a European call option with payoff function $h(x) = (e^x - e^k)^+$ has a generalized Fourier transform

$$\widehat{h}(\xi) = \int_{\mathbb{R}} dx \tfrac{1}{\sqrt{2\pi}} e^{-i\xi x}(e^x - e^k)^+ = \frac{-e^{k-ik\xi}}{\sqrt{2\pi}\,(i\xi + \xi^2)}, \quad \xi = \xi_r + i\xi_i, \quad \xi_r \in \mathbb{R}, \quad \xi_i \in (-\infty, -1).$$

In any practical scenario, one can only compute a finite number of terms in (4.24). Thus, we define $v^{(N)}$, the Nth order approximation of v by

$$v^{(N)}(t,x) = \sum_{n=0}^{N} v_n(t,x) = \int_{\mathbb{R}} d\xi \frac{1}{\sqrt{2\pi}} e^{i\xi x} \widehat{v}^{(n)}(t,\xi), \qquad \widehat{v}^{(N)}(t,\xi) := \sum_{n=0}^{N} \widehat{v}_n(t,\xi),$$

The function $u^{(N)}(t,x)$ (which we use for time-homogeneous cases) and the approximate FK transition density $p^{(N)}(t,x;T,y)$ are defined in an analogous fashion.

5.5 Pointwise error bounds for Gaussian models

In this section we prove some pointwise error estimates for $p^{(N)}(t,x;T,y)$, the Nth order approximation of the FK density of $(\partial_t + \mathcal{A})$ with $\mathcal{A}$ as in (3.7). Throughout this Section, we assume Gaussian-type jumps with x-dependent mean and variance, and (t,x)-dependent jump intensities. Furthermore, we work specifically with the Taylor series expansion of Example 4.1. That is, we use basis functions $B_n(x) = (x - \bar{x})^n$.

Theorem 5.1. *Let $N \geq 1$, and assume that*

$$m \leq a(t,x) \equiv a(x) \leq M, \qquad \gamma(t,x) = 0, \qquad t \in [0,T], \; x \in \mathbb{R},$$

for some positive constants m and M, and that

$$\nu(t,x,dz) \equiv \nu(x,dz) = \lambda(x)\,\mathcal{N}_{0,\delta^2}(dz) := \frac{\lambda(x)}{\sqrt{2\pi}\delta} e^{-\frac{z^2}{2\delta^2}} dz, \tag{5.33}$$

with

$$m \leq \delta^2 \leq M, \qquad 0 \leq \lambda(x) \leq M, \qquad x \in \mathbb{R}.$$

Moreover assume that $a, \lambda \in C^{N+1}(\mathbb{R})$ with bounded derivatives, and let $\bar{x} = y$ in (4.16). Then, we have[4]

$$\left| p(t,x;T,y) - p^{(N)}(t,x;T,y) \right| \leq g_N(T-t) \left(\bar{\Gamma}(t,x;T,y) + \|\partial_x \nu\|_\infty \widetilde{\Gamma}(t,x;T,y) \right), \tag{5.34}$$

[4] Here $\|\partial_x \nu\|_\infty := \max\{\|\partial_x \lambda\|_\infty, \|\partial_x \delta\|_\infty, \|\partial_x \mu\|_\infty\}$, where $\|\cdot\|_\infty$ denotes the sup-norm on $(0,T) \times \mathbb{R}$. Note that $\|\partial_x \nu\|_\infty = 0$ if λ, δ, μ are constants.

for any $x, y \in \mathbb{R}$ and $t < T$, where

$$g_N(s) = \mathcal{O}(s), \quad \text{as } s \to 0^+.$$

Here, the function $\bar{\Gamma}$ is the fundamental solution of the constant coefficients jump-diffusion operator

$$\partial_t u(t,x) + \frac{\bar{M}}{2}\partial_{xx} + \bar{M}\int_{\mathbb{R}} \left(u(t,x+z) - u(t,x)\right) \mathcal{N}_{\bar{M},\bar{M}}(dz),$$

where $\bar{M}$ is a suitably large constant, and $\widetilde{\Gamma}$ is defined as

$$\widetilde{\Gamma}(t,x;T,y) = \sum_{k=0}^{\infty} \frac{\bar{M}^{k/2}(T-t)^{k/2}}{\sqrt{k!}} \mathcal{C}^{k+1}\bar{\Gamma}(t,x;T,y),$$

and where $\mathcal{C}$ is the convolution operator acting as

$$\mathcal{C}f(x) = \int_{\mathbb{R}} f(x+z)\mathcal{N}_{\bar{M},\bar{M}}(dz).$$

Proof. See Appendix 5.9.3. □

Remark 5.2. *Note that the estimate (5.34) is only proved under the assumption of time-independent diffusion and jump intensity, constant variance and null mean for the jumps, and null default intensity. Nevertheless, in analogy with the purely diffusion case (see Theorem 2.3 and Remark 2.4), the same arguments used in the Appendix to prove Theorem 5.1 can be extended to prove the same result holding when*

$$m \le a(t,x) \le M, \qquad 0 \le \gamma(t,x) \le M, \qquad t \in [0,T], \ x \in \mathbb{R},$$

$$\nu(t,x,dz) = \lambda(t,x)\,\mathcal{N}_{\mu(x),\delta^2(x)}(dz) := \frac{\lambda(t,x)}{\sqrt{2\pi}\delta(x)} e^{-\frac{(z-\mu(x))^2}{2\delta^2(x)}} dz, \tag{5.35}$$

with

$$m \le \delta^2(x) \le M, \qquad 0 \le \lambda(t,x), |\mu(x)| \le M, \qquad t \in [0,T], \ x \in \mathbb{R},$$

and by assuming $a, \gamma, \lambda \in C^{0,N+1}([0,T]\times\mathbb{R})$, $\delta, \mu \in C^{N+1}(\mathbb{R})$, with x-derivatives uniformly bounded with respect to $t \in [0,T]$.

Remark 5.3. *In the proof of Theorem 5.1, we see that functions $\mathcal{C}^k\bar{\Gamma}$ take the following form*

$$\mathcal{C}^k\bar{\Gamma}(t,x;T,y) = e^{-\bar{M}(T-t)} \sum_{n=0}^{\infty} \frac{\left(\bar{M}(T-t)\right)^n}{n!\sqrt{2\pi\bar{M}(T-t+n+k)}} \exp\left(-\frac{\left(x-y+\bar{M}(n+k)\right)^2}{2\bar{M}(T-t+n+k)}\right) \tag{5.36}$$

for any $k \ge 0$, and therefore $\widetilde{\Gamma}$ can be explicitly written as

$$\begin{aligned}\widetilde{\Gamma}(t,x;T,y) =& e^{-\bar{M}(T-t)} \sum_{n,k=0}^{\infty} \frac{\left(\bar{M}(T-t)\right)^{n+\frac{k}{2}}}{n!\sqrt{k!}\sqrt{2\pi\bar{M}(T-t+n+k+1)}} \\ & \cdot \exp\left(-\frac{\left(x-y+\bar{M}(n+k+1)\right)^2}{2\bar{M}(T-t+n+k+1)}\right).\end{aligned}$$

By Remark 5.3, it follows that, when $k = 0$ and $x \neq y$, the asymptotic behaviour as $t \to T$ of the sum in (5.36) depends only on the $n = 1$ term. Consequently, we have $\bar{\Gamma}(t, x; T, y) = \mathcal{O}(T - t)$ as $(T - t)$ tends to 0. On the other hand, for $k \geq 1$, $\mathcal{C}^k \bar{\Gamma}(t, x; T, y)$, and thus also $\widetilde{\Gamma}(t, x; T, y)$, tends to a positive constant as $(T - t)$ goes to 0. It is then clear by (5.34) that, with $x \neq y$ fixed, the asymptotic behavior of the error, when t tends to T, changes from $(T-t)$ to $(T-t)^2$ depending on whether the Lévy measure is locally-dependent or not.

Remark 5.4. *The proof of Theorem 5.1 is also interesting for theoretical purposes. Indeed, it actually represents a procedure to construct $p(t, x; T, y)$. Note that with $p^{(N)}(t, x; T, y)$ being known explicitly, equation* (5.34) *provides pointwise upper bounds for the FK density as well.*

Theorem 5.1 extends Theorem 2.3 where only the purely diffusive case (i.e $\lambda \equiv 0$) is considered. In that case an estimate analogous to (5.34) holds with

$$g_N(s) = \mathcal{O}\left(s^{\frac{N+1}{2}}\right), \quad \text{as } s \to 0^+.$$

Theorem 5.1 shows that for jump processes, increasing the order of the expansion for N greater than one, theoretically does not give any gain in the rate of convergence of the asymptotic expansion as $t \to T^-$; this is due to the fact that the expansion is based on a local (Taylor) approximation while the PIDE contains a non-local part. This estimate is in accord with the results in [25] where only the case of constant Lévy measure is considered. Extensive numerical tests showed that the first order approximation gives extremely accurate results and the precision seems to be further improved by considering higher order approximations. For example, in Figure 5.2 we plot the approximate transition density $p^{(N)}(t, x; T, y)$ for different values of N for the Lévy-type model considered in Section 5.7.1. We note that, for $T - t \leq 5$, the transition densities $p^{(4)}(t, x; T, y)$ and $p^{(3)}(t, x; T, y)$ are nearly identical. This is typical in our numerical experiments.

Corollary 5.5. *Under the assumptions of Theorem 5.1, we have the following estimate for the error on the approximate prices:*

$$\left|v(t, x) - v^{(N)}(t, x)\right| \leq g_N(T - t) \int_{\mathbb{R}} |h(y)| \left(\bar{\Gamma}(t, x; T, y) + \|\partial_x \nu\|_\infty \widetilde{\Gamma}(t, x; T, y)\right) dy,$$

for any $x \in \mathbb{R}$ and $t < T$.

Some possible extensions of these asymptotic error bounds to general Lévy measures are possible, though they are certainly not straightforward. Indeed, the proof of Theorem 5.1 is based on some pointwise uniform estimates for the fundamental solution of the constant coefficient operator, i.e. the transition density of a compound Poisson process with Gaussian jumps. When considering other Lévy measures these estimates would be difficult to carry out, especially in the case of jumps with infinite activity, but they might be obtained in some suitable normed functional space. This might lead to error bounds for short maturities, which are expressed in terms of a suitable norm, as opposed to uniform pointwise bounds.

Remark 5.6. *Since, in general, it is hard to derive the truncation error bound, the reader may wonder how to determine the number of terms to include in the asymptotic expansion. Though we provide a general expression for the n-th term, realistically, only the fourth order term can be computed. That said, as we shall see from the examples in Section 5.7,* *in practice, three terms provide an approximation which is accurate enough for most applications (i.e., the resulting approximation error is smaller than the bid-ask spread typically quoted on the market). Since, $v^{(n)}$ only requires only a single Fourier integration, there is no numerical advantage for choosing smaller n. As such, for financial applications we suggest using $n = 3$ or $n = 4$.*

5.6 Implied volatility for local volatility models

For European calls and puts, it is the implied volatility induced by an option price – rather than the option price itself – that is the quantity of primary concern. In this section we derive an implied volatility expansion for time-homogeneous local volatility models (i.e., for models with no jumps and no possibility of default).

Assumption 6.1. *In this section only, we assume*

$$S_t = e^{X_t}, \qquad dX_t = -a(X_t)dt + \sqrt{2a(X_t)}dW_t,$$

so that $\phi_0(\xi) = a_0(-\xi^2 - i\xi)$ in (4.12), with $a_0 > 0$.

Remark 6.2. *Note that in the Taylor expansion of Example 4.1, we have $a_0 = a(\bar{x})$; in the two-point Taylor expansion of Example 4.2, we have $a_0 = MA$; in the Hermite expansion of Example 4.3, we have $a_0 = \int_{\mathbb{R}} a(x)e^{-(x-\bar{x})^2}dx$.*

Throughout this section, we fix an initial value of the underlying $X_0 = x$, a time to maturity t and a call option payoff $h(X_t) = (e^{X_t} - e^k)^+$. The price of this option $u(t, x) = u(t, x; k)$ can be computed (approximately) using Corollary 4.6. To simplify notation, throughout this section we will suppress all dependence on (t, x, k). However, the reader should keep in mind that option prices and their corresponding implied volatilities do depend on these variables. We begin our analysis with a definition of the Black-Scholes price.

Definition 6.3. *The Black-Scholes Price $u^{BS} : \mathbb{R}^+ \to \mathbb{R}^+$, defined as a function of volatility σ, is given by*

$$u^{BS}(\sigma) := \int d\xi\, e^{t\phi^{BS}(\xi;\sigma)}\widehat{h}(\xi)\psi_\xi(x), \qquad \phi^{BS}(\xi;\sigma) = \frac{1}{2}\sigma^2(-\xi^2 - i\xi).$$

Note that $\phi^{BS}(\cdot;\sigma)$ is simply the Lévy exponent of a Brownian motion with volatility σ and drift $-\frac{1}{2}\sigma^2$. Thus, this is simply the Fourier representation of the usual Black-Scholes price.

Remark 6.4. *Note that, by equations (4.29) and (4.30), under Assumption 6.1, we have $u_0 = u^{BS}(\sqrt{2a_0})$.*

Definition 6.5. *For fixed (t, x, k), the implied volatility corresponding to a call price $u \in ((e^x - e^k)^+, e^x)$ is defined as the unique strictly positive real solution σ of the equation*

$$u^{BS}(\sigma) = u. \tag{6.37}$$

We are now in position to derive an asymptotic expansion for implied volatility. We begin by writing the option price and implied volatility as follows

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n, \qquad \varepsilon = 1, \tag{6.38}$$
$$\sigma = \sigma_0 + \delta^\varepsilon, \qquad \delta^\varepsilon = \sum_{n=1}^{\infty} \varepsilon^n \sigma_n,$$

We introduce the constant $\varepsilon = 1$ purely for the purposes of accounting; ε plays no role in our final result. Next, we expand u^{BS} as a power series about the point σ_0. We have

$$u^{\text{BS}}(\sigma) = u^{\text{BS}}(\sigma_0 + \delta^\varepsilon) = \sum_{n=0}^{\infty} \frac{1}{n!}(\delta^\varepsilon \partial_\sigma)^n u^{\text{BS}}(\sigma_0) = u^{\text{BS}}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \varepsilon^k \sigma_k \right)^n \partial_\sigma^n u^{\text{BS}}(\sigma_0)$$

$$\begin{aligned}
&= u^{\text{BS}}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{k=n}^{\infty} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \varepsilon^k \right] \partial_\sigma^n u^{\text{BS}}(\sigma_0) \\
&= u^{\text{BS}}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sum_{n=1}^{k} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n \right] u^{\text{BS}}(\sigma_0) \\
&= u^{\text{BS}}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sigma_k \partial_\sigma + \sum_{n=2}^{k} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n \right] u^{\text{BS}}(\sigma_0).
\end{aligned} \tag{6.39}$$

Now, we insert expansions (6.38) and (6.39) into (6.37) and collect terms of like order in ε

$$\begin{aligned}
\mathcal{O}(1): &\qquad u_0 = u^{\text{BS}}(\sigma_0), \\
\mathcal{O}(\varepsilon^k): &\qquad u_k = \sigma_k \partial_\sigma u^{\text{BS}}(\sigma_0) + \sum_{n=2}^{k} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n u^{\text{BS}}(\sigma_0), \qquad k \geq 1.
\end{aligned}$$

Solving the above equations for $\{\sigma_k\}_{k=0}^{\infty}$ we find

$$\mathcal{O}(1): \qquad \sigma_0 = \sqrt{2a_0}, \tag{6.40}$$

$$\mathcal{O}(\varepsilon^k): \qquad \sigma_k = \frac{1}{\partial_\sigma u^{\text{BS}}(\sigma_0)} \left(u_k - \sum_{n=2}^{k} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n u^{\text{BS}}(\sigma_0) \right), \qquad k \geq 1, \tag{6.41}$$

where we have used $u_0 = u^{\text{BS}}(\sqrt{2a_0})$ to deduce that $\sigma_0 = \sqrt{2a_0}$ (see Remark 6.4). Observe, the right hand side of (6.41) involves only σ_j for $j \leq k-1$. Thus, the $\{\sigma_k\}_{k=1}^{\infty}$ can be found recursively. Explicitly, up to $\mathcal{O}(\varepsilon^4)$ we have

$$
\begin{aligned}
\mathcal{O}(\varepsilon): &\quad \sigma_1 = \frac{u_1}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^2): &\quad \sigma_2 = \frac{u_2 - \frac{1}{2!}\sigma_1^2 \partial_\sigma^2 u_0}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^3): &\quad \sigma_3 = \frac{u_3 - (\sigma_2\sigma_1\partial_\sigma^2 + \frac{1}{3!}\sigma_1^3\partial_\sigma^3)u_0}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^4): &\quad \sigma_4 = \frac{u_4 - (\sigma_3\sigma_1\partial_\sigma^2 + \frac{1}{2}\sigma_2^2\partial_\sigma^2 + \frac{1}{2}\sigma_2\sigma_1^2\partial_\sigma^3 + \frac{1}{24}\sigma_1^4\partial_\sigma^4)u_0}{\partial_\sigma u_0},
\end{aligned}
\tag{6.42}
$$

where

$$\partial_\sigma^n u_0 = \partial_\sigma^n u^{\mathrm{BS}}(\sigma)|_{\sigma=\sqrt{2a_0}}.$$

Theoretically, one can use recursion relation (6.41) ad infinitum to find σ_n for arbitrarily large n. To sum up, our nth order asymptotic approximation for implied volatility is as follows:

$$\sigma^{(n)} = \sum_{k=0}^{n} \sigma_k,$$

where the $\{\sigma_k\}$ are given by (6.40) and (6.41).

As written, equations (6.42) are not very convenient. Indeed, for each u_n and for all terms of the form $\partial_\sigma^n u_0$, one must be compute Fourier integrals. However, with a little more work, we can compute the $\{\sigma_i\}$ explicitly with no integrals and no special functions. The first step is to express u_n as

$$u_n = \mathcal{L}u_0, \qquad\qquad \mathcal{L} = \sum_{k\geq 0} b_k(t,x)\partial_x^k(\partial_x^2 - \partial_x),$$

where the $\{b_k\}$ are functions of (t,x).

Assumption 6.6. *For the remainder of Section 5.6, we assume $B_n(x)$ and $\phi_n(\xi) \equiv \phi_n(t,\xi)$ as in Example 4.1. That is, $B_n(x) = (x-\bar{x})^n$ and $\phi_n(\xi) = a_n(-\xi^2 - i\xi)$ with $a_n = \frac{1}{n!}\partial_x^n a(\bar{x})$. This choice is not necessary, but simplifies the computations that follow. Similar implied volatility results can be derived for other choices of $B_n(x)$ and $\phi_n(\xi)$.*

As shown in Appendix 5.9.4, under Assumption 6.6, for u_1 we have

$$u_1(t,x) = \Big(t(x-\bar{x}) + \tfrac{1}{2}t^2 a_0(2\partial_x - 1)\Big)a_1(\partial_x^2 - \partial_x)u_0(t,x). \tag{6.43}$$

and for u_2 we have

$$
\begin{aligned}
u_2(t,x) = &\Big(t(x-\bar{x})^2 + t^2(x-\bar{x})a_0(2\partial_x - 1) + \tfrac{1}{3}t^3 a_0^2(2\partial_x - 1)^2 + t^2 a_0\Big)a_2(\partial_x^2 - \partial_x)u_0(t,x) \\
&+ \Big(\tfrac{1}{2}t^2(x-\bar{x})^2 a_1(\partial_x^2 - \partial_x) + \tfrac{1}{2}t^3\,(x-\bar{x})\,a_1(\partial_x^2 - \partial_x)a_0(2\partial_x - 1) \\
&\quad + \tfrac{1}{8}t^4 a_1(\partial_x^2 - \partial_x)a_0^2(2\partial_x - 1)^2 + \tfrac{1}{2}t^2\,(x-\bar{x})\,a_1(2\partial_x - 1)
\end{aligned}
$$

$$+\tfrac{1}{6}t^3 a_0(2\partial_x-1)a_1(2\partial_x-1)+\tfrac{1}{3}t^3 a_1(\partial_x^2-\partial_x)a_0\Big)a_1(\partial_x^2-\partial_x)u_0(t,x). \qquad (6.44)$$

Now, using equations (6.43)-(6.44), as well as

$$\partial_\sigma u^{\mathrm{BS}}(\sigma)=t\sigma(\partial_x^2-\partial_x)u^{\mathrm{BS}}(\sigma), \qquad \frac{\partial_x^m(\partial_x^2-\partial_x)u^{\mathrm{BS}}(\sigma)}{(\partial_x^2-\partial_x)u^{\mathrm{BS}}(\sigma)}=\frac{\partial_x^m \exp\left(x-\frac{d_+^2}{2}\right)}{\exp\left(x-\frac{d_+^2}{2}\right)},$$

where $d_+=\frac{1}{\sigma\sqrt{t}}\left(x-k+\frac{1}{2}t\,\sigma^2\right)$, a straightforward (but tedious) algebraic computation shows that

$$\begin{aligned}
\sigma_1 &= \frac{u_1}{\partial_\sigma u_0}=\frac{a_1}{\sqrt{2a_0}}\left(\frac{1}{2}(x+k)-\bar{x}\right),\\
\sigma_2 &= \frac{u_2-\frac{1}{2!}\sigma_1^2\partial_\sigma^2 u_0}{\partial_\sigma u_0}\\
&= \frac{a_2}{3\sqrt{2a_0}}\Big(k^2+kx+x^2+ta_0-3(k+x)\bar{x}+3\bar{x}^2\Big)\\
&\quad -\frac{a_1^2}{48a_0\sqrt{2a_0}}\Big(ta_0(6+ta_0)+6\left(k^2+x^2-2(k+x)\bar{x}+2\bar{x}^2\right)\Big).
\end{aligned}$$

If we set $\bar{x}\to x$ then we have the following second order approximation for implied volatility

$$\begin{aligned}
\sigma^{(2)} &= \sigma_0+\sigma_1+\sigma_2\\
&= \sqrt{2a_0}+\frac{a_1}{2\sqrt{2a_0}}(k-x)+\frac{a_2}{3\sqrt{2a_0}}\Big((k-x)^2+ta_0\Big)\\
&\quad -\frac{a_1^2}{48a_0\sqrt{2a_0}}\Big(6(k-x)^2+ta_0\,(6+ta_0)\Big),
\end{aligned} \qquad (6.45)$$

which is quadratic in $(k-x)$ and in t.

5.7 Examples

In this section, in order to illustrate the versatility of our asymptotic expansion, we apply our approximation technique to a variety of different Lévy-type models. We consider both finite and infinite activity Lévy-type measures, models with and without a diffusion component, and models with and without jumps. We study not only option prices, but also implied volatilities and credit spreads. In each setting, if the exact or approximate option price/implied volatility/credit spread has been computed by a method other than our own, we compare this to the option price/implied volatility/credit spread obtained by our approximation. For cases where the exact or approximate density/option price is not analytically available, we use Monte Carlo methods to verify the accuracy of our method.

Note that, some of the examples considered below do not satisfy the conditions listed in Section 5.2. In particular, we will consider coefficients (a,γ,ν) that are not bounded. Nevertheless, the formal results of Section 5.4 work well in the examples considered.

5.7.1 CEV-like Lévy-type processes

We consider a Lévy-type process of the form (2.1) with CEV-like volatility and jump-intensity. Specifically, the log-price dynamics are given by

$$a(x) = \frac{1}{2}\delta^2 e^{2(\beta-1)x}, \quad \nu(x, dz) = e^{2(\beta-1)x}\mathcal{N}(dz), \quad \gamma(x) = 0, \quad \delta \geq 0, \beta \in [0, 1], \qquad (7.46)$$

where $\mathcal{N}(dx)$ is a Lévy measure. When $\mathcal{N} \equiv 0$, this model reduces to the CEV model of [65]. Note that, with $\beta \in [0, 1)$, the volatility and jump-intensity increase as $x \to -\infty$, which is consistent with the leverage effect (i.e., a decrease in the value of the underlying is often accompanied by an increase in volatility/jump intensity). This characterization will yield a negative skew in the induced implied volatility surface. As the class of models described by (7.46) is of the form (4.21) with $f(x) = e^{2(\beta-1)x}$, this class naturally lends itself to the two-point Taylor series approximation of Example 4.2. Thus, for certain numerical examples in this Section, we use basis functions B_n given by (4.22). In this case we choose expansion points $\bar{x}_1$ and $\bar{x}_2$ in a symmetric interval around X_0 and in (4.19) we choose $M = f(X_0) = e^{2(\beta-1)X_0}$. For other numerical examples, we use the (usual) one-point Taylor series expansion $B_n(x) = (x - \bar{x})^n$. In this cases, we choose $\bar{x} = X_0$.

We will consider two different characterizations of $\mathcal{N}(dz)$:

$$\text{Gaussian:} \qquad \mathcal{N}(dz) = \lambda \frac{1}{\sqrt{2\pi\eta^2}} \exp\left(\frac{-(z-m)^2}{2\eta^2}\right) dz, \qquad (7.47)$$

$$\text{Variance-Gamma:} \qquad \mathcal{N}(dz) = \left(\frac{e^{-\lambda_-|z|}}{\kappa|z|}\mathbb{I}_{\{z<0\}} + \frac{e^{-\lambda_+ z}}{\kappa z}\mathbb{I}_{\{z>0\}}\right) dz, \qquad (7.48)$$

$$\lambda_\pm = \left(\sqrt{\frac{\theta^2\kappa^2}{4} + \frac{\rho^2\kappa}{2}} \pm \frac{\theta\kappa}{2}\right)^{-1}$$

Note that the Gaussian measure is an example of a finite-activity Lévy measure (i.e., $\mathcal{N}(\mathbb{R}) < \infty$), whereas the Variance-Gamma measure, due to [172], is an infinite-activity Lévy measure (i.e., $\mathcal{N}(\mathbb{R}) = \infty$). As far as the authors of this paper are aware, there is no closed-form expression for option prices (or the transition density) in the setting of (7.46), regardless of the choice of $\mathcal{N}(dz)$. As such, we will compare our pricing approximation to prices of options computed via standard Monte Carlo methods.

Remark 7.1. [5] *Note, the CEV model typically includes an absorbing boundary condition at $S = 0$. A more rigorous way to deal with degenerate dynamics, as in the CEV model, would be to approximate the solution of the Cauchy problem related to the process S_t (as apposed to $X_t = \log S_t$). One would then equip the Cauchy problem with suitable Dirichlet conditions on the boundary $s = 0$, and work directly in the variable $s \in \mathbb{R}_+$ as opposed to the log-price on $x \in \mathbb{R}$. Indeed, this is the approach followed by [115] who approximate the true density p by a Gaussian density p_0 through a heat kernel expansion: note that the supports of p and p_0 are $\mathbb{R}_+$ and $\mathbb{R}$ respectively. In order to take into account of the boundary behavior of the true density p, an improved approximation could be achieved by using the Green function of the heat operator for $\mathbb{R}_+$ instead of the Gaussian kernel: this will be object of further research in a forthcoming paper.*

[5] We would like to thank an anonymous referee for bringing the issue of boundary conditions to our attention.

We would also like to remark explicitly that our methodology is very general and works with different choices for the leading operator of the expansion, such as the constant-coefficient PIDEs we consider in the case of jumps. Nevertheless, in the present paper, when purely diffusive models are considered, we always take the heat operator as the leading term of our expansion. The main reasons are that (i) the heat kernel is convenient for its computational simplicity and (ii) the heat kernel allows for the possibility of passing directly from a Black-Scholes-type price expansion to an implied vol expansion.

Gaussian Lévy Measure

In our first numerical experiment, we consider the case of Gaussian jumps. That is, $\mathcal{N}(dz)$ is given by (7.47). We fix the following parameters

$$\delta = 0.20, \quad \beta = 0.25, \quad \lambda = 0.3, \quad m = -0.1, \quad \eta = 0.4, \quad S_0 = e^x = 1. \tag{7.49}$$

Using Corollary 4.6, we compute the approximate prices $u^{(0)}(t, x; K)$ and $u^{(3)}(t, x; K)$ of a series of European puts over a range of strikes K and with times to maturity $t = \{0.25, 1.00, 3.00, 5.00\}$ (we add the parameter K to the arguments of $u^{(n)}$ to emphasize the dependence of $u^{(n)}$ on the strike price K). To compute $u^{(i)}(t, x; K)$, $i = \{0, 3\}$ we use the we the usual one-point Taylor series expansion (Example 4.1). We also compute the price $u(t, x; K)$ of each put by Monte Carlo simulation. For the Monte Carlo simulation, we use a standard Euler scheme with a time-step of 10^{-3} years, and we simulate 10^6 sample paths. We denote by $u^{(MC)}(t, x; K)$ the price of a put obtained by Monte Carlo simulation. As prices are often quoted in implied volatilities, we convert prices to implied volatilities by inverting the Black-Scholes formula numerically. That is, for a given put price $u(t, x; K)$, we find $\sigma(t, K)$ such that

$$u(t, x; K) = u^{\mathrm{BS}}(t, x; K, \sigma(t, K)), \tag{7.50}$$

where $u^{\mathrm{BS}}(t, x; K, \sigma)$ is the Black-Scholes price of the put as computed assuming a Black-Scholes volatility of σ. For convenience, we introduce the notation

$$\mathrm{IV}[u(t, x; K)] := \sigma(t, K)$$

to indicate the implied volatility induced by option price $u(t, x; K)$. The results of our numerical experiments are plotted in Figure 5.3. We observe that $\mathrm{IV}[u^{(3)}(t, x; K)]$ agrees almost exactly with $\mathrm{IV}[u^{(MC)}(t, x; K)]$. The computed prices $u^{(3)}(t, x; K)$ and their induced implied volatilities $\mathrm{IV}[u^{(3)}(t, x; K)]$, as well as 95% confidence intervals resulting from the Monte Carlo simulations can be found in Table 5.1.

Comparing one-point Taylor and Hermite expansions

As choosing different basis functions results in distinct option-pricing approximations, one might wonder: which choice of basis functions provides the most accurate approximation of option prices and implied volatilities? We investigate this question in Figure 5.4. In the left column, using the parameters in (7.49), we plot $\mathrm{IV}[u^{(n)}(t, x; K)]$, $t = 0.5$, $n = \{0, 1, 2, 3, 4\}$ where $u^{(n)}(t, x; K)$ is computed using both the one-point Taylor series basis functions (Example 4.1) and the Hermite polynomial basis functions (Example 4.3). We also plot $\mathrm{IV}[u^{(MC)}(t, x; K)]$, the implied volatility obtained by Monte Carlo simulation.

For comparison, in the right column, we plot the function f as well as $f^{(n)}_{\text{Taylor}}$ and $f^{(n)}_{\text{Hermite}}$ where

$$\begin{aligned} f(x) &= e^{2(\beta-1)x}, \\ f^{(n)}_{\text{Taylor}}(x) &:= \sum_{m=0}^{n} \frac{1}{m!} \partial^m f(\bar{x})(x-\bar{x})^m, \\ f^{(n)}_{\text{Hermite}}(x) &:= \sum_{m=0}^{n} \frac{1}{m!} \langle H_m, f \rangle H_m(x). \end{aligned} \tag{7.51}$$

From Figure 5.4, we observe that, for every n, the Taylor series expansion $f^{(n)}_{\text{Taylor}}$ provides a better approximation of the function f (at least locally) than does the Hermite polynomial expansion $f^{(n)}_{\text{Hermite}}$. In turn, the implied volatilities resulting from the Taylor series basis functions $\text{IV}[u^{(n)}(t,x;K)]$ more accurately approximate $\text{IV}[u^{(MC)}(t,x;K)]$ than do the implied volatilities resulting from the Hermite basis functions. The implied volatilities resulting from the two-point Taylor series price approximation (not shown in the Figure for clarity), are nearly indistinguishable implied volatilities induced by the (usual) one-point Taylor series price approximation.

Computational speed, accuracy and robustness

In order for a method of obtaining approximate option prices to be useful to practitioners, the method must be fast, accurate and work over a wide range of model parameters. In order to test the speed, accuracy and robustness of our method, we select model parameters at random from uniform distributions within the following ranges

$$\delta \in [0.0, 0.6], \quad \beta \in [0.0, 1.0], \quad \lambda \in [0.0, 1.0], \quad m \in [-1.0, 0.0], \quad \eta \in [0.0, 1.0].$$

Using the obtained parameters, we then compute approximate option prices $u^{(3)}$ and record computation times over a fixed range of strikes using our third order one-point Taylor expansion (Example 4.1). As the exact price of a call option is not available, we also compute option prices by Monte Carlo simulation. The results are displayed in Tables 5.2 and 5.3. The tables show that our third order price approximation $u^{(3)}$ consistently falls within the 95% confidence interval obtained from the Monte Carlo simulation. Moreover, using a 2.4 GHz laptop computer, an approximate call price $u^{(3)}$ can be computed in only ≈ 0.05 seconds. This is only four to five times larger than the amount of time it takes to compute a similar option price using standard Fourier methods in an exponential Lévy setting.

Variance Gamma Lévy Measure

In our second numerical experiment, we consider the case of Variance Gamma jumps. That is, $\mathcal{N}(dz)$ given by (7.48). We fix the following parameters:

$$\delta = 0.0, \quad \beta = 0.25, \quad \theta = -0.3, \quad \rho = 0.3, \quad \kappa = 0.15, \quad S_0 = e^x = 1.$$

Note that, by letting $\delta = 0$, we have set the diffusion component of X to zero: $a(x) = 0$. Thus, X is a pure-jump Lévy-type process. Using Corollary 4.6, we compute the approximate prices $u^{(0)}(t,x;K)$ and $u^{(2)}(t,x;K)$ of a series of European puts over a range of strikes and with maturities $t \in \{0.5, 1.0\}$. To compute $u^{(i)}$, $i \in \{0,2\}$, we use the two-point Taylor series expansion (Example 4.2). We also compute the put prices by Monte Carlo simulation. For the Monte Carlo simulation, we use a time-step of 10^{-3} years and we simulate 10^6 sample paths. At each time-step, we update X using the following algorithm

$$\begin{aligned} X_{t+\Delta t} &= X_t + b(X_t)\Delta t + \gamma^+(X_t) - \gamma^-(X_t), & I(x) &= e^{2(\beta-1)x}, \\ b(x) &= -\frac{I(x)}{\kappa}\left(\log\left(\frac{\lambda_-}{1+\lambda_-}\right) + \log\left(\frac{\lambda_+}{\lambda_+ - 1}\right)\right), & \gamma^\pm(x) &\sim \Gamma(I(x)\cdot\Delta t/\kappa, 1/\lambda_\pm), \end{aligned}$$

where $\Gamma(a,b)$ is a Gamma-distributed random variable with shape parameter a and scale parameter b. Note that this is equivalent to considering a VG-type process with state-dependent parameters

$$\kappa'(x) := \kappa/I(x), \qquad \theta'(x) := \theta I(x), \qquad \rho'(x) := \rho\sqrt{I(x)}.$$

These state-dependent parameters result in state-independent $\lambda_\pm$ (i.e., $\lambda_\pm$ remain constant). Once again, since implied volatilities rather than prices are the quantity of primary interest, we convert prices to implied volatilities by inverting the Black-Scholes formula numerically. The results are plotted in Figure 5.5. We observe that $\mathrm{IV}[u^{(2)}(t,x;K)]$ agrees almost exactly with $\mathrm{IV}[u^{(MC)}(t,x;K)]$. Values for $u^{(2)}(t,x;K)$, the associated implied volatilities $\mathrm{IV}[u^{(2)}(t,x;K)]$ and the 95% confidence intervals resulting from the Monte Carlo simulation can be found in table 5.4.

5.7.2 Comparison with [165]

In [165], the author considers a class of time-homogeneous Lévy-type processes of the form:

$$\begin{aligned} a(x) &= \frac{1}{2}\Big(b_0^2 + \varepsilon b_1^2 \eta(x)\Big), \\ \gamma(x) &= c_0 + \varepsilon c_1 \eta(x), \\ \nu(x,dz) &= \nu_0(dz) + \varepsilon\eta(x)\nu_1(dz). \end{aligned} \tag{7.52}$$

Here, $(b_0, b_1, c_0, c_1, \varepsilon)$ are non-negative constants, the function $\eta \geq 0$ is smooth and ν_0 and ν_1 are Lévy measures. When $\eta(x) = e_\beta(x) := e^{\beta x}$, the author obtains the following expression for European-style options written on X

$$\begin{aligned} u(t,x) &= \sum_{n=0}^{\infty} \varepsilon^n w_n(t,x), \\ w_n(t,x) &= e_{n\beta}(x)\int_{\mathbb{R}} d\xi \left(\sum_{k=0}^{n} \frac{e^{t\pi_{\xi-ik\beta}}}{\prod_{j\neq k}^{n}(\pi_{\xi-ik\beta} - \pi_{\xi-ij\beta})}\right)\left(\prod_{k=0}^{n-1}\chi_{\xi-ik\beta}\right)\widehat{h}(\xi)\psi_\xi(x). \end{aligned} \tag{7.53}$$

where t is the time to maturity, x is the present value of X and

$$\begin{aligned} \pi_\xi &= \frac{1}{2}b_0^2\left(-\xi^2 - i\xi\right) + c_0(i\xi - 1) - \int_{\mathbb{R}} \nu_0(dz)\Big(e^z - 1 - z\Big)i\xi + \int_{\mathbb{R}} \nu_0(dz)\Big(e^{i\xi z} - 1 - i\xi z\Big), \\ \chi_\xi &= \frac{1}{2}b_1^2\left(-\xi^2 - i\xi\right) + c_1(i\xi - 1) - \int_{\mathbb{R}} \nu_1(dz)\Big(e^z - 1 - z\Big)i\xi + \int_{\mathbb{R}} \nu_1(dz)\Big(e^{i\xi z} - 1 - i\xi z\Big). \end{aligned}$$

As in Corollary 4.6, $\widehat{h}(\xi)$ is the (possibly generalized) Fourier transform of the option payoff $h(x)$, and ψ_ξ is as given in (3.9).

Now consider the following model:

$$a(x) = Af(x), \quad \gamma(x) = \Gamma f(x), \quad \nu(x, dz) = f(x)\mathcal{N}(dz) \quad f(x) = a_0 + \varepsilon a_1 \eta(x), \qquad (7.54)$$

The models described by (7.52) and (7.54) coincide if we choose

$$\frac{1}{2} b_i^2 = a_i A, \qquad c_i = a_i \Gamma, \qquad \nu_i(dz) = a_i \mathcal{N}(dz), \qquad i = \{0, 1\}.$$

Furthermore, comparing equations (4.21) with (7.54), we see that (7.54) is precisely the form considered in Example 4.2. Thus, in this Section we use the two-point Taylor series approximation of Example 4.2 with basis functions B_n given by (4.22). We choose expansion points $\bar{x}_1$ and $\bar{x}_2$ in a symmetric interval around X_0 and in (4.19) we choose $M = f(X_0) = e^{\beta X_0}$.

In our numerical experiment, we consider Gaussian jumps (i.e., $\mathcal{N}(dz)$ given by (7.47)) and we fix the following parameters:

$$\begin{aligned} &\varepsilon = a_0 = a_1 = 1, && \beta = -2.0, && A = \frac{1}{2} 0.15^2, && \Gamma = 0.0, && \lambda = s = 0.2, \\ &m = -0.2, && t = 0.5, && X_0 = 0.0, && \bar{x}_1 = -0.3, && \bar{x}_2 = 0.3. \end{aligned}$$

Using Corollary 4.6, we compute the approximate prices $u^{(0)}(t, x; K)$ and $u^{(2)}(t, x; K)$ of a series of European puts with strike prices $K \in [0.5, 1.5]$ (once again, we add the parameter K to the arguments of $u^{(n)}$ to emphasize the dependence of $u^{(n)}$ on the strike price K). We also compute the price $u(t, x; K)$ using (7.53). In (7.53), we truncate the infinite sum at $n = 8$. As in Section 5.7.1, we convert option prices to implied volatilities. The results are plotted in Figure 5.6. We observe a nearly exact match between the induced implied volatilities $\text{IV}[u^{(2)}(t, x; K)]$ and $\text{IV}[u(t, x; K)]$, where $u(t, x; K)$ (with no superscript) is computed by truncating (7.53) at $n = 8$.

5.7.3 Comparison to NIG-type processes

A Normal Inverse Gaussian (NIG) (see [18]) is a Lévy process with characteristic Lévy exponent $\phi(\xi)$ given by

$$\phi(\xi) = i\mu\xi - \delta \left(\sqrt{\alpha^2 - (\beta + i\xi)^2} - \sqrt{\alpha^2 - \beta^2} \right).$$

In Chapter 14, equation 14.1 of [36], that authors consider NIG-like Feller processes with symbol

$$\phi(x, \xi) = i\mu(x)\xi - \delta(x) \left(\sqrt{\alpha^2(x) - (\beta(x) + i\xi)^2} - \sqrt{\alpha^2(x) - \beta^2(x)} \right),$$

where $\mu, \delta, \alpha, \beta \in C_b^\infty(\mathbb{R})$, $\delta, \alpha > 0$, $\mu, \beta \in \mathbb{R}$, and where there exist constants c and C such that $\delta(x) > c$, $\alpha(x) - |\beta(x)| > c$ and $|\mu(x)| \leq C$. Note that if X is a NIG-type process with symbol $\phi(x, \xi)$, then $S = e^X$ is a martingale if and only if $\phi(x, -i) = 0$. Thus, the triple (α, β, δ) fixes μ.

[36] deduce the following asymptotic expansion for $u(t,x)$ (see the equations following (14.27) and equation (16.40)).

$$u(t,x) := \mathbb{E}_x h(X_t) = \int_{\mathbb{R}} d\xi \frac{1}{\sqrt{2\pi}} e^{i\xi x} e^{t\phi(x,\xi)} \left(1 + \frac{1}{2} t^2 \left(i\partial_x \phi(x,\xi)\right)\left(\partial_\xi \phi(x,\xi)\right) + \cdots \right) \widehat{h}(\xi), \tag{7.55}$$

We note that, if one chooses basis functions $B_n(x) = (x-\bar{x})^n$ as in Example 4.1 with $\bar{x} = x$, then $\phi(x,\xi) = \phi_0(\xi)$ and $\partial_x\phi(x,\xi)\partial_\xi\phi(x,\xi) = \phi_1(\xi)\phi_0'(\xi)$. Thus, from Corollary 4.6 and equations (4.30) and (4.31), it is easy to see that expansion (7.55) is contained within the first two terms of the (one-point) Taylor expansion obtained in this paper.

5.7.4 Yields and credit spreads in the JDCEV setting

Consider a defautable bond, written on S, that pays one dollar at time $T > t$ if no default occurs prior to maturity (i.e., $S_T > 0$, $\zeta > T$) and pays zero dollars otherwise. Then the time t value of the bond is given by

$$V_t = \mathbb{E}[\mathbb{I}_{\{\zeta>T\}}|\mathcal{F}_t] = \mathbb{I}_{\{\zeta>t\}} v(t,X_t;T), \qquad v(t,X_t;T) = \mathbb{E}[e^{-\int_t^T \gamma(s,X_s)ds}|X_t].$$

We add the parameter T to the arguments of v to indicate dependence of v on the maturity date T. Note that $v(t,x;T)$ is both the price of a bond and the conditional survival probability: $\mathbb{Q}(\zeta > T|X_t = x, \zeta > t)$. The yield $Y(t,x;T)$ of such a bond, on the set $\{\zeta > t\}$, is defined as

$$Y(t,x;T) := \frac{-\log v(t,x;T)}{T-t}. \tag{7.56}$$

The credit spread is defined as the yield minus the risk-free rate of interest. Obviously, in the case of zero interest rates, we have: yield = credit spread.

In [49], the authors introduce a class of unified credit-equity models known as Jump to Default Constant Elasticity of Variance or JDCEV. Specifically, in the time-homogeneous case, the underlying S is described by (2.1) with

$$a(x) = \frac{1}{2}\delta^2 e^{2\beta x}, \qquad \gamma(x) = b + c\,\delta^2 e^{2\beta x}, \qquad \nu(x,dz) = 0,$$

where $\delta > 0$, $b \geq 0$, $c \geq 0$. We will restrict our attention to cases in which $\beta < 0$. From a financial perspective, this restriction makes sense, as it results in volatility and default intensity increasing as $S \to 0^+$, which is consistent with the leverage effect. Note that when $c > 0$, the asset S may only go to zero via a jump from a strictly positive value. That is, according to the Feller boundary classification for one-dimensional diffusions (see [35], p.14), the endpoint $-\infty$ is a natural boundary for the killed diffusion X (i.e., the probability that X reaches $-\infty$ in finite time is zero). The survival probability $v(t,x;T)$ in this setting is computed in [175], equation (8.13). We have

$$\begin{aligned} v(t,x;T) &= u(T-t,x) \\ &= \sum_{n=0}^{\infty} \Bigg(e^{-(b+\omega n)(T-t)} \frac{\Gamma(1+c/|\beta|)\Gamma(n+1/(2|\beta|))}{\Gamma(\nu+1)\Gamma(1/(2|\beta|))n!} \\ &\qquad \times A^{1/(2|\beta|)} e^x \exp\left(-Ae^{-2\beta x}\right) {}_1F_1(1-n+c/|\beta|;\nu+1;Ae^{-2\beta x}) \Bigg) \end{aligned} \tag{7.57}$$

where ${}_1F_1$ is the Kummer confluent hypergeometric function, $\Gamma(x)$ is a Gamma function and

$$\nu = \frac{1+2c}{2|\beta|}, \qquad A = \frac{b}{\delta^2|\beta|}, \qquad \omega = 2|\beta|b.$$

We compute $u(T-t,x)$ using both equation (7.57) (truncating the infinite series at $n = 70$) as well as using Corollary 4.6. We use basis functions from the Taylor series expansion of Example 4.1: $B_n(x) = (x-\bar{x})^n$. After computing bond prices, we then calculate the corresponding credit spreads using (7.56). Approximate spreads are denoted

$$Y^{(n)}(t,x;T) := \frac{-\log v^{(n)}(t,x;T)}{T-t}.$$

The survival probabilities are and the corresponding yields are plotted in Figure 5.7. Values for the yields from Figure 5.7 can also be found in Table 5.5.

Remark 7.2. *To compute survival probabilities $v(t,x;T)$, one assumes a payoff function $h(x) = 1$. Note that the Fourier transform of a constant is simply a Dirac delta function: $\widehat{h}(\xi) = \delta(\xi)$. Thus, when computing survival probabilities and/or credit spreads, no numerical integration is required. Rather, one simply uses the following identity*

$$\int_{\mathbb{R}} \widehat{u}(\xi)\partial_\xi^n \delta(\xi) d\xi = (-1)^n \partial_\xi^n \widehat{u}(\xi)|_{\xi=0}.$$

to compute inverse Fourier transforms. From the above identity and equations (4.31)*-*(4.32) *one easily obtains*

$$\begin{aligned}
u_0(t,x) &= e^{-\left(b+\delta^2 c e^{2x\beta}\right)t},\\
u_1(t,x) &= e^{-\left(b+\delta^2 c e^{2x\beta}\right)t}\Big(-\delta^2 bce^{2x\beta}t^2\beta + \frac{1}{2}\delta^4 ce^{4x\beta}t^2\beta - \delta^4 c^2 e^{4x\beta}t^2\beta\Big),\\
u_2(t,x) &= e^{-\left(b+\delta^2 c e^{2x\beta}\right)t}\Big(-\delta^4 ce^{4x\beta}t^2\beta^2 - \frac{2}{3}\delta^2 b^2 ce^{2x\beta}t^3\beta^2 + \delta^4 bce^{4x\beta}t^3\beta^2 - 2\delta^4 bc^2 e^{4x\beta}t^3\beta^2\\
&\quad - \frac{1}{3}\delta^6 ce^{6x\beta}t^3\beta^2 + 2\delta^6 c^2 e^{6x\beta}t^3\beta^2 - \frac{4}{3}\delta^6 c^3 e^{6x\beta}t^3\beta^2 + \frac{1}{2}\delta^4 b^2 c^2 e^{4x\beta}t^4\beta^2\\
&\quad - \frac{1}{2}\delta^6 bc^2 e^{6x\beta}t^4\beta^2 + \delta^6 bc^3 e^{6x\beta}t^4\beta^2 + \frac{1}{8}\delta^8 c^2 e^{8x\beta}t^4\beta^2 - \frac{1}{2}\delta^8 c^3 e^{8x\beta}t^4\beta^2\\
&\quad + \frac{1}{2}\delta^8 c^4 e^{8x\beta}t^4\beta^2\Big).
\end{aligned}$$

5.7.5 Implied Volatility Expansion for CEV

In this Section we apply the implied volatility expansion of Section 5.6 to the CEV model of [65]. The log-price dynamics are given by

$$a(x) = \frac{1}{2}\delta^2 e^{2(\beta-1)x}, \qquad \nu(x,dz) = 0, \qquad \gamma(x) = 0, \qquad \delta > 0, \qquad \beta \in [0,1],$$

In this setting the exact price of a call option with strike $K = e^k$ can be expressed as follows:

$$
\begin{aligned}
u(t,x;K) &= e^x Q(\kappa, 2+\tfrac{2}{2-\beta}, 2\chi) - e^k \left(1 - Q(2\chi, \tfrac{2}{2-\beta}, 2\kappa)\right), \qquad (7.58)\\
Q(w,v,\mu) &= \sum_{n=0}^{\infty} \left(\frac{(\mu/2)^n e^{-\mu/2}}{n!} \frac{\Gamma(v/2+n, w/2)}{\Gamma(v/2+n)} \right),\\
\chi &= \frac{2e^{(2-\beta)x}}{\delta^2(2-\beta)^2 t},\\
\kappa &= \frac{2e^{(2-\beta)k}}{\delta^2(2-\beta)^2 t},
\end{aligned}
$$

where $\Gamma(a)$ and $\Gamma(a,b)$ are the complete and incomplete Gamma functions respectively. For a given call option, the (almost) exact implied volatility σ corresponding to the CEV model can be obtained by solving (7.50) numerically, with $u(t,x;K)$ given by (7.58).

