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CICLO XXXI

Regularizing properties of the double layer heat potential and shape analysis of a periodic problem

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Riassunto

Questa Tesi è dedicata allo studio di alcuni operatori integrali della teoria del potenziale parabolica rilevanti al fine di analizzare problemi al contorno per l'equazione del calore soggetti a perturbazioni singolari del dominio con un approccio funzionale analitico già noto per i problemi ellittici, e all'applicazione di metodi di teoria del potenziale allo studio di problemi di perturbazione di tipo ellittico. La Tesi è divisa in due parti indipendenti. Nella prima parte (Capitoli 1–3) dimostriamo nuovi risultati di teoria del potenziale parabolica e, in particolare, studiamo le proprietà di alcuni operatori integrali associati ai potenziali di strato calorici, mentre nella seconda parte (Capitolo 4) investighiamo il comportamento di un problema al contorno ellittico rispetto a perturbazioni del dominio, utilizzando metodi di teoria del potenziale.

La Tesi è organizzata come segue. Nel Capitolo 1 introduciamo uno spazio normato di nuclei debolmente singolari che dipendono dalla variabile temporale, e proviamo alcuni risultati di continuità per operatori integrali parabolici rispetto a variazioni sia del nucleo nella classe di cui sopra, sia della funzione densità. Nel Capitolo 2 proviamo una formula esplicita per le derivate tangenziali del potenziale di doppio strato calorico, e proviamo delle proprietà di regolarizzazione per l'operatore integrale associato al potenziale di doppio strato calorico. Nel Capitolo 3 consideriamo i potenziali di strato calorici periodici nello spazio, e risolviamo alcuni problemi al contorno periodici per l'equazione del calore. Infine, il Capitolo 4 è dedicato allo studio del comportamento della permeabilità longitudinale di un materiale periodicamente perforato rispetto a perturbazioni della struttura di periodicità e della forma dei fori.

Alla fine della Tesi abbiamo incluso delle Appendici con alcuni risultati utilizzati.

Abstract

This Dissertation is devoted to the study of some integral operators arising in parabolic potential theory which are relevant in order to analyze boundary value problems for the heat equation subject to a singular perturbation of the domain by exploiting a known functional analytic approach for elliptic problems, and to the analysis of some elliptic perturbation problems with a potential theoretic approach. The Dissertation is divided into two independent parts. In the first part (Chapters 1–3) we produce new results in parabolic potential theory and, in particular, we study the mapping properties of some integral operators associated with layer heat potentials, while in the second part (Chapter 4) we investigate the behavior of an elliptic boundary value problem under domain perturbation with a potential theoretic approach.

The Dissertation is organized as follows. In Chapter 1 we introduce a normed class of time dependent weakly singular kernels and we prove results of joint continuity of some parabolic integral operators upon variation both of the kernel in the above class and of the density function. Moreover we apply these results to some integral operators related to layer heat potentials. In Chapter 2 we prove an explicit formula for the tangential derivatives of the double layer heat potentials and we prove a regularizing property of the integral operator associated with the double layer heat potential. In Chapter 3 we consider space-periodic layer heat potentials and we solve some periodic boundary value problems for the heat equation. Finally, Chapter 4 is devoted to the study of the behavior of the longitudinal permeability of a periodic array of cylinders upon the perturbation of the periodicity structure and of the cross sections of the cylinders.

At the end of the Dissertation we have enclosed some Appendices with some results that we have exploited.

Contents

Introduction

The Dissertation consists of two almost independent parts, which we now introduce.

First part

The first part of this Dissertation is mainly devoted to the study of integral operators arising in parabolic potential theory which are relevant in order to analyze boundary value problems for the heat equation subject to a singular perturbation of the domain by exploiting a known functional analytic approach for elliptic problems, and it is contained in Chapters 1–3. We introduce some notation in order to explain the content of this first part. Let

$$n \in \mathbb{N} \setminus \{0, 1\}, \qquad \alpha \in]0, 1[, \qquad m \in \mathbb{N} \setminus \{0\}, \qquad T \in]-\infty, +\infty].$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. We denote

$$\Omega_T \equiv \overline{]-\infty, T[} \times \Omega, \qquad \partial_T\Omega \equiv \overline{]-\infty, T[} \times \partial\Omega.$$

Let $\nu \equiv (\nu_l)_{l=1,\dots,n}$ denote the outward unit normal field to $\partial\Omega$. Let Φ_n denote the fundamental solution of the heat equation in $\mathbb{R}^{1+n}$.

We are mainly interested in the mapping properties in parabolic Schauder spaces of the boundary integral operator associated with the double layer heat potential, *i.e.*, the operator which takes a density function μ from $\partial_T\Omega$ to $\mathbb{C}$ and maps it to the function $w[\partial_T\Omega, \mu]_{|\partial_T\Omega}$ defined by

$$w[\partial_T\Omega, \mu]_{|\partial_T\Omega}(t, x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \Phi_n(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau \qquad \forall\, (t, x) \in \partial_T\Omega.$$

Layer heat potentials have been systematically exploited in the analysis of boundary value problems for parabolic equations. Without being exhaustive we mention, *e.g.*, Pogorzelski [90] where the author, by means of layer heat potential, solves some basic boundary value problems for the heat equation by solving the related integral equation with the Neumann series method. Moreover, we mention the well-known monographs Ladyženskaja, Solonnikov and Ural'ceva [58] and Friedman [38] where a large variety of parabolic problems are solved by means of layer heat potentials. Miranda [80] has used the double layer heat potential to solve the Dirichlet problem for the heat equation in a domain with a corner singularity. Baderko [8] has solved the Dirichlet and the Neumann problems for a second order parabolic operator in Schauder spaces defined on parabolic cylinders that can be unbounded in time. Fabes and Rivière [34] has solved the Dirichlet and Neumann problems for the heat equation with data in Lebesgue spaces in the case of C^1 cylinders, and later on Brown [11, 12] has extended their results to the case of Lipschitz cylinders. Costabel [17] has obtained the solvability of some boundary value problem for the heat equation on Lipschitz cylinders with data in anisotropic Sobolev spaces. Hsiao and Saranen [49] solved an exterior Dirichlet boundary value problem for the non-homogeneous heat equation in the case of $n = 2$. For time dependent Lipschitz domains we mention the works of Lewis and Murray [73] and Hofmann and Lewis [46].

For this reason, many authors have investigated the properties of the integral operators associated with layer heat potentials in several function space settings.

A first systematic treatment of the properties of layer heat potentials can be found in the works of Gevrey [39, 40], where the author has studied the properties of heat potentials in case $n = 1$.

Then Van Tun [104, 105, 106] has developed the work of Gevrey in a series of papers and has obtained some results on the Schauder regularity of heat potentials, still in dimension $n = 1$. In particular, Van Tun has proved that the integral operator associated with the double layer heat potential defined on the boundary of a parabolic cylinder improves by $1/2$ the Hölder exponent of the density.

In case $m \in \mathbb{N}$ and Ω is of class $C^{m,\alpha}$, it has long been known that if the density μ is of class $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$, then the restriction of the double layer potential to the set Ω_T can be extended to a function of $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$ (cf., *e.g.*, Ladyženskaja, Solonnikov and Ural'ceva [58] and Appendix B).

In case $m \in \mathbb{N}$ and Ω is of class $C^{m+2,\alpha}$, Kamynin [50, 51, 52, 53] has proved that the integral operator associated with the double layer heat potential is bounded from the Schauder space $C^{\frac{m+\alpha}{2};m+\alpha}([0,T]\times\partial\Omega)$ to $C^{\frac{m+1+\alpha'}{2};m+1+\alpha'}([0,T]\times\partial\Omega)$ for all $\alpha' \in]0,\alpha[$, when $T < +\infty$.

Properties of layer heat potentials have also been considered in the framework of Lebesgue and Sobolev spaces. For example, Noon [87] and Arnold and Noon [7] have proved a coercivity property for the single layer heat potential in anisotropic Sobolev spaces in case $n = 3$. Costabel [17] has proved some mapping properties of layer heat potentials in anisotropic Sobolev spaces on Lipschitz domains. Hofmann [45] has proved the boundedness in Lebesgue spaces of parabolic singular integral operators of Calderòn type, which in particular include, as a special case, the double layer heat potential. Moreover, Hofmann, Lewis and Mitrea [47] have proved some spectral properties of layer heat potential in Lebesgue spaces on Lipschitz cylinders.

Finally we mention the work of Konënkov [56], where the author has studied the mapping properties of the double layer heat potential in Zygmund spaces and has applied these results to prove the solvability in Zygmund spaces of the Dirichlet problem for the heat equation.

It is worth noting that layer potential methods have been extensively used also for solving heat diffusion problems numerically, and accordingly they turn out to be a very useful tool also in the applications. One of the main advantages of such an approach is to reduce the boundary value problem to an integral equation on the boundary and thus reducing the dimension of the problem by one unit. For example, without being exhaustive, we mention that McIntyre [78] has developed a Galerkin method to solve numerically the boundary integral equation corresponding to a mixed boundary value problem for the heat equation. Costabel, Onishi and Wendland [19] have shown the stability and the convergence of a boundary element collocation method for the Neumann problem for the heat equation in a piecewise smooth cylinder. Costabel [17] has used the theory he has developed for layer potential in order to obtain error estimates for various Galerkin boundary element methods, and he has applied such results to the numerical solution of an eddy current problem. Hamina [42] has developed a numerical method to solve an hypersingular integral equation of the first kind related to a Neumann problem for the heat equation.

Our main result regarding the integral operator associated with the double layer heat potential is the following. If Ω is of class $C^{m,\alpha}$ and $\beta \in]0,\alpha[$, then

i) $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is linear and continuous from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$;

ii) $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is linear and continuous from $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$.

These results are achieved exploiting two tools. The first tool consists in the results of Chapter 1 of this Dissertation, where we analyze a class of general weakly singular time dependent integral operators. More precisely, in this chapter we introduce a normed space of time dependent kernels $\mathcal{K}_{\gamma,a}$, and we prove results of joint continuity for integral operators defined on the boundary of parabolic cylinders of the form

$$u[\partial_T\Omega, K, \mu](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega} K(t,x,\tau,y)\mu(\tau,y)\,d\sigma_y d\tau \qquad \forall\,(t,x)\in\partial_T\Omega$$

upon variation both of the kernel K in $\mathcal{K}_{\gamma,a}$ and of the density function μ in $L^\infty(\partial\Omega)$ and in parabolic Hölder spaces. We want to stress the fact that, since the results of this chapter are proved for integral operators with a general kernel in $\mathcal{K}_{\gamma,a}$, they can be applied to prove new mapping properties not only for the double layer heat potential but also for other integral operators arising in parabolic potential theory, for instance layer potentials corresponding to a general parabolic equation. Moreover, we note that these results are of joint continuity upon variation both of the kernel and of the density, and accordingly they can be used in order to study problems which involve a perturbation of the kernel. The second tool consists in new explicit formula for the tangential derivatives of the double layer heat potential, which allows us to exploit an inductive argument in order to reduce the proof of our main result to the cases already implied by the results of Chapter 1. If $0 < T < +\infty$, the above mapping properties i) and ii) also imply that the map $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is compact in $C_0^{\frac{m}{2};m}(\partial_T\Omega)$, and in $C_0^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$, and in $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$, where the subscript 0 in the spaces means that they are spaces of functions which vanish before 0.

Regularizing and compactness properties for layer potentials have been also considered in the elliptic framework. Without being exhaustive, me mention the works of Schauder [97, 98] and Miranda [79], where the authors prove that the double layer potential associated with the fundamental solution of the Laplace equation is compact in $C^{1,\alpha}(\partial\Omega)$ and is continuous from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$, under the assumption that $n = 3$ and Ω is of class $C^{1,\alpha}$.

Moreover, Fabes, Jodeit and Rivière [35] have proved that if Ω is of the class C^1, then the double layer potential associated with the Laplace operator is compact in $L^p(\partial\Omega)$ for all $p \in]1,+\infty[$. Later, Hofmann, Mitrea and Taylor [48] have proved the same compactness result under more general conditions on $\partial\Omega$.

Wiegner [111] has proved that if $\gamma \in \mathbb{N}$ has odd length and Ω is of the class $C^{m,\alpha}$, then the integral operator with kernel $(x-y)^\gamma|x-y|^{-(n-1)-|\gamma|}$ is continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\mathrm{cl}\,\Omega)$ (and a corresponding result holds for the exterior of Ω).

von Wahl [109] has considered the case of Sobolev spaces and proved that if Ω is of the class C^∞, then the double layer potential associated with the Laplace operator improves the regularity by one unit on the boundary.

Maz'ya and Shaposhnikova [77] have proved that the double layer associated with the Laplace operator is continuous in fractional Sobolev spaces under optimal regularity assumptions on the boundary of Ω.

Mitrea [81] has proved that the double layer potential associated with second order elliptic equations and systems is compact in $C^{0,\beta}(\partial\Omega)$ for $\beta \in]0,\alpha[$ and bounded in $C^{0,\alpha}(\partial\Omega)$ under the assumption that Ω is of the class $C^{1,\alpha}$. Then, by exploiting a formula for the tangential derivatives such results have been extended in order to prove that corresponding compactness and boundedness results holds also in $C^{1,\beta}(\partial\Omega)$ and in $C^{1,\alpha}(\partial\Omega)$, respectively.

Finally, Dondi and Lanza de Cristoforis [32] have proved that if Ω is of class $C^{m,\alpha}$, then the double layer potential associated with a general second order elliptic operator is bounded from $C^m(\partial\Omega)$ to the generalized Schauder space $C^{m,\omega_\alpha}(\partial\Omega)$ of functions with m-th order derivatives satisfying a ω_α-Hölder condition with

$$\omega_\alpha(r) \sim r^\alpha |\log(r)| \qquad \text{as } r \to 0,$$

and is bounded from $C^{m,\beta}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ for $\beta \in]0,\alpha[$. In this work the authors exploit a formula for the tangential derivatives in order to prove the result by induction. In a sense, the regularizing properties of $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ that we prove in this Dissertation can be seen as a generalization of the results of [32] to the case of the double layer potential associated with the heat equation. For a more complete exposition of the previous contribution in the elliptic framework we refer to [32].

For what concerns periodic potential theory, there exists a quite vast literature for the elliptic case. Indeed, periodic analogs of layer potentials have been successfully exploited to analyze a large variety of elliptic boundary value problems in periodic domains. Instead, for the parabolic case, to the best of our knowledge, far less is known.

For example in the elliptic framework Shcherbina [100] has introduced periodic layer potentials to solve periodic boundary value problems for the Laplace equation. Ammari, Kang and Touibi [6] have used periodic layer potentials for deriving the effective properties of isotropic composite materials, while the anisotropic case is considered in Ammari, Kang and Kim [3]. Potential theoretic methods to study singular perturbation problems for the Laplace equation in periodically perforated domains have been used for example in Musolino [82] and Lanza de Cristoforis and Musolino [67]. These methods have been extended also to treat different partial differential equations: see, *e.g.*, Ammari, Kang and Lim [5] and Dalla Riva and Musolino [24] for the Lamé equations and Lanza de Cristoforis and Musolino [68] for a quasi-linear differential equation.

Moreover, Lanza de Cristoforis and collaborators have investigated in some works new properties of periodic potentials (see Lanza de Cristoforis and Musolino [65] for periodic layer potentials associated with a general second order elliptic differential operators and Dalla Riva, Lanza de Cristoforis and Musolino [22] for periodic volume potentials).

Periodic layer potentials are constructed by replacing the fundamental solution by a periodic analog of the fundamental solution in the definition of classical layer potential. Therefore, a key step is the definition of such an analog. For the Laplace equation, such a function has been introduced in a seminal paper by Hasimoto [43] (see also Ammari and Kang [4], Lanza de Cristoforis and Musolino [65], and Shcherbina [100]), whereas for the Lamé equations we refer, for example, to Ammari, Kang, and Lim [5]. For our work on periodic layer potentials for the heat equation we use the q-periodization of the heat kernel Φ_n, *i.e.*, the function $\Phi_{q,n}$ defined by

$$\Phi_{q,n}(t,x) \equiv \begin{cases} \sum_{z\in\mathbb{Z}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} & \text{if } (t,x) \in]0,+\infty[\times\mathbb{R}^n\,, \\ 0 & \text{if } (t,x) \in (]-\infty,0]\times\mathbb{R}^n) \setminus (\{0\}\times q\mathbb{Z}^n)\,, \end{cases}$$

where q is a $n \times n$ diagonal matrix with strictly positive entries in the diagonal.

Our aim is to introduce space-periodic layer heat potentials and to study their properties. In particular, we are interested in proving the periodic analog of the mapping properties we have proved for the (non-periodic) double layer heat potential $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$. The boundary integral operator corresponding to the q-periodic double layer heat potential is the map which takes a function μ from $\partial_T\Omega$ to $\mathbb{C}$ and maps it to the function $w_q[\partial_T\Omega, \mu]_{|\partial_T\Omega}$ defined by

$$w_q[\partial_T\Omega, \mu]_{|\partial_T\Omega}(t,x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \Phi_{q,n}(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau \qquad \forall\,(t,x) \in \partial_T\Omega.$$

We are able to prove that if Ω is of class $C^{m,\alpha}$, then

j) $w_q[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ is linear and continuous from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$;

jj) $w_q[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ is linear and continuous from $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$.

As before, the above mapping properties j) and jj) imply the compactness of $w_q[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ in $C_0^{\frac{m}{2};m}(\partial_T\Omega)$, and in $C_0^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$, and in $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$, if $0 < T < +\infty$.

Finally, we show how to employ periodic layer heat potentials in order to solve some initial-boundary value problems for the heat equation in the periodic setting.

We would like to point out the motivations of the work of this first part of the Dissertation. From one side we continue the study of integral operators arising in parabolic potential theory, which has an interest on its own since potential theory is a well established and important part of mathematical analysis. From the other side, we would like to develop a sufficient set of tools in order to consider singularly perturbed problems for the heat equation with the *Functional Analytic Approach (FAA)* introduced by Lanza de Cristoforis and then developed further by Lanza de Cristoforis and collaborators. This method makes use of potential theory and of functional analysis, and has shown to be powerful to investigate the dependence of the solution of elliptic boundary value problems upon regular and singular domain perturbations. Therefore, we plan to extend it for the first time to parabolic problems, starting from the heat equation. The results of this first part of the Dissertation are a first necessary step in this direction.

Regarding this potential theoretic method for perturbation problems we mention the analysis of a regular perturbation problem for the Laplace equation carried out in Lanza de Cristoforis [59], and the analysis of a singular perturbation problem for the Laplace equation in a domain with a small hole worked out in Lanza de Cristoforis [61]. This approach has been also extended to singular perturbation problems with nonlinear boundary conditions (see, *e.g.*, Lanza de Cristoforis [60, 62] for a nonlinear Robin problem and a nonlinear transmission problem in a domain with a small hole and Dalla Riva and Lanza de Cristoforis [21] for a singularly perturbed nonlinear traction problem for linearized elastostatics), and to the Stokes flow (see, *e.g.*, Dalla Riva [20] for a singularly perturbed boundary value problem for the steady state Stokes flow), and to singular perturbation problems in periodically perforated domains (see, *e.g.*, Musolino [82] for a singularly perturbed Dirichlet problem in a periodically perforated domain, Dalla Riva and Musolino [24] for a nonlinear traction problem for linearized elastostatics in a periodically perforated domain, Lanza de Cristoforis and Musolino [67, 68] for a nonlinear Robin problem and for a quasi-linear transmission problem in periodically perforated domains, and Pukhtaievych [93] for a transmission problem in a periodically perforated domain), and to other geometric settings (cf., *e.g.*, Dalla Riva and Musolino [25, 26] for multiple holes shrinking to a point, and Costabel, Dalla Riva, Dauge and Musolino [18] for a hole shrinking to a corner point of the boundary, and Bonnaillie-Noël, Dalla Riva, Dambrine, and Musolino [10] for a hole shrinking to a regular point of the boundary).

The first part of the Dissertation is organized as follows. In Chapter 1 we introduce a normed space of time dependent kernels for integral operators defined on the boundary of infinite parabolic cylinders. We prove results of joint continuity of such integral operators upon variation both of the kernel and of the density function. Moreover, we apply these results to some integral operators related to layer heat potentials. Chapter 2 is devoted to the mapping properties in parabolic Schauder spaces of the integral operator associated with the double layer heat potential. More precisely, we prove an explicit formula for the tangential derivatives of the double layer heat potential and we prove some regularizing properties exploiting such a formula together with the results of the previous Chapter 1. Finally, Chapter 3 is devoted to space-periodic layer heat potentials. We prove basic properties and some regularizing properties and we apply these results to solve some periodic boundary value problems for the heat equation. More precisely, we consider a periodic Dirichlet problem, a periodic Neumann problem, and a periodic non-ideal transmission problem. Finally, we have collected in the Appendices A and B some results on layer heat potentials that we have exploited in this part of the Dissertation. In particular, in the Appendix B we prove some regularity results for the layer heat potentials which may be considered as *folklore* and for which we could not provide a proper reference.

Note: Some of the results presented in this first part of the Dissertation have appeared or will appear in papers by the author (see [75]), and by Lanza de Cristoforis and the author (see [63, 64]).

Second part

The second part of the Dissertation is inserted in the framework of the *Functional Analytic Approach (FAA)* developed by Lanza de Cristoforis and collaborators that we have discussed above, and it is contained in Chapter 4. More precisely, this part of the Dissertation is devoted to the study of the behavior of the longitudinal permeability of a periodic array of cylinders upon perturbation of the shape of the cross section of the cylinders and upon perturbation of the periodicity structure, when a Newtonian fluid is flowing at low Reynolds numbers around the cylinders. The shape of the cross section of the cylinders is determined by the image of a base domain through a diffeomorphism ϕ and the periodicity cell is a rectangle of sides of length l and $1/l$, where l is a positive parameter. We also assume that the pressure gradient is parallel to the cylinders. Under such assumptions, the velocity field has only one non-zero component which, by the Stokes equations, satisfies a Poisson equation. Then, by integrating the longitudinal component of the velocity field, for each pair (l, ϕ), one defines the longitudinal permeability $K_{II}[l, \phi]$. In this part of the Dissertation, we are interested in studying the behavior of $K_{II}[l, \phi]$ upon the pair (l, ϕ).

In the literature, the fluid flow through periodic structures has been studied by several authors by exploiting different methods. For example, Hasimoto has investigated in [43] the viscous flow past a cubic array of spheres and he has applied his results to the two-dimensional flow past a square array of circular cylinders. His techniques are based on the construction of a spatially periodic fundamental solution for the Stokes' system and apply to specific shapes (circular/spherical obstacles and square/cubic arrays). Schmid [99] has investigated the longitudinal laminar flow in an infinite square array of circular cylinders. Sangani and Yao [95, 96] have studied the permeability of random arrays of infinitely long cylinders. Mityushev and Adler [85, 86] have considered the longitudinal permeability of periodic rectangular arrays of circular cylinders. By means of complex variable techniques, they have transformed the boundary value problem defining the permeability into a functional equation and then they have derived a formula for the longitudinal permeability as a logarithmic term and a power series in the radius of the cylinder. Finally, in Musolino and Mityushev [84] the asymptotic behavior of the longitudinal permeability of thin cylinders of arbitrary shape has been considered.

Here, instead, we are interested into the dependence of the longitudinal permeability upon the variation of the sides of the rectangular array and upon the variation of the shape of the cross section of the cylinders. In particular, in contrast with other approaches in the literature, we do not need to restrict ourselves to particular shapes, as circles or ellipses.

In our main result we prove that the map

$$(l, \phi) \mapsto K_{II}[l, \phi] \tag{1}$$

is real analytic. Such a result implies, in particular, that if we have a one-parameter analytic family of pairs $(l_\delta, \phi_\delta)_{\delta \in]-\delta_0, \delta_0[}$, then we can deduce the possibility to expand the permeability as a convergent power series, *i.e.*,

$$K_{II}[l_\delta, \phi_\delta] = \sum_{j=0}^{+\infty} c_j \delta^j \tag{2}$$

for δ close to zero. Moreover, by the analyticity of the map in (1), the coefficients $(c_j)_{j\in\mathbb{N}}$ in (2) can be constructively determined by computing the differentials of $K_{II}[\cdot,\cdot]$ (see Dalla Riva, Musolino, and Rogosin [27] for the solution of the Dirichlet problem for the Laplace equation in a planar domain with a small hole and Dalla Riva, Musolino, and Pukhtaievych [28] for the effective conductivity of a periodic composites with small inclusions). Furthermore, another important consequence of our high regularity result is that it allows to apply differential calculus in order to find critical *rectangle-shape* pairs (l, ϕ) as a first step to find optimal configurations.

In order to introduce in a more precise way the mathematical problem, we now set some notation. For $l \in]0, +\infty[$, we define

$$Q_l \equiv]0, l[\times]0, 1/l[\,, \qquad q_l \equiv \begin{pmatrix} l & 0 \\ 0 & 1/l \end{pmatrix}.$$

Moreover, we find convenient to set

$$\widetilde{Q} \equiv Q_1\,, \qquad \tilde{q} \equiv q_1\,.$$

We fix $\alpha \in]0,1[$ and a bounded open connected subset Ω of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Then we consider a class of diffeomorphisms $\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$ from $\partial\Omega$ into $\widetilde{Q}$. If $\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^2 \setminus \phi(\partial\Omega)$ has exactly two open connected components, and we denote by $\mathbb{I}[\phi]$ the bounded open connected components of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$. Then we consider the following two periodic domains

$$\begin{aligned} \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]] &\equiv \bigcup_{z\in\mathbb{Z}^2} (q_l z + q_l\mathbb{I}[\phi])\,, \\ \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- &\equiv \mathbb{R}^2 \setminus \mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]\,. \end{aligned}$$

The set $\mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]] \times \mathbb{R}$ represents an infinite array of parallel cylinders. Instead, the set $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \times \mathbb{R}$ is the region where a Newtonian fluid is flowing at low Reynolds number. Then we assume that the driving pressure gradient is constant and parallel to the cylinders. As a consequence, by a standard argument based on the particular geometry of the problem (cf., *e.g.*, Adler [1, Chapter 4], Sangani and Yao [96], and Mityushev and Adler [85, 86]), one reduces the Stokes system to a Poisson equation for the non-zero component of the velocity field. Since we are working with dimensionless quantities, we may assume that the viscosity of the fluid and the pressure gradient are both set equal to one. Accordingly, if $l \in]0,+\infty[$ and $\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$, we consider the following Dirichlet problem for the Poisson equation:

$$\begin{cases} \Delta u = 1 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- , \\ u(x + q_l z) = u(x) & \forall\, x \in \mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \forall\, z \in \mathbb{Z}^2 , \\ u(x) = 0 & \forall\, x \in \partial\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- . \end{cases} \tag{3}$$

The solution of problem (3) in the space $C^{1,\alpha}_{q_l}(\mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-)$ of q_l-periodic functions in $\mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$ of class $C^{1,\alpha}$ is unique and we denote it by $u[l,\phi]$. From the physical point of view, the function $u[l,\phi]$ represents the non-zero component of the velocity field. By means of the function $u[l,\phi]$, we can introduce the effective longitudinal permeability $K_{II}[l,\phi]$ which we define as the integral of the opposite of the flow velocity over the unit cell, *i.e.*,

$$K_{II}[l,\phi] \equiv -\int_{Q_l\setminus q_l\mathbb{I}[\phi]} u[l,\phi](x)\,dx \qquad \forall\, l \in]0,+\infty[,\ \forall\, \phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2),$$

and we pose the following question:

$$\text{What can be said on the regularity of the map } (l,\phi) \mapsto K_{II}[l,\phi]? \tag{4}$$

Shape analysis of functionals related to partial differential equations or quantities of physical relevance has been carried out by several authors and it is impossible to provide a complete list of contributions. Here we mention, for example, the monographs by Henrot and Pierre [44], by Novotny and Sokołowski [88], and by Sokołowski and Zolésio [101].

Most of the works in the literature deal with differentiability properties. Here, instead, we are interested into proving higher regularity and we answer the question in (4) by showing that $K_{II}[l,\phi]$ depends analytically on (l,ϕ). Our analysis is based on the study of a boundary value problem in a periodic domain by means of (periodic) potential theory. We now briefly outline our strategy. First of all, by means of periodic potential theory, we convert our boundary value problem into an integral equation defined on the (l,ϕ)-dependent boundary $q_l\phi(\partial\Omega)$. Next, we transform such an equation into an equivalent integral equation, which is now defined on the fixed domain $\partial\Omega$. Finally, we analyze the dependence of the solution of the integral equation upon (l,ϕ) exploiting the Implicit Function Theorem for real analytic maps in Banach spaces.

Finally, we want to mention also that boundary value problems in periodic domains have been analyzed with the method of functional equations: see, *e.g.*, Castro and Pesetskaya [13], Castro, Pesetskaya, and Rogosin [14], Kapanadze, Mishuris, and Pesetskaya [54, 55], Rogosin, Dubatovskaya, and Pesetskaya [94].

Note: Some of the results presented in this second part of the thesis will appear in a paper by Musolino, Pukhtaievych and the author [76].

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Notation

The symbol $\mathbb{N}$ denotes the set of natural numbers including 0, the symbol $\mathbb{Z}$ denotes the set of integer numbers, the symbol $\mathbb{R}$ denotes the set of real numbers, and the symbol $\mathbb{C}$ denotes the set of complex numbers. We fix once for all here and throughout all the Dissertation

$$n \in \mathbb{N} \setminus \{0, 1\}$$

that will be the dimension of the space $\mathbb{R}^n$. We denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$. If A is a matrix, then we denote by A^t the transpose matrix of A, and by A^{-1} the inverse matrix of A, and by A_{ij} the (i, j) entry of A. Let $k \in \mathbb{N} \setminus \{0\}$. We denote with $(\delta_{i,j})_{i,j \in \{1,\ldots,k\}}$, the Kronecker symbol, that is

$$\delta_{i,j} \equiv \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise }. \end{cases}$$

We denote by Γ the Euler Γ-function (cf., *e.g.*, Folland [37, p. 58].)

We denote the norm on a normed space $\mathcal{X}$ by $\| \cdot \|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by

$$\|(x, y)\|_{\mathcal{X} \times \mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}} \qquad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the Banach space of linear and continuous maps of $\mathcal{X}$ to $\mathcal{Y}$, equipped with the usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. We denote by I the identity operator from $\mathcal{X}$ to $\mathcal{X}$. For standard definitions of Calculus in normed spaces, we refer to Deimling [29].

Let $s \in \mathbb{N} \setminus \{0\}$. Let $\mathbb{D} \subseteq \mathbb{R}^s$. Then $\mathrm{cl}\, \mathbb{D}$ denotes the closure of $\mathbb{D}$, and $\partial \mathbb{D}$ denotes the boundary of $\mathbb{D}$, and $\mathrm{diam}(\mathbb{D})$ denotes the diameter of $\mathbb{D}$. If f is a function from $\mathbb{D}$ to $\mathbb{C}$, then

$$\mathrm{supp}\,(f)$$

denotes the support of f in $\mathbb{D}$. The inverse of an invertible function f is denoted by $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, which is denoted g^{-1}.

In the case of time intervals $]t_1, t_2[$, $t_1 \in [-\infty, +\infty[$, $t_2 \in] - \infty, +\infty]$, we will denote by

$$\overline{]t_1, t_2[}$$

its closure. If $x \in [0, +\infty[$, when there is no ambiguity using this notation, we denote by $[x]$ the integer part of x, that is

$$[x] \equiv \max\{l \in \mathbb{Z} : l \leq x\},$$

and by $\{x\}$ the fractional part of x, that is

$$\{x\} \equiv x - [x].$$

The symbol $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^s$ or in $\mathbb{C}$, while a dot '$\cdot$' denotes the inner product in $\mathbb{R}^s$. For all $R \in]0, +\infty[$, $x \in \mathbb{R}^s$, x_j denotes the j-th coordinate of x for all $j \in \{1, \ldots, s\}$, and $\mathbb{B}_s(x, R)$ denotes the ball $\{y \in \mathbb{R}^s : |x - y| < R\}$ and we set

$$\mathbb{B}_s \equiv \mathbb{B}_s(0, 1).$$

The symbols $B(\mathbb{D}, \mathcal{X})$ and $C^0(\mathbb{D}, \mathcal{X})$ denote the space of bounded and continuous functions from $\mathbb{D}$ to $\mathcal{X}$, respectively. We endow $B(\mathbb{D}, \mathcal{X})$ with the sup-norm and we set

$$C_b^0(\mathbb{D}, \mathcal{X}) \equiv C^0(\mathbb{D}, \mathcal{X}) \cap B(\mathbb{D}, \mathcal{X}).$$

Let Ω be an open subset of $\mathbb{R}^n$. We denote the exterior of Ω by

$$\Omega^- \equiv \mathbb{R}^n \setminus \operatorname{cl}\Omega.$$

Let $m \in \mathbb{N}\setminus\{0\}$. The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega, \mathbb{C})$, or more simply by $C^m(\Omega)$. We denote $C^\infty(\Omega)$ the set of functions f such that $f \in C^k(\Omega)$ for all $k \in \mathbb{N}$. Let $f \in C^m(\Omega)$. Then Df denotes the Jacobian matrix of f. We set

$$|\eta|_1 \equiv \sum_{i=1}^n |\eta_i| \qquad \forall \eta \in \mathbb{Z}^n.$$

If $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$, then

$$D^\eta f \equiv \frac{\partial^{|\eta|_1} f}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}\,,$$
$$\eta! \equiv \prod_{i=1}^n \eta_i!\,,$$
$$x^\eta \equiv \prod_{i=1}^n x_i{}^\eta \qquad \forall x \in \mathbb{R}^n.$$

The symmetric Hessian matrix of the second order partial derivatives of f is denoted $D^2 f$. Moreover the subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \le m$ can be extended with continuity to $\operatorname{cl}\Omega$ is denoted $C^m(\operatorname{cl}\Omega)$. The space $C^m(\operatorname{cl}\Omega)$ endowed with the norm

$$\|f\|_{C^m(\operatorname{cl}\Omega)} \equiv \sum_{\substack{\eta\in\mathbb{N}^n \\ |\eta|_1 \le m}} \sup_{\operatorname{cl}\Omega} |D^\eta f| \qquad \forall f \in C^m(\operatorname{cl}\Omega),$$

is a Banach space. For the definition of open subsets of $\mathbb{R}^n$ of class C^m we refer, *e.g*, to Gilbarg and Trudinger [41, pp. 94–95]. Moreover we denote by

$$\nu_\Omega$$

the outward unit normal to Ω, where it is defined. Sometimes, when there is no ambiguity, we will drop the subscript Ω in ν_Ω. We denote by $d\sigma$ the standard surface measure on a manifold of codimension 1 of $\mathbb{R}^n$. We will sometimes attach to $d\sigma$ a subscript to indicate the integration variable. If $\mathbb{D}$ is a measurable subset of $\mathbb{R}^n$, and $k \in \mathbb{N}$, the k-dimensional measure of the set $\mathbb{D}$ is denoted by $m_k(\mathbb{D})$.

We retain the standard notation of L^p spaces. In particular, if Ω is a measurable nonempty subset of $\mathbb{R}^n$ and $1 \le p < +\infty$ (resp. $p = +\infty$), we write $L^p(\Omega)$ for the space of (equivalence classes of) p-summable (resp. essentially bounded) measurable functions with complex values, endowed with the usual norm. We denote by $L^p_{\mathrm{loc}}(\Omega)$ the set of (equivalence classes of) functions f of Ω to $\mathbb{C}$ such that $f \in L^p(K)$ for each compact $K \subset \Omega$.

We note that throughout the Dissertation ‘analytic’ means ‘real analytic’. For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [92, p. 89] and to Deimling [29].

Part I

First part

CHAPTER 1

Time dependent boundary norms for kernels and time dependent integral operators

This chapter is mainly devoted to continuity and regularizing properties of some time dependent integral operators defined on the boundary of infinite parabolic cylinders upon variation both of the kernel and of the density (or moment) function. In addition, we apply these results in order to recover some regularizing properties in Schauder spaces for certain integral operators related to layer heat potentials.

First of all, in Section 1.1, we recall the definitions of the classical Schauder spaces and of their parabolic counterpart, *i.e.*, parabolic Schauder spaces. In Section 1.2 we collect some certainly known preliminary inequalities that we will exploit in the present chapter. Then in Sections 1.3 and 1.4 we prove some estimates for the fundamental solution of the heat equation, its derivatives, and the kernel of the double layer heat potential. In Section 1.5 we introduce a class of function spaces and norms for time dependent kernels of integral operators defined on the boundary of parabolic cylinders. Moreover, exploiting the results of Sections 1.3 and 1.4, we verify that the kernel associated with the fundamental solution of the heat equation, and its first order space and time derivatives, and the kernel of the double layer heat potential belong to such classes. In Section 1.6, we estimate the norm of an integral operator with kernel K applied to a density μ in terms of the norm of K in the above classes and of the L^∞-norm of μ, and in Section 1.7 we apply the results of Section 1.6 to some integral operators related to layer heat potentials. In Section 1.8 we consider the case of integral operators acting on Schauder spaces. More precisely we estimate the norm of an integral operator with kernel K applied to a density μ in terms of the norm of K in the above classes and of the Hölder norm of μ. Finally, in Section 1.9 we apply the results of Section 1.8 to some integral operators related to layer heat potentials.

The results of the present chapter will be also exploited in the next Chapter 2 in order to recover certain regularizing properties of the double layer heat potential. We want to stress the fact that, since the results of this chapter are proved for general time dependent integral operators with kernels in the classes that we have defined, in further works we plan to apply them to recover similar mapping properties for other integral operators arising in parabolic potential theory, for instance layer potentials corresponding to a general parabolic equation. Furthermore, we believe that the methods of Sections 1.6 and 1.8 may be applied to simplify also the exposition of other classical proofs of properties of layer potentials. Finally, we note that the type of results we prove in Sections 1.6 and 1.8 are of joint continuity upon variation both of the kernel and of the density, and accordingly they can be used in order to study problems which involve a perturbation of the kernel.

The results contained in the present chapter can be found in a paper by Lanza de Cristoforis and the author [63].

1.1 Classical and parabolic Schauder spaces

In this section we recall the definitions of classical Schauder spaces and of parabolic Schauder spaces. Let ω be a function from $]0,+\infty[$ to itself such that

$$\begin{aligned} &\text{i) } \omega \text{ is increasing,} \\ &\text{ii) } \lim_{r\to 0^+} \omega(r) = 0, \\ &\text{iii) } \sup_{(a,r)\in[1,+\infty[\times]0,+\infty[} \frac{\omega(ar)}{a\omega(r)} < +\infty, \\ &\text{iv) } \sup_{r\in]0,1[} \frac{r}{\omega(r)} < +\infty. \end{aligned} \tag{1.1}$$

Let $s \in \mathbb{N}\setminus\{0\}$. If f is a function from a subset $\mathbb{D}$ of $\mathbb{R}^s$ to a normed space $\mathcal{X}$, we set

$$|f : \mathbb{D}|_{\omega(\cdot)} \equiv \sup\left\{ \frac{\|f(x)-f(y)\|_{\mathcal{X}}}{\omega(|x-y|)} : x,y\in\mathbb{D}, x\neq y \right\}.$$

If $|f : \mathbb{D}|_{\omega(\cdot)} < \infty$, we say that f is $\omega(\cdot)$-Hölder continuous. Sometimes, we simply write $|f|_{\omega(\cdot)}$ instead of $|f : \mathbb{D}|_{\omega(\cdot)}$. We set

$$C^{0,\omega(\cdot)}(\mathbb{D},\mathcal{X}) \equiv \{f \in C^0(\mathbb{D},\mathcal{X}) : |f : \mathbb{D}|_{\omega(\cdot)} < +\infty\}$$

to be the subspace of $C^0(\mathbb{D},\mathcal{X})$ whose functions are $\omega(\cdot)$-Hölder continuous. The space

$$C_b^{0,\omega(\cdot)}(\mathbb{D},\mathcal{X}) \equiv C^{0,\omega(\cdot)}(\mathbb{D},\mathcal{X}) \cap B(\mathbb{D},\mathcal{X})$$

endowed with the norm

$$\|f\|_{C_b^{0,\omega(\cdot)}(\mathbb{D},\mathcal{X})} \equiv \sup_{\mathbb{D}} \|f\|_{\mathcal{X}} + |f : \mathbb{D}|_{\omega(\cdot)}$$

is well known to be a Banach space. If $\mathcal{X} = \mathbb{C}$, we simply write $C^0(\mathbb{D})$, $C^{0,\omega(\cdot)}(\mathbb{D})$, $C_b^{0,\omega(\cdot)}(\mathbb{D})$ instead of $C^0(\mathbb{D},\mathbb{C})$, $C^{0,\omega(\cdot)}(\mathbb{D},\mathbb{C})$, $C_b^{0,\omega(\cdot)}(\mathbb{D},\mathbb{C})$, respectively.

Particularly important is the case in which the modulus of continuity $\omega(\cdot)$ is the function r^α for some fixed $\alpha \in]0,1]$. One can easily check that with such a choice the conditions i)–iv) on $\omega(\cdot)$ in (1.1) hold. In this case, we simply write $C^{0,\alpha}(\mathbb{D})$, $C_b^{0,\alpha}(\mathbb{D})$, $|\cdot : \mathbb{D}|_\alpha$ instead of $C^{0,r^\alpha}(\mathbb{D})$, $C_b^{0,r^\alpha}(\mathbb{D})$, $|\cdot : \mathbb{D}|_{r^\alpha}$, respectively.

Then we have the following remark of immediate verification, which shows that the difficulty of estimating the Hölder quotient $\frac{\|f(x)-f(y)\|_{\mathcal{X}}}{\omega(|x-y|)}$ of a bounded function f lies entirely in case when $|x-y|$ is small, that is in the case $0 < |x-y| < a$ for a fixed $a \in]0,+\infty[$.

Remark 1.1. Let $s \in \mathbb{N}\setminus\{0\}$. Let ω satisfy conditions i)–iv) in (1.1). Let $\mathbb{D}$ be a subset of $\mathbb{R}^s$. Let $\mathcal{X}$ be a normed space. Let $a \in]0,+\infty[$ and $f \in C_b^0(\mathbb{D},\mathcal{X})$. Then,

$$\sup_{\substack{x,y\in\mathbb{D} \\ |x-y|\geq a}} \frac{\|f(x)-f(y)\|_{\mathcal{X}}}{\omega(|x-y|)} \leq \frac{2}{\omega(a)} \sup_{\mathbb{D}} \|f\|_{\mathcal{X}}\,.$$

Moreover, it is known that the pointwise product in Hölder spaces is continuous. In particular, the following result holds.

Lemma 1.2. *Let $s \in \mathbb{N} \setminus \{0\}$. Let $\mathbb{D}$ be a subset of $\mathbb{R}^s$. Let ψ_1, ψ_2, ψ_3 satisfy the conditions i)–iv) in (1.1). Let*

$$\sup_{j=1,2} \sup_{r \in]0,1[} \psi_j(r)\psi_3^{-1}(r) < +\infty.$$

Then the pointwise product is bilinear and continuous from $C_b^{0,\psi_1(\cdot)}(\mathbb{D}) \times C_b^{0,\psi_2(\cdot)}(\mathbb{D})$ to $C_b^{0,\psi_3(\cdot)}(\mathbb{D})$.

Proof. Let $f \in C_b^{0,\psi_1(\cdot)}(\mathbb{D})$ and $g \in C_b^{0,\psi_2(\cdot)}(\mathbb{D})$. Let $x, y \in \mathbb{D}$, $x \neq y$. Then

$$\begin{aligned}
|f(x)g(x) - & f(y)g(y)| \\
&= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\
&\leq \|f\|_{C_b^{0,\psi_1(\cdot)}(\mathbb{D})} \|g\|_{C_b^{0,\psi_2(\cdot)}(\mathbb{D})} \psi_2(|x-y|) + \|f\|_{C_b^{0,\psi_1(\cdot)}(\mathbb{D})} \|g\|_{C_b^{0,\psi_2(\cdot)}(\mathbb{D})} \psi_1(|x-y|) \\
&= \|f\|_{C_b^{0,\psi_1(\cdot)}(\mathbb{D})} \|g\|_{C_b^{0,\psi_2(\cdot)}(\mathbb{D})} (\psi_2(|x-y|) + \psi_1(|x-y|)).
\end{aligned}$$

Accordingly, Remark 1.1 implies the validity of the statement. □

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1]$. Let Ω be an open subset of $\mathbb{R}^n$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have m-th order derivatives that are α-Hölder continuous is denoted

$$C^{m,\alpha}(\mathrm{cl}\,\Omega).$$

Then $C_b^{m,\alpha}(\mathrm{cl}\,\Omega)$ denotes the space of m-times continuously differentiable functions f in Ω such that

$$\|f\|_{C_b^{m,\alpha}(\mathrm{cl}\,\Omega)} \equiv \|f\|_{C_b^m(\mathrm{cl}\,\Omega)} + \sum_{|\eta|=m} |D^\eta f : \Omega|_\alpha < +\infty.$$

The space $(C_b^{m,\alpha}(\mathrm{cl}\,\Omega), \|\cdot\|_{C_b^{m,\alpha}(\mathrm{cl}\,\Omega)})$ is well known to be a Banach space (see, *e.g.*, Gilbarg and Trudinger [41, pp. 52–53]). Obviously, if Ω is bounded then $C_b^{m,\alpha}(\mathrm{cl}\,\Omega) = C^{m,\alpha}(\mathrm{cl}\,\Omega)$ (and in this case we always drop the subscript b). The subspace of $C^m(\mathrm{cl}\,\Omega)$ of those functions f such that $f_{|\mathrm{cl}(\Omega \cap \mathbb{B}_n(0,R))} \in C^{m,\alpha}(\mathrm{cl}\,(\Omega \cap \mathbb{B}_n(0,R)))$ for all $R \in]0,+\infty[$ is denoted

$$C_{\mathrm{loc}}^{m,\alpha}(\mathrm{cl}\,\Omega).$$

For the definition of open subsets of $\mathbb{R}^n$ of class $C^{m,\alpha}$ for some $\alpha \in]0,1]$, we refer to Gilbarg and Trudinger [41, pp. 94–95].

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. By means of local parametrization it is possible to define the normed space

$$(C^{m,\alpha}(\partial\Omega), \|\cdot\|_{C^{m,\alpha}(\partial\Omega)}).$$

It is well known that $C^{m,\alpha}(\partial\Omega)$ is a Banach space (see, *e.g.*, Gilbarg and Trudinger [41, pp. 94–95]).

Then we have the following well known extension result. For a proof, we refer to Troianiello [103, Theorem 1.3, Lemma 1.5] (see also Gilbarg and Trudinger [41, Lemma 6.38, p. 137]).

Lemma 1.3. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1]$, $j \in \{0,\dots,m\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Then there exists a linear and continuous extension operator '$\tilde{\cdot}$' of $C^{j,\alpha}(\partial\Omega)$ to $C^{j,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R))$, which takes $f \in C^{j,\alpha}(\partial\Omega)$ to a map $\tilde{f} \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R))$ such that $\tilde{f}_{|\partial\Omega} = f$ and such that the support of $\tilde{f}$ is compact and contained in $\mathbb{B}_n(0,R)$. The same statement holds by replacing $C^{m,\alpha}$ and $C^{j,\alpha}$ by C^m and C^j, respectively.*

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. We denote by $\nu \equiv (\nu_l)_{l=1,\dots,n}$ the external unit normal field to $\partial\Omega$. Let $i,j \in \{1,\dots,n\}$. Let $f \in C^1(\partial\Omega)$. Then the M_{ij}-tangential derivative of f is defined as

$$M_{ij}[f] \equiv \nu_i \frac{\partial \tilde{f}}{\partial x_j} - \nu_j \frac{\partial \tilde{f}}{\partial x_i} \qquad \text{on } \partial\Omega, \tag{1.2}$$

and the tangential gradient $D_{\partial\Omega} f$ of f is defined as

$$D_{\partial\Omega} f \equiv D\tilde{f} - (\nu \cdot D\tilde{f})\nu \qquad \text{on } \partial\Omega,$$

where $\tilde{f}$ is an extension of f of class C^1 in an open neighborhood of $\partial\Omega$ as in Lemma 1.3. It is easy to verify that $M_{ij}[f]$ and $D_{\partial\Omega} f$ are independent on the specific choice of the extension $\tilde{f}$ of f. Moreover, a simple computation shows that the following formula for the r-th component of $D_{\partial\Omega} f$ hold.

$$\frac{\partial \tilde{f}}{\partial x_r} - (\nu \cdot D\tilde{f})\nu_r = \sum_{l=1}^{n} M_{lr}[f]\nu_l \qquad \text{on } \partial\Omega,\ \forall\, r \in \{1,\dots,n\}. \tag{1.3}$$

We will need the following well known consequence of the Divergence Theorem.

Lemma 1.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. If $f,g \in C^1(\partial\Omega)$, then*

$$\int_{\partial\Omega} M_{ij}[f] g \, d\sigma = -\int_{\partial\Omega} f M_{ij}[g] \, d\sigma,$$

for all $i,j \in \{1,\dots,n\}$.

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. We note that, exploiting the definition of norm in Schauder spaces, it is possible to prove that the norm $\|\cdot\|_{C^{m,\alpha}(\partial\Omega)}$ is equivalent to the norm on $C^{m,\alpha}(\partial\Omega)$ defined by

$$\|f\|_{C^0(\partial\Omega)} + \sum_{i,j=1}^{n} \|M_{ij}[f]\|_{C^{m-1,\alpha}(\partial\Omega)} \qquad \forall\, f \in C^{m,\alpha}(\partial\Omega).$$

Next we turn to recall the definitions of parabolic Hölder spaces defined on cylindrical domains. If $T \in]-\infty,+\infty]$ and if $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then we set

$$\mathbb{D}_T \equiv \overline{]-\infty,T[} \times \mathbb{D}\,, \qquad \partial_T\mathbb{D} \equiv (\partial\mathbb{D})_T = \overline{]-\infty,T[} \times \partial\mathbb{D}\,.$$

Clearly $\overline{]-\infty,T[} =]-\infty,T]$ if $T \in \mathbb{R}$ and $\overline{]-\infty,T[} =]-\infty,+\infty[$ if $T = +\infty$. We also note that

$$(\mathrm{cl}\,\mathbb{D})_T = \mathrm{cl}\,\mathbb{D}_T\,.$$

Remark 1.5. As is well known, the map Ξ from the vector space $\mathbb{C}^{\mathbb{D}_T}$ of functions from $\mathbb{D}_T$ to $\mathbb{C}$ to the vector space $(\mathbb{C}^{\mathbb{D}})^{\overline{]-\infty,T[}}$ of functions from $\overline{]-\infty,T[}$ to $\mathbb{C}^{\mathbb{D}}$, which takes a function f to the function Ξf from $\overline{]-\infty,T[}$ to $\mathbb{C}^{\mathbb{D}}$ which takes t to $f(t,\cdot)$ is an isomorphism. As a rule, we omit to write the canonical identification map Ξ.

We now introduce the definition of parabolic Hölder spaces, which are spaces of functions with an anisotropy in the Hölder regularity with respect to the time direction and with respect to the space direction.

Definition 1.6. Let $\alpha',\alpha''\in]0,1]$, $T\in]-\infty,+\infty]$. Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$. Then

$$C^{0,\alpha';0,\alpha''}(\mathbb{D}_T)$$

denotes the space of bounded continuous functions u from $\mathbb{D}_T$ to $\mathbb{C}$ such that

$$\begin{aligned}\|u\|_{C^{0,\alpha';0,\alpha''}(\mathbb{D}_T)} \equiv \sup_{\mathbb{D}_T}|u| + \sup_{\substack{t_1,t_2\in\overline{]-\infty,T[}\\ t_1\neq t_2}} \frac{\|u(t_1,\cdot)-u(t_2,\cdot)\|_{C^0_b(\mathbb{D})}}{|t_1-t_2|^{\alpha'}} \\ + \sup_{t\in\overline{]-\infty,T[}} |u(t,\cdot):\mathbb{D}|_{\alpha''} < +\infty\,.\end{aligned}$$

It is well known that $(C^{0,\alpha';0,\alpha''}(\mathbb{D}_T),\|\cdot\|_{C^{0,\alpha';0,\alpha''}(\mathbb{D}_T)})$ is a Banach space. By Remark 1.5, $u\in C^{0,\alpha';0,\alpha''}(\mathbb{D}_T)$ if and only if the canonically identified map Ξu belongs to

$$C^{0,\alpha'}\left(\overline{]-\infty,T[},C^0_b(\mathbb{D})\right)\cap B\left(\overline{]-\infty,T[},C^{0,\alpha''}_b(\mathbb{D})\right).$$

In the parabolic setting, one is usually interested in the parabolic Hölder spaces where the regularity in time is half of the regularity in space. In this case, in the literature it is common to find another definition of the parabolic Hölder norm, which can be easily proved to be equivalent to the previous one in the case of cylinders. More precisely, we have the following.

Proposition 1.7. *Let $\alpha\in]0,1[$, $T\in]-\infty,+\infty]$. Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$. Then a function u is in $C^{0,\frac{\alpha}{2};0,\alpha}(\mathbb{D}_T)$ if and only if*

$$\|u\|'_{C^{0,\frac{\alpha}{2};0,\alpha}(\mathbb{D}_T)} \equiv \sup_{\mathbb{D}_T}|u| + \sup_{\substack{(t_1,x_1),(t_2,x_2)\in\mathbb{D}_T\\ (t_1,x_1)\neq(t_2,x_2)}} \frac{|u(t_1,x_1)-u(t_2,x_2)|}{(|t_1-t_2|^{\frac{1}{2}}+|x_1-x_2|)^{\alpha}} < +\infty\,.$$

Moreover the norms $\|\cdot\|_{C^{0,\frac{\alpha}{2};0,\alpha}(\mathbb{D}_T)}$ and $\|\cdot\|'_{C^{0,\frac{\alpha}{2};0,\alpha}(\mathbb{D}_T)}$ in $C^{0,\frac{\alpha}{2};0,\alpha}(\mathbb{D}_T)$ are equivalent.