[115] derive the following implied volatility expansion for the CEV model

$$
\begin{aligned}
\sigma^{HW} &= \frac{\delta}{f^{1-\beta}} \left(1 + \frac{(1-\beta)(2+\beta)}{24} \left(\frac{e^x - e^k}{f} \right)^2 + \frac{(1-\beta)^2}{24} \frac{\delta^2 t}{f^{2(1-\beta)}} + \cdots \right),\\
f &= \frac{1}{2}(e^x + e^k). \qquad (7.59)
\end{aligned}
$$

Our second order implied volatility expansion can be computed using equation (6.45). We compare the 2nd order implied volatility expansion to the Hagan-Woodward expansion (7.59) in Figures 5.8 and 5.9. Over the strikes and maturities tested, our implied volatility expansion performs favorably.

5.8 Conclusion

In this chapter, we have considered an asset whose risk-neutral dynamics are described by an exponential Lévy-type martingale subject to default. This class includes nearly all non-negative Markov processes. In this very general setting, we provide a family of approximations – one for each choice of the basis functions (i.e. Taylor, two-point Taylor, L^2 basis, etc.) – for (i) the transition density of the underlying (ii) European-style option prices and their sensitivities and (iii) defaultable bond prices and their credit spreads. For the transition densities, and thus for option and bond prices as well, we establish the accuracy of our asymptotic expansion. We also derive, for local volatility models, an expansion for the implied volatility of European calls/puts. Finally, we provide extensive numerical examples illustrating both the versatility and effectiveness of our methods.

5.9 Appendix

5.9.1 Proof of Theorem 4.4

By hypothesis $v_n \in L^1(\mathbb{R}, dx)$, and thus, by standard Fourier transform properties we the following relation holds:

$$
\mathcal{F}(\mathcal{A}_k v_n(t,\cdot))(\xi) = \phi_k(t,\xi)\widehat{v}_n(t,\xi), \quad n,k \geq 0. \qquad (9.60)
$$

We now Fourier transform equation (4.26). At the left-hand side we have

$$\mathcal{F}((\partial_t + \mathcal{A}_0)\, v_n(t,\cdot))(\xi) = (\partial_t + \phi_0(t,\xi))\, \widehat{v}_n(t,\xi).$$

Next, for the right-hand side of (4.26) we get

$$\begin{aligned}
-\sum_{k=1}^{n} \int_{\mathbb{R}} dx \left(\frac{e^{-i\xi x}}{\sqrt{2\pi}} B_k(x) \right) \mathcal{A}_k v_{n-k}(t,x) &= -\sum_{k=1}^{n} \int_{\mathbb{R}} dx \left(B_k(i\partial_\xi) \frac{e^{-i\xi x}}{\sqrt{2\pi}} \right) \mathcal{A}_k v_{n-k}(t,x) \\
&= -\sum_{k=1}^{n} B_k(i\partial_\xi) \mathcal{F}(\mathcal{A}_k v_{n-k}(t,\cdot))(\xi)
\end{aligned}$$

(by (9.60))

$$= -\sum_{k=1}^{n} B_k(i\partial_\xi) \left(\phi_k(t,\xi) \widehat{v}_{n-k}(t,\xi) \right).$$

Thus, we have the following ODEs (in t) for $\widehat{v}_n(t,\xi)$

$$(\partial_t + \phi_0(t,\xi))\, \widehat{v}_0(t,\xi) = 0, \qquad \widehat{v}_0(T,\xi) = \widehat{h}(\xi), \tag{9.61}$$

$$(\partial_t + \phi_0(t,\xi))\, \widehat{v}_n(t,\xi) = -\sum_{k=1}^{n} B_k(i\partial_\xi) \left(\phi_k(t,\xi) \widehat{v}_{n-k}(t,\xi) \right) \qquad \widehat{v}_n(T,\xi) = 0. \tag{9.62}$$

One can easily verify (e.g., by substitution) that the solutions of (9.61) and (9.62) are given by (4.27) and (4.28) respectively.

5.9.2 Mathematica code

The following Mathematica code can be used to generate the $\widehat{u}_n(t,\xi)$ automatically for Taylor series basis functions: $B_n(x) = (x - x_0)^n$. We have

```
B[n_, x_, x0_] = (x − x0)^n;
Bop[n_, ξ_, x0_, ff_]:=Module[{mat, dim, x},
        mat = CoefficientList[B[n, x, x0], x];
        dim = Dimensions[mat];
        Sum[mat[[m]](i)^(m − 1)D[ff, {ξ, m − 1}], {m, 1, dim[[1]]}]
    ];
u[n_, t_, ξ_, x0_, k_]:=Exp[tϕ[0, ξ, x0]] Sum[
        Integrate[Exp[−sϕ[0, ξ, x0]](Bop[m, ξ, x0, ϕ[m, ξ, x0]u[n − m, s, ξ, x0, k]]), {s, 0, t}], {m, 1, n}];
u[0, t_, ξ_x0_, k_] = Exp[tϕ[0, ξ, x0]]h[ξ, k];
```

The function $\widehat{u}_n(t,\xi)$ is now computed explicitly by typing $\texttt{u[n\_, t\_, }\xi\texttt{\_, x0\_, k\_]}$ and pressing Shift+Enter. Note that the function $\widehat{u}_n(t,\xi)$ can depend on a parameter k (e.g., log-strike) through the Fourier transform of the payoff function $\widehat{h}(\xi,k)$. To compute $\widehat{u}_n(t,\xi)$ using other basis functions, one simply has to replace the first line in the code. For example, for Hermite polynomial basis functions, one re-writes the top line as

$$\texttt{B[n\_, x\_, x0\_]} = \frac{1}{\sqrt{\texttt{(2n)!!}\sqrt{\pi}}}\texttt{HermiteH[n, x} - \texttt{x0];}$$

where $\texttt{HermiteH[n, x]}$ is the Mathematica command for the n-th Hermite polynomial $H_n(x)$ (note that Mathematica does not normalize the Hermite polynomials as we do in equation (4.23)).

5.9.3 Proof of Theorem 5.1

Proof. For the convenience of the reader we recall some of the assumptions of Theorem 5.1: the default intensity and mean jump size are zero $\gamma = \mu = 0$, the jump intensity and diffusion component are time-independent $a(t,x) \equiv a(x)$, $\lambda(t,x) \equiv \lambda(x)$, and the standard deviation of the jumps is constant $\delta(t,x) \equiv \delta$. Thus we consider the integro-differential operator

$$\begin{aligned} Lu(t,x) = \partial_t u(t,x) + \frac{a(x)}{2}(\partial_{xx} - \partial_x)u(t,x) - \lambda(x)\left(e^{\frac{\delta^2}{2}} - 1\right)\partial_x u(t,x) \\ + \lambda(x)\int_{\mathbb{R}} \big(u(t,x+z) - u(t,x)\big)\nu_{\delta^2}(dz), \end{aligned}$$

with

$$\nu_{\delta^2}(dz) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{z^2}{2\delta^2}} dz.$$

Our idea is to use our expansion as a parametrix. That is, our expansion will as the starting point of the classical iterative method introduced by [151] to construct the fundamental solution $p(t,x;T,y)$ of L. Specifically, as in the proof of Theorem 2.3, we take as a parametrix our N-th order approximation $p^{(N)}(t,x;T,y)$ with basis functions $B_n = (x-\bar{x})^n$ and with $\overline{x} = y$. We first prove the case $N = 1$. By analogy with the classical approach (see, for instance, [101] and [71], [188] for the pure diffusive case, or [104] for the integro-differential case), we have

$$p(t,x;T,y) = p^{(1)}(t,x;T,y) + \int_t^T \int_{\mathbb{R}} p^{(0)}(t,x;s,\xi)\Phi(s,\xi;T,y)d\xi ds, \tag{9.63}$$

where Φ is determined by imposing the condition

$$0 = Lp(t,x;T,y) = Lp^{(1)}(t,x;T,y) + \int_t^T \int_{\mathbb{R}} Lp^{(0)}(t,x;s,\xi)\Phi(s,\xi;T,y)d\xi ds - \Phi(t,x;T,y).$$

Equivalently, we have

$$\Phi(t,x;T,y) = Lp^{(1)}(t,x;T,y) + \int_t^T \int_{\mathbb{R}} Lp^{(0)}(t,x;s,\xi)\Phi(s,\xi;T,y)d\xi ds,$$

and therefore by iteration

$$\Phi(t,x;T,y) = \sum_{n=0}^{\infty} Z_n(t,x;T,y), \tag{9.64}$$

where

$$Z_0(t,x;T,y) := Lp^{(1)}(t,x;T,y), \tag{9.65}$$

$$Z_{n+1}(t,x;T,y) := \int_t^T \int_{\mathbb{R}} Lp^{(0)}(t,x;s,\xi) Z_n(s,\xi;T,y) d\xi ds. \tag{9.66}$$

The proof of Theorem 5.1 is based on several technical lemmas which provide pointwise bounds of each term Z_n in (9.64). These bounds combined with formula (9.63) give the estimate of $\left|p(t,x;T,y) - p^{(1)}(t,x;T,y)\right|$.

For any $\alpha, \theta > 0$ and $\ell \geq 0$, consider the integro-differential operators

$$\begin{aligned} L^{\alpha,\theta,\ell} u(t,x) &= \partial_t u(t,x) + \frac{\alpha}{2}(\partial_{xx} - \partial_x) u(t,x) - \ell \left(e^{\frac{\theta}{2}} - 1 \right) \partial_x u(t,x) \\ &\quad + \ell \int_{\mathbb{R}} \big(u(t,x+z) - u(t,x) \big) \nu_\theta(dz), \\ \overline{L}^{\alpha,\theta,\ell} u(t,x) &= \partial_t u(t,x) + \frac{\alpha}{2} \partial_{xx} u(t,x) + \ell \int_{\mathbb{R}} \big(u(t,x+z) - u(t,x) \big) \nu_\theta(dz). \end{aligned}$$

The function

$$\Gamma^{\alpha,\theta,\ell}(t,x;T,y) := \Gamma^{\alpha,\theta,\ell}(T-t, x-y),$$

where

$$\begin{aligned} \Gamma^{\alpha,\theta,\ell}(t,x) &:= e^{-\ell t} \sum_{n=0}^{\infty} \frac{(\ell t)^n}{n!} \Gamma_n^{\alpha,\theta,\ell}(t,x), \\ \Gamma_n^{\alpha,\theta,\ell}(t,x) &:= \frac{1}{\sqrt{2\pi(\alpha t + n\theta)}} \exp\left(- \frac{\left(x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell \right) t \right)^2}{2(\alpha t + n\theta)} \right), \end{aligned}$$

is the fundamental solution of $L^{\alpha,\theta,\ell}$. Analogously, the function

$$\bar{\Gamma}^{\alpha,\theta,\ell}(t,x;T,y) := \bar{\Gamma}^{\alpha,\theta,\ell}(T-t, x-y),$$

where

$$\begin{aligned} \bar{\Gamma}^{\alpha,\theta,\ell}(t,x) &:= e^{-\ell t} \sum_{n=0}^{\infty} \frac{(\ell t)^n}{n!} \bar{\Gamma}_n^{\alpha,\theta}(t,x), \\ \bar{\Gamma}_n^{\alpha,\theta}(t,x) &:= \frac{1}{\sqrt{2\pi(\alpha t + n\theta)}} \exp\left(- \frac{x^2}{2(\alpha t + n\theta)} \right), \end{aligned}$$

is the fundamental solution of $\overline{L}^{\alpha,\theta,\ell}$. Note that under our assumptions, at order zero we have

$$p^{(0)}(t,x;T,y) = \Gamma^{a(y),\delta^2,\lambda(y)}(t,x;T,y). \tag{9.67}$$

We also introduce the convolution operator $\mathcal{C}_\theta$ defined as

$$\mathcal{C}_\theta f(x) = \int_{\mathbb{R}} f(x+z) \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{z^2}{2\theta}} dz. \tag{9.68}$$

Note that, for any $\theta > 0$, we have

$$\mathcal{C}_\theta \Gamma^{\alpha,\theta,\ell}(t,\cdot)(x) = e^{-\ell t} \sum_{n=0}^{\infty} \frac{(\ell t)^n}{n!} \Gamma_{n+1}^{\alpha,\theta,\ell}(t,x),$$
$$\mathcal{C}_\theta \bar{\Gamma}^{\alpha,\theta,\ell}(t,\cdot)(x) = e^{-\ell t} \sum_{n=0}^{\infty} \frac{(\ell t)^n}{n!} \bar{\Gamma}_{n+1}^{\alpha,\theta}(t,x).$$

In the rest of the section, we will always assume that

$$m \le \alpha, \theta \le M, \qquad 0 \le \ell \le M. \tag{9.69}$$

Even if not explicitly stated, all the constants appearing in the estimates (9.70), (9.71), (9.72), (9.75), (9.76) and (9.80) of the following lemmas will depend also on m and M.

Lemma 9.1. *For any $T > 0$ and $c > 1$ there exists a positive constant C such that*[6]

$$\mathcal{C}_\theta^N \Gamma^{\alpha,\theta,\ell}(t,x) \le C\, \mathcal{C}_{cM}^N \bar{\Gamma}^{cM,cM,cM}(t,x), \tag{9.70}$$

for any $t \in (0,T]$, $x \in \mathbb{R}$ and $N \ge 0$.

Proof. For any $n \ge 0$ we have

$$\Gamma_n^{\alpha,\theta,\ell}(t,x) \le \sqrt{\frac{cM}{m}} q_n(t,x) \bar{\Gamma}_n^{cM,cM}(t,x),$$

where

$$q_n(t,x) = \exp\left(-\frac{\left(x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t\right)^2}{2(\alpha t + n\theta)} + \frac{x^2}{2cM(t+n)} \right).$$

A direct computation shows that

$$\max_{x \in \mathbb{R}} q_n(t,x) = \exp\left(\frac{s^2 \left(\alpha + 2\left(e^{\frac{\theta}{2}} - 1\right)\ell\right)^2}{8\left(cM(n+s) - s\alpha - n\delta^2\right)} \right) \le \exp\left(\frac{T\left(\alpha + 2\left(e^{\frac{\theta}{2}} - 1\right)\ell\right)^2}{8(cM - \alpha)} \right),$$

for any $t \in (0,T]$, $n \ge 0$ and α, θ, ℓ in (9.69). Then the thesis is a straightforward consequence of the fact that $q_n(t,x)$ is bounded on $(0,T] \times \mathbb{R}$, uniformly with respect to $n \ge 0$ and α, θ, ℓ in (9.69). □

Lemma 9.2. *For any $T > 0$, $k \in \mathbb{N}$ and $c > 1$, there exists a positive constant C such that*

$$\left| \partial_x^k \Gamma_n^{\alpha,\theta,\ell}(t,x) \right| \le \frac{C}{(\alpha t + n\theta)^{k/2}} \Gamma_n^{c\alpha, c\theta, \ell}(t,x), \tag{9.71}$$

for any $x \in \mathbb{R}$, $t \in]0,T]$ and $n \ge 0$.

[6] Here $\mathcal{C}_\theta^0$ denotes the identity operator.

Proof. For any $k \geq 1$ we have

$$\partial_x^k \Gamma_n^{\alpha,\theta,\ell}(t,x) = \frac{1}{(\alpha t + n\theta)^{k/2}} \Gamma_n^{\alpha,\theta,\ell}(t,x) p_k \left(\frac{x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t}{\sqrt{\alpha t + n\theta}} \right),$$

where p_k is a polynomial of degree k. To prove the Lemma we will show that there exists a positive constant C, which depends only on m, M, T, c and k, such that

$$\left(\frac{\left| x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t \right|}{\sqrt{\alpha t + n\theta}} \right)^j \Gamma_n^{\alpha,\theta,\ell}(t,x) \leq C\, \Gamma_n^{c\alpha,c\theta,\ell}(t,x), \qquad j \leq k.$$

Proceeding as above, we set

$$\left(\frac{\left| x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t \right|}{\sqrt{\alpha t + n\theta}} \right)^j \Gamma_n^{\alpha,\theta,\ell}(t,x) = \Gamma_n^{c\alpha,c\theta,\ell}(t,x) q_{n,j}(t,x),$$

where

$$\begin{aligned} q_{n,j}(t,x) = & \left(\frac{\left| x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t \right|}{\sqrt{\alpha t + n\theta}} \right)^j \\ & \cdot \exp\left(- \frac{\left(x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t\right)^2}{2(\alpha t + n\theta)} + \frac{\left(x - \left(\frac{c\alpha}{2} + \ell e^{\frac{c\theta}{2}} - \ell\right) t\right)^2}{2(c\alpha t + nc\theta)} \right). \end{aligned}$$

Then the thesis follows from the boundedness of $q_{n,j}$ on $(0,T] \times \mathbb{R}$, uniformly with respect to $n \geq 0$ and α, θ, ℓ in (9.69). Indeed the maximum of $q_{n,j}$ can be computed explicitly and we have

$$\lim_{n\to\infty} \left(\max_{x\in\mathbb{R},\ t\in]0,T]} q_{n,j}(t,x) \right) = \left(\frac{cj}{(c-1)e} \right)^{\frac{j}{2}}.$$

□

Lemma 9.3. *For any $T > 0$ and $N \in \mathbb{N}$, there exists a positive constant C such that*

$$\ell t\, \mathcal{C}_\theta^N \Gamma^{\alpha,\theta,\ell}(t,x) \leq C\, \Gamma^{\alpha,2(N+1)\theta,\ell}(t,x) \tag{9.72}$$

for any $t \in (0,T]$ and $x \in \mathbb{R}$.

Proof. We first prove there exists a constant C_0, which depends only on m, M, T and N, such that

$$\Gamma_{n+N}^{\alpha,\theta,\ell}(t,x) \leq C_0\, \Gamma_n^{\alpha,2(N+1)\theta,\ell}(t,x), \tag{9.73}$$

$$\Gamma_N^{\alpha,\theta,\ell}(t,x) \leq C_0\, \Gamma_1^{\alpha,2(N+1)\theta,\ell}(t,x), \tag{9.74}$$

for any $t \in]0,T]$, $x \in \mathbb{R}$, $n \geq 1$ and α, θ, ℓ in (9.69). To prove (9.73) we observe that

$$\Gamma_{n+N}^{\alpha,\theta,\ell}(t,x) \leq \frac{1}{\sqrt{2\pi(\alpha t + (n+N)\theta)}} \exp\left(- \frac{\left(x - \left(\frac{\alpha}{2} + \ell e^{(N+1)\theta} - \ell\right) t\right)^2}{2(\alpha t + 2n(N+1)\theta)} \right) q_n(t,x),$$

where

$$q_n(t,x) = \exp\left(-\frac{\left(x - \left(\frac{\alpha}{2} + \ell e^{\frac{\theta}{2}} - \ell\right) t\right)^2}{2(\alpha t + (n+N)\theta)} + \frac{\left(x - \left(\frac{\alpha}{2} + \ell e^{(N+1)\theta} - \ell\right) t\right)^2}{2(\alpha t + 2n(N+1)\theta)} \right).$$

Now it is easy to check that

$$\max_{x\in\mathbb{R}} q_n(t,x) = \exp\left(\frac{\left(e^{(1+N)\theta} - e^{\frac{\theta}{2}}\right)^2 t^2\ell^2}{2(n - N + 2nN)\theta} \right) \leq \exp\left(\frac{\left(e^{(1+N)\theta} - e^{\frac{\theta}{2}}\right)^2 t^2\ell^2}{2N\theta} \right).$$

for any $t \geq 0$. Thus q_n is bounded on $(0,T]\times\mathbb{R}$, uniformly with respect to $n \in \mathbb{N}$ and α,θ,ℓ in (9.69). To see the above bound, simply observe that

$$\frac{\sqrt{\alpha t + 2n(N+1)\theta}}{\sqrt{\alpha t + (N+n)\theta}} \leq \sqrt{2(N+1)}.$$

The proof of (9.74) is completely analogous. Finally, by (9.73)-(9.74) we have

$$\begin{aligned}
\ell t\, \mathcal{C}_\theta^N \Gamma^{\alpha,\theta,\ell}(t,x) &= e^{-\ell t}\ell t\, \Gamma_N^{\alpha,\theta,\ell}(t,x) + \ell t\, e^{-\ell t} \sum_{n=1}^{\infty} \frac{(\ell t)^n}{n!} \Gamma_{n+N}^{\alpha,\theta,\ell}(t,x) \\
&\leq C_0 \left(e^{-\ell t}\ell t\, \Gamma_1^{\alpha,2(N+1)\theta,\ell}(t,x) + \ell t\, e^{-\ell t} \sum_{n=1}^{\infty} \frac{(\ell t)^n}{n!} \Gamma_n^{\alpha,2(N+1)\theta,\ell}(t,x) \right) \\
&\leq C_0(1 + MT)\ \Gamma^{\alpha,2(N+1)\theta,\ell}(t,x).
\end{aligned}$$

□

Lemma 9.4. *For any $T > 0$ and $N \geq 2$, there exists a positive constant C such that*

$$\mathcal{C}_\theta^N \Gamma^{\alpha,\theta,\ell}(t,x) \leq C\, \mathcal{C}_{2N\theta} \Gamma^{\alpha,2N\theta,\ell}(t,x) \tag{9.75}$$

for any $t \in (0,T]$ and $x \in \mathbb{R}$.

Proof. By (9.73) we have

$$\begin{aligned}
\mathcal{C}_\theta^N \Gamma^{\alpha,\theta,\ell}(t,x) &= e^{-\ell t} \sum_{n=0}^{\infty} \frac{(\ell t)^n}{n!} \Gamma_{n+1+(N-1)}^{\alpha,\theta,\ell}(t,x) \leq C\, e^{-\ell t} \sum_{n=0}^{\infty} \frac{(\ell t)^n}{n!} \Gamma_{n+1}^{\alpha,2N\theta,\ell}(t,x) \\
&= C\, \mathcal{C}_{2N\theta} \Gamma^{\alpha,2N\theta,\ell}(t,x).
\end{aligned}$$

□

Lemma 9.5. *For any $T > 0$, $N \geq 1$ and $c > 1$, there exists a positive constant C such that*

$$\left(\frac{|x|}{\sqrt{\alpha t + n\theta}} \right)^N \Gamma_n^{\alpha,\theta,\ell}(t,x) \leq C \Gamma_n^{c\alpha,c\theta,\ell}(t,x), \tag{9.76}$$

for any $x \in \mathbb{R}$, $t \in (0,T]$ and $n \geq 0$.

Proof. We first show that there exist three constants $C_1 = C_1(m, M, T, N, c)$, $C_2 = C_2(N, c)$ and $C_3 = C_3(m, M, T, N, c)$ such that

$$e^{-\frac{\left(x-\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)t\right)^2}{2(\alpha t+n\theta)}} \le C_1 e^{-\frac{x^2}{2c^{1/3}(\alpha T+n\theta)}}, \tag{9.77}$$

$$\left(\frac{|x|}{\sqrt{\alpha t+n\theta}}\right)^N e^{-\frac{x^2}{2c^{1/3}(\alpha T+n\theta)}} \le C_2 e^{-\frac{x^2}{2c^{2/3}(\alpha T+n\theta)}}, \tag{9.78}$$

$$e^{-\frac{x^2}{2c^{2/3}(\alpha T+n\theta)}} \le C_3 e^{-\frac{\left(x-\left(\frac{c\alpha}{2}+\ell e^{\frac{c\theta}{2}}-\ell\right)t\right)^2}{2c(\alpha t+n\theta)}}, \tag{9.79}$$

for any $x \in \mathbb{R}$, $t \in (0, T]$ and $n \ge 0$. In order to prove (9.77) we consider

$$q_n(t, x) = \exp\left(-\frac{\left(x-\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)t\right)^2}{2(\alpha t+n\theta)} + \frac{x^2}{2c^{1/3}(\alpha t+n\theta)}\right),$$

and show that

$$\max_{x\in\mathbb{R}} q_n(t, x) = \exp\left(\frac{\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)^2 t^2}{2(c^{1/3}-1)(t\alpha+n\theta)}\right) \le \exp\left(\frac{\left(\frac{\alpha}{2}+\ell e^{\frac{\theta}{2}}-\ell\right)^2 T}{2(c^{1/3}-1)}\right),$$

for any $t \in (0, T]$. Thus q_n is bounded on $(0, T] \times \mathbb{R}$, uniformly in $n \ge 0$ and α, θ, ℓ in (9.69). The proof of (9.79) is completely analogous. Equation (9.78) comes directly by setting

$$C_2 = \max_{a\in\mathbb{R}^+}\left(a^N e^{-\frac{a^2}{2c^{1/3}}+\frac{a^2}{2c^{2/3}}}\right) = e^{-\frac{N}{2}}\left(\frac{c^{1/3}\sqrt{N}}{\sqrt{c^{1/3}-1}}\right)^N.$$

Now, by (9.77) we have

$$\left(\frac{|x|}{\sqrt{\alpha t+n\theta}}\right)^N \Gamma_n^{\alpha,\theta,\ell}(t, x) \le C_1\left(\frac{|x|}{\sqrt{\alpha t+n\theta}}\right)^N \frac{e^{-\frac{x^2}{2c^{1/3}(\alpha T+n\theta)}}}{\sqrt{2\pi(\alpha T+n\theta)}}$$

(by (9.78))

$$\le C_1 C_2 \frac{e^{-\frac{x^2}{2c^{2/3}(\alpha T+n\theta)}}}{\sqrt{2\pi(\alpha T+n\theta)}}$$

(by (9.79))

$$\le C_1 C_2 C_3 \sqrt{c}\, \Gamma_n^{c\alpha,c\theta,\ell}(t, x).$$

□

Lemma 9.6. *For any $T > 0$, $c > 1$ and $j \in \mathbb{N} \cup \{0\}$ there exists a positive constant C such that*

$$|x| \mathcal{C}_\theta^j \Gamma^{\alpha,\theta,\ell}(t, x) \le C\left(\mathcal{C}_{2jc\theta}\Gamma^{c\alpha,c\theta,\ell}(t, x) + \mathcal{C}_{2(j+1)c\theta}\Gamma^{c\alpha,4c\theta,\ell}(t, x)\right), \tag{9.80}$$

for any $t \in (0, T]$ and $x \in \mathbb{R}$.

Proof. By Lemma 9.5 there is a constant C_0, only dependent on m, M, T and c, such that

$$
\begin{aligned}
|x|\mathcal{C}_\theta^j\Gamma^{\alpha,\theta,\ell}(t,x) &\le C_0 e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}\sqrt{\alpha t+(n+j)\theta}\,\Gamma^{c\alpha,c\theta,\ell}_{n+j}(t,x)\\
&\le C_0\sqrt{M}(\sqrt{T}+j)\mathcal{C}_{c\theta}^j\Gamma^{c\alpha,c\theta,\ell}(t,x)+C_0\sqrt{M}e^{-\ell t}\sum_{n=0}^{\infty}\frac{(\ell t)^n}{n!}n\Gamma^{c\alpha,c\theta,\ell}_{n+j}(t,x)\\
&\le C_0\sqrt{M}(\sqrt{T}+j)\mathcal{C}_{c\theta}^j\Gamma^{c\alpha,c\theta,\ell}(t,x)+C_0 M^{\frac{3}{2}}T\mathcal{C}_{c\theta}^{j+1}\Gamma^{c\alpha,c\theta,\ell}(t,x),
\end{aligned}
$$

for any $t\in(0,T]$ and $x\in\mathbb{R}$ and α,θ,ℓ in (9.69). Therefore, the thesis follows from Lemma 9.3 for $j=0$ and from Lemma 9.4 for $j\ge 1$. □

Lemma 9.7. *For any $T>0$ and $N,k\ge 1$ we have*

$$
\mathcal{C}_\theta^N\bar{\Gamma}^{\alpha,\theta,\ell}(t,x)\le\sqrt{k+1}\,\mathcal{C}_\theta^{N+k}\bar{\Gamma}^{\alpha,\theta,\ell}(t,x),\qquad t\in]0,T],\ x\in\mathbb{R}.
$$

Proof. A direct computation shows that

$$
\max_{x\in\mathbb{R}}\frac{\bar{\Gamma}^{\alpha,\theta}_{n+N}(t,x)}{\bar{\Gamma}^{\alpha,\theta}_{n+N+k}(t,x)}=\frac{\sqrt{\alpha t+(n+N+k)\theta}}{\sqrt{\alpha t+(n+N)\theta}}\le\sqrt{k+1},
$$

for any $t\le T$, $n\ge 0$, $N\ge 1$ and α,θ,ℓ in (9.69). This concludes the proof. □

Proposition 9.8. *The solution of* (4.26) *with $n=1$, $\mathcal{A}_1$ as in* (4.14)-(4.13) *and $h=\delta_y$, takes the form*

$$
v_1(t,x;T,y)=\left((T-t)(x-y)+\frac{(T-t)^2}{2}J\right)\mathcal{A}_1 p^{(0)}(t,x;T,y),
$$

where J is the operator

$$
J=a_0(2\partial_x-1)-\lambda_0\left(e^{\frac{\delta^2}{2}}-1+\delta^2\partial_x\mathcal{C}_{\delta^2}\right), \tag{9.81}
$$

and $\mathcal{C}_{\delta^2}$ is the convolution operator defined in (9.68).

Proof. Under the assumptions in this appendix, ϕ_n in (4.13) takes the form

$$
\phi_n(\xi)=a_n(-\xi^2-i\xi)-i\xi\lambda_n\left(e^{\frac{\delta^2}{2}}-1\right)+\lambda_n\left(e^{-\frac{\delta^2\xi^2}{2}}-1\right), \tag{9.82}
$$

with

$$
a_n=\frac{\partial^n a(y)}{n!},\qquad \lambda_n=\frac{\partial^n\lambda(y)}{n!}.
$$

Now, by (4.28) with $n=1$, we have

$$
\begin{aligned}
\widehat{v}_1(t,\xi;T,y)&=\int_t^T e^{(s-t)\phi_0(\xi)}\left(i\partial_\xi-y\right)\left(\phi_1(\xi)\widehat{p}^{(0)}(s,\xi;T,y)\right)ds\\
&=\left(i\partial_\xi-y\right)\int_t^T e^{(s-t)\phi_0(\xi)}\phi_1(\xi)\widehat{p}^{(0)}(s,\xi;T,y)ds
\end{aligned}
$$

$$- i \int_t^T \left(\partial_\xi e^{(s-t)\phi_0(\xi)} \right) \phi_1(\xi) \widehat{p}^{(0)}(s,\xi;T,y) ds.$$

Then, recalling that

$$\widehat{p}^{(0)}(s,\xi;T,y) = \frac{1}{\sqrt{2\pi}} e^{(T-s)\phi_0(\xi) - i\xi y},$$

we get

$$\widehat{v}_1(t,\xi;T,y) = \left((T-t)\left(i\partial_\xi - y\right) - \frac{i(T-t)^2}{2}\phi_0'(\xi) \right) \phi_1(\xi) \widehat{p}^{(0)}(t,\xi;T,y).$$

The Proposition follows by (9.82) and by inverse Fourier transforming from ξ into the original coordinate x. □

Proposition 9.9. *For any $c > 1$ and $\tau > 0$, there exists a positive constant C, only dependent on $c, \tau, m, M, \|\lambda_1\|_\infty$ and $\|a_1\|_\infty$, such that*

$$\left|(x-y)^{2-n}(\partial_{xx} - \partial_x) v_n(t,x;T,y)\right| \le C(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM})\ \bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \qquad (9.83)$$

for any $n \in \{0,1\}$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \le \tau$.

Proof. Recalling the expression of $v_0(t,x;T,y) \equiv p^{(0)}(t,x;T,y)$ in (9.67), the case $n = 0$ directly follows from Lemmas 9.2, 9.5 and 9.1 with $N = 0$. For the case $n = 1$, by Proposition 9.8 we have

$$\begin{aligned}(x-y)(\partial_{xx} - \partial_x)v_1(t,x;T,y) = (T-t)(x-y)\big((x-y)(\partial_{xx} - \partial_x) + 2\partial_x - 1\big)\mathcal{A}_1 p^{(0)}(t,x;T,y) \\ + \frac{(T-t)^2}{2}(x-y)J(\partial_{xx} - \partial_x)\mathcal{A}_1 p^{(0)}(t,x;T,y),\end{aligned}$$

with J as in (9.81) and $\mathcal{A}_1$ acting as

$$\mathcal{A}_1 u(x) = a_1(\partial_{xx} - \partial_x)u(x) - \lambda_1 \left(\left(e^{\frac{\delta^2}{2}} - 1 \right) \partial_x u(x) - \mathcal{C}_{\delta^2} u(x) + u(x) \right).$$

In the computations that follow below, in order to shorten notation, we omit the dependence of t, x, T, y in any function. By the commutative property of the operators ∂_x and $\mathcal{C}$, and by applying Lemmas 9.2, 9.3 and 9.5 with $N = 1$, there exists a positive constant C_1 only dependent on $c, \tau, m, M, \|\lambda_1\|_\infty$ and $\|a_1\|_\infty$ such that

$$|(T-t)(x-y)\big((x-y)(\partial_{xx} - \partial_x) + 2\partial_x - 1\big)a_1(\partial_{xx} - \partial_x)p^{(0)}| \le C_1 \Gamma^{ca(y), c\delta^2, \lambda(y)},$$

$$\begin{aligned}\left|(T-t)(x-y)\big((x-y)(\partial_{xx} - \partial_x) + 2\partial_x - 1\big)\lambda_1 \left(\left(e^{\frac{\delta^2}{2}} - 1 \right) \partial_x + 1 \right) p^{(0)} \right| \\ \le C_1 \big(\Gamma^{ca(y), c\delta^2, \lambda(y)} + \Gamma^{ca(y), 4c\delta^2, \lambda(y)}\big),\end{aligned}$$

$$\frac{(T-t)^2}{2}|(x-y)J(\partial_{xx} - \partial_x)a_1(\partial_{xx} - \partial_x)p^{(0)}| \le C_1 \big(\Gamma^{ca(y), c\delta^2, \lambda(y)} + \Gamma^{ca(y), 4c\delta^2, \lambda(y)}\big),$$

$$\frac{(T-t)^2}{2} \left|(x-y)J(\partial_{xx} - \partial_x)\lambda_1 \left(\left(e^{\frac{\delta^2}{2}} - 1 \right) \partial_x + 1 \right) p^{(0)} \right|$$

$$\leq C_1\big(\Gamma^{ca(y),c\delta^2,\lambda(y)} + \Gamma^{ca(y),4c\delta^2,\lambda(y)}\big), \tag{9.84}$$

for any $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$. Analogously, by the commutative property of ∂_x and $\mathcal{C}$, and by applying Lemmas 9.5, 9.2, 9.6 and 9.4 with $N = 2$, there exists a positive constant C_2 only dependent on $c, \tau, m, M, \|\lambda_1\|_\infty$ and $\|a_1\|_\infty$ such that

$$|(T-t)(x-y)\big((x-y)(\partial_{xx}-\partial_x)+2\partial_x-1\big)\lambda_1\mathcal{C}_{\delta^2}p^{(0)}|\,\|\lambda_1\|_\infty \leq C_2\left(\mathcal{C}_{c\delta^2}\Gamma^{ca(y),c\delta^2,\lambda(y)} + \mathcal{C}_{4c\delta^2}\Gamma^{ca(y),4c\delta^2,\lambda(y)}\right),$$

$$\frac{(T-t)^2}{2}|(x-y)J(\partial_{xx}-\partial_x)\lambda_1\mathcal{C}_{\delta^2}p^{(0)}| \leq \|\lambda_1\|_\infty\, C_2\big(\mathcal{C}_{c\delta^2}\Gamma^{ca(y),c\delta^2,\lambda(y)} + \mathcal{C}_{4c\delta^2}\Gamma^{ca(y),4c\delta^2,\lambda(y)}\big), \tag{9.85}$$

for any $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$. Then, (9.83) follows from (9.84) and (9.85) by applying Lemma 9.1 with $N = 0$ and $N = 1$ respectively. □

Proposition 9.10. *For any $c > 1$ and $\tau > 0$, there exists a positive constant C, only dependent on $c, \tau, m, M, \|\lambda_1\|_\infty$ and $\|a_1\|_\infty$, such that*

$$\left|(x-y)^{2-n}\left(\left(e^{\frac{\delta^2}{2}}-1\right)\partial_x + \mathcal{C}_{\delta^2} - 1\right)v_n(t,x;T,y)\right| \leq C(1+\mathcal{C}_{cM})\,\bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \tag{9.86}$$

for any $n \in \{0,1\}$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$.

Proof. For simplicity we only prove the thesis for $n = 0$. The proof for $n = 1$ is entirely analogous to that of Proposition 9.9. Once again, hereafter we omit the dependence of t, x, T, y in any function we consider. Recalling the expression of $v_0(t,x;T,y) \equiv p^{(0)}(t,x;T,y)$ in (9.67), by Lemmas 9.2, 9.5 and 9.6, there exists a positive constant C_1 only dependent on c, τ, m, M such that

$$\left|(x-y)^2\left(\left(e^{\frac{\delta^2}{2}}-1\right)\partial_x + \mathcal{C}_{\delta^2} - 1\right)v_0\right| \leq C_1\big(\Gamma^{ca(y),4c\delta^2,\lambda(y)} + (1+\mathcal{C}_{16c\delta^2})\Gamma^{ca(y),16c\delta^2,\lambda(y)}\big),$$

for any $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$. Then, (9.86) follows from Lemma 9.1 with $N = 0$ and with $N = 1$. □

Proposition 9.11. *For any $c > 1$ and $\tau > 0$, there exists a positive constant C, only dependent on $c, \tau, m, M, \|\lambda_1\|_\infty, \|\lambda_2\|_\infty$, $\|a_1\|_\infty$ and $\|a_2\|_\infty$, such that*

$$|Z_n(t,x;T,y)| \leq \frac{C^{n+1}(T-t)^{\frac{n}{2}}}{\sqrt{n!}}\left(1+\|\lambda_1\|_\infty\mathcal{C}_{cM}^{n+1}\right)\bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \tag{9.87}$$

for any $n \geq 0$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$.

Proof. Let us define the operators

$$L_0 = \partial_t + \mathcal{A}_0, \qquad L_1 = \partial_t + \mathcal{A}_0 + (x-y)\mathcal{A}_1.$$

Let us recall that, by (4.25) and (4.26) with $n = 1$, we have

$$L_0 v_0 = 0, \qquad L_0 v_1 = -(L_1 - L_0) v_0.$$

Thus, by (9.65) we have

$$\begin{aligned} Z_0(t, x; T, y) = Lp^{(1)}(t, x; T, y) &= Lv_0(t, x; T, y) + Lv_1(t, x; T, y) \\ &= (L - L_1) v_0(t, x; T, y) + (L - L_0) v_1(t, x; T, y), \end{aligned}$$

where $(L - L_0)$ and $(L - L_1)$ are explicitly given by

$$\begin{aligned} (L - L_0) &= (a(x) - a(y))(\partial_{xx} - \partial_x) + (\lambda(x) - \lambda(y)) \left(\left(e^{\frac{\delta^2}{2}} - 1 \right) \partial_x + \mathcal{C}_{\delta^2} - 1 \right), \\ (L - L_1) &= (a(x) - a(y) - a'(y)(x - y))(\partial_{xx} - \partial_x) \\ &\quad + (\lambda(x) - \lambda(y) - \lambda'(y)(x - y)) \left(\left(e^{\frac{\delta^2}{2}} - 1 \right) \partial_x + \mathcal{C}_{\delta^2} - 1 \right). \end{aligned} \tag{9.88}$$

Thus, by the Lipschitz assumptions on a, λ and their first order derivatives, we obtain

$$\begin{aligned} |Z_0(t, x; T, y)| \leq\ & \|a_2\|_\infty |x - y|^2 |(\partial_{xx} - \partial_x) v_0(t, x; T, y)| \\ &+ \|a_1\|_\infty |x - y| |(\partial_{xx} - \partial_x) v_1(t, x; T, y)| \\ &+ \|\lambda_2\|_\infty |x - y|^2 \left| \left(\left(e^{\frac{\delta^2}{2}} - 1 \right) \partial_x + \mathcal{C}_{\delta^2} - 1 \right) v_0(t, x; T, y) \right| \\ &+ \|\lambda_1\|_\infty |x - y| \left| \left(\left(e^{\frac{\delta^2}{2}} - 1 \right) \partial_x + \mathcal{C}_{\delta^2} - 1 \right) v_1(t, x; T, y) \right|. \end{aligned}$$

and, as $\|\lambda_1\|_\infty = 0$ implies $\|\lambda_2\|_\infty = 0$, by Propositions 9.9 and 9.10 there exists a positive constant C, only dependent on c, τ, m, M,$\|\lambda_1\|_\infty$,$\|\lambda_2\|_\infty$, $\|a_1\|_\infty$ and $\|a_2\|_\infty$, such that (9.87) holds for $n = 0$. To prove the general case, we proceed by induction on n. First note that, by (4.25) we have

$$|Lp^{(0)}(t, x; T, y)| = |(L - L_0) p^{(0)}(t, x; T, y)|$$

(and by (9.88) and the Lipschitz property of α, λ)

$$\begin{aligned} &\leq \|a_1\|_\infty |x - y| |(\partial_{xx} - \partial_x) p^{(0)}(t, x; T, y)| \\ &\quad + \|\lambda_1\|_\infty |x - y| \left| \left(\left(e^{\frac{\delta^2}{2}} - 1 \right) \partial_x + \mathcal{C}_{\delta^2} - 1 \right) p^{(0)}(t, x; T, y) \right| \end{aligned}$$

(and by applying Lemmas 9.1, 9.2, 9.5 and 9.6 with $N = 0, 1$)

$$\leq C_0 \left(\frac{1}{\sqrt{T - t}} + \|\lambda_1\|_\infty \mathcal{C}_{cM} \right) \bar{\Gamma}^{cM,cM,cM}(t, x; T, y), \tag{9.89}$$

for any $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$, and where C_0 is a positive constant only dependent on $c, \tau, m, M, \|\lambda_1\|_\infty$ and $\|a_1\|_\infty$. Assume now (9.87) holds for $n \geq 0$. Then by (9.66) we obtain

$$|Z_{n+1}(t, x; T, y)| \leq \int_t^T \int_{\mathbb{R}} |Lp^{(0)}(t, x; s, \xi)| |Z_n(s, \xi; T, y)| d\xi ds$$

(and by inductive hypothesis and by (9.89))

$$\leq \frac{C^{n+1}C_0}{\sqrt{n!}} \int_t^T (T-s)^{\frac{n}{2}} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{s-t}} + \|\lambda_1\|_\infty \mathcal{C}_{cM} \right) \bar{\Gamma}^{cM,cM,cM}(t,x;s,\xi) \cdot$$

$$\cdot \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \bar{\Gamma}^{cM,cM,cM}(s,\xi;T,y) d\xi ds.$$

Now, by the semigroup property

$$\int_{\mathbb{R}} \mathcal{C}_\theta^k \bar{\Gamma}^{\alpha,\theta,\ell}(t,x;s,\xi) \mathcal{C}_\theta^N \bar{\Gamma}^{\alpha,\theta,\ell}(s,\xi;T,y)\, d\xi = \mathcal{C}_\theta^{k+N} \bar{\Gamma}^{\alpha,\theta,\ell}(t,x;T,y), \qquad k,N \geq 0, \tag{9.90}$$

and by the fact that[7]

$$\int_t^T \frac{(T-s)^{\frac{n}{2}}}{\sqrt{s-t}} ds = \frac{\sqrt{\pi}(T-t)^{\frac{n+1}{2}} \Gamma_E\left(\frac{2+n}{2}\right)}{\Gamma_E\left(\frac{3+n}{2}\right)} \leq \frac{\kappa (T-t)^{\frac{n+1}{2}}}{\sqrt{n+1}},$$

with $\kappa = \sqrt{2}\pi$, we obtain

$$\begin{aligned} |Z_{n+1}(t,x;T,y)| \leq{}& \frac{C^{n+1}C_0}{\sqrt{n!}} \left(\frac{\kappa (T-t)^{\frac{n+1}{2}}}{\sqrt{n+1}} \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) \\ &+ \frac{C^{n+1}C_0}{\sqrt{n!}} \left(\frac{2(T-t)^{\frac{n+2}{2}}}{n+2} \|\lambda_1\|_\infty \left(\mathcal{C}_{cM} + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+2}\right) \right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y). \end{aligned} \tag{9.91}$$

Now, by Lemma 9.7 we have

$$\begin{aligned} \mathcal{C}_{cM}^{n+1} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) &\leq 2\, \mathcal{C}_{cM}^{n+2} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \\ \mathcal{C}_{cM} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) &\leq \sqrt{n+2}\, \mathcal{C}_{cM}^{n+2} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y). \end{aligned}$$

Inserting the above results into (9.91) we obtain

$$\begin{aligned} |Z_{n+1}(t,x;T,y)| &\leq \frac{C^{n+1}C_0}{\sqrt{n!}} \frac{(T-t)^{\frac{n+1}{2}}}{\sqrt{n+1}} \left(\kappa + 2\|\lambda_1\|_\infty \left(\kappa + \sqrt{\tau}(1 + \|\lambda_1\|_\infty)\right) \mathcal{C}_{cM}^{n+2}\right) \\ &\quad \cdot \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) \\ &\leq \frac{C^{n+1}C_1 (T-t)^{\frac{n+1}{2}}}{\sqrt{(n+1)!}} \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+2}\right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y), \end{aligned}$$

where

$$C_1 = 2C_0 \left(\kappa + \sqrt{\tau}(1 + \|\lambda_1\|_\infty)\right).$$

Now, without loss of generality we can assume $C_1 \leq C$, and thus we obtain (9.87) for $n+1$. □

[7] Here Γ_E represents the Euler Gamma function.

We are now in position to prove Theorem 5.1 for $N = 1$. Indeed, by equations (9.63), (9.64) and Proposition 9.11 we have

$$|p(t,x;T,y) - p^{(1)}(t,x;T,y)|$$
$$\leq \sum_{n=0}^{\infty} \frac{C^{n+1}}{\sqrt{n!}} \int_t^T (T-s)^{\frac{n}{2}} \int_{\mathbb{R}} p^{(0)}(t,x;s,\xi) \ \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \bar{\Gamma}^{cM,cM,cM}(s,\xi;T,y) d\xi ds$$

(and by Lemma 9.1 with $N = 0$ and $N = 1$ respectively)

$$\leq \sum_{n=0}^{\infty} \frac{C^{n+1}}{\sqrt{n!}} \int_t^T (T-s)^{\frac{n}{2}} \int_{\mathbb{R}} \bar{\Gamma}^{cM,cM,cM}(t,x;s,\xi) \ \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \bar{\Gamma}^{cM,cM,cM}(s,\xi;T,y) d\xi ds$$

(and by (9.90))

$$= 2(T-t) \left(\sum_{n=0}^{\infty} \frac{C^{n+1}(T-t)^{\frac{n}{2}}}{\sqrt{n!}} \left(1 + \|\lambda_1\|_\infty \mathcal{C}_{cM}^{n+1}\right) \bar{\Gamma}^{cM,cM,cM}(t,x;T,y) \right),$$

for any $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \leq \tau$. Since

$$\sum_{n=0}^{\infty} \frac{C^{n+1}(T-t)^{\frac{n}{2}}}{\sqrt{n!}} \mathcal{C}_{cM}^{n+1} \bar{\Gamma}^{cM,cM,cM}(t,x;T,y)$$

can be easily checked to be convergent, this concludes the proof of Theorem 5.1 for $N = 1$. The proof for the general case is based on the same arguments. However, in the general case the technical details become significantly more complicated. Therefore, for sake of simplicity, we present here only a sketch with the main steps of the proof. We repeat the same iterative construction of $p(t,x;T,y)$, but using our N-th order approximation as a starting point. Thus, we replace $p^{(1)}(t,x;T,y)$ with $p^{(N)}(t,x;T,y)$ in (9.63), where Φ is now defined as

$$\Phi(t,x;T,y) = \sum_{n=0}^{\infty} Z_n^N(t,x;T,y),$$

where

$$Z_0^N(t,x;T,y) := Lp^{(N)}(t,x;T,y),$$
$$Z_{n+1}^N(t,x;T,y) := \int_t^T \int_{\mathbb{R}} Lp^{(0)}(t,x;s,\xi) Z_n^N(s,\xi;T,y) d\xi ds.$$

Now, proceeding by induction, one can extend Propositions 9.9 and Proposition 9.10 to a general $n \in \mathbb{N}$. Eventually, after proving the identity

$$Lp^{(N)}(t,x;T,y) = \sum_{n=0}^{N} (L - L_n) v^{(N-n)}(t,x;T,y) + (L - L_0) v^{(1)}(t,x;T,y),$$

one will be able to prove the estimate (9.87) for $\left|Z_n^N(t,x;T,y)\right|$, from which Theorem 5.1 would follow exactly as in the case $N = 1$.

□

5.9.4 Proof of Equations (6.43) and (6.44)

Below, we will use the following repeatedly

$$\phi_n(\xi)e^{i\xi x} = a_n(\partial_x^2 - \partial_x)e^{i\xi x}, \quad \phi_n'(\xi)e^{i\xi x} = ia_n(2\partial_x - 1)e^{i\xi x}, \quad \phi_n''(\xi)e^{i\xi x} = -2a_n e^{i\xi x}.$$

Note that the above relations hold only when $\phi_n(\xi) = a_n(-\xi^2 - i\xi)$, as is the case when X is a diffusion with not jumps and no possibility of default. Now, using the above relations, we find

$$\begin{aligned}
u_1(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi e^{i\xi x}\widehat{u}_1(t,\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds e^{i\xi x+(t-s)\phi_0(\xi)}(i\partial_\xi - \bar{x})\phi_1(\xi)\widehat{u}_0(s,\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds e^{i\xi x+(t-s)\phi_0(\xi)}(i\partial_\xi - \bar{x})\phi_1(\xi)e^{s\phi_0(\xi)}\widehat{h}(\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds\phi_1(\xi)e^{s\phi_0(\xi)}\widehat{h}(\xi)(-i\partial_\xi - \bar{x})e^{i\xi x+(t-s)\phi_0(\xi)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \Big(t(x-\bar{x}) - \tfrac{1}{2}it^2\phi_0'(\xi)\Big)\phi_1(\xi)e^{i\xi x+t\phi_0(\xi)}\widehat{h}(\xi) \\
&= \Big(t(x-\bar{x}) + \tfrac{1}{2}t^2 a_0(2\partial_x - 1)\Big)a_1(\partial_x^2 - \partial_x)u_0(t,x),
\end{aligned}$$

which establishes (6.43). Likewise, for u_2 we have

$$\begin{aligned}
u_2(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi\, e^{i\xi x}\widehat{u}_2(t,\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds\, e^{i\xi x+(t-s)\phi_0(\xi)}(i\partial_\xi - \bar{x})^2\phi_2(\xi)\widehat{u}_0(s,\xi) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds\, e^{i\xi x+(t-s)\phi_0(\xi)}(i\partial_\xi - \bar{x})\phi_1(\xi)\widehat{u}_1(s,\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds\, e^{i\xi x+(t-s)\phi_0(\xi)}(i\partial_\xi - \bar{x})^2\phi_2(\xi)e^{s\phi_0(\xi)}\widehat{h}(\xi) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds \int_0^s dr\, e^{i\xi x+(t-s)\phi_0(\xi)}(i\partial_\xi - \bar{x})\phi_1(\xi)e^{(s-r)\phi_0(\xi)}(i\partial_\xi - \bar{x}) \\
&\qquad\qquad \cdot \phi_1(\xi)e^{r\phi_0(\xi)}\widehat{h}(\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds\, \phi_2(\xi)e^{s\phi_0(\xi)}\widehat{h}(\xi)(-i\partial_\xi - \bar{x})^2 e^{i\xi x+(t-s)\phi_0(\xi)} \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \int_0^t ds \int_0^s dr\, \phi_1(\xi)e^{r\phi_0(\xi)}\widehat{h}(\xi)(-i\partial_\xi - \bar{x})\phi_1(\xi)e^{(s-r)\phi_0(\xi)}(-i\partial_\xi - \bar{x}) \\
&\qquad\qquad \cdot e^{i\xi x+(t-s)\phi_0(\xi)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \bigg(t(x-\bar{x})^2 - it^2(x-\bar{x})\phi_0'(\xi) - \tfrac{1}{3}t^3(\phi_0'(\xi))^2 - \tfrac{1}{2}t^2\phi_0''(\xi)\bigg)\phi_2(\xi) \\
&\qquad \cdot e^{i\xi x+t\phi_0(\xi)}\widehat{h}(\xi) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\xi \bigg(\tfrac{1}{2}t^2(x-\bar{x})^2\phi_1(\xi) - \tfrac{1}{2}it^3\,(x-\bar{x})\,\phi_1(\xi)\phi_0'(\xi) - \tfrac{1}{8}t^4\phi_1(\xi)(\phi_0'(\xi))^2
\end{aligned}$$

$$
\begin{aligned}
&- \tfrac{1}{2}it^2\left(x-\bar{x}\right)\phi_1'(\xi) - \tfrac{1}{6}t^3\phi_0'(\xi)\phi_1'(\xi) - \tfrac{1}{6}t^3\phi_1(\xi)\phi_0''(\xi)\Big)\phi_1(\xi)e^{i\xi x+t\phi_0(\xi)}\widehat{h}(\xi) \\
&= \Big(t(x-\bar{x})^2 + t^2(x-\bar{x})a_0(2\partial_x-1) + \tfrac{1}{3}t^3a_0^2(2\partial_x-1)^2 + t^2a_0\Big)a_2(\partial_x^2-\partial_x)u_0(t,x) \\
&\quad + \Big(\tfrac{1}{2}t^2(x-\bar{x})^2a_1(\partial_x^2-\partial_x) + \tfrac{1}{2}t^3\left(x-\bar{x}\right)a_1(\partial_x^2-\partial_x)a_0(2\partial_x-1) \\
&\qquad + \tfrac{1}{8}t^4a_1(\partial_x^2-\partial_x)a_0^2(2\partial_x-1)^2 + \tfrac{1}{2}t^2\left(x-\bar{x}\right)a_1(2\partial_x-1) \\
&\qquad + \tfrac{1}{6}t^3a_0(2\partial_x-1)a_1(2\partial_x-1) + \tfrac{1}{3}t^3a_1(\partial_x^2-\partial_x)a_0\Big)a_1(\partial_x^2-\partial_x)u_0(t,x),
\end{aligned}
$$

which establishes (6.44).

5.10 Figures and tables

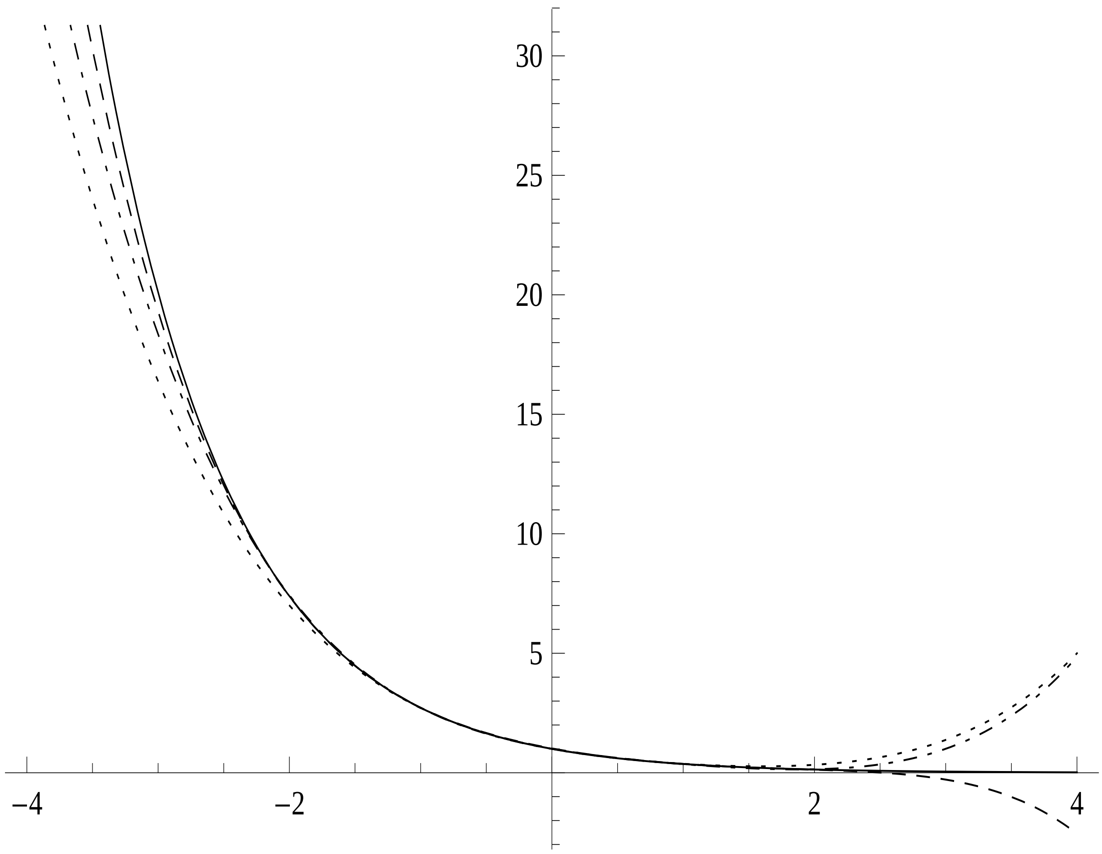

Figure 5.1: A comparison of the exact (solid), 2nd order two-point Taylor series approximation (dashed), 4th order Taylor series approximation (dotted), and 4th order Hermite polynomial expansion of the function $\exp(-x)$. For the two-point Taylor series approximation, we expand about $\bar{x}_1 = -1$ and $\bar{x}_2 = 1$. For the (usual) Taylor series and the Hermite polynomial approximations we expand about $\bar{x} = 0$.

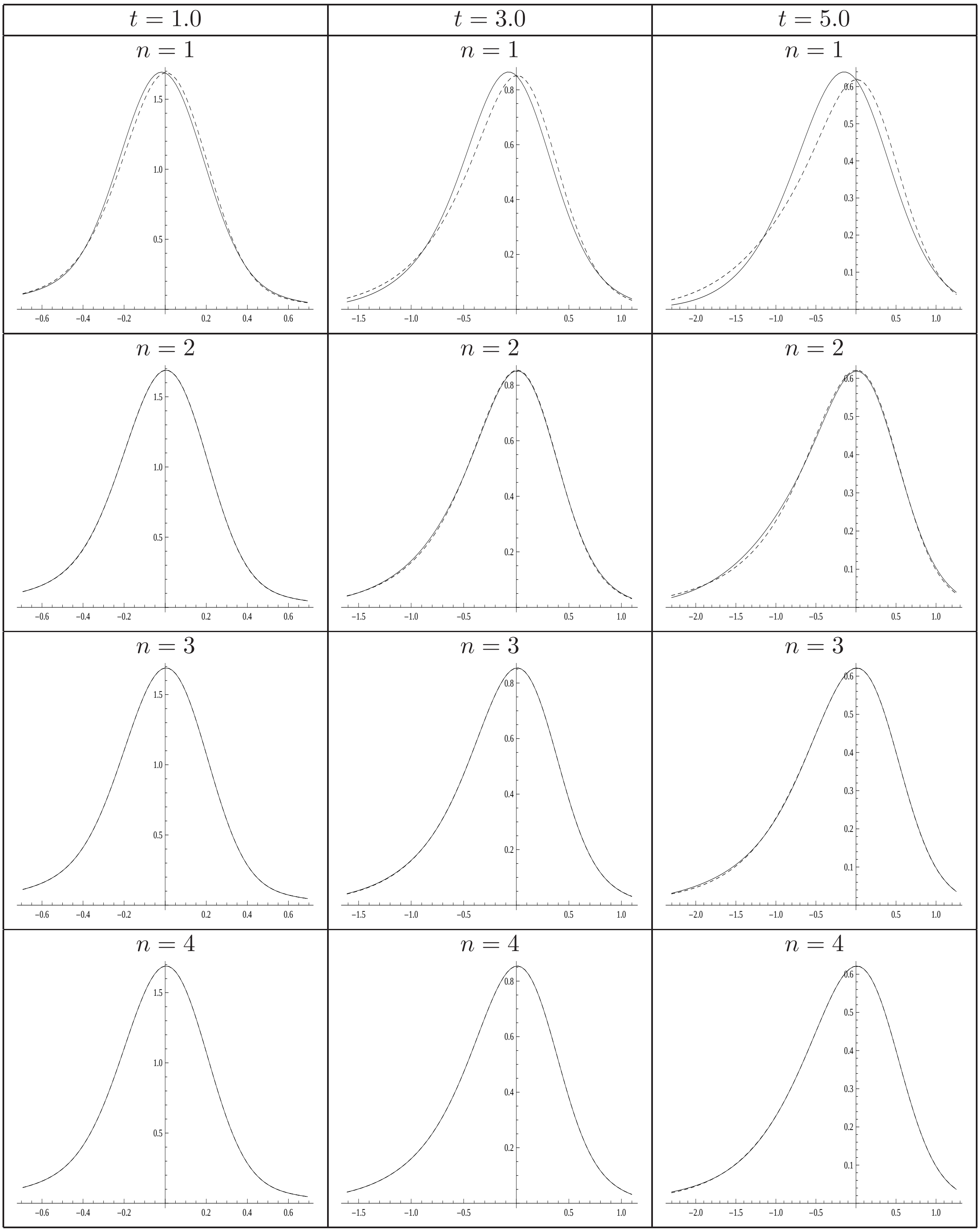

Figure 5.2: Using the model considered in Section 5.7.1 we plot $p^{(n)}(t, x; T, y)$ (solid black) and $p^{(n-1)}(t, x; T, y)$ (dashed black) as a function of y for $n = \{1, 2, 3, 4\}$ and $t = \{1.0, 3.0, 5.0\}$ years. For all plots we use the Taylor series expansion of Example 4.1. Note that as n increases $p^{(n)}$ and $p^{(n-1)}$ become nearly indistinguishable. In these plots we set $\beta = 0.7$ All other parameter values are those listed in Section 5.7.1.

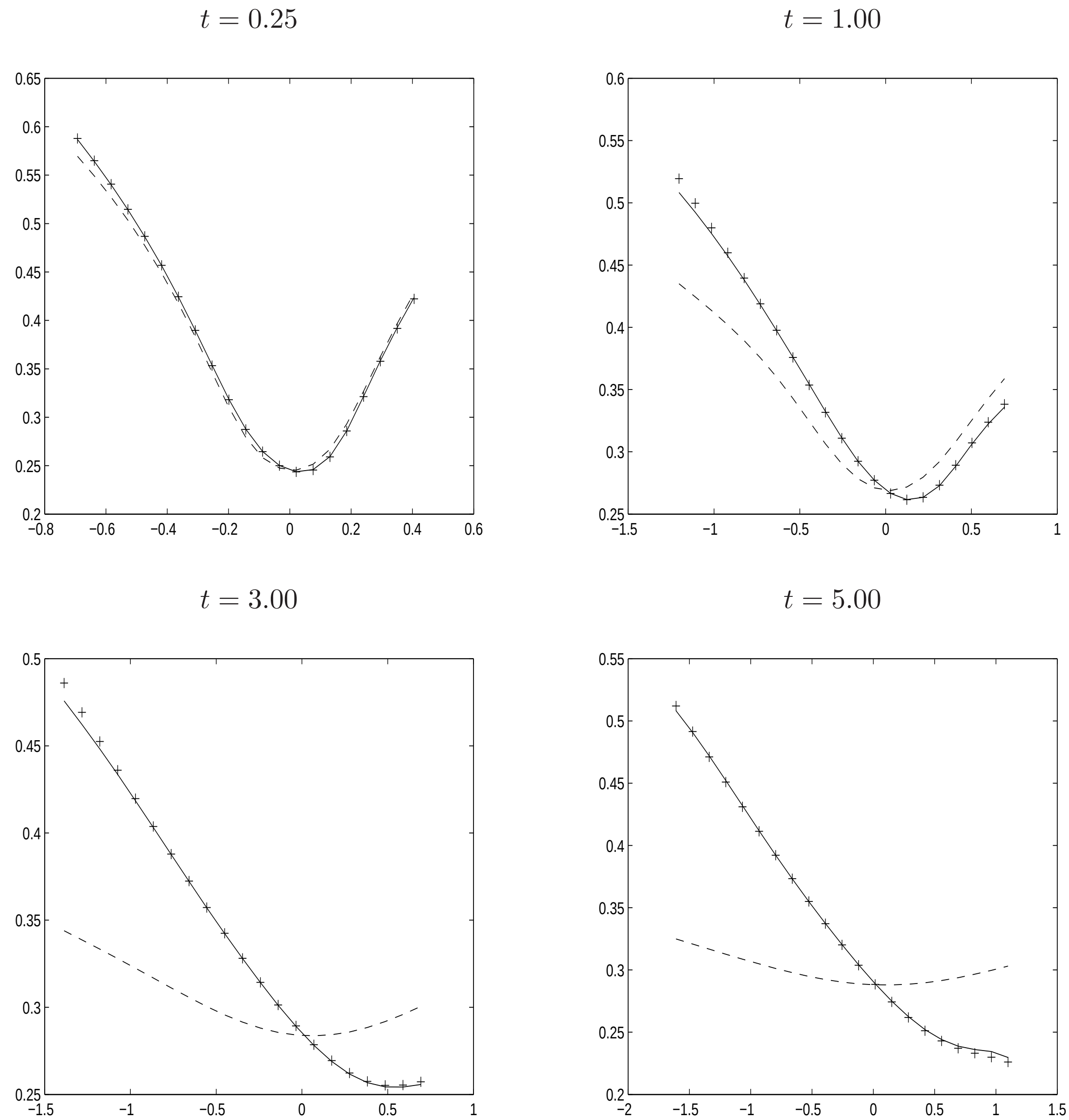

Figure 5.3: Implied volatility (IV) is plotted as a function of log-strike $k := \log K$ for the CEV-like model with Gaussian-type jumps of Section 5.7.1. The solid lines corresponds to the IV induced by $u^{(3)}(t,x)$, which is computed using the one-point Taylor expansion (see Example 4.1). The dashed lines corresponds to the IV induced by $u^{(0)}(t,x)$ (again, using the usual one-point Taylor series expansion). The crosses correspond to the IV induced by $u^{(MC)}(t,x)$, which is the price obtained from the Monte Carlo simulation.

t	k	$u^{(3)}$	u MC-95% c.i.	IV$[u^{(3)}]$	IV MC-95% c.i.
	-0.6931	0.0006	0.0006 - 0.0007	0.5864	0.5856 - 0.5901
	-0.4185	0.0024	0.0024 - 0.0025	0.4563	0.4553 - 0.4583
0.2500	-0.1438	0.0111	0.0110 - 0.0112	0.2875	0.2865 - 0.2883
	0.1308	0.1511	0.1508 - 0.1513	0.2595	0.2573 - 0.2608
	0.4055	0.5028	0.5024 - 0.5030	0.4238	0.4152 - 0.4288
	-1.2040	0.0009	0.0009 - 0.0010	0.5115	0.5176 - 0.5210
	-0.7297	0.0046	0.0047 - 0.0048	0.4174	0.4178 - 0.4199
1.0000	-0.2554	0.0314	0.0313 - 0.0316	0.3109	0.3102 - 0.3117
	0.2189	0.2781	0.2775 - 0.2784	0.2638	0.2620 - 0.2649
	0.6931	1.0034	1.0030 - 1.0041	0.3358	0.3296 - 0.3459
	-1.3863	0.0074	0.0081 - 0.0083	0.4758	0.4851 - 0.4870
	-0.8664	0.0224	0.0224 - 0.0227	0.4031	0.4029 - 0.4045
3.0000	-0.3466	0.0776	0.0773 - 0.0779	0.3280	0.3274 - 0.3288
	0.1733	0.3097	0.3094 - 0.3107	0.2690	0.2685 - 0.2703
	0.6931	1.0155	1.0150 - 1.0169	0.2558	0.2540 - 0.2604
	-1.6094	0.0160	0.0164 - 0.0166	0.5082	0.5111 - 0.5128
	-0.9324	0.0439	0.0436 - 0.0440	0.4118	0.4107 - 0.4121
5.0000	-0.2554	0.1504	0.1497 - 0.1507	0.3203	0.3194 - 0.3208
	0.4216	0.6139	0.6123 - 0.6142	0.2521	0.2500 - 0.2524
	1.0986	2.0050	2.0032 - 2.0057	0.2297	0.2163 - 0.2342

Table 5.1: Prices (u) and Implied volatility (IV$[u]$) as a function of time to maturity t and log-strike $k := \log K$ for the CEV-like model with Gaussian-type jumps of Section 5.7.1. The approximate price $u^{(3)}$ is computed using the (usual) one-point Taylor expansion (see Example 4.1). For comparison, we provide the 95% confidence intervals for prices and implied volatilities, which we obtain from the Monte Carlo simulation.

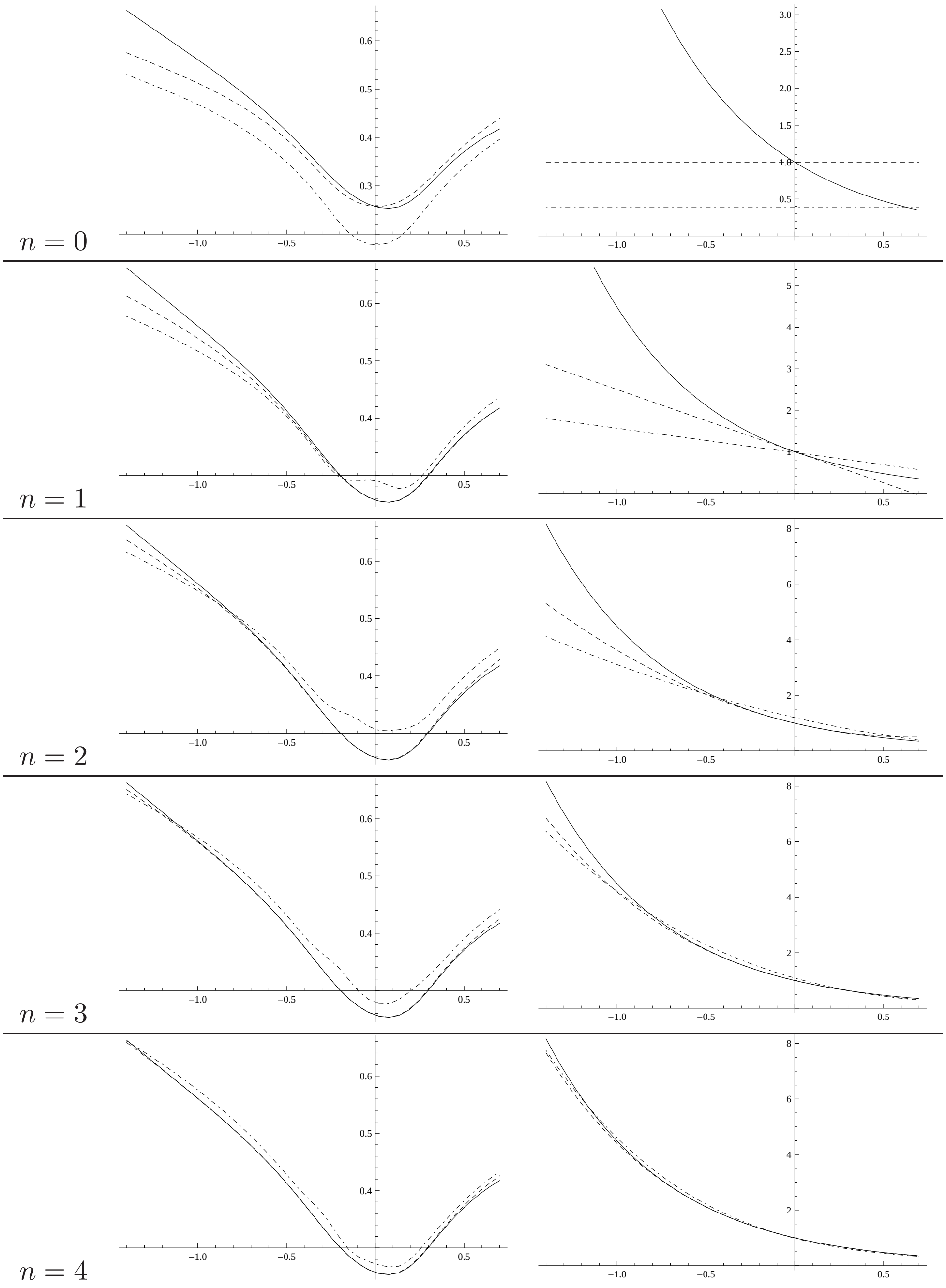

Figure 5.4: Left: for the model considered in Section 5.7.1 and for a fixed maturity $t = 0.5$, implied volatility is plotted as a function of log-strike. The dashed line corresponds to $\mathrm{IV}[u^{(n)}]$ where $u^{(n)}$ is computed using Taylor series basis functions (Example 4.1). The dot-dashed line corresponds to $\mathrm{IV}[u^{(n)}]$ where $u^{(n)}$ is computed using Hermite polynomial basis functions (Example 4.3). The solid line corresponds to $\mathrm{IV}[u^{(MC)}]$. Right: $f(x) = e^{2(\beta-1)x}$ (solid) and its n-th order Taylor series and Hermite polynomial approximations $f^{(n)}_{\text{Taylor}}(x)$ (dotted) and $f^{(n)}_{\text{Hermite}}(x)$ (dot-dashed); see equation (7.51).

$t = 0.25$ years

Parameters	$k = \log K$	$u^{(3)}$	u MC-95% c.i.	IV[$u^{(3)}$]	IV MC-95% c.i.	$\tau^{(3)}/\tau^{(0)}$
$\delta = 0.5432$	-0.6000	0.4552	0.4552 - 0.4553	0.6849	0.6836 - 0.6869	
$\beta = 0.3756$	-0.3500	0.3123	0.3122 - 0.3124	0.6230	0.6217 - 0.6242	
$\lambda = 0.0518$	-0.1000	0.1621	0.1618 - 0.1623	0.5704	0.5687 - 0.5714	4.9787
$m = -0.5013$	0.1500	0.0496	0.0492 - 0.0500	0.5240	0.5222 - 0.5266	
$\eta = 0.3839$	0.4000	0.0059	0.0057 - 0.0067	0.4821	0.4787 - 0.4950	
$\delta = 0.1182$	-0.6000	0.4566	0.4566 - 0.4567	0.7257	0.7239 - 0.7271	
$\beta = 0.9960$	-0.3500	0.3137	0.3136 - 0.3139	0.6391	0.6378 - 0.6405	
$\lambda = 0.8938$	-0.1000	0.1431	0.1429 - 0.1434	0.4615	0.4602 - 0.4630	4.77419
$m = -0.4486$	0.1500	0.0032	0.0030 - 0.0037	0.2013	0.1970 - 0.2073	
$\eta = 0.2619$	0.4000	0.0000	0.0000 - 0.0000	0.2510	0.2567 - 0.2616	
$\delta = 0.3376$	-0.6000	0.4621	0.4619 - 0.4621	0.8462	0.8439 - 0.8478	
$\beta = 0.4805$	-0.3500	0.3190	0.3189 - 0.3192	0.6949	0.6933 - 0.6968	
$\lambda = 0.9610$	-0.1000	0.1578	0.1575 - 0.1581	0.5457	0.5444 - 0.5476	4.31915
$m = -0.2420$	0.1500	0.0451	0.0448 - 0.0456	0.4990	0.4974 - 0.5021	
$\eta = 0.5391$	0.4000	0.0155	0.0152 - 0.0162	0.6006	0.5981 - 0.6080	
$\delta = 0.2469$	-0.6000	0.4592	0.4591 - 0.4593	0.7871	0.7857 - 0.7900	
$\beta = 0.1875$	-0.3500	0.3100	0.3099 - 0.3102	0.5965	0.5950 - 0.5986	
$\lambda = 0.4229$	-0.1000	0.1341	0.1338 - 0.1343	0.4083	0.4069 - 0.4096	4.46032
$m = -0.2823$	0.1500	0.0306	0.0302 - 0.0309	0.4149	0.4126 - 0.4168	
$\eta = 0.7564$	0.4000	0.0176	0.0171 - 0.0179	0.6213	0.6171 - 0.6244	

Table 5.2: After selecting model parameters randomly, we compute call prices (u) for the CEV-like model with Gaussian-type jumps discussed in Section 5.7.1. For each strike, the approximate call price $u^{(3)}$ is computed using the (usual) one-point Taylor expansion (see Example 4.1) as well as by Monte Carlo simulation. The obtained prices, as well as the associated implied volatilities (IV[u]) are displayed above. Note that, the approximate price $u^{(3)}$ (and corresponding implied volatility) consistently falls within the 95% confidence interval obtained from the Monte Carlo simulation. We denote by $\tau^{(n)}$ the total time it takes to compute the n-th order approximation of option prices $u^{(n)}$ at the five strikes displayed in the table. Because total computation time depends on processor speed, in the last column, we give the ratio $\tau^{(3)}/\tau^{(0)}$. Note that $\tau^{(0)}$ is a useful benchmark, as it corresponds to the total time it takes to compute the five call in an Exponential Lévy setting (i.e., option prices with no local dependence) using standard Fourier techniques.

$t = 1.00$ years

Parameters	$k = \log K$	$u^{(3)}$	u MC-95% c.i.	IV[$u^{(3)}$]	IV MC-95% c.i.	$\tau^{(3)}/\tau^{(0)}$
$\delta = 0.5806$	-1.0000	0.6487	0.6486 - 0.6488	0.7306	0.7294 - 0.7319	
$\beta = 0.5829$	-0.6000	0.5001	0.5000 - 0.5004	0.6719	0.6711 - 0.6734	
$\lambda = 0.0367$	-0.2000	0.3220	0.3216 - 0.3224	0.6167	0.6157 - 0.6182	4.97872
$m = -0.6622$	0.2000	0.1512	0.1507 - 0.1520	0.5649	0.5636 - 0.5671	
$\eta = 0.2984$	0.6000	0.0413	0.0408 - 0.0428	0.5166	0.5145 - 0.5219	
$\delta = 0.3921$	-1.0000	0.6556	0.6555 - 0.6561	0.8022	0.8014 - 0.8075	
$\beta = 0.1271$	-0.6000	0.5012	0.5011 - 0.5018	0.6779	0.6772 - 0.6809	
$\lambda = 0.4176$	-0.2000	0.3052	0.3051 - 0.3060	0.5655	0.5651 - 0.5678	4.54839
$m = -0.1661$	0.2000	0.1188	0.1184 - 0.1198	0.4832	0.4822 - 0.4858	
$\eta = 0.5823$	0.6000	0.0299	0.0296 - 0.0315	0.4708	0.4694 - 0.4772	
$\delta = 0.5803$	-1.0000	0.6679	0.6677 - 0.6681	0.9122	0.9108 - 0.9140	
$\beta = 0.2426$	-0.6000	0.5237	0.5236 - 0.5243	0.7916	0.7913 - 0.7943	
$\lambda = 0.5926$	-0.2000	0.3436	0.3431 - 0.3441	0.6830	0.6814 - 0.6845	4.3125
$m = -0.0877$	0.2000	0.1592	0.1581 - 0.1596	0.5851	0.5823 - 0.5862	
$\eta = 0.3236$	0.6000	0.0373	0.0358 - 0.0379	0.5009	0.4949 - 0.5033	
$\delta = 0.3096$	-1.0000	0.6323	0.6323 - 0.6324	0.36740	0.3680 - 0.3708	
$\beta = 0.6417$	-0.6000	0.4554	0.4553 - 0.4554	0.34493	0.3442 - 0.3456	
$\lambda = 0.3806$	-0.2000	0.2283	0.2281 - 0.2284	0.32159	0.3208 - 0.3221	4.9257
$m = -0.02824$	0.2000	0.0495	0.0491 - 0.0500	0.29930	0.2980 - 0.3006	
$\eta = 0.0122$	0.6000	0.0021	0.0015 - 0.0027	0.27807	0.2655 - 0.2888	

Table 5.3: After selecting model parameters randomly, we compute call prices (u) for the CEV-like model with Gaussian-type jumps discussed in Section 5.7.1. For each strike, the approximate call price $u^{(3)}$ is computed using the (usual) one-point Taylor expansion (see Example 4.1) as well as by Monte Carlo simulation. The obtained prices, as well as the associated implied volatilities (IV[u]) are displayed above. Note that, the approximate price $u^{(3)}$ (and corresponding implied volatility) consistently falls within the 95% confidence interval obtained from the Monte Carlo simulation. We denote by $\tau^{(n)}$ the total time it takes to compute the n-th order approximation of option prices $u^{(n)}$ at the five strikes displayed in the table. Because total computation time depends on processor speed, in the last column, we give the ratio $\tau^{(3)}/\tau^{(0)}$. Note that $\tau^{(0)}$ is a useful benchmark, as it corresponds to the total time it takes to compute the five call in an Exponential Lévy setting (i.e., option prices with no local dependence) using standard Fourier techniques.

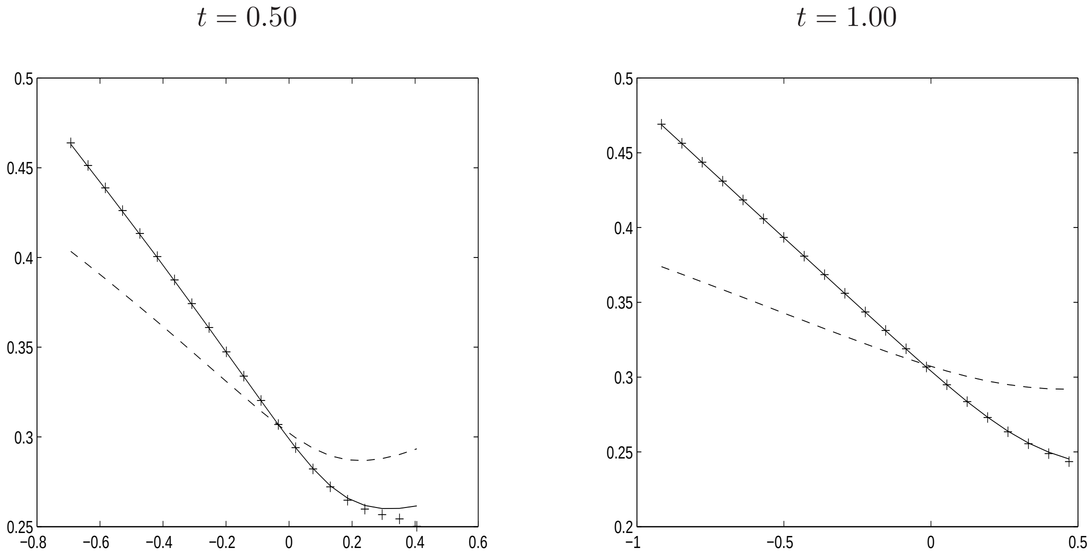

Figure 5.5: Implied volatility (IV) is plotted as a function of log-strike $k := \log K$ for the CEV-like model with Variance Gamma-type jumps of Section 5.7.1. The solid lines corresponds to the IV induced by $u^{(2)}(t,x)$, which is computed using the two-point Taylor expansion (see Example 4.2). The dashed lines corresponds to the IV induced by $u^{(0)}(t,x)$ (again, computed using the two-point Taylor series expansion). The crosses correspond to the IV induced by $u^{(MC)}(t,x)$, which is the price obtained from the Monte Carlo simulation.

t	k	$u^{(2)}$	u MC-95% c.i.	IV$[u^{(2)}]$	IV MC-95% c.i.
0.5000	-0.6931	0.0014	0.0014 - 0.0015	0.4631	0.4624 - 0.4652
	-0.4185	0.0070	0.0070 - 0.0071	0.4000	0.3995 - 0.4014
	-0.1438	0.0363	0.0362 - 0.0365	0.3336	0.3331 - 0.3346
	0.1308	0.1702	0.1697 - 0.1704	0.2727	0.2707 - 0.2736
	0.4055	0.5011	0.5004 - 0.5012	0.2615	0.2291 - 0.2646
1.0000	-0.9163	0.0028	0.0027 - 0.0028	0.4687	0.4678 - 0.4702
	-0.5697	0.0109	0.0109 - 0.0110	0.4057	0.4050 - 0.4068
	-0.2231	0.0473	0.0472 - 0.0476	0.3434	0.3428 - 0.3444
	0.1234	0.1970	0.1965 - 0.1974	0.2836	0.2825 - 0.2847
	0.4700	0.6033	0.6025 - 0.6037	0.2452	0.2355 - 0.2506

Table 5.4: Prices (u), Implied volatilities (IV$[u]$) and the corresponding confidence intervals from Figure 5.5.

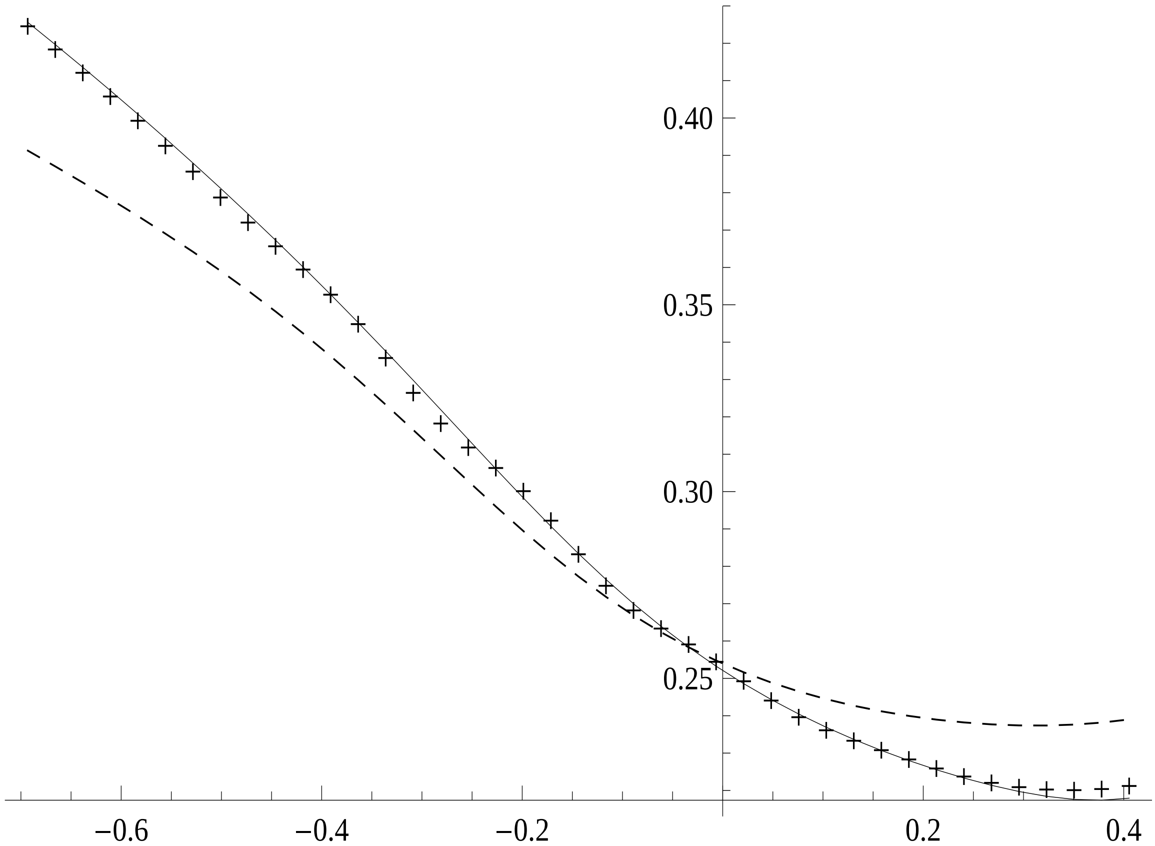

Figure 5.6: Implied volatility (IV) is plotted as a function of log-strike $k := \log K$ for the model of Section 5.7.2. The dashed line corresponds to the IV induced by $u^{(0)}(t, x)$. The solid line corresponds to the IV induced by $u^{(2)}(t, x)$. To compute $u^{(i)}(t, x)$, $i \in \{0, 2\}$, we use the two-point Taylor series expansion of Example 4.2 The crosses correspond to the IV induced by the exact price, which is computed by truncating (7.53) at $n = 8$.

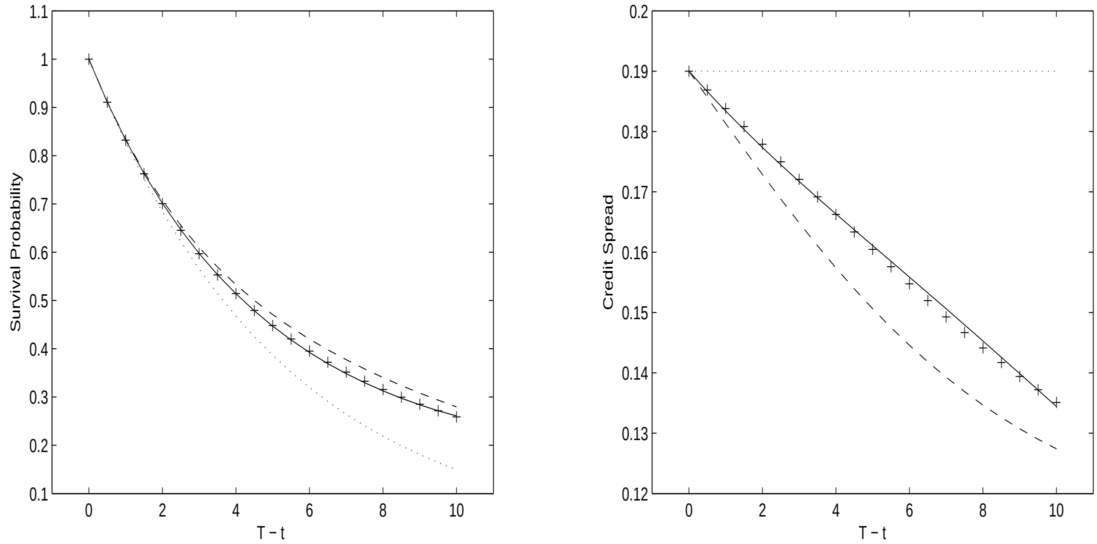

Figure 5.7: Left: survival probabilities $u(T-t,x) := \mathbb{Q}_x[\zeta > T|\zeta > t]$ for the JDCEV model described in Section 5.7.4. The dotted line, dashed line and solid line correspond to the approximations $u^{(0)}(T-t,x)$, $u^{(1)}(T-t,x)$ and $u^{(2)}(T-t,x)$ respectively, all of which are computed using Corollary 4.6. The crosses indicate the exact survival probability, computed by truncating equation (7.57) at $n = 70$. Right: the corresponding yields $Y^{(n)}(t,x;T) := -\log(u^{(n)}(T-t,x))/(T-t)$ on a defaultable bond. The parameters used in the plot are as follows: $x = \log(1)$, $\beta = -1/3$, $b = 0.01$, $c = 2$ and $a = 0.3$.

$T-t$	Y	$Y - Y^{(0)}$	$Y - Y^{(1)}$	$Y - Y^{(2)}$
1.0	0.1835	-0.0065	0.0022	0.0001
2.0	0.1777	-0.0123	0.0048	0.0003
3.0	0.1720	-0.0180	0.0071	0.0003
4.0	0.1663	-0.0237	0.0089	-0.0001
5.0	0.1605	-0.0295	0.0099	-0.0006
6.0	0.1548	-0.0352	0.0102	-0.0011
7.0	0.1493	-0.0407	0.0101	-0.0013
8.0	0.1442	-0.0458	0.0095	-0.0011
9.0	0.1394	-0.0506	0.0087	-0.0005
10.0	0.1351	-0.0549	0.0077	0.0007

Table 5.5: The yields $Y(t,x;T)$ on the defaultable bond described in Section 5.7.4: exact (Y) and nth order approximation ($Y^{(n)}$). We use the following parameters: $x = \log(1)$, $\beta = -1/3$, $b = 0.01$, $c = 2$ and $\delta = 0.3$.

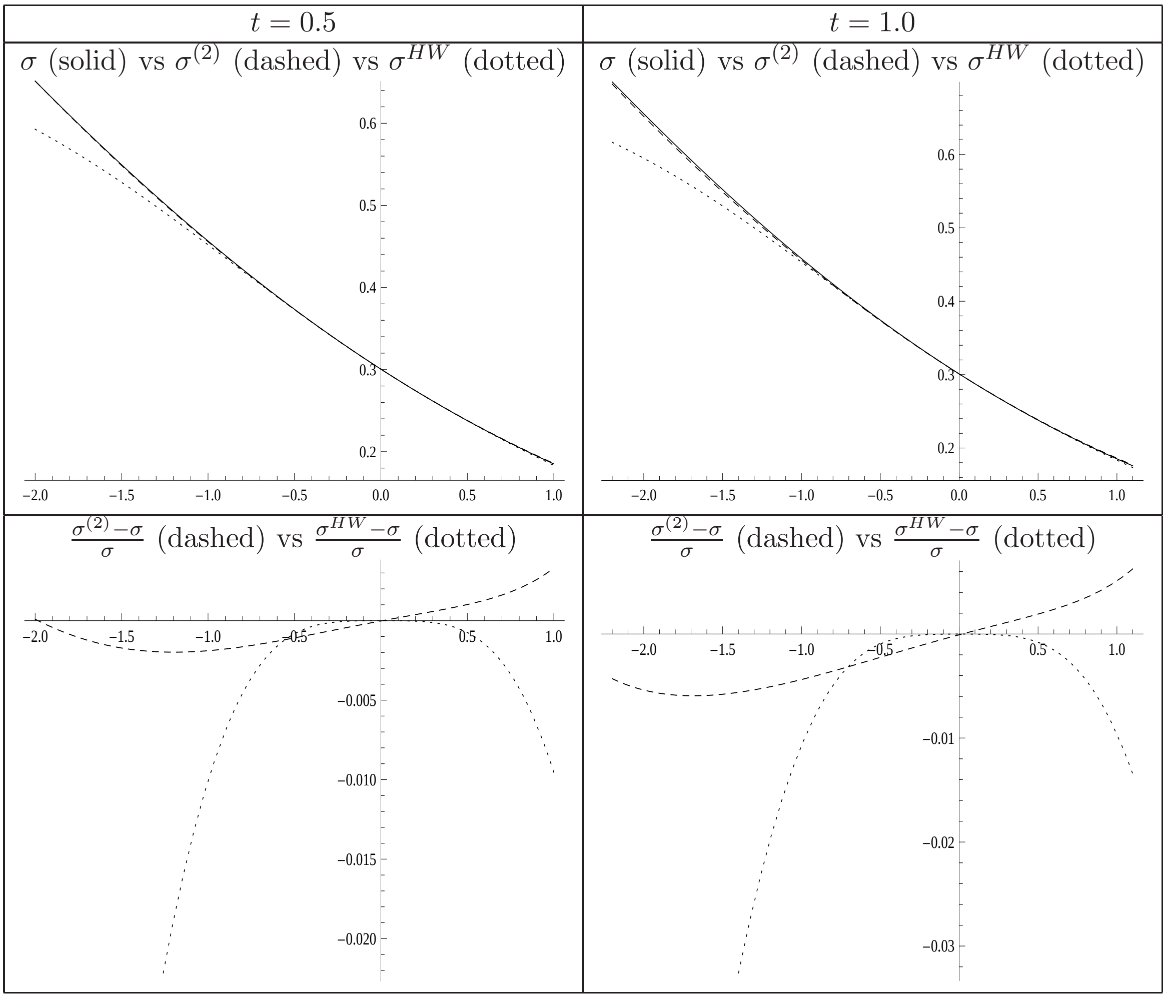

Figure 5.8: Top: implied volatility for the CEV model for maturities $t = 0.5$ and $t = 1.0$. The solid line is the exact implied volatility σ, computed by truncating Q in equation (7.58) at $n = 100$ terms and then inverting Black-Scholes numerically. The dashed line is the 2nd order implied volatility expansion $\sigma^{(2)}$, computed using equation (6.45). The dotted line is the Hagan-Woodward expansion of implied volatility σ^{HW}, computed using equation (7.59). Bottom: relative errors of the 2nd order (dashed) and Hagan-Woodward (dotted) implied volatility expansions. In all plots we use the following parameters: $\beta = 0.1$, $\delta = 0.3$, $X_0 = \log(1.0)$. Units of the horizontal axis are log-strike: $\log K$.