We now define spaces of functions with higher order derivatives in parabolic Hölder spaces, *i.e*, parabolic Schauder spaces. For this purpose we set

$$\begin{aligned}&C^{0,0;0,0}(\mathbb{D}_T)\equiv C^0_b(\mathbb{D}_T),\\ &C^{0,\alpha';0,0}(\mathbb{D}_T)\equiv C^{0,\alpha'}_b\left(\overline{]-\infty,T[},C^0(\mathbb{D})\right),\\ &C^{0,0;0,\alpha''}(\mathbb{D}_T)\equiv C^0_b\left(\overline{]-\infty,T[},C^{0,\alpha''}(\mathbb{D})\right),\end{aligned}$$

for all subset $\mathbb{D}$ of $\mathbb{R}^n$.

Definition 1.8. Let $\alpha', \alpha'' \in [0,1]$, $T \in]-\infty, +\infty]$. Let Ω be an open subset of $\mathbb{R}^n$. Then $C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\mathrm{cl}\,\Omega_T)$ denotes the space of bounded continuous functions from $\mathrm{cl}\,\Omega_T$ to $\mathbb{C}$ such that $\partial_{x_i} u$ is a bounded continuous function from Ω_T to $\mathbb{C}$ which admits a continuous extension to $\mathrm{cl}\,\Omega_T$ for all $i \in \{1, \dots, n\}$, and

$$\|u\|_{C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\mathrm{cl}\,\Omega_T)} \equiv \sup_{\mathrm{cl}\,\Omega_T} |u| + \sum_{i=1}^{n} \|\partial_{x_i} u\|_{C^{0,\frac{\alpha'}{2};0,\alpha''}(\mathrm{cl}\,\Omega_T)} + \sup_{\substack{t_1,t_2 \in \overline{]-\infty,T[},\\ t_1 \neq t_2}} \frac{\|u(t_1,\cdot) - u(t_2,\cdot)\|_{C^0(\mathrm{cl}\,\Omega)}}{|t_1 - t_2|^{\frac{1+\alpha'}{2}}} < +\infty.$$

In particular, $u \in C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\mathrm{cl}\,\Omega_T)$ if and only if $\partial_{x_i} u \in C^{0,\frac{\alpha'}{2};0,\alpha''}(\mathrm{cl}\,\Omega_T)$ for all $i \in \{1, \dots, n\}$ and $u \in C_b^{0,\frac{1+\alpha'}{2}}\left(\overline{]-\infty,T[}, C_b^0(\mathrm{cl}\,\Omega)\right)$.

Let $m \in \mathbb{N} \setminus \{0,1\}$. We recall that by $[h], \{h\}$ we denote the integer and the fractional parts of a real number $h \in \mathbb{R}$, respectively. Then $C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T)$ denotes the space of bounded continuous functions from $\mathrm{cl}\,\Omega_T$ to $\mathbb{C}$ such that $\partial_{x_i} u$ and $\partial_t u$ are bounded continuous from Ω_T to $\mathbb{C}$ which admit a continuous extension to $\mathrm{cl}\,\Omega_T$ and

$$\begin{aligned} &\partial_{x_i} u \in C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\mathrm{cl}\,\Omega_T) \qquad \forall\, i \in \{1, \dots, n\}, \\ &\partial_t u \in C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\mathrm{cl}\,\Omega_T). \end{aligned}$$

Moreover we set

$$\|u\|_{C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T)} \equiv \sup_{\mathrm{cl}\,\Omega_T} |u| + \sum_{i=1}^{n} \|\partial_{x_i} u\|_{C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\mathrm{cl}\,\Omega_T)} + \|\partial_t u\|_{C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\mathrm{cl}\,\Omega_T)}.$$

It is well known that $(C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T), \|\cdot\|_{C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T)})$ is a Banach space. We note that if $m \in \mathbb{N} \setminus \{0\}$ and if Ω is a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$, then one can define the space $C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega)$ by means of local parametrizations. The corresponding norm can be proved to be equivalent to the following norm

$$\|u\|_{C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\partial_T\Omega)} \equiv \sup_{\partial_T\Omega} |u| + \sum_{i,j=1}^{n} \|M_{ij}[u]\|_{C^{0,\frac{\alpha'}{2};0,\alpha''}(\partial_T\Omega)} + \sup_{\substack{t_1,t_2 \in \overline{]-\infty,T[}\\ t_1 \neq t_2}} \frac{\|u(t_1,\cdot) - u(t_2,\cdot)\|_{C^0(\partial\Omega)}}{|t_1 - t_2|^{\frac{1+\alpha'}{2}}} \qquad \forall\, u \in C^{0,\frac{1+\alpha'}{2};1,\alpha''}(\partial_T\Omega),$$

for $m = 1$, and

$$\|u\|_{C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega)} \equiv \sup_{\partial_T\Omega} |u| + \sum_{i,j=1}^{n} \|M_{ij}[u]\|_{C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\partial_T\Omega)} + \|\partial_t u\|_{C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\partial_T\Omega)}$$

$$\forall\, u \in C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega),$$

for $m \in \mathbb{N} \setminus \{0,1\}$.

For the sake of brevity, in the case $\alpha \in [0,1[$, we use the following notation.

$$C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\Omega_T) \equiv C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha}{2};m,\alpha}(\mathrm{cl}\Omega_T) \left(= C^{[\frac{m+\alpha}{2}],\{\frac{m+\alpha}{2}\};m,\alpha}(\mathrm{cl}\Omega_T)\right),$$
$$C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega) \equiv C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha}{2};m,\alpha}(\partial_T\Omega) \left(= C^{[\frac{m+\alpha}{2}],\{\frac{m+\alpha}{2}\};m,\alpha}(\partial_T\Omega)\right).$$

From the definition of the norms in parabolic Schauder spaces, one immediately has the following.

Remark 1.9. Let $\alpha', \alpha'' \in [0,1]$, $T \in]-\infty,+\infty]$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $i,j \in \{1,\dots,n\}$. Then the operators

$$\partial_{x_i} : C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T) \to C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\mathrm{cl}\,\Omega_T),$$
$$\partial_t : C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\mathrm{cl}\,\Omega_T) \to C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\mathrm{cl}\,\Omega_T) \quad \text{if } m \geq 2,$$

are linear and continuous. Moreover, if Ω is of class $C^{m,\alpha}$ the operators

$$M_{ij} : C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega) \to C^{[\frac{m-1}{2}],\{\frac{m-1}{2}\}+\frac{\alpha'}{2};m-1,\alpha''}(\partial_T\Omega),$$
$$\partial_t : C^{[\frac{m}{2}],\{\frac{m}{2}\}+\frac{\alpha'}{2};m,\alpha''}(\partial_T\Omega) \to C^{[\frac{m-2}{2}],\{\frac{m-2}{2}\}+\frac{\alpha'}{2};m-2,\alpha''}(\partial_T\Omega) \quad \text{if } m \geq 2,$$

are linear and continuous.

Finally, we note that parabolic Schauder spaces of higher order are continuously embedded in the spaces with lower order.

Remark 1.10. Let $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. We note that, for parabolic Schauder spaces, the same embeddings as in the classical case hold true. More precisely, if $m', m'' \in \mathbb{N}$, $\alpha, \beta \in [0,1]$, $m'+\alpha > m''+\beta$, then $C^{\frac{m'+\alpha}{2};m'+\alpha}(\mathrm{cl}\,\Omega_T)$ is continuously embedded into $C^{\frac{m''+\beta}{2};m''+\beta}(\mathrm{cl}\,\Omega_T)$. The same embeddings hold true for parabolic Schauder spaces on $\partial_T\Omega$ provided that Ω is of class $C^{m',\alpha}$.

1.2 Preliminary inequalities

In this section we collect some inequalities that we exploit during the present chapter. We start with the following elementary lemma, which collects either known inequalities or variants of known inequalities, which we need in the sequel.

Lemma 1.11. *Let $s \in \mathbb{N} \setminus \{0\}$. The following statements hold.*

(i)

$$\frac{1}{2}|x'-y| \leq |x''-y| \leq 2|x'-y| \qquad \forall\, y \in \mathbb{R}^s \setminus \mathbb{B}_s(x', 2|x'-x''|),$$

for all $x', x'' \in \mathbb{R}^s$, $x' \neq x''$.

(ii) *Let* $h \in]-\infty, 0]$. *Then*

$$\left| e^{h|x'-y|^2} - e^{h|x''-y|^2} \right| \leq 2|h|\rho_{2,y}(x',x'')e^{h\rho_{1,y}^2(x',x'')}|x'-x''|\,,$$

for all $x', x'', y \in \mathbb{R}^s$, *where*

$$\rho_{1,y}(x',x'') \equiv \min\{|x'-y|, |x''-y|\}\,, \quad \rho_{2,y}(x',x'') \equiv \max\{|x'-y|, |x''-y|\}\,.$$

(iii)

$$\frac{1}{2}|x'-y| \leq \rho_{1,y}(x',x'') \leq \rho_{2,y}(x',x'') \leq 2|x'-y| \qquad \forall\, y \in \mathbb{R}^s \setminus \mathbb{B}_s(x', 2|x'-x''|)\,,$$

for all $x', x'' \in \mathbb{R}^s$, $x' \neq x''$.

Proof. Statement (i) follows by the triangular inequality. Statement (ii) follows by applying the Mean Value Theorem to the function e^{hs^2} of $s \in [0, +\infty[$. Statement (iii) is an immediate consequence of statement (i). □

Then we have the following well known statement, that will be useful in order to estimate the kernel of the double layer heat potential. For a proof we refer, *e.g.*, to Colton and Kress [16, Theorem 2.2, p. 35] for the case of Ω of class C^2 and $\alpha = 1$, or to Cialdea [15, Section 2, p. 13] for the general case.

Lemma 1.12. *Let* $\alpha \in]0,1]$. *Let* Ω *be a bounded open connected subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Then there exists a constant* $c_{\Omega,\alpha} \in]0, +\infty[$ *such that*

$$|\nu(y)^t(x-y)| \leq c_{\Omega,\alpha}|x-y|^{1+\alpha} \qquad \forall\, x, y \in \partial\Omega\,.$$

Next we introduce a list of classical inequalities which can be verified by exploiting the local parametrizations of $\partial\Omega$.

Lemma 1.13. *Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$. *Then the following statements hold.*

(i) *Let* $\gamma \in]-\infty, n-1[$. *Then*

$$c'_{\Omega,\gamma} \equiv \sup_{x \in \partial\Omega} \int_{\partial\Omega} \frac{1}{|x-y|^\gamma}\, d\sigma_y < +\infty\,.$$

(ii) *Let* $\gamma \in]-\infty, n-1[$. *Then*

$$c''_{\Omega,\gamma} \equiv \sup_{\substack{x',x'' \in \partial\Omega \\ x' \neq x''}} |x'-x''|^{-(n-1)+\gamma} \int_{\mathbb{B}_n(x',3|x'-x''|)\cap\partial\Omega} \frac{1}{|x'-y|^\gamma}\, d\sigma_y < +\infty\,.$$

(iii) *Let* $\gamma \in]n-1, +\infty[$. *Then*

$$c'''_{\Omega,\gamma} \equiv \sup_{\substack{x',x'' \in \partial\Omega \\ x' \neq x''}} |x'-x''|^{-(n-1)+\gamma} \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{1}{|x'-y|^\gamma}\, d\sigma_y < +\infty\,.$$

(iv)

$$c^{iv}_{\Omega} \equiv \sup_{\substack{x',x'' \in \partial\Omega \\ 0<|x'-x''|<1/e}} |\log|x'-x''||^{-1} \int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{1}{|x'-y|^{n-1}}\, d\sigma_y < +\infty\,.$$

Finally, we have the following technical elementary lemma, which collects either known inequalities or variants of known inequalities, which we need in the sequel.

Lemma 1.14.

(i) Let $s \in]1,+\infty[$. Let F_s be the function from $]0,+\infty[$ to itself defined by

$$F_s(\xi) \equiv \int_\xi^{+\infty} e^{-\frac{1}{u}} u^{-s}\, du \qquad \forall\, \xi \in]0,+\infty[\,.$$

If $\gamma \in]0, s-1]$, then

$$D_{s,\gamma} \equiv \sup_{\xi \in]0,+\infty[} \xi^\gamma F_s(\xi) < +\infty\,.$$

(ii) Let $s \in]1,+\infty[$. Let $\tilde{F}_s$ be the function from $]0,+\infty[$ to itself defined by

$$\tilde{F}_s(\xi) \equiv \int_0^{\xi} e^{-\frac{1}{u}} u^{-s}\, du \qquad \forall\, \xi \in]0,+\infty[\,.$$

If $\gamma \in]0,+\infty[$, then

$$\tilde{D}_{s,\gamma} \equiv \sup_{\xi \in]0,+\infty[} \xi^{-\gamma} \tilde{F}_s(\xi) < +\infty\,.$$

(iii) Let $s \in]1,+\infty[$. Then $M_s \equiv \int_0^{+\infty} e^{-\frac{1}{u}} u^{-s}\, du < +\infty$.

(iv) Let $b_1 \in]0,+\infty[$, $b_2 \in]b_1,+\infty[$, $m \in \mathbb{N} \setminus \{0\}$. Then

$$C(b_1,b_2,m) \equiv \sup_{\eta \in]0,+\infty[} e^{\frac{\eta}{b_2} - \frac{\eta}{b_1}} \sum_{j=0}^{m} \eta^j < +\infty\,.$$

Proof. We first consider statement (i). Since $s > 1$, the function $e^{-\frac{1}{u}} u^{-s}$ is integrable in $]0,+\infty[$. The assumption $\gamma > 0$ implies that

$$\lim_{\xi \to 0^+} \xi^\gamma F_s(\xi) = 0,$$

and the assumption $\gamma \leq s-1$ and de l'Hôpital rule imply that

$$\lim_{\xi \to +\infty} \xi^\gamma F_s(\xi) \in \mathbb{R}.$$

Hence, statement (i) holds true.

Next we consider statement (ii). The integrability of function $e^{-\frac{1}{u}} u^{-s}$ in $]0,+\infty[$ and the assumption $\gamma > 0$ imply that

$$\lim_{\xi \to +\infty} \xi^{-\gamma} \tilde{F}_s(\xi) = 0$$

By de l'Hôpital rule, we have

$$\lim_{\xi \to 0^+} \xi^{-\gamma} \tilde{F}_s(\xi) = 0,$$

and thus statement (ii) follows.

Statement (iii) follows by the well known integrability of $e^{-\frac{1}{u}} u^{-s}$ in $]0,+\infty[$ when $s > 1$.

In order to prove statement (iv) it suffices to note that the argument of the supremum is a continuous function in η and has limiting values 1 and 0 at 0 and $+\infty$, respectively. □

1.3 Preliminary inequalities for the fundamental solution of the heat operator

The function Φ_n from $\mathbb{R}^{1+n} \setminus \{(0,0)\}$ to $\mathbb{R}$ defined by

$$\Phi_n(t,x) \equiv \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{if } (t,x) \in]0,+\infty[\times\mathbb{R}^n\,, \\ 0 & \text{if } (t,x) \in (]-\infty,0]\times\mathbb{R}^n) \setminus \{(0,0)\}\,, \end{cases}$$

is well known to be the fundamental solution of the heat operator

$$\partial_t - \Delta$$

in $\mathbb{R}^{1+n}$. It is well known that Φ_n is of class $C^\infty(\mathbb{R}^{1+n} \setminus \{(0,0)\})$ and solves the heat equation in $\mathbb{R}^{1+n} \setminus \{(0,0)\}$ (see, *e.g.*, Evans [33, pp. 45–49]). Moreover, for all $\eta \in \mathbb{N}^n$ and for all $h \in \mathbb{N}$ there exists a constant $K_{\eta,h} \in]0,+\infty[$ such that the inequality

$$\left| D_x^\eta \partial_t^h \Phi_n(t,x) \right| \leq K_{\eta,h} t^{-\frac{n}{2}-\frac{|\eta|_1}{2}-h} e^{-\frac{|x|^2}{8t}}\,, \tag{1.4}$$

holds for all $(t,x) \in \mathbb{R}^{n+1} \setminus \{(0,0)\}$ (see Ladyzhenskaja, Solonnikov and Ural'ceva [58, p. 274]).

The fundamental solution of the heat equation Φ_n (and its derivatives) will be used as the kernel of some integral operators related to layer heat potentials, then in this section we collect some estimates fro Φ_n and its derivatives. We start with the following lemma.

Lemma 1.15. *Let $T \in]-\infty,+\infty]$. Let G be a nonempty subset of $\mathbb{R}^n$. Then the following statements hold.*

(i)

$$C_{0,G} \equiv \sup_{\substack{(t,x)\in]0,+\infty[\times\mathbb{R}^n \\ |x|\leq \mathrm{diam}\,(G)}} |\Phi_n(t,x)| t^{\frac{n}{2}} e^{\frac{|x|^2}{4t}} < +\infty\,.$$

(ii)

$$\begin{aligned}\tilde{C}_{0,G} \equiv \sup\Bigg\{ |\Phi_n(t-\tau,x'-y) - \Phi_n(t-\tau,x''-y)| \frac{|t-\tau|^{\frac{n}{2}+1}}{|x'-y|\,|x'-x''|} e^{\frac{|x'-y|^2}{16(t-\tau)}} : \\ x',x'' \in G,\ x' \neq x'',\ y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|), \\ t,\tau \in \overline{]-\infty,T[},\ \tau < t \Bigg\} < +\infty\,.\end{aligned}$$

(iii) Let $a \in]8,+\infty[$. Then

$$\begin{aligned}\tilde{C}'_{0,a,G} \equiv \sup\Bigg\{ |\Phi_n(t'-\tau,x-y) - \Phi_n(t''-\tau,x-y)| \frac{|t'-\tau|^{\frac{n}{2}+1}}{|t'-t''|} e^{\frac{|x-y|^2}{a(t'-\tau)}} : \\ x,y \in G,\ x \neq y,\ t',t'' \in \overline{]-\infty,T[},\ t' < t'', \\ \tau < t' - 2|t'-t''| \Bigg\} < +\infty\,.\end{aligned}$$

Proof. Statement (i) is an immediate consequence of the definition of Φ_n.

We now consider statement (ii). Let $t,\tau \in \overline{]-\infty,T[}$, $\tau < t$, $x',x'' \in G$, $x' \neq x''$, $y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|)$. Then Lemma 1.11 implies that

$$
\begin{aligned}
&|\Phi_n(t-\tau, x'-y) - \Phi_n(t-\tau, x''-y)| \\
&\qquad = \frac{1}{(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}}} \left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \\
&\qquad \leq \frac{1}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}} \rho_{2,y}(x',x'') e^{-\frac{\rho^2_{1,y}(x',x'')}{4(t-\tau)}} |x'-x''| \\
&\qquad \leq \frac{1}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}} 2|x'-y| e^{-\frac{|x'-y|^2}{16(t-\tau)}} |x'-x''| ,
\end{aligned}
$$

and accordingly, (ii) follows.

Next we consider statement (iii). Let $t',t'' \in \overline{]-\infty,T[}$, $t' < t''$, $\tau < t' - 2|t'-t''|$, $x,y \in G$, $x \neq y$. By the Mean Value Theorem there exists $\xi \in]t',t''[$ such that

$$
\begin{aligned}
&|\Phi_n(t'-\tau, x-y) - \Phi_n(t''-\tau, x-y)| \\
&\quad = \frac{1}{(4\pi)^{\frac{n}{2}}} \left| \frac{1}{(t'-\tau)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t'-\tau)}} - \frac{1}{(t''-\tau)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t''-\tau)}} \right| \\
&\quad = \frac{|t'-t''|}{(4\pi)^{\frac{n}{2}}} \left| \frac{-n/2}{(\xi-\tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi-\tau)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} \frac{|x-y|^2}{4(\xi-\tau)^2} \right| .
\end{aligned}
$$

Then by Lemma 1.11 (i), and by the inequalities $|t'-\tau| \leq |\xi-\tau| \leq |t''-\tau|$, and by Lemma 1.14 (iv), we have

$$
\begin{aligned}
&\frac{|t'-t''|}{(4\pi)^{\frac{n}{2}}} \left| \frac{-n/2}{(\xi-\tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi-\tau)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} \frac{|x-y|^2}{4(\xi-\tau)^2} \right| \\
&\quad \leq \frac{|t'-t''|}{(4\pi)^{\frac{n}{2}}} \left[\frac{n/2}{(t'-\tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4(t''-\tau)}} + \frac{1}{(t'-\tau)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t''-\tau)}} \frac{|x-y|^2}{4(t'-\tau)^2} \right] \\
&\quad \leq \frac{|t'-t''|}{(4\pi)^{\frac{n}{2}}} \left[\frac{n/2}{(t'-\tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{8(t'-\tau)}} + \frac{1}{(t'-\tau)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{8(t'-\tau)}} \frac{|x-y|^2}{4(t'-\tau)^2} \right] \\
&\quad \leq \frac{(n/2)|t'-t''|}{(4\pi)^{\frac{n}{2}}(t'-\tau)^{\frac{n}{2}+1}} C(8,a,1) e^{-\frac{|x-y|^2}{a(t'-\tau)}} ,
\end{aligned}
$$

and thus statement (iii) holds true. □

Next we consider the space gradient of the fundamental solution. One can easily check that

$$
D_x\Phi_n(t,x) \equiv \begin{cases} \frac{-x}{2(4\pi)^{\frac{n}{2}} t^{\frac{n}{2}+1}} e^{-\frac{|x|^2}{4t}} & \text{if } (t,x) \in]0,+\infty[\times\mathbb{R}^n , \\ 0 & \text{if } (t,x) \in (]-\infty,0]\times\mathbb{R}^n) \setminus \{(0,0)\} . \end{cases} \tag{1.5}
$$

We have the following lemma regarding the estimate of $D_x\Phi_n$.

Lemma 1.16. *Let $T \in]-\infty,+\infty]$. Let G be a nonempty subset of $\mathbb{R}^n$. Then the following statements hold.*

(i)

$$C_{0,1,G} \equiv \sup_{\substack{(t,x)\in]0,+\infty[\times\mathbb{R}^n \\ |x|\leq \mathrm{diam}\,(G)}} |D_x\Phi_n(t,x)| \frac{t^{\frac{n}{2}+1}}{|x|} e^{\frac{|x|^2}{4t}} < +\infty\,.$$

(ii) Let $a\in]16,+\infty[$. Then

$$\begin{aligned}\tilde{C}_{0,1,a,G} & \\ \equiv \sup\Big\{ & |D_x\Phi_n(t-\tau,x'-y) - D_x\Phi_n(t-\tau,x''-y)| \frac{|t-\tau|^{\frac{n}{2}+1} e^{\frac{|x'-y|^2}{a(t-\tau)}}}{|x'-x''|} : \\ & x',x''\in G,\, x'\neq x'',\, y\in G\setminus \mathbb{B}_n(x',2|x'-x''|)\,, \\ & t,\tau \in \overline{]-\infty,T[},\, \tau<t \Big\} < +\infty\,.\end{aligned}$$

(iii) Let $a\in]8,+\infty[$. Then

$$\begin{aligned}\tilde{C}'_{0,1,a,G} \equiv & \\ \sup\Big\{ & |D_x\Phi_n(t'-\tau,x-y) - D_x\Phi_n(t''-\tau,x-y)| \frac{|t'-\tau|^{\frac{n}{2}+2}}{|x-y|\,|t'-t''|} e^{\frac{|x-y|^2}{a(t'-\tau)}} : \\ & x,y\in G,\, x\neq y,\, t',t''\in \overline{]-\infty,T[},\, t'<t'', \\ & \tau < t'-2|t'-t''| \Big\} < +\infty\,.\end{aligned}$$

Proof. Statement (i) is an immediate consequence of the formula (1.5) for $D_x\Phi_n$.

We now consider statement (ii). Let $t,\tau\in\overline{]-\infty,T[}$, $\tau<t$, $x',x''\in G$, $x'\neq x''$, $y\in G\setminus\mathbb{B}_n(x',2|x'-x''|)$. By the triangular inequality, we have

$$\begin{aligned} & |D_x\Phi_n(t-\tau,x'-y) - D_x\Phi_n(t-\tau,x''-y)| \qquad (1.6) \\ & \quad = \frac{1}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}} \left| (x'-y)e^{-\frac{|x'-y|^2}{4(t-\tau)}} - (x''-y)e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \\ & \quad \leq \frac{1}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}} \Big\{ e^{-\frac{|x'-y|^2}{4(t-\tau)}} |x'-x''| \\ & \qquad + |x''-y| \left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \Big\}\,. \end{aligned}$$

Now Lemma 1.11 implies that

$$\begin{aligned} & \left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \qquad (1.7) \\ & \quad \leq \frac{2\rho_{2,y}(x',x'')}{4(t-\tau)} e^{-\frac{\rho^2_{1,y}(x',x'')}{4(t-\tau)}} |x'-x''| \leq \frac{|x'-y|\,|x'-x''|}{(t-\tau)} e^{-\frac{|x'-y|^2}{16(t-\tau)}}\,. \end{aligned}$$

Hence, Lemma 1.11 (i) and Lemma 1.14 (iv) imply that the right hand side of (1.6) is less or equal than

$$\frac{e^{-\frac{|x'-y|^2}{16(t-\tau)}}}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}} \left\{ |x'-x''| + 2\frac{|x'-y|^2}{t-\tau}|x'-x''| \right\}$$

$$\leq C(16,a,1)\frac{e^{-\frac{|x'-y|^2}{a(t-\tau)}}}{(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}|x'-x''|\,,$$

and thus statement (ii) holds true.

Next we consider statement (iii). Let $x,y\in G$, $x\neq y$, $t',t''\in\overline{]-\infty,T[}$, $t'<t''$, $\tau<t'-2|t'-t''|$. By the Mean Value Theorem, there exists $\xi\in]t',t''[$ such that

$$\begin{aligned}&|D_x\Phi_n(t'-\tau,x-y)-D_x\Phi_n(t''-\tau,x-y)|\\&\quad=\frac{|x-y|}{2(4\pi)^{\frac{n}{2}}}\left|\frac{1}{(t'-\tau)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(t'-\tau)}}-\frac{1}{(t''-\tau)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(t''-\tau)}}\right|\\&\quad=\frac{|t'-t''|\,|x-y|}{2(4\pi)^{\frac{n}{2}}}\left|\frac{-(n/2)-1}{(\xi-\tau)^{\frac{n}{2}+2}}e^{-\frac{|x-y|^2}{4(\xi-\tau)}}+\frac{1}{(\xi-\tau)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(\xi-\tau)}}\frac{|x-y|^2}{4(\xi-\tau)^2}\right|\,.\end{aligned}$$

Then by Lemma 1.11 (i), and by the inequalities $t'-\tau\leq\xi-\tau\leq t''-\tau$, and by Lemma 1.14 (iv), we have

$$\begin{aligned}&\left|\frac{-(n/2)-1}{(\xi-\tau)^{\frac{n}{2}+2}}e^{-\frac{|x-y|^2}{4(\xi-\tau)}}+\frac{1}{(\xi-\tau)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(\xi-\tau)}}\frac{|x-y|^2}{4(\xi-\tau)^2}\right| \qquad (1.8)\\&\qquad\leq\frac{n/2+1}{(t'-\tau)^{\frac{n}{2}+2}}e^{-\frac{|x-y|^2}{4(t''-\tau)}}+\frac{1}{(t'-\tau)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(t''-\tau)}}\frac{|x-y|^2}{4(t'-\tau)^2}\\&\qquad=\frac{e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{(t'-\tau)^{\frac{n}{2}+2}}\left(\frac{n}{2}+1+\frac{1}{4}\frac{|x-y|^2}{t'-\tau}\right)\\&\qquad\leq\frac{e^{-\frac{|x-y|^2}{8(t'-\tau)}}}{(t'-\tau)^{\frac{n}{2}+2}}\left(\frac{n}{2}+1\right)\left(1+\frac{|x-y|^2}{t'-\tau}\right)\\&\qquad\leq C(8,a,1)\frac{e^{-\frac{|x-y|^2}{a(t'-\tau)}}}{(t'-\tau)^{\frac{n}{2}+2}}\left(\frac{n}{2}+1\right)\,,\end{aligned}$$

and thus statement (iii) holds true. □

Next, we consider the time derivative of the fundamental solution. One can easily check that

$$\partial_t\Phi_n(t,x)\equiv\begin{cases}\frac{e^{-\frac{|x|^2}{4t}}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x|^2}{t}\right)}{(4\pi)^{\frac{n}{2}}t^{\frac{n}{2}+1}} & \text{if } (t,x)\in]0,+\infty[\times\mathbb{R}^n\,,\\ 0 & \text{if } (t,x)\in(]-\infty,0]\times\mathbb{R}^n)\setminus\{(0,0)\}\,.\end{cases} \qquad (1.9)$$

We have the following lemma regarding the estimate of $\partial_t\Phi_n$.

Lemma 1.17. *Let $T\in]-\infty,+\infty]$. Let G be a nonempty subset of $\mathbb{R}^n$.*

(i) Let $a\in]4,+\infty[$. Then

$$C_{1,0,G}\equiv\sup_{\substack{(t,x)\in]0,+\infty[\times\mathbb{R}^n\\|x|\leq\mathrm{diam}\,(G)}}|\partial_t\Phi_n(t,x)|t^{\frac{n}{2}+1}e^{\frac{|x|^2}{at}}<+\infty\,.$$

(ii) Let $a \in]16, +\infty[$. Then

$$\tilde{C}_{1,0,a,G} \equiv \sup\Big\{ |\partial_t \Phi_n(t-\tau, x'-y) - \partial_t \Phi_n(t-\tau, x''-y)| \frac{|t-\tau|^{\frac{n}{2}+2} e^{\frac{|x'-y|^2}{a(t-\tau)}}}{|x'-y|\,|x'-x''|} : x', x'' \in G,\, x' \neq x'',\, y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|),\, t, \tau \in \overline{]-\infty, T[},\, \tau < t \Big\} < +\infty\,.$$

(iii) Let $a \in]8, +\infty[$. Then

$$\tilde{C}'_{1,0,a,G} \equiv \sup\Big\{ |\partial_t \Phi_n(t'-\tau, x-y) - \partial_t \Phi_n(t''-\tau, x-y)| \frac{|t'-\tau|^{\frac{n}{2}+2}}{|t'-t''|} e^{\frac{|x-y|^2}{a(t'-\tau)}} : x, y \in G,\, x \neq y,\, t', t'' \in \overline{]-\infty, T[},\, t' < t'',\, \tau < t' - 2|t'-t''| \Big\} < +\infty\,.$$

Proof. Statement (i) is an immediate consequence of the formula (1.9) for $\partial_t \Phi_n$ and of Lemma 1.14 (iv).

We now consider statement (ii). Let $t, \tau \in \overline{]-\infty, T[}$, $\tau < t$, $x', x'' \in G$, $x' \neq x''$, $y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|)$. By the triangular inequality, we have

$$\begin{aligned}
&|\partial_t \Phi_n(t-\tau, x'-y) - \partial_t \Phi_n(t-\tau, x''-y)| \\
&= \frac{1}{(4\pi)^{\frac{n}{2}} (t-\tau)^{\frac{n}{2}+1}} \left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} \left(-\frac{n}{2} + \frac{1}{4} \frac{|x'-y|^2}{t-\tau} \right) - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \left(-\frac{n}{2} + \frac{1}{4} \frac{|x''-y|^2}{t-\tau} \right) \right| \\
&\leq \frac{e^{-\frac{|x'-y|^2}{4(t-\tau)}}}{(4\pi)^{\frac{n}{2}} (t-\tau)^{\frac{n}{2}+1}} \frac{1}{4(t-\tau)} \left| |x'-y|^2 - |x''-y|^2 \right| \\
&\quad + \frac{\frac{n}{2}\left(1 + \frac{|x''-y|^2}{4(t-\tau)}\right)}{(4\pi)^{\frac{n}{2}} (t-\tau)^{\frac{n}{2}+1}} \left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right|.
\end{aligned}$$

By the Mean Value Theorem and by Lemma 1.11 (iii), we have

$$\left| |x'-y|^2 - |x''-y|^2 \right| \leq 2\rho_{2,y}(x', x'')|x'-x''| \leq 4|x'-y|\,|x'-x''|\,,$$

and by Lemma 1.11 (ii), (iii), we have

$$\left| e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}} \right| \leq \frac{\rho_{2,y}(x', x'')}{2(t-\tau)} e^{-\frac{\rho_{1,y}^2(x', x'')}{4(t-\tau)}} |x'-x''| \leq \frac{|x'-y|}{(t-\tau)} e^{-\frac{|x'-y|^2}{16(t-\tau)}} |x'-x''|\,.$$

Hence, Lemma 1.11 (i) and Lemma 1.14 (iv) imply that

$$|\partial_t \Phi_n(t-\tau, x'-y) - \partial_t \Phi_n(t-\tau, x''-y)|$$

$$
\begin{aligned}
&\leq \frac{e^{-\frac{|x'-y|^2}{4(t-\tau)}}}{(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}} \frac{1}{4(t-\tau)} 4|x'-y|\,|x'-x''| \\
&\quad + \frac{\frac{n}{2}\left(1+\frac{|x'-y|^2}{t-\tau}\right)}{(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}} \frac{|x'-y|}{(t-\tau)} e^{-\frac{|x'-y|^2}{16(t-\tau)}} |x'-x''| \\
&\leq \frac{|x'-y|\,|x'-x''|}{(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+2}} \left\{ e^{-\frac{|x'-y|^2}{4(t-\tau)}} + (n/2)C(16,a,1) e^{-\frac{|x'-y|^2}{a(t-\tau)}} \right\}.
\end{aligned}
$$

and thus statement (ii) holds true.

Next we consider statement (iii). Let $x, y \in G$, $x \neq y$, $t', t'' \in \overline{]-\infty, T[}$, $t' < t''$, $\tau < t' - 2|t' - t''|$. By the Mean Value Theorem, there exists $\xi \in]t', t''[$ such that

$$
\begin{aligned}
&|\partial_t \Phi_n(t'-\tau, x-y) - \partial_t \Phi_n(t''-\tau, x-y)| \\
&= \frac{1}{(4\pi)^{\frac{n}{2}}} \left| \frac{e^{-\frac{|x-y|^2}{4(t'-\tau)}}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x-y|^2}{(t'-\tau)}\right)}{(t'-\tau)^{\frac{n}{2}+1}} - \frac{e^{-\frac{|x-y|^2}{4(t''-\tau)}}\left(-\frac{n}{2}+\frac{1}{4}\frac{|x-y|^2}{(t''-\tau)}\right)}{(t''-\tau)^{\frac{n}{2}+1}} \right| \\
&\leq \frac{1}{(4\pi)^{\frac{n}{2}}} \left| \frac{(n/2+1)\, e^{-\frac{|x-y|^2}{4(\xi-\tau)}}}{(\xi-\tau)^{\frac{n}{2}+2}} \left(-\frac{n}{2}+\frac{1}{4}\frac{|x-y|^2}{(\xi-\tau)}\right) \right| |t'-t''| \\
&\quad + \frac{1}{(4\pi)^{\frac{n}{2}}} \left| \frac{e^{-\frac{|x-y|^2}{4(\xi-\tau)}}}{(\xi-\tau)^{\frac{n}{2}+1}} \frac{1}{4}\frac{|x-y|^2}{(\xi-\tau)^2} \left(-\frac{n}{2}+\frac{1}{4}\frac{|x-y|^2}{(\xi-\tau)}\right) \right| |t'-t''| \\
&\quad + \frac{1}{(4\pi)^{\frac{n}{2}}} \left| \frac{e^{-\frac{|x-y|^2}{4(\xi-\tau)}}}{(\xi-\tau)^{\frac{n}{2}+1}} \left(-\frac{1}{4}\frac{|x-y|^2}{(\xi-\tau)^2}\right) \right| |t'-t''|\,.
\end{aligned}
$$

Then by Lemma 1.11 (i), and by the inequalities $t'-\tau \leq \xi - \tau \leq t'' - \tau$, and by Lemma 1.14 (iv), we have

$$
\begin{aligned}
&|\partial_t \Phi_n(t'-\tau, x-y) - \partial_t \Phi_n(t''-\tau, x-y)| \\
&\leq \frac{|t'-t''|}{(4\pi)^{\frac{n}{2}}} \left\{ \frac{(n/2+1)\,(n/2) e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{(t'-\tau)^{\frac{n}{2}+2}} \left(1+\frac{|x-y|^2}{t'-\tau}\right) \right. \\
&\quad \left. + \frac{(n/2)|x-y|^2 e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{4(t'-\tau)^{\frac{n}{2}+3}} \left(1+\frac{|x-y|^2}{t'-\tau}\right) + \frac{|x-y|^2 e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{4(t'-\tau)^{\frac{n}{2}+3}} \right\} \\
&\leq \frac{|t'-t''|}{(4\pi)^{\frac{n}{2}}} \frac{3(n/2)\,(n/2+1)}{(t'-\tau)^{\frac{n}{2}+2}} \left[1+\frac{|x-y|^2}{t'-\tau}+\left(\frac{|x-y|^2}{t'-\tau}\right)^2\right] e^{-\frac{|x-y|^2}{8(t'-\tau)}} \\
&\leq \frac{|t'-t''|}{(4\pi)^{\frac{n}{2}}} \frac{3(n/2)\,(n/2+1)}{(t'-\tau)^{\frac{n}{2}+2}} C(8,a,2) e^{-\frac{|x-y|^2}{a(t'-\tau)}}\,,
\end{aligned}
$$

and thus statement (iii) holds true. □

1.4 Preliminary inequalities on the kernel of the double layer heat potential

In this section we prove some inequalities for the kernel of the double layer heat potential. We do so by means of the following.

Lemma 1.18. *Let* $\alpha \in]0,1]$, $T \in]-\infty,+\infty]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Then the following statements hold.*

(i)

$$b_{\Omega,\alpha} \equiv \sup\biggl\{ \left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right| \frac{|t-\tau|^{\frac{n}{2}+1}}{|x-y|^{1+\alpha}} e^{\frac{|x-y|^2}{4(t-\tau)}} : \\ x,y\in\partial\Omega,\ x\neq y,\ t,\tau\in\overline{]-\infty,T[},\ \tau<t\biggr\} < +\infty\,.$$

Here

$$\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y) \equiv -D_x\Phi_n(t-\tau,x-y)\nu(y)\,,$$

where $D_x\Phi_n$ *denotes the gradient of* Φ_n *with respect to the (spatial) second variable.*

(ii) Let $a \in]16,+\infty[$. *Then*

$$\tilde{b}_{\Omega,\alpha} \equiv \sup\biggl\{ \left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x'-y) - \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x''-y)\right| \\ \times \frac{|t-\tau|^{\frac{n}{2}+1} e^{\frac{|x'-y|^2}{a(t-\tau)}}}{|x'-y|^{\alpha}|x'-x''|} : x',x''\in\partial\Omega,\ x'\neq x'', \\ y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)\,, t,\tau\in\overline{]-\infty,T[},\ \tau<t\biggr\} < +\infty\,.$$

(iii) Let $a \in]8,+\infty[$. *Then*

$$\tilde{b}'_{\Omega,\alpha} \equiv \sup\biggl\{ \left|\frac{\partial}{\partial\nu(y)}\Phi_n(t'-\tau,x-y) - \frac{\partial}{\partial\nu(y)}\Phi_n(t''-\tau,x-y)\right| \\ \times \frac{|t'-\tau|^{\frac{n}{2}+2}}{|x-y|^{1+\alpha}\,|t'-t''|} e^{\frac{|x-y|^2}{a(t'-\tau)}} : \\ x,y\in\partial\Omega,\ x\neq y,\ t',t''\in\overline{]-\infty,T[},\ t'<t'', \\ \tau<t'-2|t'-t''|\biggr\} < +\infty\,.$$

Proof. Let $x,y\in\partial\Omega$, $x\neq y$, $t,\tau\in\overline{]-\infty,T[}$, $\tau<t$. By Lemma 1.12, we have

$$\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right| = \left|\frac{(x-y)^t\nu(y)e^{-\frac{|x-y|^2}{4(t-\tau)}}}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}\right| \\ \leq \frac{c_{\Omega,\alpha}|x-y|^{1+\alpha}e^{-\frac{|x-y|^2}{4(t-\tau)}}}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}\,,$$

and thus statement (i) holds true.

We now consider statement (ii). Let $t, \tau \in \overline{]-\infty, T[}$, $\tau < t$, $x', x'' \in \partial\Omega$, $x' \neq x''$, $y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|)$. Then the triangular inequality implies that

$$\begin{aligned}
&\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau, x'-y) - \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau, x''-y)\right| \qquad (1.10)\\
&\quad = \frac{1}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}\left|(x'-y)^t\nu(y)e^{-\frac{|x'-y|^2}{4(t-\tau)}} - (x''-y)^t\nu(y)e^{-\frac{|x''-y|^2}{4(t-\tau)}}\right|\\
&\quad \leq \frac{e^{-\frac{|x'-y|^2}{4(t-\tau)}}}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}\left|(x'-x'')^t\nu(y)\right|\\
&\qquad + \frac{|(x''-y)^t\nu(y)|}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}\left|e^{-\frac{|x'-y|^2}{4(t-\tau)}} - e^{-\frac{|x''-y|^2}{4(t-\tau)}}\right|.
\end{aligned}$$

By the inequality $|x'-x''| \leq |x'-y|$, and by the membership of $\nu \in C^{0,\alpha}(\partial\Omega)$, which is implied by the fact that Ω is of class $C^{1,\alpha}$, and by Lemma 1.12, we have that

$$\begin{aligned}
|(x'-x'')^t\nu(y)| &\leq |(x'-x'')^t(\nu(y)-\nu(x'))| + |(x'-x'')^t\nu(x')|\\
&\leq |x'-x''|\,|\nu|_\alpha|x'-y|^\alpha + c_{\Omega,\alpha}|x'-x''|^{1+\alpha}\\
&\leq |x'-x''|\,|x'-y|^\alpha(|\nu|_\alpha + c_{\Omega,\alpha}).
\end{aligned}$$

Moreover, Lemma 1.12 and Lemma 1.11 (i) imply that

$$|(x''-y)^t\nu(y)| \leq c_{\Omega,\alpha}|x''-y|^{1+\alpha} \leq c_{\Omega,\alpha}2^{1+\alpha}|x'-y|^{1+\alpha}.$$

Accordingly, Lemmas 1.11 (i) and 1.14 (iv) and inequality (1.7) imply that the right hand side of (1.10) is less or equal than

$$\begin{aligned}
&\frac{e^{-\frac{|x'-y|^2}{4(t-\tau)}}}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}|x'-x''|\,|x'-y|^\alpha(|\nu|_\alpha + c_{\Omega,\alpha})\\
&\qquad + \frac{c_{\Omega,\alpha}2^{1+\alpha}|x'-y|^{1+\alpha}e^{-\frac{|x'-y|^2}{16(t-\tau)}}}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}}\frac{|x'-y|\,|x'-x''|}{(t-\tau)}\\
&\quad \leq \max\left\{\frac{(|\nu|_\alpha + c_{\Omega,\alpha})}{2(4\pi)^{\frac{n}{2}}}, \frac{c_{\Omega,\alpha}2^{1+\alpha}}{2(4\pi)^{\frac{n}{2}}}\right\}\frac{|x'-x''|\,|x'-y|^\alpha e^{-\frac{|x'-y|^2}{16(t-\tau)}}}{(t-\tau)^{\frac{n}{2}+1}}\\
&\qquad \times\left\{1 + \frac{|x'-y|^2}{t-\tau}\right\}\\
&\quad \leq \max\left\{\frac{(|\nu|_\alpha + c_{\Omega,\alpha})}{2(4\pi)^{\frac{n}{2}}}, \frac{c_{\Omega,\alpha}2^{1+\alpha}}{2(4\pi)^{\frac{n}{2}}}\right\}C(16,a,1)\frac{|x'-x''|\,|x'-y|^\alpha e^{-\frac{|x'-y|^2}{a(t-\tau)}}}{(t-\tau)^{\frac{n}{2}+1}},
\end{aligned}$$

and thus statement (ii) holds true.

Next we consider statement (iii). Let $x, y \in \partial\Omega$, $x \neq y$, $t', t'' \in \overline{]-\infty, T[}$, $t' < t''$, $\tau < t' - 2|t'-t''|$. By the Mean Value Theorem, there exists $\xi \in]t', t''[$ such that

$$\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t'-\tau, x-y) - \frac{\partial}{\partial\nu(y)}\Phi_n(t''-\tau, x-y)\right|$$

$$
\begin{aligned}
&= \frac{|(x-y)^t \nu(y)|}{2(4\pi)^{\frac{n}{2}}} \left| \frac{e^{-\frac{|x-y|^2}{4(t'-\tau)}}}{(t'-\tau)^{\frac{n}{2}+1}} - \frac{e^{-\frac{|x-y|^2}{4(t''-\tau)}}}{(t''-\tau)^{\frac{n}{2}+1}} \right| \\
&= \frac{|(x-y)^t \nu(y)|\, |t'-t''|}{2(4\pi)^{\frac{n}{2}}} \\
&\quad \times \left| \frac{-(n/2)-1}{(\xi-\tau)^{\frac{n}{2}+2}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi-\tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} \frac{|x-y|^2}{4(\xi-\tau)^2} \right| .
\end{aligned}
$$

Then by Lemma 1.11 (i), and by Lemma 1.12, and by inequality (1.8), we have

$$
\begin{aligned}
&\left| \frac{\partial}{\partial \nu(y)} \Phi_n(t'-\tau, x-y) - \frac{\partial}{\partial \nu(y)} \Phi_n(t''-\tau, x-y) \right| \\
&\leq \frac{c_{\Omega,\alpha} |x-y|^{1+\alpha}\, |t'-t''|}{2(4\pi)^{\frac{n}{2}}} \\
&\quad \times \left| \frac{-(n/2)-1}{(\xi-\tau)^{\frac{n}{2}+2}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} + \frac{1}{(\xi-\tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4(\xi-\tau)}} \frac{|x-y|^2}{4(\xi-\tau)^2} \right| \\
&\leq \frac{c_{\Omega,\alpha}[(n/2)+1]}{2(4\pi)^{\frac{n}{2}}} C(8,a,1) \frac{|x-y|^{1+\alpha} |t'-t''|}{(t'-\tau)^{\frac{n}{2}+2}} e^{-\frac{|x-y|^2}{a(t'-\tau)}} .
\end{aligned}
$$

and thus statement (iii) holds true. □

1.5 A special class of time dependent boundary kernels

For each subset A of $\mathbb{R} \times \mathbb{R}^n$, we find convenient to set

$$
\Delta_A \equiv \{(t,x,\tau,y) \in A \times A : \, t = \tau, x = y\} .
$$

For each $T \in]-\infty,+\infty]$ and $G \subseteq \mathbb{R}^n$, we now introduce a normed space of functions on $(G_T)^2 \setminus \Delta_{G_T}$ which may carry a singularity as the variable tends to a point of the diagonal, just as in the case of the kernels of integral operators corresponding to layer heat potentials. This space will serve as the space of kernels of the integral operators that we will consider in the following sections.

Definition 1.19. Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let G be a nonempty subset of $\mathbb{R}^n$. Let

$$
\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'') \in \mathbb{R}^8 . \tag{1.11}
$$

We denote by $\mathcal{K}_{\gamma,a}(G_T)$ the set of continuous functions K from $(G_T)^2 \setminus \Delta_{G_T}$ to $\mathbb{C}$ such that

$$
K(t,x,\tau,y) = 0 \qquad \text{if } (t,x,\tau,y) \in (G_T)^2 \setminus \Delta_{G_T},\, \tau \geq t ,
$$

and such that

$$
\begin{aligned}
&\|K\|_{\mathcal{K}_{\gamma,a}(G_T)} \\
&\equiv \sup \left\{ |K(t,x,\tau,y)| \frac{|t-\tau|^{\gamma_1}}{|x-y|^{\gamma_2}} e^{\frac{|x-y|^2}{a(t-\tau)}} : \, x,y \in G, x \neq y, t,\tau \in \overline{]-\infty,T[}, \tau < t \right\} \\
&\quad + \sup \left\{ \, |K(t,x',\tau,y) - K(t,x'',\tau,y)| \, \frac{|t-\tau|^{\gamma_1'} e^{\frac{|x'-y|^2}{a(t-\tau)}}}{|x'-y|^{\gamma_2'} |x'-x''|^{\gamma_l'}} : \right.
\end{aligned}
$$

$$x', x'' \in G, x' \neq x'', y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|)\,, t, \tau \in \overline{]-\infty, T[}, \ \tau < t \Big\}$$

$$+ \sup\Big\{ \big|K(t', x, \tau, y) - K(t'', x, \tau, y)\big| \frac{|t'-\tau|^{\gamma_1''}}{|x-y|^{\gamma_2''}\,|t'-t''|^{\gamma_l''}} e^{\frac{|x-y|^2}{a(t'-\tau)}} :$$

$$x, y \in G, x \neq y, t', t'' \in \overline{]-\infty, T[}, t' < t'', \tau < t' - 2|t'-t''| \Big\} < +\infty\,.$$

One can easily verify that $\|\cdot\|_{\mathcal{K}_{\gamma,a}(G_T)}$ is actually a norm and that $(\mathcal{K}_{\gamma,a}(G_T), \|\cdot\|_{\mathcal{K}_{\gamma,a}(G_T)})$ is a Banach space.

The estimates we have proved in Lemmas 1.15, 1.16, 1.17 and 1.18 show that the functions

$$\Phi_n(t-\tau, x-y), \quad \partial_{x_r}\Phi_n(t-\tau, x-y), \quad \partial_t\Phi_n(t-\tau, x-y), \quad \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau, x-y)$$

belong to the above class of kernels $\mathcal{K}_{\gamma,a}$ for some $\gamma \in \mathbb{R}^8$ and for some $a \in]0, +\infty[$. More precisely we have the following.

Remark 1.20. Let $a \in]16, +\infty[$, $T \in]-\infty, +\infty]$.

(i) Let G be a nonempty subset of $\mathbb{R}^n$. Then Lemma 1.15 implies that the kernel

$$\Phi_n(t-\tau, x-y)$$

belongs to $\mathcal{K}_{\gamma,a}(G_T)$, with $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'')$ and

$$\gamma_1 = \frac{n}{2}\,, \quad \gamma_2 = 0\,, \quad \gamma_1' = \frac{n}{2} + 1\,, \quad \gamma_2' = 1\,, \quad \gamma_l' = 1\,,$$
$$\gamma_1'' = \frac{n}{2} + 1\,, \quad \gamma_2'' = 0\,, \quad \gamma_l'' = 1\,.$$

(ii) Let G be a nonempty subset of $\mathbb{R}^n$. Then Lemma 1.17 implies that the kernel

$$\partial_t\Phi_n(t-\tau, x-y)$$

belongs to $\mathcal{K}_{\gamma,a}(G_T)$, with $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'')$ and

$$\gamma_1 = \frac{n}{2} + 1\,, \quad \gamma_2 = 0\,, \quad \gamma_1' = \frac{n}{2} + 2\,, \quad \gamma_2' = 1\,, \quad \gamma_l' = 1\,,$$
$$\gamma_1'' = \frac{n}{2} + 2\,, \quad \gamma_2'' = 0\,, \quad \gamma_l'' = 1\,.$$

(iii) Let G be a nonempty subset of $\mathbb{R}^n$. Let $r \in \{1, \ldots, n\}$. Then Lemma 1.16 implies that the kernel

$$\partial_{x_r}\Phi_n(t-\tau, x-y)$$

belongs to $\mathcal{K}_{\gamma,a}(G_T)$, with $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'')$ and

$$\gamma_1 = \frac{n}{2} + 1\,, \quad \gamma_2 = 1\,, \quad \gamma_1' = \frac{n}{2} + 1\,, \quad \gamma_2' = 0\,, \quad \gamma_l' = 1\,,$$
$$\gamma_1'' = \frac{n}{2} + 2\,, \quad \gamma_2'' = 1\,, \quad \gamma_l'' = 1\,.$$

(iv) Let $\alpha \in]0,1]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then Lemma 1.18 implies that the kernel

$$\frac{\partial}{\partial \nu(y)} \Phi_n(t-\tau, x-y)$$

belongs to $\mathcal{K}_{\gamma,a}(\partial_T\Omega)$, with $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'')$ and

$$\gamma_1 = \frac{n}{2} + 1\,, \quad \gamma_2 = 1+\alpha\,, \quad \gamma_1' = \frac{n}{2}+1\,, \quad \gamma_2' = \alpha\,, \quad \gamma_l' = 1\,,$$
$$\gamma_1'' = \frac{n}{2} + 2\,, \quad \gamma_2'' = 1+\alpha\,, \quad \gamma_l'' = 1\,.$$

1.6 Integral operators on the space of essentially bounded functions

For each $\theta \in]0,1]$, we define the function $\omega_\theta(\cdot)$ from $]0,+\infty[$ to itself by setting

$$\omega_\theta(r) \equiv \begin{cases} r^\theta |\log r| & \text{if } r \in]0, r_\theta]\,, \\ r_\theta^\theta |\log r_\theta| & \text{if } r \in]r_\theta, +\infty[\,, \end{cases}$$

where

$$r_\theta \equiv e^{-\frac{1}{\theta}} \qquad \forall\, \theta \in]0,1]\,.$$

Obviously, $\omega_\theta(\cdot)$ satisfies the conditions i)–iv) of (1.1). We also note that if $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then the following continuous embedding holds

$$C_b^{0,\omega_\theta(\cdot)}(\mathbb{D}) \subseteq C_b^{0,\theta'}(\mathbb{D})$$

for all $\theta' \in]0,\theta[$. In this section we consider the mapping properties of an integral operator with a kernel in the class $\mathcal{K}_{\gamma,a}(\partial_T\Omega)$ and acting on the space of essentially bounded functions on $\partial_T\Omega$. We start with the following proposition, regarding the Hölder continuity with respect to the space variables of such an integral operator.

Proposition 1.21. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\gamma \equiv (\gamma_1, \gamma_2, \gamma_1', \gamma_2', \gamma_l', \gamma_1'', \gamma_2'', \gamma_l'') \in \mathbb{R}^8$. Then the following statements hold.*

(i) Let $\gamma_1 > 1$ and $2\gamma_1 - \gamma_2 - 2 < n-1$. If $(K,\mu) \in \mathcal{K}_{\gamma,a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$, and if $(t,x) \in \partial_T\Omega$, then the function $K(t,x,\cdot,\cdot)\mu(\cdot,\cdot)$ is integrable in $\partial_T\Omega$ and the function $u[\partial_T\Omega, K, \mu]$ from $\partial_T\Omega$ to $\mathbb{C}$ defined by

$$u[\partial_T\Omega, K, \mu](t,x) \equiv \int_{-\infty}^t \int_{\partial\Omega} K(t,x,\tau,y)\mu(\tau,y)\, d\sigma_y d\tau \qquad \forall\,(t,x) \in \partial_T\Omega\,, \tag{1.12}$$

is bounded. Moreover, the map from $\mathcal{K}_{\gamma,a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$ to $B(\partial_T\Omega)$, which takes (K,μ) to $u[\partial_T\Omega, K, \mu]$ is bilinear and continuous.

(ii) Let $\gamma_1 > 1$ and $2\gamma_1 - \gamma_2 - 2 \in [n-2, n-1[$. Moreover, let $\gamma_1' > 1$, $\gamma_l' \in]0,1]$, and

$$\gamma_l' + (n-1) - (2\gamma_1' - \gamma_2' - 2) > 0.$$

Let

$$\omega(r) \equiv \begin{cases} r^{\min\{(n-1)-(2\gamma_1-\gamma_2-2),\gamma_l'+(n-1)-(2\gamma_1'-\gamma_2'-2)\}} & \text{if } 2\gamma_1' - \gamma_2' - 2 > n-1\,, \\ \max\{r^{(n-1)-(2\gamma_1-\gamma_2-2)}, \omega_{\gamma_l'}(r)\} & \text{if } 2\gamma_1' - \gamma_2' - 2 = n-1\,, \\ r^{\min\{(n-1)-(2\gamma_1-\gamma_2-2),\gamma_l'\}} & \text{if } 2\gamma_1' - \gamma_2' - 2 < n-1\,, \end{cases}$$

for all $r \in]0,+\infty[$*. Then the map from* $\mathcal{K}_{\gamma,a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$ *to*

$$B\left(\overline{]-\infty,T[}, C^{0,\omega(\cdot)}(\partial\Omega)\right)$$

which takes (K,μ) *to* $u[\partial_T\Omega, K, \mu]$ *is bilinear and continuous (cf. Remark 1.5.)*

Proof. Let $(t,x) \in \partial_T\Omega$. Then we have

$$\begin{aligned}
&\left|\int_{-\infty}^{t}\int_{\partial\Omega} K(t,x,\tau,y)\mu(\tau,y)\,d\sigma_y d\tau\right| \\
&\quad \leq \int_{-\infty}^{t}\int_{\partial\Omega} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)} \frac{|x-y|^{\gamma_2}}{|t-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau \\
&\quad = \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)} \int_{\partial\Omega}\int_0^{+\infty} \frac{|x-y|^{\gamma_2+2}a^{-1+\gamma_1}}{u^{\gamma_1}|x-y|^{2\gamma_1}} e^{-\frac{1}{u}}\,du d\sigma_y \\
&\quad = \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)} a^{-1+\gamma_1} \int_0^{+\infty} u^{-\gamma_1} e^{-\frac{1}{u}}\,du \int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}},
\end{aligned}$$

and the integrals in the right hand side converges for $2\gamma_1 - \gamma_2 - 2 < n-1$ and $\gamma_1 > 1$. Then Lemma 1.13 (i) implies the validity of statement (i).

Next we consider statement (ii). Let $t \in \overline{]-\infty,T[}$, $x', x'' \in \partial\Omega$. By statement (i) and Remark 1.1, there is no loss of generality in assuming that $0 < |x'-x''| \leq r_{\gamma_l'}$. Then the inclusion $\mathbb{B}_n(x', 2|x'-x''|) \subseteq \mathbb{B}_n(x'', 3|x'-x''|)$ and the triangular inequality imply that

$$\begin{aligned}
&|u[\partial_T\Omega, K, \mu](t,x') - u[\partial_T\Omega, K, \mu](t,x'')| \qquad (1.13)\\
&\quad \leq \|\mu\|_{L^\infty(\partial_T\Omega)} \Bigg\{ \int_{-\infty}^{t}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(t,x',\tau,y)|\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(t,x'',\tau,y)|\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} |K(t,x',\tau,y) - K(t,x'',\tau,y)|\,d\sigma_y d\tau \Bigg\} \\
&\quad \leq \|\mu\|_{L^\infty(\partial_T\Omega)}\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \Bigg\{ \int_{-\infty}^{t}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{|x'-y|^{\gamma_2}}{(t-\tau)^{\gamma_1}} e^{-\frac{|x'-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{|x''-y|^{\gamma_2}}{(t-\tau)^{\gamma_1}} e^{-\frac{|x''-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau \\
&\qquad + |x'-x''|^{\gamma_l'} \int_{-\infty}^{t}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x'-y|^{\gamma_2'}}{(t-\tau)^{\gamma_1'}} e^{-\frac{|x'-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau \Bigg\}.
\end{aligned}$$

Then by setting $a(t-\tau) = u|x'-y|^2$, $a(t-\tau) = u|x''-y|^2$, $a(t-\tau) = u|x'-y|^2$ in the first, and second and third integrals in the right hand side of (1.13), respectively, we deduce that the right hand side of (1.13) equals

$$\begin{aligned}
&\|\mu\|_{L^\infty(\partial_T\Omega)}\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \Bigg\{ \int_0^{+\infty}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{|x'-y|^{\gamma_2+2}a^{-1+\gamma_1}}{u^{\gamma_1}|x'-y|^{2\gamma_1}} e^{-\frac{1}{u}}\,d\sigma_y du \\
&\qquad + \int_0^{+\infty}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{|x''-y|^{\gamma_2+2}a^{-1+\gamma_1}}{u^{\gamma_1}|x''-y|^{2\gamma_1}} e^{-\frac{1}{u}}\,d\sigma_y du \\
&\qquad + |x'-x''|^{\gamma_l'} \int_0^{+\infty}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x'-y|^{\gamma_2'+2}a^{-1+\gamma_1'}}{u^{\gamma_1'}|x'-y|^{2\gamma_1'}} e^{-\frac{1}{u}}\,d\sigma_y du \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\mu\|_{L^\infty(\partial_T\Omega)}\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\Bigg\{\frac{\Gamma(\gamma_1-1)}{a^{1-\gamma_1}}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega}\frac{d\sigma_y}{|x'-y|^{2\gamma_1-\gamma_2-2}}\\
&+\frac{\Gamma(\gamma_1-1)}{a^{1-\gamma_1}}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega}\frac{d\sigma_y}{|x''-y|^{2\gamma_1-\gamma_2-2}}\\
&+\frac{\Gamma(\gamma_1'-1)}{a^{1-\gamma_1'}}|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\Bigg\}\\
&\leq \|\mu\|_{L^\infty(\partial_T\Omega)}\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\Bigg\{2\frac{\Gamma(\gamma_1-1)}{a^{1-\gamma_1}}c''_{\Omega,\alpha}|x'-x''|^{(n-1)-(2\gamma_1-\gamma_2-2)}\\
&+\frac{\Gamma(\gamma_1'-1)}{a^{1-\gamma_1'}}|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\Bigg\},
\end{aligned}
$$

where we have used Lemma 1.13 (ii). We now distinguish three cases. In case $2\gamma_1'-\gamma_2'-2>n-1$, Lemma 1.13 (iii) implies that

$$
|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\leq c'''_{\Omega,2\gamma_1'-\gamma_2'-2}|x'-x''|^{\gamma_l'+(n-1)-(2\gamma_1'-\gamma_2'-2)}\,.
$$

In case $2\gamma_1'-\gamma_2'-2=n-1$, Lemma 1.13 (iv) implies that

$$
|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\leq |x'-x''|^{\gamma_l'}c^{iv}_{\Omega}\left|\log|x'-x''|\right|\,.
$$

In case $2\gamma_1'-\gamma_2'-2<n-1$, Lemma 1.13 (i) implies that

$$
|x'-x''|^{\gamma_l'}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2}}\leq c'_{\Omega,2\gamma_1'-\gamma_2'-2}|x'-x''|^{\gamma_l'}\,.
$$

Then the above inequalities imply the validity of statement (ii). □

Next, we consider the time Hölder regularity of the same integral operator of Proposition 1.21. More precisely, the following proposition holds.

Proposition 1.22. *Let $a\in]0,+\infty[$, $T\in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\gamma\equiv(\gamma_1,\gamma_2,\gamma_1',\gamma_2',\gamma_l',\gamma_1'',\gamma_2'',\gamma_l'')\in\mathbb{R}^8$. Let $\gamma_1>1$, $2\gamma_1-\gamma_2-2<n-1$. Let*

$$
h\in\left]0,\frac{(n-1)-(2\gamma_1-\gamma_2-2)}{2}\right[\cap]0,1]\,.
$$

Let $\gamma_1''>1$, $\gamma_l''\in]0,1]$,

$$
\max\left\{0,\frac{(2\gamma_1''-\gamma_2''-2)-(n-1)}{2}\right\}<\min\{\gamma_1''-1,\gamma_l''\},
$$

$$
h'\in\left]\max\left\{0,\frac{(2\gamma_1''-\gamma_2''-2)-(n-1)}{2}\right\},\min\{\gamma_1''-1,\gamma_l''\}\right]\,.
$$

Then the bilinear map from $\mathcal{K}_{\gamma,a}(\partial_T\Omega)\times L^\infty(\partial_T\Omega)$ to $C_b^{0,\min\{h,\gamma_l''-h'\}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$, which takes (K,μ) to $u[\partial_T\Omega,K,\mu]$ is continuous (cf. Remark 1.5).

Proof. By Proposition 1.21 (i), it suffices to estimate the Hölder quotient of $u[\partial_T\Omega, K, \mu]$ in the time variable. Let $x \in \partial\Omega$, $t', t'' \in \overline{]-\infty, T[}$, $t' < t''$. By Remark 1.1 and Proposition 1.21 (i), there is no loss of generality in assuming that $0 < |t' - t''| \leq 1/e$. Then the inclusion of intervals

$$]t' - 2|t' - t''|, t' + 2|t' - t''|[\subseteq]t'' - 3|t' - t''|, t'' + 3|t' - t''|[$$

and the triangular inequality imply that

$$\begin{aligned} &|u[\partial_T\Omega, K, \mu](t', x) - u[\partial_T\Omega, K, \mu](t'', x)| \qquad (1.14)\\ &\leq \int_{t'-2|t'-t''|}^{t'+2|t'-t''|} \int_{\partial\Omega} \left|K(t', x, \tau, y) - K(t'', x, \tau, y)\right| |\mu(\tau, y)| \, d\sigma_y d\tau \\ &\quad + \int_{-\infty}^{t'-2|t'-t''|} \int_{\partial\Omega} \left|K(t', x, \tau, y) - K(t'', x, \tau, y)\right| |\mu(\tau, y)| \, d\sigma_y d\tau \\ &\leq \|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \left\{ \int_{t'-2|t'-t''|}^{t'} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2}}{|t'-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t'-\tau)}} \, d\sigma_y d\tau \right. \\ &\quad + \int_{t''-3|t'-t''|}^{t''} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2}}{|t''-\tau|^{\gamma_1}} e^{-\frac{|x-y|^2}{a(t''-\tau)}} \, d\sigma_y d\tau \\ &\quad \left. + \int_{\partial\Omega} \int_{-\infty}^{t'-2|t'-t''|} \frac{|x-y|^{\gamma_2''}}{|t'-\tau|^{\gamma_1''}} |t' - t''|^{\gamma_l''} e^{-\frac{|x-y|^2}{a(t'-\tau)}} \, d\sigma_y d\tau \right\}. \end{aligned}$$

Then by setting $a(t' - \tau) = u|x - y|^2$, $a(t'' - \tau) = u|x - y|^2$, $a(t' - \tau) = u|x - y|^2$ in the first, and second and third integrals in the right hand side of (1.14), respectively, we deduce that the right hand side of (1.14) equals

$$\begin{aligned} &\|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \left\{ \int_{\partial\Omega} \int_0^{2a\frac{|t'-t''|}{|x-y|^2}} \frac{|x-y|^{\gamma_2+2} a^{\gamma_1-1}}{u^{\gamma_1} |x-y|^{2\gamma_1}} e^{-\frac{1}{u}} \, du \, d\sigma_y \right. \\ &\quad + \int_{\partial\Omega} \int_0^{3a\frac{|t'-t''|}{|x-y|^2}} \frac{|x-y|^{\gamma_2+2} a^{\gamma_1-1}}{u^{\gamma_1} |x-y|^{2\gamma_1}} e^{-\frac{1}{u}} \, du \, d\sigma_y \\ &\quad \left. + \int_{\partial\Omega} \int_{2a\frac{|t'-t''|}{|x-y|^2}}^{+\infty} \frac{|x-y|^{\gamma_2''+2} a^{\gamma_1''-1}}{u^{\gamma_1''} |x-y|^{2\gamma_1''}} |t' - t''|^{\gamma_l''} e^{-\frac{1}{u}} \, du \, d\sigma_y \right\}. \end{aligned}$$

Next we note that our assumptions of h and h' imply that $2\gamma_1 - \gamma_2 - 2 + 2h < n - 1$ and that $2\gamma_1'' - \gamma_2'' - 2 - 2h' < n - 1$. Then by Lemmas 1.13 (i) and 1.14 (i), (ii), the right hand side of (1.14) is less or equal to

$$\begin{aligned} &\|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \left\{ \int_{\partial\Omega} \tilde{D}_{\gamma_1,h} a^{\gamma_1-1} \left(2a\frac{|t'-t''|}{|x-y|^2}\right)^h \frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}} \right. \\ &\quad + \int_{\partial\Omega} \tilde{D}_{\gamma_1,h} a^{\gamma_1-1} \left(3a\frac{|t'-t''|}{|x-y|^2}\right)^h \frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}} \\ &\quad \left. + \int_{\partial\Omega} D_{\gamma_1'',h'} \left(2a\frac{|t'-t''|}{|x-y|^2}\right)^{-h'} a^{\gamma_1''-1} |t' - t''|^{\gamma_l''} \frac{d\sigma_y}{|x-y|^{2\gamma_1''-\gamma_2''-2}} \right\} \\ &\leq \|\mu\|_{L^\infty(\partial_T\Omega)} \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \left\{ \tilde{D}_{\gamma_1,h} a^{\gamma_1-1} (2a)^h c'_{\Omega,2\gamma_1-\gamma_2-2+2h} \right. \\ &\quad \left. + \tilde{D}_{\gamma_1,h} a^{\gamma_1-1} (3a)^h c'_{\Omega,2\gamma_1-\gamma_2-2+2h} + D_{\gamma_1'',h'} a^{\gamma_1''-1} (2a)^{-h'} c'_{\Omega,2\gamma_1''-\gamma_2''-2-2h'} \right\} \end{aligned}$$

$$\times|t'-t''|^{\min\{h,\gamma_l''-h'\}}\,,$$

and thus the statement holds true. □

We will also need to understand the behavior away from the boundary $\partial_T\Omega$ of the integral operators we are dealing with. For this reason, we prove the following lemma which concerns an integral operator which is not necessarily defined on $\partial_T\Omega$.

Lemma 1.23. *Let $a\in]0,+\infty[$, $T\in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let G be a subset of $\mathbb{R}^n$. Let $\gamma\equiv(\gamma_1,\gamma_2,\gamma_1',\gamma_2',\gamma_l',\gamma_1'',\gamma_2'',\gamma_l'')\in\mathbb{R}^8$. Let $\gamma_1>1$, $2\gamma_1-\gamma_2-2<n-1$. Let $K\in C^0((G_T\times\partial_T\Omega)\setminus\Delta_{\partial_T\Omega})$ be such that*

$$K(t,x,\tau,y)=0 \qquad \text{if } (t,x,\tau,y)\in(G_T\times\partial_T\Omega)\setminus\Delta_{\partial_T\Omega},\ \tau\geq t\,,$$

and such that

$$\kappa_{\gamma_1,\gamma_2}\equiv\sup\left\{|K(t,x,\tau,y)|\frac{|t-\tau|^{\gamma_1}}{|x-y|^{\gamma_2}}e^{\frac{|x-y|^2}{a(t-\tau)}}\,:\,(t,x,\tau,y)\in(G_T\times\partial_T\Omega)\setminus\Delta_{\partial_T\Omega}\right\}<+\infty\,.$$

If $\mu\in L^\infty(\partial_T\Omega)$, and if $(t,x)\in G_T$, then the function $K(t,x,\cdot,\cdot)\mu(\cdot,\cdot)$ is Lebesgue integrable in $\partial_T\Omega$. Moreover, the function $u^\sharp[G_T,\partial_T\Omega,K,\mu]$ from G_T to $\mathbb{C}$ defined by

$$u^\sharp[G_T,\partial_T\Omega,K,\mu](t,x)\equiv\int_{-\infty}^t\int_{\partial\Omega}K(t,x,\tau,y)\mu(\tau,y)\,d\sigma_y d\tau \qquad \forall\,(t,x)\in G_T,$$

is continuous. If $\sup_{x\in G}\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}}<+\infty$, then the following inequality holds

$$\begin{aligned}&|u^\sharp[G_T,\partial_T\Omega,K,\mu](t,x)| \\ &\qquad\leq\Gamma(\gamma_1-1)a^{\gamma_1-1}\sup_{x\in G}\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}}\kappa_{\gamma_1,\gamma_2}\|\mu\|_{L^\infty(\partial_T\Omega)}\,,\end{aligned}\tag{1.15}$$

for all $(t,x)\in G_T$.

Proof. Let $(t,x)\in G_T$. Then we have

$$\begin{aligned}&\int_{-\infty}^t\int_{\partial\Omega}|K(t,x,\tau,y)\mu(\tau,y)|\,d\sigma_y d\tau\\ &\quad\leq\kappa_{\gamma_1,\gamma_2}\|\mu\|_{L^\infty(\partial_T\Omega)}\int_{-\infty}^t\int_{\partial\Omega}\frac{|x-y|^{\gamma_2}}{|t-\tau|^{\gamma_1}}e^{-\frac{|x-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau\\ &\quad=\kappa_{\gamma_1,\gamma_2}\|\mu\|_{L^\infty(\partial_T\Omega)}\int_{\partial\Omega}\int_0^{+\infty}\frac{|x-y|^{\gamma_2+2}a^{\gamma_1-1}}{u^{\gamma_1}|x-y|^{2\gamma_1}}e^{-\frac{1}{u}}\,du\,d\sigma_y\\ &\quad=\kappa_{\gamma_1,\gamma_2}\|\mu\|_{L^\infty(\partial_T\Omega)}a^{\gamma_1-1}\int_0^{+\infty}u^{-\gamma_1}e^{-\frac{1}{u}}\,du\int_{\partial\Omega}\frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}}\,.\end{aligned}$$

Our assumptions imply the convergence of the integrals in the right hand side and the validity of inequality (1.15). The continuity of $u^\sharp[G_T,\partial_T\Omega,K,\mu]$ follows by the Vitali Convergence Theorem. □

1.7 Applications to some integral operators related to layer heat potentials with essentially bounded densities

In this section we apply the results of the previous Section 1.6 in order to obtain some mapping properties for some integral operators related to layer heat potentials. The results of this section will be applied in the next Chapter 2 in order to study the mapping properties of the double layer heat potential. We start with the analysis of a class of integral operators which we need to study the properties of an integral operator related to the kernel $D_x\Phi_n(t-\tau, x-y)$, and we introduce the following two statements. The first one is needed to understand the behavior of such a class of integral operators in the whole domain Ω_T, whereas the second one investigates the corresponding boundary integral operators.

Lemma 1.24. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. Let $Z \in C^0((\mathrm{cl}\,\Omega_T \times \partial_T\Omega) \setminus \Delta_{\partial_T\Omega})$ be such that*

$$Z(t,x,\tau,y) = 0 \qquad \text{if } (t,x,\tau,y) \in (\mathrm{cl}\,\Omega_T \times \partial_T\Omega) \setminus \Delta_{\partial_T\Omega}, \tau \geq t\,,$$

and such that

$$\zeta \equiv \sup\biggl\{ |Z(t,x,\tau,y)| \frac{|t-\tau|^{\frac{n}{2}+1}}{|x-y|} e^{\frac{|x-y|^2}{a(t-\tau)}} : \\ (t,x,\tau,y) \in (\mathrm{cl}\,\Omega_T \times \partial_T\Omega) \setminus \Delta_{\partial_T\Omega} \biggr\} < +\infty\,.$$

Let $f \in C^{0,\theta}(\mathrm{cl}\,\Omega)$. Let $H^\sharp[Z,f]$ be the function from $(\mathrm{cl}\,\Omega_T \times \partial_T\Omega) \setminus \Delta_{\partial_T\Omega}$ to $\mathbb{C}$ defined by

$$H^\sharp[Z,f](t,x,\tau,y) \equiv (f(x)-f(y))Z(t,x,\tau,y) \quad \forall\, (t,x,\tau,y) \in (\mathrm{cl}\,\Omega_T \times \partial_T\Omega) \setminus \Delta_{\partial_T\Omega}\,.$$

If $\mu \in L^\infty(\partial_T\Omega)$ and if $(t,x) \in \mathrm{cl}\,\Omega_T$, then the function $H^\sharp[Z,f](t,x,\cdot,\cdot)\mu(\cdot,\cdot)$ is Lebesgue integrable in $\partial_T\Omega$ and the function $Q^\sharp[Z,f,\mu]$ from $\mathrm{cl}\,\Omega_T$ to $\mathbb{C}$ defined by

$$Q^\sharp[Z,f,\mu](t,x) \equiv \int_{-\infty}^t \int_{\partial\Omega} H^\sharp[Z,f](t,x,\tau,y)\mu(\tau,y)\,d\sigma_y d\tau \qquad \forall (t,x) \in \mathrm{cl}\,\Omega_T\,,$$

is continuous and bounded.

Proof. We plan to apply Lemma 1.23. By definition of ζ, and by the Hölder continuity of f, we have

$$\Bigl| H^\sharp[Z,f](t,x,\tau,y) \Bigr| \leq \frac{|f|_\theta |x-y|^{1+\theta}}{|t-\tau|^{\frac{n}{2}+1}} \zeta e^{-\frac{|x-y|^2}{a(t-\tau)}}$$

for all $(t,x,\tau,y) \in (\mathrm{cl}\,\Omega_T \times \partial_T\Omega) \setminus \Delta_{\partial_T\Omega}$. Next we note that

$$(n/2)+1 > 1\,, \qquad 2\,((n/2)+1) - (1+\theta) - 2 = n-1-\theta < n-1\,,$$

and that the Vitali Convergence Theorem implies the continuity of the function $\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{(n-1)-\theta}}$ in the variable $x \in \mathrm{cl}\,\Omega$ and accordingly that $\sup_{x\in \mathrm{cl}\,\Omega} \int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{(n-1)-\theta}} < +\infty$. Then Lemma 1.23 implies the validity of the statement. □

Next we have the following.

Lemma 1.25. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. Let*

$$\begin{aligned} \gamma_n &\equiv ((n/2)+1, 1, (n/2)+1, 0, 1, (n/2)+2, 1, 1) \,, \\ \gamma_{n,\theta} &\equiv ((n/2)+1, 1+\theta, (n/2)+1, 1, \theta, (n/2)+2, 1+\theta, 1) \,, \end{aligned} \tag{1.16}$$

(cf. Remark 1.20 (iii)). Then the following statements hold.

(i) *The map H from $\mathcal{K}_{\gamma_n,a}(\partial_T\Omega) \times C^{0,\theta}(\partial\Omega)$ to $\mathcal{K}_{\gamma_{n,\theta},4a}(\partial_T\Omega)$, which takes (Z,g) to the function $H[Z,g]$ from $(\partial_T\Omega)^2 \setminus \Delta_{\partial_T\Omega}$ to $\mathbb{C}$ defined by*

$$H[Z,g](t,x,\tau,y) \equiv (g(x)-g(y))Z(t,x,\tau,y) \quad \forall\, (t,x,\tau,y) \in (\partial_T\Omega)^2 \setminus \Delta_{\partial_T\Omega}\,,$$

is bilinear and continuous.

(ii) *Let $\theta_1 \in]0,\theta[$. The map Q from $\mathcal{K}_{\gamma_n,a}(\partial_T\Omega) \times C^{0,\theta}(\partial\Omega) \times L^\infty(\partial_T\Omega)$ to*

$$B\left(\overline{]-\infty,T[}, C^{0,\omega_\theta(\cdot)}(\partial\Omega)\right) \cap C_b^{0,\frac{\theta_1}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right),$$

which takes (Z,g,μ) to the function $Q[Z,g,\mu]$ from $\partial_T\Omega$ to $\mathbb{C}$ defined by

$$Q[Z,g,\mu](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega} H[Z,g](t,x,\tau,y)\mu(\tau,y)\,d\sigma_y d\tau \quad \forall\,(t,x) \in \partial_T\Omega\,, \tag{1.17}$$

is trilinear and continuous (cf. Remark 1.5).

Proof. We first consider statement (i). Let $x,y \in \partial\Omega$, $x \neq y$, $t,\tau \in \overline{]-\infty,T[}$, $\tau < t$. The membership of Z in $\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)$ and of g in $C^{0,\theta}(\partial\Omega)$ implies that

$$|(g(x)-g(y))Z(t,x,\tau,y)| \leq |g|_\theta |x-y|^\theta \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \frac{|x-y|}{|t-\tau|^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{a(t-\tau)}}\,. \tag{1.18}$$

Let $t,\tau \in \overline{]-\infty,T[}$, $\tau < t$, $x',x'' \in \partial\Omega$, $x' \neq x''$, $y \in \partial\Omega \setminus \mathbb{B}_n(x', 2|x'-x''|)$. Lemma 1.11 (i) and the membership of Z in $\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)$ and of g in $C^{0,\theta}(\partial\Omega)$ imply that

$$\begin{aligned} &\left|(g(x')-g(y))Z(t,x',\tau,y) - (g(x'')-g(y))Z(t,x'',\tau,y)\right| \\ &\quad \leq |g(x')-g(y)|\,|Z(t,x',\tau,y)-Z(t,x'',\tau,y)| + |g(x')-g(x'')|\,|Z(t,x'',\tau,y)| \\ &\quad \leq |g|_\theta \frac{|x'-y|^\theta |x'-x''|}{|t-\tau|^{\frac{n}{2}+1}} e^{-\frac{|x'-y|^2}{a(t-\tau)}} \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \\ &\qquad + \frac{|x'-x''|^\theta |x''-y|}{|t-\tau|^{\frac{n}{2}+1}} e^{-\frac{|x''-y|^2}{a(t-\tau)}} |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \\ &\quad \leq |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \left\{ \frac{|x'-y|^\theta}{|t-\tau|^{\frac{n}{2}+1}} |x'-x''| + |x'-x''|^\theta \frac{2|x'-y|}{|t-\tau|^{\frac{n}{2}+1}} \right\} e^{-\frac{|x'-y|^2}{4a(t-\tau)}}\,. \end{aligned}$$

Since $|x'-x''| \leq |x'-y|$, we have $|x'-x''|^{1-\theta} \leq |x'-y|^{1-\theta}$. Hence,

$$\begin{aligned} &|x'-y|^\theta |x'-x''| + |x'-x''|^\theta 2|x'-y| \\ &\qquad \leq |x'-y|\,|x'-x''|^\theta + |x'-x''|^\theta 2|x'-y| = 3|x'-y|\,|x'-x''|^\theta\,, \end{aligned}$$

and accordingly

$$\left|(g(x')-g(y))Z(t,x',\tau,y) - (g(x'')-g(y))Z(t,x'',\tau,y)\right| \tag{1.19}$$

$$\leq |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} 3\frac{|x'-y|}{|t-\tau|^{\frac{n}{2}+1}}|x'-x''|^\theta e^{-\frac{|x'-y|^2}{4a(t-\tau)}} .$$

Now let $x, y \in \partial\Omega$, $x \neq y$, $t', t'' \in \overline{]-\infty, T[}$, $t' < t''$, $\tau < t' - 2|t' - t''|$. The membership of Z in $\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)$ and of g in $C^{0,\theta}(\partial\Omega)$ implies that

$$\begin{aligned} &\big|(g(x)-g(y))Z(t',x,\tau,y)-(g(x)-g(y))Z(t'',x,\tau,y)\big| \\ &\qquad \leq |x-y|^\theta |g|_\theta |Z(t',x,\tau,y)-Z(t'',x,\tau,y)| \\ &\qquad \leq |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} |x-y|^\theta \frac{|x-y|}{|t'-\tau|^{\frac{n}{2}+2}}|t'-t''|e^{-\frac{|x-y|^2}{a(t'-\tau)}} . \end{aligned} \tag{1.20}$$

Inequalities (1.18), (1.19) and (1.20) imply the validity of statement (i).

Next we consider statement (ii). By Proposition 1.21 (ii) with $\gamma = \gamma_{n,\theta}$, the map $u[\partial_T\Omega, \cdot, \cdot]$ is bilinear and continuous from $\mathcal{K}_{\gamma_{n,\theta},4a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$ to

$$B\left(\overline{]-\infty,T[}, C^{0,\max\{r^\theta,\omega_\theta(\cdot)\}}(\partial\Omega)\right) .$$

Indeed,

$$\begin{aligned} &\gamma_1 = (n/2)+1 > 1 , \\ &2\gamma_1 - \gamma_2 - 2 = 2((n/2)+1) - (1+\theta) - 2 = (n-1) - \theta \in [n-2, n-1[, \\ &\gamma_1' = (n/2)+1 > 1 , \\ &\gamma_l' = \theta \in]0,1] , \\ &2\gamma_1' - \gamma_2' - 2 = 2((n/2)+1) - 1 - 2 = n-1 , \\ &(n-1) - (2\gamma_1 - \gamma_2 - 2) = (n-1) - [2((n/2)+1) - (1+\theta) - 2] = \theta , \\ &\gamma_l' + (n-1) - (2\gamma_1' - \gamma_2' - 2) = \theta + (n-1) - (n-1) = \theta > 0 , \end{aligned}$$

and $C^{0,\max\{r^\theta,\omega_\theta(\cdot)\}}(\partial\Omega) = C^{0,\omega_\theta(\cdot)}(\partial\Omega)$.

Next we wish to apply Proposition 1.22 with $\gamma = \gamma_{n,\theta}$. Clearly, $\gamma_1 = (n/2)+1 > 1$. Moreover, we have seen above that $2\gamma_1 - \gamma_2 - 2 = (n-1) - \theta \in [n-2, n-1[$. Then we can choose

$$h \equiv \theta_1/2 \in]0, \theta/2[= \left]0, \frac{(n-1) - [(n-1) - \theta]}{2}\right[\cap]0,1] .$$

Next we observe that $\gamma_1'' = (n/2)+2 > 1$, $\gamma_l'' = 1$ and that

$$\begin{aligned} &\frac{(2\gamma_1'' - \gamma_2'' - 2) - (n-1)}{2} = \frac{2((n/2)+2) - (1+\theta) - 2 - (n-1)}{2} = 1 - (\theta/2) > 0, \\ &\gamma_1'' - 1 = ((n/2)+2) - 1 = (n/2)+1 > 1 , \\ &\min\{\gamma_1'' - 1, \gamma_l''\} = \min\{(n/2)+1, 1\} = 1 > 1 - (\theta/2) . \end{aligned}$$

Since $1 - (\theta/2) < 1 - (\theta_1/2) = 1 - h < 1$, we can choose

$$h' \in]1 - (\theta/2), 1[,$$

close enough to $1 - (\theta/2)$ so that $h' < 1 - (\theta_1/2) = 1 - h$, *i.e.*, such that $h < 1 - h'$. Then $\min\{h, \gamma_l'' - h'\} = \min\{h, 1 - h'\} = h$. Accordingly, Proposition 1.22 implies that the map $u[\partial_T\Omega, \cdot, \cdot]$ is bilinear and continuous from $\mathcal{K}_{\gamma_{n,\theta},4a}(\partial_T\Omega) \times L^\infty(\partial_T\Omega)$ to $C_b^{0,\frac{\theta_1}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$. Hence, statement (i) implies the validity of statement (ii). □

Remark 1.26. Under the assumptions of the previous proposition, one can readily prove that $B\left(\overline{]-\infty,T[},C^{0,\omega_\theta(\cdot)}(\partial\Omega)\right)\cap C_b^{0,\frac{\theta_1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ is continuously embedded into $C^{\frac{\theta_1}{2},\theta_1}(\partial_T\Omega)$.

Then Remark 1.20 (iii), Lemma 1.25 and Remark 1.26 immediately imply the validity of the following theorem regarding the mapping properties of an integral operator with $\partial_{x_r}\Phi_n(t-\tau,x-y)$ as kernel.

Theorem 1.27. *Let $T\in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta\in]0,1]$, $\theta_1\in]0,\theta[$. Let $r\in\{1,\dots,n\}$. Then the map $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from $C^{0,\theta}(\partial\Omega)\times L^\infty(\partial_T\Omega)$ to $C^{\frac{\theta_1}{2},\theta_1}(\partial_T\Omega)$ which takes (g,μ) to the function*

$$
\begin{aligned}
&Q[\partial_{x_r}\Phi_n(t-\tau,x-y),g,\mu](t,x) \\
&\quad=\int_{-\infty}^t\int_{\partial\Omega}(g(x)-g(y))\partial_{x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \quad \forall\,(t,x)\in\partial_T\Omega\,,
\end{aligned}
\tag{1.21}
$$

is bilinear and continuous (cf. Remark 1.5).

Next we turn to analyze a class of integral operators which we need to study the properties of class of integral operators related to the kernel $\partial_t\Phi_n(t-\tau,x-y)$, and we introduce the following two statements. Again, the first one is needed to understand the behavior of such a class of integral operators in the whole domain Ω_T, whereas the second one investigates the mapping properties of the corresponding boundary integral operators.

Lemma 1.28. *Let $a\in]0,+\infty[$, $T\in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta\in]0,1]$. Let $\tilde{Z}\in C^0((\mathrm{cl}\,\Omega_T\times\partial_T\Omega)\setminus\Delta_{\partial_T\Omega})$ be such that*

$$
\tilde{Z}(t,x,\tau,y)=0 \qquad \text{if } (t,x,\tau,y)\in(\mathrm{cl}\,\Omega_T\times\partial_T\Omega)\setminus\Delta_{\partial_T\Omega},\ \tau\geq t\,,
$$

and such that

$$
\tilde{\zeta}\equiv\sup\left\{|\tilde{Z}(t,x,\tau,y)||t-\tau|^{\frac{n}{2}+1}e^{\frac{|x-y|^2}{a(t-\tau)}}\,:\,(t,x,\tau,y)\in(\mathrm{cl}\,\Omega_T\times\partial_T\Omega)\setminus\Delta_{\partial_T\Omega}\right\}<+\infty\,.
$$

Let $f\in C^{0,\theta}(\mathrm{cl}\,\Omega)$ and $\mu\in C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$. Let $\tilde{H}^\sharp[\tilde{Z},f,\mu]$ be the function from $(\mathrm{cl}\,\Omega_T\times\partial_T\Omega)\setminus\Delta_{\partial_T\Omega}$ to $\mathbb{C}$ defined by

$$
\begin{aligned}
\tilde{H}^\sharp[\tilde{Z},f,\mu](t,x,\tau,y)&\equiv(f(x)-f(y))\tilde{Z}(t,x,\tau,y)(\mu(\tau,y)-\mu(t,y))\\
&\qquad\forall\,(t,x,\tau,y)\in(\mathrm{cl}\,\Omega_T\times\partial_T\Omega)\setminus\Delta_{\partial_T\Omega}\,.
\end{aligned}
$$

If $(t,x)\in\mathrm{cl}\,\Omega_T$, then the function $\tilde{H}^\sharp[\tilde{Z},f,\mu](t,x,\cdot,\cdot)$ is Lebesgue integrable in $\partial_T\Omega$ and the function $\tilde{Q}^\sharp[Z,f,\mu]$ from $\mathrm{cl}\,\Omega_T$ to $\mathbb{C}$ defined by

$$
\tilde{Q}^\sharp[\tilde{Z},f,\mu](t,x)\equiv\int_{-\infty}^t\int_{\partial\Omega}\tilde{H}^\sharp[\tilde{Z},f,\mu](t,x,\tau,y)\,d\sigma_y d\tau \qquad \forall(t,x)\in\mathrm{cl}\,\Omega_T\,,
$$

is continuous and bounded.

Proof. We plan to apply Lemma 1.23. By definition of $\tilde{\zeta}$ and of $\tilde{H}[\tilde{Z},f,\mu]$, we have

$$
\left|\tilde{H}^\sharp[\tilde{Z},\tilde{f}](t,x,\tau,y)\right|\leq\frac{|f|_\theta\|\mu\|_{C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)}|x-y|^\theta}{|t-\tau|^{\frac{n+1}{2}}}\tilde{\zeta}e^{-\frac{|x-y|^2}{a(t-\tau)}}\,,
$$

for all $(t,x,\tau,y) \in \mathrm{cl}\,\Omega_T \times \partial_T\Omega \setminus \Delta_{\partial_T\Omega}$. Next we note that

$$(n+1)/2 > 1\,, \qquad 2((n+1)/2) - \theta - 2 = n - 1 - \theta < n-1\,,$$

and that Vitali Convergence Theorem implies the continuity of the function $\int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{(n-1)-\theta}}$ in the variable $x \in \mathrm{cl}\,\Omega$ and accordingly that $\sup_{x\in \mathrm{cl}\,\Omega} \int_{\partial\Omega} \frac{d\sigma_y}{|x-y|^{(n-1)-\theta}} < +\infty$. Then Lemma 1.23 with $\mu \equiv 1$ implies the validity of the statement. □

Next we have the following.

Proposition 1.29. *Let $a \in]0,+\infty[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. Let $b \in [1/2,1]$, $b_1 \in]0,b[$. If $n=2$, then we further assume that $b<1$. Let*

$$\begin{aligned} \gamma_n^\sharp &\equiv ((n/2)+1, 0, (n/2)+2, 1, 1, (n/2)+2, 0, 1)\,, \\ \gamma_{n,\theta,b}^\sharp &\equiv ((n/2)+1-b, \theta, (n/2)+1-b, 0, \theta, (n/2)+1-(b-b_1), \theta, b_1)\,, \end{aligned} \tag{1.22}$$

(cf. Remark 1.20 (ii)). Then the following statements hold.

(i) The map $\tilde{H}$ from $\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega) \times C^{0,\theta}(\partial\Omega) \times C^{0,b;0,1}(\partial_T\Omega)$ to $\mathcal{K}_{\gamma_{n,\theta,b}^\sharp,5a}(\partial_T\Omega)$, which takes (Z,g,μ) to the function $\tilde{H}[Z,g,\mu]$ from $(\partial_T\Omega)^2 \setminus \Delta_{\partial_T\Omega}$ to $\mathbb{C}$ defined by

$$\tilde{H}[Z,g,\mu](t,x,\tau,y) \equiv (g(x)-g(y))Z(t,x,\tau,y)(\mu(\tau,y)-\mu(t,y))\,,$$

for all $(t,x,\tau,y) \in (\partial_T\Omega)^2 \setminus \Delta_{\partial_T\Omega}$, is trilinear and continuous.

(ii) Let $2b+\theta \leq 2$. Let

$$\tilde{\omega}_{b,\theta}(r) \equiv \begin{cases} r^\theta & \text{if } b \in]1/2,1]\,, \\ \omega_\theta(r) & \text{if } b = 1/2\,. \end{cases}$$

The map $\tilde{Q}$ from $\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega) \times C^{0,\theta}(\partial\Omega) \times C^{0,b;0,1}(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[}, C^{0,\tilde{\omega}_{b,\theta}(\cdot)}(\partial\Omega)\right)$ which takes (Z,g,μ) to the function $\tilde{Q}[Z,g,\mu]$ from $\partial_T\Omega$ to $\mathbb{C}$ defined by

$$\tilde{Q}[Z,g,\mu](t,x) \equiv \int_{-\infty}^t \int_{\partial\Omega} \tilde{H}[Z,g,\mu](\tau,y)\,d\sigma_y d\tau\,,$$

for all $(t,x) \in \partial_T\Omega$, is trilinear and continuous.

(iii) The interval $\left]\max\left\{0, \frac{1-\theta-2(b-b_1)}{2}\right\}, \min\{(n/2)-(b-b_1), b_1\}\right]$ is not empty and the map from $\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega) \times C^{0,\theta}(\partial\Omega) \times C^{0,b;0,1}(\partial_T\Omega)$ to $C_b^{0,\min\{h,b_1-b_2\}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ which takes (Z,g,μ) to $\tilde{Q}[Z,g,\mu]$ is trilinear and continuous for all

$$\begin{aligned} h &\in \left]0, \frac{2b+\theta-1}{2}\right[\,, \\ b_2 &\in \left]\max\left\{0, \frac{1-\theta-2(b-b_1)}{2}\right\}, \min\{(n/2)-(b-b_1), b_1\}\right]\,. \end{aligned}$$

Proof. We first consider statement (i). Let $x,y \in \partial\Omega$, $x \neq y$, $t,\tau \in \overline{]-\infty,T[}$, $\tau < t$. Then we have

$$|(g(x)-g(y))Z(t,x,\tau,y)|\,|\mu(\tau,y)-\mu(t,y)| \tag{1.23}$$

$$
\begin{aligned}
&\leq |g|_\theta \frac{|x-y|^\theta}{|t-\tau|^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{a(t-\tau)}} \|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} |\tau-t|^b \\
&= |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} \frac{|x-y|^\theta}{|t-\tau|^{\frac{n}{2}+1-b}} e^{-\frac{|x-y|^2}{a(t-\tau)}} .
\end{aligned}
$$

Let $t,\tau \in \overline{]-\infty,T[}$, $\tau<t$, $x',x''\in\partial\Omega$, $x'\neq x''$, $y\in\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)$. Then Lemma 1.11 (i), and the definition of $\|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)}$, and the Hölder continuity of g, μ, and the triangular inequality imply that

$$
\begin{aligned}
&|(g(x')-g(y))Z(t,x',\tau,y)(\mu(\tau,y)-\mu(t,y)) \\
&\qquad -(g(x'')-g(y))Z(t,x'',\tau,y)(\mu(\tau,y)-\mu(t,y))| \\
&\leq \Big\{ |g(x')-g(y)|\,|Z(t,x',\tau,y)-Z(t,x'',\tau,y)| \\
&\qquad +|g(x')-g(x'')|\,|Z(t,x'',\tau,y)| \Big\} |\mu(\tau,y)-\mu(t,y)| \\
&\leq |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} \Big\{ \frac{|x'-y|^{\theta+1}}{|t-\tau|^{\frac{n}{2}+2}} |x'-x''| e^{-\frac{|x'-y|^2}{a(t-\tau)}} \\
&\qquad + \frac{|x'-x''|^\theta}{|t-\tau|^{\frac{n}{2}+1}} e^{-\frac{|x''-y|^2}{a(t-\tau)}} \Big\} |t-\tau|^b .
\end{aligned}
\tag{1.24}
$$

Since $|x'-x''|\leq|x'-y|$, we have $|x'-x''|^{1-\theta}\leq|x'-y|^{1-\theta}$. Moreover, Lemma 1.11 (i) implies that $|x''-y|\geq\frac{1}{2}|x'-y|$. Then Lemma 1.14 (iv) implies that the right hand side of (1.24) is less or equal to

$$
\begin{aligned}
&|g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} \Big\{ \frac{|x'-y|^2|x'-x''|^\theta}{|t-\tau|^{\frac{n}{2}+2}} e^{-\frac{|x'-y|^2}{a(t-\tau)}} \\
&\qquad + \frac{|x'-x''|^\theta}{|t-\tau|^{\frac{n}{2}+1}} e^{-\frac{|x'-y|^2}{4a(t-\tau)}} \Big\} |t-\tau|^b \\
&\leq |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} \frac{|x'-x''|^\theta}{|t-\tau|^{\frac{n}{2}+1-b}} \Big\{ \frac{|x'-y|^2}{t-\tau}+1 \Big\} e^{-\frac{|x'-y|^2}{4a(t-\tau)}} \\
&\leq |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} \frac{|x'-x''|^\theta}{|t-\tau|^{\frac{n}{2}+1-b}} C(4a,5a,1) e^{-\frac{|x'-y|^2}{5a(t-\tau)}} .
\end{aligned}
\tag{1.25}
$$

Let $x,y\in\partial\Omega$, $x\neq y$, $t',t''\in\overline{]-\infty,T[}$, $t'<t''$, $\tau<t'-2|t'-t''|$. Then Lemma 1.11 (i) and the triangular inequality imply that

$$
\begin{aligned}
&|(g(x)-g(y))Z(t',x,\tau,y)(\mu(\tau,y)-\mu(t',y)) \\
&\qquad -(g(x)-g(y))Z(t'',x,\tau,y)(\mu(\tau,y)-\mu(t'',y))| \\
&\leq |(g(x)-g(y))Z(t',x,\tau,y)|\,|(\mu(\tau,y)-\mu(t',y))-(\mu(\tau,y)-\mu(t'',y))| \\
&\quad +|\mu(\tau,y)-\mu(t'',y)|\,|(g(x)-g(y))Z(t',x,\tau,y)-(g(x)-g(y))Z(t'',x,\tau,y)| \\
&\leq |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} \Big\{ \frac{|x-y|^\theta|t'-t''|^b}{(t'-\tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{a(t'-\tau)}} \\
&\quad + \frac{|\tau-t''|^b|x-y|^\theta|t'-t''|}{(t'-\tau)^{\frac{n}{2}+2}} e^{-\frac{|x-y|^2}{a(t'-\tau)}} \Big\} \\
&\leq |g|_\theta \|Z\|_{\mathcal{K}_{\gamma_n^\sharp,a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} |t'-t''|^{b_1} \Big\{ \frac{|x-y|^\theta|t'-t''|^{b-b_1}}{(t'-\tau)^{\frac{n}{2}+1}}
\end{aligned}
\tag{1.26}
$$

$$
\begin{aligned}
&+2^b \frac{|x-y|^\theta |t'-t''|^{1-b_1}}{(t'-\tau)^{\frac{n}{2}+2-b}} \Big\} e^{-\frac{|x-y|^2}{a(t'-\tau)}} \\
&\le 2|g|_\theta \|Z\|_{\mathcal{K}_{\gamma^\sharp_{n},a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} |t'-t''|^{b_1} \Big\{ \frac{|x-y|^\theta}{(t'-\tau)^{\frac{n}{2}+1-(b-b_1)}} \\
&+ \frac{|x-y|^\theta}{(t'-\tau)^{\frac{n}{2}+2-b-(1-b_1)}} \Big\} e^{-\frac{|x-y|^2}{a(t'-\tau)}} \\
&\le 4|g|_\theta \|Z\|_{\mathcal{K}_{\gamma^\sharp_{n},a}(\partial_T\Omega)} \|\mu\|_{C^{0,b;0,1}(\partial_T\Omega)} |t'-t''|^{b_1} \frac{|x-y|^\theta}{(t'-\tau)^{\frac{n}{2}+1-(b-b_1)}} e^{-\frac{|x-y|^2}{a(t'-\tau)}} .
\end{aligned}
$$

Then inequalities (1.23)–(1.26) imply the validity of statement (i).

We now prove statement (ii). We distinguish case $b \in]1/2, 1]$ and case $b = 1/2$ and we first consider case $b \in]1/2, 1]$. By Proposition 1.21 (ii) with $\gamma = \gamma^\sharp_{n,\theta,b}$, the map $u[\partial_T\Omega, \cdot, 1]$ is linear and continuous from $\mathcal{K}_{\gamma^\sharp_{n,\theta,b},5a}(\partial_T\Omega)$ to

$$
\begin{aligned}
&B\left(\overline{]-\infty,T[}, C^{0,\min\{(n-1)-(n-2b-\theta),\theta\}}(\partial\Omega)\right) \\
&\qquad = B\left(\overline{]-\infty,T[}, C^{0,\min\{2b+\theta-1,\theta\}}(\partial\Omega)\right) = B\left(\overline{]-\infty,T[}, C^{0,\theta}(\partial\Omega)\right) .
\end{aligned}
$$

Indeed, by our assumptions we have $1 < 2b+\theta \le 2$ and

$$
\begin{aligned}
&\gamma_1 = (n/2) + 1 - b > 1 , \qquad (1.27)\\
&2\gamma_1 - \gamma_2 - 2 = 2[(n/2)+1-b] - \theta - 2 \\
&\qquad\qquad = n - 2b - \theta = (n-1) - (2b+\theta-1) \in [n-2, n-1[, \\
&\gamma'_1 = (n/2) + 1 - b > 1 , \\
&\gamma'_l = \theta \in]0,1] , \\
&\gamma'_l + (n-1) - (2\gamma'_1 - \gamma'_2 - 2) = \theta + (n-1) - (n-2b) = \theta - 1 + 2b > 0 .
\end{aligned}
$$

Moreover, assumption $b > 1/2$ implies that

$$
2\gamma'_1 - \gamma'_2 - 2 = 2((n/2)+1-b) - 2 = n - 2b < n-1 ,
$$

and that

$$
(n-1) - (2\gamma_1 - \gamma_2 - 2) = 2b + \theta - 1 > \theta = \gamma'_l .
$$

Hence, statement (i) implies the validity of statement (ii) for $b \in]1/2, 1]$. Next we consider the case $b = 1/2$. Since

$$
2\gamma'_1 - \gamma'_2 - 2 = n - 1 , \qquad (n-1) - (2\gamma_1 - \gamma_2 - 2) = \theta ,
$$

statement (i) and Proposition 1.21 (ii) imply the validity of statement (ii).

We now turn to prove statement (iii). We first note that the assumptions $b \in [1/2, 1]$ and $\theta > 0$ imply that $\frac{1-\theta-2(b-b_1)}{2} < b_1$ and that accordingly the interval

$$
\left]\max\left\{0, \frac{1-\theta-2(b-b_1)}{2}\right\}, \min\{(n/2)-(b-b_1), b_1\}\right]
$$

is not empty. Indeed, $[1-\theta-2(b-b_1)]/2 < (n/2) - (b-b_1)$. Next we plan to exploit Proposition 1.22 with $\gamma = \gamma^\sharp_{n,\theta,b}$. By assumption and by the equalities in (1.27), we have

$$
h \in \big]0, (2b+\theta-1)/2\big[= \big]0, [(n-1) - (2\gamma_1 - \gamma_2 - 2)]/2\big[\cap \big]0,1\big] .
$$

Next we observe that $\gamma''_1 = (n/2)+1-(b-b_1) > 1$, $\gamma''_l = b_1 \in]0,1]$ and that

$$\begin{aligned}\frac{(2\gamma''_1-\gamma''_2-2)-(n-1)}{2} &= \frac{n+2-2(b-b_1)-\theta-2-(n-1)}{2}\\ &= \frac{(1-\theta)-2(b-b_1)}{2} < \gamma''_1 - 1 = (n/2)+1-(b-b_1)-1\\ &= (n/2)-(b-b_1) > 0\,,\\ \frac{(2\gamma''_1-\gamma''_2-2)-(n-1)}{2} &= \frac{(1-\theta)-2(b-b_1)}{2} < \gamma''_l = b_1 > 0\,.\end{aligned}$$

Indeed, $(1-\theta)-2(b-b_1) < n-2(b-b_1)$ and $(1-\theta)-2(b-b_1) < 2b_1$. By assumption,

$$b_2 \in \left]\max\left\{0, \frac{1-\theta-2(b-b_1)}{2}\right\}, \min\{(n/2)-(b-b_1), b_1\}\right]\,.$$

Then the map $u[\partial_T\Omega,\cdot,1]$ from $\mathcal{K}_{\gamma^\sharp_{n,\theta,b},5a}(\partial_T\Omega)$ to $C_b^{0,\min\{h,b_1-b_2\}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ is linear and continuous. Hence, statement (i) implies the validity of statement (iii). □

Finally, we specialize the previous proposition to case in which the kernel Z of the integral operator $\tilde{Q}[Z,\cdot,\cdot]$ is exactly $\partial_t\Phi_n(t-\tau,x-y)$.

Theorem 1.30. *Let $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$. Then the following statements hold.*

(i) The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ from $C^{0,\alpha}(\partial\Omega)\times C^{0,\frac{1+\beta}{2};0,1}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ which takes (g,μ) to the function

$$\begin{aligned}&\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),g,\mu](t,x) \qquad (1.28)\\ &\quad = \int_{-\infty}^{t}\int_{\partial\Omega}(g(x)-g(y))\partial_t\Phi_n(t-\tau,x-y)(\mu(\tau,y)-\mu(t,y))\,d\sigma_y d\tau\,,\end{aligned}$$

for all $(t,x)\in\partial_T\Omega$, is bilinear and continuous.

(ii) The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,\frac{1}{2};0,1}(\partial_T\Omega)$ to $C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$.

(iii) The map $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,1}(\partial\Omega)\times C^{0,1;0,1}(\partial_T\Omega)$ to $C_b^{0,\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$.

Proof. Let $a \in]16,+\infty[$. Then Remark 1.20 (ii) implies that $\partial_t\Phi_n(t-\tau,x-y)$ belongs to $\mathcal{K}_{\gamma^\sharp_n,a}(\partial_T\Omega)$ with $\gamma^\sharp_n$ as in (1.22). We now prove statement (i). If $\beta_1 \in]0,\beta]$, then $C^{0,\frac{1+\beta}{2};0,1}(\partial_T\Omega)$ is continuously imbedded into $C^{0,\frac{1+\beta_1}{2};0,1}(\partial_T\Omega)$. Thus there is no loss of generality in assuming that $\alpha+\beta<1$. Then we can apply Proposition 1.29 with

$$b = \frac{1+\beta}{2}, \qquad \theta = \alpha, \qquad b_1 \in \left]\frac{\alpha}{2}, \frac{\alpha+\beta}{2}\right[\subseteq \left]\frac{\alpha}{2}, \frac{1+\beta}{2}\right[.$$

Indeed, $2b+\theta = 1+\beta+\alpha \in]1,2]$. Proposition 1.29 (ii) implies that $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega)\times C^{0,\frac{1+\beta}{2};0,1}(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[}, C^{0,\alpha}(\partial\Omega)\right)$. Then we note that

$$h \equiv \frac{\alpha}{2} \in \left]0, \frac{2b+\theta-1}{2}\right[= \left]0, \frac{\alpha+\beta}{2}\right[\,,$$

and that

$$1 \geq b - b_1 \geq \frac{1-\alpha}{2},$$

$$(n/2) - [2^{-1}(1+\beta) - b_1] \geq (n/2) - 2^{-1}(1+\beta) + (\alpha/2) \geq (\alpha/2) - (\beta/2),$$

and that accordingly

$$\left]\max\left\{0, \frac{1-\theta-2(b-b_1)}{2}\right\}, \min\{(n/2) - [b-b_1], b_1\}\right[$$
$$=]0, \min\{(n/2) - [2^{-1}(1+\beta) - b_1], b_1\}[\supseteq]0, 2^{-1}(\alpha-\beta)[\,.$$

Since $b_1 > \alpha/2$, we can choose $b_2 \in]0, 2^{-1}(\alpha-\beta)[$ such that $b_1 - b_2 > \alpha/2$, and thus $\min\{h, b_1 - b_2\} = \alpha/2$, and Proposition 1.29 (iii) implies that $\tilde{Q}[\partial_t \Phi_n(t-\tau, x-y), \cdot, \cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\frac{1+\beta}{2};0,1}(\partial_T\Omega)$ to $C_b^{0,\frac{\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$. Then statement (i) holds true.