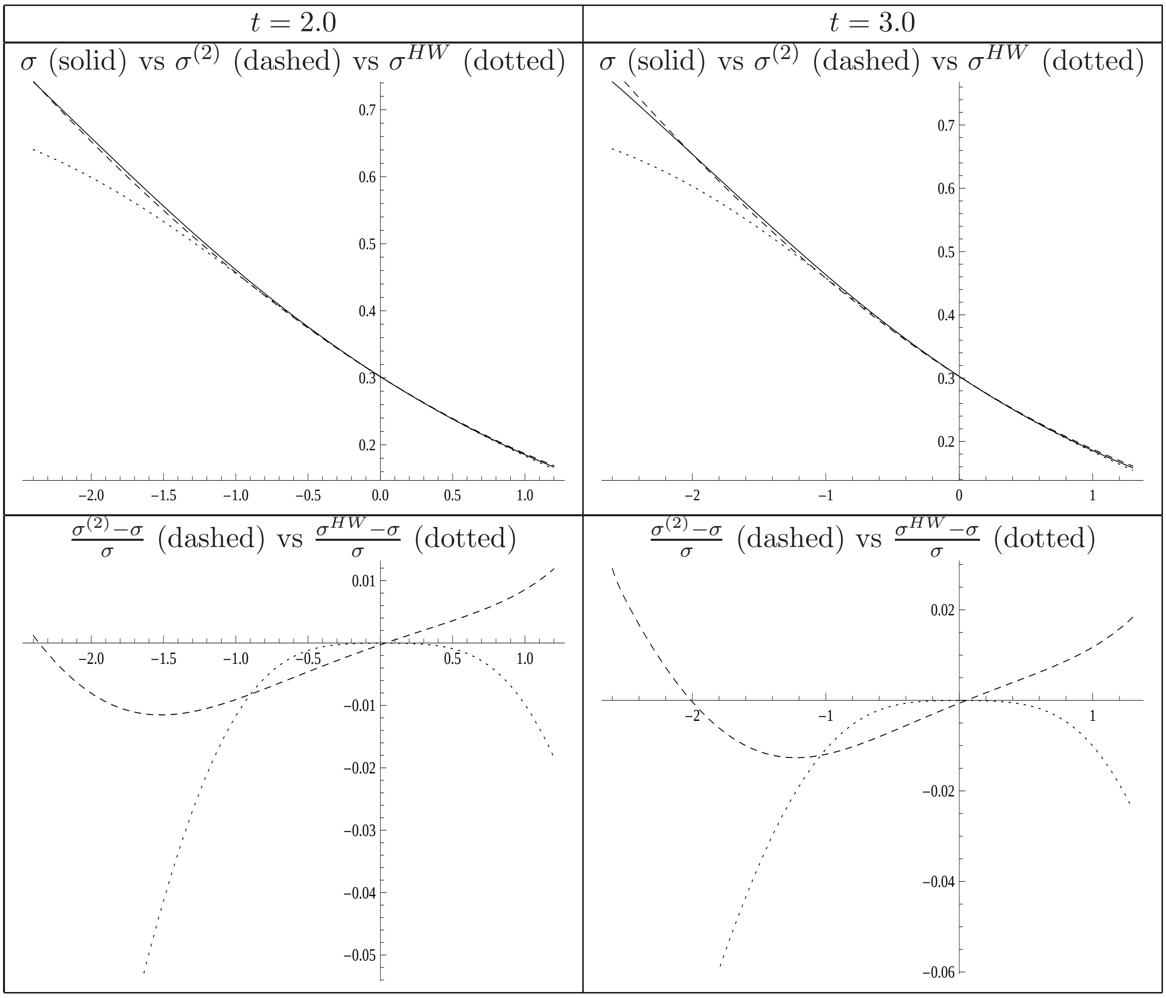

Figure 5.9: Top: implied volatility for the CEV model for maturities $t = 2.0$ and $t = 3.0$. The solid line is the exact implied volatility σ, computed by truncating Q in equation (7.58) at $n = 100$ terms and then inverting Black-Scholes numerically. The dashed line is the 2nd order implied volatility expansion $\sigma^{(2)}$, computed using equation (6.45). The dotted line is the Hagan-Woodward expansion of implied volatility σ^{HW}, computed using equation (7.59). Bottom: relative errors of the 2nd order (dashed) and Hagan-Woodward (dotted) implied volatility expansions. In all plots we use the following parameters: $\beta = 0.1$, $\delta = 0.3$, $X_0 = \log(1.0)$. Units of the horizontal axis are log-strike: $\log K$.

Chapter 6

Implied volatility for any local-stochastic volatility model

Based on a joint work ([169]) with Dr. Matthew Lorig and Prof. A. Pascucci.

Abstract: we consider an asset whose risk-neutral dynamics are described by a general local-stochastic volatility model. In this setting, we derive a family of asymptotic expansions for the transition density of the underlying as well as for European-style option prices and for implied volatilities. Our expansions are numerically efficient. Approximate transition densities and implied volatilities are explicit; they do not require any special functions nor do they require numerical integration. Approximate option prices require only a Normal CDF (as is the case of the Black-Scholes setting). Additionally, we establish rigorous error bounds for our transition density expansion. To illustrate the accuracy and versatility of our implied volatility expansion, we implement this expansion under five different model dynamics: CEV local volatility, quadratic local volatility, Heston stochastic volatility, 3/2 stochastic volatility, and SABR local-stochastic volatility. Our implied volatility expansion is found to perform favorably compared to other well-known expansions for these models.

Keywords: implied volatility, local-stochastic volatility, CEV, Heston, SABR.

6.1 Introduction

Neither local volatility (LV) nor stochastic volatility (SV) models are able to fit empirically observed implied volatility levels over the full range of strikes an maturities. This has led to the development of local-stochastic volatility (LSV) models, which combine the features of LV and SV models by describing the instantaneous volatility of an underlying S by a function $f(S_t, Z_t)$ where Z is some auxiliary, possibly multidimensional, stochastic process (see, for instance, [159], [2], [87], [120] and [58]). Compared to their LV and SV counterparts, LSV models produce implied volatility surfaces that more closely match those observed in the market. However, LSV models rarely allow for exact formulas for option prices. Thus, LSV models present two challenges. First, given an LSV model, can one find accurate closed-form approximations for option prices? Second, given approximate option prices, can one find accurate closed-form approximations for implied volatilities?

In the area of pricing, there have been a number or recent developments. An exhaustive review of LSV pricing approximations would be prohibitive. To cite one, in [164] the author adds multiscale stochastic volatility to general scalar diffusions, and thus obtains analytically tractable eigenfunction approximations for options prices. Note also that in Chapter 4 (see also [184]) we present a local volatility-enhanced Heston model, for which we provide a Fourier-like representation for approximate option prices.

Typically, unobservable LSV (or SV or LV) model parameters are obtained by calibrating these models to implied volatilities that are observed on the market. To do this, one must find model-induced implied volatilities over a range of strikes and maturities. Computing model-induced implied volatilities from option prices by inverting the Black-Scholes formula numerically is a computationally intensive task, and therefore, not suitable for the purposes of calibration. For this reason closed-form approximations for model-induced implied volatilities are needed. A number of different approach have been taken for computing approximate implied volatilities in LV, SV and LSV models. We review some of these approaches below. Concerning LV models, perhaps the earliest and most well-known implied volatility result is due to Hagan et al (see [115]), who use singular perturbation methods to obtain an implied volatility expansion for general LV models. For certain models (e.g., CEV) they obtain closed-form approximations. More recently, in [166] the author uses regular perturbation methods to obtain an implied volatility expansion when a LV model can be written as a regular perturbation around Black-Scholes. In [130] they extend and refine the results in [166] to find closed-form approximations of implied volatility for local Lévy-type models with jumps. In [106] the small-time asymptotics of implied volatility for LV models are considered by using heat kernel methods. There is no shortage of implied volatility results for SV models either. In [98] (see also [99]), the authors derive an asymptotic expansion for general multiscale stochastic volatility models using combined singular and regular perturbation theory. In [93] they use the Freidlin-Wentzell theory of large deviations for SDEs to obtain the small-time behavior of implied volatility for general stochastic volatility models with zero correlation. Their work adds mathematical rigor to previous work in [154].Large deviation techniques are used in [92] to obtain the small-time behavior of implied volatility in the Heston model (with correlation). They further refine these results in [94]. Concerning LSV models, perhaps the most well-known implied volatility result is due to Hagan et al (see [114]), who use WKB approximation methods to obtain implied volatility asymptotics in a LSV model with a CEV-like factor of local volatility and a GBM-like factor of non-local volatility (i.e., the SABR model). More recently, in

[117] the author uses a heat kernel expansion on a Riemann manifold to derive first order asymptotics for implied volatility for any LSV model. As an example, he introduces the λ-SABR model, which is a LSV model with a mean reverting non-local factor of volatility, and obtains closed form asymptotic formulas for implied volatility in this setting (see also [119]). There are also some model-free results concerning the extreme-strike behavior of implied volatility. Most notably, we mention the work in [150] and [103].

In this chapter, we consider general LSV models. For these models, we derive a family of closed-form asymptotic expansions for transition densities, option prices and implied volatilities. Our method extends the one presented in Chapter 5, which is itself and extension of the technique presented in Chapter 3. The major contributions of this chapter are as follows:

- As we have done in Chapter 5, in order to achieve our approximation result, we expand the diffusion coefficients of a multi-dimensional diffusion in an arbitrary basis, i.e. $f(x,y) = \sum_n \sum_h c_{n,h} B_{n,h}(x,y)$. Thus, we not only extend the results in Chapter 3 (see also [183] and [185]) from one to multiple dimensions, but we also consider more general expansions.

- We provide an explicit formula for the nth term in our transition density and option-price expansions. The terms in the density expansion appear as Hermite polynomials multiplied by Gaussian kernels and thus, can be computed extremely quickly. Note that in Chapter 5 (see also [171]) the nth term of the transition density is given as a Fourier transform, which is computationally more intensive.

- We provide closed-form approximations for implied volatility in a general local-stochastic volatility setting. We show (through a series of numerical experiments) that our implied volatility approximation performs favorably when compared to other well-known implied volatility approximations (e.g., [115] for CEV, [94] for Heston, and [114] for SABR).

- Many of the above-mentioned implied volatility approximations rely on some special structure for the underlying diffusion (e.g., fast- or slow-varying volatility, or some particular Riemannian geometry which allows for closed-form computation of geodesics). When these structures are absent, the associated implied volatility expansions will not work. By contrast, our implied volatility approximation works for any LSV model (actually, by the Adjoint Expansion method, jumps can be added as well). Thus, in addition to being highly accurate, our approach is quite general and includes several models of great interest for the financial industry. For instance, to the best of our knowledge, we give the first approximation formula for implied volatilities in the 3/2 stochastic volatility model. Of late, the 3/2 model has attracted much interest due to its ability match market prices for both European-style options as well as variance and volatility derivatives (see [16]).

- We provide a general result showing how to pass in a model-free way from a price expansion to an implied volatility expansion.

The rest of this chapter proceeds as follows: In Section 6.2 we present the general class of local-stochastic volatility models. We also list some technical model assumptions. Next, in Section 6.3 we derive the option-pricing PDE. In Section 6.4 we derive a formal asymptotic expansion (in fact, a family of asymptotic expansions) for the function that solves the option-pricing PDE. The main result of this Section is Theorem 4.8, which shows that every term in our price (density) expansion can be written as a differential operator acting on a Black-Scholes price (Gaussian density). We also establish error bounds for our asymptotic price and density expansions in Section 6.4. In Section 6.5 we derive our implied volatility results. We do this in two steps. First, in Section 6.5.1 we show how one can pass in a model-free way from a price expansion to an implied volatility expansion. Next, in Section 6.5.2 we show that, when the price expansion is as given in Theorem 4.8, the implied volatility expansion is explicit. That is, the expansion does not require any integration or special functions. In Section 6.6 we implement our implied volatility expansion under four different model dynamics: quadratic local volatility, Heston stochastic volatility, 3/2 stochastic volatility, and SABR local-stochastic volatility. Section 6.7 reviews our results and suggests directions for future research. The explicit representation for the implied volatility expansion, as well as long proofs are given in the Appendix.

6.2 General local-stochastic volatility models

For simplicity, we assume a frictionless market, no arbitrage, zero interest rates and no dividends. We take, as given, an equivalent martingale measure $\mathbb{Q}$, chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{Q})$. The filtration $\{\mathcal{F}_t, t \geq 0\}$ represents the history of the market. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{Q}$. We consider a strictly positive asset S whose risk-neutral dynamics are given by

$$\begin{cases} S_t = \exp(X_t), & \\ \mathrm{d}X_t = -\frac{1}{2}\sigma^2(X_t, Y_t)\mathrm{d}t + \sigma(X_t, Y_t)\mathrm{d}W_t, & X_0 = x \in \mathbb{R}, \\ \mathrm{d}Y_t = \alpha(X_t, Y_t)\mathrm{d}t + \beta(X_t, Y_t)\mathrm{d}B_t, & Y_0 = y \in \mathbb{R}, \\ \mathrm{d}\langle W, B\rangle_t = \rho(X_t, Y_t)\,\mathrm{d}t, & |\rho| < 1. \end{cases} \tag{2.1}$$

We assume that SDE (2.1) has a unique strong solution, that σ and β are strictly positive functions and that σ, β, ρ and α are smooth. Sufficient conditions for the existence of a unique strong solution can be found, for example, in [127]. We also assume that the coefficients are such that $\mathbb{E}\, S_t < \infty$ for all $t \in [0, \infty)$. The class of models described by (2.1) enjoys the following features:

- **Local-stochastic volatility**. The diffusion coefficient of X depends both locally on X and non-locally on an auxiliary driving process Y through the function $\sigma(x, y)$.

- **Martingale property**. The drift $-\frac{1}{2}\sigma^2(X, Y)$ of X is chosen so as to ensure that $S = e^X$ is a martingale (as it must be to rule out arbitrage).

- **Arbitrary Y dynamics**. Both the drift $\alpha(X, Y)$ and diffusion coefficient $\beta(X, Y)$ of the auxiliary driving process Y are allowed to depend on both X and Y.

- **Arbitrary correlation**. The correlation $\rho(X, Y)$ between the Brownian motions W and B is allowed to depend on both X and Y.

Equation (2.1) includes virtually all one-factor stochastic volatility models, all local stochastic volatility models, and all one-factor local-stochastic volatility models.

Remark 2.1 (Multi-factor local-stochastic volatility models and time-dependent coefficients)**.** *The results of this paper can be extended in a straightforward fashion to include models with n non-local factors of volatility and time-dependent drift and diffusion coefficients:*

$$\begin{aligned} S_t &= \exp(X_t), \\ \mathrm{d}X_t &= -\frac{1}{2}\sigma^2(t, X_t, \mathbf{Y}_t)\mathrm{d}t + \sigma(t, X_t, \mathbf{Y}_t)\mathrm{d}W_t, && X_0 = x \in \mathbb{R}, \\ \mathrm{d}Y_t^{(i)} &= \alpha^{(i)}(t, X_t, \mathbf{Y}_t)\mathrm{d}t + \sum_{i=1}^{n} \beta^{(i,j)}(t, X_t, \mathbf{Y}_t)\mathrm{d}B_t^{(j)}, && \mathbf{Y}_0 = \mathbf{y} \in \mathbb{R}^n, \\ \mathrm{d}\langle W, B^{(i)}\rangle_t &= \rho^{(i)}(t, X_t, \mathbf{Y}_t)\,\mathrm{d}t, && |\rho^{(i)}| < 1. \end{aligned}$$

Though, for simplicity, we restrict our analysis to the case of time-homogenous coefficients and $n = 1$.

6.3 Transition density and option pricing PDE

Let V_t be the time t value of a European derivative, expiring at time $T > t$ with payoff $H(X_T, Y_T)$. Using risk-neutral pricing, the value V_t of the derivative at time t is given by the conditional expectation of the option payoff

$$V_t = \mathbb{E}[H(X_T, Y_T)|\mathcal{F}_t] = \mathbb{E}[H(X_T, Y_T)|X_t, Y_t].$$

Note that we have used the Markov property of the process (X, Y) to replace the filtration $\mathcal{F}_t$ by the sigma-algebra generated by (X_t, Y_t). Thus, to value a European-style option we must compute functions of the form

$$v(t, x, y) := \mathbb{E}[H(X_T, Y_T)|X_t = x, Y_t = y] = \int_{\mathbb{R}^2} \mathrm{d}w\,\mathrm{d}z\, p(t, x, y; T, w, z)H(w, z). \quad (3.2)$$

Here, $p(t, x, y; T, w, z)$ is the transition density of the process (X, Y). Note that, by setting $H = \delta_{w,z}$ (the Dirac mass at (w, z)) the function $v(t, x, y)$ becomes the transition density $p(t, x, y; T, w, z)$ since

$$\int_{\mathbb{R}^2} \mathrm{d}w'\mathrm{d}z'\, p(t, x, y; T, w', z')\delta_{w,z}(w', z') = p(t, x, y; T, w, z).$$

If the function v, defined by (3.2), is $C^{1,2}([0, T), \mathbb{R}^2)$, then v satisfies the Kolmogorov Backward equation

$$(\partial_t + \mathcal{A})v = 0, \qquad\qquad v(T, x, y) = H(x, y),$$

where the operator $\mathcal{A}$ is the infinitesimal generator of the process (X, Y), given explicitly by

$$\mathcal{A} = a(x, y)(\partial_x^2 - \partial_x) + \alpha(x, y)\partial_y + b(x, y)\partial_y^2 + c(x, y)\partial_x\partial_y, \quad (3.3)$$

and where the functions a, b and c are defined as

$$a(x,y) := \frac{1}{2}\sigma^2(x,y), \qquad b(x,y) := \frac{1}{2}\beta^2(x,y), \qquad c(x,y) := \rho(x,y)\sigma(x,y)\beta(x,y).$$

At this stage, it is convenient to define

$$t(s) := T - s, \qquad u(t(s),x,y) := v(s,x,y).$$

Then, a simple application of the chain rule shows

$$(-\partial_t + \mathcal{A})u = 0, \qquad u(0,x,y) = H(x,y). \tag{3.4}$$

In what follows, it will be convenient to characterize the differential operator $\mathcal{A}$ by its action on <u>oscillating exponential functions</u> $\psi_{\lambda,\omega}(x,y) := \frac{1}{2\pi}e^{i\lambda x + i\omega y}$. Indeed, observe that

$$\mathcal{A}\psi_{\lambda,\omega}(x,y) = \phi(x,y,\lambda,\omega)\psi_{\lambda,\omega}(x,y), \qquad \psi_{\lambda,\omega}(x,y) := \frac{1}{2\pi}e^{i\lambda x + i\omega y}$$

where $\phi(x,y,\lambda,\omega)$, referred to as the <u>symbol</u> of $\mathcal{A}$, is given by

$$\phi(x,y,\lambda,\omega) = a(x,y)(-\lambda^2 - i\lambda) + \alpha(x,y)i\omega - b(x,y)\omega^2 - c(x,y)\lambda\omega.$$

The symbol of $\mathcal{A}$ appears naturally in connection with the Fourier transform as follows. For any $f \in \mathcal{S}(\mathbb{R}^2)$, the Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^2$, we define

$$\text{Fourier Transform}: \qquad [\mathcal{F} f](\lambda,\omega) = \widehat{f}(\lambda,\omega) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\, e^{-i\lambda x - i\omega y} f(x,y),$$

$$\text{Inverse Transform}: \qquad [\mathcal{F}^{-1}\widehat{f}](x,y) = f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda\mathrm{d}\omega\, e^{i\lambda x + i\omega y}\widehat{f}(\lambda,\omega).$$

Note that

$$\mathcal{A}f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda\mathrm{d}\omega\, \phi(x,y,\lambda,\omega)\, e^{i\lambda x + i\omega y}\widehat{f}(\lambda,\omega). \tag{3.5}$$

6.4 Density and option price expansions

Our goal is to construct an approximate solution of Cauchy problem (3.4). Extending the approach of [183] and [171] for scalar Markov processes to the present multi-dimensional setting, we assume that the symbol of $\mathcal{A}$ admits an expansion of the form

$$\phi(x,y,\lambda,\omega) = \sum_{n=0}^{\infty}\sum_{h=0}^{n} B_{n-h,h}(x,y)\phi_{n-h,h}(\lambda,\omega), \tag{4.6}$$

where $(B_{i,j})$ is a sequence of analytic <u>basis functions</u> satisfying $B_{0,0} = 1$ and where each $\phi_{i,j}(\lambda,\omega)$ is of the form

$$\phi_{i,j}(\lambda,\omega) = a_{i,j}(-\lambda^2 - i\lambda) + \alpha_{i,j}\, i\omega - b_{i,j}\,\omega^2 - c_{i,j}\,\lambda\,\omega.$$

Observe that each $\phi_{i,j}(\lambda, \omega)$ is the symbol of a differential operator $\mathcal{A}_{i,j}$ where

$$\mathcal{A}_{i,j} := \phi_{i,j}(\mathcal{D}_x, \mathcal{D}_y), \qquad \mathcal{D}_x := -i\partial_x, \qquad \mathcal{D}_y := -i\partial_y,$$

which is the infinitesimal generator of a constant coefficient diffusion in $\mathbb{R}^2$. Noting that

$$\mathcal{A}_{i,j}\psi_{\lambda,\omega}(x, y) = \phi_{i,j}(\lambda, \omega)\psi_{\lambda,\omega}(x, y),$$

we see that, formally, the generator $\mathcal{A}$ can be written as follows

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n, \qquad \mathcal{A}_n = \sum_{h=0}^{n} B_{n-h,h}(x, y)\phi_{n-h,h}(\mathcal{D}_x, \mathcal{D}_y). \tag{4.7}$$

Note that, as the basis functions $(B_{i,j})$ are assumed to be analytic, they can be seen as symbols of the differential operators $B_{i,j}(-i\partial_\lambda, -i\partial_\omega)$. Indeed, we have

$$B_{i,j}(-i\partial_\lambda, -i\partial_\omega)\psi_{x,y}(\lambda, \omega) = B_{i,j}(x, y)\psi_{x,y}(\lambda, \omega).$$

Remark 4.1. *More generally, one could consider a decomposition of ϕ as follows*

$$\phi(x, y, \lambda, \omega) = \sum_{n=0}^{\infty} \sum_{h=0}^{n} \sum_{i+j=1}^{2} B^{i,j}_{n-h,h}(x, y)a^{i,j}_{n-h,h}(i\lambda)^i(i\omega)^j.$$

However, because this generalization brings with it a significant notational cost (i.e., it introduces two new indices, which one must keep track of), we restrict our analysis to the case where ϕ is given by (4.6)*.*

Below, we illustrate a few useful choices for basis functions.

Example 4.2 (Taylor series)**.** *In [185], the authors expand the drift and diffusion coefficients of a scalar diffusion as a power series about an arbitrary point. Extending this idea to the multiple dimensions, we fix a point $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ and we expand*

$$\begin{aligned}
\alpha(x, y) &= \sum_{n=0}^{\infty} \sum_{h=0}^{n} \alpha_{n-h,h}(x - \bar{x})^{n-h}(y - \bar{y})^h, & \alpha_{n-h,h} &:= \frac{1}{(n-h)!\,h!}\partial_x^{n-h}\partial_y^h \alpha(\bar{x}, \bar{y}), \\
a(x, y) &= \sum_{n=0}^{\infty} \sum_{h=0}^{n} a_{n-h,h}(x - \bar{x})^{n-h}(y - \bar{y})^h, & a_{n-h,h} &:= \frac{1}{(n-h)!\,h!}\partial_x^{n-h}\partial_y^h a(\bar{x}, \bar{y}), \\
b(x, y) &= \sum_{n=0}^{\infty} \sum_{h=0}^{n} b_{n-h,h}(x - \bar{x})^{n-h}(y - \bar{y})^h, & b_{n-h,h} &:= \frac{1}{(n-h)!\,h!}\partial_x^{n-h}\partial_y^h b(\bar{x}, \bar{y}), \\
c(x, y) &= \sum_{n=0}^{\infty} \sum_{h=0}^{n} c_{n-h,h}(x - \bar{x})^{n-h}(y - \bar{y})^h, & c_{n-h,h} &:= \frac{1}{(n-h)!\,h!}\partial_x^{n-h}\partial_y^h c(\bar{x}, \bar{y}).
\end{aligned} \tag{4.8}$$

Setting $B_{n-h,h}(x, y) = (x-\bar{x})^{n-h}(y-\bar{y})^h$ we observe that (4.6) *and* (4.7) *become, respectively*

$$\phi(x, y, \lambda, \omega) = \sum_{n=0}^{\infty} \sum_{h=0}^{n} (x - \bar{x})^{n-h}(y - \bar{y})^h \phi_{n-h,h}(\lambda, \omega),$$

$$\mathcal{A} = \sum_{n=0}^{\infty} \sum_{h=0}^{n} (x - \bar{x})^{n-h}(y - \bar{y})^h \phi_{n-h,h}(\mathcal{D}_x, \mathcal{D}_y).$$

where the coefficients $\alpha_{n-h,h}$, $a_{n-h,h}$, $b_{n-h,h}$ and $c_{n-h,h}$ of $\phi_{n-h,h}$ are given in (4.8)*.*

Example 4.3 (Two-Point Taylor Series)**.** *Consider a local volatility model*

$$\mathrm{d}X_t = -\frac{1}{2}\sigma^2(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

with generator $\mathcal{A}$ *and symbol* ϕ *given by*

$$\mathcal{A} = a(x)(\partial_x^2 - \partial_x), \qquad \phi(x,\lambda) = a(x)(-\lambda^2 - i\lambda), \qquad a(x) := \frac{1}{2}\sigma^2(x). \quad (4.9)$$

For fixed $\bar{x}_0, \bar{x}_1, \bar{x}_2 \in \mathbb{R}$, *the function* a *can be expanded as a* <u>*two-point Taylor series*</u> *as follows*

$$a(x) = a(\bar{x}_0) + \sum_{n=0}^{\infty} \left(a_n(\bar{x}_0, \bar{x}_1, \bar{x}_2)(x - \bar{x}_1) + a_n(\bar{x}_0, \bar{x}_2, \bar{x}_1)(x - \bar{x}_2)\right)(x - \bar{x}_1)^n (x - \bar{x}_2)^n, \quad (4.10)$$

where

$$a_0(\bar{x}_0, \bar{x}_1, \bar{x}_2) = \frac{a(\bar{x}_2) - a(\bar{x}_0)}{\bar{x}_2 - \bar{x}_1},$$

$$a_n(\bar{x}_0, \bar{x}_1, \bar{x}_2) = \sum_{h=0}^{n} \frac{(h+n-1)!}{h!n!(n-h)!} \frac{(-1)^h h \partial_{\bar{x}_1}^{n-h}[a(\bar{x}_1) - a(\bar{x}_0)] + (-1)^{n+1} n \partial_{\bar{x}_2}^{n-h}[a(\bar{x}_2) - a(\bar{x}_0)]}{(\bar{x}_1 - \bar{x}_2)^{h+n+1}}.$$

For the derivation of this result we refer the reader to [86, 162]. Note that truncating the two-point Taylor series expansion (4.10) *at* $n = m$ *results in an expansion for* a *which is of order* $\mathcal{O}(x^{2m+1})$*. The advantage of using a two-point Taylor series is that, by considering the first* n *derivatives of a function* a *at two points* $\bar{x}_1$ *and* $\bar{x}_2$*, one can achieve a more accurate approximation of* a *over a wider range of values than if one were to approximate* a *using* $2n$ *derivatives at a single point (i.e., the usual Taylor series approximation).*

Using (4.9) *and* (4.10)*, we can formally express the symbol* ϕ *as*

$$\phi(x,\lambda) = \sum_{n=0}^{\infty} B_n(x)\phi_n(\lambda),$$

where $B_0(x) = 1$, $\phi_0 = -a(\bar{x}_0)(\lambda^2 + i\lambda)$ *and*

$$B_n(x) = \left(a_{n-1}(\bar{x}_0, \bar{x}_1, \bar{x}_2)(x - \bar{x}_1) + a_{n-1}(\bar{x}_0, \bar{x}_2, \bar{x}_1)(x - \bar{x}_2)\right)(x - \bar{x}_1)^{n-1}(x - \bar{x}_2)^{n-1}, \quad n \geq 1,$$

$$\phi_n(\lambda) = -(\lambda^2 + i\lambda), \qquad n \geq 1.$$

Example 4.4 (Non-local approximation in a weighted L^2-space)**.** *Let* $(B_{i,j})$ *be an orthonormal basis in the weighted space* $L^2(\mathbb{R}^2, \mathfrak{m}(x,y)\mathrm{d}x\mathrm{d}y)$*. Then* $\phi_{i,j}(\lambda,\omega)$ *is given by*

$$\phi_{i,j}(\lambda,\omega) = \langle \phi(\cdot,\cdot,\lambda,\omega), B_{i,j}(\cdot,\cdot)\rangle_{\mathfrak{m}}.$$

For instance, one could choose the <u>*Hermite polynomials*</u> $\mathbf{H}_n$ *centered at* $(\bar{x}, \bar{y})$ *as basis functions*

$$B_{n,h}(x,y) = \frac{\mathbf{H}_n(x - \bar{x})}{\sqrt{(2n)!!\sqrt{\pi}}} \frac{\mathbf{H}_h(y - \bar{y})}{\sqrt{(2h)!!\sqrt{\pi}}}, \qquad \mathbf{H}_n(x) := (-1)^n \frac{\partial_x^n \exp(-x^2)}{\exp(-x^2)}. \quad (4.11)$$

Such basis functions are orthonormal under a Gaussian weighting

$$\langle B_{i,j}, B_{h,l} \rangle_{\mathfrak{m}} := \delta_{i,j}\delta_{h,l}, \qquad \mathfrak{m}(x,y) = \exp\left(-(x-\bar{x})^2 - (y-\bar{y})^2\right).$$

Having discussed some useful basis functions, we now return to Cauchy problem (3.4). We re-write the operator $\mathcal{A}$ in (4.7) as

$$\mathcal{A} = \sum_{n=0}^{\infty} \varepsilon^n \mathcal{A}_n, \qquad \varepsilon = 1, \tag{4.12}$$

where we have introduced ε, which serves merely as an accounting feature. Next, we suppose that the solution u can be written as a sum of the form

$$u = u^{\varepsilon} := \sum_{n=0}^{\infty} \varepsilon^n u_n, \qquad \varepsilon = 1. \tag{4.13}$$

We insert expansions (4.12) and (4.13) into PDE (3.4) and collect terms of like order in ε. We find

$$\mathcal{O}(1): \qquad (-\partial_t + \mathcal{A}_0)u_0 = 0, \qquad u_0(0,x,y) = H(x,y), \tag{4.14}$$

$$\mathcal{O}(\varepsilon^n): \qquad (-\partial_t + \mathcal{A}_0)u_n = -\sum_{h=1}^{n} \mathcal{A}_h u_{n-h}, \qquad u_n(0,x,y) = 0. \tag{4.15}$$

Having served its purpose, we set ε to the side. Our goal is to solve Cauchy problems (4.14) and (4.15). Observe that u_0, the solution of (4.14) is well-known

$$u_0(t,x,y) = \int_{\mathbb{R}^2} \mathrm{d}w\mathrm{d}z\, p_0(0,x,y;t,w,z)H(w,z), \tag{4.16}$$

where p_0 is the <u>fundamental solution</u> of PDE (4.14), which is simply the density $f_{\mu,\Sigma}(x,y)$ of a two-dimensional Gaussian random vector with mean vector μ and covariance matrix Σ given by

$$\mu = \begin{pmatrix} w + a_{0,0}t \\ z - \alpha_{0,0}t \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 2a_{0,0}t & c_{0,0}t \\ c_{0,0}t & 2b_{0,0}t \end{pmatrix}.$$

Remark 4.5. *In the case of Examples 4.2, 4.3 and 4.4 the matrix Σ is positive definite for any $t > 0$. For instance, consider the Taylor series expansion (Example 4.2). Using* (4.8) *we have* $\det(\Sigma) = t\sigma^2(\bar{x},\bar{y})\beta^2(\bar{x},\bar{y})(1-\rho(\bar{x},\bar{y})) > 0$ *by the assumptions on the coefficients σ, β and ρ.*

In order to find an explicit expression for the sequence of higher order terms (u_i) we shall first derive an explicit expression for $\widehat{u}_i$, the Fourier transform u_i. We will then use the Fourier representation $\widehat{u}_i$ to show that each u_i can be expressed as a differential operator acting on u_0.

Proposition 4.6. *Suppose $H \in L^1(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y)$ and let $\widehat{H}$ denote its Fourier transform. Suppose further that u_n and $\widehat{u}_n$ exist. Then $\widehat{u}_0$ is given by*

$$\widehat{u}_0(t,\lambda,\omega) = e^{t\phi_{0,0}(\lambda,\omega)}\widehat{H}(\lambda,\omega), \tag{4.17}$$

and $\widehat{u}_n$ $(n \geq 1)$ *is given by*

$$\widehat{u}_n(t,\lambda,\omega) = \sum_{h=1}^{n}\sum_{l=0}^{h}\int_0^t \mathrm{d}s\, e^{(t-s)\phi_{0,0}(\lambda,\omega)} B_{h-l,l}(i\partial_\lambda, i\partial_\omega)\phi_{h-l,l}(\lambda,\omega)\widehat{u}_{n-h}(s,\lambda,\omega), \quad n \geq 1. \tag{4.18}$$

Note that the operator $B_{h-l,l}(i\partial_\lambda, i\partial_\omega)$ *acts on* <u>*everything*</u> *to the right of it.*

Proof. See Appendix 6.8.2. □

Remark 4.7. *Proposition 4.6 is the two-dimensional extension of Corollary 10 from [171]. In that paper, the authors focus on scalar Lévy-type processes. In fact, although we have only considered two-dimensional diffusions in this paper, Proposition 4.6 remains valid if* $\phi(x,y,\lambda,\omega)$ *is the symbol of the generator of a two-dimensional Lévy-type process. In the Lévy-type case, due to the complications that arise from jumps,* u_n *must be expressed as an inverse Fourier transform of* $\widehat{u}_n$*; it is not possible to find* u_n *directly. However, because we limit the analysis in this paper to models without jumps, as the following Theorem shows, we are able to find an explicit expression for* u_n *as a differential operator acting on* u_0*.*

Theorem 4.8. *For every* $n \geq 1$*, define*

$$\mathcal{L}_n(t,x,y,\lambda,\omega) = \sum_{h=1}^{n}\int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{k-1}} \mathrm{d}t_h \sum_{\pi\in\Pi_h(n)} \Phi_{\pi(h)}(t_h,t)\cdots\Phi_{\pi(1)}(t_1,t), \tag{4.19}$$

where $\Pi_h(n)$ *is the set of permutations* π *such that*

$$\Pi_h(n) = \{\pi : \mathbb{N} \to \mathbb{Z}^+ \ : \sum_{l=1}^{h}\pi(l) = n\},$$

and $\Phi_h(s,t)$ *is an abbreviation for the operator*

$$\begin{aligned}\Phi_h(s,t) &= \Phi_h(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)\\ &:= \sum_{l=0}^{h}\phi_{h-l,l}(\lambda,\omega)\frac{B_{h-l,l}(-i\partial_\lambda,-i\partial_\omega)e^{i\lambda x+i\omega y+(t-s)\phi_{0,0}(\lambda,\omega)}}{e^{i\lambda x+i\omega y+(t-s)\phi_{0,0}(\lambda,\omega)}}, \quad k \geq 1.\end{aligned} \tag{4.20}$$

Then $u_n(t,x,y)$*, the solution of* (4.15)*, is given by*

$$u_n(t,x,y) = \mathcal{L}_n(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0(t,x,y), \quad n \geq 1, \tag{4.21}$$

where u_0 *is the solution of Cauchy problem* (4.14)*. Note that the operators* $(\Phi_h)_{h\geq 1}$ *act on* <u>*everything*</u> *to the right of them.*

Proof. See Appendix 6.8.3. □

Remark 4.9. *In the novel paper [168] the authors gave a different, and more explicit representation for the operators* $\mathcal{L}_n$*, in a multi-dimensional defaultable setting with time-dependent coefficients.*

Remark 4.10. *Note that p_n, the n-th order term in the transition density expansion $p = \sum_{n=0}^{\infty} p_n$, is expressed as a differential operator acting on p_0, which is simply a two-dimensional Gaussian density. Thus, from (4.11), we see that, independent of the choice of basis functions $(B_{i,j})$, each p_n can be written as a sum of Hermite polynomials multiplied by a Gaussian density.*

We now state an asymptotic convergence theorem which extends the results in [185]. Define our n-th order approximation for the prices as

$$v^{(n)}(t,x,y) := \sum_{h=0}^{n} u_h(T-t,x,y),$$

where the sequence of (u_h) is as given in Theorem 4.8. The n-th order approximation of the transition density $p^{(n)}(t,x,y;T,z,w)$ is defined as the special case where $H = \delta_{w,z}$.

The following theorem provides an asymptotic pointwise estimate as $t \to T^-$ for the error encountered by replacing the exact transition density p with the n-th order approximation $p^{(n)}$.

Theorem 4.11. *Let the operator $\mathcal{A}$ be expanded through the Taylor basis functions as described in Example 4.2. Assume that the functions $a = a(x,y)$, $\alpha = \alpha(x,y)$, $b = b(x,y)$ and $c = c(x,y)$ are differentiable up to order n with bounded and Lipschitz continuous derivatives. Assume that the covariance matrix is bounded and uniformly positive definite. That is,*

$$M^{-1}|\xi|^2 < \begin{pmatrix} \xi_1 \xi_2 \end{pmatrix} \begin{pmatrix} 2a(x,y) & c(x,y) \\ c(x,y) & 2b(x,y) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} < M|\xi|^2$$

for any $(x,y) \in \mathbb{R}^2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, where M is a positive constant. If $(\bar{x},\bar{y}) = (x,y)$ or $(\bar{x},\bar{y}) = (z,w)$ in (4.8), then we have

$$\left| p(t,x,y;T,z,w) - p^{(n)}(t,x,y;T,z,w) \right| \leq g_n(T-t)\Gamma^M(t,x,y;T,z,w),$$

for any $x,y,z,w \in \mathbb{R}$ and $t \in [0,T)$, where Γ^M denotes the Gaussian fundamental solution of the heat operator $M(\partial_{xx} + \partial_{yy}) + \partial_t$ and $g_n(s) = \mathcal{O}\left(s^{\frac{n+1}{2}}\right)$ as $s \to 0^+$.

Proof. It is based on the *parametrix method* (see, for instance, [188]), and it is analogous to the one dimensional case (Theorem 3.2). For a detailed proof we refer to the novel paper [168], where a more general result is obtained in the time-dependent and multi-dimensional case. □

Note that we obtain the same order of convergence for short maturities that we previously obtained in the one-dimensional case. As a direct corollary, we also have the following asymptotic estimate for option prices.

Corollary 4.12. *Under the assumptions of Theorem 4.11, for any $n \in \mathbb{N}$ we have*

$$\left| v(t,x,y) - v^{(n)}(t,x,y) \right| \leq g_n(T-t) \int_{\mathbb{R}^2} \mathrm{d}w\mathrm{d}z\, H(w,z)\, \Gamma^M(t,x,y;T,w,z)$$

for $x,y \in \mathbb{R}$ and $t \in [0,T)$.

6.5 Implied volatility expansions

European call and put prices are commonly quoted in units of implied volatility rather than in units of currency. In fact, in the financial industry, model parameters for the risk-neutral dynamics of a security are routinely obtained by calibrating to the market's implied volatility surface. Because calibration requires computing implied volatilities across a range of strikes and maturities and over a large set of model parameters, it is extremely useful to have a method of computing implied volatilities quickly.

We shall break this Section into two parts. First, in Section 6.5.1, we show how to pass in a general and model-independent way from an expansion of option prices to an expansion of implied volatilities. Then, in Section 6.5.2, we show that when call option prices can be computed as a series whose terms are as given in Theorem 4.8, the terms in the corresponding implied volatility expansion can be computed explicitly (i.e., without special functions or integrals). As such, approximate implied volatilities can be computed even faster than approximate option prices, which require the special function $\mathcal{N}$, the standard normal CDF.

6.5.1 Implied volatility expansions from price expansions – the general case

To begin our analysis, we assume that one has a model for the log of the underlying $X = \log S$. We fix a time to maturity $t > 0$, an initial value $X_0 = x$ and a call option payoff $H(X_t) = (e^{X_t} - e^k)^+$. Our goal is to find the implied volatility for this particular call option. To ease notation, we will suppress much of the dependence on (t, x, k). However, the reader should keep in mind that the implied volatility of the option under consideration does depend on (t, x, k), even if this is not explicitly indicated. Below, we provide definitions of the Black-Scholes price and implied volatility, which will be fundamental throughout this Section.

Definition 5.1. *For a fixed (t, x, k), the Black-Scholes price $u^{\text{BS}} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by*

$$u^{\text{BS}}(\sigma) := e^x \mathcal{N}(d_+(\sigma)) - e^k \mathcal{N}(d_-(\sigma)), \qquad d_\pm(\sigma) := \frac{1}{\sigma\sqrt{t}}\left(x - k \pm \frac{\sigma^2 t}{2}\right), \quad (5.22)$$

Where $\mathcal{N}$ is the CDF of a standard normal random variable.

Definition 5.2. *For fixed (t, x, k), the implied volatility corresponding to a call price $u \in ((e^x - e^k)^+, e^x)$ is defined as the unique strictly positive real solution σ of the equation*

$$u^{\text{BS}}(\sigma) = u. \quad (5.23)$$

Notice that $[u^{\text{BS}}]^{-1}$ is an analytic function on its domain $((e^x - e^k)^+, e^x)$. For any $u \in ((e^x - e^k)^+, e^x)$, we denote by ρ_u the radius of convergence of the Taylor series of $[u^{\text{BS}}]^{-1}$ about u.

The main result of the Section is the following Theorem:

Theorem 5.3. *Assume that the call price u admits an expansion of the form*

$$u = u^{\text{BS}}(\sigma_0) + \sum_{n=1}^{\infty} u_n, \quad (5.24)$$

for some positive σ_0 *and some sequence* $(u_n)_{n\geq 1}$ *where* $u_n \in \mathbb{R}$ *for all* n*. If*

$$|u - u^{\mathrm{BS}}(\sigma_0)| < \rho_{u^{\mathrm{BS}}(\sigma_0)}, \tag{5.25}$$

then the implied volatility $\sigma := [u^{\mathrm{BS}}]^{-1}(u)$ *is given by*

$$\sigma = \sigma_0 + \sum_{n=1}^{\infty} \sigma_n, \tag{5.26}$$

where the sequence $(\sigma_n)_{n\geq 1}$ *is defined recursively by*

$$\sigma_n = U_n(\sigma_0) - \frac{1}{n!}\sum_{h=2}^{n} A_h(\sigma_0)\mathbf{B}_{n,h}\big(\sigma_1, 2!\sigma_2, 3!\sigma_1, \ldots, (n-h+1)!\sigma_{n-h+1}\big). \tag{5.27}$$

In (5.27)*,* $\mathbf{B}_{n,h}$ *denotes the* (n,h)*-th partial Bell polynomial*[1] *and*

$$U_n(\sigma_0) := \frac{u_n}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)}, \quad n \geq 1, \tag{5.28}$$

$$A_n(\sigma_0) := \frac{\partial_\sigma^n u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)}, \quad n \geq 2. \tag{5.29}$$

Proof. We define $u(\varepsilon)$ an analytic function of ε by

$$u(\varepsilon) := u^{\mathrm{BS}}(\sigma_0) + \sum_{n=1}^{\infty} \varepsilon^n u_n, \quad \varepsilon \in [0,1]. \tag{5.30}$$

Note that $\sigma(\varepsilon) := [u^{\mathrm{BS}}]^{-1}(u(\varepsilon))$ is the composition of two analytic functions; it is therefore an analytic function of ε and admits an expansion about $\varepsilon = 0$ of the form

$$\sigma(\varepsilon) = \sigma_0 + \sum_{n=1}^{\infty} \varepsilon^n \sigma_n, \qquad \sigma_n = \frac{1}{n!}\partial_\varepsilon^n \sigma(\varepsilon)|_{\varepsilon=0}, \tag{5.31}$$

which by (5.25) is convergent for any $\varepsilon \in [0,1]$. By (5.30) we also have

$$u_n = \frac{1}{n!}\partial_\varepsilon^n u^{\mathrm{BS}}(\sigma(\varepsilon))|_{\varepsilon=0}. \tag{5.32}$$

We compute the n-th derivative of the composition of the two functions in (5.32) by applying the Bell polynomial version of the Faa di Bruno's formula, which can be found in [195] and [134]. We have

$$u_n = \frac{1}{n!}\sum_{h=1}^{n} \partial_\sigma^h u^{\mathrm{BS}}(\sigma_0)\mathbf{B}_{n,h}\left(\partial_\varepsilon\sigma(\varepsilon), \partial_\varepsilon^2\sigma(\varepsilon), \ldots, \partial_\varepsilon^{n-h+1}\sigma(\varepsilon)\right)|_{\varepsilon=0}. \tag{5.33}$$

Theorem 5.3 follows by inserting (5.31) into (5.33) and solving for σ_n. □

In the following Proposition, we will show that the coefficients A_n in (5.29) can be computed explicitly using an iterative algorithm. In particular, each $A_n(\sigma)$ is a rational function of σ and no special functions appear in its expression.

[1] Partial Bell polynomials are already implemented in Mathematica as $\mathrm{BellY[n, h, \{x_1, \ldots, x_{n-h+1}\}]}$.