Next we consider statement (ii). We apply Proposition 1.29 (ii) with $b = 1/2, \theta = \alpha$. Then the map $\tilde{Q}[\partial_t \Phi_n(t-\tau, x-y), \cdot, \cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\frac{1}{2};0,1}(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[}, C^{0,\omega_\alpha(\cdot)}(\partial\Omega)\right)$. By the continuity of the embedding of $C^{0,\omega_\alpha(\cdot)}(\partial\Omega)$ into $C^{0,\beta}(\partial\Omega)$, the same map is continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\frac{1}{2};0,1}(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[}, C^{0,\beta}(\partial\Omega)\right)$. Then we plan to apply Proposition 1.29 (iii) with $b = 1/2$, $\theta = \alpha$, $b_1 \in]\beta/2, \alpha/2[$. We take

$$h \equiv \frac{\beta}{2} \in \left]0, \frac{2b+\theta-1}{2}\right[= \left]0, \frac{\alpha}{2}\right[\,,$$

and we choose

$$b_2 \in \left]\max\left\{0, \frac{1-\theta-2(b-b_1)}{2}\right\}, \min\{(n/2) - (b-b_1), b_1\}\right]$$
$$=]\max\{0, b_1 - \alpha/2\}, \min\{(n-1)/2 + b_1, b_1\}]$$
$$=]0, b_1[\,,$$

where we have exploited the membership of b_1 in $]\beta/2, \alpha/2[$. We note that since $b_1 > \beta/2$ we can choose $b_2 \in]0, b_1[$ such that $b_1 - b_2 > \beta/2$. Hence, $\tilde{Q}[\partial_t \Phi_n(t-\tau, x-y), \cdot, \cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\frac{1}{2};0,1}(\partial_T\Omega)$ to $C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T]}, C^0(\partial\Omega)\right)$.Then we can conclude that $\tilde{Q}[\partial_t \Phi_n(t-\tau, x-y), \cdot, \cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega) \times C^{0,\frac{1}{2};0,1}(\partial_T\Omega)$ to $C^{\frac{\beta}{2},\beta}(\partial_T\Omega)$. Hence, statement (ii) holds true.

Next we turn to prove statement (iii). We plan to apply Proposition 1.29 (iii) with $b \in]2^{-1}(1+\alpha), 1[$, $\theta = 1$, $b_1 \in]2^{-1}(1+\alpha), b[$. We note that

$$h \equiv \frac{1+\alpha}{2} \in \left]0, \frac{2b+\theta-1}{2}\right[=]0, b[\,,$$

and that $b - b_1 \leq 1 - 2^{-1}(1+\alpha) = 2^{-1}(1-\alpha)$, and that accordingly

$$\left]\max\left\{0, \frac{1-\theta-2(b-b_1)}{2}\right\}, \min\{(n/2) - [b-b_1], b_1\}\right[$$
$$=]0, \min\{(n/2) - [b-b_1], b_1\}[\supseteq]0, \min\{1 - 2^{-1}(1-\alpha), b_1\}[$$
$$=]0, 2^{-1}(1+\alpha)[\,.$$

Since $b_1 > 2^{-1}(1+\alpha)$, we can choose $b_2 \in]0, 2^{-1}(1+\alpha)[$ such that $b_1 - b_2 > 2^{-1}(1+\alpha)$ and thus $\min\{h, b_1 - b_2\} = 2^{-1}(1+\alpha)$. Hence, Proposition 1.29 (iii) implies that statement (iii) holds true. □

1.8 Integral operators on the space of Hölder continuous functions

Next we consider the action of the integral operator $u[\partial_T\Omega,\cdot,\cdot]$ on (K,μ), in case K is in the class of kernels $\mathcal{K}_{\gamma,a}$ that we have introduced in Definition 1.19 and the functional variable μ is Hölder continuous. We start with the following result which collects two statements. The first one regards the Hölder regularity in space and the second one regards the Hölder regularity in time of $u[\partial_T\Omega,K,\mu]$.

Proposition 1.31. *Let $a\in]0,+\infty[$, $T\in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\gamma\in\mathbb{R}^8$ be as in (1.11). Let γ_l', $\gamma_l''\in]0,1]$. Let $\gamma_1>1$, $2\gamma_1-\gamma_2-2\in[n-2,n-1[$. Then the following statements hold.*

(i) Let $\eta_2\in]0,1[$. Let $\gamma_1'-(\eta_2/2)>1$, $(n-1)-(2\gamma_1'-\gamma_2'-2-\eta_2)+\gamma_l'>0$. Let

$$\omega(r)\equiv\begin{cases} r^{\min\{(n-1)-(2\gamma_1-\gamma_2-2),\gamma_l'\}} & \text{if } 2\gamma_1'-\gamma_2'-2-\eta_2<n-1\,,\\ \max\{r^{(n-1)-(2\gamma_1-\gamma_2-2)},\omega_{\gamma_l'}(r)\} & \text{if } 2\gamma_1'-\gamma_2'-2-\eta_2=n-1\,,\\ r^{\min\{(n-1)-(2\gamma_1-\gamma_2-2),(n-1)-(2\gamma_1'-\gamma_2'-2-\eta_2)+\gamma_l'\}} & \text{if } 2\gamma_1'-\gamma_2'-2-\eta_2>n-1\,,\end{cases}$$

for all $r\in]0,+\infty[$. Then there exists $c_1\in]0,+\infty[$ such that the function $u[\partial_T\Omega,K,\mu]$ defined by (1.12) satisfies the following inequality

$$\begin{aligned}|u[\partial_T\Omega,K,\mu](t,x')-u[\partial_T\Omega,K,\mu](t,x'')|&\\ \le c_1\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C^{\frac{\eta_2}{2};\eta_2}(\partial_T\Omega)}\omega(|x'-x''|)&\\ +\|\mu\|_{L^\infty(\partial_T\Omega)}|u[\partial_T\Omega,K,1](t,x')-u[\partial_T\Omega,K,1](t,x'')|\,,&\end{aligned}\tag{1.29}$$

for all $x',x''\in\partial\Omega$, $t\in\overline{]-\infty,T[}$, and for all $(K,\mu)\in\mathcal{K}_{\gamma,a}(\partial_T\Omega)\times C^{\frac{\eta_2}{2};\eta_2}(\partial_T\Omega)$.

(ii) Let $\eta_1\in]0,2[$, $\eta_2\in]0,\eta_1]$. Let

$$\begin{aligned}&\gamma_1-(\eta_2/2)>1\,,\\ &2\gamma_1-\gamma_2-2+(\eta_1-\eta_2)<n-1\,,\\ &\gamma_1''-(\eta_2/2)>1\,,\\ &\gamma_l''<\gamma_1''-1+2^{-1}(\eta_1-\eta_2)\,,\\ &\eta_1<2\gamma_l''\,,\\ &2\gamma_1''-\gamma_2''-2-2\gamma_l''+(\eta_1-\eta_2)<(n-1)\,.\end{aligned}$$

Then there exists $c_2\in]0,+\infty[$ such that the function $u[\partial_T\Omega,K,\mu]$ defined by (1.12) satisfies the following inequality

$$\begin{aligned}|u[\partial_T\Omega,K,\mu](t',x)-u[\partial_T\Omega,K,\mu](t'',x)|&\\ \le c_2\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{\frac{\eta_1}{2}}&\\ +|u[\partial_T\Omega,K,\mu(t',\cdot)](t',x)-u[\partial_T\Omega,K,\mu(t',\cdot)](t'',x)|\,,&\end{aligned}\tag{1.30}$$

for all $x\in\partial\Omega$, $t',t''\in\overline{]-\infty,T[}$, $t'<t''$, and for all $(K,\mu)\in\mathcal{K}_{\gamma,a}(\partial_T\Omega)\times C_b^{0,\frac{\eta_2}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$.

Proof. We first consider inequality (1.29). Let $x', x'' \in \partial\Omega$, $t \in \overline{]-\infty, T[}$. By Remark 1.1 and by Proposition 1.21 (i), it suffices to consider case $0 < |x'-x''| < r_{\gamma_l'}$. By the triangular inequality and by the inclusion $\mathbb{B}_n(x', 2|x'-x''|) \subseteq \mathbb{B}_n(x'', 3|x'-x''|)$, we have

$$\begin{aligned}
&|u[\partial_T\Omega, K, \mu](t,x') - u[\partial_T\Omega, K, \mu](t,x'')| \\
&\quad\leq \Big| \big(u[\partial_T\Omega, K, \mu](t,x') - \mu(t,x')u[\partial_T\Omega, K, 1](t,x')\big) \\
&\qquad - \big(u[\partial_T\Omega, K, \mu](t,x'') - \mu(t,x')u[\partial_T\Omega, K, 1](t,x'')\big)\Big| \\
&\qquad + |\mu(t,x')|\,|u[\partial_T\Omega, K, 1](t,x') - u[\partial_T\Omega, K, 1](t,x'')| \\
&\quad\leq \int_{-\infty}^{t}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(t,x',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(t,x'',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} |K(t,x',\tau,y)-K(t,x'',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad + \|\mu\|_{L^\infty(\partial_T\Omega)}|u[\partial_T\Omega, K, 1](t,x') - u[\partial_T\Omega, K, 1](t,x'')|\,.
\end{aligned} \tag{1.31}$$

We now estimate the sum of the first two terms in the right hand side of (1.31). By Lemma 1.13 (ii), we have

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} |K(t,x',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} |K(t,x'',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\quad\leq 2\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)}\Big\{\int_{-\infty}^{t}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{|x'-y|^{\gamma_2}}{|t-\tau|^{\gamma_1}}e^{-\frac{|x'-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{|x''-y|^{\gamma_2}}{|t-\tau|^{\gamma_1}}e^{-\frac{|x''-y|^2}{a(t-\tau)}}\,d\sigma_y d\tau\Big\} \\
&\quad\leq 2\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)} \\
&\qquad\times\Big\{\int_{0}^{+\infty}\int_{\mathbb{B}_n(x',2|x'-x''|)\cap\partial\Omega} \frac{|x'-y|^{\gamma_2+2}a^{-1+\gamma_1}}{|u|^{\gamma_1}|x'-y|^{2\gamma_1}}e^{-\frac{1}{u}}\,d\sigma_y du \\
&\qquad + \int_{0}^{+\infty}\int_{\mathbb{B}_n(x'',3|x'-x''|)\cap\partial\Omega} \frac{|x''-y|^{\gamma_2+2}a^{-1+\gamma_1}}{|u|^{\gamma_1}|x''-y|^{2\gamma_1}}e^{-\frac{1}{u}}\,d\sigma_y du\Big\} \\
&\quad\leq 4\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{L^\infty(\partial_T\Omega)}\frac{\Gamma(\gamma_1-1)}{a^{\gamma_1-1}}c''_{\Omega,2\gamma_1-\gamma_2-2}|x'-x''|^{(n-1)-(2\gamma_1-\gamma_2-2)}\,.
\end{aligned} \tag{1.32}$$

We now consider the third term in the right hand side of inequality (1.31).

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} |K(t,x',\tau,y)-K(t,x'',\tau,y)|\,|\mu(\tau,y)-\mu(t,x')|\,d\sigma_y d\tau \\
&\quad\leq \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\int_{-\infty}^{t}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)} \frac{|x'-y|^{\gamma_2'}}{|t-\tau|^{\gamma_1'}}|x'-x''|^{\gamma_l'}e^{-\frac{|x'-y|^2}{a(t-\tau)}} \\
&\qquad\times\big[|\mu(\tau,y)-\mu(\tau,x')| + |\mu(\tau,x')-\mu(t,x')|\big]\,d\sigma_y d\tau \\
&\quad\leq \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C^{\frac{\eta_2}{2};\eta_2}(\partial_T\Omega)}
\end{aligned} \tag{1.33}$$

$$
\begin{aligned}
&\times\Big\{\int_0^{+\infty}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{|x'-y|^{\gamma_2'+2+\eta_2}a^{-1+\gamma_1'}}{u^{\gamma_1'}|x'-y|^{2\gamma_1'}}|x'-x''|^{\gamma_l'}e^{-\frac{1}{u}}\,d\sigma_y du\\
&+\int_0^{+\infty}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{|x'-y|^{\gamma_2'+2}a^{-1+\gamma_1'-(\frac{\eta_2}{2})}}{u^{\gamma_1'-(\frac{\eta_2}{2})}|x'-y|^{2\gamma_1'-\eta_2}}|x'-x''|^{\gamma_l'}e^{-\frac{1}{u}}\,d\sigma_y du\Big\}\\
&\leq 2\|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C^{\frac{\eta_2}{2};\eta_2}(\partial_T\Omega)}|x'-x''|^{\gamma_l'}\\
&\times\max\Big\{\frac{\Gamma(\gamma_1'-1)}{a^{\gamma_1'-1}},\frac{\Gamma(\gamma_1'-\frac{\eta_2}{2}-1)}{a^{\gamma_1'-\frac{\eta_2}{2}-1}}\Big\}\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\,.
\end{aligned}
$$

At this point we distinguish three cases. If $2\gamma_1'-\gamma_2'-2-\eta_2<n-1$, then Lemma 1.13 (i) implies that

$$
\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\leq\int_{\partial\Omega}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\leq c'_{\Omega,2\gamma_1'-\gamma_2'-2-\eta_2}\,,
$$

and thus inequalities (1.31)–(1.33) imply that there exists $c_1>0$ such that inequality (1.29) holds with $\omega(r)$ as in statement (i). If $2\gamma_1'-\gamma_2'-2-\eta_2=n-1$, then Lemma 1.13 (iv) implies that

$$
\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\leq c_\Omega^{iv}|\log|x'-x''||\,,
$$

and thus inequalities (1.31)–(1.33) imply that there exists $c_1>0$ such that inequality (1.29) holds with $\omega(r)$ as in statement (i). If $2\gamma_1'-\gamma_2'-2-\eta_2>n-1$, then Lemma 1.13 (iii) implies that

$$
\int_{\partial\Omega\setminus\mathbb{B}_n(x',2|x'-x''|)}\frac{d\sigma_y}{|x'-y|^{2\gamma_1'-\gamma_2'-2-\eta_2}}\leq c'''_{\Omega,2\gamma_1'-\gamma_2'-2-\eta_2}|x'-x''|^{(n-1)-(2\gamma_1'-\gamma_2'-2-\eta_2)}\,,
$$

and thus inequalities (1.31)–(1.33) imply that there exists $c_1>0$ such that inequality (1.29) holds with $\omega(r)$ as in statement (i).

Next we consider statement (ii). Let $x\in\partial\Omega$, $t',t''\in\overline{]-\infty,T[}$, $t'<t''$. By Remark 1.1 and by Proposition 1.21 (i), it suffices to consider case $0<|t'-t''|<1$. By Lemma 1.11 (i), and by the inclusion of intervals

$$
\big]t'-2|t'-t''|,t'+2|t'-t''|\big[\subseteq\big]t''-3|t'-t''|,t''+3|t'-t''|\big[,
$$

we have

$$
\begin{aligned}
&|u[\partial_T\Omega,K,\mu](t',x)-u[\partial_T\Omega,K,\mu](t'',x)| \qquad (1.34)\\
&\leq\Big|\big(u[\partial_T\Omega,K,\mu](t',x)-u[\partial_T\Omega,K,\mu(t',\cdot)](t',x)\big)\\
&\quad-\big(u[\partial_T\Omega,K,\mu](t'',x)-u[\partial_T\Omega,K,\mu(t',\cdot)](t'',x)\big)\Big|\\
&\quad+|u[\partial_T\Omega,K,\mu(t',\cdot)](t',x)-u[\partial_T\Omega,K,\mu(t',\cdot)](t'',x)|\\
&\leq\int_{t'-2|t'-t''|}^{t'+2|t'-t''|}\int_{\partial\Omega}|K(t',x,\tau,y)|\,|\mu(\tau,y)-\mu(t',y)|\,d\sigma_y d\tau\\
&\quad+\int_{t''-3|t'-t''|}^{t''+3|t'-t''|}\int_{\partial\Omega}|K(t'',x,\tau,y)|\,|\mu(\tau,y)-\mu(t',y)|\,d\sigma_y d\tau\\
&\quad+\int_{-\infty}^{t'-2|t'-t''|}\int_{\partial\Omega}|(K(t',x,\tau,y)-K(t'',x,\tau,y))(\mu(\tau,y)-\mu(t',y))|\,d\sigma_y d\tau
\end{aligned}
$$

$$+|u[\partial_T\Omega, K, \mu(t',\cdot)](t',x) - u[\partial_T\Omega, K, \mu(t',\cdot)](t'',x)| .$$

We now estimate the first two summands in the right hand side of (1.34). By Lemmas 1.13 (i) and 1.14 (ii), and by the elementary inequality $|t'-\tau|^{\frac{\eta_2}{2}} \le |t'-t''|^{\frac{\eta_2}{2}} + |t''-\tau|^{\frac{\eta_2}{2}}$, we have

$$\begin{aligned}
&\int_{t'-2|t'-t''|}^{t'+2|t'-t''|}\int_{\partial\Omega} |K(t',x,\tau,y)|\,|\mu(\tau,y)-\mu(t',y)|\,d\sigma_y d\tau \qquad (1.35)\\
&\quad+\int_{t''-3|t'-t''|}^{t''+3|t'-t''|}\int_{\partial\Omega} |K(t'',x,\tau,y)|\,|\mu(\tau,y)-\mu(t',y)|\,d\sigma_y d\tau\\
&\le \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}\\
&\quad\times\Bigg\{\int_{t'-2|t'-t''|}^{t'}\int_{\partial\Omega}\frac{|x-y|^{\gamma_2}}{|t'-\tau|^{\gamma_1}}e^{-\frac{|x-y|^2}{a(t'-\tau)}}|t'-\tau|^{\frac{\eta_2}{2}}\,d\sigma_y d\tau\\
&\quad+|t'-t''|^{\frac{\eta_2}{2}}\int_{t''-3|t'-t''|}^{t''}\int_{\partial\Omega}\frac{|x-y|^{\gamma_2}}{|t''-\tau|^{\gamma_1}}e^{-\frac{|x-y|^2}{a(t''-\tau)}}\,d\sigma_y d\tau\\
&\quad+\int_{t''-3|t'-t''|}^{t''}\int_{\partial\Omega}\frac{|x-y|^{\gamma_2}}{|t''-\tau|^{\gamma_1}}e^{-\frac{|x-y|^2}{a(t''-\tau)}}|t''-\tau|^{\frac{\eta_2}{2}}\,d\sigma_y d\tau\Bigg\}\\
&\le \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}\\
&\quad\times\Bigg\{\int_{\partial\Omega}\int_0^{\frac{2a|t'-t''|}{|x-y|^2}}\frac{|x-y|^{\gamma_2+2}a^{-1+\gamma_1-\frac{\eta_2}{2}}}{u^{\gamma_1-\frac{\eta_2}{2}}|x-y|^{2\gamma_1-\eta_2}}e^{-\frac{1}{u}}\,du\,d\sigma_y\\
&\quad+|t'-t''|^{\frac{\eta_2}{2}}\int_{\partial\Omega}\int_0^{\frac{3a|t'-t''|}{|x-y|^2}}\frac{|x-y|^{\gamma_2+2}}{u^{\gamma_1}|x-y|^{2\gamma_1}}a^{-1+\gamma_1}e^{-\frac{1}{u}}\,du\,d\sigma_y\\
&\quad+\int_{\partial\Omega}\int_0^{\frac{3a|t'-t''|}{|x-y|^2}}\frac{|x-y|^{\gamma_2+2}a^{-1+\gamma_1-\frac{\eta_2}{2}}}{u^{\gamma_1-\frac{\eta_2}{2}}|x-y|^{2\gamma_1-\eta_2}}e^{-\frac{1}{u}}\,du\,d\sigma_y\Bigg\}\\
&\le \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}\max\{a^{-1+\gamma_1-\frac{\eta_2}{2}},a^{-1+\gamma_1}\}\\
&\quad\times\Bigg\{2\int_{\partial\Omega}\tilde{D}_{\gamma_1-\frac{\eta_2}{2},\frac{\eta_1}{2}}\left(\frac{3a|t'-t''|}{|x-y|^2}\right)^{\frac{\eta_1}{2}}\frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2-\eta_2}}\\
&\quad+|t'-t''|^{\frac{\eta_2}{2}}\int_{\partial\Omega}\tilde{D}_{\gamma_1,\frac{\eta_1-\eta_2}{2}}\left(\frac{3a|t'-t''|}{|x-y|^2}\right)^{\frac{\eta_1-\eta_2}{2}}\frac{d\sigma_y}{|x-y|^{2\gamma_1-\gamma_2-2}}\Bigg\}\\
&\le \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}\max\{a^{-1+\gamma_1-\frac{\eta_2}{2}},a^{-1+\gamma_1}\}\\
&\quad\times\Bigg\{2\tilde{D}_{\gamma_1-\frac{\eta_2}{2},\frac{\eta_1}{2}}(3a)^{\frac{\eta_1}{2}}c'_{\Omega,2\gamma_1-\gamma_2-2+(\eta_1-\eta_2)}\\
&\quad+(3a)^{\frac{\eta_1-\eta_2}{2}}\tilde{D}_{\gamma_1,\frac{\eta_1-\eta_2}{2}}c'_{\Omega,2\gamma_1-\gamma_2-2+(\eta_1-\eta_2)}\Bigg\}|t'-t''|^{\frac{\eta_1}{2}} .
\end{aligned}$$

We now estimate the third term in the right hand side of (1.34). By Lemmas 1.13 (i) and 1.14 (i), we have

$$\begin{aligned}
&\int_{-\infty}^{t'-2|t'-t''|}\int_{\partial\Omega}|(K(t',x,\tau,y)-K(t'',x,\tau,y))(\mu(\tau,y)-\mu(t',y))|\,d\sigma_y d\tau \qquad (1.36)\\
&\le \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)}\|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{-\infty}^{t'-2|t'-t''|} \int_{\partial\Omega} \frac{|x-y|^{\gamma_2''}}{|t'-\tau|^{\gamma_1''-\frac{\eta_2}{2}}} |t'-t''|^{\gamma_l''} e^{-\frac{|x-y|^2}{a(t'-\tau)}}\, d\sigma_y d\tau \\
&\le \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} |t'-t''|^{\gamma_l''} \\
&\times \int_{\partial\Omega} \int_{\frac{2a|t'-t''|}{|x-y|^2}}^{+\infty} \frac{|x-y|^{\gamma_2''+2} a^{-1+\gamma_1''-\frac{\eta_2}{2}}}{u^{\gamma_1''-\frac{\eta_2}{2}} |x-y|^{2\gamma_1''-\eta_2}} e^{-\frac{1}{u}}\, d\sigma_y du \\
&\le \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} D_{\gamma_1''-\frac{\eta_2}{2},r} |t'-t''|^{\gamma_l''} a^{-1+\gamma_1''-\frac{\eta_2}{2}} \\
&\times \int_{\partial\Omega} \left(\frac{2a|t'-t''|}{|x-y|^2}\right)^{-r} \frac{d\sigma_y}{|x-y|^{2\gamma_1''-\gamma_2''-2-\eta_2}} \\
&\le \|K\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\frac{\eta_2}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} D_{\gamma_1''-\frac{\eta_2}{2},r} |t'-t''|^{\gamma_l''-r} \\
&\times (2a)^{-r} a^{-1+\gamma_1''-\frac{\eta_2}{2}} c'_{\Omega,2\gamma_1''-\gamma_2''-2-\eta_2-2r}\,,
\end{aligned}
$$

for all $r \in]0, \gamma_1'' - (\eta_2/2) - 1[$, provided that $2\gamma_1'' - \gamma_2'' - 2 - \eta_2 - 2r < (n-1)$. We now wish to select r so that $\gamma_l'' - r = \eta_1/2$, *i.e.*, $r = \gamma_l'' - \eta_1/2$. To do so, we must verify that

$$
0 < \gamma_l'' - (\eta_1/2)\,, \qquad \gamma_l'' - (\eta_1/2) < \gamma_1'' - (\eta_2/2) - 1\,, \tag{1.37}
$$

and that $2\gamma_1'' - \gamma_2'' - 2 - \eta_2 - 2r < (n-1)$. We can rewrite inequalities (1.37) as

$$
\eta_1 < 2\gamma_l''\,, \qquad \gamma_l'' < \gamma_1'' - 1 + 2^{-1}(\eta_1 - \eta_2)\,.
$$

and we observe that such inequalities hold by assumption. Moreover, if we set $r = \gamma_l'' - \eta_1/2$, then our assumptions imply that

$$
2\gamma_1'' - \gamma_2'' - 2 - \eta_2 - 2r = 2\gamma_1'' - \gamma_2'' - 2 - \eta_2 - 2\gamma_l'' + \eta_1 < (n-1)\,,
$$

Hence, we conclude that we can choose r as above, and that accordingly inequalities (1.34)–(1.36) imply the validity of statement (ii). □

1.9 Applications to integral operators related to layer heat potentials with Hölder continuous densities

In this last section of the present chapter, we apply the results of the previous Section 1.8 in order to obtain some mapping properties for some integral operators related to layer heat potentials. We start with the analysis of a class of integral operators which we need to study the properties of an integral operator related to the kernel $D_x\Phi_n(t-\tau, x-y)$, and we introduce the following statement.

Lemma 1.32. *Let $a \in]0, +\infty[$, $T \in]-\infty, +\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\alpha \in]0,1[$, $\beta \in]0, \alpha[$. Let γ_n be defined as in (1.16). Then the following statements hold.*

(i) There exists a constant $q_1 \in]0, +\infty[$ such that

$$
\begin{aligned}
&|Q[Z,g,\mu](t,x') - Q[Z,g,\mu](t,x'')| \\
&\qquad \le q_1 \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \|g\|_{C^{0,\alpha}(\partial\Omega)} \|\mu\|_{C^{\frac{\beta}{2};\beta}(\partial_T\Omega)} |x'-x''|^{\alpha}
\end{aligned} \tag{1.38}
$$

$$+\|\mu\|_{L^\infty(\partial_T\Omega)}|Q[Z,g,1](t,x')-Q[Z,g,1](t,x'')|\,,$$

for all $x',x''\in\partial\Omega$, $t\in\overline{]-\infty,T[}$, *and for all* $(Z,g,\mu)\in\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)\times C^{0,\alpha}(\partial\Omega)\times C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$ *(cf. (1.17)).*

(ii) There exists a constant $q_2\in]0,+\infty[$ *such that*

$$\begin{aligned}|Q[Z,g,\mu](t',x)-Q[Z,g,\mu](t'',x)|&\\ \leq q_2\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\mu\|_{C_b^{0,\frac{\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{\frac{\alpha}{2}}&\\ +|Q[Z,g,\mu(t',\cdot)](t',x)-Q[Z,g,\mu(t',\cdot)](t'',x)|\,,&\end{aligned}\tag{1.39}$$

for all $x\in\partial\Omega$, $t',t''\in\overline{]-\infty,T[}$, $t'<t''$, *and for all* $(Z,g,\mu)\in\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)\times C^{0,\alpha}(\partial\Omega)\times C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$.

(iii) There exists a constant $q_3\in]0,+\infty[$ *such that*

$$\begin{aligned}|Q[Z,g,\mu](t',x)-Q[Z,g,\mu](t'',x)|&\\ \leq q_3\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\mu\|_{C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{\frac{1+\alpha}{2}}&\\ +|Q[Z,g,\mu(t',\cdot)](t',x)-Q[Z,g,\mu(t',\cdot)](t'',x)|\,,&\end{aligned}\tag{1.40}$$

for all $x\in\partial\Omega$, $t',t''\in\overline{]-\infty,T[}$, $t'<t''$, *and for all* $(Z,g,\mu)\in\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)\times C^{0,\alpha}(\partial\Omega)\times C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$.

(iv) There exists a constant $q_4\in]0,+\infty[$ *such that*

$$\begin{aligned}|Q[Z,g,\mu](t',x)-Q[Z,g,\mu](t'',x)|&\\ \leq q_4\|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\mu\|_{C_b^{0,\frac{1}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}|t'-t''|^{\frac{1+\beta}{2}}&\\ +|Q[Z,g,\mu(t',\cdot)](t',x)-Q[Z,g,\mu(t',\cdot)](t'',x)|,&\end{aligned}\tag{1.41}$$

for all $x\in\partial\Omega$, $t',t''\in\overline{]-\infty,T[}$, $t'<t''$, *and for all* $(Z,g,\mu)\in\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)\times C^{0,\alpha}(\partial\Omega)\times C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$.

Proof. We first consider statement (i). Let $\gamma_{n,\theta}$, be defined as in (1.16) with $\theta=\alpha$. By Proposition 1.31 (i) with $\gamma=\gamma_{n,\theta}$, $\theta=\alpha$, $\eta_2=\beta$, there exists $c_1>0$ such that inequality (1.29) holds with $\omega(r)\equiv r^\alpha$ for all $(K,\mu)\in\mathcal{K}_{\gamma_n,\alpha,4a}(\partial_T\Omega)\times C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$. Indeed, $\gamma_l'=\alpha$, $\gamma_l''=1$,

$$\begin{aligned}&\gamma_1=(n/2)+1>1\,,\\ &2\gamma_1-\gamma_2-2=2((n/2)+1)-(1+\alpha)-2=(n-1)-\alpha\in[n-2,n-1[\,,\\ &\gamma_1'-(\eta_2/2)=(n/2)+1-(\beta/2)>1\,,\\ &(n-1)-(2\gamma_1'-\gamma_2'-2-\eta_2)+\gamma_l'\\ &\qquad=(n-1)-[2((n/2)+1)-1-2]+\alpha+\beta=\alpha+\beta>0\,,\\ &2\gamma_1'-\gamma_2'-2-\eta_2=2((n/2)+1)-1-2-\beta=(n-1)-\beta<(n-1)\,.\end{aligned}$$

Then inequality (1.38) follows by Lemma 1.25 (i) with $\theta=\alpha$ and by the equality

$$u[\partial_T\Omega,H[Z,g],\mu]=Q[Z,g,\mu].$$

Next we consider statement (ii). By Proposition 1.31 (ii) with $\gamma = \gamma_{n,\theta}$, $\theta = \alpha$, $\eta_1 = \alpha$, $\eta_2 = \beta$, there exists $c_2 > 0$ such that inequality (1.30) holds for all $(K, \mu) \in \mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega) \times C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$. Indeed, $\gamma_l' = \alpha$, $\gamma_l'' = 1$,

$$
\begin{aligned}
&\gamma_1 - (\eta_2/2) = (n/2) + 1 - (\beta/2) > 1\,,\\
&2\gamma_1 - \gamma_2 - 2 + (\eta_1 - \eta_2) = (n-1) - \alpha + (\alpha - \beta) = (n-1) - \beta < (n-1)\,,\\
&\gamma_1'' - (\eta_2/2) = (n/2) + 2 - (\beta/2) > 1\,,\\
&\gamma_l'' - \gamma_1'' + 1 - 2^{-1}(\eta_1 - \eta_2)\\
&\qquad = 1 - ((n/2)+2) + 1 - 2^{-1}(\alpha - \beta) = -(n/2) - 2^{-1}(\alpha - \beta) < 0\,,\\
&\eta_1 - 2\gamma_l'' = \alpha - 2 < 0\\
&2\gamma_1'' - \gamma_2'' - 2 - 2\gamma_l'' + (\eta_1 - \eta_2)\\
&\qquad = 2((n/2)+2) - (1+\alpha) - 2 - 2 + (\alpha - \beta) = (n-1) - \beta < (n-1)\,.
\end{aligned}
$$

Then inequality (1.39) follows by Lemma 1.25 (i) with $\theta = \alpha$ and by the equality

$$u[\partial_T\Omega, H[Z,g], \mu] = Q[Z,g,\mu].$$

Next we consider statement (iii). Let $\gamma_{n,\alpha}$ be as in (1.16) with $\theta = \alpha$. By Proposition 1.31 (ii) with $\gamma = \gamma_{n,\alpha}$, $\eta_1 = (1+\alpha)$, $\eta_2 = (1+\beta)$, there exists $c_3 > 0$ such that

$$
\begin{aligned}
&|u[\partial_T\Omega, K, \mu](t', x) - u[\partial_T\Omega, K, \mu](t'', x)| \\
&\quad \leq c_3 \|K\|_{\mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)} |t' - t''|^{\frac{1+\alpha}{2}}\\
&\qquad + |u[\partial_T\Omega, K, \mu(t',\cdot)](t', x) - u[\partial_T\Omega, K, \mu(t',\cdot)](t'', x)|\,,
\end{aligned}
\tag{1.42}
$$

for all $x \in \partial\Omega$, $t', t'' \in \overline{]-\infty,T[}$, $t' < t''$, and for all

$$(K, \mu) \in \mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega) \times C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right).$$

Indeed,

$$
\begin{aligned}
&\gamma_1 - (\eta_2/2) = (n/2) + 1 - 2^{-1}(1+\beta) > 1\,,\\
&2\gamma_1 - \gamma_2 - 2 + (\eta_1 - \eta_2)\\
&\qquad = 2((n/2)+1) - 1 - \alpha - 2 + (1+\alpha) - (1+\beta) = n - 1 - \alpha + (\alpha - \beta)\\
&\qquad < (n-1)\,,\\
&\gamma_1'' - (\eta_2/2) = (n/2) + 2 - 2^{-1}(1+\beta) > 1\,,\\
&\gamma_l'' - \gamma_1'' + 1 - 2^{-1}(\eta_1 - \eta_2)\\
&\qquad = 1 - ((n/2)+2) + 1 - 2^{-1}(\alpha - \beta) = -(n/2) - 2^{-1}(\alpha - \beta) < 0\,,\\
&\eta_1 - 2\gamma_l'' = (1+\alpha) - 2 < 0\,,\\
&2\gamma_1'' - \gamma_2'' - 2 - 2\gamma_l'' + ((1+\alpha) - (1+\beta))\\
&\qquad = 2((n/2)+2) - 1 - \alpha - 2 - 2 + ((1+\alpha) - (1+\beta))\\
&\qquad = (n-1) - \alpha + (\alpha - \beta) < (n-1)\,.
\end{aligned}
$$

Then inequality (1.40) follows by Lemma 1.25 (i) with $\theta = \alpha$ and by inequality (1.42) and by the equality

$$u[\partial_T\Omega, H[Z,g], \mu] = Q[Z,g,\mu].$$

Finally we consider statement (iv). We plan to apply Proposition 1.31 (ii) with $\eta_1 = 1+\beta, \eta_2 = 1$, $\gamma = \gamma_{n,\alpha}$. As above we can verify that all the assumption of Proposition 1.31 (ii) are satisfied and that accordingly there exists $q_4 > 0$ such that

$$\begin{aligned} &|u[\partial_T\Omega, K, \mu](t', x) - u[\partial_T\Omega, K, \mu](t'', x)| \\ &\quad \leq q_4 \|K\|_{\mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)} |t'-t''|^{\frac{1+\beta}{2}} \\ &\quad +|u[\partial_T\Omega, K, \mu(t',\cdot)](t', x) - u[\partial_T\Omega, K, \mu(t',\cdot)](t'', x)|, \end{aligned} \tag{1.43}$$

for all $x \in \partial\Omega, t', t'' \in \overline{]-\infty, T[}, t' < t''$, and for all

$$(K, \mu) \in \mathcal{K}_{\gamma_{n,\alpha},4a}(\partial_T\Omega) \times C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right).$$

Then the inequality (1.41) follows by Lemma 1.25 (i) and by inequality (1.43) and by the equality

$$u[\partial_T\Omega, H[Z, g], \mu] = Q[Z, g, \mu].$$

□

Finally, we specialize the previous Lemma 1.32 in the case in which the kernel Z is exactly $\partial_{x_r}\Phi_n(t-\tau, x-y)$ with $r \in \{1, \dots, n\}$, and we prove the following.

Theorem 1.33. *Let $T \in]-\infty, +\infty]$. Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$. Let $r \in \{1,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) The map $Q[\partial_{x_r}\Phi_n(t-\tau, x-y), \cdot, \cdot]$ from $C^{0,\alpha}(\partial\Omega) \times C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ which takes (g, μ) to $Q[\partial_{x_r}\Phi_n(t-\tau, x-y), g, \mu]$ is bilinear and continuous (cf. (1.21).)

(ii) The map $Q[\partial_{x_r}\Phi_n(t-\tau, x-y), \cdot, \cdot]$ from $C^{0,\alpha}(\partial\Omega) \times C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C_b^{0,\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ which takes (g, μ) to $Q[\partial_{x_r}\Phi_n(t-\tau, x-y), g, \mu]$ is bilinear and continuous.

(iii) The map $Q[\partial_{x_r}\Phi_n(t-\tau, x-y), \cdot, \cdot]$ from $C^{0,\alpha}(\partial\Omega) \times C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to the space $C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ which takes (g, μ) to $Q[\partial_{x_r}\Phi_n(t-\tau, x-y), g, \mu]$ is bilinear and continuous.

Proof. Let $Z(t, x, \tau, y) \equiv \partial_{x_r}\Phi_n(t-\tau, x-y)$. Let $a \in]16, +\infty[$. We now prove statement (i). By Theorem 1.27 the map $Q[Z, \cdot, \cdot]$ is bilinear and continuous from $C^{0,\alpha}(\partial\Omega) \times C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C^0(\partial_T\Omega)$. By Remark 1.20 (iii), we have $Z \in \mathcal{K}_{\gamma_n,a}(\partial_T\Omega)$ with γ_n as in (1.16). Then Lemma 1.32 (i) implies that there exists a constant $q_1 > 0$ such that

$$\begin{aligned} &|Q[Z, g, \mu](t, x') - Q[Z, g, \mu](t, x'')| \\ &\quad \leq q_1 \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \|g\|_{C^{0,\alpha}(\partial\Omega)} \|\mu\|_{C^{\frac{\beta}{2};\beta}(\partial_T\Omega))} |x'-x''|^{\alpha} \\ &\quad + \|\mu\|_{L^\infty(\partial_T\Omega)} |Q[Z, g, 1](t, x') - Q[Z, g, 1](t, x'')|, \end{aligned} \tag{1.44}$$

for all $x', x'' \in \partial\Omega$, $t \in \overline{]-\infty, T[}$, and for all $(g, \mu) \in C^{0,\alpha}(\partial\Omega) \times C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$. By performing the change of variables $(t-\tau) = u|x-y|^2$, we have

$$Q[Z, g, 1](t, x) = \int_{-\infty}^{t} \int_{\partial\Omega} (g(x) - g(y)) \frac{\partial}{\partial x_r} \Phi_n(t-\tau, x-y)\, d\sigma_y d\sigma_\tau \tag{1.45}$$

$$
\begin{aligned}
&= \int_{-\infty}^{t} \int_{\partial\Omega} (g(x)-g(y)) \frac{x_r - y_r}{2(4\pi)^{\frac{n}{2}}(t-\tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4(t-\tau)}}\, d\sigma_y d\tau \\
&= \frac{1}{2\pi^{\frac{n}{2}}} \int_0^{+\infty} u^{-\frac{n}{2}-1} e^{-\frac{1}{u}}\, du \int_{\partial\Omega} (g(x)-g(y)) \frac{x_r-y_r}{|x-y|^n}\, d\sigma_y \\
&= \frac{1}{2\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right) \int_{\partial\Omega} (g(x)-g(y)) \frac{x_r-y_r}{|x-y|^n}\, d\sigma_y \\
&= \frac{1}{s_n} \int_{\partial\Omega} (g(x)-g(y)) \frac{x_r-y_r}{|x-y|^n}\, d\sigma_y \\
&= -\int_{\partial\Omega} (g(x)-g(y)) \frac{\partial}{\partial x_r} S_n(x-y)\, d\sigma_y \qquad \forall\, (t,x) \in \partial_T\Omega\,,
\end{aligned}
$$

where s_n in the $n-1$ dimensional measure of $\partial\mathbb{B}_n$ and S_n is the fundamental solution of the Laplace equation (see Appendix A and in particular the definition (A.1) and the Lemmas A.3 and A.4). Then known properties of harmonic layer potentials imply that there exists $q_1' > 0$ such that

$$
|Q[Z,g,1](t,x') - Q[Z,g,1](t,x'')| \le q_1' |x'-x''|^\alpha, \tag{1.46}
$$

for all $x', x'' \in \partial\Omega$, $t \in \overline{]-\infty, T[}$ (cf. *e.g.* Schauder [97, Hilfsatz VII, p. 112], and Dondi and Lanza de Cristoforis [32, §8]).

Next we turn to consider the time Hölder quotient. We apply Lemma 1.32 (ii), which implies that there exists a constant $q_2 > 0$ such that

$$
\begin{aligned}
&|Q[Z,g,\mu](t',x) - Q[Z,g,\mu](t'',x)| \qquad (1.47)\\
&\qquad \le q_2 \|Z\|_{\mathcal{K}_{\gamma_n,a}(\partial_T\Omega)} \|g\|_{C^{0,\alpha}(\partial\Omega)} \|\mu\|_{C_b^{0,\frac{\beta}{2}}(\overline{]-\infty,T[}, C^0(\partial\Omega))} |t'-t''|^{\frac{\alpha}{2}} \\
&\qquad\quad + |Q[Z,g,\mu(t',\cdot)](t',x) - Q[Z,g,\mu(t',\cdot)](t'',x)|,
\end{aligned}
$$

for all $x \in \partial\Omega$, $t', t'' \in \overline{]-\infty, T[}, t' < t''$, and for all

$$
(g,\mu) \in C^{0,\alpha}(\partial\Omega) \times C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right).
$$

By exploiting the same change of variables of (1.45) we note that

$$
\begin{aligned}
Q[Z,g,\mu(t,\cdot)](t,x) &= \int_{-\infty}^{t} \int_{\partial\Omega} (g(x)-g(y)) \frac{\partial}{\partial x_r} \Phi_n(t-\tau, x-y) \mu(t,y)\, d\sigma_y d\tau \\
&= -\int_{\partial\Omega} (g(x)-g(y)) \frac{\partial}{\partial x_r} S_n(x-y) \mu(t,y)\, d\sigma_y \qquad \forall\, (t,x) \in \partial_T\Omega\,,
\end{aligned}
$$

(see Appendix A and in particular the definition (A.1) and the Lemmas A.3 and A.4). Accordingly,

$$
|Q[Z,g,\mu(t',\cdot)](t',x) - Q[Z,g,\mu(t',\cdot)](t'',x)| = 0\,, \tag{1.48}
$$

for all $x \in \partial\Omega$, $t', t'' \in \overline{]-\infty, T[}$, $t' < t''$. Then by inequalities (1.44), (1.46), (1.47) and equality (1.48), we conclude that statement (i) holds true.

Similarly, statement (ii) is a consequence of Lemma 1.32 (iii) and of equality (1.48), and statement (iii) is a consequence of Lemma 1.32 (iv) and of equality (1.48). □

CHAPTER 2

Regularizing properties of the double layer heat potential

This second chapter of the Dissertation is devoted to the mapping properties in parabolic Schauder spaces of the boundary integral operator associated with the double layer heat potential and of a boundary integral operator related to the normal derivative of the single layer heat potential.

Let

$$\alpha \in]0,1[, \qquad m \in \mathbb{N} \setminus \{0\}, \qquad T \in]-\infty, +\infty].$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\nu \equiv (\nu_l)_{l=1,\dots,n}$ denote the external unit normal to Ω. Let $\mu \in L^\infty(\partial\Omega)$. Then the double layer heat potential is the map from $(\mathbb{R}^n)_T$ to $\mathbb{C}$ defined by

$$w[\partial_T\Omega, \mu](t,x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \Phi_n(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau \qquad \forall\,(t,x) \in (\mathbb{R}^n)_T. \tag{2.1}$$

Moreover we set

$$w_*[\partial_T\Omega, \mu](t,x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial\nu(x)} \Phi_n(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau \qquad \forall\,(t,x) \in \partial_T\Omega. \tag{2.2}$$

The boundary integral operator $w_*[\partial_T\Omega, \cdot]$ is an operator related the the normal derivative of the single layer heat potential. As the main result of the present chapter we prove that, if Ω is of class $C^{m,\alpha}$ and if $\beta \in]0,\alpha[$, then the following mapping properties for the operators $w[\partial_T\Omega, \cdot]_{|\partial\Omega}$ and $w_*[\partial_T\Omega, \cdot]$ hold (see Theorems 2.15 and 2.16 below).

i) $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ is linear and continuous from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$;

ii) $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ is linear and continuous from $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$;

iii) $w_*[\partial_T\Omega, \cdot]$ is linear and continuous from $C^{\frac{m-1}{2};m-1}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$;

iv) $w_*[\partial_T\Omega, \cdot]$ is linear and continuous from $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$.

The mapping properties i)–iv) can be seen as the fact that the boundary integral operators $w[\partial_T\Omega, \cdot]_{|\partial\Omega}$ and $w_*[\partial_T\Omega, \cdot]$ have a regularizing effect. The proof of i) and ii) is mainly done exploiting two tools. The first consists in the results of the previous Chapter 1, which are used in order to recover some regularizing properties for the double layer heat potential in lower order parabolic Schauder spaces. The second tool is a new explicit formula for the tangential derivatives of the double layer heat potentials, which allows us to deduce the general case by an induction argument. The properties iii), iv) are instead deduced as a corollary of i), ii).

This chapter is organized as follows. In Section 2.1 we introduce the single and the double layer heat potentials and we collect some of their classical properties, like continuity, regularity and jump formulas. In order not to make the presentation of this chapter heavy, we postpone parts of the proofs of these properties to the Appendix B. In Section 2.2 we exploit the results of Chapter 1 in order to deduce the regularizing properties of $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ for the case $m=0$, and for the case $m=1$ in the time variable. In Section 2.3 we introduce some auxiliary integral operators, and we study their mapping properties. In Section 2.4 we prove a new explicit formula for the tangential derivatives of the double layer heat potential (see (1.2) for the definition of the M_{ij}-tangential derivative). More precisely, we prove an explicit formula for

$$M_{ij}[w[\partial_T\Omega,\mu]_{|\partial_T\Omega}] - w[\partial_T\Omega, M_{ij}[\mu]]_{|\partial_T\Omega} \qquad \text{on } \partial_T\Omega,$$

in terms of the auxiliary integral operators of Section 2.3 (see Theorem 2.13 below). Section 2.5 contains the main results of this chapter. By means of the formula for the tangential derivatives, we can prove the regularizing properties i) and ii) mentioned above exploiting an induction argument and the results of the previous sections. We note that with our strategy we can avoid to flatten the boundary of Ω with parametrization functions as done by other authors when they consider certain mapping properties of boundary integral operators. In Section 2.6 we deduce, as a corollary of the previous results, the mapping properties iii)-iv) mentioned above. Finally, in Section 2.7 we deduce some compactness results for the operators $w[\partial_T\Omega,\cdot]_{|\partial\Omega}$ and $w_*[\partial_T\Omega,\cdot]$, which are a consequence of the regularizing properties i)–iv).

The results contained in the present chapter can be found in two papers by Lanza de Cristoforis and the author [63, 64].

2.1 Classical properties of layer heat potentials

In this section we collect some known properties about the double and single layer heat potentials. We start with the following statement about the classical properties of the double layer heat potential.

Theorem 2.1 (Properties of the double layer heat potential)**.** *Let $\alpha \in]0,1[$. Let $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) If $\mu \in C^0_b(\partial_T\Omega)$, then the function $w[\partial_T\Omega,\mu]$ defined by (2.1) is of class $C^\infty((\mathbb{R}^n \setminus \partial\Omega)_T)$ and solves the heat equation in $(\mathbb{R}^n \setminus \partial\Omega)_T$. The restriction $w[\partial_T\Omega,\mu]_{|\Omega_T}$ can be extended uniquely to a continuous function $w^+[\partial_T\Omega,\mu]$ from $\mathrm{cl}\,\Omega_T$ to $\mathbb{C}$. The restriction $w[\partial_T\Omega,\mu]_{|\Omega^-_T}$ can be extended uniquely to a continuous function $w^-[\partial_T\Omega,\mu]$ from $\mathrm{cl}\,\Omega^-_T$ to $\mathbb{C}$. Moreover, the following jump formula holds.

$$w^\pm[\partial_T\Omega,\mu](t,x) = \mp\frac{1}{2}\mu(t,x) + w[\partial_T\Omega,\mu](t,x), \qquad \forall\,(t,x)\in\partial_T\Omega. \tag{2.3}$$

(ii) Let $m \in \mathbb{N}\setminus\{0\}$. Let Ω be of class $C^{m,\alpha}$. Let $R \in]0,+\infty[$ such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Then the map from $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$ which takes μ to $w^+[\partial_T\Omega,\mu]$ is linear and continuous. Moreover, the map from $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R) \setminus \Omega)_T)$ which takes μ to $w^-[\partial_T\Omega,\mu]_{|(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T}$ is linear and continuous.

Proof. By the definition of double layer heat potential we have that

$$
\begin{aligned}
w[\partial_T\Omega](t,x) = &\int_{t-1}^{t}\int_{\partial\Omega}\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau\\
&+\int_{-\infty}^{t-1}\int_{\partial\Omega}\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau,
\end{aligned}
$$

for all $(t,x)\in\mathbb{R}^n$. We note that the second term in the right hand side of the previous equality is an integral operator with a smooth kernel which does not display a singularity. Then statement (i) follows by the properties of the fundamental solution Φ_n, by classical differentiation theorems for integrals depending on a parameter, and by the jump formula for the double layer heat potential which can be found, *e.g.*, in Ladyženskaja, Solonnikov and Ural'tseva [58, p. 407] or in Watson [110, Lemma 2.7, p. 41].

The proof of statement (ii), that is the proof of the Schauder regularity properties for the double layer heat potential, is postponed to Appendix B, and in particular to Theorem B.9 (ii) (see also Ladyženskaja, Solonnikov and Ural'tseva [58, Chapter 4.2], for the Schauder regularity of the double layer heat potential in the case Ω is replaced by a half space). □

Then we have a corresponding statement for the single layer heat potential.

Theorem 2.2 (Properties of the single layer heat potential)**.** *Let* $\alpha\in]0,1[$. *Let* $T\in]-\infty,+\infty]$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Let* $R\in]0,+\infty[$ *such that* $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. *Then the following statements hold.*

(i) Let $n\geq 3$. *Let* $\mu\in C^0_b(\partial_T\Omega)$. *Then the function* $v[\partial_T\Omega,\mu]$ *from* $(\mathbb{R}^n)_T$ *to* $\mathbb{C}$ *defined by*

$$
v[\partial_T\Omega,\mu](t,x)\equiv\int_{-\infty}^{t}\int_{\partial\Omega}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau\qquad\forall\,(t,x)\in(\mathbb{R}^n)_T,\quad(2.4)
$$

i.e., the (n-dimensional) single layer heat potential, is continuous in $(\mathbb{R}^n)_T$, *is of class* $C^\infty((\mathbb{R}^n\setminus\partial\Omega)_T)$ *and solves the heat equation in* $(\mathbb{R}^n\setminus\partial\Omega)_T$.

Let $n=2$. *Let* $\mu\in C^0_b(\partial_T\Omega)$. *Let* $x_0\in\Omega$. *Then the function* $v[\partial_T\Omega,\mu]$ *from* $(\mathbb{R}^n)_T$ *to* $\mathbb{C}$ *defined by*

$$
v[\partial_T\Omega,\mu](t,x)\equiv\int_{-\infty}^{+\infty}\int_{\partial\Omega}(\Phi_n(t-\tau,x-y)-\Phi_n(0-\tau,x_0-y))\mu(\tau,y)\,d\sigma_y d\tau,\quad(2.5)
$$

for all $(t,x)\in(\mathbb{R}^2)_T$, *i.e., the (2-dimensional) single layer heat potential, is continuous in* $(\mathbb{R}^2)_T$, *is of class* $C^\infty((\mathbb{R}^n\setminus\partial\Omega)_T)$ *and solves the heat equation in* $(\mathbb{R}^2\setminus\partial\Omega)_T$.

Both in case $n\geq 3$ *and in case* $n=2$, *we denote by* $v^+[\partial_T\Omega,\mu]$ *and* $v^-[\partial_T\Omega,\mu]$ *the restriction of* $v[\partial_T\Omega,\mu]$ *to* $\mathrm{cl}\,\Omega_T$ *and to* $\mathrm{cl}\,\Omega^-_T$, *respectively.*

(ii) Let $m\in\mathbb{N}\setminus\{0\}$ *and* $r\in\{1,\dots,n\}$. *Let* Ω *be of class* $C^{m,\alpha}$. *Then the map from* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$ *which takes* μ *to* $\frac{\partial}{\partial x_r}v^+[\partial_T\Omega,\mu]$ *is linear and continuous. Moreover, the map from the space* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T)$ *which takes* μ *to* $\frac{\partial}{\partial x_r}v^-[\partial_T\Omega,\mu]_{|(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T}$ *is linear and continuous.*

(iii) Let $m\in\mathbb{N}\setminus\{0\}$. *Let* Ω *be of class* $C^{m,\alpha}$. *Then the map from* $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$ *which takes* μ *to* $\frac{\partial}{\partial t}v^+[\partial_T\Omega,\mu]$ *is linear and continuous. Moreover, the map from* $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T)$ *which takes* μ *to the restriction* $\frac{\partial}{\partial t}v^-[\partial_T\Omega,\mu]_{|(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T}$ *is linear and continuous.*

(iv) *Let $n \geq 3$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be of class $C^{m,\alpha}$. Then the map from $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$ which takes μ to $v^+[\partial_T\Omega,\mu]$ is linear and continuous. Moreover, the map from $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T)$ which takes μ to $v^-[\partial_T\Omega,\mu]_{|(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T}$ is linear and continuous.*

(v) *Let $\mu \in C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. Let $r \in \{1,\dots,n\}$. Then the following jump relations hold.*

$$
\begin{aligned}
&\frac{\partial}{\partial\nu(x)}v^{\pm}[\partial_T\Omega,\mu](t,x) = \pm\frac{1}{2}\mu(t,x) + w_*[\partial_T\Omega,\mu](t,x), \qquad (2.6)\\
&\frac{\partial}{\partial x_r}v^{\pm}[\partial_T\Omega,\mu](t,x) = \pm\frac{1}{2}\mu(t,x)\nu_r(x) + \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau,
\end{aligned}
$$

for all $(t,x) \in \partial_T\Omega$.

Proof. By the definition of single layer potential we have that

$$
\begin{aligned}
v[\partial_T\Omega](t,x) = &\int_{t-1}^{t}\int_{\partial\Omega}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \qquad (2.7)\\
&+\int_{t-1}^{0}\int_{\partial\Omega}\delta_{2,n}\Phi_n(-\tau,x_0-y)\mu(\tau,y)\,d\sigma_y d\tau\\
&+\int_{-\infty}^{t-1}\int_{\partial\Omega}(\Phi_n(t-\tau,x-y)-\delta_{2,n}\Phi_n(-\tau,x_0-y))\mu(\tau,y)\,d\sigma_y d\tau,
\end{aligned}
$$

for all $(t,x) \in (\mathbb{R}^n)_T$. Where we recall that by $(\delta_{i,j})_{i,j\in\{1,\dots,n\}}$ we denote the Kronecher symbol. We note that the second and the third term in the right hand side of equality (2.7) are integral operators with smooth kernels which do not display singularities. Then statement (i) follows by the properties of the fundamental solution Φ_n, by classical differentiation theorems for integral depending on a parameter and by the continuity in $(\mathbb{R}^n)_T$ of the first term in the right hand side of the equality (2.7) (see, *e.g.*, Friedman [38, Chapter 5, Section 2]).

The proof of statements (ii), (iii) and (iv) is postponed to Appendix B, in particular to Theorem B.9 (i) (see also Ladyženskaja, Solonnikov and Ural'tseva [58, Chapter 4.2], for the Schauder regularity of the single layer heat potential in the case Ω is replaced by a half space).

Statement (v) is a consequence of equality (2.7) and of the jump formula for the normal derivative of the single layer heat potential, which can be found, *e.g.*, in Ladyženskaja, Solonnikov and Ural'tseva [58, p. 405] or in Friedman [38, Theorem 1, p. 137] □

Remark 2.3. The above definition of single layer heat potential $v[\partial_T\Omega,\mu]$ in the case $n=2$ clearly depends on the choice of $x_0 \in \Omega$, even if we don't make it explicit in the notation. Indeed, a different choice would define a single layer which differs from that with x_0 by a constant. We have defined the single layer potential in such a way because, in case $n=2$, the kernel

$$\Phi_n(t-\cdot,x-\cdot)$$

is not integrable in $]-\infty,+\infty[\times\partial\Omega$, instead the kernel

$$\Phi_n(t-\cdot,x-\cdot)-\Phi_n(0-\cdot,x_0-\cdot)$$

is integrable in $]-\infty,+\infty[\times\partial\Omega$.

However, we note that if $T \in]0,+\infty]$ and $\mathrm{supp}\,(\mu) \subseteq \overline{[0,T[}\times\partial\Omega$ (and this is the case when one considers an initial-boundary value problem for the heat equation in $\overline{[0,T[}\times\mathrm{cl}\,\Omega$ with zero initial condition at $t = 0$), then the single layer potential $v[\partial_T\Omega, \mu]$ no longer depends on x_0, and

$$v[\partial_T\Omega, \mu](t,x) = \int_0^t \int_{\partial\Omega} \Phi_n(t-\tau, x-y)\mu(\tau,y)d\sigma_y d\tau \qquad \forall\,(t,x) \in \overline{[0,T[} \times \mathbb{R}^2,$$

which is the classical definition of single layer heat potential (cf., *e.g.*, Friedman [38, p. 136]).

2.2 Lower order regularizing properties of the double layer heat potential

In this section we prove some regularizing properties for the integral operator $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ acting on the spaces of essentially bounded functions and Hölder continuous functions on $\partial_T\Omega$. Such mapping properties are proved by applying the results of Chapter 1, when the kernel K of the integral operator $u[\partial_T\Omega, K, \cdot]$ is the kernel of the double layer heat potential, *i.e*, $K(t,x,\tau,y) = \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau, x-y)$. We start with the following theorem which regards the double layer heat potential acting on the space of essentially bounded functions.

Theorem 2.4. *Let $\alpha \in]0,1]$, $\beta \in]0,\alpha[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$.*

(i) The operator from $L^\infty(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[}, C^{0,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega)\right)$ which takes μ to $w[\partial_T\Omega, \mu]_{|\partial_T\Omega}$ is linear and continuous.

(ii) The operator from $L^\infty(\partial_T\Omega)$ to $C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ which takes μ to $w[\partial_T\Omega, \mu]_{|\partial_T\Omega}$ is linear and continuous.

(iii) The operator from $L^\infty(\partial_T\Omega)$ to $C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$ which takes μ to $w[\partial_T\Omega, \mu]_{|\partial_T\Omega}$ is linear and continuous.

Proof. We first consider statement (i). Let $a \in]16,+\infty[$. We already know that the kernel $\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)$ belongs to $\mathcal{K}_{\gamma,a}(\partial_T\Omega)$ with γ as in Remark 1.20 (iv). Clearly,

$$\gamma_1 = (n/2)+1 > 1\,, \qquad 2\gamma_1 - \gamma_2 - 2 = (n-1) - \alpha \in [n-2, n-1[\,,$$

and

$$\gamma_1' = (n/2)+1 > 1\,, \qquad 2\gamma_1' - \gamma_2' - 2 = n - \alpha \begin{cases} > (n-1) & \text{if } \alpha < 1\,, \\ = (n-1) & \text{if } \alpha = 1\,. \end{cases}$$

$$\gamma_l' + (n-1) - (2\gamma_1' - \gamma_2' - 2) = 1 + (n-1) - (n-\alpha) = \alpha > 0\,, \qquad \gamma_l' = 1\,.$$

If $\alpha < 1$, then Proposition 1.21 (ii) implies that $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to

$$B\left(\overline{]-\infty,T[}, C^{0,\min\{(n-1)-[(n-1)-\alpha],\alpha\}}(\partial\Omega)\right)$$
$$= B\left(\overline{]-\infty,T[}, C^{0,\alpha}(\partial\Omega)\right) = B\left(\overline{]-\infty,T[}, C^{0,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega)\right)\,.$$

If $\alpha = 1$, then Proposition 1.21 (ii) implies that $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to

$$B\left(\overline{]-\infty,T[}, C^{0,\omega_1(r)}(\partial\Omega)\right) = B\left(\overline{]-\infty,T[}, C^{0,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega)\right)\,.$$

Hence, statement (i) follows.

Next we consider statement (ii). We plan to apply Proposition 1.22. We note that $\gamma_1 > 1$ and that

$$\left]0, \frac{(n-1)-(2\gamma_1-\gamma_2-2)}{2}\right[\cap]0,1] =]0,\alpha/2[\,.$$

Then we can choose $h \equiv \beta/2$. Next we note that $\gamma_1'' > 1$, $\gamma_l'' = 1$ and that

$$\begin{aligned}
&(2\gamma_1''-\gamma_2''-2) = n+1-\alpha\,,\\
&\frac{(2\gamma_1''-\gamma_2''-2)-(n-1)}{2} = \frac{2-\alpha}{2} = 1-\frac{\alpha}{2} < 1\,,\\
&\min\{\gamma_1''-1,\gamma_l''\} = \min\{(n/2)+1,1\} = 1\,.
\end{aligned}$$

Next we choose $h' \equiv 1-h = 1-(\beta/2)$. Clearly $h' \in]1-(\alpha/2),1[$. Then Proposition 1.22 implies that $w[\partial_T\Omega,\cdot]$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to

$$C_b^{0,\min\{h,\gamma_l''-h'\}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right) = C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)\,,$$

and thus statement (ii) holds true.

Finally statements (i), (ii) and the continuity of the embedding of $C^{0,,\max\{r^\alpha,\omega_1(r)\}}(\partial\Omega)$ into $C^{0,\beta}(\partial\Omega)$, imply the validity of statement (iii). □

Then we have the following statement regarding the double layer heat potential acting on the space of Hölder continuous functions.

Theorem 2.5. *Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) The operator from $C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $C_b^{0,\frac{\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ which takes μ to $w[\partial_T\Omega,\mu]_{|\partial_T\Omega}$ is linear and continuous.

(ii) The operator from $C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ which takes μ to $w[\partial_T\Omega,\mu]_{|\partial_T\Omega}$ is linear and continuous.

(iii) The operator from $C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $C_b^{0,\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ which takes μ to $w[\partial_T\Omega,\mu]_{|\partial_T\Omega}$ is linear and continuous.

(iv) The operator from $C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ which takes μ to $w[\partial_T\Omega,\mu]_{|\partial_T\Omega}$ is linear and continuous.

Proof. We first consider statement (i). By Theorem 2.4 (i), we already know that $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to $B\left(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega)\right)$, and accordingly from $C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $B\left(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega)\right)$.

Next we plan to apply Proposition 1.31 (ii) with $\eta_1 = \alpha$, $\eta_2 = \beta$, $a \in]16,+\infty[$. By Remark 1.20 (iv), we already know that the kernel $\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y) \in \mathcal{K}_{\gamma,a}(\partial_T\Omega)$, with γ as in Remark 1.20 (iv). Then we observe that

$$\begin{aligned}
&\gamma_1-(\eta_2/2) = (n/2)+1-(\beta/2) > 1\,,\\
&2\gamma_1-\gamma_2-2 = 2((n/2)+1)-(1+\alpha)-2 = (n-1)-\alpha \in [n-2,(n-1)[\,,
\end{aligned}$$

$$
\begin{aligned}
&2\gamma_1 - \gamma_2 - 2 + (\eta_1 - \eta_2) = (n-1) - \alpha + (\alpha - \beta) = (n-1) - \beta < (n-1)\,,\\
&\gamma_1'' - (\eta_2/2) = ((n/2)+2) - (\beta/2) > 1\,,\\
&\gamma_l'' - \gamma_1'' + 1 - 2^{-1}(\eta_1 - \eta_2) = 1 - ((n/2)+2) + 1 - 2^{-1}(\alpha - \beta) < 0\,,\\
&\eta_1 - 2\gamma_l'' = \alpha - 2 < 0\,,\\
&2\gamma_1'' - \gamma_2'' - 2 - 2\gamma_l'' + (\eta_1 - \eta_2)\\
&\qquad = 2((n/2)+2) - (1+\alpha) - 2 - 2 + (\alpha - \beta) = (n-1) - \beta < (n-1)\,.
\end{aligned}
$$

Then Proposition 1.31 (ii) implies the existence of a constant $c_2 > 0$ such that

$$
\begin{aligned}
|w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t',x) - w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t'',x)| \qquad (2.8)\\
\leq c_2 \left\| \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\frac{\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} |t'-t''|^{\frac{\alpha}{2}}\\
+ |w[\partial_T\Omega,\mu(t',\cdot)]_{|\partial_T\Omega}(t',x) - w[\partial_T\Omega,\mu(t',\cdot)]_{|\partial_T\Omega}(t'',x)|\,,
\end{aligned}
$$

for all $x \in \partial\Omega$, $t',t'' \in \overline{]-\infty,T[}$, $t' < t''$, and for all $\mu \in C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$. Lemma A.4 of Appendix A, which says that the double layer heat potential with time independent densities coincides up to a minus sign with the harmonic double layer potential, implies that

$$
\begin{aligned}
w[\partial_T\Omega,\mu(t',\cdot)]_{|\partial_T\Omega}(t,x) &= -\int_{\partial\Omega}\frac{\partial}{\partial\nu(y)}S_n(x-y)\mu(t',y)\,d\sigma_y \qquad (2.9)\\
&= \tilde{w}[\partial\Omega,\mu(t',\cdot)]_{|\partial\Omega}(x),
\end{aligned}
$$

for all $x \in \partial\Omega$, $t,t' \in \overline{]-\infty,T[}$ and for all $\mu \in C_b^{0,\frac{\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$. We recall that S_n is the fundamental solution of the Laplace equation, and $\tilde{w}[\partial\Omega,\cdot]$ is the harmonic double layer potential (see Appendix A for the definitions). Then the second summand in the right hand side of (2.8) equals 0 and inequality (2.8) implies that statement (i) holds true.

Statement (ii) is an immediate consequence of statement (i) and of Theorem 2.4 (i).

We now consider statement (iii). Since $C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ is continuously embedded into $L^\infty(\partial_T\Omega)$, Theorem 2.4 (i) implies that $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is linear and continuous from $C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $B\left(\overline{]-\infty,T[},C^{0,\alpha}(\partial\Omega)\right)$. Next we plan to apply Proposition 1.31 (ii) with $\eta_1 = 1+\alpha$, $\eta_2 = 1+\beta$, $a \in]16,+\infty[$, γ as in Remark 1.20 (iv). As above, we can verify that all the assumptions of Proposition 1.31 (ii) are satisfied and that accordingly there exists $\tilde{c}_2 > 0$ such that

$$
\begin{aligned}
|w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t',x) - w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t'',x)| \qquad (2.10)\\
\leq \tilde{c}_2 \left\| \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} |t'-t''|^{\frac{1+\alpha}{2}}\\
+ |w[\partial_T\Omega,\mu(t',\cdot)]_{|\partial_T\Omega}(t',x) - w[\partial_T\Omega,\mu(t',\cdot)]_{|\partial_T\Omega}(t'',x)|\,,
\end{aligned}
$$

for all $x \in \partial\Omega$, $t',t'' \in \overline{]-\infty,T[}$, $t' < t''$, and for all $\mu \in C_b^{0,\frac{1+\beta}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$. Then again by equality (2.9), we conclude that the second summand in the right hand side of (2.10) vanishes and that accordingly statement (iii) holds true.

Finally, We consider statement (iv). Since $C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ is continuously embedded into $L^\infty(\partial_T\Omega)$, Theorem 2.4 (i) implies that $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is linear and continuous from

$C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $B\left(\overline{]-\infty,,T[},C^{0,\alpha}(\partial\Omega)\right)$. Next we plan to apply Proposition 1.31 (ii) with $\eta_1 = 1+\beta, \eta_2 = 1, a \in]16,+\infty[, \gamma$ as in Remark 1.20 (iv). As above, we can verify that all the assumptions of Proposition 1.31 (ii) are satisfied and that accordingly there exists $\hat{c}_2 > 0$ such that

$$\begin{aligned}&|w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t',x) - w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t'',x)| \qquad (2.11)\\ &\quad\le \hat{c}_2 \left\|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right\|_{\mathcal{K}_{\gamma,a}(\partial_T\Omega)} \|\mu\|_{C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)} |t'-t''|^{\frac{1+\beta}{2}}\\ &\quad + |w[\partial_T\Omega,\mu(t',\cdot)]_{|\partial_T\Omega}(t',x) - w[\partial_T\Omega,\mu(t',\cdot)]_{|\partial_T\Omega}(t'',x)|,\end{aligned}$$

for all $x \in \partial\Omega, t', t'' \in \overline{]-\infty,T[}, t' < t''$, and for all $\mu \in C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$. Then again by equality (2.9), we conclude that the second summand in the right hand side of (2.11) vanishes and that accordingly statement (iv) holds true. □

2.3 Auxiliary integral operators

In order to compute the tangential derivatives of the double layer heat potential, we now introduce some auxiliary integral operators and we analyze their mapping properties. We start with the following statement about an integral operator related to the kernel $\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)$.

Lemma 2.6. *Let $T \in]-\infty,+\infty]$. Let $r \in \{1,\dots,n\}$. Then the following statements hold.*

(i) Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. If $(f,\mu) \in C^{0,\theta}(\mathrm{cl}\,\Omega) \times L^\infty(\partial_T\Omega)$, then the function defined by

$$Q_r^\sharp[f,\mu](t,x) \equiv \int_{-\infty}^t \int_{\partial\Omega} (f(x)-f(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \quad \forall\,(t,x) \in \mathrm{cl}\,\Omega_T$$

is continuous and bounded.

(ii) Let $\alpha \in]0,1[, \beta,\theta \in]0,1[$. Let $m \in \mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the map $Q_r^\sharp[\cdot,\cdot]$ from $C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ to $C^{\frac{m-1+\min\{\theta,\alpha,\beta\}}{2};m-1+\min\{\theta,\alpha,\beta\}}(\mathrm{cl}\,\Omega_T)$ which takes (f,μ) to $Q_r^\sharp[f,\mu]$ is bilinear and continuous.

Proof. By Remark 1.20 (iii), the kernel $\frac{\partial}{\partial x_r}\Phi(t-\tau,x-y)$ satisfies the assumptions of Lemma 1.24, which implies the validity of statement (i).

We now consider statement (ii). If $(t,x) \in \Omega_T$, then classical differentiations theorems for integrals depending on a parameter implies that

$$Q_r^\sharp[f,\mu](t,x) = f(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega,\mu](t,x) - \frac{\partial}{\partial x_r}v^+[\partial_T\Omega,f\mu](t,x),$$

where we recall that $v^+[\partial_T\Omega,\cdot]$ is the interior single layer heat potential (see Theorem 2.2 (i)). If $(t,x) \in \partial_T\Omega$, then the jump formula (2.6) for the derivatives of single layer heat potential of Theorem Theorem 2.2 (v) implies that

$$Q_r^\sharp[f,\mu](t,x) = f(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega,\mu](t,x) - f(x)\frac{1}{2}\mu(x)\nu_r(x)$$

$$
\begin{aligned}
&- \frac{\partial}{\partial x_r} v^+[\partial_T \Omega, f\mu](t,x) + \frac{1}{2} f(x)\mu(x)\nu_r(x) \\
&= f(x)\frac{\partial}{\partial x_r} v^+[\partial_T \Omega, \mu](t,x) - \frac{\partial}{\partial x_r} v^+[\partial_T \Omega, f\mu](t,x).
\end{aligned}
$$

Accordingly,

$$
Q_r^\sharp[f,\mu](t,x) = f(x)\frac{\partial}{\partial x_r} v^+[\partial_T \Omega, \mu](t,x) - \frac{\partial}{\partial x_r} v^+[\partial_T \Omega, f\mu](t,x) \qquad \forall\, (t,x) \in \mathrm{cl}\,\Omega_T,
$$

for all $(f,\mu) \in C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$. Then the statement follows by the mapping properties of the single layer heat potential of Theorem 2.2 (ii) and by the continuity of pointwise product in Schauder spaces. □

Then we have a corresponding lemma concerning an integral operator related to the kernel $\frac{\partial}{\partial t}\Phi_n(t-\tau, x-y)$.

Lemma 2.7. *Let $T \in]-\infty,+\infty]$. Then the following statements hold.*

(i) Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\theta \in]0,1]$. If (f,μ) belongs to $C^{0,\theta}(\mathrm{cl}\,\Omega) \times C_b^{0,\frac{1}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$, then the function defined by

$$
\tilde{Q}_t^\sharp[f,\mu](t,x) \equiv \int_{-\infty}^{t}\int_{\partial\Omega} (f(x)-f(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)(\mu(\tau,y)-\mu(t,y))\,d\sigma_y d\tau,
$$

for all $(t,x) \in \mathrm{cl}\,\Omega_T$, is continuous and bounded.

(ii) Let $\alpha \in]0,1[$, $\beta,\theta \in]0,1[$. Let $m \in \mathbb{N}\setminus\{0,1\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the map $\tilde{Q}_t^\sharp[\cdot,\cdot]$ from $C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ to $C^{\frac{m-1+\min\{\theta,\alpha,\beta\}}{2};m-1+\min\{\theta,\alpha,\beta\}}(\mathrm{cl}\,\Omega_T)$ which takes (f,μ) to $\tilde{Q}_t^\sharp[f,\mu]$ is bilinear and continuous.

Proof. By Remark 1.20 (ii), the kernel $\frac{\partial}{\partial t}\Phi(t-\tau,x-y)$ satisfies the assumptions of Lemma 1.28, which implies the validity of statement (i).

Now we consider statement (ii). Let $(t,x) \in \Omega_T$. Lemma A.5 of Appendix A implies that

$$
\begin{aligned}
\tilde{Q}_t^\sharp[f,\mu](t,x) = &\int_{-\infty}^{t}\int_{\partial\Omega} (f(x)-f(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)(\mu(\tau,y)-\mu(t,y))\,d\sigma_y d\tau \qquad (2.12)\\
=&f(x)\frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu](t,x) - \frac{\partial}{\partial t} v^+[\partial_T\Omega, f\mu](t,x),
\end{aligned}
$$

for all $(f,\mu) \in C^{m-1,\theta}(\mathrm{cl}\,\Omega) \times C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$. Theorem 2.2 (iii) implies that the right hand side of equality (2.12) defines a continuous function of $(t,x) \in \mathrm{cl}\,\Omega_T$. By statement (i) the left hand side of (2.12) is also continuous in $(t,x) \in \mathrm{cl}\,\Omega_T$. Hence, the above equality must hold in $\mathrm{cl}\,\Omega_T$. More precisely,

$$
\tilde{Q}_t^\sharp[f,\mu](t,x) = f(x)\frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu](t,x) - \frac{\partial}{\partial t} v^+[\partial_T\Omega, f\mu](t,x) \qquad \forall\,(t,x) \in \mathrm{cl}\,\Omega_T.
$$

Accordingly, the statement follows by the mapping properties of the single layer heat potential of Theorem 2.2 (iii) and by the continuity of the pointwise product in Schauder spaces. □

We now prove explicit formulas for time and tangential derivatives of the boundary integral operators $Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot]$ and $\tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]$ (for the definitions see (1.17) and (1.21), respectively). In order to shorten the notation, from now on we use the following abbreviations

$$Q_r[\cdot,\cdot] \equiv Q[\partial_{x_r}\Phi_n(t-\tau,x-y),\cdot,\cdot],$$

for all $r \in \{1,\dots,n\}$, and

$$\tilde{Q}_t[\cdot,\cdot] \equiv \tilde{Q}[\partial_t\Phi_n(t-\tau,x-y),\cdot,\cdot]. \tag{2.13}$$

It is immediate from the definitions that the operators $Q_r[\cdot,\cdot]$ and $\tilde{Q}_t[\cdot,\cdot]$ are nothing more than the restrictions on $\partial_T\Omega$ of the operators $Q_r^\sharp[\cdot,\cdot]$ and $\tilde{Q}_t^\sharp[\cdot,\cdot]$ of Theorems 2.6 and 2.7, respectively. More precisely

$$Q_r^\sharp[\cdot,\cdot]_{|\partial_T\Omega} = Q_r[\cdot,\cdot], \tag{2.14}$$

and

$$\tilde{Q}_t^\sharp[\cdot,\cdot]_{|\partial_T\Omega} = \tilde{Q}_t[\cdot,\cdot]. \tag{2.15}$$

Then we have the following lemma about an explicit formula for the tangential derivatives of $Q_r[\cdot,\cdot]$.

Lemma 2.8. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let $r \in \{1,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{2,\alpha}$. Let $g \in C^{1,\alpha}(\partial\Omega)$ and $\mu \in C^{\frac{1}{2};1}(\partial_T\Omega)$. Then $Q_r[g,\mu](t,\cdot) \in C^1(\partial\Omega)$ for all $t \in \overline{]-\infty,T[}$. Moreover, the following formula holds.*

$$\begin{aligned}
&M_{ij}[Q_r[g,\mu]](t,x) \qquad\qquad (2.16)\\
&= \nu_i(x)Q_r\left[\frac{\partial g}{\partial x_j}-(Dg\cdot\nu)\nu_j,\mu\right](t,x)-\nu_j(x)Q_r\left[\frac{\partial g}{\partial x_i}-(Dg\cdot\nu)\nu_i,\mu\right](t,x)\\
&\quad+\nu_i(x)Q_r\left[g,\sum_{h=1}^n M_{hj}[\nu_h\mu]\right](t,x)-\nu_j(x)Q_r\left[g,\sum_{h=1}^n M_{hi}[\nu_h\mu]\right](t,x)\\
&\quad+\nu_i(x)\left\{\sum_{s=1}^n Q_s\left[\nu_j,M_{sr}[g]\mu\right](t,x)+\sum_{s=1}^n Q_s\left[g,M_{sr}[\nu_j\mu]\right](t,x)\right\}\\
&\quad-\nu_j(x)\left\{\sum_{s=1}^n Q_s\left[\nu_i,M_{sr}[g]\mu\right](t,x)+\sum_{s=1}^n Q_s\left[g,M_{sr}[\nu_i\mu]\right](t,x)\right\}\\
&\quad+\nu_i(x)\tilde{Q}_t\left[g,\nu_r\nu_j\mu\right](t,x)-\nu_j(x)\tilde{Q}_t\left[g,\nu_r\nu_i\mu\right](t,x),
\end{aligned}$$

for all $(t,x) \in \partial_T\Omega$ and for all $i,j \in \{1,\dots,n\}$.

Proof. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let '$\tilde{\cdot}$' be an extension operator as in Lemma 1.3, defined on $C^{1,\alpha}(\partial\Omega)$.

First we fix $\beta \in]0,\alpha[$ and we prove formula (2.16) under the additional assumption that

$$\mu \in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega).$$

By Lemma 2.6 (ii) we already know that $Q_r^\sharp[\tilde{g},\mu](t,\cdot)$ belongs to $C^1(\mathrm{cl}\,\Omega)$ for all $t \in \overline{]-\infty,T[}$. We find convenient to introduce the notation

$$M_{ij}^\sharp[f](x) \equiv \tilde{\nu}_i(x)\frac{\partial f}{\partial x_j}(x)-\tilde{\nu}_j(x)\frac{\partial f}{\partial x_i}(x), \tag{2.17}$$

for all $f \in C^1(\mathrm{cl}\,\Omega)$ and for all $x \in \mathrm{cl}\,\Omega$, $i, j \in \{1, \dots, n\}$. If necessary, we write $M^\sharp_{ij,x}$ to emphasize that we are taking x as variable of the differential operator $M^\sharp_{ij}$. Next we fix $(t, x) \in \Omega_T$ and we compute

$$M^\sharp_{ij}[Q^\sharp_r[\tilde{g}, \mu]](t, x) = \tilde{\nu}_i(x)\frac{\partial}{\partial x_j}Q^\sharp_r[\tilde{g}, \mu](t, x) - \tilde{\nu}_j(x)\frac{\partial}{\partial x_i}Q^\sharp_r[\tilde{g}, \mu](t, x).$$

By classical differentiation theorems for integrals depending on a parameter, we have

$$\begin{aligned}\frac{\partial}{\partial x_i}Q^\sharp_r[\tilde{g}, \mu](t, x) = &\int_{-\infty}^t \int_{\partial\Omega} \frac{\partial \tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau \\ &+ \int_{-\infty}^t \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y))\frac{\partial^2}{\partial x_i \partial x_r}\Phi_n(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau\,.\end{aligned}$$

Since $\sum_{h=1}^n \nu_h^2 = 1$ on $\partial\Omega$, we have

$$\begin{aligned}&\frac{\partial}{\partial x_i}Q^\sharp_r[\tilde{g}, \mu](t, x) \qquad (2.18)\\
&\quad = \int_{-\infty}^t \int_{\partial\Omega} \frac{\partial \tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau \\
&\quad - \int_{-\infty}^t \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y))\sum_{h=1}^n \nu_h^2(y)\frac{\partial}{\partial y_i}\left(\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\right)\mu(\tau, y)\, d\sigma_y d\tau \\
&\quad = \int_{-\infty}^t \int_{\partial\Omega} \frac{\partial \tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau \\
&\quad - \int_{-\infty}^t \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y))\sum_{h=1}^n \left[\left(\nu_h(y)\frac{\partial}{\partial y_i} - \nu_i(y)\frac{\partial}{\partial y_h}\right)\left(\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\right)\right] \\
&\qquad\qquad \times \mu(\tau, y)\nu_h(y)\, d\sigma_y d\tau \\
&\quad - \int_{-\infty}^t \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y))\sum_{h=1}^n \left[\nu_h(y)\frac{\partial}{\partial y_h}\left(\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\right)\right] \\
&\qquad\qquad \times \nu_i(y)\mu(\tau, y)\, d\sigma_y d\tau.\end{aligned}$$

By Lemma 1.4 on the consequence of the Divergence Theorem for the tangential derivatives, the second integral in the right hand side of formula (2.18) takes the following form

$$\begin{aligned}&\int_{-\infty}^t \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y))\sum_{h=1}^n \left[\left(\nu_h(y)\frac{\partial}{\partial y_i} - \nu_i(y)\frac{\partial}{\partial y_h}\right)\left(\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\right)\right] \\
&\quad \times \mu(\tau, y)\nu_h(y)\, d\sigma_y d\tau \\
&= -\sum_{h=1}^n \int_{-\infty}^t \int_{\partial\Omega} M_{hi,y}\left[(\tilde{g}(x) - \tilde{g}(y))\mu(\tau, y)\nu_h(y)\right]\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\, d\sigma_y d\tau \\
&= \sum_{h=1}^n \int_{-\infty}^t \int_{\partial\Omega} M_{hi,y}[\tilde{g}(y)]\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)\mu(\tau, y)\nu_h(y)\, d\sigma_y d\tau \\
&\quad - \sum_{h=1}^n \int_{-\infty}^t \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau, x-y)M_{hi,y}[\mu(\tau, y)\nu_h(y)]\, d\sigma_y d\tau.\end{aligned}$$

Accordingly, we have

$$
\begin{aligned}
&\frac{\partial}{\partial x_i} Q_r^\sharp[\tilde{g},\mu](t,x) \\
&\quad= \frac{\partial \tilde{g}}{\partial x_i}(x) \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\quad - \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega} M_{hi,y}[\tilde{g}(y)]\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&\quad + \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{hi,y}[\mu(\tau,y)\nu_h(y)]\,d\sigma_y d\tau \\
&\quad - \int_{-\infty}^{t}\int_{\partial\Omega} (\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_r}\left(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right)\nu_i(y)\mu(\tau,y)\,d\sigma_y d\tau.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&M_{ij}^\sharp[Q_r^\sharp[\tilde{g},\mu]](t,x) \qquad (2.19)\\
&\quad = M_{ij}^\sharp[\tilde{g}](x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\quad + \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}\{-\tilde{\nu}_i(x)M_{hj,y}[\tilde{g}(y)]+\tilde{\nu}_j(x)M_{hi,y}[\tilde{g}(y)]\} \\
&\qquad \times \frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&\quad + \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y) \\
&\qquad \times\{\tilde{\nu}_i(x)M_{hj,y}[\mu(\tau,y)\nu_h(y)]-\tilde{\nu}_j(x)M_{hi,y}[\mu(\tau,y)\nu_h(y)]\}\,d\sigma_y d\tau \\
&\quad + \int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_r}\left(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right) \\
&\qquad \times\{\tilde{\nu}_j(x)\nu_i(y)-\tilde{\nu}_i(x)\nu_j(y)\}\mu(\tau,y)\,d\sigma_y d\tau\,.
\end{aligned}
$$

We now consider the first two terms in the right hand side of formula (2.19). By the obivious identity

$$
M_{ij}^\sharp[\tilde{g}] = \tilde{\nu}_i\left(\frac{\partial\tilde{g}}{\partial x_j}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_j\right)-\tilde{\nu}_j\left(\frac{\partial\tilde{g}}{\partial x_i}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_i\right)\qquad \text{in } \mathrm{cl}\,\Omega, \qquad (2.20)
$$

and by the identity

$$
\begin{aligned}
&\sum_{h=1}^{n}\{-\tilde{\nu}_i(x)M_{hj,y}[\tilde{g}(y)]+\tilde{\nu}_j(x)M_{hi,y}[\tilde{g}(y)]\}\nu_h(y) \qquad (2.21)\\
&\qquad = -\,\tilde{\nu}_i(x)\left(\frac{\partial\tilde{g}}{\partial x_j}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\right) \\
&\qquad\quad + \tilde{\nu}_j(x)\left(\frac{\partial\tilde{g}}{\partial x_i}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\right)\qquad \forall\, y\in\partial\Omega,
\end{aligned}
$$

which follows by formula (1.3), we rewrite the sum of the first two terms in the right hand side of (2.19) in the following form

$$
M_{ij}^\sharp[\tilde{g}](x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \qquad (2.22)
$$

$$
\begin{aligned}
&+\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}\{-\tilde{\nu}_i(x)M_{hj,y}[\tilde{g}(y)]+\tilde{\nu}_j(x)M_{hi,y}[\tilde{g}(y)]\}\\
&\quad\times\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau\\
=&\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau\\
&\quad\times\tilde{\nu}_i(x)\left(\frac{\partial\tilde{g}}{\partial x_j}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_j(x)\right)\\
&-\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau\\
&\quad\times\tilde{\nu}_j(x)\left(\frac{\partial\tilde{g}}{\partial x_i}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_i(x)\right)\\
&-\tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}\left(\frac{\partial\tilde{g}}{\partial x_j}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\right)\\
&\quad\times\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau\\
&+\tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\left(\frac{\partial\tilde{g}}{\partial x_i}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\right)\\
&\quad\times\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau\\
=&\tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}\left[\left(\frac{\partial\tilde{g}}{\partial x_j}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_j(x)\right)\right.\\
&-\left.\left(\frac{\partial\tilde{g}}{\partial x_j}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\right)\right]\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau\\
&-\tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\left[\left(\frac{\partial\tilde{g}}{\partial x_i}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_i(x)\right)\right.\\
&-\left.\left(\frac{\partial\tilde{g}}{\partial x_i}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\right)\right]\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau\\
=&\tilde{\nu}_i(x)Q_r^{\sharp}\left[\frac{\partial\tilde{g}}{\partial x_j}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_j,\mu\right](t,x)-\tilde{\nu}_j(x)Q_r^{\sharp}\left[\frac{\partial\tilde{g}}{\partial x_i}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_i,\mu\right](t,x).
\end{aligned}
$$

We now consider the third term in the right hand side of formula (2.19). Namely,

$$
\begin{aligned}
&\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y) \qquad (2.23)\\
&\quad\times\left\{\tilde{\nu}_i(x)M_{hj,y}[\mu(\tau,y)\nu_h(y)]-\tilde{\nu}_j(x)M_{hi,y}[\mu(\tau,y)\nu_h(y)]\right\}d\sigma_y d\tau\\
=&\tilde{\nu}_i(x)Q_r^{\sharp}\left[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h\mu]\right](t,x)-\tilde{\nu}_j(x)Q_r^{\sharp}\left[\tilde{g},\sum_{h=1}^{n}M_{hi}[\nu_h\mu]\right](t,x).
\end{aligned}
$$

Next we consider the last integral in the right hand side of formula (2.19). Since Φ_n solves the heat equation in $\mathbb{R}\times\mathbb{R}^n\setminus\{(0,0)\}$, if $(t,x)\in\Omega_T$ is fixed as above and $(\tau,y)\in\partial_T\Omega$, we have that

$$\frac{\partial}{\partial x_r}\left(\frac{\partial}{\partial \nu(y)}\Phi_n(t-\tau,x-y)\right)=\sum_{s=1}^{n}\left(\nu_s(y)\frac{\partial}{\partial y_r}-\nu_r(y)\frac{\partial}{\partial y_s}\right)\frac{\partial}{\partial x_s}\Phi_n(t-\tau,x-y) \qquad (2.24)$$
$$-\nu_r(y)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y).$$

Accordingly, equality (2.24), and the consequence of the Divergence Theorem for the tangential derivatives of Lemma 1.4, and Lemma A.5 imply that

$$\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_r}\left(\frac{\partial}{\partial \nu(y)}\Phi_n(t-\tau,x-y)\right) \qquad (2.25)$$
$$\times\{\tilde{\nu}_j(x)\nu_i(y)-\tilde{\nu}_i(x)\nu_j(y)\}\mu(\tau,y)\,d\sigma_y d\tau$$
$$=-\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\left[\sum_{s=1}^{n}\left(\nu_s(y)\frac{\partial}{\partial y_r}-\nu_r(y)\frac{\partial}{\partial y_s}\right)\frac{\partial}{\partial x_s}\Phi_n(t-\tau,x-y)\right]$$
$$\times\Big\{\tilde{\nu}_i(x)(\nu_j(y)-\tilde{\nu}_j(x))+\tilde{\nu}_j(x)(\tilde{\nu}_i(x)-\nu_i(y))\Big\}\mu(\tau,y)\,d\sigma_y d\tau$$
$$+\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\nu_r(y)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)$$
$$\times\{\tilde{\nu}_i(x)\nu_j(y)-\tilde{\nu}_j(x)\nu_i(y)\}\mu(\tau,y)\,d\sigma_y d\tau$$
$$=\tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{\nu}_j(x)-\nu_j(y))\frac{\partial}{\partial x_s}\Phi_n(t-\tau,x-y)M_{sr}[g](y)\mu(\tau,y)\,d\sigma_y d\tau$$
$$+\tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_s}\Phi_n(t-\tau,x-y)M_{sr}[\nu_j\mu](y)\,d\sigma_y d\tau$$
$$-\tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{\nu}_i(x)-\nu_i(y))\frac{\partial}{\partial x_s}\Phi_n(t-\tau,x-y)M_{sr}[g](y)\mu(\tau,y)\,d\sigma_y d\tau$$
$$-\tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial x_s}\Phi_n(t-\tau,x-y)M_{sr}[\nu_i\mu](y)\,d\sigma_y d\tau$$
$$+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\left[\tilde{g},\nu_r\nu_j\mu\right](t,x)-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp\left[\tilde{g},\nu_r\nu_i\mu\right](t,x)$$
$$=\tilde{\nu}_i(x)\left\{\sum_{s=1}^{n}Q_s^\sharp[\tilde{\nu}_j,M_{sr}[g]\mu](t,x)+\sum_{s=1}^{n}Q_s^\sharp[\tilde{g},M_{sr}[\nu_j\mu]](t,x)\right\}$$
$$-\tilde{\nu}_j(x)\left\{\sum_{s=1}^{n}Q_s^\sharp[\tilde{\nu}_i,M_{sr}[g]\mu](t,x)+\sum_{s=1}^{n}Q_s^\sharp[\tilde{g},M_{sr}[\nu_i\mu]](t,x)\right\}$$
$$+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\left[\tilde{g},\nu_r\nu_j\mu\right](t,x)-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp\left[\tilde{g},\nu_r\nu_i\mu\right](t,x).$$

Then by combining formulas (2.19), (2.22), (2.23) and (2.25), we obtain that the following formula

$$M_{ij}^\sharp[Q_r^\sharp[\tilde{g},\mu]](t,x) \qquad (2.26)$$
$$=\ \tilde{\nu}_i(x)Q_r^\sharp\left[\frac{\partial\tilde{g}}{\partial x_j}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_j,\mu\right](t,x)-\tilde{\nu}_j(x)Q_r^\sharp\left[\frac{\partial\tilde{g}}{\partial x_i}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_i,\mu\right](t,x)$$
$$+\tilde{\nu}_i(x)Q_r^\sharp\left[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h\mu]\right](t,x)-\tilde{\nu}_j(x)Q_r^\sharp\left[\tilde{g},\sum_{h=1}^{n}M_{hi}[\nu_h\mu]\right](t,x)$$

$$
\begin{aligned}
&+\tilde{\nu}_i(x)\left\{\sum_{s=1}^{n} Q_s^\sharp\left[\tilde{\nu}_j, M_{sr}[g]\mu\right](t,x)+\sum_{s=1}^{n} Q_s^\sharp\left[\tilde{g}, M_{sr}[\nu_j\mu]\right](t,x)\right\}\\
&-\tilde{\nu}_j(x)\left\{\sum_{s=1}^{n} Q_s^\sharp\left[\tilde{\nu}_i, M_{sr}[g]\mu\right](t,x)+\sum_{s=1}^{n} Q_s^\sharp\left[\tilde{g}, M_{sr}[\nu_i\mu]\right](t,x)\right\}\\
&+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp[\tilde{g},\nu_r\nu_j\mu](t,x)-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp[\tilde{g},\nu_r\nu_i\mu](t,x),
\end{aligned}
$$

holds for all $(t,x)\in\Omega_T$. Now, under our assumptions, the first argument of the terms $Q_r^\sharp$, $Q_s^\sharp$, $\tilde{Q}_t^\sharp$, which appear in the right hand side of formula (2.26) belongs to $C^{0,\alpha}(\mathrm{cl}\,\Omega)$, and the second argument of the terms $Q_r^\sharp$, $Q_s^\sharp$, which appear in the right hand side of formula (2.26) belongs to $L^\infty(\partial_T\Omega)$, and the second argument of the terms $\tilde{Q}_t^\sharp$, which appear in the right hand side of formula (2.26) belongs to $C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$. Then Lemma 2.6 (i) and Lemma 2.7 (i) imply that the right hand side of formula (2.26) defines a continuous function of the variable $(t,x)\in\mathrm{cl}\,\Omega_T$. Since Ω is of class $C^{2,\alpha}$ and $\tilde{g}\in C^{1,\alpha}(\mathrm{cl}\,\Omega)$, and since we are assuming that $\mu\in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$, Lemma 2.6 (ii) implies that $M_{ij}^\sharp[Q_r^\sharp[\tilde{g},\mu]]\in C^0(\mathrm{cl}\,\Omega_T)$. Hence, the equality of formula (2.26) must hold for all $(t,x)\in\mathrm{cl}\,\Omega_T$ and thus in particular for all $(t,x)\in\partial_T\Omega$. Clearly $M_{ij}^\sharp[\cdot]_{|\partial\Omega}=M_{ij}[\cdot]$. Moreover $Q_r^\sharp[\cdot,\cdot]_{|\partial_T\Omega}=Q_r[\cdot,\cdot]$ and $\tilde{Q}_t^\sharp[\cdot,\cdot]_{|\partial_T\Omega}=\tilde{Q}_t[\cdot,\cdot]$ (cf. equalities (2.14) and (2.15), respectively). Then we can conclude that formula (2.16) holds under the assumption that $\mu\in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$.

Now let $\mu\in C^{\frac{1}{2};1}(\partial_T\Omega)$. We consider only the case $T=+\infty$. Indeed the case $T<+\infty$ can be treated similarly. We fix $t\in\,]-\infty,T[$ and we consider

$$
\eta_1,\eta_2,\eta_3\in C_b^\infty(\mathbb{R})
$$

such that

i) $\sum_{i=1}^3\eta_i=1$;

ii) $0\le\eta_i\le1$ for all $i=1,2,3$;

iii) $\mathrm{supp}\,(\eta_1)\subseteq]-\infty,t-1[$, $\mathrm{supp}\,(\eta_2)\subseteq]t-2,t+2[$ and $\mathrm{supp}\,(\eta_3)\subseteq]t+1,+\infty[$.

Then we set

$$
\mu_i(\tau,x)=\mu(\tau,x)\eta_i(\tau)\qquad\forall\,(\tau,x)\in\partial_T\Omega,\ \forall\,i=1,2,3.
$$

Clearly

$$
\mu(\tau,x)=\mu_1(\tau,x)+\mu_2(\tau,x)+\mu_3(\tau,x)\qquad\forall\,(\tau,x)\in\partial_T\Omega.
$$

We denote by $P_{ijr}[g,\mu]$ the right hand side of (2.16). We now show that the weak M_{ij}-derivative of $Q_r[g,\mu_2]$ coincides with $P_{ijr}[g,\mu_2]$. Since μ_2 has compact support, by considering an extension of μ_2 of class $C^{\frac{1}{2};1}$ with compact support in $\mathbb{R}^{n+1}$ (see Ladyženskaja, Solonnikov and Ural'ceva [58, Chapter 1.1, pp 9–10]) and by considering a sequence of mollifiers of such an extension, and by taking the restriction to $\partial_T\Omega$, we conclude that there exists a sequence of functions $\{\mu_{2l}\}_{l\in\mathbb{N}}$ in $C^{1;2}(\partial_T\Omega)$ such that μ_{2l} converges to μ_2 in $C^{\frac{1}{2};1}(\partial_T\Omega)$. Next we note that Remarks 1.9, 1.10 and Theorems 1.27 and 1.30 (ii) and the continuity of the pointwise product in Schauder spaces, imply that the operators $Q_r[g,\cdot]$ and $P_{ijr}[g,\cdot]$ are continuous from $C^{\frac{1}{2};1}(\partial_T\Omega)$ to $C_b^0(\partial_T\Omega)$. Moreover we note that (2.16) holds for μ_{2l}. Then, if $\psi\in C^1(\partial\Omega)$, we have that

$$
\int_{\partial\Omega} Q_r[g,\mu_2](t,y)M_{ij}[\psi](y)\,d\sigma_y
$$

$$
\begin{aligned}
&= \lim_{l\to\infty} \int_{\partial\Omega} Q_r[g,\mu_{2l}](t,y) M_{ij}[\psi](y)\, d\sigma_y \\
&= -\lim_{l\to\infty} \int_{\partial\Omega} M_{ij}[Q_r[g,\mu_{2l}]](t,y)\psi(y)\, d\sigma_y \\
&= -\lim_{l\to\infty} \int_{\partial\Omega} P_{ijr}[g,\mu_{2l}](t,y)\psi(y)\, d\sigma_y \\
&= -\int_{\partial\Omega} P_{ijr}[g,\mu_2](t,y)\psi(y)\, d\sigma_y.
\end{aligned}
$$

Hence $P_{ijr}[g,\mu_2](t,\cdot)$ coincides with the weak M_{ij}-derivative of $Q_r[g,\mu_2](t,\cdot)$ for all $i,j \in \{1,\dots,n\}$. Since both $P_{ijr}[g,\mu_2]$ and $Q_r[g,\mu_2]$ are continuous functions, it follows that $Q_r[g,\mu_2](t,\cdot) \in C^1(\partial\Omega)$ and that $M_{ij}[Q_r[g,\mu_2]](t,\cdot) = P_{ijr}[g,\mu_2](t,\cdot)$ classically on $\partial\Omega$.

Moreover $M_{ij}[Q_r[g,\mu_1]](t,\cdot) = P_{ijr}[g,\mu_1](t,\cdot)$ on $\partial\Omega$. Indeed, $\mu_1(\tau,\cdot) = 0$ for all $\tau \in]t-1,+\infty[$ and then the integral operators involved show no singularities and then formulas (2.18)–(2.26) hold with μ replaced with μ_1 by classical differentiation theorems for integral depending on a parameter.

Finally, since $\mu_3(\tau,\cdot) = 0$ for all $\tau \in]-\infty, t+1[$ the definition of $Q_r[\cdot,\cdot]$ and $\tilde{Q}_t[\cdot,\cdot]$ implies that $M_{ij}[Q_r[g,\mu_3]](t,\cdot) = P_{ijr}[g,\mu_3](t,\cdot) = 0$ on $\partial\Omega$. Accordingly, we can conclude that $M_{ij}[Q_r[g,\mu]] = P_{ijr}[g,\mu]$ on $\partial_T\Omega$. □

Then we have the following lemma about the time derivative of $Q_r[\cdot,\cdot]$.

Lemma 2.9. *Let $\alpha \in]0,1[$. Let $T \in]-\infty,+\infty]$. Let $r \in \{1,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^2. Let $g \in C^{0,\alpha}(\partial\Omega)$ and $\mu \in C^{1;2}(\partial_T\Omega)$. Then $Q_r[g,\mu]$ is continuously differentiable with respect to t and the following formula holds.*

$$
\frac{\partial}{\partial t} Q_r[g,\mu](t,x) = Q_r\left[g, \partial_t\mu\right](t,x) \qquad \forall\, (t,x) \in \partial_T\Omega. \tag{2.27}
$$

Proof. We note that

$$
\begin{aligned}
Q_r[g,\mu](t,x) &= \int_{-\infty}^{t} \int_{\partial\Omega} (g(x)-g(y)) \frac{\partial}{\partial x_r} \Phi_n(t-\tau, x-y)\mu(\tau,y) d\sigma_y d\tau \\
&= \int_0^{+\infty} \int_{\partial\Omega} (g(x)-g(y)) \frac{\partial}{\partial x_r} \Phi_n(\tau, x-y)\mu(t-\tau,y) d\sigma_y d\tau,
\end{aligned}
$$

for all $(t,x) \in \partial_T\Omega$. Lemma 1.16 (i) implies that

$$
C_{0,1,\partial\Omega} = \sup_{\substack{(t,x)\in]0,+\infty[\times\mathbb{R}^n \\ |x|\le \operatorname{diam}(\partial\Omega)}} |D_x\Phi_n(t,x)| \frac{t^{\frac{n}{2}+1}}{|x|} e^{\frac{|x|^2}{4t}} < +\infty\,,
$$

and thus

$$
\begin{aligned}
&\left| (g(x)-g(y)) \frac{\partial}{\partial x_r} \Phi_n(\tau, x-y) \partial_t\mu(t-\tau,y) \right| \\
&\qquad \le C_{0,1,\partial\Omega} \|g\|_{C^{0,\alpha}(\partial\Omega)} \|\partial_t\mu\|_{C^0_b(\partial_T\Omega)} |x-y|^{1+\alpha} \tau^{-\frac{n}{2}-1} e^{-\frac{|x-y|^2}{4\tau}},
\end{aligned}
$$

for all $(t,x) \in \partial_T\Omega$, and for all $(\tau,y) \in]0,+\infty[\times\partial\Omega$. Moreover, the change of variable $u|x-y|^2 = 4\tau$ implies that

$$
\int_0^{+\infty} \int_{\partial\Omega} |x-y|^{1+\alpha} \tau^{-\frac{n}{2}-1} e^{-\frac{|x-y|^2}{4\tau}} d\sigma_y d\tau
$$

$$=4^{\frac{n}{2}}\int_0^{+\infty} u^{-\frac{n}{2}-1}e^{-\frac{1}{u}}du\int_{\partial\Omega}\frac{1}{|x-y|^{n-1-\alpha}}d\sigma_y,$$

for all $x\in\partial\Omega$. Accordingly, Lemma 1.13 implies that there exists a constant $c'_{\Omega,n-1-\alpha}>0$ such that

$$\int_0^{+\infty}\int_{\partial\Omega}|x-y|^{1+\alpha}\tau^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4\tau}}d\sigma_y d\tau\le 4^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)c'_{\Omega,n-1-\alpha}\,,$$

for all $x\in\partial\Omega$. Then the statement follows by classical differentiation theorems for integrals depending on a parameter. □

Now we prove a lemma which contains an explicit formula for the tangential derivatives of $\tilde{Q}_t[\cdot,\cdot]$ (for the definition, see (1.28) and (2.13)).

Lemma 2.10. *Let $\alpha\in]0,1[$. Let $T\in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{2,\alpha}$. Let $g\in C^{1,\alpha}(\partial\Omega)$ and $\mu\in C^{1;2}(\partial_T\Omega)$. Then $\tilde{Q}_t[g,\mu](t,\cdot)\in C^1(\partial\Omega)$ for all $t\in\overline{]-\infty,T[}$. Moreover, the following formula holds.*

$$\begin{aligned}
&M_{ij}[\tilde{Q}_t[g,\mu]](t,x) \qquad (2.28)\\
&= \nu_i(x)\tilde{Q}_t\left[\frac{\partial g}{\partial x_j}-(Dg\cdot\nu)\nu_j,\mu\right](t,x)-\nu_j(x)\tilde{Q}_t\left[\frac{\partial g}{\partial x_i}-(Dg\cdot\nu)\nu_i,\mu\right](t,x)\\
&+\nu_i(x)\tilde{Q}_t\left[g,\sum_{h=1}^n\nu_h M_{hj}[\mu]\right](t,x)-\nu_j(x)\tilde{Q}_t\left[g,\sum_{h=1}^n\nu_h M_{hi}[\mu]\right](t,x)\\
&-\nu_i(x)g(x)\tilde{Q}_t\left[\sum_{h=1}^n M_{hj}[\nu_h],\mu\right](t,x)+\nu_j(x)g(x)\tilde{Q}_t\left[\sum_{h=1}^n M_{hi}[\nu_h],\mu\right](t,x)\\
&+\nu_i(x)\tilde{Q}_t\left[\sum_{h=1}^n M_{hj}[\nu_h],g\mu\right](t,x)-\nu_j(x)\tilde{Q}_t\left[\sum_{h=1}^n M_{hi}[\nu_h],g\mu\right](t,x)\\
&+\nu_i(x)\sum_{h=1}^n M_{hj}[\nu_h](x)\tilde{Q}_t[g,\mu](t,x)-\nu_j(x)\sum_{h=1}^n M_{hi}[\nu_h](x)\tilde{Q}_t[g,\mu](t,x)\\
&-\nu_i(x)\sum_{s=1}^n Q_s\left[g,\nu_s\nu_j\frac{\partial\mu}{\partial t}\right](t,x)+\nu_j(x)\sum_{s=1}^n Q_s\left[g,\nu_s\nu_i\frac{\partial\mu}{\partial t}\right](t,x),
\end{aligned}$$

for all $(t,x)\in\partial_T\Omega$ and for all $i,j\in\{1,\dots,n\}$.

Proof. Let $R\in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Let '$\tilde{\ }$' be an extension operator as in Lemma 1.3, defined on $C^{1,\alpha}(\partial\Omega)$.