Proposition 5.4. *Define the differential operator*

$$\mathcal{J} := t(\partial_x^2 - \partial_x). \tag{5.34}$$

Then

$$A_n(\sigma) = \frac{P_n(\mathcal{J})u^{\text{BS}}(\sigma)}{\partial_\sigma u^{\text{BS}}(\sigma)}, \tag{5.35}$$

where P_n is a polynomial function of order n defined recursively by

$$\begin{aligned}
&P_0(\mathcal{J}) = 1, \\
&P_1(\mathcal{J}) = \sigma\mathcal{J}, \\
&P_n(\mathcal{J}) = \sigma\mathcal{J}P_{n-1}(\mathcal{J}) + (n-1)\mathcal{J}P_{n-2}(\mathcal{J}), \qquad n \geq 2.
\end{aligned}$$

Moreover, the coefficients $A_n(\sigma_0)$ can be expressed explicitly in terms of Hermite[2] *polynomials.*

Proof. First, we recall the classical relation between the Delta, Gamma and Vega for European options in the Black-Scholes setting

$$\partial_\sigma u^{\text{BS}}(\sigma) = \sigma\mathcal{J}u^{\text{BS}}(\sigma). \tag{5.36}$$

Next, using the product rule for derivatives we compute

$$\begin{aligned}
\partial_\sigma^{n+1}u^{\text{BS}} &= \partial_\sigma^n\left(\partial_\sigma u^{\text{BS}}\right) = \partial_\sigma^n\left(\sigma\mathcal{J}u^{\text{BS}}\right) = \sum_{h=0}^n \binom{n}{h}\left(\partial_\sigma^h\sigma\right)\left(\mathcal{J}\partial_\sigma^{n-h}u^{\text{BS}}\right) \\
&= \left(\sigma\mathcal{J}\partial_\sigma^n + n\partial_\sigma^{n-1}\mathcal{J}\right)u^{\text{BS}}.
\end{aligned} \tag{5.37}$$

Equation (5.35) follows from (5.29) and (5.37). Now, to show that each of the $A_n(\sigma)$ can be expressed as a sum of Hermite polynomials, we observe that

$$\frac{\partial_x^n \exp\left(-\left(\frac{x-a}{b}\right)^2\right)}{\exp\left(-\left(\frac{x-a}{b}\right)^2\right)} = \frac{(-1)^n}{b^n}\mathbf{H}_n\left(\frac{x-a}{b}\right), \qquad a \in \mathbb{R}, b > 0, \tag{5.38}$$

where $\mathbf{H}_n$ is the n-th Hermite polynomial, defined in (4.11). Moreover using the Black-Scholes formula for call options (5.22) a direct computation shows

$$\mathcal{J}u^{\text{BS}}(\sigma) = \frac{e^k\sqrt{t}}{\sigma\sqrt{2\pi}}\exp\left(-\left(\frac{x-k-\sigma^2t/2}{\sigma\sqrt{2t}}\right)^2\right). \tag{5.39}$$

Thus, using (5.36) and (5.39) we obtain

$$\frac{\mathcal{J}^{n+1}u^{\text{BS}}(\sigma)}{\partial_\sigma u^{\text{BS}}(\sigma)} = \frac{\mathcal{J}^n\mathcal{J}u^{\text{BS}}(\sigma)}{\sigma\mathcal{J}u^{\text{BS}}(\sigma)} = \frac{\mathcal{J}^n\exp\left(-\left(\frac{x-k-\sigma^2t/2}{\sigma\sqrt{2t}}\right)^2\right)}{\sigma\exp\left(-\left(\frac{x-k-\sigma^2t/2}{\sigma\sqrt{2t}}\right)^2\right)}$$

[2] Our thanks to Peter Carr for pointing out the connection to Hermite polynomials.

$$= \frac{t^n}{\sigma}\sum_{h=0}^{n}\binom{n}{h}(-1)^h \frac{\partial_x^{2n-h}\exp\left(-\left(\frac{x-k-\sigma^2 t/2}{\sigma\sqrt{2t}}\right)^2\right)}{\exp\left(-\left(\frac{x-k-\sigma^2 t/2}{\sigma\sqrt{2t}}\right)^2\right)},$$

where, in the last equality, we have used the binomial expansion of $(\partial_{xx} - \partial_x)^n$. Finally, using (5.38) with $a = k + \frac{\sigma^2 t}{2}$ and $b = \sigma\sqrt{2t}$, we obtain

$$\frac{\partial^n u^{\mathrm{BS}}(\sigma)}{\partial_\sigma u^{\mathrm{BS}}(\sigma)} = \sum_{h=0}^{n-1}\binom{n-1}{h}\frac{t^{\frac{h}{2}}}{\sigma\left(\sigma\sqrt{2}\right)^{2(n-1)-h}}\mathbf{H}_{2(n-1)-h}\left(\frac{x-k-\sigma^2 t/2}{\sigma\sqrt{2t}}\right), \qquad n \geq 1, \tag{5.40}$$

Combining (5.35) with (5.40), we conclude that $A_n(\sigma)$ can be expressed as a sum of Hermite polynomials. In particular, computing $A_n(\sigma)$ does not involve any special functions or integration. □

Below, using (5.27) and Proposition 5.4, we provide explicit expressions for σ_n for $n \leq 3$. For simplicity, we remove the argument σ_0 from $U_n(\sigma_0)$. We have

$$\begin{aligned}
\sigma_1 &= U_1, \\
\sigma_2 &= U_2 - \frac{1}{2}\left(\frac{(k-x)^2}{t\sigma_0^3} - \frac{t\sigma_0}{4}\right)U_1^2, \\
\sigma_3 &= U_3 + \frac{1}{48}\left(2tU_1^3 + t^2\sigma_0^2 U_1^3 + 12t\sigma_0 U_1 U_2\right) \\
&\quad + \frac{1}{6t\sigma_0^4}\left(3U_1^3 - t\sigma_0^2 U_1^3 - 6\sigma_0 U_1 U_2\right)(k-x)^2 + \frac{1}{3t^2\sigma_0^6}U_1^3(k-x)^4,
\end{aligned}$$

where the (U_n) are as given in (5.28).

6.5.2 Implied volatility when option prices are given by Theorem 4.8.

We now consider the specific case where the sequence of (u_n) is as given in Theorem 4.8. We will show that, in this particular setting, the expansion (5.26) is convergent and approximate implied volatilities can be computed without any numerical integration or special functions. We begin with the following observation:

Remark 5.5. *From* (4.16)*, one can easily show that* $u_0 = u^{\mathrm{BS}}(\sqrt{2a_{0,0}})$*. Then, our expansion for the price of a European call option* (4.13) *in the general local-stochastic volatility setting* (2.1) *becomes*

$$u = u^{\mathrm{BS}}(\sigma_0) + \sum_{n=1}^{\infty} u_n, \qquad \sigma_0 = \sqrt{2a_{0,0}}. \tag{5.41}$$

From (5.41)*, it is clear that our option price expansion is of the form* (5.24)*. Therefore, we can use Theorem 5.3 to find approximate implied volatilities.*

Note that, in general, computing approximate implied volatilities using Theorem 5.3 requires numerical integration, as U_n appearing on the right-hand side of (5.27) contains u_n, which usually must be computed as a numerical integral. However, as the following Proposition shows, when the sequence of (u_n) are as given in Theorem 4.8, the sequence of

(U_n) appearing in (5.27) can be computed explicitly, with no numerical integration and no special functions.

Proposition 5.6. *Let the sequence of* (u_n) *be as given in Theorem 4.8. Then* U_n*, defined in* (5.28)*, are given by*

$$U_n(\sigma_0) = \sum_{h=0}^{N^{(n)}} D_h^{(n)} \mathbf{H}_h \left(\frac{x - k - \sigma_0^2 t/2}{\sigma\sqrt{2t}} \right).$$

where $\sigma_0 = \sqrt{2a_{0,0}}$*, the sequence of coefficients* $(D_h^{(n)})$ *are* (t,x,y)*-dependent constants, and each* $N^{(n)}$ $(n \in \mathbb{N})$ *is a finite positive integer.*

Proof. From Theorem 4.8, one can deduce that every u_n is of the form

$$u_n = \sum_{h=0}^{N^{(n)}} C_h^{(n)} \partial_x^h \left(\partial_x^2 - \partial_x\right) u^{\mathrm{BS}}(\sigma_0), \qquad \sigma_0 = \sqrt{2a_{0,0}}, \tag{5.42}$$

where the sequence of $(C_h^{(n)})$ are (t,x,y)-dependent constants and $N^{(n)}$ is a finite positive integer for every n. Both the sequence of coefficients $(C_h^{(n)})$ and the limit of the sum $N^{(n)}$ depend on the choice of basis functions $(B_{i,j}(x,y))$ and can be computed explicitly using (4.21). However (and we shall emphasize the following) independent of the choice of basis function, the general form (5.42) always holds; this is due to the fact that $B_{0,0}(x,y) = 1$. Now, using (5.42) we compute

$$\begin{aligned}
U_n(\sigma_0) &= \sum_{h=0}^{N^{(n)}} C_h^{(n)} \frac{\partial_x^h \left(\partial_x^2 - \partial_x\right) u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)} && \text{(by (5.28))} \\
&= \sum_{h=0}^{N^{(n)}} C_h^{(n)} \frac{\partial_x^h \daleth u^{\mathrm{BS}}(\sigma_0)}{t\sigma_0 \daleth u^{\mathrm{BS}}(\sigma_0)} && \text{(by (5.34))} \\
&= \sum_{h=0}^{N^{(n)}} C_h^{(n)} \frac{\partial_x^h \exp\left(- \left(\frac{x-k-\sigma_0^2 t/2}{\sigma\sqrt{2t}} \right)^2 \right)}{t\sigma_0 \exp\left(- \left(\frac{x-k-\sigma_0^2 t/2}{\sigma_0\sqrt{2t}} \right)^2 \right)} && \text{(by (5.39))} \\
&= \sum_{h=0}^{N^{(n)}} D_h^{(n)} \mathbf{H}_h \left(\frac{x - k - \frac{\sigma_0^2 t}{2}}{\sigma_0\sqrt{2t}} \right), && \text{(by (5.38))}
\end{aligned}$$

where we have absorbed some powers of t and σ_0 into $D_h^{(n)}$. □

To review, when the sequence of (u_n) is as given in Theorem 4.8, then using Theorem 5.3 and Propositions 5.4 and 5.6, approximate implied volatilities can be computed as a sum of Hermite polynomials in log-moneyness: $(k-x)$. We emphasize: No numerical integration or special functions are required. Approximate implied volatilities can therefore be computed even more quickly than approximate option prices (which require a normal CDF).

Remark 5.7. *Proposition 5.6 holds for any choice of the basis functions $B_{i,j}(x,y)$. However, for the Taylor expansion basis of Example 4.2, Corollary 4.12 ensures that condition* (5.25) *is satisfied for any t small enough. Therefore the expansion* (5.26) *is convergent for short maturities.*

We define the n-th order approximation of implied volatility as

$$\sigma^{(n)} := \sum_{h=0}^{n} \sigma_h. \tag{5.43}$$

For a given sequence of basis functions $(B_{i,j})$ explicit expressions for each σ_h in the sequence $(\sigma_h)_{h\geq 1}$ can be computed using a using a computer algebra program such as Wolfram's Mathematica. In Appendix 6.8.1, we provide explicit expressions for σ_h for $h \leq 2$ when the basis functions are given by $B_{n,m}(x,y) = (x-\bar{x})^n(y-\bar{y})^m$ (as in Example 4.2). On the authors' websites, we also provide a Mathematica notebook which contains the expressions for σ_h for $h \leq 3$.

6.6 Implied volatility examples

In this Section we use the results of Section 6.5.2 to compute approximate model-induced implied volatilities (5.43) under four different model dynamics in which European option prices can be computed explicitly.

- Section 6.6.1: Quadratic local volatility model
- Section 6.6.2: Heston stochastic volatility model
- Section 6.6.3: 3/2 stochastic volatility model
- Section 6.6.4: SABR local-stochastic volatility model

Assumption 6.1. *In all of the examples that follow we assume basis functions $B_{n,h}(x,y) = (x-\bar{x})^n(y-\bar{y})^h$ (as in Example 4.2) with $(\bar{x},\bar{y}) = (X_0, Y_0)$. Thus, approximate implied volatilities can be computed using the formulas given in Appendix 6.8.1 as well as the Mathematica notebook available on the authors' websites.*

6.6.1 Quadratic local volatility model

In the Quadratic local volatility model, the dynamics of the underlying S are given by

$$\mathrm{d}S_t = \left(\frac{\delta}{S_t}\frac{(e^R - S_t)(e^L - S_t)}{e^R - e^L}\right) S_t \mathrm{d}W_t, \qquad S_0 = s > 0, \qquad s < e^L < e^R.$$

Note that volatility increases as $S \to 0^+$, which is consistent with the leverage effect and which results in a negative at-the-money skew in the model-induced implied volatility surface. The left-hand root e^L of the polynomial $(e^R - s)(e^L - s)$ is an unattainable boundary for S. The origin, however, is attainable. In order to prevent the process S from taking negative values,one typically specifies zero as an absorbing boundary. Hence, the state space of S is $[0, e^L)$. In log notation $X := \log S$, we have the following dynamics

$$\mathrm{d}X_t = -\frac{1}{2}\left(\frac{\delta}{e^{X_t}}\frac{(e^R - e^{X_t})(e^L - e^{X_t})}{e^R - e^L}\right)^2 \mathrm{d}t + \frac{\delta}{e^{X_t}}\frac{(e^R - e^{X_t})(e^L - e^{X_t})}{e^R - e^L}\mathrm{d}W_t,$$

$$X_0 = x := \log s. \tag{6.44}$$

The generator of X is given by

$$\mathcal{A} = \frac{1}{2}\left(\frac{\delta}{e^x}\frac{(e^R - e^x)(e^L - e^x)}{e^R - e^L}\right)^2 (\partial_x^2 - \partial_x).$$

Thus, from (3.3) we identify

$$a(x,y) = \frac{1}{2}\left(\frac{\delta}{e^x}\frac{(e^R - e^x)(e^L - e^x)}{e^R - e^L}\right)^2, \qquad b(x,y) = 0, \qquad c(x,y) = 0, \qquad \alpha(x,y) = 0.$$

We fix a time to maturity t and log-strike k. Using the formulas from Appendix 6.8.1, as well as the Mathematica notebook provided on the authors' websites we compute explicitly

$$\begin{aligned}
\sigma_0 &= \frac{\delta}{e^x}\frac{(e^R - e^x)(e^L - e^x)}{e^R - e^L}, \\
\sigma_1 &= \left(\frac{a_{1,0}}{2\sigma_0}\right)(k - x), \\
\sigma_2 &= \left(-\frac{t\left(12 + t\sigma_0^2\right)a_{1,0}^2}{96\sigma_0} + \frac{1}{6}t\sigma_0 a_{2,0}\right) + \left(\frac{-3a_{1,0}^2 + 4\sigma_0^2 a_{2,0}}{12\sigma_0^3}\right)(k - x)^2, \\
\sigma_3 &= \frac{-t}{192\sigma_0^3}\left(\left(-12 + t\sigma_0^2\right)a_{1,0}^3 + 4\sigma_0^2\left(8 + t\sigma_0^2\right)a_{1,0}a_{2,0} - 48\sigma_0^4 a_{3,0}\right)(k - x) \\
&\quad + \frac{1}{12\sigma_0^5}\left(3a_{1,0}^3 - 5\sigma_0^2 a_{1,0}a_{2,0} + 3\sigma_0^4 a_{3,0}\right)(k - x)^3,
\end{aligned} \tag{6.45}$$

where

$$\begin{aligned}
a_{1,0} &= \frac{\delta^2\left(-\sinh(L + R - 2x) + \sinh(L - x) + \sinh(R - x)\right)}{\cosh(L - R) - 1}, \\
a_{2,0} &= \frac{1}{4}\delta^2\left(2\cosh(L + R - 2x) - \cosh(L - x) - \cosh(R - x)\right)\operatorname{csch}^2\left(\frac{L - R}{2}\right), \\
a_{3,0} &= \frac{e^{L+R}\delta^2\left(-4\sinh(L + R - 2x) + \sinh(L - x) + \sinh(R - x)\right)}{3\left(e^L - e^R\right)^2}.
\end{aligned}$$

The exact price of a call option is computed in [3] Lemma 3.1. Assuming $k < L$ we have:

$$\begin{aligned}
u(t,x) &= e^{k_1}\mathcal{N}(-d_-^{(1)}) - e^{x_2}\mathcal{N}(d_+^{(2)}) - e^{x_1}\mathcal{N}(-d_+^{(1)}) + e^{k_2}\mathcal{N}(d_-^{(2)}), \qquad d_\pm^{(i)} = \frac{x_i - k_i \pm \frac{1}{2}\delta^2 t}{\sqrt{\delta^2 t}}, \\
e^{k_1} &= \frac{(e^L - e^k)(e^R - e^x)}{e^R - e^L}, \qquad e^{x_1} = \frac{(e^L - e^x)(e^R - e^k)}{e^R - e^L}, \\
e^{k_2} &= \frac{(e^R - e^k)(e^R - e^x)}{e^R - e^L}, \qquad e^{x_2} = \frac{(e^L - e^x)(e^L - e^k)}{e^R - e^L}.
\end{aligned} \tag{6.46}$$

Thus, the exact implied volatility σ can be obtained by solving (5.23) numerically.

In Figure 6.1 we plot our third order implied volatility approximation $\sigma^{(3)}$ and the exact implied volatility σ. Relative error of the approximation is given in Figure 6.2. In order to visualize the range of strikes and maturities over which our implied volatility expansion accurately approximates the exact implied volatility, we provide in Figure 6.3 a contour plot of the absolute value of the relative error $|\sigma^{(3)} - \sigma|/\sigma$ of our third order implied volatility approximation as a function of time to maturity t and log-moneyness $(k-x)$. From Figure 6.3, we observe a relative error of less than 1% for nearly all strikes k maturities t such that $(k-x) \in (-1.5, 1.5)$ and $t < 4$. A relative error of less than 3% is observed for nearly all strikes k maturities t such that $(k-x) \in (-1.5, 1.5)$ and $t < 10$.

6.6.2 Heston stochastic volatility model

Perhaps the most well-known stochastic volatility model is that of [121]. In the Heston model, the dynamics of the underlying S are given by

$$\begin{aligned} \mathrm{d}S_t &= \sqrt{Z_t} S_t \mathrm{d}W_t, & S_0 &= s > 0, \\ \mathrm{d}Z_t &= \kappa(\theta - Z_t)\mathrm{d}t + \delta\sqrt{Z_t}\mathrm{d}B_t, & Z_0 &= z > 0, \\ \mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t. \end{aligned}$$

Although it is not required, one typically sets $\rho < 0$ in order to capture the leverage effect. In log notation $(X, Y) := (\log S, \log Z)$ we have the following dynamics

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}e^{Y_t}\mathrm{d}t + e^{\frac{1}{2}Y_t}\mathrm{d}W_t, & X_0 &= x := \log s, \\ \mathrm{d}Y_t &= \left((\kappa\theta - \tfrac{1}{2}\delta^2)e^{-Y_t} - \kappa\right)\mathrm{d}t + \delta\, e^{-\frac{1}{2}Y_t}\mathrm{d}B_t, & Y_0 &= y := \log z, \\ \mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t. \end{aligned} \tag{6.47}$$

The generator of (X, Y) is given by

$$\mathcal{A} = \frac{1}{2}e^y\left(\partial_x^2 - \partial_x\right) + \left((\kappa\theta - \tfrac{1}{2}\delta^2)e^{-y} - \kappa\right)\partial_y + \frac{1}{2}\delta^2 e^{-y}\partial_y^2 + \rho\,\delta\partial_x\partial_y.$$

Thus, using (3.3), we identify

$$a(x,y) = \frac{1}{2}e^y, \quad b(x,y) = \frac{1}{2}\delta^2 e^{-y}, \quad c(x,y) = \rho\,\delta, \quad \alpha(x,y) = \left((\kappa\theta - \tfrac{1}{2}\delta^2)e^{-y} - \kappa\right).$$

We fix a time to maturity t and log-strike k. Using the formulas from Appendix 6.8.1 as well as the Mathematica notebook provided on the authors' websites, we compute explicitly

$$\begin{aligned} \sigma_0 &= e^{y/2}, \\ \sigma_1 &= \frac{1}{8}e^{-y/2}t\left(-\delta^2 + 2\left(-e^y + \theta\right)\kappa + e^y\delta\rho\right) + \frac{1}{4}e^{-y/2}\delta\rho(k-x), \\ \sigma_2 &= \left(\frac{-e^{-3y/2}}{128}t^2\left(\delta^2 - 2\theta\kappa\right)^2 + \frac{e^{y/2}}{96}t^2\left(5\kappa^2 - 5\delta\kappa\rho + \delta^2\left(-1 + 2\rho^2\right)\right)\right. \\ &\quad \left. + \frac{e^{-y/2}}{192}t\left(-4t\theta\kappa^2 - t\delta^3\rho + 2t\delta\theta\kappa\rho + 2\delta^2\left(8 + t\kappa + \rho^2\right)\right)\right) \\ &\quad + \frac{1}{96}e^{-3y/2}t\delta\rho\left(5\delta^2 + 2\left(e^y - 5\theta\right)\kappa - e^y\delta\rho\right)(k-x) + \frac{1}{48}e^{-3y/2}\delta^2\left(2 - 5\rho^2\right)(k-x)^2, \end{aligned}$$

$$
\begin{aligned}
\sigma_3 = \Big(&- \frac{e^{-5y/2} t^3 \left(\delta^2 - 2\theta\kappa\right)^3}{1024} + \frac{e^{y/2} t^3 (-2\kappa + \delta\rho) \left(6\kappa^2 - 6\delta\kappa\rho + \delta^2 \left(-6 + 5\rho^2\right)\right)}{1536} \\
&+ \frac{e^{-3y/2} t^2 \left(\delta^2 - 2\theta\kappa\right) \left(4t\theta\kappa^2 + t\delta^3\rho - 2t\delta\theta\kappa\rho + \delta^2 \left(16 - 2t\kappa + 20\rho^2\right)\right)}{3072} \\
&+ \frac{1}{768} e^{-y/2} t^2 \left(3\delta^2\rho^2(-2\kappa + \delta\rho) + t\kappa \left(\delta^2 - 2\theta\kappa\right) (-\kappa + \delta\rho)\right) \Big) \\
&+ \Big(\frac{7t^2\delta\rho e^{-5y/2}}{512} \left(\delta^2 - 2\theta\kappa\right)^2 + \frac{t^2\delta\rho e^{-y/2}}{384} \left(-3\kappa^2 + \delta \left(\delta + 3\kappa\rho - 2\delta\rho^2\right)\right) \\
&- \frac{e^{-3y/2}}{768} t\delta\rho \left(20t\theta\kappa^2 + 5t\delta^3\rho - 10t\delta\theta\kappa\rho + 2\delta^2 \left(8 - 5t\kappa + 9\rho^2\right)\right) \Big)(k - x) \\
&+ \frac{e^{-5y/2} t\delta^2}{384} \left(e^y(-2\kappa + \delta\rho) \left(-2 + 7\rho^2\right) - \left(\delta^2 - 2\theta\kappa\right) \left(-8 + 23\rho^2\right)\right) (k - x)^2 \\
&+ \frac{e^{-5y/2}\delta^3\rho}{96} \left(-5 + 8\rho^2\right) (k - x)^3.
\end{aligned} \tag{6.48}
$$

The characteristic function of X_t is computed explicitly in [121]

$$
\begin{aligned}
\eta(t, x, y, \lambda) := \log \mathbb{E}_{x,y} e^{i\lambda X_t} &= i\lambda x + C(t, \lambda) + D(t, \lambda) e^y, \\
C(t, \lambda) &= \frac{\kappa\theta}{\delta^2} \left((\kappa - \rho\delta i\lambda + d(\lambda))t - 2 \log \left[\frac{1 - f(\lambda) e^{d(\lambda)t}}{1 - f(\lambda)} \right] \right), \\
D(t, \lambda) &= \frac{\kappa - \rho\delta i\lambda + d(\lambda)}{\delta^2} \frac{1 - e^{d(\lambda)t}}{1 - f(\lambda) e^{d(\lambda)t}}, \\
f(\lambda) &= \frac{\kappa - \rho\delta i\lambda + d(\lambda)}{\kappa - \rho\delta i\lambda - d(\lambda)}, \\
d(\lambda) &= \sqrt{\delta^2(\lambda^2 + i\lambda) + (\kappa - \rho i\lambda\delta)^2}.
\end{aligned}
$$

Thus, the price of a European call option can be computed using standard Fourier methods

$$
u(t, x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}\lambda_r \, e^{\eta(t,x,y,\lambda)} \widehat{h}(\lambda), \quad \widehat{h}(\lambda) = \frac{-e^{k - ik\lambda}}{i\lambda + \lambda^2}, \quad \lambda = \lambda_r + i\lambda_i, \quad \lambda_i < -1. \tag{6.49}
$$

Note, since the call option payoff $h(x) = (e^x - e^k)^+$ is not in $L^1(\mathbb{R})$, its Fourier transform $\widehat{h}(\lambda)$ must be computed in a generalized sense by fixing an imaginary component of the Fourier variable $\lambda_i < -1$. Using (6.49) the exact implied volatility σ can be computed to solving (5.23) numerically. In Figure 6.4 we plot our third order implied volatility approximation $\sigma^{(3)}$ and the exact implied volatility σ. For comparison, we also plot the small-time near-the-money implied volatility expansion of [94] (see Theorem 3.2 and Corollary 4.3)

$$
\begin{aligned}
\sigma^{\mathrm{FJL}} &= \left(g_0^2 + g_1 t + o(t)\right)^{1/2}, \\
g_0 &= e^{y/2} \left(1 + \frac{1}{4}\rho\delta(k - x)e^{-y} + \frac{1}{24}\left(1 - \frac{5\rho^2}{2}\right)\delta^2(k - x)^2 e^{-2y}\right) + \mathcal{O}((k - x)^3), \\
g_1 &= -\frac{\delta^2}{12}\left(1 - \frac{\rho^2}{4}\right) + \frac{e^y\rho\delta}{4} + \frac{\kappa}{2}(\theta - e^y) + \frac{1}{24}\rho\delta e^{-y}(\delta^2\overline{\rho}^2 - 2\kappa(\theta + e^y) + \rho\delta e^y)(k - x) \\
&\quad + \frac{\delta^2 e^{-2y}}{7680}\left(176\delta^2 - 480\kappa\theta - 712\rho^2\delta^2 + 521\rho^4\delta^2 + 40\rho^3\delta e^y + 1040\kappa\theta\rho^2 - 80\kappa\rho^2 e^y\right)
\end{aligned} \tag{6.50}
$$

$$\cdot (k-x)^2 + \mathcal{O}\big((k-x)^3\big), \qquad\qquad \overline{\rho} = \sqrt{1-\rho^2}.$$

Relative errors of the two approximations are given in Figure 6.5. It is clear from the Figures that our third order implied volatility expansions $\sigma^{(3)}$ provides a better approximation of the true implied volatility σ than does the implied volatility expansion σ^{FJL}. The improvement marked by $\sigma^{(3)}$ is particularly noticeable at the largest strikes and at longer maturities.

We are interested in learning the range of strikes and maturities over which our implied volatility expansion accurately approximates the exact implied volatility. Thus, in Figure 6.6 we provide a contour plot of the absolute value of the relative error $|\sigma^{(3)} - \sigma|/\sigma$ of our third order implied volatility approximation as a function of time to maturity t and log-moneyness $(k-x)$. From the Figure, we observe an absolute relative error of less than 2% for most options satisfying $(k-x) \in (-0.75, 0.75)$ and $t \in (0.0, 2.7)$ years.

6.6.3 3/2 stochastic volatility model

We consider now the 3/2 stochastic volatility model. The risk-neutral dynamics of the underlying S in this setting are given by

$$\begin{aligned}
\mathrm{d}S_t &= \sqrt{Z_t} S_t \mathrm{d}W_t, & S_0 &= s > 0,\\
\mathrm{d}Z_t &= Z_t \left(\kappa(\theta - Z_t)\mathrm{d}t + \delta\sqrt{Z_t}\mathrm{d}B_t\right), & Z_0 &= z > 0,\\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned}$$

As in all stochastic volatility models, one typically sets $\rho < 0$ in order to capture the leverage effect. The 3/2 model is noteworthy in that it does not fall into the affine class of [75], and yet it still allows for European option prices to be computed in semi-closed form (as a Fourier integral). Notice however that the characteristic function (given in (6.53) below) involves special functions such as the Gamma and the confluent hypergeometric functions. Therefore, Fourier pricing methods are not an efficient means of computed prices. The importance of the 3/2 model in the pricing of options on realized variance is well documented by [74]. In particular, the 3/2 model allows for upward-sloping implied volatility of variance smiles while Heston's model leads to downward-sloping volatility of variance smiles, in disagreement with observed skews in variance markets.

In log notation $(X, Y) := (\log S, \log Z)$ we have the following dynamics

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}e^{Y_t}\mathrm{d}t + e^{\frac{1}{2}Y_t}\mathrm{d}W_t, & X_0 &= x := \log s,\\
\mathrm{d}Y_t &= \left(\kappa(\theta - e^{Y_t}) - \frac{1}{2}\delta^2 e^{Y_t}\right)\mathrm{d}t + \delta\, e^{\frac{1}{2}Y_t}\mathrm{d}B_t, & Y_0 &= y := \log z,\\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned} \tag{6.51}$$

The generator of (X, Y) is given by

$$\mathcal{A} = \frac{1}{2}e^y\left(\partial_x^2 - \partial_x\right) + \left(\kappa(\theta - e^y) - \frac{1}{2}\delta^2 e^y\right)\partial_y + \frac{1}{2}\delta^2 e^y \partial_y^2 + \rho\,\delta\, e^y \partial_x\partial_y.$$

Thus, using (3.3), we identify

$$a(x,y) = \frac{1}{2}e^y, \quad b(x,y) = \frac{1}{2}\delta^2 e^y, \quad c(x,y) = \rho\,\delta\, e^y, \quad \alpha(x,y) = \kappa(\theta - e^y) - \frac{1}{2}\delta^2 e^y.$$

We fix a time to maturity t and log-strike k. Using the formulas from Appendix 6.8.1 as well as the Mathematica notebook provided on the authors' websites, we compute explicitly

$$
\begin{aligned}
\sigma_0 &= e^{y/2}, \\
\sigma_1 &= -\frac{1}{8}e^{y/2}t\left(-2\theta\kappa + e^y\left(\delta^2 + 2\kappa - \delta\rho\right)\right) + \frac{1}{4}e^{y/2}\delta\rho(k-x), \\
\sigma_2 &= e^{y/2}\left(\frac{5}{96}t^2\theta^2\kappa^2\right) + e^{3y/2}\left(-\frac{1}{96}t\left(18t\theta\kappa^2 - 9t\delta\theta\kappa\rho + \delta^2\left(-8 + 9t\theta\kappa + 7\rho^2\right)\right)\right) \\
&\quad + e^{6y/2}\left(\frac{1}{384}t^2\left(13\delta^4 + 52\kappa^2 - 26\delta^3\rho - 52\delta\kappa\rho + 4\delta^2\left(-1 + 13\kappa + 4\rho^2\right)\right)\right) \\
&\quad + \frac{1}{96}e^{y/2}t\delta\rho\left(6\theta\kappa - 7e^y\left(\delta^2 + 2\kappa - \delta\rho\right)\right)(k-x) - \frac{1}{48}e^{y/2}\delta^2\left(-2+\rho^2\right)(k-x)^2, \qquad (6.52) \\
\sigma_3 &= \frac{1}{3072}\Big(e^{y/2}\left(24t^3\theta^3\kappa^3\right) + e^{3y/2}\left(-12t^2\theta\kappa\left(22t\theta\kappa^2 - 11t\delta\theta\kappa\rho + \delta^2\left(-16 + 11t\theta\kappa + 14\rho^2\right)\right)\right) \\
&\quad + e^{5y/2}\big(-240t^2\delta^4 - 480t^2\delta^2\kappa - 40t^3\delta^2\theta\kappa + 130t^3\delta^4\theta\kappa + 520t^3\delta^2\theta\kappa^2 + 520t^3\theta\kappa^3 + 240t^2\delta^3\rho \\
&\quad - 260t^3\delta^3\theta\kappa\rho - 520t^3\delta\theta\kappa^2\rho + 180t^2\delta^4\rho^2 + 360t^2\delta^2\kappa\rho^2 + 160t^3\delta^2\theta\kappa\rho^2 - 180t^2\delta^3\rho^3\big) \\
&\quad + e^{7y/2}\left(-t^3\left(\delta^2 + 2\kappa - \delta\rho\right)\left(35\delta^4 + 140\kappa^2 - 70\delta^3\rho - 140\delta\kappa\rho + 2\delta^2\left(-16 + 70\kappa + 29\rho^2\right)\right)\right)\Big) \\
&\quad + \frac{1}{1536}\Big(e^{y/2}\left(20t^2\delta\theta^2\kappa^2\rho\right) + e^{3y/2}\left(-12t\delta\rho\left(14t\theta\kappa^2 - 7t\delta\theta\kappa\rho + \delta^2\left(-4 + 7t\theta\kappa + 3\rho^2\right)\right)\right) \\
&\quad + e^{5y/2}\left(t^2\delta\rho\left(45\delta^4 + 180\kappa^2 - 90\delta^3\rho - 180\delta\kappa\rho + 4\delta^2\left(-4 + 45\kappa + 14\rho^2\right)\right)\right)\Big)(k-x) \\
&\quad + \frac{1}{384}e^{y/2}t\delta^2\left(e^y\left(\delta^2 + 2\kappa - \delta\rho\right)\left(-8+\rho^2\right) - 2\theta\kappa\left(-2+\rho^2\right)\right)(k-x)^2.
\end{aligned}
$$

To the best of our knowledge, the above formula is the first explicit implied volatility expansion for the 3/2 model. The characteristic function of X_t is given, for example, in Proposition 3.2 of [16]. We have

$$
\mathbb{E}_{x,y}e^{i\lambda X_t} = e^{i\lambda x}\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\left(\frac{2}{\delta^2 z}\right)^{\alpha} M\left(\alpha, \gamma, \frac{-2}{\delta^2 z}\right) \qquad (6.53)
$$

with

$$
\begin{aligned}
z &= \frac{e^y}{\kappa\theta}(e^{\kappa\theta t} - 1), \quad \gamma = 2\left(\alpha + 1 - \frac{p}{\delta^2}\right), \\
\alpha &= -\left(\frac{1}{2} - \frac{p}{\delta^2}\right) + \left(\left(\frac{1}{2} - \frac{p}{\delta^2}\right)^2 + 2\frac{q}{\delta^2}\right)^{1/2}, \quad p = -\kappa + i\delta\rho\lambda, \quad q = \frac{1}{2}(i\lambda + \lambda^2),
\end{aligned}
$$

where Γ is a Gamma function and M is a confluent hypergeometric function. Thus, the price of a European call option can be computed using standard Fourier methods

$$
u(t,x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\lambda_r\, \widehat{h}(\lambda)\mathbb{E}_{x,y}e^{i\lambda X_t}, \qquad \lambda = \lambda_r + i\lambda_i, \qquad \lambda_i < -1, \quad (6.54)
$$

where $\widehat{h}(\lambda)$ is given in (6.49). Using (6.54) the exact implied volatility σ can be computed to solving (5.23) numerically.

In Figure 6.7 we plot our third order implied volatility approximation $\sigma^{(3)}$ and the exact implied volatility σ. Relative error of the approximation is given in Figure 6.8. In order to visualize the range of strikes and maturities over which our implied volatility expansion accurately approximates the exact implied volatility, we provide in Figure 6.9 a contour plot of the absolute value of the relative error $|\sigma^{(3)} - \sigma|/\sigma$ of our third order implied volatility approximation as a function of time to maturity t and log-moneyness $(k-x)$. From Figures 6.9, we observe a relative error of less than 1% for nearly all strikes k maturities t such that $(k-x) \in (-1.0, 0.8)$ and $t < 1.5$ years. A relative error of less than 3% is observed for nearly all strikes k maturities t such that $(k-x) \in (-1.0, 0.8)$ and $t < 2.5$ years.

6.6.4 SABR local-stochastic volatility

The SABR model of [114] is a local-stochastic volatility model in which the risk-neutral dynamics of S are given by

$$\begin{aligned}
\mathrm{d}S_t &= Z_t S_t^\beta \mathrm{d}W_t, && S_0 = s > 0,\\
\mathrm{d}Z_t &= \delta Z_t \mathrm{d}B_t, && Z_0 = z > 0,\\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned}$$

Modeling the non-local component of volatility Z as a geometric Brownian motion results in a true implied volatility smile (i.e., upward sloping implied volatility for high strikes); this is in contrast to the CEV model, for which the model-induced implied volatility is monotone decreasing (for $\beta < 1$). In log notation $(X, Y) := (\log S, \log Z)$ we have, we have the following dynamics:

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}e^{2Y_t+2(\beta-1)X_t}\mathrm{d}t + e^{Y_t+(\beta-1)X_t}\mathrm{d}W_t, && X_0 = x := \log s,\\
\mathrm{d}Y_t &= -\frac{1}{2}\delta^2\mathrm{d}t + \delta\,\mathrm{d}B_t, && Y_0 = y := \log z,\\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t.
\end{aligned} \tag{6.55}$$

The generator of (X, Y) is given by

$$\mathcal{A} = \frac{1}{2}e^{2y+2(\beta-1)x}(\partial_x^2 - \partial_x) - \frac{1}{2}\delta^2\partial_y + \frac{1}{2}\delta^2\partial_y^2 + \rho\,\delta\, e^{y+(\beta-1)x}\partial_x\partial_y.$$

Thus, using (3.3), we identify

$$a(x,y) = \frac{1}{2}e^{2y+2(\beta-1)x}, \quad b(x,y) = \frac{1}{2}\delta^2, \quad c(x,y) = \rho\,\delta\, e^{y+(\beta-1)x}, \quad \alpha(x,y) = -\frac{1}{2}\delta^2.$$

We fix a time to maturity t and log-strike k. Using the formulas from Appendix 6.8.1 as well as the Mathematica notebook provided on the authors' websites, we compute explicitly

$$\sigma_0 = e^{y+(\beta-1)x}, \quad \sigma_1 = \sigma_{1,0} + \sigma_{0,1}, \quad \sigma_2 = \sigma_{2,0} + \sigma_{1,1} + \sigma_{0,1}, \quad \sigma_3 = \sigma_{3,0} + \sigma_{2,1} + \sigma_{1,2} + \sigma_{0,3}, \tag{6.56}$$

where

$$\sigma_{1,0} = \frac{1}{2}(k-x)(-1+\beta)\sigma_0,$$

$$
\begin{aligned}
\sigma_{0,1} &= \frac{1}{4}\delta\left(2(k-x)\rho + t\sigma_0\left(-\delta+\rho\sigma_0\right)\right), \\
\sigma_{2,0} &= \frac{1}{96}(-1+\beta)^2\sigma_0\left(8(k-x)^2 + t\sigma_0^2\left(4-t\sigma_0^2\right)\right), \\
\sigma_{1,1} &= -\frac{1}{48}t(-1+\beta)\delta\sigma_0\left(6(k-x)\delta - 2(6+5k-5x)\rho\sigma_0 + t\rho\sigma_0^3\right), \\
\sigma_{0,2} &= \frac{1}{96}t\delta^2\sigma_0\left(32+5t\delta^2-12\rho^2+2t\sigma_0\left(-7\delta\rho+\left(-2+6\rho^2\right)\sigma_0\right)\right) \\
&\quad - \frac{1}{24}t\delta^2\rho\left(\delta-3\rho\sigma_0\right)(k-x) + \frac{\delta^2\left(2-3\rho^2\right)}{12\sigma_0}(k-x)^2, \\
\sigma_{3,0} &= -\frac{1}{192}t(k-x)(-1+\beta)^3\sigma_0^3\left(-12+5t\sigma_0^2\right), \\
\sigma_{2,1} &= \frac{1}{384}t^2(-1+\beta)^2\delta\sigma_0^3\left(-12\delta+28\rho\sigma_0+t\sigma_0^2\left(5\delta-7\rho\sigma_0\right)\right) \\
&\quad - \frac{13}{192}t(-1+\beta)^2\delta\rho\sigma_0^2\left(-4+t\sigma_0^2\right)(k-x) - \frac{1}{48}t(-1+\beta)^2\delta\sigma_0\left(\delta-3\rho\sigma_0\right)(k-x)^2,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{1,2} &= \frac{1}{192}t^2(-1+\beta)\delta^2\rho\sigma_0^2\left(-28\delta+52\rho\sigma_0+t\sigma_0^2\left(5\delta-7\rho\sigma_0\right)\right) \\
&\quad + \frac{1}{192}t(-1+\beta)\delta^2\sigma_0\left(32+5t\delta^2+12\rho^2-22t\delta\rho\sigma_0+4t\left(-3+5\rho^2\right)\sigma_0^2\right)(k-x) \\
&\quad + \frac{1}{24}t(-1+\beta)\delta^2\rho^2\sigma_0(k-x)^2 + \frac{(-1+\beta)\delta^2\left(-2+3\rho^2\right)}{24\sigma_0}(k-x)^3, \\
\sigma_{0,3} &= -\sigma_0\frac{1}{128}t^2\delta^4\left(16+t\delta^2-4\rho^2\right) + \sigma_0^2\frac{1}{384}t^2\delta^3\rho\left(104+19t\delta^2-36\rho^2\right) \\
&\quad + \sigma_0^3\frac{1}{192}t^3\delta^4\left(8-21\rho^2\right) + \sigma_0^4\frac{1}{192}t^3\delta^3\rho\left(-11+15\rho^2\right) \\
&\quad - \frac{1}{192}t\delta^3\rho\left(8+12x+t\delta^2-12\rho^2+6t\sigma_0\left(\delta\rho+\left(1-2\rho^2\right)\sigma_0\right)\right)(k-x) \\
&\quad - \frac{1}{16}t\delta^3\rho\left(-1+\rho^2\right)(k-x)^2 + \frac{\delta^3\rho\left(-5+6\rho^2\right)}{24\sigma_0^2}(k-x)^3.
\end{aligned}
$$

There is no formula for European option prices in the general SABR setting. However, for the special zero-correlation case $\rho = 0$ the exact price of a European call is computed in [10]:

$$
\begin{aligned}
u(t,x) &= e^{(x+k)/2}\frac{e^{-\delta^2 t/8}}{\sqrt{2\pi\delta^2 t}}\Bigg\{\frac{1}{\pi}\int_0^\infty \mathrm{d}V\int_0^\pi \mathrm{d}\phi\frac{1}{V}\left(\frac{V}{V_0}\right)^{-1/2}\frac{\sin\phi\sin(|\nu|\phi)}{b-\cos\phi}\exp\left(\frac{\xi_\phi^2}{2\delta^2 t}\right) \\
&\quad + \frac{\sin(|\nu|\pi)}{\pi}\int_0^\infty \mathrm{d}V\int_0^\infty \mathrm{d}\psi\frac{1}{V}\left(\frac{V}{V_0}\right)^{-1/2}\frac{\sinh\psi}{b-\cosh\psi}e^{-|\nu|\psi}\exp\left(\frac{\xi_\psi^2}{2\delta^2 t}\right)\Bigg\} \\
&\quad + (e^x - e^k)^+, \qquad (6.57) \\
\xi_\phi &= \arccos\left(\frac{q_h^2+q_x^2+V^2+V_0^2}{2VV_0} - \frac{q_h q_x}{VV_0}\cos\phi\right), \\
\xi_\psi &= \arccos\left(\frac{q_h^2+q_x^2+V^2+V_0^2}{2VV_0} + \frac{q_h q_x}{VV_0}\cosh\psi\right),
\end{aligned}
$$

$$b = \frac{q_h^2 + q_x^2}{2 q_h q_x}, \qquad q_h = \frac{e^{(1-\beta)k}}{1-\beta}, \qquad q_x = \frac{e^{(1-\beta)x}}{1-\beta}, \qquad \nu = \frac{-1}{2(1-\beta)}, \qquad V_0 = \frac{e^y}{\delta}.$$

Thus, in the zero-correlation setting, the exact implied volatility σ can be obtained by using the above formula and then by solving (5.23) numerically. In Figure 6.10 we plot our third order implied volatility approximation $\sigma^{(3)}$ and the exact implied volatility σ. For comparison, we also plot the implied volatility expansion of [114]

$$\sigma^{\text{HKLW}} = \delta \frac{x-k}{D(\zeta)} \left\{ 1 + t\delta^2 \left[\frac{2\gamma_2 - \gamma_1^2 + 1/f^2}{24} \left(\frac{e^{y+\beta f}}{\delta} \right)^2 + \frac{\rho \gamma_1 e^{y+\beta f}}{4\delta} + \frac{2-3\rho^2}{24} \right] \right\}, \quad (6.58)$$

with

$$f = \frac{1}{2}(e^x + e^k), \quad \zeta = \frac{\delta\, e^{-y}}{\beta - 1} \left(e^{(1-\beta)k} - e^{(1-\beta)x} \right), \quad \gamma_1 = \beta/f,$$
$$\gamma_2 = \beta(\beta-1)/f^2, \quad D(\zeta) = \log \left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1-\rho} \right).$$

Note that we use the "corrected" SABR formula, which appears in [180]. Relative errors of the two approximations are given in Figure 6.11. From the Figures we observe that both expansions $\sigma^{(3)}$ and σ^{HKLW} provide excellent approximations of the true implied volatility σ for options with maturities of ~ 1.5 years or less. However, for longer maturities $t > 2.0$, it is clear that $\sigma^{(3)}$ more closely approximates σ than does σ^{HKLW}.

We are interested in learning the range of strikes and maturities over which our implied volatility expansion accurately approximates the exact implied volatility. Thus, in Figure 6.12 we provide a contour plot of the absolute value of the relative error $|\sigma^{(3)} - \sigma|/\sigma$ of our third order implied volatility approximation as a function of time to maturity t and log-moneyness $(k - x)$. From the Figure, we observe an absolute relative error of less than 2% for most options satisfying $(k - x) \in (-1.5, 1.4)$ and $t < 5.0$ years.

6.7 Conclusions

In this paper we consider general local-stochastic volatility models. In this setting, we provide a family of approximations – one for each choice of the basis functions (i.e. Taylor series, Two-point Taylor series, L^2 basis, etc.) – for (i) the transition density of the underlying (ii) European-style option prices and (iii) implied volatilities. Our density expansions require no integration; every term can be written as a sum of Hermite polynomials multiplied by a Gaussian density. The terms in our option price expansions are expressed as a differential operator acting on the Black-Scholes price. Thus, to compute approximate prices, one requires only a normal CDF. Our implied volatility expansion is explicit; it requires no special functions nor does it require any numerical integration. Thus, approximate implied volatilities can be computed even faster than option prices.

We carry out extensive computations using the Taylor series basis functions. In particular, we establish the rigorous error bounds of our transition density expansion. We also implement our implied volatility approximation under five separate model dynamics: CEV local volatility, Quadratic local volatility, Heston stochastic volatility, 3/2 stochastic volatility, and SABR local-stochastic volatility. In each setting we demonstrate that our implied volatility expansion provides an excellent approximation of the true implied volatility over a large range of strikes and maturities.

Looking forward, we are currently working to extend our density, pricing and implied volatility approximations to Lévy-type local-stochastic volatility models. We are also examining how our approximation techniques can be applied to a variety of exotic options. Finally, we are investigating how different basis functions can be used advantageously in different settings.