First we fix $\beta\in]0,\alpha[$ and we prove formula (2.28) under the additional assumption that

$$\mu\in C^{\frac{2+\beta}{2};2+\beta}(\partial_T\Omega).$$

By Lemma 2.7 (ii) we already know that $\tilde{Q}^\sharp_t[\tilde{g},\mu](t,\cdot)$ belongs to $C^1(\mathrm{cl}\,\Omega)$ for all $t\in\overline{]-\infty,T[}$. Next we fix $(t,x)\in\Omega_T$ and we compute

$$M^\sharp_{ij}[\tilde{Q}^\sharp_t[\tilde{g},\mu]](t,x)=\tilde{\nu}_i(x)\frac{\partial}{\partial x_j}\tilde{Q}^\sharp_t[\tilde{g},\mu](t,x)-\tilde{\nu}_j(x)\frac{\partial}{\partial x_i}\tilde{Q}^\sharp_t[\tilde{g},\mu](t,x)\,,$$

(see (2.17) for the definition of $M^\sharp_{ij}[\cdot]$). First we note that Lemma A.5 of Appendix A implies that

$$\int_{-\infty}^t\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(t,y)\,d\sigma_y d\tau=0\,.$$

Accordingly, by classical differentiation theorems for integrals depending on a parameter, we have

$$
\begin{aligned}
&\frac{\partial}{\partial x_i}\tilde{Q}_t^\sharp[\tilde{g},\mu](t,x) \\
&\quad = \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial \tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial^2}{\partial x_i\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau.
\end{aligned}
\tag{2.29}
$$

Since $\sum_{h=1}^{n}\nu_h^2 = 1$ on $\partial\Omega$, we have

$$
\begin{aligned}
&\frac{\partial}{\partial x_i}\tilde{Q}_t^\sharp[\tilde{g},\mu](t,x) \\
&\quad = \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial \tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\qquad - \int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^{n}\nu_h^2(y)\frac{\partial}{\partial y_i}\left(\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\right)\mu(\tau,y)\,d\sigma_y d\tau \\
&\quad = \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial \tilde{g}}{\partial x_i}(x)\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\qquad - \int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^{n}\left[\left(\nu_h(y)\frac{\partial}{\partial y_i}-\nu_i(y)\frac{\partial}{\partial y_h}\right)\left(\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\right)\right] \\
&\qquad\quad \times \mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&\qquad - \int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^{n}\left[\nu_h(y)\frac{\partial}{\partial y_h}\left(\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\right)\right] \\
&\qquad\quad \times \nu_i(y)\mu(\tau,y)\,d\sigma_y d\tau.
\end{aligned}
$$

By the consequence of the Divergence Theorem of Lemma 1.4, the second integral in the right hand side takes the following form

$$
\begin{aligned}
&\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^{n}\left[\left(\nu_h(y)\frac{\partial}{\partial y_i}-\nu_i(y)\frac{\partial}{\partial y_h}\right)\left(\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\right)\right] \\
&\qquad\qquad \times \mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&\quad = \int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\sum_{h=1}^{n}M_{hi,y}\left[\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\right]\mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&\quad = \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}M_{hi,y}[\tilde{g}(y)]\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&\qquad - \sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)M_{hi,y}\left[\mu(\tau,y)\nu_h(y)\right]\,d\sigma_y d\tau.
\end{aligned}
$$

Accordingly, we have that

$$
\frac{\partial}{\partial x_i}\tilde{Q}_t^\sharp[\tilde{g},\mu](t,x)
$$

$$
\begin{aligned}
&= \frac{\partial \tilde g}{\partial x_i}(x) \int_{-\infty}^t \int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\quad - \sum_{h=1}^n \int_{-\infty}^t \int_{\partial\Omega} M_{hi,y}[\tilde g(y)] \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&\quad + \sum_{h=1}^n \int_{-\infty}^t \int_{\partial\Omega} (\tilde g(x)-\tilde g(y)) \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y) M_{hi,y}[\mu(\tau,y)\nu_h(y)]\,d\sigma_y d\tau \\
&\quad - \int_{-\infty}^t \int_{\partial\Omega} (\tilde g(x)-\tilde g(y)) \frac{\partial}{\partial t}\left(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right)\nu_i(y)\mu(\tau,y)\,d\sigma_y d\tau.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&M^\sharp_{ij}[\tilde Q^\sharp_t[\tilde g,\mu]](t,x) \qquad (2.30)\\
&\quad = M^\sharp_{ij}[\tilde g](x) \int_{-\infty}^t \int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\qquad + \sum_{h=1}^n \int_{-\infty}^t \int_{\partial\Omega} \{-\tilde\nu_i(x) M_{hj,y}[\tilde g(y)] + \tilde\nu_j(x) M_{hi,y}[\tilde g(y)]\} \\
&\qquad\qquad \times \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&\qquad + \sum_{h=1}^n \int_{-\infty}^t \int_{\partial\Omega} (\tilde g(x)-\tilde g(y)) \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y) \\
&\qquad\qquad \times \Big\{\tilde\nu_i(x) M_{hj,y}[\mu(\tau,y)\nu_h(y)] - \tilde\nu_j(x) M_{hi,y}[\mu(\tau,y)\nu_h(y)]\Big\}\,d\sigma_y d\tau \\
&\qquad + \int_{-\infty}^t \int_{\partial\Omega} (\tilde g(x)-\tilde g(y)) \frac{\partial}{\partial t}\left(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right) \\
&\qquad\qquad \times \{\tilde\nu_j(x)\nu_i(y) - \tilde\nu_i(x)\nu_j(y)\}\,\mu(\tau,y)\,d\sigma_y d\tau.
\end{aligned}
$$

By the identities (2.20) and (2.21) we rewrite the sum of the first two terms in the right hand side of (2.30) in the following form

$$
\begin{aligned}
&M^\sharp_{ij}[\tilde g](x) \int_{-\infty}^t \int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \qquad (2.31)\\
&\quad + \sum_{h=1}^n \int_{-\infty}^t \int_{\partial\Omega} \{-\tilde\nu_i(x) M_{hj,y}[\tilde g(y)] + \tilde\nu_j(x) M_{hi,y}[\tilde g(y)]\} \\
&\qquad \times \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\nu_h(y)\,d\sigma_y d\tau \\
&= \int_{-\infty}^t \int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\qquad \times \tilde\nu_i(x)\left(\frac{\partial \tilde g}{\partial x_j}(x) - (D\tilde g(x)\cdot\tilde\nu(x))\tilde\nu_j(x)\right) \\
&\quad - \int_{-\infty}^t \int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\qquad \times \tilde\nu_j(x)\left(\frac{\partial \tilde g}{\partial x_i}(x) - (D\tilde g(x)\cdot\tilde\nu(x))\tilde\nu_i(x)\right) \\
&\quad - \tilde\nu_i(x) \int_{-\infty}^t \int_{\partial\Omega} \left(\frac{\partial \tilde g}{\partial x_j}(y) - (D\tilde g(y)\cdot\tilde\nu(y))\tilde\nu_j(y)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{\partial}{\partial t}\Phi_n(t-\tau, x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&+\tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\left(\frac{\partial \tilde{g}}{\partial x_i}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\right) \\
&\times \frac{\partial}{\partial t}\Phi_n(t-\tau, x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
=&\tilde{\nu}_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}\Bigg[\left(\frac{\partial \tilde{g}}{\partial x_j}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_j(x)\right) \\
&-\left(\frac{\partial \tilde{g}}{\partial x_j}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_j(y)\right)\Bigg] \\
&\times \frac{\partial}{\partial t}\Phi_n(t-\tau, x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&-\tilde{\nu}_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\Bigg[\left(\frac{\partial \tilde{g}}{\partial x_i}(x)-(D\tilde{g}(x)\cdot\tilde{\nu}(x))\tilde{\nu}_i(x)\right) \\
&-\left(\frac{\partial \tilde{g}}{\partial x_i}(y)-(D\tilde{g}(y)\cdot\tilde{\nu}(y))\tilde{\nu}_i(y)\right)\Bigg] \\
&\qquad\times \frac{\partial}{\partial t}\Phi_n(t-\tau, x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
=&\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\left[\frac{\partial \tilde{g}}{\partial x_j}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_j,\mu\right](t,x)-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp\left[\frac{\partial \tilde{g}}{\partial x_i}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_i,\mu\right](t,x).
\end{aligned}
$$

Now we consider the third term in the right hand side of formula (2.30).

$$
\begin{aligned}
&\sum_{h=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y) \\
&\quad\times\{\tilde{\nu}_i(x)M_{hj,y}[\mu(\tau,y)\nu_h(y)]-\tilde{\nu}_j(x)M_{hi,y}[\mu(\tau,y)\nu_h(y)]\}\,d\sigma_y d\tau \\
&=\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\left[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h\mu]\right](t,x)-\tilde{\nu}_j(x)\tilde{Q}_t^\sharp\left[\tilde{g},\sum_{h=1}^{n}M_{hi}[\nu_h\mu]\right](t,x).
\end{aligned} \tag{2.32}
$$

Next we consider the first term in the right hand side of (2.32) and we note that

$$
\begin{aligned}
&\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\left[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h\mu]\right](t,x) \\
&=\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\left[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h]\mu\right](t,x)+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\left[\tilde{g},\sum_{h=1}^{n}M_{hj}[\mu]\nu_h\right](t,x) \\
&\quad-\tilde{\nu}_i(x)\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h](x)\tilde{Q}_t^\sharp[\tilde{g},\mu](t,x)+\tilde{\nu}_i(x)\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h](x)\tilde{Q}_t^\sharp[\tilde{g},\mu](t,x),
\end{aligned}
$$

and that the definition of $\tilde{Q}_t^\sharp[\cdot,\cdot]$ implies that

$$
\begin{aligned}
&\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\left[\tilde{g},\sum_{h=1}^{n}M_{hj}[\nu_h]\mu\right](t,x)-\tilde{\nu}_i(x)\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h](x)\tilde{Q}_t^\sharp[\tilde{g},\mu](t,x) \\
&\qquad=-\tilde{\nu}_i(x)\tilde{g}(x)\tilde{Q}_t^\sharp\left[\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h],\mu\right](t,x)+\tilde{\nu}_i(x)\tilde{Q}_t^\sharp\Big[\sum_{h=1}^{n}M_{hj}^\sharp[\tilde{\nu}_h],g\mu\Big](t,x),
\end{aligned}
$$

and that corresponding equalities can be written for the second term in the right hand side of (2.32). Hence, we deduce that the right hand side of formula (2.32) equals

$$\begin{aligned}
&\tilde{\nu}_i(x)\tilde{Q}^\sharp_t\left[\tilde{g},\sum_{h=1}^n \nu_h M_{hj}[\mu]\right](t,x)-\tilde{\nu}_j(x)\tilde{Q}^\sharp_t\left[\tilde{g},\sum_{h=1}^n \nu_h M_{hi}[\mu]\right](t,x) \\
&-\tilde{\nu}_i(x)\tilde{g}(x)\tilde{Q}^\sharp_t\left[\sum_{h=1}^n M^\sharp_{hj}[\tilde{\nu}_h],\mu\right](t,x)+\tilde{\nu}_j(x)\tilde{g}(x)\tilde{Q}^\sharp_t\left[\sum_{h=1}^n M^\sharp_{hi}[\tilde{\nu}_h],\mu\right](t,x) \\
&+\tilde{\nu}_i(x)\tilde{Q}^\sharp_t\Big[\sum_{h=1}^n M^\sharp_{hj}[\nu_h],g\mu\Big](t,x)-\tilde{\nu}_j(x)\tilde{Q}^\sharp_t\Big[\sum_{h=1}^n M^\sharp_{hi}[\nu_h],g\mu\Big](t,x) \\
&+\tilde{\nu}_i(x)\sum_{h=1}^n M^\sharp_{hj}[\tilde{\nu}_h](x)\tilde{Q}^\sharp_t[\tilde{g},\mu](t,x)-\tilde{\nu}_j(x)\sum_{h=1}^n M^\sharp_{hi}[\tilde{\nu}_h](x)\tilde{Q}^\sharp_t[\tilde{g},\mu](t,x).
\end{aligned} \tag{2.33}$$

Next we consider the last integral in the right hand side of formula (2.30). By integration by parts, we have

$$\begin{aligned}
&\int_{-\infty}^t\int_{\partial\Omega}(\tilde{g}(x)-\tilde{g}(y))\frac{\partial}{\partial t}\left(\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right) \\
&\qquad\times\{\tilde{\nu}_j(x)\nu_i(y)-\tilde{\nu}_i(x)\nu_j(y)\}\,\mu(\tau,y)\,d\sigma_y d\tau \\
&\qquad=\tilde{\nu}_j(x)\sum_{s=1}^n Q^\sharp_s[\tilde{g},\nu_s\nu_i\partial_t\mu](t,x)-\tilde{\nu}_i(x)\sum_{s=1}^n Q^\sharp_s[\tilde{g},\nu_s\nu_j\partial_t\mu](t,x).
\end{aligned} \tag{2.34}$$

Then by combining formulas (2.30), (2.31), (2.33) and (2.34), we obtain that the formula

$$\begin{aligned}
&M^\sharp_{ij}[\tilde{Q}^\sharp_t[\tilde{g},\mu]](t,x) \\
&=\tilde{\nu}_i(x)\tilde{Q}^\sharp_t\left[\frac{\partial\tilde{g}}{\partial x_j}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_j,\mu\right](t,x)-\tilde{\nu}_j(x)\tilde{Q}^\sharp_t\left[\frac{\partial\tilde{g}}{\partial x_i}-(D\tilde{g}\cdot\tilde{\nu})\tilde{\nu}_i,\mu\right](t,x) \\
&+\tilde{\nu}_i(x)\tilde{Q}^\sharp_t\left[\tilde{g},\sum_{h=1}^n \nu_h M_{hj}[\mu]\right](t,x)-\tilde{\nu}_j(x)\tilde{Q}^\sharp_t\left[\tilde{g},\sum_{h=1}^n \nu_h M_{hi}[\mu]\right](t,x) \\
&-\tilde{\nu}_i(x)\tilde{g}(x)\tilde{Q}^\sharp_t\left[\sum_{h=1}^n M^\sharp_{hj}[\tilde{\nu}_h],\mu\right](t,x)+\tilde{\nu}_j(x)\tilde{g}(x)\tilde{Q}^\sharp_t\left[\sum_{h=1}^n M^\sharp_{hi}[\tilde{\nu}_h],\mu\right](t,x) \\
&+\tilde{\nu}_i(x)\tilde{Q}^\sharp_t\left[\sum_{h=1}^n M^\sharp_{hj}[\tilde{\nu}_h],g\mu\right](t,x)-\tilde{\nu}_j(x)\tilde{Q}^\sharp_t\left[\sum_{h=1}^n M^\sharp_{hi}[\tilde{\nu}_h],g\mu\right](t,x) \\
&+\tilde{\nu}_i(x)\sum_{h=1}^n M^\sharp_{hj}[\tilde{\nu}_h](x)\tilde{Q}^\sharp_t[\tilde{g},\mu](t,x)-\tilde{\nu}_j(x)\sum_{h=1}^n M^\sharp_{hi}[\tilde{\nu}_h](x)\tilde{Q}^\sharp_t[\tilde{g},\mu](t,x) \\
&-\tilde{\nu}_i(x)\sum_{s=1}^n Q^\sharp_s[\tilde{g},\nu_s\nu_j\partial_t\mu](t,x)+\tilde{\nu}_j(x)\sum_{s=1}^n Q^\sharp_s[\tilde{g},\nu_s\nu_i\partial_t\mu](t,x),
\end{aligned} \tag{2.35}$$

holds for all $(t,x)\in\Omega_T$. Now, under our assumptions, the first argument of the terms $\tilde{Q}^\sharp_t$, $Q^\sharp_s$, which appear in the right hand side of formula (2.35) belongs to $C^{0,\alpha}(\mathrm{cl}\,\Omega)$, and the second argument of the terms $Q^\sharp_s$, which appear in the right hand side of formula (2.35) belongs to $C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$, and the second argument of the terms $\tilde{Q}^\sharp_t$, which appear in the right hand side of formula (2.26) belongs to $C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$. Then Lemma 2.6 (i) and Lemma 2.7 (i) imply that the right hand side of formula (2.35) defines a continuous function of the variable $(t,x) \in \mathrm{cl}\,\Omega_T$. Since Ω is of class $C^{2,\alpha}$ and $\tilde{g} \in C^{1,\alpha}(\mathrm{cl}\,\Omega)$, and since we are assuming that $\mu \in C^{\frac{2+\beta}{2};2+\beta}(\partial_T\Omega)$, Lemma 2.7 (ii) implies that $M^\sharp_{ij}[\tilde{Q}^\sharp_t[\tilde{g},\mu]] \in C^0(\mathrm{cl}\,\Omega_T)$. Hence, formula (2.35) must hold for all $(t,x) \in \mathrm{cl}\,\Omega_T$ and thus in particular for all $(t,x) \in \partial_T\Omega$. Clearly $M^\sharp_{ij}[\cdot]_{|\partial\Omega} = M_{ij}[\cdot]$. Moreover $Q^\sharp_r[\cdot,\cdot]_{|\partial_T\Omega} = Q_r[\cdot,\cdot]$ and $\tilde{Q}^\sharp_t[\cdot,\cdot]_{|\partial_T\Omega} = \tilde{Q}_t[\cdot,\cdot]$ (cf. equalities (2.14) and (2.15), respectively). Then we can conclude that (2.28) holds under the assumption that $\mu \in C^{\frac{2+\beta}{2};2+\beta}(\partial_T\Omega)$.

Now let $\mu \in C^{1;2}(\partial_T\Omega)$. We consider only the case $T = +\infty$. Indeed the case $T < +\infty$ can be treated similarly. We fix $t \in]-\infty, T[$ and we consider

$$\eta_1, \eta_2, \eta_3 \in C^\infty_b(\mathbb{R})$$

such that

i) $\sum_{i=1}^3 \eta_i = 1$;

ii) $0 \leq \eta_i \leq 1$ for all $i = 1,2,3$;

iii) $\mathrm{supp}\,(\eta_1) \subseteq]-\infty, t-1[$, $\mathrm{supp}\,(\eta_2) \subseteq]t-2, t+2[$ and $\mathrm{supp}\,(\eta_3) \subseteq]t+1, +\infty[$.

Then we set

$$\mu_i(\tau,x) = \mu(\tau,x)\eta_i(\tau) \qquad \forall\,(\tau,x) \in \partial_T\Omega,\ \forall\, i = 1,2,3.$$

Clearly

$$\mu(\tau,x) = \mu_1(\tau,x) + \mu_2(\tau,x) + \mu_3(\tau,x) \qquad \forall\,(\tau,x) \in \partial_T\Omega.$$

We denote by $\tilde{P}_{ijt}[g,\mu]$ the right hand side of (2.28). We now show that the weak M_{ij}-derivative of $\tilde{Q}_t[g,\mu_2]$ coincides with $\tilde{P}_{ijt}[g,\mu_2]$. Since μ_2 has compact support, by considering an extension of μ_2 of class $C^{1;2}$ with compact support in $\mathbb{R}^{n+1}$ (see Ladyženskaja, Solonnikov and Ural'ceva [58, Chapter 1.1, pp 9–10]) and by considering a sequence of mollifiers of such an extension, and by taking the restriction to $\partial_T\Omega$, we conclude that there exists a sequence of functions $\{\mu_{2l}\}_{l\in\mathbb{N}}$ in $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ such that μ_{2l} converges to μ_2 in $C^{1;2}(\partial_T\Omega)$. Next we note that Remarks 1.9, 1.10 and Theorems 1.27 and 1.30 (ii) and the continuity of the pointwise product in Schauder spaces, imply that the operators $\tilde{Q}_t[g,\cdot]$ and $\tilde{P}_{ijt}[g,\cdot]$ are continuous from $C^{1;2}(\partial_T\Omega)$ to $C^0_b(\partial_T\Omega)$. Moreover we note that (2.28) holds for μ_{2l}. Then, if $\psi \in C^1(\partial\Omega)$, we have that

$$\begin{aligned}
&\int_{\partial\Omega} \tilde{Q}_t[g,\mu_2](t,y) M_{ij}[\psi](y)\, d\sigma_y \\
&\qquad = \lim_{l\to\infty} \int_{\partial\Omega} \tilde{Q}_t[g,\mu_{2l}](t,y) M_{ij}[\psi](y)\, d\sigma_y \\
&\qquad = -\lim_{l\to\infty} \int_{\partial\Omega} M_{ij}[\tilde{Q}_t[g,\mu_{2l}]](t,y)\psi(y)\, d\sigma_y \\
&\qquad = -\lim_{l\to\infty} \int_{\partial\Omega} \tilde{P}_{ijt}[g,\mu_{2l}](t,y)\psi(y)\, d\sigma_y \\
&\qquad = -\int_{\partial\Omega} \tilde{P}_{ijt}[g,\mu_2](t,y)\psi(y) d\sigma_y.
\end{aligned}$$

Hence $\tilde{P}_{ijt}[g,\mu_2](t,\cdot)$ coincides with the weak M_{ij}-derivative of $\tilde{Q}_t[g,\mu_2](t,\cdot)$ for all $i,j \in \{1,\ldots,n\}$. Since both $\tilde{P}_{ijt}[g,\mu_2]$ and $\tilde{Q}_t[g,\mu_2]$ are continuous functions, it follows that $\tilde{Q}_t[g,\mu_2](t,\cdot) \in C^1(\partial\Omega)$ and that $M_{ij}[\tilde{Q}_t[g,\mu_2]](t,\cdot) = \tilde{P}_{ijt}[g,\mu_2](t,\cdot)$ classically on $\partial\Omega$.

Moreover $M_{ij}[\tilde{Q}_t[g,\mu_1]](t,\cdot) = \tilde{P}_{ijt}[g,\mu_1](t,\cdot)$ on $\partial\Omega$. Indeed $\mu_1(\tau,\cdot) = 0$ for all $\tau \in]t-1,+\infty[$ and then the integral operators involved show no singularities and accordingly formulas (2.29)–(2.35) hold with μ replaced with μ_1 by classical differentiation theorems for integrals depending on a parameter.

Finally, since $\mu_3(\tau,\cdot) = 0$ for all $\tau \in]-\infty,t+1[$ the definition of $Q_r[\cdot,\cdot]$ and $\tilde{Q}_t[\cdot,\cdot]$ implies that $M_{ij}[\tilde{Q}_t[g,\mu_3]](t,\cdot) = \tilde{P}_{ijt}[g,\mu_3](t,\cdot) = 0$ on $\partial\Omega$. Accordingly, we can conclude that $M_{ij}[\tilde{Q}_t[g,\mu]] = \tilde{P}_{ijt}[g,\mu]$ on $\partial_T\Omega$. □

Then we have the following lemma about the time derivative of $\tilde{Q}_t[\cdot,\cdot]$.

Lemma 2.11. *Let $\alpha \in]0,1[$. Let $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^3. Let $g \in C^{0,\alpha}(\partial\Omega)$ and $\mu \in C^{\frac{3}{2};3}(\partial_T\Omega)$. Then $\tilde{Q}_t[g,\mu]$ is continuously differentiable with respect to t and the following formula holds.*

$$\frac{\partial}{\partial t}\tilde{Q}_t[g,\mu](t,x) = \tilde{Q}_t[g,\partial_t\mu](t,x) \qquad \forall\,(t,x) \in \partial_T\Omega. \tag{2.36}$$

Proof. We note that

$$\begin{aligned}
&\tilde{Q}_t[g,\mu](t,x)\\
&\quad= \int_{-\infty}^{t}\int_{\partial\Omega}(g(x)-g(y))\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)(\mu(\tau,y)-\mu(t,y))\,d\sigma_y d\tau\\
&\quad= \int_{0}^{+\infty}\int_{\partial\Omega}(g(x)-g(y))\frac{\partial}{\partial t}\Phi_n(\tau,x-y)(\mu(t-\tau,y)-\mu(t,y))\,d\sigma_y d\tau,
\end{aligned}$$

for all $(t,x) \in \partial_T\Omega$. Lemma 1.17 (i) implies that

$$C_{1,0,\partial\Omega} = \sup_{\substack{(t,x)\in]0,+\infty[\times\mathbb{R}^n,\\ |x|\le \mathrm{diam}\,(\partial\Omega)}} |\partial_t\Phi_n(t,x)|t^{\frac{n}{2}+1}e^{\frac{|x|^2}{8t}} < +\infty\,,$$

and thus

$$\begin{aligned}
&\left|(g(x)-g(y))\frac{\partial}{\partial t}\Phi_n(\tau,x-y)(\partial_t\mu(t-\tau,y)-\partial_t\mu(t,y))\right|\\
&\qquad\le C_{1,0,\partial\Omega}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\partial_t\mu\|_{C^{\frac{1}{2};1}(\partial_T\Omega)}|x-y|^{\alpha}\tau^{-\frac{n+1}{2}}e^{-\frac{|x-y|^2}{8\tau}}\,,
\end{aligned}$$

for all $(t,x) \in \partial_T\Omega$, and for all $(\tau,y) \in]0,+\infty[\times\partial\Omega$. Moreover, the change of variable $u|x-y|^2 = 4\tau$ implies that

$$\begin{aligned}
&\int_{0}^{+\infty}\int_{\partial\Omega}|x-y|^{\alpha}\tau^{-\frac{n+1}{2}}e^{-\frac{|x-y|^2}{8\tau}}\,d\sigma_y d\tau\\
&\qquad=8^{\frac{n-1}{2}}\int_{0}^{+\infty}u^{-\frac{n+1}{2}}e^{-\frac{1}{u}}du\int_{\partial\Omega}\frac{1}{|x-y|^{n-1-\alpha}}\,d\sigma_y,
\end{aligned}$$

for all $x \in \partial\Omega$. Accordingly, Lemma 1.13 (i) implies that there exists a constant $c'_{\Omega,n-1-\alpha} > 0$ such that

$$\int_{0}^{+\infty}\int_{\partial\Omega}|x-y|^{\alpha}\tau^{-\frac{n+1}{2}}e^{-\frac{|x-y|^2}{8\tau}}\,d\sigma_y d\tau \le 8^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)c'_{\Omega,n-1-\alpha}\,,$$

for all $x \in \partial\Omega$. Then the statement follows by classical differentiation theorems for integrals depending on a parameter. □

Exploiting the lower order mapping properties of the operators $Q_r[\cdot,\cdot]$ and $\tilde{Q}_t[\cdot,\cdot]$ of Theorems 1.27, 1.30 and 1.33, and the formulas for their time and tangential derivatives (2.16), (2.27), (2.28) and (2.36), we can prove the following result about the mapping properties in parabolic Schauder spaces of the operators $Q_r[\cdot,\cdot]$ and $\tilde{Q}_t[\cdot,\cdot]$. The proof is based on an induction argument.

Theorem 2.12. *Let $r \in \{1,\dots,n\}$. Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$, $m \in \mathbb{N}\setminus\{0\}$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.*

(i) The operator Q_r from $C^{m-1,\alpha}(\partial\Omega)\times C^{\frac{m-1}{2};m-1}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$, which takes (g,μ) to $Q_r[g,\mu]$, is bilinear and continuous.

(ii) The operator $\tilde{Q}_t$ from $C^{m-1,\alpha}(\partial\Omega)\times C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$, which takes (g,μ) to $\tilde{Q}_t[g,\mu]$, is bilinear and continuous.

(iii) The operator Q_r from $C^{m-1,\alpha}(\partial\Omega)\times C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$, which takes (g,μ) to $Q_r[g,\mu]$, is bilinear and continuous.

(iv) The operator $\tilde{Q}_t$ from $C^{m-1,\alpha}(\partial\Omega)\times C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$, which takes (g,μ) to $\tilde{Q}_t[g,\mu]$, is bilinear and continuous.

Proof. We first prove both statement (i) and statement (ii) at the same time. We proceed by induction on m. The case $m=1$ for the operator Q_r follows from Theorem 1.27, and for the operator $\tilde{Q}_t$ follows from Theorem 1.30 (ii).

We now consider case $m=2$. The continuity of Q_r follows by the continuity of Q_r with values into $C^0_b(\partial_T\Omega)$ which follows by case $m=1$, and by the continuity of the operator Q_r with values in $C_b^{0,\frac{1+\beta}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))$ which follows by Theorem 1.33 (iii), and by the continuity of $M_{ij}[Q_r]$ with values in $C^{\frac{\beta}{2};\beta}(\partial_T\Omega)$, which follows by the formula (2.16) of Lemma 2.8, and by case $m=1$ for Q_r and $\tilde{Q}_t$ and by the continuity of the pointwise product in Schauder spaces and by Remarks 1.9 and 1.10. By the same arguments, the continuity of the pointwise product in Schauder spaces, Remarks 1.9 and 1.10, Theorem 1.30 (iii), and formula (2.28) of Lemma 2.10 and case $m=1$ imply the validity of the statement for the operator $\tilde{Q}_t$.

We now prove that if statements (i) and (ii) hold for all $m'\leq m$ and $m\geq 2$, then they hold for $m+1$. It suffices to prove that the following three statements hold.

(j) Q_r is continuous from $C^{m,\alpha}(\partial\Omega)\times C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^0_b(\partial_T\Omega)$ and $\tilde{Q}_t$ is continuous from $C^{m,\alpha}(\partial\Omega)\times C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^0_b(\partial_T\Omega)$.

(jj) $M_{ij}[Q_r]$ is continuous from $C^{m,\alpha}(\partial\Omega)\times C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$, and $M_{ij}[\tilde{Q}_t]$ is continuous from $C^{m,\alpha}(\partial\Omega)\times C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ for all $i,j\in\{1,\dots,n\}$.

(jjj) $\frac{\partial}{\partial t}Q_r$ is continuous from $C^{m,\alpha}(\partial\Omega)\times C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m-2+\beta}{2};m-2+\beta}(\partial_T\Omega)$, and $\frac{\partial}{\partial t}\tilde{Q}_t$ is continuous from $C^{m,\alpha}(\partial\Omega)\times C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^{\frac{m-2+\beta}{2};m-2+\beta}(\partial_T\Omega)$.

Statement (j) holds by statements (i), (ii) with $m=1$ and by Remark 1.10. We now consider statement (jj). The continuity of the pointwise product in Schauder spaces, Remarks 1.9 and 1.10, Lemma 2.8 and inductive assumption imply the validity of the statement (jj) for the operator Q_r. By the same argument, the continuity of the pointwise product in Schauder spaces, Remarks 1.9 and 1.10, Lemma 2.10 and the inductive assumption imply the validity of the statement (jj) for the operator $\tilde{Q}_t$.

Next we consider statement (jjj). Remarks 1.9 and 1.10 and Lemma 2.9 imply the validity of statement (jjj) for the operator Q_r. By the same argument, Remarks 1.9 and 1.10 and Lemma 2.11 imply the validity of statement (jjj) for the operator $\tilde{Q}_t$. Accordingly the proof of statements (i) and (ii) is complete.

The proof of statements (iii) and (iv) follows the line of the proof of statements (i) and (ii), by replacing the use of Theorems 1.27, 1.33 (iii), 1.30 (ii) by that of Theorems 1.33 (i), 1.33 (ii), 1.30 (i), respectively. □

2.4 Tangential derivatives of the double layer heat potential

In this section we prove a new explicit formula for the tangential derivatives of the double layer heat potential, in terms of the operators $Q_r[\cdot,\cdot]$ and $\tilde{Q}_t[\cdot,\cdot]$. Such a formula enables to exploit an induction argument in order to show the regularizing properties for $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ that we prove in the next section.

Theorem 2.13. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in C^{\frac{1}{2};1}(\partial_T\Omega)$. Then $w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t,\cdot) \in C^1(\partial\Omega)$ for all $t \in \overline{]-\infty,T[}$. Moreover the following formula holds.*

$$\begin{aligned}M_{ij}[w[\partial_T\Omega,\mu]_{|\partial_T\Omega}](t,x) = &\sum_{r=1}^{n}\big\{Q_r[\nu_i, M_{jr}[\mu]](t,x) - Q_r[\nu_j, M_{ir}[\mu]](t,x)\big\} \\ &+ \nu_i(x)\tilde{Q}_t[\nu_j,\mu](t,x) - \nu_j(x)\tilde{Q}_t[\nu_i,\mu](t,x) \\ &+ w[\partial_T\Omega, M_{ij}[\mu]]_{|\partial_T\Omega}(t,x),\end{aligned} \tag{2.37}$$

for all $(t,x) \in \partial_T\Omega$ and for all $i,j \in \{1,\ldots,n\}$.

Proof. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let '$\tilde{\cdot}$' be an extension operator as in Lemma 1.3, defined on $C^{1,\alpha}(\partial\Omega)$.

First we fix $\beta \in]0,\alpha[$ and we prove formula (2.37) under the additional assumption that

$$\mu \in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega).$$

We first note that by Theorem 2.1 (ii) we have that $w^+[\partial_T\Omega,\mu] \in C^{\frac{1+\beta}{2};1+\beta}(\mathrm{cl}\,\Omega_T)$. Next we fix $(t,x) \in \Omega_T$ and we compute

$$M^\sharp_{ij}[w^+[\partial_T\Omega,\mu]](t,x) = \tilde{\nu}_i(x)\frac{\partial}{\partial x_j}w^+[\partial_T\Omega,\mu](t,x) - \tilde{\nu}_j(x)\frac{\partial}{\partial x_i}w^+[\partial_T\Omega,\mu](t,x).$$

By formula (2.24), by the consequence of the Divergence Theorem of Lemma 1.4 and by classical differentiation theorems for integrals depending on a parameter, we have that

$$\frac{\partial}{\partial x_i}w^+[\partial_T\Omega,\mu](t,x) = \sum_{r=1}^{n}\frac{\partial}{\partial x_r}v^+[\partial_T\Omega, M_{ir}[\mu]](t,x) - \frac{\partial}{\partial t}v^+[\partial_T\Omega,\nu_i\mu](t,x).$$

Then by the obvious identity

$$-\tilde{\nu}_i(x)\nu_j(y) + \tilde{\nu}_j(x)\nu_i(y) = \tilde{\nu}_i(x)(\tilde{\nu}_j(x) - \nu_j(y)) - \tilde{\nu}_j(x)(\tilde{\nu}_i(x) - \nu_i(y)),$$

for all $y \in \partial\Omega$, and by Lemma A.5, we have that

$$
\begin{aligned}
&M^{\sharp}_{ij}[w^+[\partial_T\Omega,\mu]](t,x) \qquad (2.38)\\
&=\sum_{r=1}^{n}\Big\{\tilde{\nu}_i(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega, M_{jr}[\mu]](t,x)-\tilde{\nu}_j(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega, M_{ir}[\mu]](t,x)\Big\}\\
&\quad+\tilde{\nu}_i(x)\tilde{Q}^{\sharp}_t[\tilde{\nu}_j,\mu](t,x)-\tilde{\nu}_j(x)\tilde{Q}^{\sharp}_t[\tilde{\nu}_i,\mu](t,x).
\end{aligned}
$$

Now under our assumptions, Theorems 2.1 (ii), 2.2 (ii), (iii) and Lemma 2.7 (i) imply that both sides of (2.38) define continuous functions in $\mathrm{cl}\,\Omega_T$ and then (2.38) must hold for all $(t,x)\in \mathrm{cl}\,\Omega_T$. In particular (2.38) holds for all $(t,x)\in\partial_T\Omega$. Now we fix $(t,x)\in\partial_T\Omega$. Then the jump relation (2.3) for the double layer heat potential and equality (2.38) in $\partial_T\Omega$ imply that

$$
\begin{aligned}
&M_{ij}[w[\partial_T\Omega,\mu]_{|\partial_T\Omega}](t,x)\\
&\quad=\frac{1}{2}M_{ij}[\mu](t,x)+M_{ij}[w^+[\partial_T\Omega,\mu]](t,x)\\
&\quad=\frac{1}{2}M_{ij}[\mu](t,x)\\
&\qquad+\sum_{r=1}^{n}\Big\{\nu_i(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega, M_{jr}[\mu]](t,x)-\nu_j(x)\frac{\partial}{\partial x_r}v^+[\partial_T\Omega, M_{ir}[\mu]](t,x)\Big\}\\
&\qquad+\nu_i(x)\tilde{Q}_t[\nu_j,\mu](t,x)-\nu_j(x)\tilde{Q}_t[\nu_i,\mu](t,x).
\end{aligned}
$$

Then the jump relation (2.6) for the derivatives of the single layer heat potential implies that

$$
\begin{aligned}
&M_{ij}[w[\partial_T\Omega,\mu]_{|\partial_T\Omega}](t,x)= \qquad (2.39)\\
&\quad=\frac{1}{2}M_{ij}[\mu](t,x)+\frac{1}{2}\Big\{\sum_{r=1}^{n}M_{jr}[\mu](t,x)\nu_r(x)\nu_i(x)-\sum_{r=1}^{n}M_{ir}[\mu](t,x)\nu_r(x)\nu_j(x)\Big\}\\
&\qquad+\sum_{r=1}^{n}\Big\{\nu_i(x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{jr}[\mu](\tau,y)d\sigma_y d\tau\\
&\qquad-\nu_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{ir}[\mu](\tau,y)d\sigma_y d\tau\Big\}\\
&\qquad+\nu_i(x)\tilde{Q}_t[\nu_j,\mu](t,x)-\nu_j(x)\tilde{Q}_t[\nu_i,\mu](t,x).
\end{aligned}
$$

We now consider the second term in the right hand side of formula (2.39).

$$
\begin{aligned}
&\sum_{r=1}^{n}M_{jr}[\mu](t,x)\nu_r(x)\nu_i(x)-\sum_{r=1}^{n}M_{ir}[\mu](t,x)\nu_r(x)\nu_j(x) \qquad (2.40)\\
&\quad=\sum_{r=1}^{n}\Big[\nu_j(x)\frac{\partial\tilde{\mu}}{\partial x_r}(t,x)\nu_r(x)\nu_i(x)-\nu_r^2(x)\frac{\partial\tilde{\mu}}{\partial x_j}(t,x)\nu_i(x)\\
&\qquad-\nu_i(x)\frac{\partial\tilde{\mu}}{\partial x_r}(t,x)\nu_r(x)\nu_j(x)+\nu_r^2(x)\frac{\partial\tilde{\mu}}{\partial x_i}(t,x)\nu_j(x)\Big]\\
&\quad=-M_{ij}[\mu](t,x).
\end{aligned}
$$

Next consider the third term in the right hand side of formula (2.39). We observe that

$$\nu_i(x)M_{jr}[\mu](\tau,y)-\nu_j(x)M_{ir}[\mu](\tau,y)$$

$$
\begin{aligned}
&= \big(\nu_i(x)M_{jr}[\mu](\tau,y) - \nu_i(y)M_{jr}[\mu](\tau,y)\big)\\
&\quad + \big(\nu_i(y)M_{jr}[\mu](\tau,y) - \nu_j(y)M_{ir}[\mu](\tau,y)\big)\\
&\quad + \big(\nu_j(y)M_{ir}[\mu](\tau,y) - \nu_j(x)M_{ir}[\mu](\tau,y)\big),
\end{aligned}
$$

for all $(\tau, y) \in \partial_T\Omega$. Moreover

$$
\begin{aligned}
\nu_i(y)M_{jr}[\mu](\tau,y) - &\nu_j(y)M_{ir}[\mu](\tau,y)\\
&= \nu_i(y)\nu_j(y)\frac{\partial\tilde{\mu}}{\partial y_r}(\tau,y) - \nu_i(y)\nu_r(y)\frac{\partial\tilde{\mu}}{\partial y_j}(\tau,y)\\
&\quad - \nu_i(y)\nu_j(y)\frac{\partial\tilde{\mu}}{\partial y_r}(\tau,y) + \nu_j(y)\nu_r(y)\frac{\partial\tilde{\mu}}{\partial y_i}(\tau,y)\\
&= -\nu_r(y)M_{ij}[\mu](\tau,y),
\end{aligned}
$$

for all $(\tau, y) \in \partial_T\Omega$. Then

$$
\begin{aligned}
\sum_{r=1}^{n}\Big\{\nu_i(x)&\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{jr}[\mu](\tau,y)d\sigma_y d\tau\\
&- \nu_j(x)\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_r}\Phi_n(t-\tau,x-y)M_{ir}[\mu](\tau,y)d\sigma_y d\tau\Big\}\\
=&\sum_{r=1}^{n}\{Q_r[\nu_i, M_{jr}[\mu]](t,x) - Q_r[\nu_j, M_{ir}[\mu]](t,x)\}\\
&+ w[\partial_T\Omega, M_{ij}[\mu]](t,x).
\end{aligned} \tag{2.41}
$$

Then equalities (2.39), (2.40) and (2.41) imply that

$$
\begin{aligned}
M_{ij}[w[\partial_T\Omega,&\mu]_{|\partial_T\Omega}](t,x)\\
&= \sum_{r=1}^{n}\big\{Q_r[\nu_i, M_{jr}[\mu]](t,x) - Q_r[\nu_j, M_{ir}[\mu]](t,x)\big\}\\
&\quad + \nu_i(x)\tilde{Q}_t[\nu_j,\mu](t,x) - \nu_j(x)\tilde{Q}_t[\nu_i,\mu](t,x)\\
&\quad + w[\partial_T\Omega, M_{ij}[\mu]_{|\partial_T\Omega}](t,x),
\end{aligned} \tag{2.42}
$$

and then (2.37) holds true for $\mu \in C^{\frac{1+\beta}{2};1+\beta}(\partial_T\Omega)$.

Now let $\mu \in C^{\frac{1}{2};1}(\partial_T\Omega)$. We consider only the case $T = +\infty$. Indeed case the $T < +\infty$ can be treated similarly. We fix $t \in \overline{]-\infty, T[}$ and we consider

$$\eta_1, \eta_2, \eta_3 \in C_b^\infty(\mathbb{R})$$

such that

i) $\sum_{i=1}^{3}\eta_i = 1$;

ii) $0 \le \eta_i \le 1$ for all $i = 1, 2, 3$;

iii) $\mathrm{supp}\,(\eta_1) \subseteq]-\infty, t-1[$, $\mathrm{supp}\,(\eta_2) \subseteq]t-2, t+2[$ and $\mathrm{supp}\,(\eta_3) \subseteq]t+1, +\infty[$.

Then we set

$$\mu_i(\tau,x) = \mu(\tau,x)\eta_i(\tau) \qquad \forall\,(\tau,x) \in \partial_T\Omega, \forall\, i = 1,2,3.$$

Clearly

$$\mu(\tau,x) = \mu_1(\tau,x) + \mu_2(\tau,x) + \mu_3(\tau,x) \qquad \forall\,(\tau,x) \in \partial_T\Omega.$$

We denote by $R_{ij}[\mu]$ the right hand side of (2.37). We note that by Theorem 1.27, Theorem 1.30 (ii) and Theorem 2.4 (iii), $R_{ij}[\cdot]$ and $w[\partial_T\Omega,\cdot]$ are continuous from $C^{\frac{1}{2};1}(\partial_T\Omega)$ to $C^0_b(\partial_T\Omega)$. We now show that the weak M_{ij}-derivative of $w[\partial_T\Omega,\mu_2]_{|\partial_T\Omega}$ coincides with $R_{ij}[\mu_2]$. Since μ_2 has compact support, by considering an extension of μ_2 of class $C^{\frac{1}{2};1}$ with compact support in $\mathbb{R}^{n+1}$ (see Ladyženskaja, Solonnikov and Ural'ceva [58, Chapter 1.1, pp 9–10]) and by considering a sequence of mollifiers of such an extension, and by taking the restriction to $\partial_T\Omega$, we conclude that there exists a sequence $\{\mu_{2l}\}_{l\in\mathbb{N}}$ in $C^{\frac{1+\alpha}{2},1+\alpha}(\partial_T\Omega)$ such that μ_{2l} converges to μ_2 in $C^{\frac{1}{2};1}(\partial_T\Omega)$. Moreover we note that (2.37) holds for μ_{2l}. Then if $\psi \in C^1(\partial\Omega)$, we have that

$$\begin{aligned}
&\int_{\partial\Omega} w[\partial_T\Omega,\mu_2]_{|\partial_T\Omega}(t,y) M_{ij}[\psi](y)\,d\sigma_y \\
&\qquad = \lim_{l\to\infty}\int_{\partial\Omega} w[\partial_T\Omega,\mu_{2l}]_{|\partial_T\Omega}(t,y) M_{ij}[\psi](y)\,d\sigma_y \\
&\qquad = -\lim_{l\to\infty}\int_{\partial\Omega} M_{ij}[w[\partial_T\Omega,\mu_{2l}]_{|\partial_T\Omega}](t,y)\psi(y)\,d\sigma_y \\
&\qquad = -\lim_{l\to\infty}\int_{\partial\Omega} R_{ij}[\mu_{2l}](t,y)\psi(y)\,d\sigma_y \\
&\qquad = -\int_{\partial\Omega} R_{ij}[\mu_2](t,y)\psi(y)\,d\sigma_y.
\end{aligned}$$

Hence $R_{ij}[\mu_2](t,\cdot)$ coincides with the weak M_{ij}-derivative of $w[\partial_T\Omega,\mu_2]_{|\partial_T\Omega}(t,\cdot)$ for all $i,j \in \{1,\dots,n\}$ on $\partial\Omega$. Since both $R_{ij}[\mu]$ and $w[\partial_T\Omega,\mu]$ are continuous functions, it follows that $w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t,\cdot) \in C^1(\partial\Omega)$ and that $M_{ij}[w[\partial_T\Omega,\mu]_{|\partial_T\Omega}](t,\cdot) = R_{ij}[\mu](t,\cdot)$ classically on $\partial\Omega$.

Moreover $M_{ij}[w[\partial_T\Omega,\mu_1]_{|\partial_T\Omega}](t,\cdot) = R_{ij}[\mu_1](t,\cdot)$ in $\partial\Omega$. Indeed $\mu_1(\tau,\cdot) = 0$ for all $\tau \in]t-1,+\infty[$ and thus the integral operators involved show no singularities and then formulas (2.38)–(2.42) hold with μ replaced with μ_1 by classical differentiation theorems for integrals depending on a parameter.

Finally, since $\mu_3(\tau,\cdot) = 0$ for all $\tau \in]-\infty,t+1[$, the definitions of $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$, $Q_r[\cdot,\cdot]$ and $\tilde{Q}_t[\cdot,\cdot]$ imply that $M_{ij}[w[\partial_T\Omega,\mu_3]_{|\partial_T\Omega}](t,\cdot) = R_{ij}[\mu_3](t,\cdot) = 0$ in $\partial\Omega$. Accordingly, we can conclude that $M_{ij}[w[\partial_T\Omega,\mu]_{|\partial_T\Omega}] = R_{ij}[\mu]$ on $\partial_T\Omega$. □

Finally, we have also the following lemma regarding the time derivative of $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$.

Theorem 2.14. *Let $\alpha \in]0,1[$. Let $T \in]-\infty,+\infty]$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^2. Let $\mu \in C^{1;2}(\partial_T\Omega)$. Then*

$$\frac{\partial}{\partial t} w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t,x) = w[\partial_T\Omega,\partial_t\mu]_{|\partial_T\Omega}(t,x) \qquad \forall\,(t,x) \in \partial_T\Omega.$$

Proof. We note that

$$\begin{aligned}
w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t,x) &= \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\mu(\tau,y)d\sigma_y d\tau \\
&= \int_{0}^{+\infty}\int_{\partial\Omega} \frac{\partial}{\partial\nu(y)}\Phi_n(\tau,x-y)\mu(t-\tau,y)d\sigma_y d\tau,
\end{aligned}$$

for all $(t,x) \in \partial_T\Omega$.

Lemma 1.18 (i) implies that there exists a constant $b_{\Omega,\alpha} > 0$ such that

$$\left|\frac{\partial}{\partial\nu(y)}\Phi_n(\tau, x-y)\partial_t\mu(t-\tau, y)\right| \le \|\partial_t\mu\|_{C^0_b(\partial_T\Omega)} b_{\Omega,\alpha}|x-y|^{1+\alpha}\tau^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4\tau}},$$

for all $(t, x) \in \partial_T\Omega$, and for all $(\tau, y) \in]0, +\infty[\times\partial\Omega$. Moreover, the change of variable $u|x-y|^2 = 4\tau$ implies that

$$\begin{aligned}\int_0^{+\infty}\int_{\partial\Omega}&|x-y|^{1+\alpha}\tau^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4\tau}}d\sigma_y d\tau\\ &=4^{\frac{n}{2}}\int_0^{+\infty}u^{-\frac{n}{2}-1}e^{-\frac{1}{u}}du\int_{\partial\Omega}\frac{1}{|x-y|^{n-1-\alpha}}d\sigma_y,\end{aligned}$$

for all $x \in \partial\Omega$. Accordingly, Lemma 1.13 (i) implies that there exists a constant $c'_{\Omega,n-1-\alpha} > 0$ such that

$$\begin{aligned}\int_0^{+\infty}\int_{\partial\Omega}&|x-y|^{1+\alpha}\tau^{-\frac{n}{2}-1}e^{-\frac{|x-y|^2}{4\tau}}d\sigma_y d\tau\\ &\le 4^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)c'_{\Omega,n-1-\alpha},\end{aligned}$$

for all $x \in \partial\Omega$. Then the statement follows by classical differentiation theorems for integrals depending on a parameter. □

2.5 Regularizing properties of the double layer heat potential $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$

Now, we are ready to prove the regularizing properties of $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ in parabolic Schauder spaces. The strategy is to start from Theorem 2.5, which we use as the base case, and then exploit the formulas for the derivatives of Theorems 2.13, 2.14, together with the mapping properties of $Q_r[\cdot,\cdot]$ and $\tilde{Q}_t[\cdot,\cdot]$ of Theorem 2.12 in order to invoke an induction argument.

Theorem 2.15. *Let $\alpha \in]0, 1[$, $\beta \in]0, \alpha[$, $m \in \mathbb{N}\setminus\{0\}$, $T \in]-\infty, +\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.*

(i) The operator from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ which takes μ to $w[\partial_T\Omega, \mu]_{|\partial_T\Omega}$ is linear and continuous.

(ii) The operator from $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ which takes μ to $w[\partial_T\Omega, \mu]_{|\partial_T\Omega}$ is linear and continuous.

Proof. We first prove statement (i). We proceed by induction on m. The case $m = 1$ follows by the continuity of the pointwise product in Schauder spaces, by Remarks 1.9 and 1.10 on the continuity of the differential operators and of the embeddings in parabolic Schauder spaces, by formula (2.37) of Theorem 2.13 and by Theorems 2.4 (iii), 2.5 (iv) and 2.12 (i),(ii).

We now prove that if statement (i) holds for all $m' \le m$ and $m \ge 1$, then it holds for $m + 1$. It suffices to prove that the following three statements hold.

(j) $w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ is continuous from $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^0_b(\partial_T\Omega)$.

(jj) $M_{ij}[w[\partial_T\Omega, \cdot]_{|\partial_T\Omega}]$ is continuous from $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$.

(jjj) $\frac{\partial}{\partial t}w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is continuous from $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$.

Statement (j) holds by case $m=1$ and by Remark 1.10 on the continuity of the embeddings in parabolic Schauder spaces. We now consider statement (jj). The continuity of the pointwise product in Schauder spaces, Remarks 1.9 and 1.10 on the continuity of the differential operators and of the embeddings in parabolic Schauder spaces, Theorems 2.13, 2.12 (i),(ii) and the inductive assumption imply the validity of the statement (jj). Next we consider statement (jjj). Remarks 1.9 and 1.10 on the continuity of the differential operators and of the embeddings in parabolic Schauder spaces, Theorem 2.14, and the inductive assumption imply the validity of statement (jjj). Accordingly the proof of statement (i) is complete.

The proof of statement (ii) follows the line of the proof of statement (i), by replacing the use of Theorems 2.4 (iii), 2.5 (iv) and 2.12 (i),(ii) by that of Theorems 2.5 (ii), 2.5 (iii), 2.12 (iii),(iv), respectively. □

2.6 Regularizing properties of the integral operator $w_*[\partial_T\Omega,\cdot]$

We now consider the regularizing properties of the operator $w_*[\partial_T\Omega,\cdot]$ (for the definition, see (2.2)).

Theorem 2.16. *Under the assumptions of Theorem 2.15, the following statements hold.*

(i) The operator from $C^{\frac{m-1}{2};m-1}(\partial_T\Omega)$ to $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ which takes μ to $w_*[\partial_T\Omega,\mu]$ is linear and continuous.

(ii) The operator from $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ which takes μ to $w_*[\partial_T\Omega,\mu]$ is linear and continuous.

Proof. First we note that

$$
\begin{aligned}
w_*[\partial_T\Omega,\mu](t,x) &= \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial\nu(x)}\Phi_n(t-\tau,x-y)\mu(\tau,y)d\sigma_y d\tau \qquad (2.43)\\
&= \sum_{i=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}\nu_i(x)\frac{\partial}{\partial x_i}\Phi_n(t-\tau,x-y)\mu(\tau,y)d\sigma_y d\tau\\
&= \sum_{i=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}(\nu_i(x)-\nu_i(y))\frac{\partial}{\partial x_i}\Phi_n(t-\tau,x-y)\mu(\tau,y)d\sigma_y d\tau\\
&\quad -\sum_{i=1}^{n}\int_{-\infty}^{t}\int_{\partial\Omega}\nu_i(y)\frac{\partial}{\partial y_i}\Phi_n(t-\tau,x-y)\mu(\tau,y)d\sigma_y d\tau\\
&= \sum_{i=1}^{n}Q_i[\nu_i,\mu](t,x)-w[\partial_T\Omega,\mu]_{|\partial_T\Omega}(t,x),
\end{aligned}
$$

for all $(t,x)\in\partial_T\Omega$.

We now consider statement (i). Case $m=1$ follows by Theorems 2.12 (i), 2.4 (iii) and by the previous formula (2.43). Case $m>1$ follows by Theorems 2.12 (i), 2.15 (i) and by the previous formula (2.43).

The proof of statement (ii) follows the lines of the proof of statement (i), by replacing the use of Theorems 2.12 (i), 2.4 (iii), 2.15 (i) by that of Theorems 2.12 (iii), 2.5 (ii), 2.15 (ii) □

2.7 Compactness results for $w[\partial_T\Omega,\cdot]_{|\partial\Omega}$ and $w_*[\partial_T\Omega,\cdot]$

As a consequence of the regularizing properties of Theorem 2.15 and 2.16, in this section we prove some compactness results for $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and for $w_*[\partial_T\Omega,\cdot]$. We start with the following definition of a subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\partial\Omega)$.

Let $m\in\mathbb{N}$, $\alpha\in]0,1[$, $T\in]0,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. We set

$$C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)\equiv\left\{u\in C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega):\,u(t,x)=0\text{ for all }t\in]-\infty,0],\,x\in\partial\Omega\right\},\tag{2.44}$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$. The following compactness results is a well known consequence of the Ascoli-Arzelà Theorem.

Lemma 2.17. *Let $m\in\mathbb{N}$, $\alpha\in]0,1[$, $\beta\in[0,\alpha[$, $T\in]0,+\infty[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the embedding of $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ in $C_0^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ is compact.*

Finally, we have the following immediate corollary of Theorems 2.15 and 2.16 and of Lemma 2.17.

Corollary 2.18. *Let $\alpha\in]0,1[$, $\beta\in]0,\alpha[$, $m\in\mathbb{N}\setminus\{0\}$, $T\in]0,+\infty[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.*

i) The linear operator $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is compact from $C_0^{\frac{m}{2};m}(\partial_T\Omega)$ to itself, and from $C_0^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ to itself, and from $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to itself.

ii) The linear operator $w_*[\partial_T\Omega,\cdot]$ is compact from $C_0^{\frac{m-1}{2};m-1}(\partial_T\Omega)$ to itself, and from $C_0^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ to itself, and from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to itself.

CHAPTER 3

Periodic layer heat potentials

This Chapter is devoted to the study of space-periodic layer heat potentials and to the application to the solution of some initial-boundary value problems for the heat equation in parabolic cylinders defined as the product of a bounded time interval and an unbounded periodically perforated domain.

In order to explain in more details the content of the present chapter we introduce some notation. We fix once for all an n-tuple of positive real numbers

$$(q_{11}, \ldots, q_{nn}) \in]0, +\infty[^n$$

and we define the periodicity cell Q as

$$Q \equiv \prod_{j=1}^{n}]0, q_{jj}[.$$

Let $\mathbb{D}_n^+$ denotes the space of $n \times n$ diagonal matrices with entries in $]0, +\infty[$. Moreover, we denote by q the element of $\mathbb{D}_n^+$ defined by

$$q \equiv \begin{pmatrix} q_{11} & 0 & \cdots & 0 \\ 0 & q_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{nn} \end{pmatrix},$$

and by $m_n(Q)$ the measure of the fundamental cell Q. Clearly

$$q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$$

is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell Q (see Figure 3.1). Then we take

$$\alpha \in]0, 1[, \qquad m \in \mathbb{N} \setminus \{0\}, \qquad T \in]-\infty, +\infty],$$

and a bounded open subset Ω of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that

$$\mathrm{cl}\,\Omega \subseteq Q\,.$$

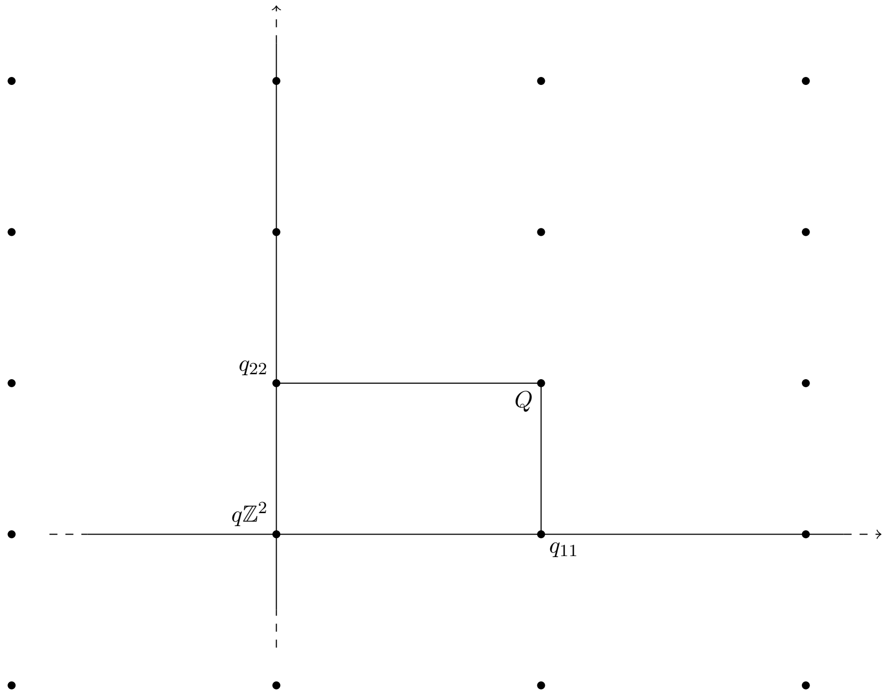

Figure 3.1: *The set $q\mathbb{Z}^n$ and the periodicity cell Q, in case $n = 2$.*

Before defining the space-periodic layer heat potentials, we need a periodic analog of the fundamental solution of the heat equation. To this aim, we introduce the function $\Phi_{q,n}$ from $(\mathbb{R} \times \mathbb{R}^n) \setminus (\{0\} \times q\mathbb{Z}^n)$ to $\mathbb{R}$ defined by

$$\Phi_{q,n}(t,x) \equiv \begin{cases} \sum_{z \in \mathbb{Z}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} & \text{if } (t,x) \in]0,+\infty[\times \mathbb{R}^n\,, \\ 0 & \text{if } (t,x) \in (]-\infty,0] \times \mathbb{R}^n) \setminus (\{0\} \times q\mathbb{Z}^n)\,, \end{cases} \tag{3.1}$$

and we refer to Theorem 3.5 (i) below for the convergence of the series. As it is known, $\Phi_{q,n}$ represents a q-periodic analog of the (classical) fundamental solution for the heat equation Φ_n (see, *e.g.*, Pinsky [89, Chapter 4.2] for the case $n = 1$ and Bernstein, Ebert and Sören Kraußhar [9] for $n \geq 2$).

Then we are in the position to introduce the q-periodic in space layer heat potentials. Let $x_0 \in \Omega$, then we set

$$v_q[\partial_T \Omega, \mu](t,x) \equiv \int_{-\infty}^{+\infty} \int_{\partial\Omega} (\Phi_{q,n}(t-\tau, x-y) - \Phi_{q,n}(0-\tau, x_0-y))\mu(\tau,y)\, d\sigma_y d\tau \tag{3.2}$$
$$\forall\, (t,x) \in (\mathbb{R}^n)_T,$$

and

$$w_q[\partial_T \Omega, \mu](t,x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(y)} \Phi_{q,n}(t-\tau, x-y)\mu(\tau,y)\, d\sigma_y d\tau \qquad \forall\, (t,x) \in (\mathbb{R}^n)_T, \tag{3.3}$$

and

$$w_{q,*}[\partial_T \Omega, \mu](t,x) \equiv \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(x)} \Phi_{q,n}(t-\tau, x-y)\mu(\tau,y)\, d\sigma_y d\tau \qquad \forall\, (t,x) \in \partial_T \Omega, \tag{3.4}$$

where the density (or moment) μ is a function in $L^\infty(\partial\Omega)$.

The functions $v_q[\partial_T\Omega,\mu]$ and $w_q[\partial_T\Omega,\mu]$ are the q-periodic in space single and double layer heat potential with density μ, respectively. The function $w_{q,*}[\partial_T\Omega,\mu]$ is instead a function related to the normal derivative of the q-periodic in space single layer potential $v_q[\partial_T\Omega,\mu]$.

Remark 3.1. We note that the above definition of single layer heat potential $v_q[\partial_T\Omega,\mu]$ clearly depends on the choice of $x_0 \in \Omega$, even if we don't make it explicit in the notation. Indeed, a different choice would provide a single layer which differs from that with x_0 by a constant. We have defined the single layer potential in such a way because the kernel

$$\Phi_{q,n}(t-\cdot, x-\cdot)$$

is not integrable in $]-\infty,t]\times\partial\Omega$, instead the kernel

$$\Phi_{q,n}(t-\cdot, x-\cdot) - \Phi_{q,n}(0-\cdot, x_0-\cdot)$$

is integrable in $]-\infty,t]\times\partial\Omega$.

However, we note that if $T\in]0,+\infty]$ and $\operatorname{supp}\mu \subseteq \overline{[0,T[}\times\partial\Omega$ (and this is the case when one considers an initial-boundary value problem for the heat equation with zero initial condition at $t=0$), then the q-periodic single layer heat potential $v_q[\partial_T\Omega,\mu]$ no longer depends on x_0 and

$$v_q[\partial_T\Omega,\mu](t,x) = \int_0^t\int_{\partial\Omega}\Phi_{q,n}(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \qquad \forall\,(t,x)\in\overline{[0,T[}\times\mathbb{R}^n,$$

that is the periodic analog of the classical definition of single layer heat potential.

This chapter is organized as follows. In Section 3.1 we provide the definition of parabolic Schauder spaces of space-periodic functions and the definition of some subspaces that we need in our analysis. In Section 3.2 we prove a characterization for parabolic Schauder spaces made of space-periodic functions. Section 3.3 is devoted to the q-periodic fundamental solution of the heat equation $\Phi_{q,n}$ and we prove some properties of $\Phi_{q,n}$. In Section 4 we consider the q-periodic layer heat potentials $v_q[\partial_T\Omega,\cdot]$ and $w_q[\partial_T\Omega,\cdot]$, and we prove some properties which are the periodic analog of the corresponding properties for classical layer heat potentials, like continuity, Schauder regularity and jump relations. More precisely we prove an analog of Theorems 2.1 and 2.2 in the periodic setting. Moreover, as the main result of this section, we prove a regularizing property of $w_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and of $w_{q,*}[\partial_T\Omega,\cdot]$ on $\partial_T\Omega$ under the assumption that Ω is of class $C^{m,\alpha}$. Such mapping properties are the periodic analog of the regularizing properties we have proved in Chapter 2 for $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and for $w_*[\partial_T\Omega,\cdot]$ (see Theorems 2.15 and 2.16). As a corollary, in Section 3.5 we deduce some compactness results for $w_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and for $w_{q,*}[\partial_T\Omega,\cdot]$ under the assumption that $T<+\infty$. Finally, in Section 3.6 we apply the results of the previous sections in order to solve three types of boundary value problems for the heat equation in an unbounded periodic domain. More precisely we consider the periodic version of a Dirichlet problem, of a Neumann problem, and of a non-ideal transmission problem.

Some of the results of this chapter can be found in a paper by the author [75].

3.1 Parabolic Schauder spaces of periodic functions

We now introduce two subspaces of parabolic Schauder spaces that are useful in order to consider initial-boundary value problems with zero initial condition at time $t=0$. If

$T \in]0, +\infty]$, $m \in \mathbb{N}$, $\alpha \in [0, 1[$, and Ω is a subset of $\mathbb{R}^n$, then we set

$$C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T) \equiv \left\{ u \in C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T) : \, u(t,x) = 0 \text{ for all } t \in]-\infty, 0], x \in \mathrm{cl}\,\Omega \right\},$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$. Moreover, as we have already introduced in (2.44), we recall that if Ω is of class $C^{m,\alpha}$, then

$$C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T \Omega) \equiv \left\{ u \in C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T \Omega) : \, u(t,x) = 0 \text{ for all } t \in]-\infty, 0], x \in \partial\Omega \right\},$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T \Omega)$.

Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$ such that

$$x + qe_i \in \mathbb{D} \qquad \forall\, x \in \mathbb{D}, \, \forall i \in \{1, \ldots, n\},$$

where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. In this case we say that the set $\mathbb{D}$ is q-periodic. We say that a function u from $\mathbb{D}_T$ to $\mathbb{C}$ is q-periodic in space, or simply q-periodic, if

$$u(t,x) = u(t, x + qe_i) \qquad \forall\, (t,x) \in \mathbb{D}_T, \, \forall\, i \in \{1, \ldots, n\}.$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,\Omega \subseteq Q$. We now introduce two q-periodic sets (see Figures 3.2, 3.4). We set

$$\mathbb{S}_q[\Omega] \equiv \bigcup_{z \in \mathbb{Z}^n} (qz + \Omega) = q\mathbb{Z}^n + \Omega.$$

Then we have that

$$\mathbb{S}_q[\Omega]^- = \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{S}_q[\Omega].$$

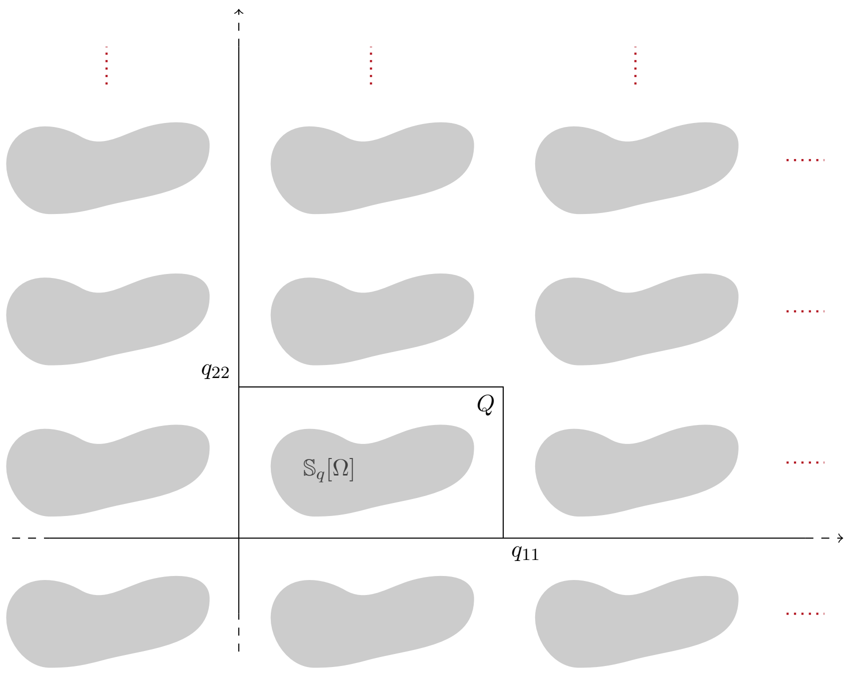

Figure 3.2: *In gray the periodic set* $\mathbb{S}_q[\Omega]$ *in case* $n = 2$.

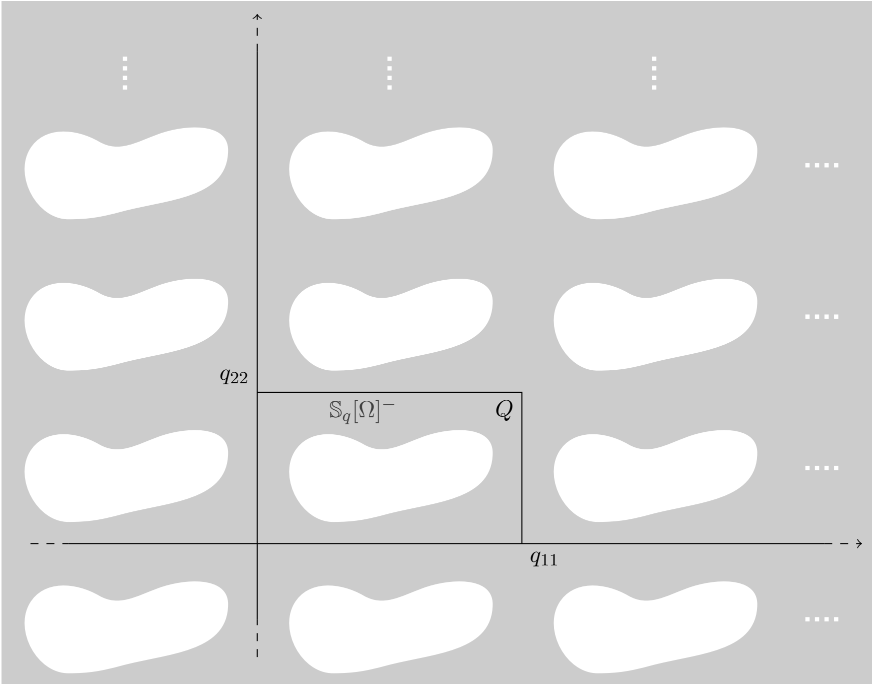

Figure 3.4: *In gray the periodic set* $\mathbb{S}_q[\Omega]^-$ *in case* $n = 2$.

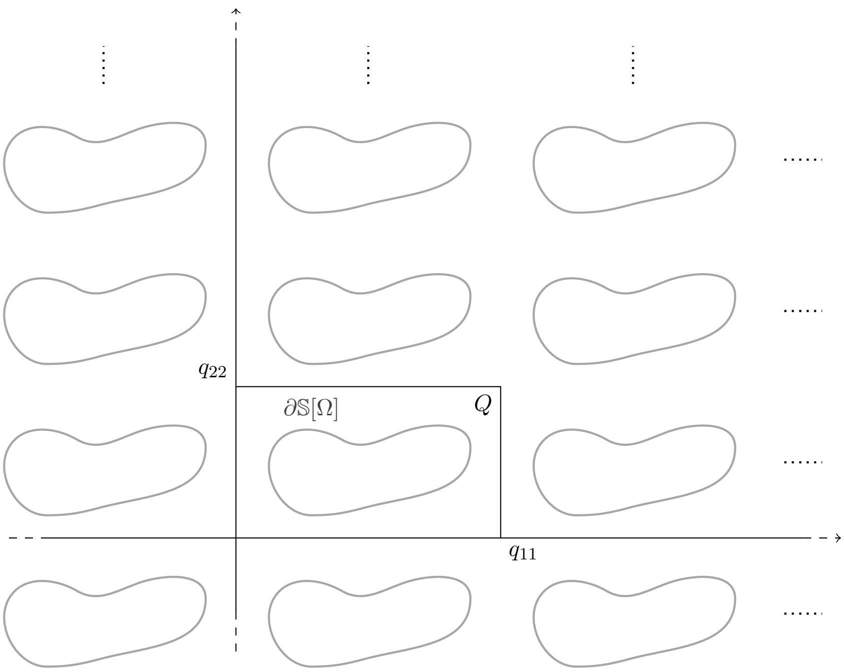

Figure 3.3: *In gray the periodic set* $\partial\mathbb{S}_q[\Omega]$ *in case* $n = 2$.

Since we will consider space-periodic problems, we introduce the following subspaces of parabolic Schauder spaces. Let $T \in]-\infty, +\infty]$, $m \in \mathbb{N}$, and $\alpha \in [0, 1[$. We set

$$C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T) \equiv \left\{u \in C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T) : u \text{ is } q\text{-periodic in space}\right\}, \qquad (3.5)$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$, and

$$C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-) \equiv \left\{u \in C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-) : u \text{ is } q\text{-periodic in space}\right\}, \tag{3.6}$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$. If $T \in]0,+\infty]$, then we can define $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ replacing the spaces $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ in the right hand side of (3.5) and (3.6), by $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$, respectively.

3.2 A characterization of parabolic Schauder spaces of periodic functions

In this section we prove two auxiliary results concerning the q-periodic parabolic Schuader spaces $C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and $C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$. In the first lemma we prove that when dealing with q-periodic Schauder functions in $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$, it suffices to consider Schauder functions in $\mathrm{cl}\,\Omega_T$. Next we prove a corresponding lemma for q-periodic Schauder functions in $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$. More precisely we show that when dealing with q-periodic Schauder functions in $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$, it suffices to work in a cylinder with as base a suitable neighborhood of the periodicity cell Q. We start with the following.

Lemma 3.2. *Let $\alpha \in [0,1[$, $T \in]-\infty,+\infty]$, $m \in \mathbb{N}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Then the restriction operator induces a linear homeomorphism from $C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ onto $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$.*

Proof. The case $m = 0$, $\alpha = 0$ is an obvious consequence of the q-periodicity of the functions in the space $C_q^{0;0}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and of the Open Mapping Theorem.

Next we consider the case $m = 0$, $\alpha \in]0,1[$. Obviously if $u \in C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$, then its restriction $u_{|\mathrm{cl}\,\Omega_T}$ belongs to $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$. Conversely, if $v \in C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$ then there exists a unique q-periodic function u from $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ to $\mathbb{C}$ such that $v = u_{|\mathrm{cl}\,\Omega_T}$ and

$$|u(t_1,x) - u(t_2,x)| \le \|v\|_{C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)}|t_1 - t_2|^{\frac{\alpha}{2}} \qquad \forall\, t_1,t_2 \in \overline{]-\infty,T[},\ \forall\, x \in \mathrm{cl}\,\Omega,$$

$$|u(t,x_1) - u(t,x_2)| \le \|v\|_{C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)}|x_1 - x_2|^{\alpha} \qquad \forall\, t \in \overline{]-\infty,T[},\ \forall\, x_1,x_2 \in \mathrm{cl}\,\Omega.$$

Next we set

$$d \equiv \inf\{|x-y| : (x,y) \in \mathrm{cl}\,\Omega \times (\mathbb{R}^n \setminus Q)\}.$$

Clearly $d > 0$. By the q-periodicity of u we have that

$$|u(t_1,x) - u(t_2,x)| \le \|v\|_{C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)}|t_1 - t_2|^{\frac{\alpha}{2}} \qquad \forall\, t_1,t_2 \in \overline{]-\infty,T[},\ \forall\, x \in \mathrm{cl}\,\mathbb{S}_q[\Omega].$$

Moreover

$$|u(t,x_1) - u(t,x_2)| \le \max\{2d^{-\alpha} \sup_{\mathrm{cl}\,\Omega_T} |v|, \|v\|_{C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)}\}|x_1 - x_2|^{\alpha}$$
$$\forall\, t \in \overline{]-\infty,T[}, \forall\, x_1,x_2 \in \mathrm{cl}\,\mathbb{S}_q[\Omega].$$

Accordingly $u \in C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$. Hence the restriction operator from $C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ to $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$ is a continuous bijection. Then the Open Mapping Theorem implies that it is a homeomorphism.

Next we consider the case $m = 1$, $\alpha \in [0,1[$. Obviously if $u \in C_q^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$, then its restriction $u_{|\mathrm{cl}\,\Omega_T}$ belongs to $C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\Omega_T)$. Conversely, if $v \in C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\Omega_T)$ then there exists a unique q-periodic function u from $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ to $\mathbb{C}$ such that $v = u_{|\mathrm{cl}\,\Omega_T}$. Moreover, following the same lines of the proof of the previous case one can prove that $\partial_{x_i} u \in C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ for all $i \in \{1, \dots, n\}$, and that

$$|u(t_1,x) - u(t_2,x)| \le \|v\|_{C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\Omega_T)} |t_1 - t_2|^{\frac{1+\alpha}{2}} \qquad \forall\, t_1, t_2 \in \overline{]-\infty, T[},\ \forall\, x \in \mathrm{cl}\,\mathbb{S}_q[\Omega].$$

Accordingly, the restriction operator from $C_q^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ to $C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\Omega_T)$ is a continuous bijection. Then the Open Mapping Theorem implies that it is a homeomorphism.

The general case follows by the previous cases and by an inductive argument. □

Then we have a corresponding lemma for q-periodic functions on $\mathbb{S}_q[\Omega]_T^-$.

Lemma 3.3. *Let $\alpha \in [0,1[$, $T \in]-\infty, +\infty]$, $m \in \mathbb{N}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Let V be an open bounded connected subset of $\mathbb{R}^n$ such that*

$$\mathrm{cl}\,Q \subseteq V, \quad \mathrm{cl}\,V \cap (qz + \mathrm{cl}\,\Omega) = \varnothing \qquad \forall\, z \in \mathbb{Z}^n \setminus \{0\}.$$

Let $W = V \setminus \mathrm{cl}\,\Omega$. Then the restriction operator induces a linear homeomorphism from $C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ onto the subspace

$$C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,W_T) \equiv \Big\{ v \in C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,W_T) : \\ \exists\, u \in \mathbb{C}^{\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-} \text{ such that } u \text{ is } q\text{-periodic}, v = u_{|\mathrm{cl}\,W_T} \Big\},$$

of $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,W_T)$.

Proof. The case $m = 0$, $\alpha = 0$ is an obvious consequence of the q-periodicity of the functions in the space $C_q^{0;0}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ and of the Open Mapping Theorem.

Next we consider the case $m = 0$, $\alpha \in]0,1[$. Obviously if $u \in C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$, then its restriction $u_{|\mathrm{cl}\,W_T}$ belongs to $C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,W_T)$. Conversely, if $v \in C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,W_T)$ then there exists a unique q-periodic function u from $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$ to $\mathbb{C}$ such that $v = u_{|\mathrm{cl}\,W_T}$ and

$$|u(t_1,x) - u(t_2,x)| \le \|v\|_{C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,W_T)} |t_1 - t_2|^{\frac{\alpha}{2}} \qquad \forall\, t_1, t_2 \in \overline{]-\infty, T[},\ \forall\, x \in \mathrm{cl}\,W,$$

$$|u(t,x_1) - u(t,x_2)| \le \|v\|_{C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,W_T)} |x_1 - x_2|^{\alpha} \qquad \forall\, t \in \overline{]-\infty, T[},\ \forall\, x_1, x_2 \in \mathrm{cl}\,W.$$

Next we set

$$d \equiv \inf\{|x - y| : (x,y) \in \mathrm{cl}\,Q \times (\mathbb{R}^n \setminus V)\}.$$

Clearly $d > 0$. By the q-periodicity of u we have that

$$|u(t_1,x) - u(t_2,x)| \le \|v\|_{C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,W_T)} |t_1 - t_2|^{\frac{\alpha}{2}} \qquad \forall\, t_1, t_2 \in \overline{]-\infty, T[},\ \forall\, x \in \mathrm{cl}\,\mathbb{S}_q[\Omega]^-.$$

Moreover

$$|u(t,x_1) - u(t,x_2)| \le \max\{2d^{-\alpha} \sup_{\mathrm{cl}\,\Omega_T} |v|, \|v\|_{C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,W_T)}\} |x_1 - x_2|^{\alpha}$$

$$\forall\, t \in \overline{]-\infty, T[}, \ \forall\, x_1, x_2 \in \mathrm{cl}\, \mathbb{S}_q[\Omega]^-.$$

Accordingly $u \in C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\, \mathbb{S}_q[\Omega]_T^-)$. Hence the restriction operator from $C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\, \mathbb{S}_q[\Omega]_T^-)$ to $C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\, W_T)$ is a continuous bijection. Then the Open Mapping Theorem implies that it is a homeomorphism.

Next we consider the case $m = 1$, $\alpha \in [0,1[$. Obviously if $u \in C_q^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\, \mathbb{S}_q[\Omega]_T^-)$, then its restriction $u_{|\mathrm{cl}\, W_T}$ belongs to $C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\, W_T)$. Conversely, if $v \in C_q^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\, W_T)$ then there exists a unique q-periodic function u from $\mathrm{cl}\, \mathbb{S}_q[\Omega]_T^-$ to $\mathbb{C}$ such that $v = u_{|\mathrm{cl}\, W_T}$. Moreover, following the same lines of the proof of the previous case one can prove that $\partial_{x_i} u \in C_q^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\, \mathbb{S}_q[\Omega]_T^-)$ for all $i \in \{1, \ldots, n\}$, and that

$$|u(t_1, x) - u(t_2, x)| \leq \|v\|_{C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\, W_T)} |t_1 - t_2|^{\frac{1+\alpha}{2}} \qquad \forall\, t_1, t_2 \in \overline{]-\infty, T[}, \forall\, x \in \mathrm{cl}\, \mathbb{S}_q[\Omega]^-.$$

Accordingly, the restriction operator from $C_q^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\, \mathbb{S}_q[\Omega]_T^-)$ to $C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\, W_T)$ is a continuous bijection. Then the Open Mapping Theorem implies that it is a homeomorphism.

The general case follows by the previous cases and by an inductive argument. □

3.3 The periodic fundamental solution

As we have already said, in order to construct periodic layer potentials we need a periodic analog of the fundamental solution of the heat equation. Therefore this section is devoted to the q-periodic fundamental solution of the heat equation $\Phi_{q,n}$ (cf. (3.1)).

We will need the following trivial lemma.

Lemma 3.4. *Let $a, b \in]0, +\infty[$. Then there exists a constant $\tilde{K}_{a,b} > 0$ such that*

$$\xi^a e^{-b\xi} \leq \tilde{K}_{a,b}, \qquad \forall\, \xi \in [0, +\infty[.$$

In the following Theorem we collect some known properties and variants of known properties of the q-periodic fundamental solution $\Phi_{q,n}$. Moreover we introduce the function $R_{q,n}$ defined by

$$R_{q,n}(t, x) \equiv \Phi_{q,n}(t, x) - \Phi_n(t, x), \qquad \forall\, (t, x) \in (\mathbb{R} \times \mathbb{R}^n) \setminus (\{0\} \times q\mathbb{Z}^n), \tag{3.7}$$

that is the difference between the periodic and the classical fundamental solution, and we study such a function.

Theorem 3.5. *Let $\Phi_{q,n}$ be the function defined in (3.1). Then the following statements hold.*

(i) The generalized series in (3.1), which defines $\Phi_{q,n}$, converges uniformly on the compact subsets of $]0, +\infty[\times \mathbb{R}^n$.

(ii) Let K be a compact subset of $\mathbb{R}^n$ such that $K \cap q\mathbb{Z}^n = \varnothing$. Then

$$\lim_{t \to 0^+} \Phi_{q,n}(t, x) = 0,$$

uniformly with respect to $x \in K$.

(iii) $\Phi_{q,n}$ is q-periodic and $\Phi_{q,n} \in C^\infty((\mathbb{R} \times \mathbb{R}^n) \setminus (\{0\} \times q\mathbb{Z}^n))$. Moreover $\Phi_{q,n}$ solves the heat equation in $(\mathbb{R} \times \mathbb{R}^n) \setminus (\{0\} \times q\mathbb{Z}^n)$.

(iv) *Let $f \in C^0(\mathbb{R}^n)$ such that f is q-periodic. Let u be the function from $]0,+\infty[\times\mathbb{R}^n$ to $\mathbb{C}$ defined by*

$$u(t,x) \equiv \int_Q \Phi_{q,n}(t,x-y) f(y)\,dy \qquad \forall\,(t,x) \in]0,+\infty[\times\mathbb{R}^n.$$

Then u belongs to $C^\infty(]0,+\infty[\times\mathbb{R}^n)$, solves the heat equation in $]0,+\infty[\times\mathbb{R}^n$, is q-periodic and

$$\lim_{t\to 0^+} u(t,x) = f(x) \qquad \forall\, x \in \mathbb{R}^n.$$

(v)

$$\Phi_{q,n}(t,x) = \sum_{z\in\mathbb{Z}^n} \frac{1}{m_n(Q)} e^{-4\pi^2|q^{-1}z|^2 t + 2\pi i (q^{-1}z)\cdot x} \qquad \forall\,(t,x) \in]0,+\infty[\times\mathbb{R}^n \tag{3.8}$$

(vi) *Let $a \in]0,+\infty[$, $\eta \in \mathbb{N}^n$, $h \in \mathbb{N}$ such that $|\eta|_1 + h \geq 1$. Then there exist two constants $C_{\eta,h,a}, c_{\eta,h} \in]0,+\infty[$ such that*

$$|\partial_t^h D^\eta \Phi_{q,n}(t,x)| \leq C_{\eta,h,a} e^{-c_{\eta,h} t} \qquad \forall\,(t,x) \in]a,+\infty[\times\mathbb{R}^n.$$

(vii) *The function $R_{q,n}$ can be extended by continuity in $(\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q(\mathbb{Z}^n\setminus\{0\}))$ and*

$$R_{q,n} \in C^\infty((\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q(\mathbb{Z}^n\setminus\{0\}))).$$

Proof. We first consider statement (i). Let $t_1, t_2 \in]0,+\infty[$, $t_1 < t_2$, $R \in]0,+\infty[$. It suffices to show the uniform convergence of the series in (3.1) in $M \equiv [t_1,t_2]\times \mathrm{cl}\,\mathbb{B}_n(0,R)$. We fix $(t,x) \in M$ and $z \in \mathbb{Z}^n$ such that $|qz| > R$. Then

$$|x+qz|^{n+2} \geq (|qz|-|x|)^{n+2} \geq (|qz|-R)^{n+2} > 0.$$

Lemma 3.4 implies

$$\begin{aligned}\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(\frac{4t}{|x+qz|^2}\right)^{\frac{n}{2}+1} \left(\frac{|x+qz|^2}{4t}\right)^{\frac{n}{2}+1} e^{-\frac{|x+qz|^2}{4t}} \\ &\leq \frac{4}{\pi^{\frac{n}{2}}} \tilde{K}_{\frac{n}{2}+1,1} \frac{t}{|x+qz|^{n+2}} \leq \frac{4}{\pi^{\frac{n}{2}}} \tilde{K}_{\frac{n}{2}+1,1} \frac{t_2}{(|qz|-R)^{n+2}}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{z\in\mathbb{Z}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} &= \sum_{\substack{z\in\mathbb{Z}^n\\|qz|\leq R}} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} + \sum_{\substack{z\in\mathbb{Z}^n\\|qz|>R}} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} \\ &\leq \sum_{\substack{z\in\mathbb{Z}^n\\|qz|\leq R}} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} + \frac{4}{\pi^{\frac{n}{2}}} \tilde{K}_{\frac{n}{2}+1,1} \sum_{\substack{z\in\mathbb{Z}^n\\|qz|>R}} \frac{t_2}{(|qz|-R)^{n+2}},\end{aligned}$$

and the right hand side of the previous inequality is bounded by a constant independent of $(t,x) \in M$.

Next we consider statement (ii). Let $R \in]0,+\infty[$ such that $K \subseteq \mathbb{B}_n(0,R)$. Since

$$I \equiv \inf\{|x+qz|^2 : x \in K,\ z \in \mathbb{Z}^n,\ |qz| \leq R\} > 0,$$

we have

$$\sum_{z\in\mathbb{Z}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} = \sum_{\substack{z\in\mathbb{Z}^n \\ |qz|>R}} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} + \sum_{\substack{z\in\mathbb{Z}^n \\ |qz|\le R}} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}}$$
$$\le \frac{4}{\pi^{\frac{n}{2}}} \tilde{K}_{\frac{n}{2}+1,1} \sum_{\substack{z\in\mathbb{Z}^n \\ |qz|>R}} \frac{t}{(|qz|-R)^{n+2}} + \sum_{\substack{z\in\mathbb{Z}^n \\ |qz|\le R}} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{I}{4t}},$$

and accordingly, (ii) follows.

The q-periodicity of $\Phi_{q,n}$ in (iii) follows by its definition. We now prove that $\Phi_{q,n} \in C^\infty((\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q\mathbb{Z}^n))$. Exploiting the estimate for the derivatives of the classical fundamental solution of the heat equation (1.4) and arguing as in (i) one can prove that the generalized series

$$\sum_{z\in\mathbb{Z}^n} \partial_t^h D^\eta \Phi_n(t, x+qz),$$

converges uniformly on the compact subsets of $]0,+\infty[\times\mathbb{R}^n$ for all $h\in\mathbb{N}, \eta\in\mathbb{N}^n$. Indeed, exploiting the estimate (1.4) and Lemma 3.4 we have that

$$\begin{aligned} \Big|\partial_t^h\, D^\eta \Phi_n\,(t, x+qz)| & \\ \le K_{\eta,h} t^{-\frac{n}{2}-\frac{|\eta|_1}{2}-h} e^{-\frac{|x+qx|^2}{8t}} & \\ = K_{\eta,h} t^{-\frac{n}{2}-\frac{|\eta|_1}{2}-h} \left(\frac{t}{|x+qz|^2}\right)^{\frac{n}{2}+\frac{|\eta|_1}{2}+h+1} \left(\frac{|x+qz|^2}{t}\right)^{\frac{n}{2}+\frac{|\eta|_1}{2}+h+1} e^{-\frac{|x+qx|^2}{8t}} & \\ \le K_{\eta,h} \tilde{K}_{\frac{n}{2}+\frac{|\eta|_1}{2}+h+1,\frac{1}{8}} \frac{t}{|x+qz|^{n+|\eta|_1+2h+2}} \qquad \forall\,(t,x)\in]0,+\infty[\times\mathbb{R}^n, & \end{aligned} \tag{3.9}$$

for all $h\in\mathbb{N}, \eta\in\mathbb{N}^n$, and then we can conclude following the same line of the proof of statement (i). Moreover, exploiting once more (1.4) and arguing as in (ii), one can prove that if K is a compact subset of $\mathbb{R}^n$ such that $K\cap q\mathbb{Z}^n = \varnothing$, then

$$\lim_{t\to 0^+} \sum_{z\in\mathbb{Z}^n} \partial_t^h D^\eta \Phi_n(t, x+qz) = 0,$$

uniformly with respect to $x\in K$. Indeed, inequality (3.9) and estimate (1.4) implies that

$$\sum_{z\in\mathbb{Z}^n} \partial_t^h D^\eta \Phi_n(t, x+qz)$$
$$\le K_{\eta,h} \tilde{K}_{\frac{n}{2}+\frac{|\eta|_1}{2}+h+1,\frac{1}{8}} \sum_{\substack{z\in\mathbb{Z}^n \\ |qz|>R}} \frac{t}{|x+qz|^{n+|\eta|_1+2h+2}} + K_{\eta,h} \sum_{\substack{z\in\mathbb{Z}^n \\ |qz|\le R}} \frac{1}{t^{\frac{n}{2}+\frac{|\eta|_1}{2}+h}} e^{-\frac{|x+qx|^2}{8t}},$$

for all $(t,x)\in]0,+\infty[\times\mathbb{R}^n$, and accordingly we can conclude following the same line of the proof of statement (ii). Then $\Phi_{q,n}\in C^\infty((\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q\mathbb{Z}^n))$. Finally, Φ_n solves the heat equation in $(\mathbb{R}\times\mathbb{R}^n)\setminus\{0,0\}$ and then $\Phi_{q,n}$ solves the heat equation in $(\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q\mathbb{Z}^n)$. Accordingly, (iii) follows.

Next we consider statement (iv). Statement (iii) and classical differentiation theorems for integrals depending on a parameter imply that u belongs to $C^\infty(]0,+\infty[\times\mathbb{R}^n)$ and solves the heat equation in $]0,+\infty[\times\mathbb{R}^n$. The q-periodicity of u follows by the q-periodicity of $\Phi_{q,n}$. By the q-periodicity of f and by the Fubini-Tonelli Theorem, we have

$$\begin{aligned}
u(t,x) &= \int_Q \Phi_{q,n}(t,x-y)f(y)\,dy \\
&= \int_Q \sum_{z\in\mathbb{Z}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y+qz|^2}{4t}} f(y)\,dy \\
&= \sum_{z\in\mathbb{Z}^n} \int_Q \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y+qz|^2}{4t}} f(y)\,dy \\
&= \sum_{z\in\mathbb{Z}^n} \int_{-qz+Q} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} f(y+qz)\,dy \\
&= \sum_{z\in\mathbb{Z}^n} \int_{-qz+Q} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy \\
&= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy \\
&= \int_{\mathbb{R}^n} \Phi_n(t,x-y)f(y)\,dy,
\end{aligned}$$

for all $(t,x)\in]0,+\infty[\times\mathbb{R}^n$. Then by classical properties of Φ_n and by the membership of f in $C^0(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, (iv) follows (see Evans [33, Theorem 1, p. 47]).

We now consider statement (v). It i well known that

$$\mathcal{F}_x\left[e^{-\frac{|x|^2}{4t}}\right](\xi) = (4\pi t)^{\frac{n}{2}} e^{-4\pi^2|\xi|^2 t} \qquad \forall\,(t,\xi)\in]0,+\infty[, \tag{3.10}$$

where $\mathcal{F}_x$ denotes the classical Fourier transform with respect to the space variables, which we recall is defined by

$$\mathcal{F}[g](\xi) \equiv \int_{\mathbb{R}^n} g(x)e^{-2\pi i\xi\cdot x}\,dx \qquad \forall\, g\in L^1(\mathbb{R}^n),\ \forall\,\xi\in\mathbb{R}^n.$$

Let now $g\in C^0(\mathbb{R}^n)$ be such that there exist two constants $C,\varepsilon\in[0,+\infty[$ such that

$$|g(x)|\le C(1+|x|)^{-n-\varepsilon}, \qquad |\mathcal{F}[g](\xi)|\le C(1+|\xi|)^{-n-\varepsilon} \qquad \forall\, x,\xi\in\mathbb{R}^n.$$

Then the Poisson summation formula says that

$$\sum_{z\in\mathbb{Z}^n} g(x+qz) = \sum_{z\in\mathbb{Z}^n} \frac{1}{m_n(Q)} \mathcal{F}_x[g](q^{-1}z)e^{2\pi i(q^{-1}z)\cdot x} \qquad \forall\, x\in\mathbb{R}^n, \tag{3.11}$$

(see, *e.g.*, Folland [37, p. 254]). Then combining (3.10) and (3.11) we obtain formula (3.8).

Next we consider statement (vi). By differentiating term by term formula (3.8), which can be done by the computations below, we have that

$$\begin{aligned}
|\partial_t^h D^\eta \Phi_{q,n}(t,x)| &= \left|\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{(-4\pi^2|q^{-1}z|^2)^h(2\pi i q^{-1}z)^\eta}{m_n(Q)} e^{-4\pi^2|q^{-1}z|^2 t+2\pi i(q^{-1}z)\cdot x}\right| \\
&\le \frac{(2\pi)^{2h+|\eta|_1}}{m_n(Q)} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} |q^{-1}z|^{2h+|\eta|_1} e^{-4\pi^2|q^{-1}z|^2 t},
\end{aligned}$$

for all $(t,x) \in]0,+\infty[\times\mathbb{R}^n$. We note that there exist two constants $d_1, d_2 \in]0,+\infty[$ such that

$$d_1|z|_1 \le |q^{-1}z| \le d_2|z|_1 \qquad \forall\, z \in \mathbb{Z}^n.$$

Then

$$\begin{aligned} |\partial_t^h D^\eta \Phi_n(t,x)| &\le \frac{(2\pi d_2)^{2h+|\eta|_1}}{m_n(Q)} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} |z|_1^{2h+|\eta|_1} e^{-4\pi^2 d_1^2 |z|_1^2 t} \\ &\le \frac{(2\pi d_2)^{2h+|\eta|_1}}{m_n(Q)} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} |z|_1^{2h+|\eta|_1} e^{-4\pi^2 d_1^2 |z|_1 t}, \end{aligned} \tag{3.12}$$

for all $(t,x) \in]0,+\infty[\times\mathbb{R}^n$. Now let $r \in \mathbb{N}\setminus\{0\}$. The number of elements of the set

$$A_r \equiv \{z \in \mathbb{Z}^n : |z|_1 = r\}$$

can be estimated by

$$2^n \binom{n+r-1}{r}.$$

Indeed

$$\binom{n+r-1}{r} = \frac{(r+1)(r+2)\cdots(r+n-1)}{(n-1)!},$$

is known to be the number of monomials of fixed degree r in n variables. Then there exists a constant $C'_n \in]0,+\infty[$ which does not depend on r such that

$$\mathrm{card}(A_r) \le C'_n r^{n-1}. \tag{3.13}$$

Let $a \in]0,+\infty[$. Inequality (3.13) and Lemma 3.4 imply that

$$\begin{aligned} \sum_{z\in\mathbb{Z}^n\setminus\{0\}} |z|_1^{2h+|\eta|_1} e^{-4\pi^2 d_1^2 |z|_1 t} &\le C'_n \sum_{r\in\mathbb{N}\setminus\{0\}} r^{2h+|\eta|_1+n-1} e^{-4\pi^2 d_1^2 r t} \\ &= C'_n \sum_{r\in\mathbb{N}\setminus\{0\}} r^{2h+|\eta|_1+n-1} e^{-2\pi^2 d_1^2 r t} e^{-2\pi^2 d_1^2 r t} \\ &\le C'_n \sum_{r\in\mathbb{N}\setminus\{0\}} r^{2h+|\eta|_1+n-1} e^{-2\pi^2 d_1^2 r t} e^{-2\pi^2 d_1^2 r a} \\ &\le C'_n \tilde{K}_{2h+|\eta|_1+n-1,2\pi^2 d_1^2 a} \sum_{r\in\mathbb{N}\setminus\{0\}} e^{-2\pi^2 d_1^2 r t}, \end{aligned} \tag{3.14}$$

for all $t \in]a,+\infty[$. Moreover there exists a constant $C'' \in]0,+\infty[$ such that

$$\sum_{r\in\mathbb{N}\setminus\{0\}} e^{-2\pi^2 d_1^2 r t} = \frac{e^{-2\pi^2 d_1^2 t}}{1-e^{-2\pi^2 d_1^2 t}} \le C'' e^{-2\pi^2 d_1^2 t}, \tag{3.15}$$

for all $t \in]a,+\infty[$. Then statement (vi) follows combining (3.12), (3.14) and (3.15).

Finally we consider statement (vii). Since

$$R_{q,n}(t,x) = \Phi_{q,n}(t,x) - \Phi_n(t,x) = \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}},$$

for all $(t,x) \in (\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q\mathbb{Z}^n)$, then $R_{q,n}$ can be extended by continuity in $(\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q(\mathbb{Z}^n\setminus\{0\}))$. Moreover, arguing as in the proof of statements (i), (ii) and (iii), one proves that $R_{q,n} \in C^\infty((\mathbb{R}\times\mathbb{R}^n)\setminus(\{0\}\times q(\mathbb{Z}^n\setminus\{0\})))$. □

By Theorem 3.5 (vii) the map $R_{q,n}$ and its derivatives are non-singular in the origin $(0,0) \in \mathbb{R} \times \mathbb{R}^n$. Hence, we can prove the following statement regarding the mapping properties of an integral operator with the derivatives of $R_{q,n}$ as kernel.

Lemma 3.6. *Let $T \in]-\infty,+\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $h \in \mathbb{N}$, $\eta \in \mathbb{N}^n$ be such that $|\eta|_1 + h \geq 1$. Let V be a bounded open connected subset of $\mathbb{R}^n$ of class C^∞ such that*

$$\mathrm{cl}\,Q \subseteq V, \quad \mathrm{cl}\,V \cap (qz + \mathrm{cl}\,\Omega) = \varnothing \qquad \forall\, z \in \mathbb{Z}^n \setminus \{0\}.$$

Let $\mu \in L^\infty(\partial_T\Omega)$. Let $u[\partial_T\Omega, \partial_t^h D^\eta R_{q,n}, \mu]$ be defined by

$$u[\partial_T\Omega, \partial_t^h D^\eta R_{q,n}, \mu](t,x) \equiv \int_{\partial\Omega}\int_{-\infty}^{t} \partial_t^h D^\eta R_{q,n}(t-\tau, x-y)\mu(\tau,y)\, d\sigma_y d\tau \qquad \forall\,(t,x) \in \mathrm{cl}\,V_T.$$

Then $u[\partial_T\Omega, \partial_t^h D^\eta R_{q,n}, \cdot]$ is linear and continuous from $L^\infty(\partial_T\Omega)$ to $C^{\frac{m}{2};m}(\mathrm{cl}\,V_T)$ for all $m \in \mathbb{N} \setminus \{0\}$.

Proof. We first note that if $x \in \mathrm{cl}\,V$ and $y \in \partial\Omega$, then

$$x - y \notin q\mathbb{Z}^n \setminus \{0\}.$$

Indeed if by contradiction $x - y \in q\mathbb{Z}^n \setminus \{0\}$, then $x \in \partial\Omega + (q\mathbb{Z}^n \setminus \{0\})$ and thus there exists $z \in \mathbb{Z}^n \setminus \{0\}$ such that $\mathrm{cl}\,V \cap (qz + \partial\Omega) \neq \varnothing$ which cannot be. Hence

$$\mathrm{cl}\,V - \partial\Omega \subseteq (\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}.$$

Then the statement follows by Theorem 3.5 (vi), (vii) and by classical differentiation theorems for integrals depending on a parameter. □

3.4 Properties of periodic layer potentials

In this section we consider the q-periodic layer heat potentials associated with the fundamental solution $\Phi_{q,n}$. First we prove those properties which are also known to hold for classical layer heat potentials. Namely Schauder regularity in $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ and in $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$, and jump formulas (Theorems 3.7 and 3.8 below). Then we prove a regularizing property of the integral operators $w_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and $w_{q,*}[\partial_T\Omega,\cdot]$ on $\partial_T\Omega$ (Theorems 3.9 and 3.10 below). The key consideration in order to prove such properties is to write the periodic layer potentials, by means of the map $R_{q,n}$, as a sum of the corresponding classical layer heat potential and of an integral operator with the smooth kernel $R_{q,n}$. We note that this idea stems from a similar idea which has been exploited in the elliptic case (see, *e.g*, Ammari and Kang [4, p. 57] and Lanza de Cristoforis and Musolino [65]).

We start with the following Theorem concerning the q-periodic single layer heat potential $v_q[\partial_T\Omega,\cdot]$ (for the definition, see (3.2)).

Theorem 3.7. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Then the following statements hold.*

(i) Let $\mu \in L^\infty(\partial_T\Omega)$. Then $v_q[\partial_T\Omega,\mu]$ is continuous, q-periodic in space and $v_q[\partial_T\Omega,\mu]$ is of class $C^\infty((\mathbb{R}^n\setminus\partial\mathbb{S}_q[\Omega])_T)$. Moreover $v_q[\partial_T\Omega,\mu]$ solves the heat equation in $(\mathbb{R}^n\setminus\partial\mathbb{S}_q[\Omega])_T$. We denote by $v_q^+[\partial\Omega_T,\mu]$ and $v_q^-[\partial\Omega_T,\mu]$ the restriction of $v_q[\partial\Omega_T,\mu]$ to $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ and $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$, respectively.

(ii) *Let $m \in \mathbb{N} \setminus \{0\}$, $i \in \{1, \dots, n\}$. Let Ω be of class $C^{m,\alpha}$. Then the map from $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C_q^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ which takes μ to $\frac{\partial}{\partial x_i} v_q^+[\partial_T\Omega, \mu]$ is linear and continuous. The same statement holds with $\frac{\partial}{\partial x_i} v_q^+[\partial_T\Omega, \mu]$ and $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ replaced by $\frac{\partial}{\partial x_i} v_q^-[\partial_T\Omega, \mu]$ and $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$, respectively.*

(iii) *Let $m \in \mathbb{N} \setminus \{0\}$. Let Ω be of class $C^{m,\alpha}$. Then the map from $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C_q^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ which takes μ to $\frac{\partial}{\partial t} v_q^+[\partial_T\Omega, \mu]$ is linear and continuous. The same statement holds with $\frac{\partial}{\partial t} v_q^+[\partial_T\Omega, \mu]$ and $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ replaced by $\frac{\partial}{\partial t} v_q^-[\partial_T\Omega, \mu]$ and $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$, respectively.*

(iv) *Let $m \in \mathbb{N} \setminus \{0\}$. Let Ω be of class $C^{m,\alpha}$. Let $T \in]0, +\infty[$. Then the map from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ which takes μ to $v_q^+[\partial_T\Omega, \mu]$ is linear and continuous. The same statement holds with $v_q^+[\partial_T\Omega, \mu]$ and $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ replaced by $v_q^-[\partial_T\Omega, \mu]$ and $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$, respectively.*

(v) *Let $\mu \in C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. Let $r \in \{1, \dots, n\}$. Then the following jump relations hold.*

$$\frac{\partial}{\partial \nu_\Omega(x)} v_q^\pm[\partial_T\Omega, \mu](t, x) = \pm\frac{1}{2}\mu(t, x) + w_{q,*}[\partial_T\Omega, \mu](t, x), \tag{3.16}$$

$$\begin{aligned}\frac{\partial}{\partial x_r} v_q^\pm[\partial_T\Omega, \mu](t, x) &= \pm\frac{1}{2}\mu(t, x)\nu_{\Omega,r}(x) \\ &\quad + \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial x_r}\Phi_{q,n}(t-\tau, x-y)\mu(\tau, y)\, d\sigma_y d\tau,\end{aligned}$$

for all $(t, x) \in \partial_T\Omega$.

Proof. First we fix a bounded open connected subset V of $\mathbb{R}^n$ of class C^∞ such that

$$\mathrm{cl}\, Q \subseteq V, \quad \mathrm{cl}\, V \cap (qz + \mathrm{cl}\,\Omega) = \varnothing, \quad \forall\, z \in \mathbb{Z}^n \setminus \{0\}.$$

Now we consider statement (i). Since $\Phi_{q,n}$ is q-periodic, then $v_q[\partial_T\Omega, \mu]$ is easily seen to be q-periodic. Moreover, the definition of $R_{q,n}$ (cf. (3.7)) implies that

$$\begin{aligned} v_q[\partial_T\Omega, \mu](t, x) &= \int_{-\infty}^{+\infty}\int_{\partial\Omega} (\Phi_n(t-\tau, x-y) - \Phi_n(0-\tau, x_0-y))\mu(\tau, y)\, d\sigma_y d\tau \\ &\quad + \int_{-\infty}^{+\infty}\int_{\partial\Omega} (R_{q,n}(t-\tau, x-y) - R_{q,n}(0-\tau, x_0-y))\mu(\tau, y)\, d\sigma_y d\tau, \end{aligned} \tag{3.17}$$

for all $(t, x) \in \mathrm{cl}\, V_T$. The continuity in $\mathrm{cl}\, V_T$ of the first term in the right hand side of equality (3.17) follows by the continuity of the classical single layer heat potential of Theorem 2.2 (i) (see also Friedman [38, p. 136]), and the continuity in $\mathrm{cl}\, V_T$ of the second term in the right hand side of equality (3.17) follows by Theorem 3.5 (vi), (vii) and by the Dominated Convergence Theorem. Hence, the continuity of $v_q[\partial_T\Omega, \mu]$ in $(\mathbb{R}^n)_T$ follows by the q-periodicity of $v_q[\partial_T\Omega, \mu]$. Next we note that

$$x - y \notin q\mathbb{Z}^n \qquad \forall\, (x, y) \in (\mathbb{R}^n \setminus \partial\mathbb{S}_q[\Omega]) \times \partial\Omega.$$

Indeed, if by contradiction $(x, y) \in (\mathbb{R}^n \setminus \partial\mathbb{S}_q[\Omega]) \times \partial\Omega$ and $x - y \in q\mathbb{Z}^n$, then $x \in \partial\Omega + q\mathbb{Z}^n = \partial\mathbb{S}_q[\Omega]$, contrary to our assumption on x. Then Theorem 3.5 (iii) and standard differentiation theorems for integrals depending on a parameter imply that $v_q[\partial_T\Omega,\mu]$ is in $C^\infty((\mathbb{R}^n\setminus\partial\mathbb{S}_q[\Omega])_T)$ and that solves the heat equation in $(\mathbb{R}^n\setminus\partial\mathbb{S}_q[\Omega])_T$.