6.8 Appendix

6.8.1 Implied volatility expressions

Assuming basis functions $B_{n,h}(x',y') = (x'-\bar{x})^n(y'-\bar{y})^h$ with $(\bar{x},\bar{y}) = (X_0,Y_0) := (x,y)$ we compute, explicitly

$$\sigma_0 = \sqrt{2a_{0,0}}, \qquad \sigma_1 = \sigma_{1,0} + \sigma_{0,1}, \qquad \sigma_2 = \sigma_{2,0} + \sigma_{1,1} + \sigma_{0,2},$$

where

$$\sigma_{1,0} = \left(\frac{a_{1,0}}{2\sigma_0}\right)(k-x), \qquad \sigma_{0,1} = \left(\frac{ta_{0,1}\left(c_{0,0}+2\alpha_{0,0}\right)}{4\sigma_0}\right) + \left(\frac{a_{0,1}c_{0,0}}{2\sigma_0^3}\right)(k-x),$$

and

$$\sigma_{2,0} = \left(-\frac{t\left(12+t\sigma_0^2\right)a_{1,0}^2}{96\sigma_0} + \frac{1}{6}t\sigma_0 a_{2,0}\right) + \left(\frac{-3a_{1,0}^2+4\sigma_0^2 a_{2,0}}{12\sigma_0^3}\right)(k-x)^2$$

$$\sigma_{1,1} = \left(\frac{t\left(8\sigma_0^2 a_{1,1}c_{0,0} + a_{0,1}\left(\left(4-t\sigma_0^2\right)a_{1,0}c_{0,0} - 8\sigma_0^2 c_{1,0}\right)\right)}{48\sigma_0^3}\right)$$
$$+ \left(\frac{t\left(4\sigma_0^2 a_{1,1}\left(c_{0,0}+2\alpha_{0,0}\right) + a_{0,1}\left(-5a_{1,0}\left(c_{0,0}+2\alpha_{0,0}\right) + 2\sigma_0^2\left(c_{1,0}+2\alpha_{1,0}\right)\right)\right)}{24\sigma_0^3}\right)(k-x)$$
$$+ \left(\frac{2\sigma_0^2 a_{1,1}c_{0,0} + a_{0,1}\left(-5a_{1,0}c_{0,0}+\sigma_0^2 c_{1,0}\right)}{6\sigma_0^5}\right)(k-x)^2$$

$$\sigma_{0,2} = \Big(-\frac{ta_{0,1}^2 b_{0,0}}{3\sigma_0^3} - \frac{t^2 a_{0,1}^2 b_{0,0}}{12\sigma_0} + \frac{ta_{0,2}b_{0,0}}{\sigma_0} + \frac{3ta_{0,1}^2 c_{0,0}^2}{8\sigma_0^5} - \frac{ta_{0,2}c_{0,0}^2}{3\sigma_0^3}$$
$$+ \frac{t^2 a_{0,2}c_{0,0}^2}{12\sigma_0} - \frac{ta_{0,1}c_{0,0}c_{0,1}}{6\sigma_0^3} + \frac{t^2 a_{0,1}c_{0,0}c_{0,1}}{24\sigma_0} - \frac{t^2 a_{0,1}^2 c_{0,0}\alpha_{0,0}}{8\sigma_0^3} + \frac{t^2 a_{0,2}c_{0,0}\alpha_{0,0}}{3\sigma_0}$$
$$+ \frac{t^2 a_{0,1}c_{0,1}\alpha_{0,0}}{12\sigma_0} - \frac{t^2 a_{0,1}^2\alpha_{0,0}^2}{8\sigma_0^3} + \frac{t^2 a_{0,2}\alpha_{0,0}^2}{3\sigma_0} + \frac{t^2 a_{0,1}c_{0,0}\alpha_{0,1}}{12\sigma_0} + \frac{t^2 a_{0,1}\alpha_{0,0}\alpha_{0,1}}{6\sigma_0}\Big)$$
$$+ \Big(-\frac{3ta_{0,1}^2 c_{0,0}^2}{8\sigma_0^5} + \frac{ta_{0,2}c_{0,0}^2}{3\sigma_0^3} + \frac{ta_{0,1}c_{0,0}c_{0,1}}{6\sigma_0^3} - \frac{3ta_{0,1}^2 c_{0,0}\alpha_{0,0}}{4\sigma_0^5}$$
$$+ \frac{2ta_{0,2}c_{0,0}\alpha_{0,0}}{3\sigma_0^3} + \frac{ta_{0,1}c_{0,1}\alpha_{0,0}}{6\sigma_0^3} + \frac{ta_{0,1}c_{0,0}\alpha_{0,1}}{6\sigma_0^3}\Big)(k-x)$$
$$+ \left(\frac{-9a_{0,1}^2 c_{0,0}^2 + 2\sigma_0^2\left(2a_{0,1}^2 b_{0,0} + 2a_{0,2}c_{0,0}^2 + a_{0,1}c_{0,0}c_{0,1}\right)}{12\sigma_0^7}\right)(k-x)^2.$$

Higher order terms are too long to reasonably include in this text. However, σ_3 and (for local volatility models) σ_4 can be computed easily using the Mathematica code provided free of charge on the authors' websites.

```
http://explicitsolutions.wordpress.com
www.princeton.edu/~mlorig
www.math.unipd.it/~stefanop
www.dm.unibo.it/~pascucci
```

6.8.2 Proof of Proposition 4.6

The formal adjoint of an operator $\mathcal{A}$ in $L^2(\mathbb{R}^2, \mathrm{d}x\mathrm{d}y)$ is the operator $\mathcal{A}^\dagger$ such that

$$\langle f, \mathcal{A}g \rangle = \langle \mathcal{A}^\dagger f, g \rangle, \qquad \langle u, v \rangle := \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y \, \overline{u(x,y)} v(x,y), \qquad u, v \in \mathcal{S}(\mathbb{R}^2).$$

Observe that

$$\mathcal{A}_h^\dagger = \sum_{l=0}^{h} \phi_{h-l,l}(\mathcal{D}_x, \mathcal{D}_y) B_{h-l,l}(x,y),$$

which can be deduced by integrating by parts. Now, we note that

$$\begin{aligned}
\langle \psi_{\lambda,\omega}, \mathcal{A}_h u(t,\cdot,\cdot) \rangle &= \langle \mathcal{A}_h^\dagger \psi_{\lambda,\omega}, u \rangle \\
&= \sum_{l=0}^{h} \langle \phi_{h-l,l}(\mathcal{D}_x, \mathcal{D}_y) B_{h-l,l} \psi_{\lambda,\omega}, u(t,\cdot,\cdot) \rangle \\
&= \sum_{l=0}^{h} B_{h-l,l}(i\partial_\lambda, i\partial_\omega) \phi_{h-l,l}(\lambda,\omega) \langle \psi_{\lambda,\omega}, u(t,\cdot,\cdot) \rangle \\
&= \sum_{l=0}^{h} B_{h-l,l}(i\partial_\lambda, i\partial_\omega) \phi_{h-l,l}(\lambda,\omega) \widehat{u}(t,\lambda,\omega).
\end{aligned}$$

We Fourier transform equation (4.15). Focusing first on the left-hand side, and using the above result we have

$$\begin{aligned}
\langle \psi_{\lambda,\omega}, (-\partial_t + \mathcal{A}_0) u_n(t,\cdot,\cdot) \rangle &= -\partial_t \langle \psi_{\lambda,\omega}, u_n(t,\cdot,\cdot) \rangle + \langle \mathcal{A}_0^\dagger \psi_{\lambda,\omega}, u_n(t,\cdot,\cdot) \rangle \\
&= \left(-\partial_t + \phi_{0,0}(\lambda,\omega) \right) \widehat{u}_n(t,\lambda,\omega).
\end{aligned}$$

Next, for the right-hand side of (4.15) we compute

$$-\sum_{h=1}^{n} \langle \psi_{\lambda,\omega}, \mathcal{A}_h u_{n-h}(t,\cdot,\cdot) \rangle = -\sum_{h=1}^{n} \sum_{l=0}^{h} B_{h-l,l}(i\partial_\lambda, i\partial_\omega) \phi_{h-l,l}(\lambda,\omega) \widehat{u}_{n-h}(t,\lambda,\omega).$$

Thus, we have the following ODE (in t) for $\widehat{u}_0(t,\lambda,\omega)$

$$\left(-\partial_t + \phi_{0,0}(\lambda,\omega) \right) \widehat{u}_0(t,\lambda,\omega) = 0, \qquad \widehat{u}_0(0,\lambda,\omega) = \widehat{H}(\lambda,\omega). \tag{8.59}$$

Likewise, for $\widehat{u}_n(t,\lambda,\omega)$ we have the following ODE in t

$$(-\partial_t + \phi_{0,0}(\lambda,\omega))\,\widehat{u}_n(t,\lambda,\omega) = -\sum_{h=1}^{n}\sum_{l=0}^{h} B_{h-l,l}(i\partial_\lambda, i\partial_\omega)\phi_{h-l,l}(\lambda,\omega)\widehat{u}_{n-h}(t,\lambda,\omega),$$
$$\widehat{u}_n(0,\lambda,\omega) = 0, \tag{8.60}$$

for any $n \geq 1$. The solutions of (8.59) and (8.60) are given by (4.17) and (4.18) respectively.

6.8.3 Proof of Theorem 4.8

Throughout this Appendix, we shall use the following identity repeatedly:

$$\widehat{u}_0(s,\lambda,\omega) = e^{i\lambda x + i\omega y}\widehat{u}_0(t,\lambda,\omega)\frac{1}{e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}}}. \tag{8.61}$$

We begin by computing $u_1(t,x,y)$. We have

$$\begin{aligned}
&u_1(t,x,y)\\
&\overset{1}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda\mathrm{d}\omega\, e^{i\lambda x + i\omega y}\,\widehat{u}_1(t,\lambda,\omega)\\
&\overset{2}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda\mathrm{d}\omega \int_0^t \mathrm{d}s \sum_{i=0}^{1} e^{i\lambda x + i\omega y + (t-s)\phi_0(\lambda,\omega)} B_{1-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{1-i,i}(\lambda,\omega)\widehat{u}_0(s,\lambda,\omega)\\
&\overset{3}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda\mathrm{d}\omega \int_0^t \mathrm{d}s \sum_{i=0}^{1} \widehat{u}_0(s,\lambda,\omega)\phi_{1-i,i}(\lambda,\omega)B_{1-i,i}(-i\partial_\lambda, -i\partial_\omega)e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)}\\
&\overset{4}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda\mathrm{d}\omega\, e^{i\lambda x + i\omega y}\widehat{u}_0(t,\lambda,\omega)\\
&\qquad \int_0^t \mathrm{d}s \sum_{i=0}^{1} \phi_{1-i,i}(\lambda,\omega)\frac{B_{1-i,i}(-i\partial_\lambda, -i\partial_\omega)e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)}}{e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)}}\\
&\overset{5}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda\mathrm{d}\omega\, e^{i\lambda x + i\omega y}\widehat{u}_0(t,\lambda,\omega)\int_0^t \mathrm{d}s\, \Phi_1(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)\\
&\overset{6}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda\mathrm{d}\omega\, e^{i\lambda x + i\omega y}\widehat{u}_0(t,\lambda,\omega)\mathcal{L}_1(t,x,y,\lambda,\omega)\\
&\overset{7}{=} \mathcal{L}_1(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0(t,x,y),
\end{aligned}$$

where $\mathcal{L}_1(t,x,y,\lambda,\omega)$ is given in (4.19). Because we shall repeat the above steps for higher order terms, we describe the above computation in detail. In the first equality we have expressed u_1 as an inverse Fourier transform of $\widehat{u}_1$. In the second equality we have used Proposition 4.6 to write out $\widehat{u}_1$ explicitly. In the third equality we have used integration by parts to replace $B_{1-i,i}(i\partial_\lambda, i\partial_\omega)$ acting on $\phi_{1-i,i}(\lambda,\omega)\widehat{u}_0(s,\lambda,\omega)$ by its adjoint $B_{1-i,i}(-i\partial_\lambda, -i\partial_\omega)$ acting on $e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)}$. In the fourth equality we have used (8.61). In the fifth equality we have used (4.20) to recognize the inner-most integrand as $\Phi_1(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)$. In the sixth step we have used (4.19) to recognize the inner-most integral as $\mathcal{L}_1(s,t,x,y,\lambda,\omega)$. Lastly, in the seventh equality, we have used (3.5) and the fact that $\mathcal{L}_1(s,t,x,y,\lambda,\omega)$ is the symbol of the differential operator $\mathcal{L}_{1,0}(s,t,x,y,\mathcal{D}_x,\mathcal{D}_y)$.

Now, we move on to $u_2(t,x,y)$. We have

$$u_2(t,x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \, e^{i\lambda x + i\omega y} \widehat{u}_2(t,\lambda,\omega) + u_2^A + u_2^B$$

with

$$u_2^A = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \sum_{i=0}^{2} e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} B_{2-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{2-i,i}(\lambda,\omega)\widehat{u}_0(s,\lambda,\omega),$$

$$u_2^B = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \sum_{i=0}^{1} e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} B_{1-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{1-i,i}(\lambda,\omega)\widehat{u}_1(s,\lambda,\omega).$$

Comparing with the expression for u_1, we see that u_2^A is given by

$$u_2^A = \mathcal{L}_2^A(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0(t,x,y),$$
$$\mathcal{L}_2^A(t,x,y,\lambda,\omega) := \int_0^t \mathrm{d}s \, \Phi_2(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega).$$

For u_2^B, we compute

$$\begin{aligned}
u_2^B &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \sum_{i=0}^{1}\sum_{j=0}^{1} e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} \dots \\
&\qquad B_{1-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{1-i,i}(\lambda,\omega) e^{(s-r)\phi_{0,0}(\lambda,\omega)} B_{1-j,j}(i\partial_\lambda, i\partial_\omega)\phi_{1-j,j}(\lambda,\omega)\widehat{u}_0(r,\lambda,\omega) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \sum_{i=0}^{1}\sum_{j=0}^{1} \widehat{u}_0(r,\lambda,\omega)\phi_{1-j,j}(\lambda,\omega) \cdots \\
&\qquad B_{1-j,j}(-i\partial_\lambda, -i\partial_\omega) e^{(s-r)\phi_{0,0}(\lambda,\omega)} \phi_{1-i,i}(\lambda,\omega) B_{1-i,i}(-i\partial_\lambda, -i\partial_\omega) e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \, e^{i\lambda x + i\omega y} \widehat{u}_0(t,\lambda,\omega) \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \cdots \\
&\qquad \sum_{j=0}^{1} \phi_{1-j,j}(\lambda,\omega) \frac{B_{1-j,j}(-i\partial_\lambda, -i\partial_\omega) e^{i\lambda x + i\omega y + (t-r)\phi_{0,0}(\lambda,\omega)}}{e^{i\lambda x + i\omega y + (t-r)\phi_{0,0}(\lambda,\omega)}} \cdots \\
&\qquad \sum_{i=0}^{1} \phi_{1-i,i}(\lambda,\omega) \frac{B_{1-i,i}(-i\partial_\lambda, -i\partial_\omega) e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)}}{e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)}} \cdots \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \, e^{i\lambda x + i\omega y} \widehat{u}_0(t,\lambda,\omega) \cdots \\
&\qquad \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \, \Phi_1(r,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)\Phi_1(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega) \\
&= \mathcal{L}_2^B(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0(t,x,y),
\end{aligned}$$

where

$$\mathcal{L}_2^B(t,x,y,\lambda,\omega) := \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \, \Phi_1(r,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)\Phi_1(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega).$$

Pulling both terms u_2^A and u_2^B together, we have

$$\begin{aligned} u_2(t,x,y) = u_2^A + u_2^B &= \Big(\mathcal{L}_2^A(t,x,y,\mathcal{D}_x,\mathcal{D}_y) + \mathcal{L}_2^B(t,x,y,\mathcal{D}_x,\mathcal{D}_y)\Big)u_0(t,x,y) \\ &= \mathcal{L}_2(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0(t,x,y), \end{aligned}$$

Next, we examine u_3. We have

$$u_3(t,x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega\, e^{i\lambda x + i\omega y}\widehat{u}_{3,0}(t,\lambda,\omega) + u_3^A + u_3^B + u_3^C$$

with

$$\begin{aligned} u_3^A &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \sum_{i=0}^{3} e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} B_{3-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{3-i,i}(\lambda,\omega)\widehat{u}_0(s,\lambda,\omega), \\ u_3^B &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \sum_{i=0}^{2} e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} B_{2-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{2-i,i}(\lambda,\omega)\widehat{u}_1(s,\lambda,\omega), \\ u_3^C &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \sum_{i=0}^{1} e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} B_{1-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{1-i,i}(\lambda,\omega)\widehat{u}_2(s,\lambda,\omega). \end{aligned}$$

Comparing with u_2^A and u_2^B we recognize

$$\begin{aligned} u_3^A &= \mathcal{L}_{3,0}^A(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0(t,x,y), \\ \mathcal{L}_3^A(t,x,y,\omega,\lambda) &:= \int_0^t \mathrm{d}s\, \Phi_3(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega) \\ u_3^B &= \mathcal{L}_3^B(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0(t,x,y), \\ \mathcal{L}_{3,0}^B(t,x,y,\lambda,\omega) &:= \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\, \Phi_1(r,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)\Phi_2(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega). \end{aligned}$$

For u_3^C, we compute

$$\begin{aligned} u_3^C = &\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \sum_{i=0}^{1}\sum_{j=0}^{2} e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} \\ &B_{1-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{1-i,i}(\lambda,\omega)e^{(s-r)\phi_{0,0}(\lambda,\omega)}B_{2-j,j}(i\partial_\lambda, i\partial_\omega)\phi_{2-j,j}(\lambda,\omega)\widehat{u}_0(s,\lambda,\omega) \\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\lambda \mathrm{d}\omega \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}q \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{h=0}^{1} e^{i\lambda x + i\omega y + (t-s)\phi_{0,0}(\lambda,\omega)} \\ &B_{1-i,i}(i\partial_\lambda, i\partial_\omega)\phi_{1-i,i}(\lambda,\omega)e^{(s-r)\phi_{0,0}(\lambda,\omega)}B_{1-j,j}(i\partial_\lambda, i\partial_\omega)\phi_{1-j,j}(\lambda,\omega) \\ &e^{(r-q)\phi_{0,0}(\lambda,\omega)}B_{1-h,h}(i\partial_\lambda, i\partial_\omega)\phi_{1-h,h}(\lambda,\omega)\widehat{u}_0(s,\lambda,\omega) \\ = &\mathcal{L}_3^C(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0, \end{aligned}$$

where

$$\mathcal{L}_3^C(t,x,y,\lambda,\omega) := \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\, \Phi_2(r,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)\Phi_1(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)$$

$$+\int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^r \mathrm{d}q \, \Phi_1(q,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)\Phi_1(r,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega)$$

$$\cdot \Phi_1(s,t,x,y,\lambda,\omega,-i\partial_\lambda,-i\partial_\omega).$$

Pulling all three terms u_3^A, u_3^B and u_3^C together, we see that

$$\begin{aligned}
u_3 &= u_3^A + u_3^B + u_3^C \\
&= \Big(\mathcal{L}_3^A(t,x,y,\mathcal{D}_x,\mathcal{D}_y) + \mathcal{L}_3^B(t,x,y,\mathcal{D}_x,\mathcal{D}_y) + \mathcal{L}_3^C(t,x,y,\mathcal{D}_x,\mathcal{D}_y)\Big)u_0(t,x,y), \\
&= \mathcal{L}_3(t,x,y,\mathcal{D}_x,\mathcal{D}_y)u_0(t,x,y),
\end{aligned}$$

Now, we compare (below, for simplicity, we remove the arguments x, y, λ, ω, $-i\partial_\lambda$ and $-i\partial_\omega$)

$$\begin{aligned}
\mathcal{L}_1(t) &= \int_0^t \mathrm{d}t_1 \, \Phi_1(t_1,t), \\
\mathcal{L}_2(t) &= \int_0^t \mathrm{d}t_1 \, \Phi_2(t_1,t) + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \, \Phi_1(t_2,t)\Phi_1(t_1,t), \\
\mathcal{L}_3(t) &= \int_0^t \mathrm{d}t_1 \, \Phi_3(t_1,t) + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \, (\Phi_1(t_2,t)\Phi_1(t_1,t) + \Phi_1(t_2,t)\Phi_1(t_1,t)) \\
&\quad + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \, \Phi_1(t_3,t)\Phi_1(t_2,t)\Phi_1(t_1,t).
\end{aligned}$$

From the above pattern, one guesses

$$\begin{aligned}
\mathcal{L}_4(t) &= \int_0^t \mathrm{d}t_1 \, \Phi_4(t_1,t) \\
&\quad + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \, (\Phi_3(t_2,t)\Phi_1(t_1,t) + \Phi_1(t_2,t)\Phi_3(t_1,t) + \Phi_2(t_2,t)\Phi_2(t_1,t)) \\
&\quad + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \, \big(\Phi_2(t_3,t)\Phi_1(t_2,t)\Phi_1(t_1,t) + \Phi_1(t_3,t)\Phi_2(t_2,t)\Phi_1(t_1,t) \\
&\qquad + \Phi_1(t_3,t)\Phi_1(t_2,t)\Phi_2(t_1,t)\big) \\
&\quad + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_4 \, \Phi_1(t_4,t)\Phi_1(t_3,t)\Phi_1(t_3,t)\Phi_1(t_1,t).
\end{aligned}$$

And, indeed, one can easily check that this is correct. The general expression for $\mathcal{L}_n$ is that given in Theorem 4.8. Indeed, one can check that the expression given for u_n in Theorem 4.8 satisfies Cauchy problem (4.15).

6.9 Figures and tables

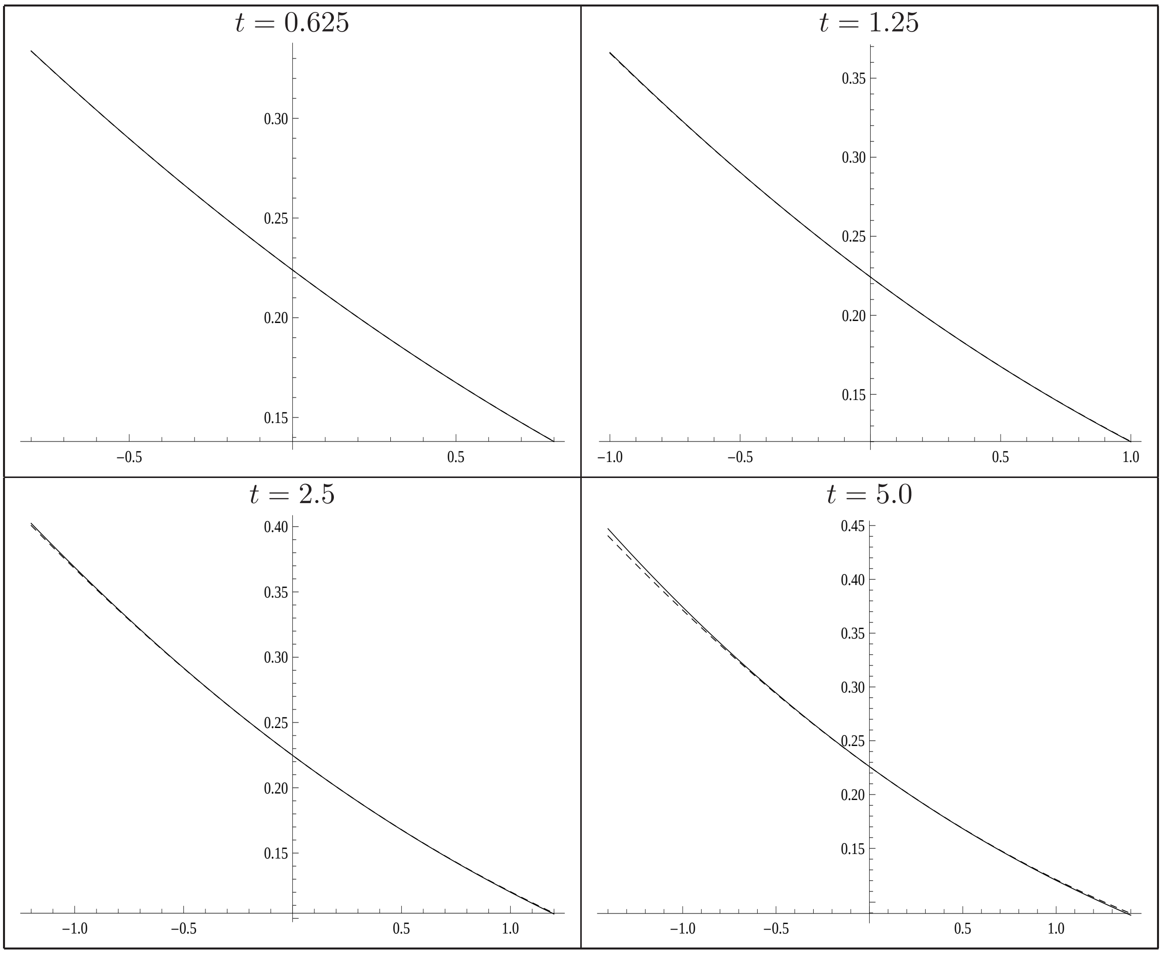

Figure 6.1: Implied volatility in the Quadratic local volatility model (6.44) is plotted as a function of log-moneyness $(k-x)$ for four different maturities t. The solid line corresponds to the exact implied volatility σ, which we obtain by computing the exact price u using (6.46) and then by solving (5.23) numerically. The dashed line (which is nearly indistinguishable from the solid line) corresponds to our third order implied volatility approximation $\sigma^{(3)}$, which we compute by summing the terms in (6.45). In all four plots we use the following parameters: $L = 2.0$, $R = 15.0$, $\delta = 0.02$, $x = 0.0$. The relative error of the approximation $\sigma^{(3)}$ is given in Figure 6.2.

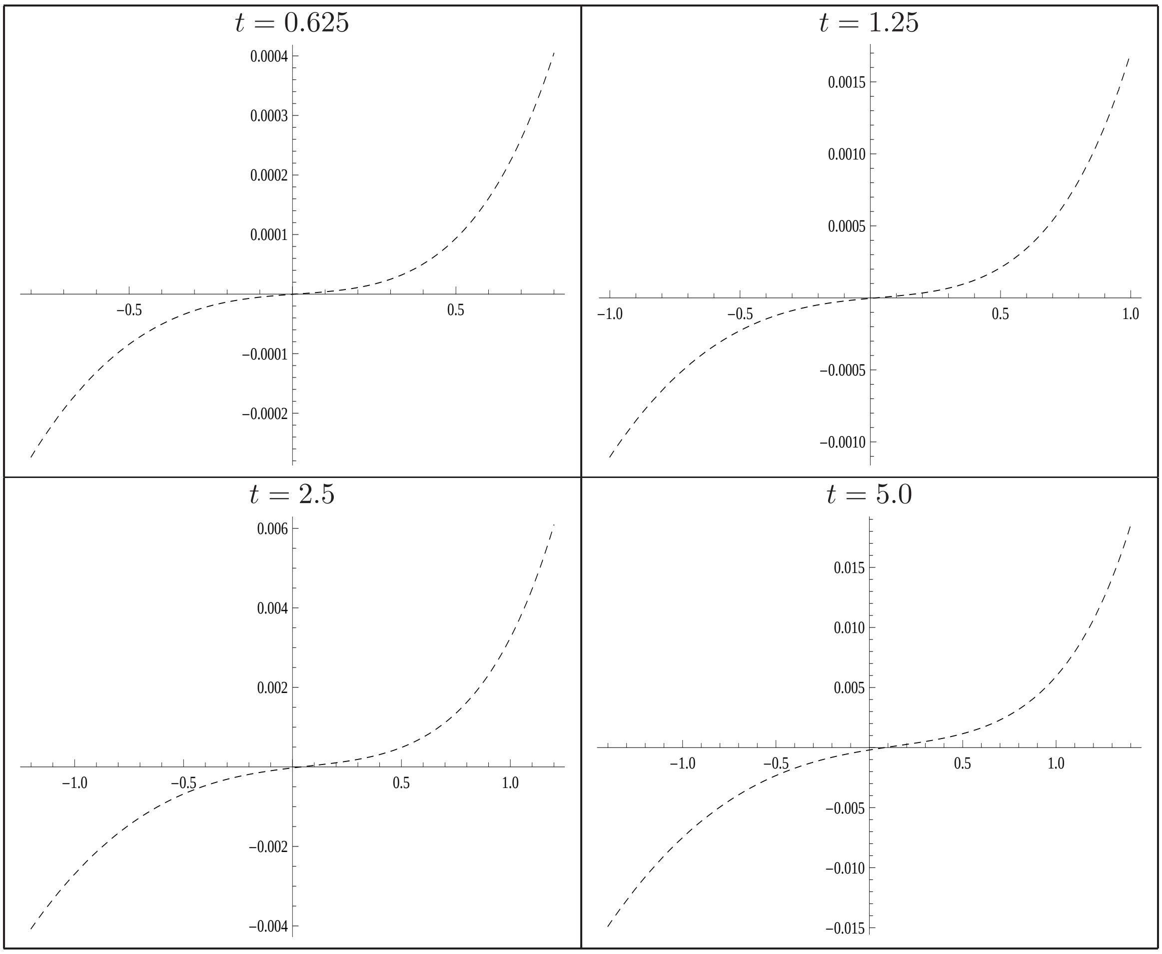

Figure 6.2: Relative error $(\sigma^{(3)} - \sigma)/\sigma$ of our third order implied volatility approximation is plotted as a function of log-moneyness $(k-x)$ for four different maturities t in the Quadratic local volatility model (6.44). The exact implied volatility σ is obtained by computing the exact price u using (6.46) and then by solving (5.23) numerically. Our third order implied volatility approximation $\sigma^{(3)}$ is computed by summing the terms in (6.45). In all four plots we use the following parameters: $L = 2.0$, $R = 15.0$, $\delta = 0.02$, $x = 0.0$.

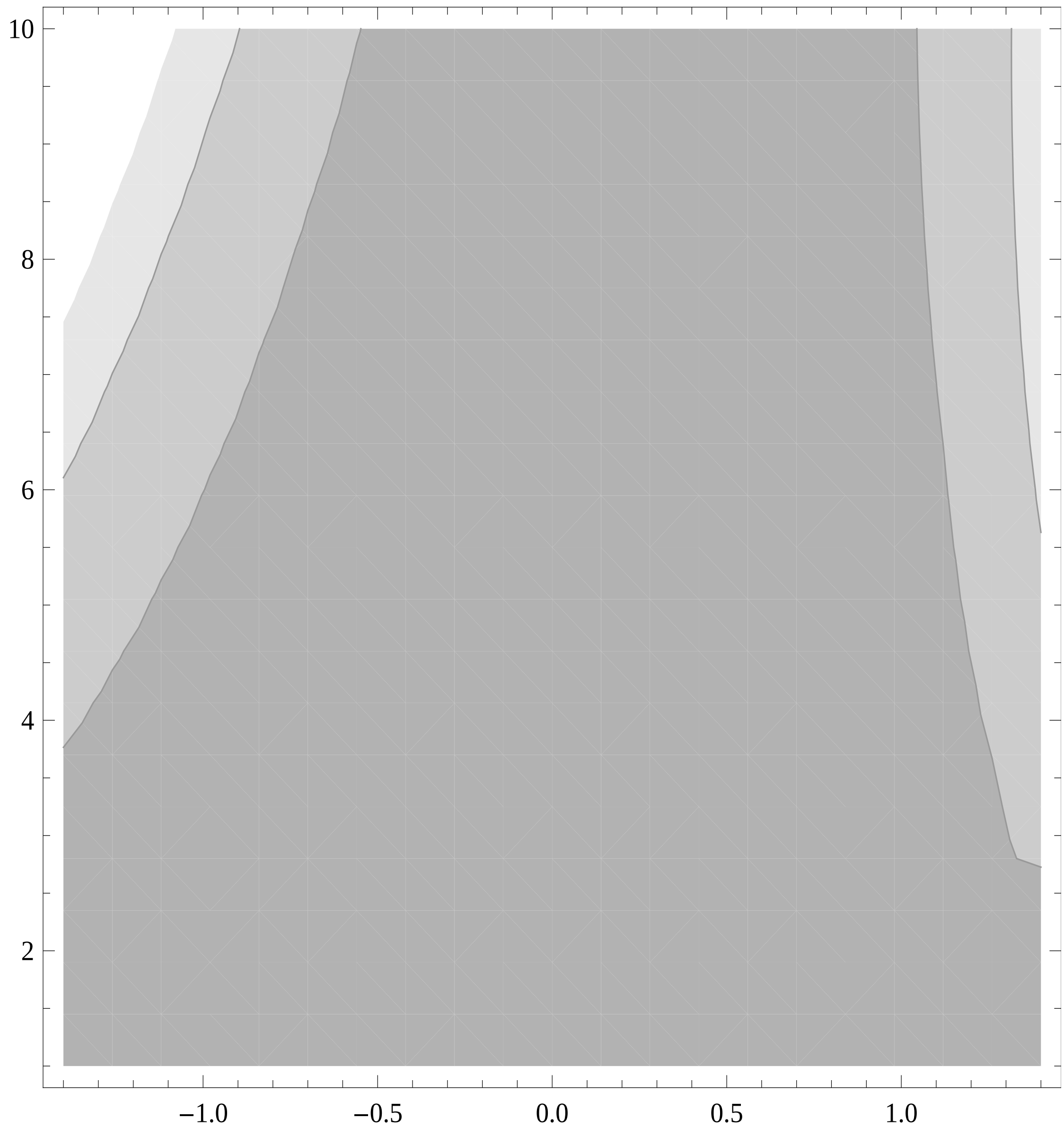

Figure 6.3: For the Quadratic local volatility model (6.44) we plot the absolute value of the relative error $|\sigma^{(3)} - \sigma|/\sigma$ of our third order implied volatility approximation as a function of log-moneyness $(k - x)$ and maturity t. The horizontal axis represents log-moneyness $(k - x)$ and the vertical axis represents maturity t. Ranging from darkest to lightest, the regions above represent relative errors of $< 1\%$, 1% to 2%, 2% to 3% and $> 3\%$. The exact implied volatility σ is obtained by computing the exact price u using (6.46) and then by solving (5.23) numerically. Our third order implied volatility approximation $\sigma^{(3)}$ is computed by summing the terms in (6.45). We use the following parameters: $L = 2.0$, $R = 15.0$, $\delta = 0.02$, $x = 0.0$.

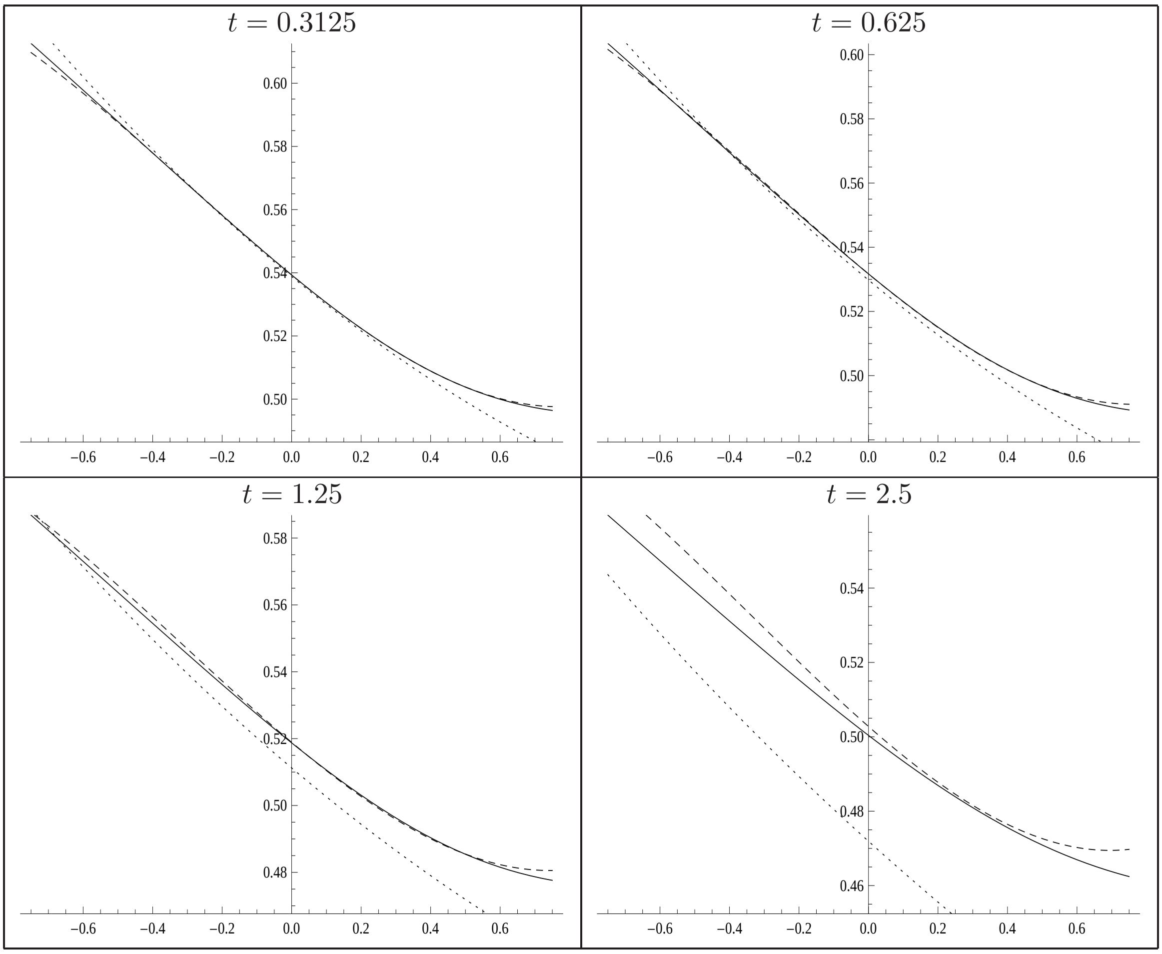

Figure 6.4: Implied volatility the Heston model (6.47) is plotted as a function of log-moneyness $(k - x)$ for four different maturities t. The solid line corresponds to the exact implied volatility σ, which we obtain by computing the exact price u using (6.49) and then by solving (5.23) numerically. The dashed line corresponds to our third order implied volatility approximation $\sigma^{(3)}$, which we compute by summing the terms in (6.48). The dotted line corresponds to the implied volatility expansion σ^{FJL} of [94], which is computed using (6.50). In all four plots we use the following parameters: $\kappa = 0.33$, $\theta = 0.3$, $\delta = 0.44$, $\rho = -0.45$ $x = 0.0$, $y = \log\theta$. Note that our third order approximation of implied volatility $\sigma^{(3)}$ captures the at-the-money level and slope of the true implied volatility, as well as the smile effect, which is seen at large strikes. Relative errors for the two approximations $\sigma^{(3)}$ and σ^{FJL} are given in Figure 6.5.

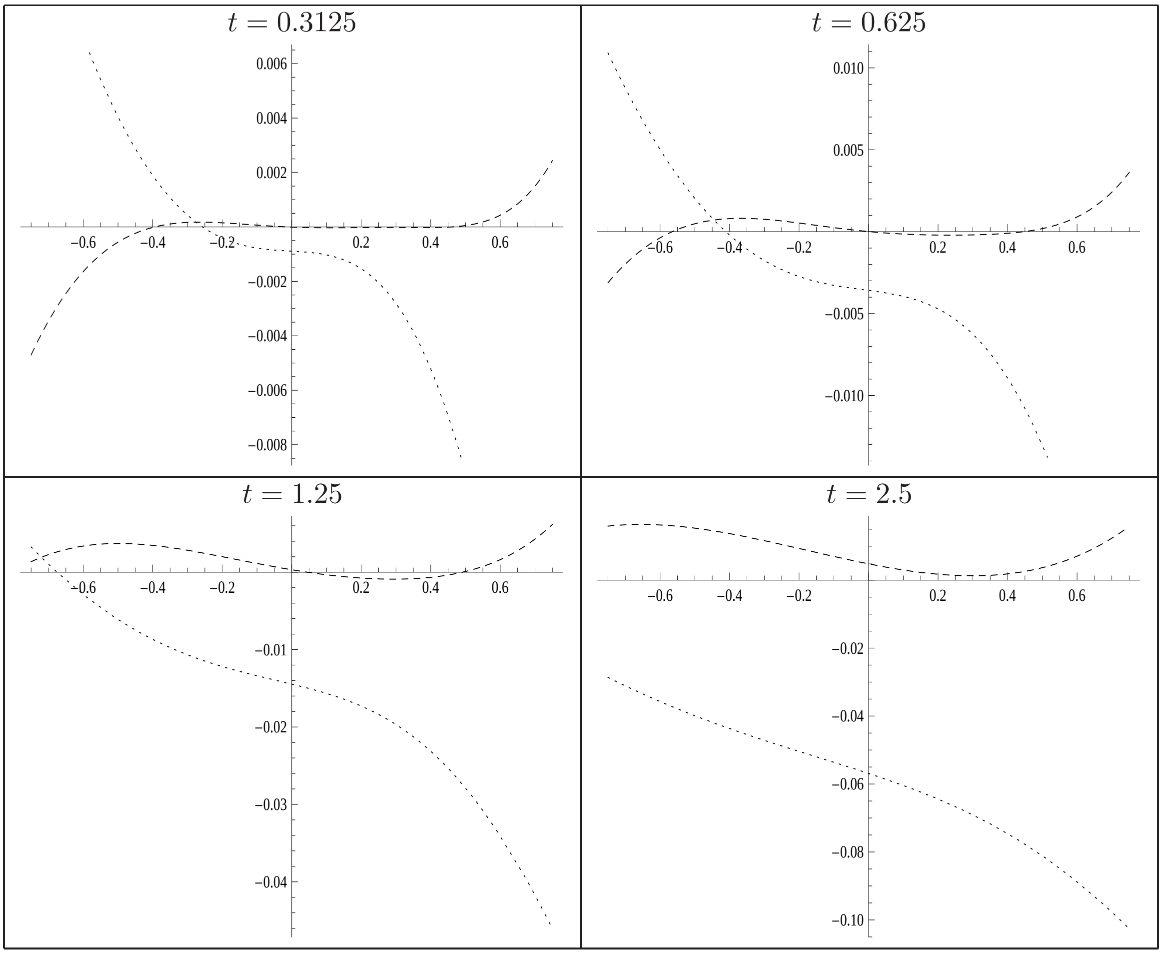

Figure 6.5: Relative error $(\sigma^{\text{Approx}} - \sigma)/\sigma$ is plotted as a function of log-moneyness $(k - x)$ for four different maturities t using two implied volatility approximations in the Heston model (6.47). The dashed line corresponds to the relative error of our third order implied volatility approximation: $\sigma^{\text{Approx}} = \sigma^{(3)}$. The dotted line corresponds to the relative error of the implied volatility approximation of [94]: $\sigma^{\text{Approx}} = \sigma^{\text{FJL}}$. The exact implied volatility σ is obtained by computing the exact price u using (6.49) and then by solving (5.23) numerically. Our third order implied volatility approximation $\sigma^{(3)}$ is computed by summing the terms in (6.48). The implied volatility expansion σ^{FJL} of [94] is computed using (6.50). In all four plots we use the following parameters: $\kappa = 0.33$, $\theta = 0.3$, $\delta = 0.44$, $\rho = -0.45$ $x = 0.0$, $y = \log\theta$. Independent of the strike an maturity, the plots demonstrate that our third order implied volatility expansion $\sigma^{(3)}$ provides a better approximation to the true implied volatility σ than does the implied volatility expansion σ^{FJL} of [94]. This difference in quality between the two implied volatility expansions is most notable at higher strikes and longer maturities.

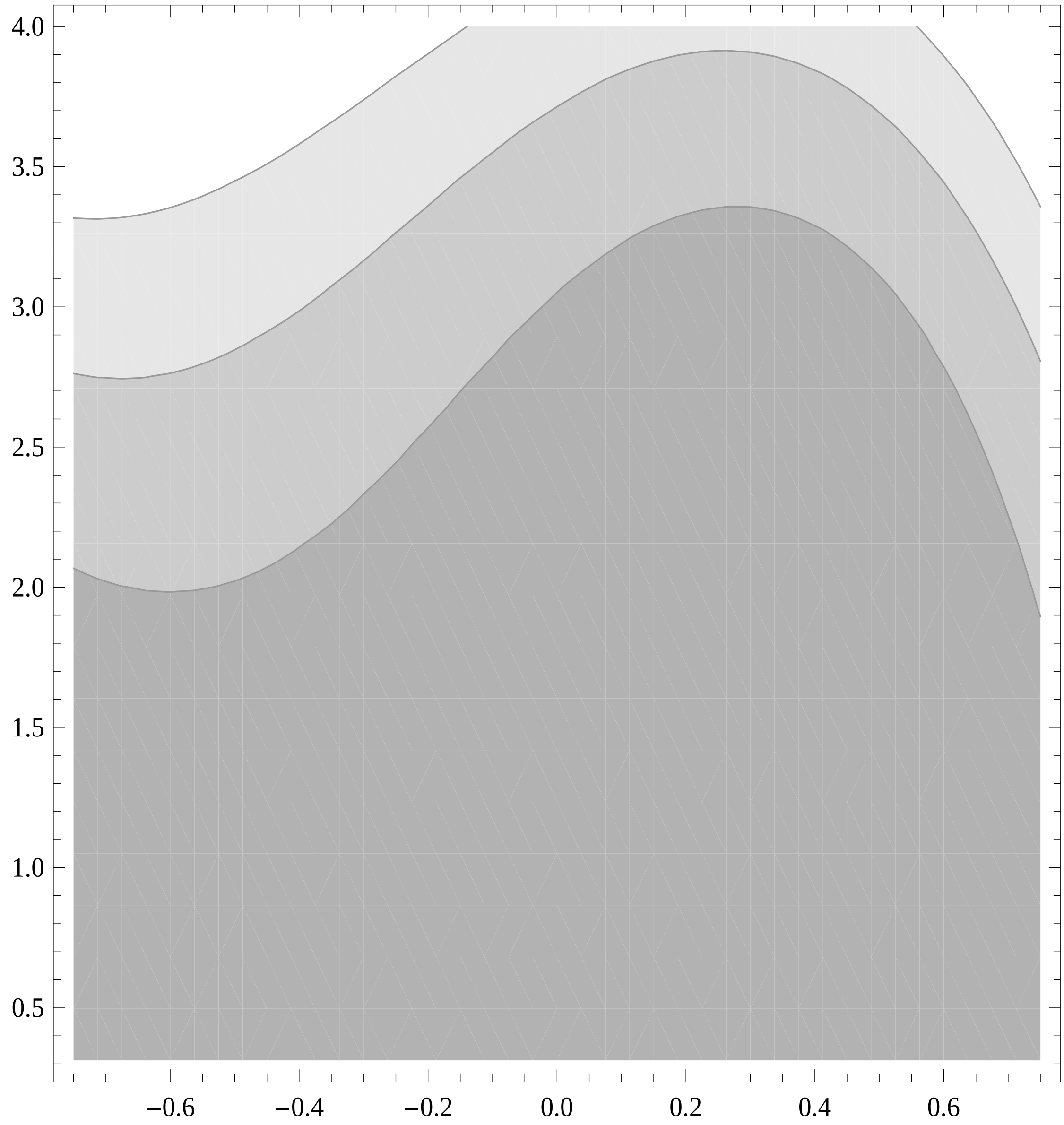

Figure 6.6: For the Heston model (6.47), we plot the absolute value of the relative error $|\sigma^{(3)} - \sigma|/\sigma$ of our third order implied volatility approximation as a function of log-moneyness $(k - x)$ and maturity t. The horizontal axis represents log-moneyness $(k - x)$ and the vertical axis represents maturity t. Ranging from darkest to lightest, the regions above represent relative errors of $< 1\%$, 1% to 2%, 2% to 3% and $> 3\%$. The exact implied volatility σ is obtained by computing the exact price u using (6.49) and then by solving (5.23) numerically. Our third order implied volatility approximation $\sigma^{(3)}$ is computed by summing the terms in (6.48). We use the following parameters: $\kappa = 0.33$, $\theta = 0.3$, $\delta = 0.44$, $\rho = -0.45$ $x = 0.0$, $y = \log \theta$.

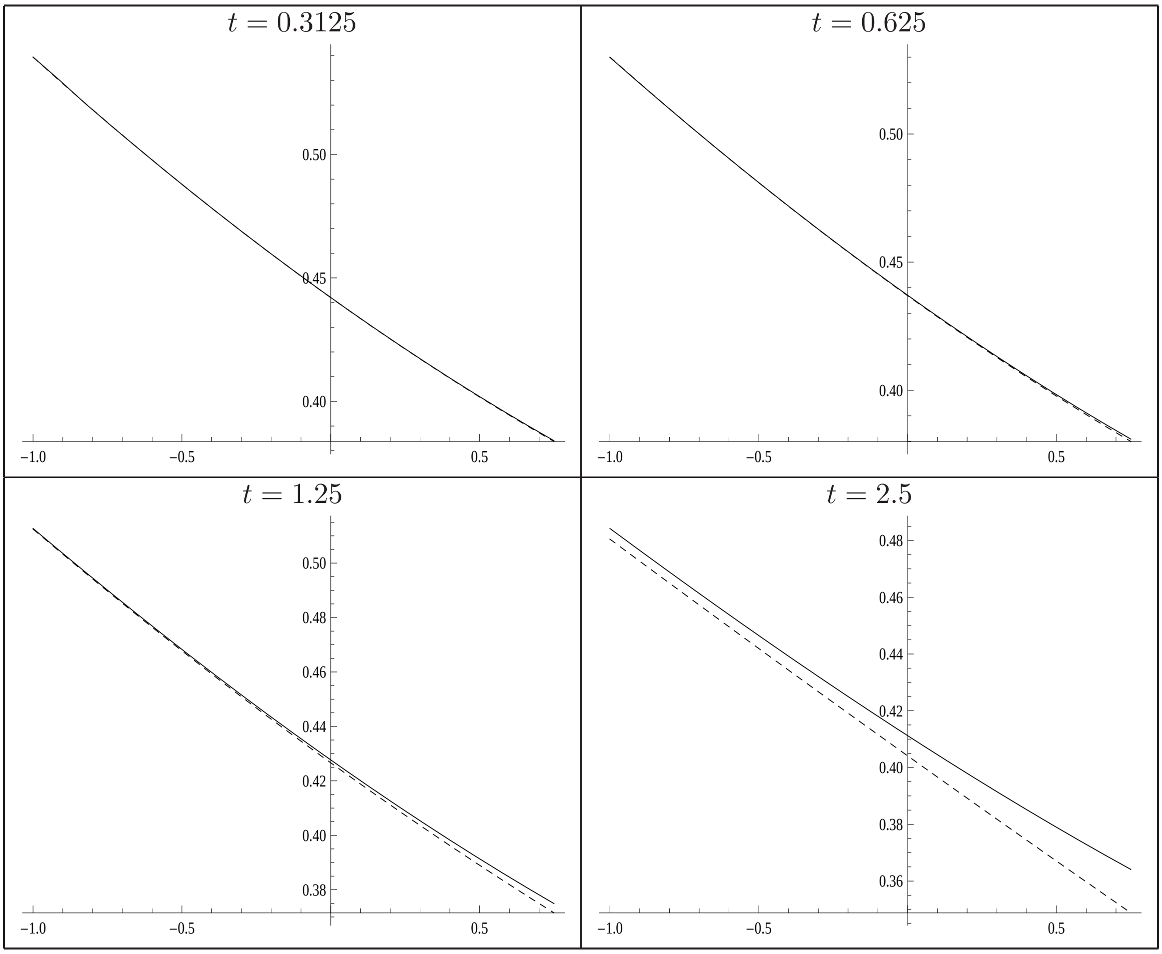

Figure 6.7: Implied volatility in the 3/2 stochastic volatility model (6.51) is plotted as a function of log-moneyness $(k-x)$ for four different maturities t. The solid line corresponds to the exact implied volatility σ, which we obtain by computing the exact price u using (6.54) and then by solving (5.23) numerically. The dashed line corresponds to our third order implied volatility approximation $\sigma^{(3)}$, which we compute by summing the terms in (6.52). In all four plots we use the following parameters: $\kappa = 0.5$, $\theta = 0.2$, $\delta = 1.00$, $\rho = -0.8$ $x = 0.0$, $y = \log\theta$. Relative error for the approximation $\sigma^{(3)}$ is given in Figure 6.8.

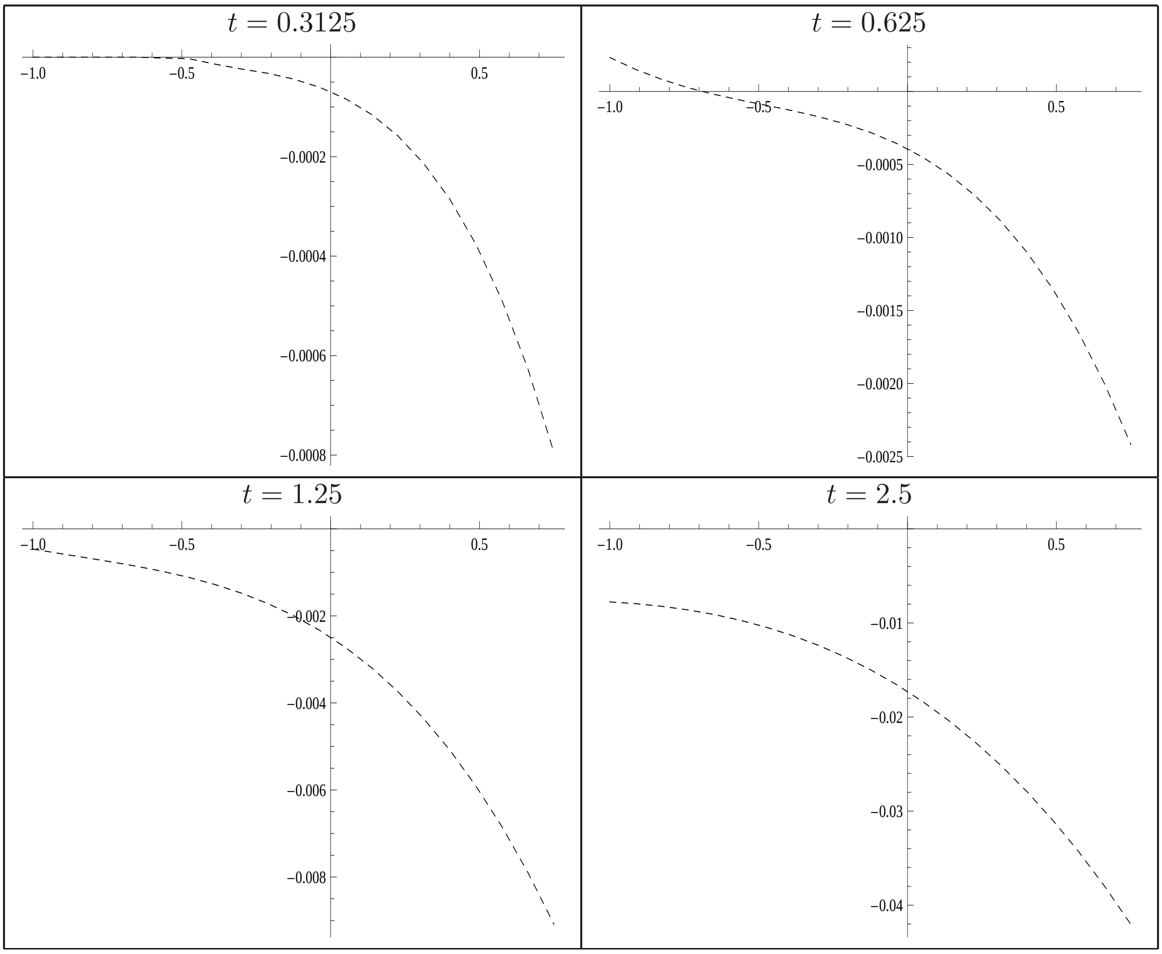

Figure 6.8: Relative error $(\sigma^{(3)} - \sigma)/\sigma$ of our third order implied volatility approximation is plotted as a function of log-moneyness $(k - x)$ for four different maturities t in the 3/2 stochastic volatility model (6.51). The exact implied volatility σ is obtained by computing the exact price u using (6.54) and then by solving (5.23) numerically. Our third order implied volatility approximation $\sigma^{(3)}$ is computed by summing the terms in (6.52). In all four plots we use the following parameters: $\kappa = 0.5$, $\theta = 0.2$, $\delta = 1.00$, $\rho = -0.8$ $x = 0.0$, $y = \log\theta$.

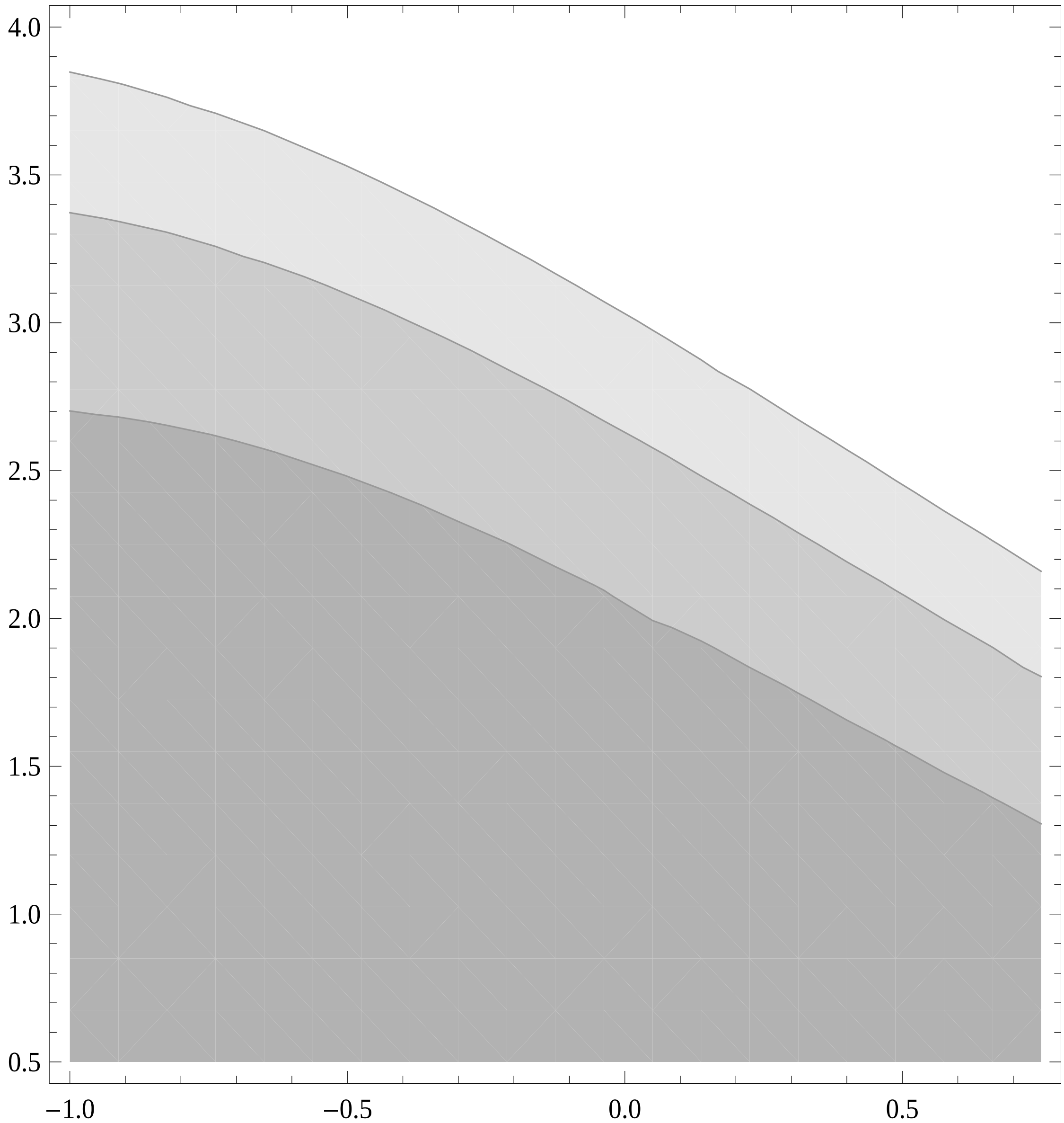

Figure 6.9: For the 3/2 stochastic volatility model (6.51), we plot the absolute value of the relative error $|\sigma^{(3)} - \sigma|/\sigma$ of our third order implied volatility approximation as a function of log-moneyness $(k - x)$ and maturity t. The horizontal axis represents log-moneyness $(k - x)$ and the vertical axis represents maturity t. Ranging from darkest to lightest, the regions above represent relative errors of $< 1\%$, 1% to 2%, 2% to 3% and $> 3\%$. The exact implied volatility σ is obtained by computing the exact price u using (6.49) and then by solving (5.23) numerically. Our third order implied volatility approximation $\sigma^{(3)}$ is computed by summing the terms in (6.48). We use the following parameters:$\kappa = 0.5$, $\theta = 0.2$, $\delta = 1.00$, $\rho = -0.8$ $x = 0.0$, $y = \log\theta$.

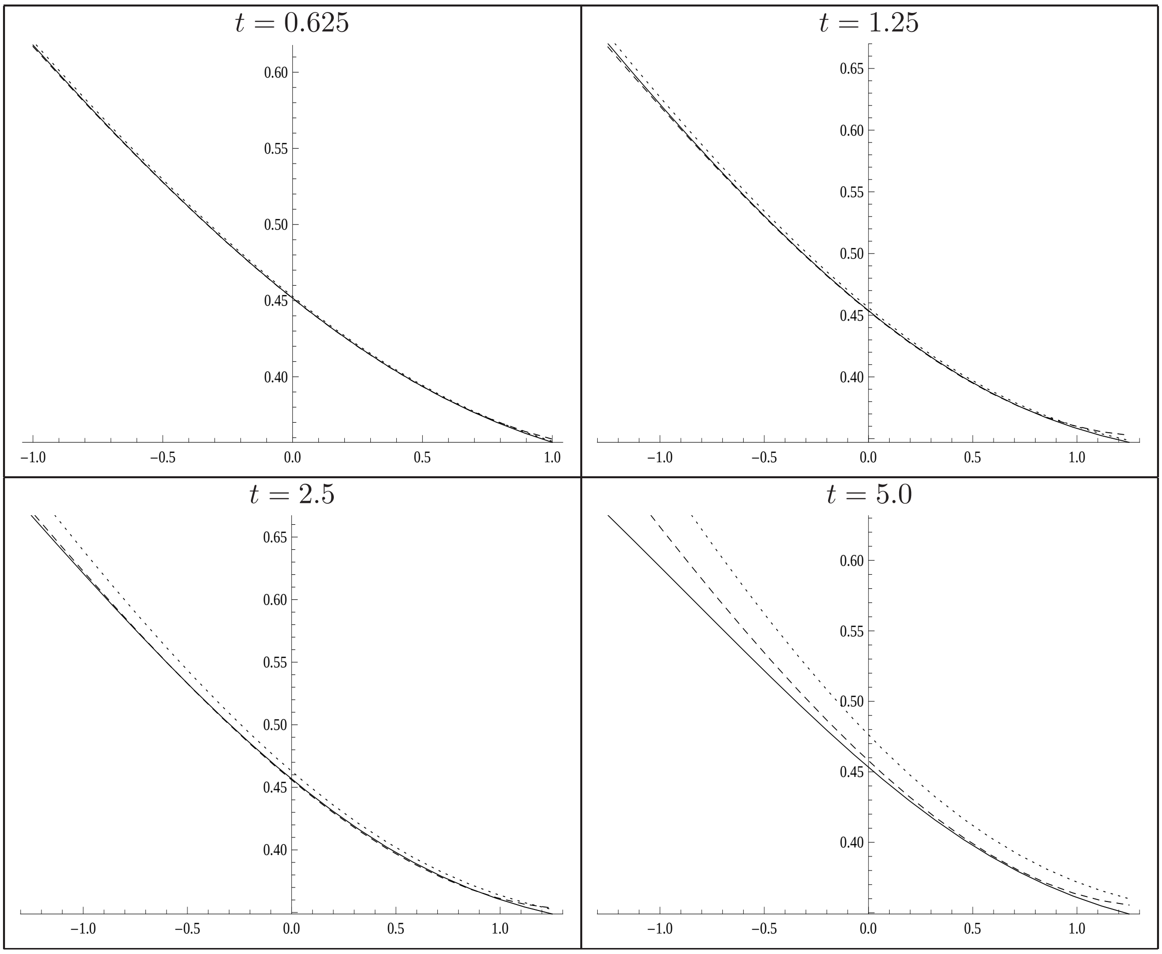

Figure 6.10: Implied volatility the SABR model (6.55) is plotted as a function of log-moneyness $(k - x)$ for four different maturities t. The solid line corresponds to the exact implied volatility σ, which we obtain by computing the exact price u using (6.57) and then by solving (5.23) numerically. The dashed line corresponds to our third order implied volatility approximation $\sigma^{(3)}$, which we compute using (6.56). The dotted line corresponds to the implied volatility expansion σ^{HKLW} of [114], which is computed using (6.58). In all four plots we use the following parameters: $\beta = 0.4$, $\delta = 0.25$, $\rho = 0.0$, $x = 0.0$, $y = -0.8$. For the two shortest maturities, both implied volatility expansions $\sigma^{(3)}$ and σ^{HKLW} provide an excellent approximation of the true implied volatility σ. However, for the two longest maturities, it is clear that our third order expansion $\sigma^{(3)}$ provides a better approximation to the true implied volatility σ than does the implied volatility expansion σ^{HKLW} of [114]. Relative errors for the two approximations $\sigma^{(3)}$ and σ^{HKLW} are given in Figure 6.11.

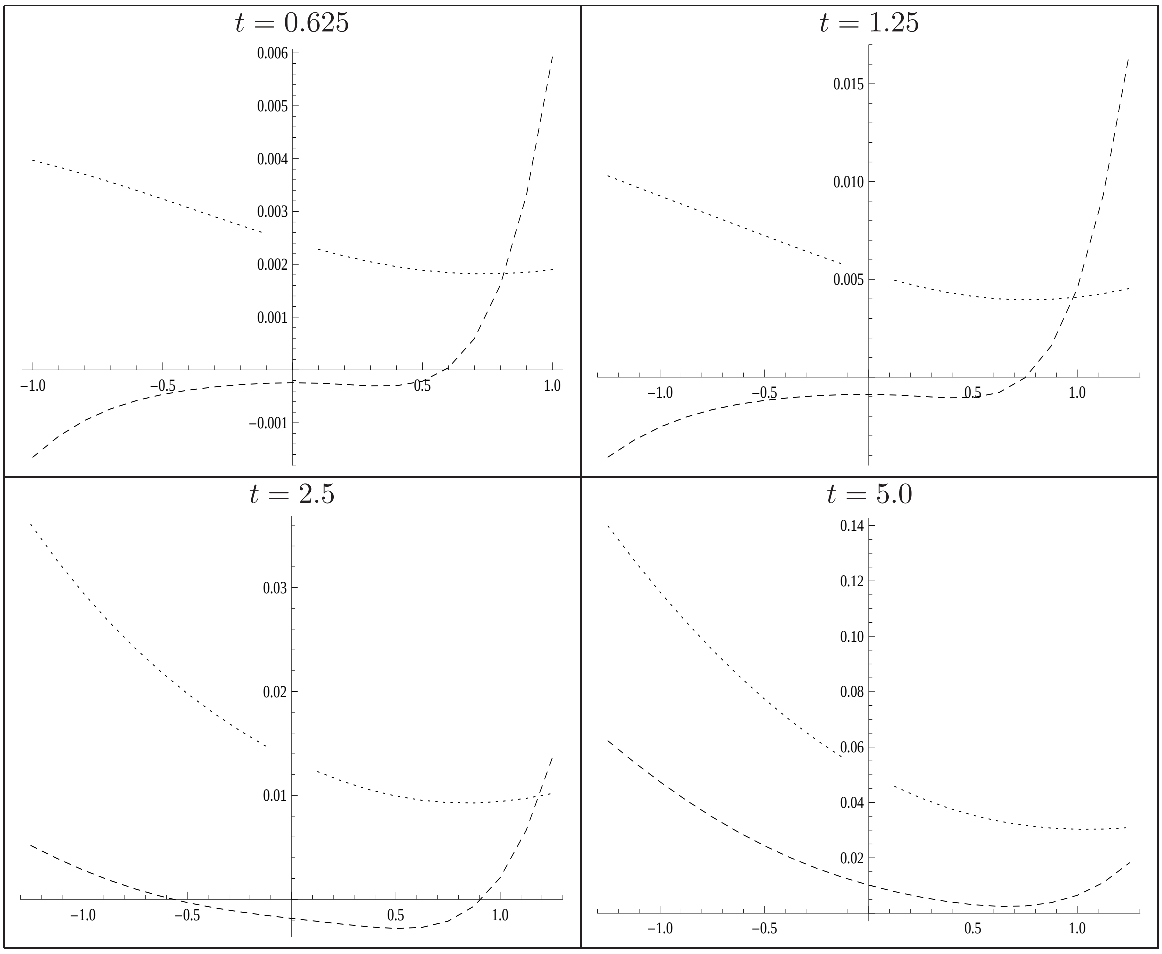

Figure 6.11: Relative error $(\sigma^{\text{Approx}} - \sigma)/\sigma$ is plotted as a function of log-moneyness $(k-x)$ for four different maturities t using two implied volatility approximations in the SABR model (6.55). The dashed line corresponds to the relative error of our third order implied volatility approximation: $\sigma^{\text{Approx}} = \sigma^{(3)}$. The dotted line corresponds to the relative error of the implied volatility approximation of [114]: $\sigma^{\text{Approx}} = \sigma^{\text{HKLW}}$. The exact implied volatility σ is obtained by computing the exact price u using (6.57) and then by solving (5.23) numerically. Our third order implied volatility approximation $\sigma^{(3)}$ is computed using (6.56). The implied volatility expansion σ^{HKLW} of [114] is computed using (6.58). In all four plots we use the following parameters: $\beta = 0.4$, $\delta = 0.25$, $\rho = 0.0$, $x = 0.0$, $y = -0.8$. For the two shortest maturities, both implied volatility approximations $\sigma^{(3)}$ and σ^{HKLW} have a relative error of less than 1% for all $(k - x) \in (-1, 1)$. However, for $t = 2.5$, the relative error of $\sigma^{(3)}$ remains less than 1% for all $(k - x) \in (-1, 1)$ whereas the relative error of σ^{HKLW} ranges from 1% to 4%. The improvement marked by $\sigma^{(3)}$ is even more pronounced at $t = 5.0$.

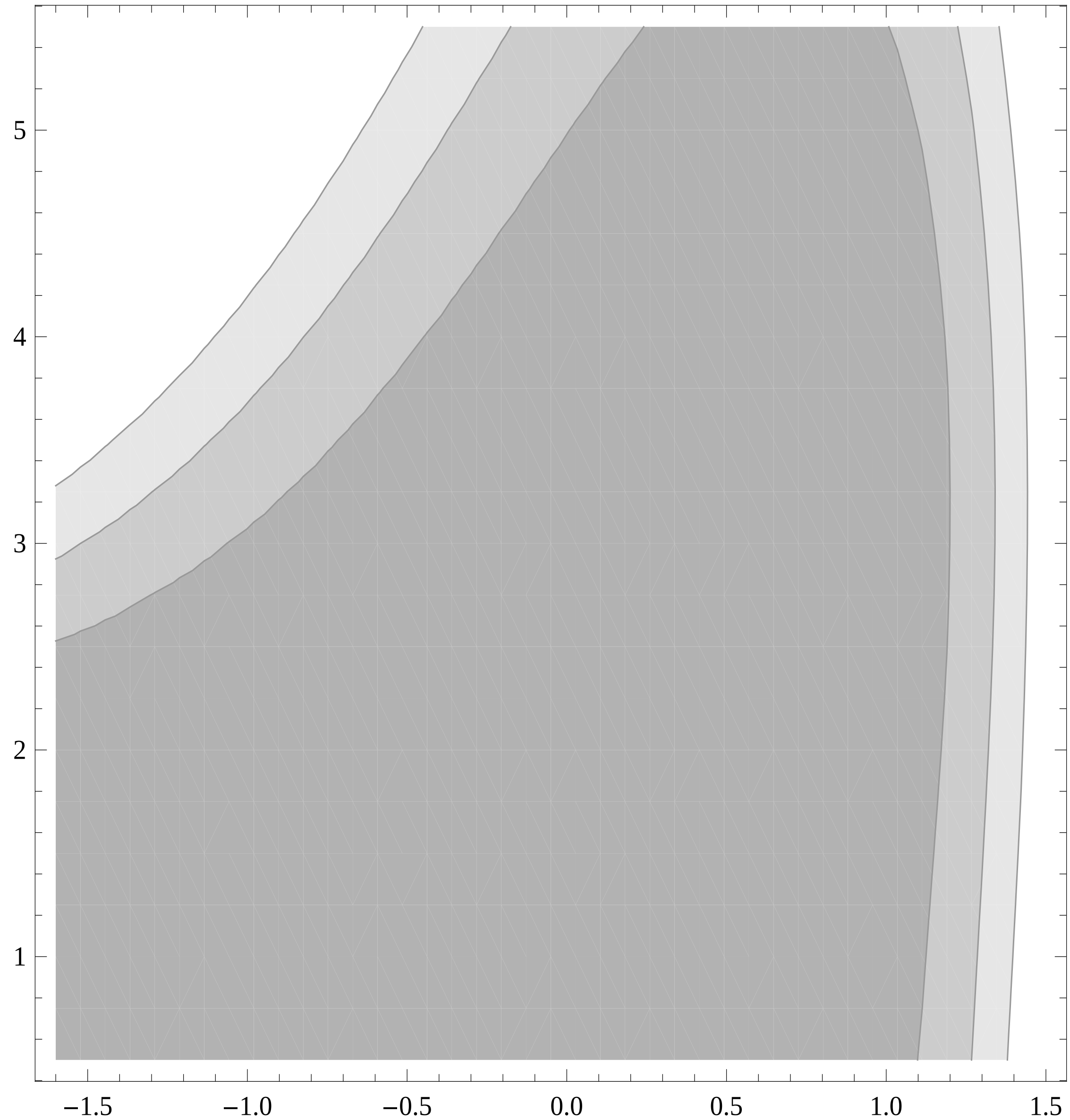

Figure 6.12: For the SABR model (6.55), we plot the absolute value of the relative error $|\sigma^{(3)} - \sigma|/\sigma$ of our third order implied volatility approximation as a function of log-moneyness $(k - x)$ and maturity t. The horizontal axis represents log-moneyness $(k - x)$ and the vertical axis represents maturity t. Ranging from darkest to lightest, the regions above represent relative errors of $< 1\%$, 1% to 2%, 2% to 3% and $> 3\%$. The exact implied volatility σ is obtained by computing the exact price u using (6.57) and then by solving (5.23) numerically. Our third order implied volatility approximation $\sigma^{(3)}$ is computed by summing the terms in (6.56). We use the following parameters: $\beta = 0.4$, $\delta = 0.25$, $\rho = 0.0$, $x = 0.0$, $y = -0.8$.

Part IV

Analytical expansions for hypoelliptic PDE's and application to pricing of Asian-style derivatives

Chapter 7

Approximations for Asian options in local volatility models

Based on a joint work ([95]) with Prof. Paolo Foschi and Prof. A. Pascucci.

Abstract: we develop approximate formulae expressed in terms of elementary functions for the density, the price and the Greeks of path dependent options of Asian style, in a general local volatility model. An algorithm for computing higher order approximations is provided. The proof is based on a Gaussian expansion method in the framework of hypoelliptic, not uniformly parabolic, partial differential equations.

Keywords: local volatility, Asian options, arithmetic average process, analytical approximation, hypoelliptic operators, ultra-parabolic operators, Black and Scholes, option pricing, Greeks.

7.1 Introduction

In this chapter we aim to adapt the methodology introduced in [183] (and extended in Chapter 3) to the case of hypoelliptic (fully degenerate-parabolic) two-dimensional operators, in order to apply it to the problem of pricing Asian-style claims under a general local volatility model.

Asian options are path dependent derivatives whose payoff depends on some form of averaging prices of the underlying asset. Asian-style derivatives are widely traded in both exchanges and over-the-counter markets and constitute an important family of contracts with several applications.

From the theoretical point of view, arithmetically-averaged Asian options have attracted an increasing interest in the last decades due to the awkward nature of the related mathematical problems. Indeed, even in the standard Black & Scholes (BS) model, when the underlying asset is a geometric Brownian motion, the distribution of the arithmetic average is not lognormal and it is quite complex to analytically characterize it. An integral representation was obtained in the pioneering work by Yor [213, 214], but with limited practical use in the valuation of Asian options.

Later on, Geman and Yor [108] gave an explicit representation of the Asian option prices in terms of the Laplace transform of hypergeometric functions. However, several authors (see Shaw [200], Fu, Madan and Wang [102], Dufresne [81]) noticed the greater difficulty of pricing Asian options with short maturities or small volatilities using the analytical method in [108]. This is also a disadvantage of the Laguerre expansion proposed by Dufresne [79]. In [201] Shaw used a contour integral approach based on Mellin transforms to improve the accuracy of the results in the case of low volatilities, albeit at a higher computational cost. Complex analysis methods were also used in [198] where series expansions for computing the Black-Scholes values based on the Geman and Yor [108] representation are derived.

Several other numerical approaches to price efficiently Asian options in the BS model have been attempted. Monte Carlo simulation techniques were discussed by Kemna and Vorst [141], Boyle, Broadie and Glasserman [37], Fu, Madan and Wang [102], Jourdain and Sbai [136] and Guasoni and Robertson [112]. Takahashi and Yoshida [204] used Monte Carlo simulation combined with an asymptotic method based on Malliavin-Watanabe calculus. Linetsky [155] analyzed the problem using the spectral theory of singular Sturm-Liouville operators and obtained an eigenfunction expansion of the Asian option pricing function in the basis of Whittaker functions: Linetsky's series formula gives very accurate results, however it may converge slowly in the case of low volatility becoming computationally expensive.

While most of the literature focuses on the log-normal dynamics and provides ad-hoc methods for pricing Asian options in the special case of the BS model, there are some notable exceptions given by the very recent papers by Hubaleck and Sgarra [126], Kim and Wee [143] (for Geometric Asian options), Bayraktar and Xing [21], Cai and Kou [39] where models with jumps are considered. Moreover Dufresne [80], Dassios and Nagaradjasarma [67] consider the square-root dynamics.

Concerning the PDE approach, the averaging price for an Asian option is usually described by introducing an additional stochastic process (cf. Dewynne and Wilmott [70]): state augmentation converts the path-dependent problem into an equivalent path-independent and Markovian problem. Increasing the dimension causes the resulting pricing PDE to be degenerate and not uniformly parabolic: theoretical results for a class of hypoelliptic PDEs, which includes Asian equations of European and American style as particular cases, were proved by Barucci, Polidoro and Vespri [19], Di Francesco, Pascucci and Polidoro [72], Pascucci [187] and Bally and Kohatsu-Higa [17]. We recall that, in the BS model and for special homogeneous payoff functions, it is possible to reduce the study of Asian options to a PDE with only one state variable: PDE reduction techniques were initiated by Ingersoll [128] and developed by Rogers and Shi [196] and Zhang [215]. Similarly, Vecer [207] used a change of numeraire technique to reduce the Asian pricing problem to a single spatial variable PDE that can be solved numerically by standard schemes; that technique was also extended in [97] to the case of a mean-reverting stochastic volatility model. Glasgow and Taylor [109], Taylor [205] and Caister, O'Hara and Govinder [40] proposed a general study of symmetries for the Asian PDE and found other nontrivial reductions of the pricing equation.

The reduced PDE formulation was used by Dewynne and Shaw [69] to derive accurate approximation formulae for Asian-rate Call options in the BS model by a matched asymptotic expansion. In general, analytical approaches based on perturbation theory and asymptotic expansions have several advantages with respect to standard numerical methods: first of all, analytical approximations give closed-form solutions that exhibit an explicit dependency of the results on the underlying parameters. Moreover analytical approaches produce much better and much faster sensitivities than numerical methods, although often accurate error estimates are not trivial to obtain. In the case of geometric Asian options under local volatility (LV) dynamics, global error bounds were recently found in [62]. Other asymptotic methods for Asian options with explicit error bounds were studied by Kunitomo and Takahashi [147], Shiraya and Takahashi [202], Shiraya, Takahashi and Toda [203] by Malliavin calculus techniques. Also Gobet and Miri [111] recently used Malliavin calculus to get analytical approximations and explicit error bounds: their approach is similar to ours as it is based on a Taylor expansion of the coefficients, but on the basis of preliminary numerical comparisons the resulting formulas seem to be different.

In this chapter we consider the pricing problem for arithmetic Asian options under a LV, possibly time-dependent, model. LV models are widely used in the industry to cope with the well-known limitations of the BS model. In this general framework, dimension reduction is not possible anymore: then our idea is to use the natural geometric-differential structure of the pricing operator regarded as a hypoelliptic (not uniformly parabolic) PDE of Kolmogorov type in $\mathbb{R}^3$. Our main results are explicit, BS-type approximation formulae not only for the option price, but also for the the terminal distribution of the asset and the average; further we also get explicit approximation formulae for the Greeks that appear to be new also in the standard log-normal case. Although we do not address the theoretical problem of the convergence to get explicit error estimates (see Remark 2.9), experimental results show that under the BS dynamics our formulae are extremely accurate if compared with other results in the literature. Under a general LV model, in comparison with Monte Carlo simulations the results are effectively exact under standard parameter regimes.

An interesting feature of our methodology is that, in the case of linear payoff functions of the form

$$\phi(S, A) = \phi_1 + \phi_2 S + \phi_3 A,$$

with $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$, the resulting approximation formula is exact at order zero and all the higher order terms are null (cf. Remark 2.4); this seems to be a significant consistency result. Since the approximation formulae for a general LV model are rather long, in this paper we only give the explicit expression in the first order case and provide a general iterative algorithm for computing the higher order approximations, which can be easily implemented in any symbolic computational software: the Mathematica notebook containing the general formulae and the experiments reported in Section 7.3 is available in the web-site of the authors (http://explicitsolutions.wordpress.com).

We also mention that our method is very general and can also be applied to other path-dependent models driven by hypoelliptic degenerate PDEs; for instance, the models proposed by Hobson and Rogers [122] and Foschi and Pascucci [96].

The remainder of the chapter is organized as follows. Section 7.2 describes arithmetic and geometric Asian options, sets up the valuation problem by PDE methods and introduces our notations. Subsection 7.2.1 presents the approximation methodology and Subsection 7.2.2 states some preliminary result on linear SDEs. Subsection 7.2.3 contains the main results of the paper and in Subsection 7.2.4 the first order approximation formulae are derived in the case of time-independent coefficients. Section 7.3 presents computational results.

7.2 Asian options and linear SDEs

We consider a standard market model where there is a risky asset S following the stochastic differential equation

$$dS_t = (r(t) - q(t))S_t dt + \sigma(t, S_t)S_t dW_t \tag{2.1}$$

under the risk-neutral measure. In (2.1), $r(t)$ and $q(t)$ denote the risk-free rate and the dividend yield at time t respectively, σ is the local volatility function and W is a standard real Brownian motion.

The averaging prices for an Asian option are usually described by the additional state process

$$dA_t = f(t, S_t)dt. \tag{2.2}$$

In particular, for the continuously sampled Asian options we typically have

$$\begin{aligned} f(t,s) &= g(t)s && \text{(arithmetic average option)},\\ f(t,s) &= g(t)\log s && \text{(geometric average option)}, \end{aligned}$$

where g is some weight function. In the sequel, for simplicity, we shall only consider the case $g \equiv 1$ even if our methodology can include a generic positive weight g. By usual no-arbitrage arguments, the price of a European Asian option with payoff function ϕ is given by

$$V(t, S_t, A_t) = e^{-\int_t^T r(\tau)d\tau} u(t, S_t, A_t)$$

where

$$u(t, S_t, A_t) = E\left[\phi\left(S_T, A_T\right) \mid S_t, A_t\right]. \tag{2.3}$$

Typical payoff functions are given by

$$\phi(S, A) = \left(\frac{A}{T} - K\right)^+ \qquad \text{(fixed strike arithmetic Call)},$$

$$\phi(S,A) = \left(S - \frac{A}{T}\right)^+ \qquad \text{(floating strike arithmetic Call)},$$
$$\phi(S,A) = \left(e^{\frac{A}{T}} - K\right)^+ \qquad \text{(fixed strike geometric Call)},$$
$$\phi(S,A) = \left(S - e^{\frac{A}{T}}\right)^+ \qquad \text{(floating strike geometric Call)}.$$

Clearly, Asian Puts can be considered as well: we recall that symmetry results, analogous to the standard Put-Call parity, between the floating and fixed-strike Asian options were proved by Henderson and Wojakowski [116].

By Feynman-Kac representation, the price function u in (2.3) is the solution to the Cauchy problem

$$\begin{cases} Lu(t,s,a) = 0, & t < T, \ s,a \in \mathbb{R}_+, \\ u(T,s,a) = \phi(s,a), & s,a \in \mathbb{R}_+, \end{cases}$$

where L is the ultra-parabolic[1] pricing operator:

$$L = \frac{\sigma^2(t,s)s^2}{2}\partial_{ss} + (r(t) - q(t))s\partial_s + f(t,s)\partial_a + \partial_t. \tag{2.4}$$

Under suitable regularity and growth conditions, existence and uniqueness of the solution to the Cauchy problem for L were proved by Barucci, Polidoro and Vespri [19].

Remark 2.1. *Consider the geometric Asian option under the BS dynamics: by the standard log-change of variable*

$$X_t = (\log S_t, A_t)$$

equations (2.1)-(2.2) *are transformed into the system of linear SDEs*

$$\begin{aligned} dX_t^1 &= \left(r(t) - q(t) - \frac{\sigma^2(t)}{2}\right)dt + \sigma(t)dW_t, \\ dX_t^2 &= X_t^1 dt. \end{aligned} \tag{2.5}$$

Thus X is a Gaussian process with 2-dimensional normal transition density Γ that is the fundamental solution of the differential operator

$$K := \frac{\sigma^2(t)}{2}\left(\partial_{x_1x_1} - \partial_{x_1}\right) + \left(r(t) - q(t)\right)\partial_{x_1} + x_1\partial_{x_2} + \partial_t, \qquad x \in \mathbb{R}^2.$$

The expression of Γ is given in Subsection 7.2.2 and explicit formulae for fixed and floating strike geometric Asian options can be easily found: Kemna and Vorst [141] have derived the first exact valuation formula for the geometric average Asian option. We also mention Angus [7] who considered more general payoffs.

7.2.1 Approximation methodology

In this subsection we derive a new expansion formula for the fundamental solution of the arithmetic Asian operator (cf. (2.4) with $f(t,s) = s$). The coefficients of the expansion will be computed explicitly in Subsection 7.2.3. We consider the operator

$$L = \frac{\alpha(t,s)}{2}\partial_{ss} + (r(t) - q(t))s\partial_s + s\partial_a + \partial_t \tag{2.6}$$

[1] L is defined on $\mathbb{R}^3$ but contains only the second order derivative w.r.t the variable s: thus L is not a uniformly parabolic operator.

where

$$\alpha(t,s)=\sigma^2(t,s)s^2 \tag{2.7}$$

We assume that α is a suitably smooth, positive function and we take the Taylor expansion of $\alpha(t,\cdot)$ about $s_0 \in \mathbb{R}_+$: then formally we get

$$L = L^0 + \sum_{k=1}^{\infty}(s-s_0)^k \alpha_k(t)\partial_{ss}$$

where, setting $\alpha_0(t)=\alpha(t,s_0)$,

$$L^0 = \frac{\alpha_0(t)}{2}\partial_{ss} + (r(t)-q(t))s\partial_s + s\partial_a + \partial_t, \tag{2.8}$$

is the leading term in the approximation of L and

$$\alpha_k(t) = \frac{1}{2k!}\,\partial_s^k \alpha(t,s_0), \qquad k \geq 1.$$

Notice that L^0 in (2.8) is the Kolmogorov operator associated to the system

$$\begin{cases} dS_t = (r(t)-q(t))\,S_t dt + \sqrt{\alpha_0(t)}dW_t, \\ dA_t = S_t dt. \end{cases} \tag{2.9}$$

Remark 2.2. *As in the geometric case,* (2.9) *is a system of linear SDEs whose solution* (S,A) *has a 2-dimensional normal transition density* Γ^0*. Moreover* Γ^0 *is the Gaussian fundamental solution of* L^0 *in* (2.8) *and its explicit expression will be given in Subsection 7.2.2.*

Following [183], the fundamental solution Γ of the pricing operator L in (2.6) admits an expansion of the form

$$\Gamma(t,s,a;T,S,A) = \sum_{n=0}^{\infty} G^n(t,s,a;T,S,A) \tag{2.10}$$

where

$$G^0(t,s,a;T,S;A) = \Gamma^0(t,s,a;T,S;A), \qquad t<T, \;\; s,a,S,A \in \mathbb{R},$$

and $G^n\,(\cdot;T,S,A)$, for any $n \geq 1$ and T,S,A, is defined recursively in terms of the following sequence of Cauchy problems posed on $]-\infty,T[\times\mathbb{R}^2$:

$$\begin{cases} L^0 G^n(t,s,a;T,S;A) = -\sum\limits_{k=1}^{n}(s-s_0)^k\alpha_k(t)\partial_{ss}G^{n-k}(t,s,a;T,S;A), \\ G^n(T,s,a;T,S;A) = 0, \qquad s,a \in \mathbb{R}. \end{cases} \tag{2.11}$$

For instance, $G^1\,(\cdot;T,S,A)$ is defined by

$$\begin{cases} L^0 G^1(t,s,a;T,S;A) = -(s-s_0)\alpha_1(t)\partial_{ss}G^0(t,s,a;T,S;A), \\ G^1(T,s,a;T,S;A) = 0, \qquad s,a \in \mathbb{R}, \end{cases} \tag{2.12}$$

and for $n=2$ we have

$$\begin{cases} L^0 G^2(t,s,a;T,S;A) = -(s-s_0)\alpha_1(t)\partial_{ss}G^1(t,s,a;T,S;A) \\ \qquad\qquad\qquad\qquad\quad -(s-s_0)^2\alpha_2(t)\partial_{ss}G^0(t,s,a;T,S;A), \\ G^2(T,s,a;T,S;A) = 0, \qquad s,a \in \mathbb{R}. \end{cases}$$

Remark 2.3. *Under the BS dynamics, the diffusion coefficient in* (2.7) *is of the form*

$$\alpha(t,s) = \sigma(t)s^2$$

where $t \mapsto \sigma(t)$ *is a deterministic function. Thus*

$$\alpha_n \equiv 0, \qquad n \geq 3,$$

and in this particular case, the sequence of Cauchy problems in (2.11) *reduces to*

$$\begin{cases} L^0 G^n(t,s,a;T,S;A) = -(s-s_0)\alpha_1(t)\partial_{ss}G^{n-1}(t,s,a;T,S;A) \\ \qquad\qquad\qquad\qquad\qquad -(s-s_0)^2\alpha_2(t)\partial_{ss}G^{n-2}(t,s,a;T,S;A), \\ G^n(T,s,a;T,S;A) = 0, \qquad s,a \in \mathbb{R}, \end{cases}$$

for $n \geq 2$. *A similar reduction holds for any diffusion coefficient of polynomial type in the variable* s.

In general, the sequence $(G^n)_{n\geq 1}$ defined by (2.11) can be computed explicitly by an iterative algorithm: this will be detailed in Subsection 7.2.3 by using the results on linear SDEs presented in Subsection 7.2.2. In particular, it turns out that

$$G^n(t,s,a;T,S;A) = J^n_{t,T,s,a}G^0(t,s,a;T,S;A), \qquad n \geq 0,$$

where $J^0_{t,T,s,a}$ is the identity operator and, for $n \geq 1$, $J^n_{t,T,s,a}$ is a differential operator, acting in the variables s, a, of the form

$$J^n_{t,T,s,a} = \sum_{k=0}^{n} s^k \sum_{i=2}^{3n}\sum_{j=0}^{i} f^n_{i-j,j,k}(t,T)\frac{\partial^i}{\partial s^{i-j}\partial a^j},$$

and the coefficients $f^n_{i-j,j,k}$ are deterministic functions whose explicit expression can be computed iteratively as in Theorem 2.7 and Remark 2.8 below. Thus, by (2.10), the N-th order approximation of Γ is given by

$$\Gamma(t,s,a;T,S,A) \approx \sum_{n=0}^{N} J^n_{t,T,s,a}G^0(t,s,a;T,S,A).$$

Moreover we have the following N-th order approximation formula for the price of an arithmetic Asian option with payoff function ϕ:

$$\begin{aligned} u(t,S_t,A_t) &= \iint \Gamma(t,s,a;T,S,A)\phi(S,A)dSdA \qquad (2.13) \\ &\approx \iint \Gamma^N(t,s,a;T,S,A)\phi(S,A)dSdA = \sum_{n=0}^{N} J^n_{t,T,s,a}C_0(t,s,a) =: u_N(t,S_t,A_t) \end{aligned}$$

where

$$C_0(t,s,a) = \iint \Gamma^0(t,s,a;T,S,A)\phi(S,A)dSdA.$$

Notice that C_0 is the price of a *geometric* Asian option under the BS dynamics and therefore for typical payoff functions it has a closed form expression. Similarly we obtain explicit approximation formulae for the Greeks and for any other payoff which admits an explicit pricing formula in the geometric case.

Remark 2.4. *Let us consider an affine payoff function of the form*

$$\phi(S, A) = \phi_1 + \phi_2 S + \phi_3 A,$$

with $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$. Then a direct computation shows that

$$C_0(t, s, a) = \phi_1 + \langle (\phi_2, \phi_3), m_{t,s,a}(T) \rangle$$

with $m_{t,s,a}(T)$ as in (2.22)*. Since $C_0(t, s, a)$ is again an affine function of (s, a), we have that*

$$J^n_{t,T,s,a} C_0(t, s, a) = 0, \qquad \forall\, n \geq 1,$$

and therefore $u_N \equiv u_0$ in (2.13)*, for any $N \in \mathbb{N}$. Moreover, by the uniqueness of the solution of the Cauchy problem for L, we also have $u_0 = u$, that is when the payoff is an affine function of S and A, then the first approximation is exact and all the higher order terms are null. Roughly speaking, this property follows from the fact that the differential operators L and L^0 have the same first order part and only differ in the coefficient of their second order derivative.*

7.2.2 Non-degeneracy conditions for linear SDEs

In this subsection we collect some preliminary results on linear SDEs that will be used in the derivation of the approximation formulae for the arithmetic density. First notice that equations (2.5) and (2.9) belong to the general class of linear SDEs

$$dX_t = (B(t)X_t + b(t))\, dt + \sigma(t) dW_t, \tag{2.14}$$

where b, B and σ are L^∞_{loc}-functions with values in the space of $(N \times 1)$, $(N \times N)$ and $(N \times d)$-dimensional matrices respectively and W is a d-dimensional uncorrelated Brownian motion, with $d \leq N$. The solution $X = X^{t,x}$ to (2.14) with initial condition $x \in \mathbb{R}^N$ at time t, is given explicitly by

$$X_T = \Phi(t, T) \left(x + \int_t^T \Phi^{-1}(t, \tau) b(\tau) d\tau + \int_t^T \Phi^{-1}(t, \tau) \sigma(\tau) dW_\tau \right),$$

where $T \mapsto \Phi(t, T)$ is the matrix-valued solution to the deterministic Cauchy problem

$$\begin{cases} \frac{d}{dT} \Phi(t, T) = B(T) \Phi(t, T), \\ \Phi(t, t) = I_N. \end{cases}$$

Moreover $X^{t,x}$ is a Gaussian process with expectation

$$m_{t,x}(T) := E\left[X^{t,x}_T \right] = \Phi(t, T) R(x, t, T),$$

where

$$R(x, t, T) = x + \int_t^T \Phi^{-1}(t, \tau) b(\tau) d\tau$$

and covariance matrix

$$C(t, T) = \operatorname{cov}\left(X^{t,x}_T \right) = \Phi(t, T) M(t, T) \Phi(t, T)^*, \tag{2.15}$$

where

$$M(t,T)=\int_t^T \Phi^{-1}(t,\tau)\sigma(\tau)\left(\Phi^{-1}(t,\tau)\sigma(\tau)\right)^* d\tau.$$

The Kolmogorov operator associated with X is

$$\begin{aligned} K &= \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}(t)\partial_{x_i x_j} + \langle b(t)+B(t)x,\nabla\rangle + \partial_t \\ &= \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}(t)\partial_{x_i x_j} + \sum_{i=1}^{N} b_i(t)\partial_{x_i} + \sum_{i,j=1}^{N} B_{ij}(t)x_j\partial_{x_i} + \partial_t, \end{aligned} \tag{2.16}$$

where $(c_{ij})=\sigma\sigma^*$.

Now we assume the following crucial condition:

[H.1] *the matrix $C(t,T)$ (or equivalently, the matrix $M(t,T)$) is positive definite for any $T>t$.*

Under this condition, $X_T^{t,x}$ has a transition density given by

$$\Gamma_K(t,x,T,y)=\frac{1}{\sqrt{(2\pi)^N \det C(t,T)}}\, e^{-\frac{1}{2}\langle C^{-1}(t,T)(y-m_{t,x}(T)),y-m_{t,x}(T)\rangle}. \tag{2.17}$$

Γ_K is also the fundamental solution of K in (2.16). Condition [H.1] can be expressed in geometric-differential terms: in fact, it is known that [H.1] is equivalent to the following condition due to Hörmander [124]

[H.2] $\qquad \operatorname{rank}\mathcal{L}(Y_1,\cdots,Y_d,Y)(t,x)=N+1, \qquad (t,x)\in\mathbb{R}^{N+1},$

where $\mathcal{L}(Y_1,\cdots,Y_d,Y)$ denotes the Lie algebra generated by the vector fields in $\mathbb{R}^{N+1}$

$$Y_i=\sum_{j=1}^{N}\sigma_{ji}\partial_{x_j}, \qquad i=1,\dots,d$$

and

$$Y=\langle B(t)x+b(t),\nabla\rangle+\partial_t.$$

In other terms, $\mathcal{L}(Y_1,\cdots,Y_d,Y)(t,x)$ is the vector space spanned by the vector fields $Y_1,\cdots,Y_d$, Y, by their first order commutators $[Y_k,Y], k=1,\dots,d$, where $[Y_k,Y]u := Y_kYu-YY_ku$ and by their higher order commutators $[Y_j,...,[Y_k,Y]...]$, evaluated at the point (t,x).

Hörmander's condition and [H.1] are also equivalent to another condition from control theory: for any $T>0$, a curve $x:[0,T]\longmapsto\mathbb{R}^N$ is called K-admissible if it is absolutely continuous and satisfies

$$x'(t)=B(t)x(t)+b(t)+\sigma(t)w(t), \qquad \text{a.e. in } [0,T],$$

for a suitable function w with values in $\mathbb{R}^d$ (notice the close analogy with the SDE (2.14)). The function w is called the *control* of the path x. A fundamental result by Kalman, Ho and Narendra [139] states that [H.1] is equivalent to the following condition:

[H.3] *for every* $x_0, x_1 \in \mathbb{R}^N$ *and* $T > 0$*, there exists a* K*-admissible path such that* $x(0) = x_0$ *and* $x(T) = x_1$.

When B and σ are constant matrices, then [H.3] is equivalent to the well known Kalman's rank condition (we also refer to LaSalle [149] where this result first appeared)

$$\operatorname{rank}\left(\sigma \;\; B\sigma \;\; \dots \;\; B^{N-1}\sigma\right) = N.$$

An analogous condition for time-dependent matrices $\sigma(t)$ and $B(t)$ was given by Coron [63], Agrachev and Sachkov [1].

The following simple result will be crucial in the sequel.

Proposition 2.5. *Under assumption [H.1], we have*

$$\nabla_y \Gamma_K(t,x,T,y) = -\left(\Phi^{-1}(t,T)\right)^* \nabla_x \Gamma_K(t,x,T,y), \tag{2.18}$$

$$y\Gamma_K(t,x,T,y) = \Phi(t,T)\left(R(x,t,T) + M(t,T)\nabla_x\right)\Gamma_K(t,x,T,y), \tag{2.19}$$

for any $x, y \in \mathbb{R}^N$ *and* $t < T$.

Proof. The density Γ_K in (2.17) can be rewritten in the equivalent form

$$\Gamma_K(t,x,T,y) = \frac{e^{-\frac{1}{2}\langle M^{-1}(t,T)\left(\Phi^{-1}(t,T)y - R(x,t,T)\right), \Phi^{-1}(t,T)y - R(x,t,T)\rangle}}{\sqrt{(2\pi)^N \det C(t,T)}}.$$

By differentiating, we get

$$\frac{\nabla_x \Gamma_K(t,x,T,y)}{\Gamma_K(t,x,T,y)} = M^{-1}(t,T)\left(\Phi^{-1}(t,T)y - R(x,t,T)\right) \tag{2.20}$$

and

$$\frac{\nabla_y \Gamma_K(t,x,T,y)}{\Gamma_K(t,x,T,y)} = -\left(\Phi^{-1}(t,T)\right)^* M^{-1}(t,T)(\Phi^{-1}(t,T)y - R(x,t,T))$$

(by (2.20))

$$= -\left(\Phi^{-1}(t,T)\right)^* \frac{\nabla_x \Gamma_K(t,x,T,y)}{\Gamma_K(t,x,T,y)},$$

and this proves (2.18). Formula (2.19) follows immediately from (2.20).

7.2.3 Approximation formulae for the density

We consider the operator

$$L^0 = \frac{\alpha_0(t)}{2}\partial_{ss} + \mu(t)s\partial_s + s\partial_a + \partial_t, \qquad (t,s,a) \in \mathbb{R}^3, \tag{2.21}$$

that is the leading term in the approximation of arithmetic Asian options, as in (2.8) with

$$\mu = r - q.$$

According to notations of Subsection 7.2.2, we have $b = 0$ and

$$B(t) = \begin{pmatrix} \mu(t) & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence we have

$$\Phi(t,T) = \begin{pmatrix} e^{\int_t^T \mu(\tau)d\tau} & 0 \\ \int_t^T e^{\int_t^{\tau_1} \mu(\tau_2)d\tau_2} d\tau_1 & 1 \end{pmatrix},$$
$$m_{t,s,a}(T) = \begin{pmatrix} s\, e^{\int_t^T \mu(\tau)d\tau} \\ a + s \int_t^T e^{\int_t^{\tau_1} \mu(\tau_2)d\tau_2} d\tau_1 \end{pmatrix},$$
$$M(t,T) = \int_t^T \alpha_0(\tau) e^{-2\int_t^\tau \mu(\tau_1)d\tau_1} \begin{pmatrix} 1 & -\int_t^\tau e^{\int_t^{\tau_1} \mu(\tau_2)d\tau_2} d\tau_1 \\ -\int_t^\tau e^{\int_t^{\tau_1} \mu(\tau_2)d\tau_2} d\tau_1 & \left(\int_t^\tau e^{\int_t^{\tau_1} \mu(\tau_2)d\tau_2} d\tau_1\right)^2 \end{pmatrix} d\tau. \tag{2.22}$$

It is easy to verify that $M(t,T)$ (and the covariance matrix $C(t,T)$) is positive definite by checking Hörmander's condition [H.2]: indeed, the commutator of the vector fields

$$Y_1 = \sqrt{\alpha_0(t)}\partial_s, \qquad Y = \mu(t)s\partial_s + s\partial_a + \partial_t$$

is equal to

$$[Y_1, Y] = \sqrt{\alpha_0(t)}\,(\mu(t)\partial_s + \partial_a)$$

and therefore, assuming that $\alpha_0 > 0$, then the rank of the Lie algebra generated by Y_1 and Y is equal to three.

If μ and α_0 are constant, all computations can be carried out more explicitly and we have

$$\begin{aligned}
\Phi(t,T) = e^{(T-t)B} &= \begin{pmatrix} e^{\mu(T-t)} & 0 \\ \frac{e^{\mu(T-t)}-1}{\mu} & 1 \end{pmatrix} \\
m_{t,s,a}(T) &= \begin{pmatrix} e^{\mu(T-t)} s \\ a + \frac{\left(e^{\mu(T-t)}-1\right)s}{\mu} \end{pmatrix} \\
M(t,T) &= \alpha_0 \begin{pmatrix} \frac{1-e^{-2\mu(T-t)}}{2\mu} & -\frac{\left(1-e^{-\mu(T-t)}\right)^2}{2\mu^2} \\ -\frac{\left(1-e^{-\mu(T-t)}\right)^2}{2\mu^2} & \frac{4e^{-\mu(T-t)}-e^{-2\mu(T-t)}+2\mu(T-t)-3}{2\mu^3} \end{pmatrix}.
\end{aligned} \tag{2.23}$$

In particular, for $\mu = 0$ we get

$$\Phi(t,T) = \begin{pmatrix} 1 & 0 \\ T-t & 1 \end{pmatrix}, \qquad M(t,T) = \alpha_0 \begin{pmatrix} T-t & -\frac{(T-t)^2}{2} \\ -\frac{(T-t)^2}{2} & \frac{(T-t)^3}{3} \end{pmatrix}.$$

Now let us recall the notation $\Gamma^0(t,s,a;T,S,A)$ for the fundamental solution of L^0 in (2.21). In Corollary 2.6 below we reformulate more explicitly the properties of Γ_0 stated in Proposition 2.5. To this end and to shorten notations, we introduce the operator

$$V_{t,T,s,a} = \left(\Phi(t,T)\left(\begin{pmatrix} s \\ a \end{pmatrix} + M(t,T)\nabla_{s,a}\right)\right)_1, \tag{2.24}$$

where in general, for a given vector Z, we use the subscript Z_1 to denote its first component. Moreover, we define the differential operator $W^{i,j}_{t,T}$ as the composition

$$W^{i,j}_{t,T,s,a} = W^i_{1,t,T,s,a} W^j_{2,t,T,s,a} \tag{2.25}$$

of the first order operators

$$W_{k,t,T,s,a} = \left(\left(\Phi^{-1}(t,T) \right)^* \nabla_{s,a} \right)_k \qquad k = 1, 2. \tag{2.26}$$

As a direct application of Proposition 2.5, we have the following results which shows how the product and the derivatives with respect to the second set of variables of $\Gamma^0(t,s,a;T,S,A)$ can be expressed in terms of the operators V in (2.24) and W and (2.25), acting in the first set of variables (ie, t, s, a).

Corollary 2.6. *For any $t < T$, $s, a, S, A \in \mathbb{R}$ and $i, j \in \mathbb{N} \cup \{0\}$ we have*

$$S^i \Gamma^0(t,s,a;T,S,A) = V^i_{t,T,s,a} \Gamma^0(t,s,a;T,S,A), \tag{2.27}$$

$$\frac{\partial^{i+j}}{\partial S^i \partial A^j} \Gamma^0(t,s,a;T,S,A) = (-1)^{i+j} W^{i,j}_{t,T,s,a} \Gamma^0(t,s,a;T,S,A). \tag{2.28}$$

Next we prove our main result.

Theorem 2.7. *For any $n \geq 0$, the solution G^n of problem* (2.11) *is given by*

$$G^n(t,s,a;T,S,A) = J^n_{t,T,s,a} \Gamma^0(t,s,a;T,S,A) \tag{2.29}$$

where Γ^0 is the fundamental solution of L^0 in (2.21)*, $J^0_{t,T,s,a}$ is the identity operator and, for $n \geq 1$, $J^n_{t,T,s,a}$ is a differential operator of the form*

$$J^n_{t,T,s,a} = \sum_{k=0}^{n} s^k \sum_{i=2}^{3n} \sum_{j=0}^{i} f^n_{i-j,j,k}(t,T) \frac{\partial^i}{\partial s^{i-j} \partial a^j}. \tag{2.30}$$

The coefficients $f^n_{i-j,j,k}(t,T)$ in (2.30) *are deterministic functions that can be determined iteratively by using the following alternative expression of J^n, $n \geq 1$, given in terms of the operators V and W in* (2.24)-(2.25)*:*

$$J^n_{t,T,s,a} = \sum_{i=1}^{n} \int_t^T \alpha_i(\tau) (V_{t,\tau,s,a} - s_0)^i W^{2,0}_{t,\tau,s,a} \hat{J}^{n-i}_{t,\tau,T,s,a} d\tau \tag{2.31}$$

where $\hat{J}^0_{t,\tau,T,s,a}$ is the identity operator and

$$\hat{J}^n_{t,\tau,T,s,a} = \sum_{k=0}^{n} \sum_{i=2}^{3n} \sum_{j=0}^{i} f^n_{i-j,j,k}(t,T) V^k_{t,\tau,s,a} W^{i-j,j}_{t,\tau,s,a}. \tag{2.32}$$

Proof. We first remark that, if we assume $J^n_{t,T,s,a}$ and $\hat{J}^n_{t,\tau,T,s,a}$ as in (2.30) and (2.32) respectively, then by Corollary 2.6, for any $\tau \in]t,T[$, we have

$$\int_{\mathbb{R}^2} \Gamma^0(t,s,a;\tau,\xi,\eta) J^n_{\tau,T,\xi,\eta} \Gamma^0(\tau,\xi,\eta;T,S,A) d\xi d\eta$$

$$= \int_{\mathbb{R}^2} \hat{J}^n_{t,\tau,T,s,a} \Gamma^0(t,s,a;\tau,\xi,\eta)\Gamma^0(\tau,\xi,\eta;T,S,A)d\xi d\eta$$

(here $\hat{J}^n_{t,\tau,T,s,a}$ plays the role of the "adjoint" operator of $J^n_{\tau,T,\xi,\eta}$)

$$= \hat{J}^n_{t,\tau,T,s,a} \int_{\mathbb{R}^2} \Gamma^0(t,s,a;\tau,\xi,\eta)\Gamma^0(\tau,\xi,\eta;T,S,A)d\xi d\eta$$

(by the semigroup property of Γ^0)

$$= \hat{J}^n_{t,\tau,T,s,a}\Gamma^0(t,s,a;T,S,A). \tag{2.33}$$

Next we prove the thesis by induction. For $n = 1$, by the representation formula for the non-homogeneous parabolic Cauchy problem (2.12) with null final condition, we have

$$\begin{aligned} &G^1(t,s,a;T,S,A) \\ &= \int_t^T \int_{\mathbb{R}^2} \Gamma^0(t,s,a;\tau,\xi,\eta)\alpha_1(\tau)(\xi - s_0)\partial_{\xi\xi}\Gamma^0(\tau,\xi,\eta;T,S,A)d\xi d\eta d\tau \end{aligned}$$

(by (2.27))

$$= \int_t^T \alpha_1(\tau)(V_{t,\tau,s,a} - s_0) \int_{\mathbb{R}^2} \Gamma^0(t,s,a;\tau,\xi,\eta)\partial_{\xi\xi}\Gamma^0(\tau,\xi,\eta;T,S,A)d\xi d\eta d\tau$$

(by parts and by (2.28))

$$= \int_t^T \alpha_1(\tau)(V_{t,\tau,s,a} - s_0)W^{2,0}_{t,\tau,s,a} \int_{\mathbb{R}^2} \Gamma^0(t,s,a;\tau,\xi,\eta)\Gamma^0(\tau,\xi,\eta;T,S,A)d\xi d\eta d\tau$$

(by the semigroup property of Γ^0)

$$= \int_t^T \alpha_1(\tau)(V_{t,\tau,s,a} - s_0)W^{2,0}_{t,\tau,s,a}\Gamma^0(t,s,a;T,S,A)d\tau.$$

This proves (2.29)-(2.31) for $n = 1$, that is

$$G^1(t,s,a;T,S,A) = J^1_{t,T,s,a}\Gamma^0(t,s,a;T,S,A),$$

where

$$J^1_{t,T,s,a} = \int_t^T \alpha_1(\tau)(V_{t,\tau,s,a} - s_0)W^{2,0}_{t,\tau,s,a}d\tau. \tag{2.34}$$

Using (2.34) and the explicit expression of the operators V, W in (2.24)-(2.25)-(2.26) given in terms of Φ, M in (2.22), we can easily rewrite $J^1_{t,T,s,a}$ in the form (2.30): we refer to Remark 2.8 below for the details and the derivation of the explicit expression of the coefficients $f^1_{i-j,j,k}$.

Now we assume that (2.29), (2.30) and (2.31) are valid for a generic but fixed n and we prove them for $n+1$. Using again the standard representation formula for non-homogeneous parabolic Cauchy problem (2.11) with null final condition, we have

$$G^{n+1}(t,s,a;T,S,A) = \sum_{i=1}^{n+1} \int_t^T \alpha_i(\tau)I_i(t,s,y,\tau,T,S,A)d\tau, \tag{2.35}$$

where

$$
\begin{aligned}
&I_i(t,s,y,\tau,T,S,A)\\
&\int_{\mathbb{R}^2}\Gamma^0(t,s,a;\tau,\xi,\eta)(\xi-s_0)^i\partial_{\xi\xi}G^{n+1-i}(\tau,\xi,\eta;T,S,A)d\xi d\eta
\end{aligned}
$$

(by Corollary 2.6)

$$
=(V_{t,\tau,s,a}-s_0)^i W^{2,0}_{t,\tau,s,a}\int_{\mathbb{R}^2}\Gamma^0(t,s,a;\tau,\xi,\eta)G^{n+1-i}(\tau,\xi,\eta;T,S,A)d\xi d\eta
$$

(by the inductive hypothesis)

$$
=(V_{t,\tau,s,a}-s_0)^i W^{2,0}_{t,\tau,s,a}\int_{\mathbb{R}^2}\Gamma^0(t,s,a;\tau,\xi,\eta)J^{n+1-i}_{\tau,T,\xi,\eta}\Gamma^0(\tau,\xi,\eta;T,S,A)d\xi d\eta
$$

(by (2.33))

$$
=(V_{t,\tau,s,a}-s_0)^i W^{2,0}_{t,\tau,s,a}\hat{J}^{n+1-i}_{t,\tau,T,s,a}\Gamma^0(t,s,a;T,S,A). \tag{2.36}
$$

Plugging (2.36) into (2.35), we obtain formulae (2.29)-(2.31) and this concludes the proof. □

Remark 2.8. *Starting from formula* (2.34)

$$
J^1_{t,T,s,a}=\int_t^T\alpha_1(\tau)(V_{t,\tau,s,a}-s_0)W^{2,0}_{t,\tau,s,a}d\tau,
$$

we find the more explicit representation of $J^1_{t,T,s,a}$ *in the form* (2.30)*, that is*

$$
J^1_{t,T,s,a}=\sum_{k=0}^{1}s^k\sum_{i=2}^{3}\sum_{j=0}^{i}f^1_{i-j,j,k}(t,T)\frac{\partial^i}{\partial s^{i-j}\partial a^j}. \tag{2.37}
$$

We first remark that, by the definition (2.24)*,* (2.25) *and* (2.26) *of the operators* V *and* W*, we have*

$$
\begin{aligned}
V_{t,\tau,s,a}&=s\Phi_{11}(t,\tau)+M_{11}(t,\tau)\Phi_{11}(t,\tau)\partial_s+M_{21}(t,\tau)\Phi_{11}(t,\tau)\partial_a\\
W^{2,0}_{t,\tau,s,a}&=\frac{1}{\Phi_{11}(t,\tau)^2}\partial_{ss}-\frac{2\Phi_{21}(t,\tau)}{\Phi_{11}(t,\tau)^2}\partial_{sa}+\frac{\Phi_{21}(t,\tau)^2}{\Phi_{11}(t,\tau)^2}\partial_{aa},
\end{aligned}
$$

where Φ_{ij} *and* M_{ij} *denote the components of the matrices* Φ *and* M *in* (2.22) *respectively. Thus we get*

$$
\begin{aligned}
(V_{t,\tau,s,a}-s_0)\,W^{2,0}_{t,\tau,s,a}=&\frac{s\Phi_{11}(t,\tau)-s_0}{\Phi_{11}(t,\tau)^2}\partial_{ss}+\frac{2(s_0-s\Phi_{11}(t,\tau))\Phi_{21}(t,\tau)}{\Phi_{11}(t,\tau)^2}\partial_{sa}\\
&+\frac{(s\Phi_{11}(t,\tau)-s_0)\Phi_{21}(t,\tau)^2}{\Phi_{11}(t,\tau)^2}\partial_{aa}+\frac{M_{11}(t,\tau)}{\Phi_{11}(t,\tau)}\partial_{sss}\\
&+\frac{M_{21}(t,\tau)-2M_{11}(t,\tau)\Phi_{21}(t,\tau)}{\Phi_{11}(t,\tau)}\partial_{ssa}
\end{aligned}
$$

$$+\frac{\Phi_{21}(t,\tau)(M_{11}(t,\tau)\Phi_{21}(t,\tau)-2M_{21}(t,\tau))}{\Phi_{11}(t,\tau)}\partial_{saa}$$
$$+\frac{M_{21}(t,\tau)\Phi_{21}(t,\tau)^2}{\Phi_{11}(t,\tau)}\partial_{aaa}.$$

Reordering all terms, we obtain the following expression for the coefficients $f^1_{i-j,j,k}$ in (2.37)*:*

$$
\begin{aligned}
f^1_{2,0,0}(t,T) &= -s_0\int_t^T \frac{\alpha_1(\tau)}{\Phi_{11}(t,\tau)^2}d\tau,\\
f^1_{1,1,0}(t,T) &= 2s_0\int_t^T \alpha_1(\tau)\frac{\Phi_{21}(t,\tau)}{\Phi_{11}(t,\tau)^2}d\tau,\\
f^1_{0,2,0}(t,T) &= -s_0\int_t^T \alpha_1(\tau)\frac{\Phi_{21}(t,\tau)^2}{\Phi_{11}(t,\tau)^2}d\tau,\\
f^1_{3,0,0}(t,T) &= \int_t^T \alpha_1(\tau)\frac{M_{11}(t,\tau)}{\Phi_{11}(t,\tau)}d\tau,\\
f^1_{2,1,0}(t,T) &= \int_t^T \alpha_1(\tau)\frac{M_{21}(t,\tau)-2M_{11}(t,\tau)\Phi_{21}(t,\tau)}{\Phi_{11}(t,\tau)}d\tau,\\
f^1_{1,2,0}(t,T) &= \int_t^T \alpha_1(\tau)\frac{\Phi_{21}(t,\tau)(M_{11}(t,\tau)\Phi_{21}(t,\tau)-2M_{21}(t,\tau))}{\Phi_{11}(t,\tau)}d\tau,\\
f^1_{0,3,0}(t,T) &= \int_t^T \alpha_1(\tau)\frac{M_{21}(t,\tau)\Phi_{21}(t,\tau)^2}{\Phi_{11}(t,\tau)}d\tau,\\
f^1_{2,0,1}(t,T) &= \int_t^T \frac{\alpha_1(\tau)}{\Phi_{11}(t,\tau)}d\tau,\\
f^1_{1,1,1}(t,T) &= -2\int_t^T \alpha_1(\tau)\frac{\Phi_{21}(t,\tau)}{\Phi_{11}(t,\tau)}d\tau,\\
f^1_{0,2,1}(t,T) &= \int_t^T \alpha_1(\tau)\frac{\Phi_{21}(t,\tau)^2}{\Phi_{11}(t,\tau)}d\tau,\\
f^1_{3,0,1}(t,T) &= f^1_{2,1,1}(t,T) = f^1_{1,2,1}(t,T) = f^1_{0,3,1}(t,T) = 0.
\end{aligned}
$$

Having the explicit representation of $J^1_{t,T,s}$, from (2.32) *we directly get the expression of $\hat{J}^1_{t,\tau,T,s,a}$:*

$$\hat{J}^1_{t,\tau,T,s,a} = \sum_{k=0}^{1}\sum_{i=2}^{3}\sum_{j=0}^{i} f^1_{i-j,j,k}(t,T)V^k_{t,\tau,s,a}W^{i-j,j}_{t,\tau,s,a}. \tag{2.38}$$

Plugging (2.38) *into* (2.31) *with $n=2$, we can easily find $J^2_{t,T,s,a}$ and $\hat{J}^2_{t,\tau,T,s,a}$. By an analogous iterative procedure, we can compute the higher order approximation formulae. In Section 7.3, we present some experiment where we computed explicitly the operators $J^n_{t,T,s,a}$ up to the third order for $r \neq q$ and up to the fifth order for $r = q$, to get very accurate results.*

Remark 2.9. *The theoretical problem of the convergence and the error estimates for the expansion will not be addressed here: Corielli, Foschi and Pascucci [62] recently found global error bounds, based on Schauder estimates, for a similar expansion for degenerate PDEs* *of Asian type. It turns out that theoretical error estimates are generally very conservative and experimental results show that the explicit formulae have very good precision even in extreme cases; further, the assumptions needed to prove the theoretical results rule out models of practical interest such as the CEV model. We notice explicitly that the result in [62] is a very particular case of the more general approach proposed here and essentially corresponds to a first order expansion while here we derive very accurate approximations up to the fifth order with a completely different technique.*

7.2.4 Time-independent coefficients

As an illustrative example, we work out the approximation formulae for the density and the fixed-strike arithmetic Asian Call in a local volatility model with time-independent coefficients: the BS and the Constant Elasticity of Variance (CEV) models are meaningful particular cases. Hence we assume the following risk-neutral dynamics for the asset

$$dS_t = (r - q)S_t dt + \sigma(S_t)S_t dW_t.$$

We set

$$\alpha(s) = \sigma^2(s)s^2, \qquad \mu = r - q,$$

and consider the pricing operator

$$L = \frac{\alpha(s)}{2}\,\partial_{ss} + \mu s\partial_s + s\partial_a + \partial_t.$$

We also fix $s_0 > 0$ and put

$$\alpha_0 = \alpha(s_0), \qquad \alpha_k = \frac{1}{2k!}\,\partial_s^k \alpha(s_0), \qquad k \geq 1. \tag{2.39}$$

Then

$$L^0 = \frac{\alpha_0}{2}\partial_{ss} + \mu s\partial_s + s\partial_a + \partial_t$$

is the leading term in the approximation of L. Since the parameters are time independent, it is not restrictive to assume $t = 0$: accordingly, we simplify the notations and we write $\Gamma(s, a; T, S, A)$ and $J^1_{T,s,a}$ instead of $\Gamma(0, s, a; T, S, A)$ and $J^1_{0,T,s,a}$ respectively. The fundamental solution of L^0 is given by

$$\Gamma^0(s, a; T, S, A) = \frac{1}{2\pi\sqrt{\det C(T)}}\, e^{-\frac{1}{2}\langle C^{-1}(T)((S,A)-m_{s,a}(T)),(S,A)-m_{s,a}(T)\rangle},$$

with $m_{s,a}(T) \equiv m_{0,s,a}(T)$ as in (2.23) and $C(T) \equiv C(0, T)$ as in (2.15)-(2.23).

For simplicity, we assume $\mu \neq 0$ and report only the first order formulae: the Mathematica notebook of higher order approximations is available in the web-site of the authors.

By Theorem 2.7 the 1-st order approximation for the density is given by

$$\Gamma^1(s, a; T, S, A) = \Gamma^0(s, a; T, S, A) + J^1_{T,s,a}\Gamma^0(s, a; T, S, A)$$

where

$$J^1_{T,s,a} = \sum_{k=0}^{1} s^k \sum_{i=2}^{3}\sum_{j=0}^{i} f^1_{i-j,j,k}(T)\frac{\partial^i}{\partial s^{i-j}\partial a^j},$$

and $f^1_{i-j,j,k}(T) \equiv f^1_{i-j,j,k}(0,T)$ are the deterministic functions defined in Remark 2.8: specifically, in the case of time-independent coefficients, we have

$$f^1_{i-j,j,k}(T) = \alpha_1 \alpha_{i-3} g_{i-j,j,k}(T)$$

with $\alpha_{-1} = 1$, α_i as in (2.39) for $i = 0, 1$ and where

$$\begin{aligned}
g_{2,0,0} &= \frac{\left(e^{-2T\mu} - 1\right) s_0}{2\mu}, \qquad g_{1,1,0} = \frac{e^{-2T\mu}\left(e^{T\mu} - 1\right)^2 s_0}{\mu^2},\\
g_{0,2,0} &= \frac{s_0\left(3 + e^{-2T\mu} - 4e^{-T\mu} - 2T\mu\right)}{2\mu^3}, \qquad g_{2,0,1} = \frac{\left(1 - e^{-T\mu}\right)}{\mu},\\
g_{1,1,1} &= -\frac{2\left(-1 + e^{-T\mu} + T\mu\right)}{\mu^2}, \qquad g_{0,2,1} = -\frac{2(T\mu - \sinh(T\mu))}{\mu^3},\\
g_{3,0,0} &= \frac{\left(2 + e^{-3T\mu} - 3e^{-T\mu}\right)}{6\mu^2},\\
g_{2,1,0} &= \frac{\left(1 - T\mu + e^{-2T\mu}(-1 - \sinh(T\mu))\right)}{\mu^3},\\
g_{1,2,0} &= \frac{e^{-3T\mu}\left(1 - e^{T\mu}\right)^4}{2\mu^4},\\
g_{0,3,0} &= -\frac{\left(-e^{-3T\mu} + 6e^{-2T\mu} - 18e^{-T\mu} + 3e^{T\mu} + 2(5 - 6T\mu)\right)}{6\mu^5},
\end{aligned} \tag{2.40}$$

and $g_{i-j,j,k} = 0$ when $i + k = 4$. Notice that *the functions in* (2.40) *are model independent:* the particular form of the volatility enters in the approximation formula only through Γ^0 and the coefficients α_n of the Taylor expansion of the volatility function.

Accordingly, the first order approximation for the fixed-strike Asian Call is given by

$$e^{-rT} u(s, a, T),$$

where

$$u(s,a,T) = C_{\mathrm{BS}}(s,a,T) + \sum_{k=0}^{1} s^k \sum_{i=2}^{3} \sum_{j=0}^{i} f^1_{i-j,j,k}(T) \frac{\partial^i}{\partial s^{i-j} \partial a^j} C_{\mathrm{BS}}(s,a,T),$$

and

$$\begin{aligned}
C_{\mathrm{BS}}(s,a,T) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma^0(s,a;T,S,A) \left(\frac{A}{T} - K\right)^+ dS dA\\
&= \frac{e^{-\frac{\mu \beta(s,a,T)^2}{\gamma(T)}} \sqrt{\mu^3 \gamma(T)}}{2T\mu^3 \sqrt{\pi}} - \frac{\mu^2 \beta(s,a,T)}{T\mu^3} \left(1 - \mathcal{N}\left(\beta(s,a,T)\sqrt{\frac{2\mu}{\gamma(T)}}\right)\right)
\end{aligned}$$

with

$$\beta(s,a,T) = s - se^{T\mu} - a\mu + KT\mu, \quad \gamma(T) = \alpha_0\left(3 - 4e^{\mu T} + e^{2\mu T} + 2\mu T\right).$$

In the above formula, numerical errors due to cancellations for short maturities can be corrected by using the resulting series expansion. We also remark that a suitable choice of s_0 may improve the accuracy of the approximation formula: as we shall see in Section 7.3, in most cases $s_0 = s$ is a convenient choice that allows to get very accurate results.

7.3 Numerical experiments

In this section our approximation formulae are tested and compared with method proposed by Linetsky [155], the Mellin transform based method (Mellin500) of Shaw [201], the PDE method of Vecer [208], the matched asymptotic expansions of Dewynne and Shaw [69] (MAE3 and MAE5) and the method of Dassios and Nagaradjasarma [67] (DN). Our 2nd, 3rd and 5th order approximations will be denoted by FPP2, FPP3 and FPP5. In the first part of this section a set of experimental results under BS dynamics are reported, then in the second part the CEV dynamics is considered.

7.3.1 Tests under Black & Scholes dynamics

In order to assess the performances of our approximations for pricing arithmetic Call options under a BS model, we used the family of tests introduced in [107], and later used in [67, 69, 79, 102, 155, 208] as a standard for this task. Table 7.1 reports the interest rate r, the volatility σ, the time to maturity T, the strike K and the initial asset price S_0 for the seven cases. In this set of tests a null dividend rate is assumed: $q = 0$.

Table 7.2 reports the results of methods Linetsky, FPP3, FPP2, Mellin500, Vecer and MAE3. The results of Linetsky, Vecer and MAE3 are taken from [155], [208] and [69], respectively.

Following [69] we repeated the same seven tests with a dividend rate equal to the interest rate (see Table 7.3). The results of Linetsky and Vecer are not reported: the former because these tests were not considered in his paper; the latter because Vecer's code cannot deal with that special case.

In that case, the discrepancies between FPP3 and MAE5 can be found only at the 5th decimal place. Furthermore, FPP5 and MAE5 columns show that the contribution of the 5th order approximations to the accuracy of the methods is not substantial.

Next, in order to address the issues raised in Shaw [200], Fu, Madan and Wang [102] and Dufresne [81], we tested our method with a low-volatility parameter $\sigma = 0.01$. Table 7.4 shows the performances of the approximations against Monte Carlo 95% confidence intervals. These intervals are computed using 500 000 Monte Carlo replication and an Euler discretization with 300 time-steps for $T = 0.25$ and $T = 1$ and 1500 time-steps for $T = 5$. In these experiments the initial asset level is $S_0 = 100$, the interest rate is $r = 0.05$ and the dividend yield is null $q = 0$.

The methods considered are Vecer, Mellin500, FPP3 and MAE3. Here, we used the Mathematica implementations of Vecer and Mellin500 provided by the authors, whereas MAE3 formula was coded by ourself. Mellin500 implementation requires a numerical integration on an unbounded domain which needs to be truncated. We have set the length of the truncated domain to 10^9 and fixed the number of recursion in Mathematica `NIntegrate` function to 100. The execution time of the Monte Carlo , Vecer and Mellin500 methods is also reported. Also here, FPP3 and MAE3 methods are almost identical and both always fall very close to Monte Carlo results: the worst case has an error of 5×10^{-3}. Notice that the Euler discretization may induce a little bias in Monte Carlo results.

We remark that, although the proposed approximations have a performance very similar to the method of Dewynne and Shaw, our approach is more flexible and capable of dealing with local volatility dynamics; moreover, our method can also produce explicit approximation formulae for the Greeks and the asset-average density.

7.3.2 Tests under CEV dynamics

In this section we test the performances of our approximation when the volatility is not constant. More specifically, we consider the CEV dynamics

$$dS_t = (r-q)dS_t dt + \sigma S^{\beta} dW_t, \qquad \beta \in]0,1[,$$

which corresponds to a local volatility model with $\sigma(t,S) = \sigma S^{\beta-1}$. Although this is a "degenerate" case, as $\sigma(t,S)$ is not bounded, the following experiments confirm that the approximation is still precise enough.

Firstly, we performed the experiments proposed by Dassios and Nagaradjasarma in [67] for the square-root model, $\beta = \frac{1}{2}$. The results on these tests are reported in Tables 7.5 and 7.6, where the 2-nd and 3-rd order approximations are compared with the results of a Monte Carlo method. The same number of Monte Carlo replications and time-steps of previous experiments was used. Here again, both FPP2 and FPP3 approximations show good performances.

Figure 7.1 and 7.2 show the cross-sections of absolute (left) and relative (right) errors of the 3-rd order approximation when $\beta = \frac{2}{3}$ and $\beta = \frac{1}{3}$, respectively. The errors are computed against prices computed by means of an Euler Monte Carlo method with 300 time-steps and 500 000 replications. The shaded bands show the 95% and 99% Monte Carlo confidence intervals for each strike. The initial stock price is $S_0 = 1$, the risk-free rate is $r = 5\%$, the dividend yield is $q = 0$ and the maturity is $T = 1$. Two levels for the volatility parameter are considered $\sigma = 10\%$ and $\sigma = 50\%$. The two figures show that the approximations have good global performances for both the CEV exponents and both the volatility levels.

Finally, since our technique provides explicit approximating formulae also for the sensitivities of option prices, we show in Figure 7.3 the graphs of the Delta, the Gamma and the Vega of an arithmetic Asian Call, with fixed strike, under the CEV model. Notice that, usual no-arbitrage bounds, like having the Delta in the interval $[0,1]$ or positive Gamma, are not violated. The Mathematica notebook containing all the explicit formulae for the prices and the Greeks is available in the web-site of the authors.

7.4 Figures and tables

Case	S_0	K	r	σ	T
1	2	2	0.02	0.1	1
2	2	2	0.18	0.3	1
3	2	2	0.0125	0.25	2
4	1.9	2	0.05	0.5	1
5	2	2	0.05	0.5	1
6	2.1	2	0.05	0.5	1
7	2	2	0.05	0.5	2

Table 7.1: Parameter values for seven test cases

Case	Linetsky	FPP3	FPP2	Mellin500	Vecer	MAE3
1	0.05598604	0.05598604	0.05598602	0.05603631	0.055986	0.05598596
2	0.21838755	0.21838706	0.21838375	0.21835987	0.218388	0.21836866
3	0.17226874	0.17226694	0.17226600	0.17236881	0.172269	0.17226265
4	0.19317379	0.19316359	0.19320627	0.19297162	0.193174	0.19318824
5	0.24641569	0.24640562	0.24640056	0.24651870	0.246416	0.24638175
6	0.30622036	0.30620974	0.30615763	0.30649701	0.306220	0.30613888
7	0.35009522	0.35003972	0.35001419	0.34892612	0.350095	0.34990862

Table 7.2: Asian Call Option Prices when $q = 0$ (parameters as in Table 7.1)

Case	FPP5	FPP3	FPP2	MAE3	MAE5
1	0.045143	0.045143	0.045143	0.045143	0.045143
2	0.115188	0.115188	0.115188	0.115188	0.115188
3	0.158380	0.158378	0.158378	0.158378	0.158380
4	0.169201	0.169192	0.169238	0.169238	0.169201
5	0.217815	0.217805	0.217805	0.217805	0.217815
6	0.272924	0.272914	0.272868	0.272869	0.272925
7	0.291316	0.291263	0.291263	0.291264	0.291316

Table 7.3: Asian Call Option Prices when $q = r$ (parameters as in Table 7.1).

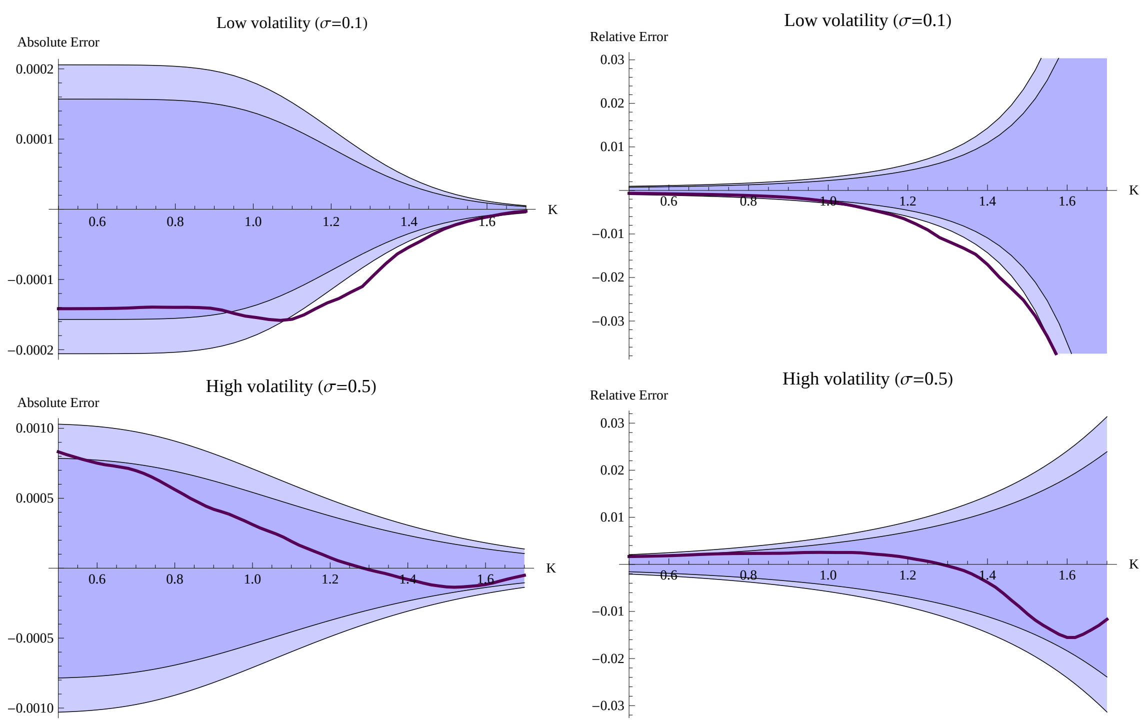

Figure 7.1: FPP3 approximation errors for Asian Call options with maturity $T = 1$, when the underlying has a CEV dynamics with exponent $\beta = \frac{2}{3}$ and parameters $S_0 = 1$, $r = 5\%$ and $q = 0$. Two levels of volatility are considered: $\sigma = 10\%$ and $\sigma = 50\%$

T	K	Euler - Monte Carlo method			Vecer		Mellin500		FPP3	MAE3
		95% c.i.		ex. time	value	ex. time	value	ex. time		
0.25	99	1.60849×10^{0}	1.61008×10^{0}	70.52	-4.18937×10^{1}	0.022	1.51718×10^{0}	4.09	1.60739×10^{0}	1.60739×10^{0}
0.25	100	6.22333×10^{-1}	6.23908×10^{-1}	70.64	5.40466×10^{-1}	0.022	6.96855×10^{-1}	4.08	6.21359×10^{-1}	6.21359×10^{-1}
0.25	101	1.39301×10^{-2}	1.42436×10^{-2}	71.02	-3.96014×10^{-2}	0.022	1.60361×10^{-1}	4.09	1.37618×10^{-2}	1.37615×10^{-2}
1.00	97	5.27670×10^{0}	5.27985×10^{0}	70.91	-9.73504×10^{0}	0.019	5.27474×10^{0}	4.38	5.27190×10^{0}	5.27190×10^{0}
1.00	100	2.42451×10^{0}	2.42767×10^{0}	70.89	2.37512×10^{0}	0.020	2.43303×10^{0}	4.26	2.41821×10^{0}	2.41821×10^{0}
1.00	103	7.44026×10^{-2}	7.54593×10^{-2}	70.61	7.25478×10^{-2}	0.020	8.50816×10^{-2}	4.24	7.26910×10^{-2}	7.24337×10^{-2}
5.00	80	2.61775×10^{1}	2.61840×10^{1}	316.62	2.52779×10^{1}	0.018	2.61756×10^{1}	4.40	2.61756×10^{1}	2.61756×10^{1}
5.00	100	1.06040×10^{1}	1.06105×10^{1}	319.28	1.05993×10^{1}	0.018	1.05993×10^{1}	4.33	1.05996×10^{1}	1.05996×10^{1}
5.00	120	1.41956×10^{-6}	1.38366×10^{-5}	284.00	1.07085×10^{-5}	0.017	1.42235×10^{-3}	2.62	2.06699×10^{-5}	5.73317×10^{-6}

Table 7.4: Tests with low volatility: $\sigma = 0.01$, $S_0 = 100$, $r = 0.05$ and $q = 0$

Case	S_0	K	r	σ	T	DN	FPP3	FPP2	MC 95% c.i.
$1^\star$	2	2	0.02	0.14	1	0.0197	0.055562	0.055562	0.055321 – 0.055732
$2^\star$	2	2	0.18	0.42	1	0.2189	0.217874	0.217875	0.218319 – 0.219678
$3^\star$	2	2	0.0125	0.35	2	0.1725	0.170926	0.170926	0.171126 – 0.172555
$4^\star$	1.9	2	0.05	0.69	1	0.1902	0.190834	0.190821	0.190303 – 0.192121
$5^\star$	2	2	0.05	0.72	1	NA	0.251121	0.251123	0.250675 – 0.252807
$6^\star$	2.1	2	0.05	0.72	1	0.3098	0.308715	0.308730	0.308791 – 0.311150
$7^\star$	2	2	0.05	0.71	2	0.3339	0.353197	0.353206	0.352269 – 0.355313

Table 7.5: Tests proposed by Dassios and Nagardjasarma [67] for the CEV model.

σ	T	DN	FPP3	FPP2	MC 95% c.i.
0.71	0.1	0.0751	0.075387	0.075387	0.075068 – 0.075689
0.71	0.5	0.1725	0.173175	0.173175	0.173265 – 0.174717
0.71	1.0	0.2468	0.248018	0.248019	0.247738 – 0.249841
0.71	2.0	0.3339	0.353197	0.353206	0.351111 – 0.354146
0.71	5.0	0.3733	0.545714	0.545800	0.545812 – 0.550679
0.1	1	0.0484	0.061439	0.061439	0.061329 – 0.061674
0.3	1	0.1207	0.120680	0.120680	0.120596 – 0.121494
0.5	1	0.1827	0.182723	0.182724	0.182814 – 0.184285
0.7	1	0.2446	0.244913	0.244914	0.244959 – 0.247030

Table 7.6: Second set of tests proposed by Dassios and Nagardjasarma [67]. The remaining parameters are set to $S_0 = K = 2$, $r = 0.05$, $q = 0$ and $\beta = \frac{1}{2}$

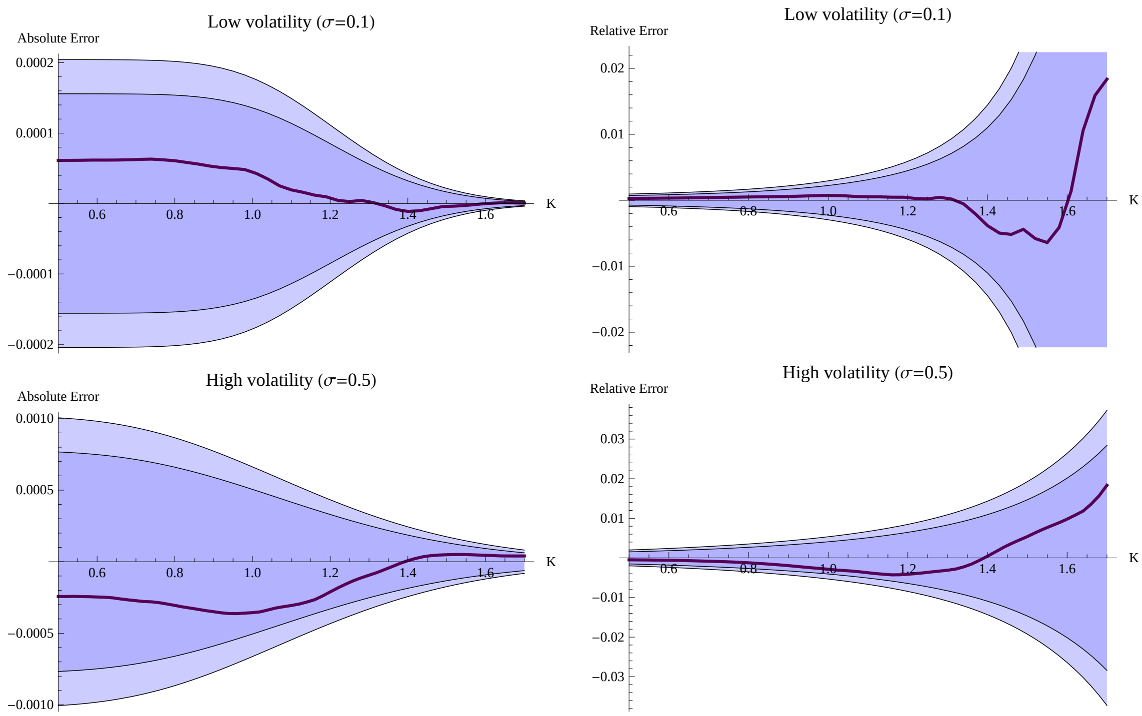

Figure 7.2: FPP3 approximation errors for Asian Call options with maturity $T = 1$, when the underlying has a CEV dynamics with exponent $\beta = \frac{1}{3}$ and parameters $S_0 = 1$, $r = 5\%$ and $q = 0$. Two levels of volatility are considered: $\sigma = 10\%$ and $\sigma = 50\%$

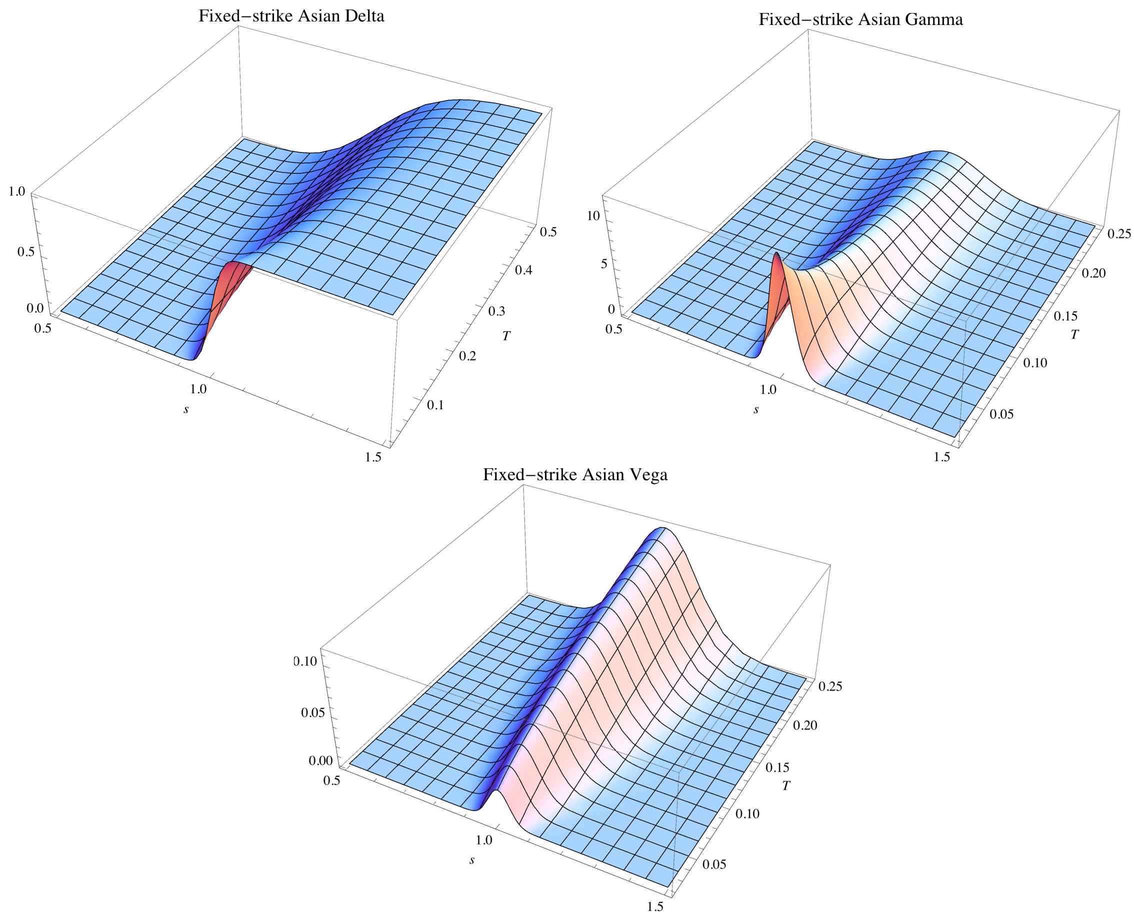

Figure 7.3: Delta, Gamma and Vega of an arithmetic Average Call with fixed strike $K = 1$, under a CEV dynamics with parameters $r = 5\%$, $q = 0$, $\sigma = 40\%$ and $\beta = \frac{1}{2}$

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