Next we consider statement (ii). The definition of $R_{q,n}$ (cf. (3.7)) and classical differentiation theorems for integrals depending on a parameter imply that

$$\frac{\partial}{\partial x_i}v_q^+[\partial_T\Omega,\mu]=\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\mu]+u[\partial_T\Omega,\partial_{x_i}R_{q,n},\mu]\qquad\text{in }\operatorname{cl}\Omega_T,\tag{3.18}$$

and

$$\frac{\partial}{\partial x_i}v_q^-[\partial_T\Omega,\mu]=\frac{\partial}{\partial x_i}v^-[\partial_T\Omega,\mu]+u[\partial_T\Omega,\partial_{x_i}R_{q,n},\mu]\qquad\text{in }(\operatorname{cl}V\setminus\Omega)_T.\tag{3.19}$$

Here $v^+[\partial_T\Omega,\mu]$ and $v^-[\partial_T\Omega,\mu]$ are respectively the restrictions to $\operatorname{cl}\Omega_T$ and to $(\operatorname{cl}V\setminus\Omega)_T$ of the single layer heat potential associated with the fundamental solution of the heat equation Φ_n (see Theorem 2.2 (i)) and $u[\partial_T\Omega,\partial_{x_i}R_{q,n},\cdot]$ is the map defined in Lemma 3.6. By known properties of classical layer potentials $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$ is linear and continuous from $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\operatorname{cl}\Omega_T)$ and $\frac{\partial}{\partial x_i}v^-[\partial_T\Omega,\cdot]_{|(\operatorname{cl}V\setminus\Omega)_T}$ is linear and continuous from $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}((\operatorname{cl}V\setminus\Omega)_T)$ (see Theorem 2.2 (ii)). Then equalities (3.18), (3.19), and Lemma 3.6, and the continuity of the embedding of $C^{\frac{m}{2};m}(\operatorname{cl}V_T)$ into $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\operatorname{cl}V_T)$, imply that $\frac{\partial}{\partial x_i}v_q^+[\partial_T\Omega,\cdot]_{|\operatorname{cl}\Omega_T}$ is linear and continuous from $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\operatorname{cl}\Omega_T)$ and $\frac{\partial}{\partial x_i}v_q^-[\partial_T\Omega,\cdot]_{|(\operatorname{cl}V\setminus\Omega)_T}$ is linear and continuous from $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}((\operatorname{cl}V\setminus\Omega)_T)$. Then the q-periodicity of $\frac{\partial}{\partial x_i}v_q^+[\partial_T\Omega,\cdot]$ and of $\frac{\partial}{\partial x_i}v_q^-[\partial_T\Omega,\cdot]$ and Lemmas 3.2, 3.3 imply the validity of the statement for $\frac{\partial}{\partial x_i}v_q^+[\partial_T\Omega,\cdot]$ and $\frac{\partial}{\partial x_i}v_q^-[\partial_T\Omega,\cdot]$, respectively.

Statement (iii) and (iv) can be proved following the same lines of the proof of statement (ii) exploiting the properties of the classical single layer heat potential of Theorem 2.2.

Finally we consider statement (v). The jump formulas (3.16) follow from equalities (3.18), (3.19), from Lemma 3.6 and from the classical jump formulas for the normal derivative and for the x_i-derivative of the single layer heat potentials $v^\pm[\partial_T\Omega,\cdot]$ (see Theorem 2.2 (v)). □

Next we prove similar properties for the q-periodic double layer heat potential $w_q[\partial_T\Omega,\cdot]$ (cf. (3.3)). Namely, we have the following.

Theorem 3.8. *Under the same assumptions of Theorem 3.7, the following statements hold.*

(i) Let $\mu\in L^\infty(\partial_T\Omega)$. Then $w_q[\partial_T\Omega,\mu]$ is q-periodic in space, $w_q[\partial_T\Omega,\mu]\in C^\infty((\mathbb{R}^n\setminus\partial\mathbb{S}_q[\Omega])_T)$, and $w_q[\partial_T\Omega,\mu]$ solves the heat equation in $(\mathbb{R}^n\setminus\partial\mathbb{S}_q[\Omega])_T$.

(ii) Let $m\in\mathbb{N}\setminus\{0\}$. Let Ω be of class $C^{m,\alpha}$. Let $\mu\in C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$. Then the restriction $w_q[\partial_T\Omega,\mu]_{|\mathbb{S}_q[\Omega]_T}$ can be extended uniquely to an element $w_q^+[\partial_T\Omega,\mu]\in C_q^{\frac{m+\alpha}{2};m+\alpha}(\operatorname{cl}\mathbb{S}_q[\Omega]_T)$ and the restriction $w_q[\partial_T\Omega,\mu]_{|\mathbb{S}_q[\Omega]^-_T}$ can be extended uniquely to an element $w_q^-[\partial_T\Omega,\mu]\in C_q^{\frac{m+\alpha}{2};m+\alpha}(\operatorname{cl}\mathbb{S}_q[\Omega]^-_T)$. Moreover the following jump formulas hold.

$$w_q^\pm[\partial_T\Omega,\mu](t,x)=\mp\frac{1}{2}\mu(t,x)+w_q[\partial_T\Omega,\mu](t,x),\tag{3.20}$$

$$\frac{\partial}{\partial\nu_\Omega(x)}w_q^+[\partial_T\Omega,\mu](t,x)-\frac{\partial}{\partial\nu_\Omega(x)}w_q^-[\partial_T\Omega,\mu](t,x)=0,\tag{3.21}$$

for all $(t,x)\in\partial_T\Omega$.

(iii) Let $m \in \mathbb{N} \setminus \{0\}$. Let Ω be of class $C^{m,\alpha}$. The operator from $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ which takes μ to the function $w_q^+[\partial_T\Omega,\mu]$ is linear and continuous. The operator from $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ which takes μ to the function $w_q^-[\partial_T\Omega,\mu]$ is linear and continuous.

Proof. First we fix a bounded open connected subset V of $\mathbb{R}^n$ of class C^∞ such that

$$\mathrm{cl}\,Q \subseteq V, \quad \mathrm{cl}\,V \cap (qz + \mathrm{cl}\,\Omega) = \varnothing, \qquad \forall\, z \in \mathbb{Z}^n \setminus \{0\}.$$

We consider statement (i). The q-periodicity of $w_q[\partial_T\Omega,\mu]$ follows by the q-periodicity of $\Phi_{q,n}$. Next we note that

$$x - y \notin q\mathbb{Z}^n \qquad \forall\,(x,y) \in (\mathbb{R}^n \setminus \partial\mathbb{S}_q[\Omega]) \times \partial\Omega.$$

Indeed if by contradiction $(x,y) \in (\mathbb{R}^n \setminus \partial\mathbb{S}_q[\Omega]) \times \partial\Omega$ and $x-y \in q\mathbb{Z}^n$, then $x \in \partial\Omega + q\mathbb{Z}^n = \partial\mathbb{S}_q[\Omega]$, contrary to our assumption on x. Then Theorem 3.5 (iii) and standard differentiation theorems for integrals depending on a parameter imply that $w_q[\partial_T\Omega,\mu]$ is in $C^\infty((\mathbb{R}^n\setminus\partial\mathbb{S}_q[\Omega])_T)$ and solves the heat equation in $(\mathbb{R}^n \setminus \partial\mathbb{S}_q[\Omega])_T$.

Next we consider statement (ii). The definition of $R_{q,n}$ (cf. (3.7)) implies that

$$w_q[\partial_T\Omega,\mu](t,x) = w[\partial_T\Omega,\mu](t,x) - \sum_{i=1}^{n} u[\partial_T\Omega, \partial_{x_i}R_{q,n}, \nu_{\Omega,i}\mu](t,x) \qquad \forall\,(t,x) \in \mathrm{cl}\,V_T. \tag{3.22}$$

Here $w[\partial_T\Omega,\cdot]$ is the double layer heat potential associated with the fundamental solution Φ_n (see [64, Theorem 2.8 (i), p.9]). By classical results in potential theory we know that $w[\partial_T\Omega,\mu]_{|\Omega_T}$ can be uniquely extended to an element $w^+[\partial_T\Omega,\mu] \in C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$ and $w[\partial_T\Omega,\mu]_{|(V\setminus\mathrm{cl}\,\Omega)_T}$ can be uniquely extended to an element $w^-[\partial_T\Omega,\mu] \in C^{\frac{m+\alpha}{2};m+\alpha}((\mathrm{cl}\,V \setminus \Omega)_T)$ (Theorem 2.1 (i), (ii), see also Watson [110, Lemma 2.7, p. 41]). Then equality (3.22) and Lemma 3.6 imply that $w_q[\partial_T\Omega,\mu]_{|\Omega_T}$ can be uniquely extended to an element in $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$. Accordingly, the q-periodicity of $w_q[\partial_T\Omega,\mu]$ and Lemma 3.2 imply that $w_q[\partial_T\Omega,\mu]_{|\mathbb{S}_q[\Omega]_T}$ admits a unique extension $w_q^+[\partial_T\Omega,\mu]$ in $C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$. Similarly equality (3.22) and Lemma 3.6 imply that $w_q[\partial_T\Omega,\mu]_{|(V\setminus\mathrm{cl}\,\Omega)_T}$ can be uniquely extended to an element in $C^{\frac{m+\alpha}{2};m+\alpha}((\mathrm{cl}\,V\setminus\Omega)_T)$. Accordingly, the q-periodicity of $w_q[\partial_T\Omega,\mu]$ and Lemma 3.3 imply that $w_q[\partial_T\Omega,\mu]_{|\mathbb{S}_q[\Omega]_T^-}$ admits a unique extension $w_q^-[\partial_T\Omega,\mu]$ in $C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$. The jump formulas (3.20), (3.21) follow immediately by equality (3.22), and by Lemma 3.6 and by the classical jump formulas for $w[\partial_T\Omega,\mu]$ (see, *e.g.*, Theorem 2.1 (i)).

Finally, the continuity of the operators in (iii) follows by equality (3.22), and by the continuity properties of the interior and exterior double layer heat potential $w^+[\partial_T\Omega,\cdot]$ and $w^-[\partial_T\Omega,\cdot]_{|(\mathrm{cl}\,V\setminus\Omega)_T}$ (see Theorem 2.1 (ii)), and by Lemmas 3.2, 3.3 and by Lemma 3.6. □

Now, exploiting the regularizing effect of the integral operators associated with the (non-periodic) layer heat potentials $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and $w_*[\partial_T\Omega,\cdot]$ of Theorems 2.15 and 2.16, respectively, we are ready to prove that the integral operators $w_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and $w_{*,q}[\partial_T\Omega,\cdot]$ (cf. (3.3), (3.4)) have the same smoothing effect on the parabolic boundary $\partial_T\Omega$.

Theorem 3.9. *Let $\alpha \in]0,1[$, $\beta \in]0,\alpha[$, $T \in]-\infty,+\infty]$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Then the following statements hold.*

(i) The operator from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ which takes μ to $w_q[\partial_T\Omega,\mu]_{|\partial_T\Omega}$ is linear and continuous.

(ii) The operator from $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ *to* $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ *which takes* μ *to* $w_q[\partial_T\Omega,\mu]_{|\partial_T\Omega}$ *is linear and continuous.*

Proof. We first consider statement (i). As we have already seen in the proof of Theorem 3.8, we have

$$w_q[\partial_T\Omega,\mu](t,x) = w[\partial_T\Omega,\mu](t,x) - \sum_{i=1}^{n} u[\partial_T\Omega, \partial_{x_i}R_{q,n}, \nu_{\Omega,i}\mu](t,x) \qquad \forall\,(t,x)\in\partial_T\Omega$$

Let V be a bounded open subset of $\mathbb{R}^n$ as in the statement of Lemma 3.6. Then Lemma 3.6 implies that the operator $u[\partial_T\Omega, \partial_{x_i}R_{q,n}, \nu_{\Omega,i}\cdot]$ is linear and continuous from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+1}{2};m+1}(\mathrm{cl}\,V_T)$. Since the restriction operator is linear and continuous from $C^{\frac{m+1}{2};m+1}(\mathrm{cl}\,V_T)$ to $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$, then the operator $u[\partial_T\Omega, \partial_{x_i}R_{q,n}, \nu_{\Omega,i}\cdot]_{|\partial_T\Omega}$ is linear and continuous from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$. Then the continuity of the embedding of $C^{\frac{m+1}{2};m+1}(\partial_T\Omega)$ in $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$, and the continuity of the operator $w[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ from $C^{\frac{m}{2};m}(\partial_T\Omega)$ to $C^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ imply the validity of statement (i) (see Theorem 2.15 (i)) .

Statement (ii) can be proved following the same lines of the proof of statement (i) by replacing the use of Theorem 2.15 (i) by that of Theorem 2.15 (ii). □

Next we consider the mapping properties of the operator $w_{*,q}[\partial_T\Omega,\cdot]$. The proof of the following Theorem follows the same lines of that of Theorem 3.9 replacing the use of Theorem 2.15 by that of Theorem 2.16

Theorem 3.10. *Under the assumptions of Theorem 3.9 the following statements hold.*

(i) The operator from the space $C^{\frac{m-1}{2};m-1}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ *which takes* μ *to* $w_{*,q}[\partial_T\Omega,\mu]$ *is linear and continuous.*

(ii) The operator from the space $C^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ *to* $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ *which takes* μ *to* $w_{*,q}[\partial_T\Omega,\mu]$ *is linear and continuous.*

3.5 Compactness results for $w_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and $w_{*,q}[\partial_T\Omega,\cdot]$

As a consequence of the regularizing properties of Theorem 3.9 and 3.10, in this section we deduce some compactness results for $w_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and for $w_{*,q}[\partial_T\Omega,\cdot]$. More precisely, we have the following immediate corollary of Theorems 3.9 and 3.10 and of Lemma 2.17 about the compactness of the embedding of parabolic Schauder spaces.

Corollary 3.11. *Let* $\alpha\in]0,1[$, $\beta\in]0,\alpha[$, $m\in\mathbb{N}\setminus\{0\}$, $T\in]0,+\infty[$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$. *Then the following statements hold.*

i) The linear operator $w_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ *is compact from* $C_0^{\frac{m}{2};m}(\partial_T\Omega)$ *to itself, and from* $C_0^{\frac{m+\beta}{2};m+\beta}(\partial_T\Omega)$ *to itself, and from* $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ *to itself.*

ii) The linear operator $w_{*,q}[\partial_T\Omega,\cdot]$ *is compact from* $C_0^{\frac{m-1}{2};m-1}(\partial_T\Omega)$ *to itself, and from* $C_0^{\frac{m-1+\beta}{2};m-1+\beta}(\partial_T\Omega)$ *to itself, and from* $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ *to itself.*

3.6 Applications to periodic boundary value problems

In this section we show how to apply the results of the previous sections in order to solve some boundary value problems for the heat equation in a periodic setting. More precisely, we consider an initial-Dirichlet problem and an initial-Neumann problem in the unbounded space-periodic domain $\mathrm{cl}\,\mathbb{S}_q[\Omega]^-_T$ (see Subsections 3.6.1 and 3.6.2 below). Moreover we consider also a non-ideal transmission problem (see Subsection 3.6.3 below).

Throughout this section we fix

$$\alpha \in]0,1[, \quad T \in]0,+\infty[, \quad m \in \mathbb{N}\setminus\{0\}.$$

We observe that the main tool that we use in order to solve the Dirichlet, the Neumann and the non-ideal transmission problem by means of space-periodic layer heat potentials is the compactness of the operators $w_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and $w_{q,*}[\partial_T\Omega,\cdot]$ in $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ and in $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$, respectively, which is a consequence of the regularizing properties of Theorems 3.9 and 3.10, and is proved in Corollary 3.11.

3.6.1 A periodic Dirichlet problem

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $f \in C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$. We consider the following Dirichlet problem for the heat equation in $]0,T]\times\mathbb{S}_q[\Omega]^-$.

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in }]0,T]\times\mathbb{S}_q[\Omega]^-, \\ u(t,x+qe_i) = u(t,x) & \forall\,(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-, \forall\, i\in\{1,\dots,n\}, \\ u = f & \text{on } [0,T]\times\partial\Omega, \\ u(0,\cdot)=0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases} \tag{3.23}$$

We start by the following consequence of the classical maximum principle for the heat equation, which will imply the uniqueness of the solution of problem (3.23).

Proposition 3.12. *Let $T\in]0,+\infty[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,\Omega\subseteq Q$ and such that $\mathbb{R}^n\setminus\mathrm{cl}\,\Omega$ is connected. Let $u\in C^0([0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)$ be one time continuously differentiable with respect to the time variable and two times continuously differentiable with respect to the space variables in $]0,T]\times\mathbb{S}_q[\Omega]^-$, and such that*

$$u(t,x+qe_i)=u(t,x),$$

for all $(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-$ and for all $i\in\{1,\dots,n\}$, and such that

$$\partial_t u(t,x)-\Delta u(t,x)=0,$$

for all $(t,x)\in]0,T]\times\mathbb{S}_q[\Omega]^-$. Then the following statements hold.

(i) If there exists a point $(t_0,x_0)\in]0,T]\times\mathbb{S}_q[\Omega]^-$ such that

$$u(t_0,x_0)=\max_{[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-} u,$$

then u is constant in $[0,t_0]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-$.

(ii) If there exists a point $(t_0, x_0) \in]0, T] \times \mathbb{S}_q[\Omega]^-$ *such that*

$$u(t_0, x_0) = \min_{[0,T]\times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-} u,$$

then u *is constant in* $[0, t_0] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-$.

(iii) u *attains its maximum and its minimum in* $(\{0\} \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-) \cup (]0, T] \times \partial\mathbb{S}_q[\Omega])$.

Proof. Statement (iii) clearly follows from statements (i) and (ii). Moreover, statement (ii) follows from statement (i) applied to the function $-u$.

Therefore it suffices to consider statement (i). Let V be a bounded open connected subset of $\mathbb{R}^n$ such that

$$\mathrm{cl}\,Q \subseteq V, \quad \mathrm{cl}\,V \cap (qz + \mathrm{cl}\,\Omega) = \varnothing, \qquad \text{for all } z \in \mathbb{Z}^n \setminus \{0\}.$$

If $(t_0, x_0) \in]0, T] \times \mathbb{S}_q[\Omega]^-$ is a maximum point for u, then the periodicity of u implies that there exist a point $\bar{x}_0 \in \mathrm{cl}\,Q \setminus \mathrm{cl}\,\Omega$ such that $(t_0, \bar{x}_0) \in]0, T] \times (\mathrm{cl}\,Q \setminus \mathrm{cl}\,\Omega)$ is a point of maximum for u. Then the classical maximum principle for the heat equation (see *e.g.* Evans [33, Theorem 4 p.54]) applied to $u_{|[0,T]\times(\mathrm{cl}\,V\setminus\Omega)}$ implies that u is constant in $[0, t_0] \times (\mathrm{cl}\,V \setminus \Omega)$. Indeed, $\mathrm{cl}\,Q \setminus \mathrm{cl}\,\Omega \subseteq V \setminus \mathrm{cl}\,\Omega$. Finally the periodicity of u implies the validity of statement (i). □

Since we plan to find a solution of problem (3.23) in the form of a space-periodic double layer heat potential, we need to solve the related boundary integral equation. We do so by means of the following.

Lemma 3.13. *Let* $\alpha \in]0, 1[$, $T \in]0, +\infty[$, $m \in \mathbb{N} \setminus \{0\}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$ *such that* $\mathrm{cl}\,\Omega \subseteq Q$. *Let* $f \in C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$. *Then the integral equation*

$$\frac{1}{2}\mu + w_q[\partial_T\Omega, \mu]_{|\partial_T\Omega} = f \quad \text{ on } \partial_T\Omega, \tag{3.24}$$

has a unique solution μ *in* $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$.

Proof. By Corollary 3.11 (i), the operator $w_q[\partial_T\Omega, \cdot]_{|\partial_T\Omega}$ is compact from $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to itself. Accordingly, the Fredholm Alternative Theorem (see, *e.g.*, Kress [57, Theorem 4.17]) implies that, in order to conclude the proof, it suffices to show that the integral equation

$$\frac{1}{2}\mu + w_q[\partial_T\Omega, \mu]_{|\partial_T\Omega} = 0 \quad \text{on } \partial_T\Omega, \tag{3.25}$$

has the unique trivial solution $\mu = 0$ in $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$. If $\mu \in C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ solves (3.25), then Theorem 3.8 (i), the jump formula (3.20) for the q-periodic double layer heat potential and the maximum principle of Proposition 3.12 imply that

$$w_q^-[\partial_T\Omega, \mu] = 0 \qquad \text{in } [0, T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \tag{3.26}$$

Formula (3.21) implies that

$$\frac{\partial}{\partial\nu_\Omega(x)} w_q^+[\partial_T\Omega, \mu](t, x) = \frac{\partial}{\partial\nu_\Omega(x)} w_q^-[\partial_T\Omega, \mu](t, x) = 0 \qquad \forall\, (t, x) \in [0, T] \times \partial\Omega.$$

Then the function $w_q^+[\partial_T\Omega,\mu]$ solves the Neumann problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in }]0,T] \times \Omega, \\ \frac{\partial}{\partial \nu_\Omega} u = 0 & \text{on } [0,T] \times \partial\Omega, \\ u(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\Omega. \end{cases}$$

By the uniqueness of the classical Neumann problem, we have that also

$$w_q^+[\partial_T\Omega,\mu] = 0 \qquad \text{in } [0,T] \times \mathrm{cl}\,\Omega. \tag{3.27}$$

Finally, by summing the two jump formula in (3.20) for the interior and exterior q-periodic double layer heat potential, and by equalities (3.26) and (3.27), we have that

$$\mu = w_q^-[\partial_T\Omega,\mu] - w_q^+[\partial_T\Omega,\mu] = 0 \quad \text{on } [0,T] \times \partial\Omega.$$

Accordingly, the statement follows. □

Now we are ready to prove the following result concerning problem (3.23).

Theorem 3.14. *Let $\alpha \in]0,1[$, $T \in]0,+\infty[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $f \in C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$. Then there exists a unique function u in $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ which is one time continuously differentiable with respect to the time variable and two times continuously differentiable with respect to the space variables in $]0,T] \times \mathbb{S}_q[\Omega]^-$ and which solves problem (3.23). Moreover*

$$u = w_q^-[\partial_T\Omega,\mu] \quad \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-, \tag{3.28}$$

where μ is the unique solution in $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ of the integral equation (3.24).

Proof. We first note that the maximum principle of Proposition 3.12 implies that problem (3.23) has at most one solution. Then we only need to show that the function defined by (3.28) is a solution of problem (3.23). By Lemma 3.13 there exists a unique solution $\mu \in C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ of the integral equation (3.24). Then by Theorem 3.8 and by the equation (3.24) the function defined by (3.28) is a q-periodic function in $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ which solves the heat equation and which satisfies the Dirichlet boundary condition in (3.23), and thus is a solution of problem (3.23). □

3.6.2 A periodic Neumann problem

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $g \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. We now consider the following Neumann problem.

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in }]0,T] \times \mathbb{S}_q[\Omega]^-, \\ u(t,x+qe_i) = u(t,x) & \forall\,(t,x) \in [0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-, \forall\, i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_\Omega} u = g & \text{on } [0,T] \times \partial\Omega, \\ u(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases} \tag{3.29}$$

In order to apply an energy argument to prove the uniqueness of the solution of problem (3.29), we need the following approximation result for the set Ω. For a proof we refer to Verchota [107, Teorem 1.12, p. 581], where the author considers the more general case of a Lipschitz domain.

Lemma 3.15. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. Then the following statements hold.*

(i) There exist a sequence $(\Omega_j)_{j\in\mathbb{N}}$ of bounded open subsets of $\mathbb{R}^n$ of class C^∞ such that $\mathrm{cl}\,\Omega \subseteq \Omega_j$ for all $j \in \mathbb{N}$, and a sequence of homeomorphisms $(\Lambda_j)_{j\in\mathbb{N}}$ from $\partial\Omega$ to $\partial\Omega_j$ such that

$$\lim_{j\to+\infty} \sup_{x\in\partial\Omega} |x - \Lambda_j(x)| = 0.$$

(ii) For all $j \in \mathbb{N}$ there exists a function $\omega_j \in L^\infty(\partial\Omega)$ such that

$$\int_{\partial\Omega} (h \circ \Lambda_j(s))\omega_j(s)\, d\sigma_s = \int_{\Lambda_j(\partial\Omega)} h(y)\, d\sigma_y = \int_{\partial\Omega_j} h(y)\, d\sigma_y, \qquad \forall\, h \in C(\partial\Omega_j).$$

Moreover, ω_j converges to 1 in $L^p(\partial\Omega)$ as j tends to $+\infty$, for all $p \in [1, +\infty[$.

(iii) $\nu_{\Omega_j} \circ \Lambda_j$ converges to ν_Ω in $(L^p(\partial\Omega))^n$ as j tends to $+\infty$, for all $p \in [1, +\infty[$.

The same statement of Lemma 3.15 holds with a sequence $(\Omega'_j)_{j\in\mathbb{N}}$ of approximating sets from the inside of Ω. We are now ready to prove the following uniqueness result for problem (3.29).

Proposition 3.16. *Let $T \in]0, +\infty[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1 such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $u \in C_{0,q}^{\frac{1}{2};1}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ be one time continuously differentiable with respect to the time variable and two times continuously differentiable with respect to the space variables in $]0, T] \times \mathbb{S}_q[\Omega]^-$ and be such that*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in }]0, T] \times \mathbb{S}_q[\Omega]^-, \\ u(t, x + qe_i) = u(t, x) & \forall\, (t, x) \in [0, T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-, \forall\, i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial\nu_\Omega} u = 0 & \text{on } [0, T] \times \partial\Omega, \\ u(0, \cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases}$$

Then $u = 0$ in $[0, T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-$.

Proof. We apply Lemma 3.15 to the set Ω. Let Ω_j, Λ_j, ω_j for all $j \in \mathbb{N}$ be as in Lemma 3.15. We can clearly assume that $\mathrm{cl}\,\Omega_j \subseteq Q$ for all $j \in \mathbb{N}$. Moreover, as a consequence of Lemma 3.15 (i), we have that

$$\lim_{j\to+\infty} \mathbb{1}_{Q\setminus\mathrm{cl}\,\Omega_j}(x) = \mathbb{1}_{Q\setminus\mathrm{cl}\,\Omega}(x) \qquad \forall x \in Q \setminus \mathrm{cl}\,\Omega. \tag{3.30}$$

Let e and e_j be the functions from $[0, T]$ to $[0, +\infty[$ defined by

$$e(t) \equiv \int_{Q\setminus\mathrm{cl}\,\Omega} (u(t, y))^2\, dy, \qquad \forall\, t \in [0, T],$$
$$e_j(t) \equiv \int_{Q\setminus\mathrm{cl}\,\Omega_j} (u(t, y))^2\, dy \qquad \forall\, t \in [0, T], \forall\, j \in \mathbb{N}.$$

By the Dominated Convergence Theorem, $e, e_j \in C^0([0, T])$ for all $j \in \mathbb{N}$. In addition, the Dominated Convergence Theorem and (3.30) imply that

$$\lim_{j\to+\infty} e_j(t) = e(t) \qquad \forall\, t \in [0, T].$$

We now note that classical differentiation theorems for integral depending on a parameter imply that $e_j \in C^1(]0,T[)$ for all $j \in \mathbb{N}$ and, exploiting the Divergence Theorem, we have that

$$\begin{aligned}
\frac{d}{dt}e_j(t) &= 2\int_{Q\setminus \mathrm{cl}\,\Omega_j} u(t,y)\partial_t u(t,y)\,dy \qquad (3.31)\\
&= 2\int_{Q\setminus \mathrm{cl}\,\Omega_j} u(t,y)\Delta u(t,y)\,dy\\
&= -2\int_{Q\setminus \mathrm{cl}\,\Omega_j} |Du(t,y)|^2\,dy + 2\int_{\partial Q} u(t,y)\frac{\partial}{\partial \nu_Q(y)}u(t,y)\,d\sigma_y\\
&\quad -2\int_{\partial\Omega_j} u(t,y)\frac{\partial}{\partial \nu_{\Omega_j}(y)}u(t,y)\,d\sigma_y\\
&= -2\int_{Q\setminus \mathrm{cl}\,\Omega_j} |Du(t,y)|^2\,dy - 2\int_{\partial\Omega_j} u(t,y)\frac{\partial}{\partial \nu_{\Omega_j}(y)}u(t,y)\,d\sigma_y,
\end{aligned}$$

for all $t \in]0,T[$ and for all $j \in \mathbb{N}$. Indeed, the q-periodicity of u implies that

$$\int_{\partial Q} u(t,y)\frac{\partial}{\partial \nu_Q(y)}u(t,y)\,d\sigma_y = 0, \qquad \forall\, t \in]0,T[.$$

Now we turn to consider the second integral in the right hand side of (3.31). Lemma 3.15 (ii) implies that

$$\begin{aligned}
&\int_{\partial\Omega_j} u(t,y)\frac{\partial}{\partial \nu_{\Omega_j(y)}}u(t,y)\,d\sigma_y\\
&\qquad = \int_{\partial\Omega_j} u(t,y)Du(t,y)\cdot \nu_{\Omega_j}(y)\,d\sigma_y\\
&\qquad = \int_{\partial\Omega} u(t,\Lambda_j(s))Du(t,\Lambda_j(s))\cdot \nu_{\Omega_j}(\Lambda_j(s))\omega_j(s)\,d\sigma_s
\end{aligned}$$

for all $t \in]0,T[$ and for all $j \in \mathbb{N}$. We note that Lemma 3.15 and the membership of u in $C^{\frac{1}{2};1}_{0,q}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-_T)$ imply that $u(t,\Lambda_j(\cdot))$ converges to $u(t,\cdot)$ in $C^0(\partial\Omega)$ as j tends to $+\infty$, and $Du(t,\Lambda_j(\cdot))$ converges to $Du(t,\cdot)$ in $(C^0(\partial\Omega))^n$ as j tends to $+\infty$, both uniformly in $t \in [0,T]$. We now fix $p \in]0,+\infty[$ and we set $p' \equiv \frac{p}{p-1} \in]1,+\infty[$. Lemma 3.15 (ii), (iii) implies that ω_j converges to 1 in $L^p(\partial\Omega)$ as j tend to $+\infty$, and $\nu_{\Omega_j}(\Lambda_j(\cdot))$ converges to ν_Ω in $(L^{p'}(\partial\Omega))^n$ as j tends to $+\infty$. Then the Hölder inequality implies that

$$u(t,\Lambda_j(\cdot))Du(t,\Lambda_j(\cdot))\cdot \nu_{\Omega_j}(\Lambda_j(\cdot))\omega_j$$

converges to

$$u(t,\cdot)Du(t,\cdot)\cdot \nu_\Omega$$

in $L^1(\partial\Omega)$ as j tends to $+\infty$, uniformly in $t \in [0,T]$. In particular we have that

$$\begin{aligned}
&\lim_{j\to+\infty}\int_{\partial\Omega} u(t,\Lambda_j(s))Du(t,\Lambda_j(s))\cdot \nu_{\Omega_j}(\Lambda_j(s))\omega_j(s)\,d\sigma_y\\
&\qquad = \int_{\partial\Omega} u(t,y)\frac{\partial}{\partial \nu_\Omega(y)}u(t,y)\,d\sigma_y = 0,
\end{aligned}$$

uniformly in $t \in [0,T]$. Moreover, we note that

$$\left|\int_{Q\setminus \mathrm{cl}\,\Omega} |Du(t,y)|^2\,dy - \int_{Q\setminus \mathrm{cl}\,\Omega_j} |Du(t,y)|^2\,dy\right|$$

$$\leq \sup_{[0,T]\times(Q\setminus\mathrm{cl}\,\Omega)} |Du|^2 \int_{Q\setminus\mathrm{cl}\,\Omega} |\mathbb{1}_{Q\setminus\mathrm{cl}\,\Omega}(y) - \mathbb{1}_{Q\setminus\mathrm{cl}\,\Omega_j}(y)|\,dy \qquad \forall t \in [0,T],\ \forall j \in \mathbb{N}.$$

Accordingly, the Dominated Convergence Theorem and (3.30) imply that

$$\lim_{j\to+\infty} \int_{Q\setminus\mathrm{cl}\,\Omega_j} |Du(t,y)|^2\,dy = \int_{Q\setminus\mathrm{cl}\,\Omega} |Du(t,y)|^2\,dy, \qquad \text{uniformly in } t \in [0,T].$$

Accordingly, $e \in C^1(]0,T[)$ and

$$\frac{d}{dt}e(t) = -2\int_{Q\setminus\mathrm{cl}\,\Omega} |Du(t,y)|^2\,dy \qquad \forall\, t \in]0,T[. \tag{3.32}$$

Equality (3.32) implies that $\frac{d}{dt}e \leq 0$ in $]0,T[$. Since $e(0) = 0$ and $e \geq 0$ in $[0,T]$, then $e = 0$ for all in $[0,T]$. Accordingly, $u = 0$ in $[0,T] \times \mathrm{cl}\,Q \setminus \Omega$ and thus the q-periodicity of u implies the validity of the statement. □

Since we plan to find a solution of problem (3.29) in the form of a space-periodic single layer heat potential, we need to solve the related boundary integral equation. We do so by means of the following.

Lemma 3.17. *Let $\alpha \in]0,1[$, $T \in]0,+\infty[$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $g \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. Then the integral equation*

$$-\frac{1}{2}\mu + w_{q,*}[\partial_T\Omega,\mu] = g \quad \text{in } \partial_T\Omega, \tag{3.33}$$

has a unique solution μ in $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$.

Proof. By Corollary 3.11 (ii), the operator $w_{*,q}[\partial_T\Omega,\cdot]$ is compact from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to itself. Accordingly, the Fredholm Alternative Theorem (see, *e.g.*, Kress [57, Theorem 4.17]) implies that, in order to conclude the proof, it suffices to show that the integral equation

$$-\frac{1}{2}\mu + w_{q,*}[\partial_T\Omega,\mu] = 0 \quad \text{on } \partial_T\Omega, \tag{3.34}$$

has the unique trivial solution $\mu = 0$ in $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. If $\mu \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ solves (3.34), then Theorem 3.7 (i), (iv), the jump formula (3.16) for the normal derivative of the q-periodic single layer heat potential and Proposition 3.16 on the uniqueness of the Neumann problem in $[0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-$ imply that

$$v_q^-[\partial_T\Omega,\mu] = 0 \qquad \text{in } [0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-.$$

The continuity of the q-periodic single layer heat potential (cf. Theorem 3.7 (i)) implies that also $v_q^+[\partial_T\Omega,\mu] = 0$ in $[0,T] \times \partial\mathbb{S}_q[\Omega]$. Hence, $v_q^+[\partial_T\Omega,\mu]$ solves the Dirichlet problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in }]0,T] \times \Omega, \\ u = 0 & \text{on } [0,T] \times \partial\Omega, \\ u(0,\cdot) = 0 & \text{in } \Omega. \end{cases}$$

By the uniqueness of the classical Dirichlet problem, we have that also

$$v_q^+[\partial_T\Omega,\mu] = 0 \qquad \text{in } [0,T] \times \mathrm{cl}\,\Omega.$$

Hence

$$\frac{\partial}{\partial\nu_\Omega}v_q^+[\partial_T\Omega,\mu]=\frac{\partial}{\partial\nu_\Omega}v_q^-[\partial_T\Omega,\mu]=0 \qquad \text{on } [0,T]\times\partial\Omega. \tag{3.35}$$

Finally, by summing up the two jump formulas (3.16) for the normal derivative of the interior and exterior q-periodic single layer heat potential, and by equality (3.35), we have that

$$\mu=\frac{\partial}{\partial\nu_\Omega}v_q^+[\partial_T\Omega,\mu]-\frac{\partial}{\partial\nu_\Omega}v_q^-[\partial_T\Omega,\mu]=0 \quad \text{on } [0,T]\times\partial\Omega.$$

Accordingly, the statement follows. □

Now we are ready to prove the following result concerning problem (3.29).

Theorem 3.18. *Let $\alpha\in]0,1[$, $T\in]0,+\infty[$, $m\in\mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega\subseteq Q$. Let $g\in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. Then there exists a unique function u in $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ which is one time continuously differentiable with respect to the time variable and two times continuously differentiable with respect to the space variables in $]0,T]\times\mathbb{S}_q[\Omega]^-$ and which solves problem (3.29). Moreover*

$$u=v_q^-[\partial_T\Omega,\mu] \quad \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-, \tag{3.36}$$

where μ is the unique solution in $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ of the integral equation (3.33).

Proof. Proposition 3.16 implies that problem (3.29) has at most one solution. Then we only need to show that the function defined by (3.36) is a solution of problem (3.29). By Lemma 3.17 there exists a unique solution $\mu\in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ of the integral equation (3.33) and then by Theorem 3.7 and by equation (3.33) the function defined by (3.36) is a q-periodic function in $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ which solves the heat equation and which satisfies the Neumann boundary condition in (3.29), and thus it is a solution of problem (3.29). □

3.6.3 A periodic non-ideal transmission problem

Finally we consider a periodic transmission problem, which models a heat diffusion in a two-composite material with thermal resistance at the interface. Let $\lambda^+,\lambda^-,\gamma\in]0,+\infty[$ and let $f,g\in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. We consider the following non-ideal transmission problem.

$$\begin{cases}
\partial_t u^+-\Delta u^+=0 & \text{in }]0,T]\times\mathbb{S}_q[\Omega],\\
\partial_t u^--\Delta u^-=0 & \text{in }]0,T]\times\mathbb{S}_q[\Omega]^-,\\
u^+(t,x+qe_i)=u^+(t,x) & \forall\,(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega],\forall\,i\in\{1,\dots,n\},\\
u^-(t,x+qe_i)=u^-(t,x) & \forall\,(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-,\forall\,i\in\{1,\dots,n\},\\
\lambda^+\frac{\partial}{\partial\nu_\Omega}u^++\gamma(u^+-u^-)=f & \text{on } [0,T]\times\partial\Omega,\\
\lambda^-\frac{\partial}{\partial\nu_\Omega}u^--\lambda^+\frac{\partial}{\partial\nu_\Omega}u^+=g & \text{on } [0,T]\times\partial\Omega,\\
u^+(0,\cdot)=0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega],\\
u^-(0,\cdot)=0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-.
\end{cases} \tag{3.37}$$

The fifth condition of system (3.37) is the non-ideal transmission (or imperfect contact) condition, which models the thermal resistance at the interface. In particular this condition says that the temperature field at the interface displays a jump proportional to the normal heat flux. This discontinuity of the temperature field is a well know phenomenon in physics which has been studied since the work of Kapitza in 1941, in which the author has studied for the first time the thermal interface behavior in liquid helium (see, e.g., Swartz and Pohl [102], Lipton [74] and references therein). We mention also the works of Donato and Jose [30], [31] for the study of the asymptotic behavior of the approximate control of a similar parabolic transmission problem and the work of Dalla Riva and Musolino [23] in which the authors consider a singularly perturbed stationary version of this transmission problem in order to study the effective conductivity of a periodic composite.

We have the following uniqueness result for the problem (3.37).

Proposition 3.19. *Let $T \in]0,+\infty[$. Let $\lambda^+, \lambda^-, \gamma \in]0,+\infty[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1 such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $u^+ \in C^{\frac{1}{2};1}_{0,q}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$, $u^- \in C^{\frac{1}{2};1}_{0,q}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-_T)$ be one time continuously differentiable with respect to the time variable and two times continuously differentiable with respect to the space variables in $]0,T]\times\mathbb{S}_q[\Omega]$ and $]0,T]\times\mathbb{S}_q[\Omega]^-$, respectively. Moreover, we assume that*

$$\begin{cases} \partial_t u^+ - \Delta u^+ = 0 & \text{in }]0,T[\times\mathbb{S}_q[\Omega], \\ \partial_t u^- - \Delta u^- = 0 & \text{in }]0,T[\times\mathbb{S}_q[\Omega]^-, \\ u^+(t,x+qe_i) = u^+(t,x) & \forall\,(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega],\ \forall\, i\in\{1,\dots,n\}, \\ u^-(t,x+qe_i) = u^-(t,x) & \forall\,(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-,\ \forall\, i\in\{1,\dots,n\}, \\ \lambda^+\frac{\partial}{\partial\nu_\Omega}u^+ + \gamma(u^+-u^-) = 0 & \text{on } [0,T]\times\partial\Omega, \\ \lambda^-\frac{\partial}{\partial\nu_\Omega}u^- - \lambda^+\frac{\partial}{\partial\nu_\Omega}u^+ = 0 & \text{on } [0,T]\times\partial\Omega, \\ u^+(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega], \\ u^-(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases} \tag{3.38}$$

Then $u^+ = 0$ in $[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]$ and $u^- = 0$ in $[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-$.

Proof. Let $(u^+,u^-) \in C^{\frac{1}{2};1}_{0,q}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T) \times C^{\frac{1}{2};1}_{0,q}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-_T)$ be a solution of problem (3.38). Let Ω_j, Λ_j, ω_j for all $j\in\mathbb{N}$ be as in Lemma 3.15 associated with an approximation of Ω from the outside, and let Ω'_j, Λ'_j, ω'_j for all $j\in\mathbb{N}$ be as in Lemma 3.15 associated with an approximation of Ω from the inside. We can clearly assume that $\mathrm{cl}\,\Omega_j \subseteq Q$ for all $j\in\mathbb{N}$. Moreover, as a consequence of Lemma 3.15 (i), we have that

$$\begin{aligned} &\lim_{j\to+\infty} \mathbb{1}_{Q\setminus\mathrm{cl}\,\Omega_j}(x) = \mathbb{1}_{Q\setminus\mathrm{cl}\,\Omega}(x) && \forall x\in Q\setminus\mathrm{cl}\,\Omega, \\ &\lim_{j\to+\infty} \mathbb{1}_{\Omega'_j}(x) = \mathbb{1}_{\Omega}(x) && \forall x\in \Omega. \end{aligned} \tag{3.39}$$

Let e^+, e^-, e^+_j, e^-_j be the functions from $[0,T]$ to $[0,+\infty[$ defined by

$$\begin{aligned} e^+(t) &\equiv \int_\Omega (u^+(t,y))^2\,dy, && \forall\, t\in[0,T], \\ e^-(t) &\equiv \int_{Q\setminus\mathrm{cl}\,\Omega} (u^-(t,y))^2\,dy, && \forall\, t\in[0,T], \\ e^+_j(t) &\equiv \int_{\Omega'_j} (u^+(t,y))^2\,dy && \forall\, t\in[0,T],\ \forall\, j\in\mathbb{N}, \\ e^-_j(t) &\equiv \int_{Q\setminus\mathrm{cl}\,\Omega_j} (u^-(t,y))^2\,dy && \forall\, t\in[0,T],\ \forall\, j\in\mathbb{N}. \end{aligned}$$

By the Dominated Convergence Theorem, $e^+, e^-, e_j^+, e_j^- \in C^0([0,T])$ for all $j \in \mathbb{N}$. In addition, the Dominated Convergence Theorem and (3.39) imply that

$$\lim_{j\to+\infty} e_j(t) = e^+(t) \qquad \forall\, t \in [0,T],$$
$$\lim_{j\to+\infty} e_j(t) = e^-(t) \qquad \forall\, t \in [0,T].$$

We now note that classical differentiation theorems for integral depending on a parameter imply that $e_j^+, e_j^- \in C^1(]0,T[)$ for all $j \in \mathbb{N}$ and, exploiting the Divergence Theorem, we have that

$$\begin{aligned}\frac{d}{dt}e_j^+(t) =& 2\int_{\Omega_j'} u^+(t,y)\partial_t u^+(t,y)\,dy = 2\int_{\Omega_j'} u^+(t,y)\Delta u(t,y)\,dy \\ =& -2\int_{\Omega_j'} |Du^+(t,y)|^2\,dy + 2\int_{\partial\Omega_j'} u^+(t,y)\frac{\partial}{\partial\nu_{\Omega_j'}(y)}u^+(t,y)\,d\sigma_y,\end{aligned} \tag{3.40}$$

for all $t \in]0,T[$ and for all $j \in \mathbb{N}$. Moreover, in a similar way, exploiting another time the Divergence Theorem we have that

$$\begin{aligned}\frac{d}{dt}e_j^-(t) =& 2\int_{Q\setminus \mathrm{cl}\,\Omega_j} u^-(t,y)\partial_t u^-(t,y)\,dy = 2\int_{Q\setminus \mathrm{cl}\,\Omega_j} u^-(t,y)\Delta u(t,y)\,dy \\ =& -2\int_{Q\setminus \mathrm{cl}\,\Omega_j} |Du^-(t,y)|^2\,dy + 2\int_{\partial Q} u^-(t,y)\frac{\partial}{\partial\nu_Q(y)}u^-(t,y)\,d\sigma_y \\ & -2\int_{\partial\Omega_j} u^-(t,y)\frac{\partial}{\partial\nu_{\Omega_j}(y)}u^-(t,y)\,d\sigma_y \\ =& -2\int_{Q\setminus \mathrm{cl}\,\Omega_j} |Du^-(t,y)|^2\,dy - 2\int_{\partial\Omega_j} u^-(t,y)\frac{\partial}{\partial\nu_{\Omega_j}(y)}u(t,y)\,d\sigma_y,\end{aligned} \tag{3.41}$$

for all $t \in]0,T[$ and for all $j \in \mathbb{N}$. Indeed, the q-periodicity of u implies that

$$\int_{\partial Q} u^-(t,y)\frac{\partial}{\partial\nu_Q(y)}u^-(t,y)\,d\sigma_y = 0, \qquad \forall\, t \in]0,T[.$$

Next, we turn to consider the second integral in the right hand side of (3.40). Lemma 3.15 (ii) implies that

$$\begin{aligned}&\int_{\partial\Omega_j'} u^+(t,y)\frac{\partial}{\partial\nu_{\Omega_j'(y)}}u^+(t,y)\,d\sigma_y \\ &\qquad = \int_{\partial\Omega_j'} u^+(t,y)Du^+(t,y)\cdot\nu_{\Omega_j'}(y)\,d\sigma_y \\ &\qquad = \int_{\partial\Omega} u^+(t,\Lambda_j(s))Du^+(t,\Lambda_j'(s))\cdot\nu_{\Omega_j'}(\Lambda_j'(s))\omega_j'(s)\,d\sigma_s\end{aligned}$$

for all $t \in]0,T[$ and for all $j \in \mathbb{N}$. We note that Lemma 3.15 and the membership of u^+ in $C_{0,q}^{\frac{1}{2};1}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ imply that $u^+(t,\Lambda_j'(\cdot))$ converges to $u^+(t,\cdot)$ in $C^0(\partial\Omega)$ as j tends to $+\infty$, and $Du^+(t,\Lambda_j'(\cdot))$ converges to $Du^+(t,\cdot)$ in $(C^0(\partial\Omega))^n$ as j tends to $+\infty$, both uniformly in $t \in [0,T]$. We now fix $p \in]0,+\infty[$ and we set $p' \equiv \frac{p}{p-1} \in]1,+\infty[$. Lemma 3.15 (ii), (iii) implies that ω_j' converges to 1 in $L^p(\partial\Omega)$ as j tend to $+\infty$, and $\nu_{\Omega_j'}(\Lambda_j'(\cdot))$ converges to ν_Ω in $(L^{p'}(\partial\Omega))^n$ as j tends to $+\infty$. Then the Hölder inequality implies that

$$u^+(t,\Lambda_j'(\cdot))Du^+(t,\Lambda_j'(\cdot))\cdot\nu_{\Omega_j'}(\Lambda_j'(\cdot))\omega_j'$$

converges to

$$u^+(t,\cdot)Du^+(t,\cdot)\cdot\nu_\Omega$$

in $L^1(\partial\Omega)$ as j tends to $+\infty$, uniformly in $t\in[0,T]$. In particular we have that

$$\begin{aligned}\lim_{j\to+\infty}\int_{\partial\Omega}u^+(t,\Lambda'_j(s))Du^+(t,\Lambda'_j(s))\cdot\nu_{\Omega'_j}(\Lambda'_j(s))\omega'_j(s)\,d\sigma_y\\ =\int_{\partial\Omega}u^+(t,y)\frac{\partial}{\partial\nu_\Omega(y)}u^+(t,y)\,d\sigma_y,\end{aligned}$$

uniformly in $t\in[0,T]$. Moreover, we note that

$$\begin{aligned}&\left|\int_\Omega|Du^+(t,y)|^2\,dy-\int_{\Omega'_j}|Du^+(t,y)|^2\,dy\right|\\ &\qquad\le\sup_{[0,T]\times\Omega}|Du^+|^2\int_\Omega|\mathbb{1}_{Q\setminus\mathrm{cl}\,\Omega}(y)-\mathbb{1}_{Q\setminus\mathrm{cl}\,\Omega'_j}(y)|\,dy\qquad\forall t\in[0,T],\ \forall j\in\mathbb{N}.\end{aligned}$$

Accordingly, the Dominated Convergence Theorem and (3.39) implies that

$$\lim_{j\to+\infty}\int_{\Omega'_j}|Du^+(t,y)|^2\,dy=\int_\Omega|Du^+(t,y)|^2\,dy,\qquad\text{uniformly in } t\in[0,T].$$

Accordingly equality (3.40) implies that $e^+\in C^1(]0,T[)$ and

$$\frac{d}{dt}e^+(t)=-2\int_\Omega|Du^+(t,y)|^2\,dy+2\int_{\partial\Omega}u^+(t,y)\frac{\partial}{\partial\nu_\Omega(y)}u^+(t,y)\,d\sigma_y\qquad\forall\,t\in]0,T[.\tag{3.42}$$

Next, we consider the second integral in the right hand side of (3.41). Lemma 3.15 (ii) implies that

$$\begin{aligned}&\int_{\partial\Omega_j}u^-(t,y)\frac{\partial}{\partial\nu_{\Omega_j(y)}}u^-(t,y)\,d\sigma_y\\ &\qquad=\int_{\partial\Omega_j}u^-(t,y)Du^-(t,y)\cdot\nu_{\Omega'_j}(y)\,d\sigma_y\\ &\qquad=\int_{\partial\Omega}u^-(t,\Lambda_j(s))Du^-(t,\Lambda_j(s))\cdot\nu_{\Omega_j}(\Lambda_j(s))\omega_j(s)\,d\sigma_s\end{aligned}$$

for all $t\in]0,T[$ and for all $j\in\mathbb{N}$. We note that Lemma 3.15 and the membership of u^- in $C^{\frac{1}{2};1}_{0,q}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-_T)$ imply that $u^-(t,\Lambda_j(\cdot))$ converges to $u^-(t,\cdot)$ in $C^0(\partial\Omega)$ as j tends to $+\infty$, and $Du^-(t,\Lambda_j(\cdot))$ converges to $Du^-(t,\cdot)$ in $(C^0(\partial\Omega))^n$ as j tends to $+\infty$, both uniformly in $t\in[0,T]$. Now we fix $p\in]0,+\infty[$ and we set $p'\equiv\frac{p}{p-1}\in]1,+\infty[$. Lemma 3.15 (ii), (iii) implies that ω_j converges to 1 in $L^p(\partial\Omega)$ as j tend to $+\infty$, and $\nu_{\Omega_j}(\Lambda_j(\cdot))$ converges to ν_Ω in $(L^{p'}(\partial\Omega))^n$ as j tends to $+\infty$. Then the Hölder inequality implies that

$$u^-(t,\Lambda_j(\cdot))Du^-(t,\Lambda_j(\cdot))\cdot\nu_{\Omega_j}(\Lambda_j(\cdot))\omega_j$$

converges to

$$u^-(t,\cdot)Du^-(t,\cdot)\cdot\nu_\Omega$$

in $L^1(\partial\Omega)$ as j tends to $+\infty$, uniformly in $t\in[0,T]$. In particular we have that

$$\lim_{j\to+\infty}\int_{\partial\Omega}u^-(t,\Lambda_j(s))Du^-(t,\Lambda_j(s))\cdot\nu_{\Omega_j}(\Lambda_j(s))\omega_j(s)\,d\sigma_y$$

$$= \int_{\partial\Omega} u^-(t,y) \frac{\partial}{\partial\nu_\Omega(y)} u^-(t,y)\, d\sigma_y,$$

uniformly in $t \in [0,T]$. Moreover, we note that

$$\left| \int_{Q\setminus \mathrm{cl}\,\Omega} |Du^-(t,y)|^2\, dy - \int_{Q\setminus \mathrm{cl}\,\Omega_j} |Du^-(t,y)|^2\, dy \right|$$
$$\leq \sup_{[0,T]\times(Q\setminus \mathrm{cl}\,\Omega)} |Du^-|^2 \int_{Q\setminus \mathrm{cl}\,\Omega} |\mathbb{1}_{Q\setminus \mathrm{cl}\,\Omega}(y) - \mathbb{1}_{Q\setminus \mathrm{cl}\,\Omega_j}(y)|\, dy \qquad \forall t \in [0,T],\ \forall j \in \mathbb{N}.$$

Thus, the Dominated Convergence Theorem and (3.39) imply that

$$\lim_{j\to+\infty} \int_{Q\setminus \mathrm{cl}\,\Omega_j} |Du^-(t,y)|^2\, dy = \int_{Q\setminus \mathrm{cl}\,\Omega} |Du^-(t,y)|^2\, dy \qquad \text{uniformly in } t \in [0,T].$$

Accordingly, equality (3.41) implies that $e^- \in C^1(]0,T[)$ and

$$\frac{d}{dt} e^-(t) = -2 \int_{Q\setminus \mathrm{cl}\,\Omega} |Du^-(t,y)|^2\, dy - 2 \int_{\partial\Omega} u^-(t,y) \frac{\partial}{\partial\nu_\Omega(y)} u^-(t,y)\, d\sigma_y \qquad \forall\, t \in]0,T[. \tag{3.43}$$

Then, if we set $e \equiv \lambda^+ e^+ + \lambda^- e^-$, equalities (3.42), (3.43) and the transmission boundary conditions in problem (3.38) imply that

$$\begin{aligned}
\frac{d}{dt} e(t) =& -2 \left(\lambda^+ \int_\Omega |Du^+(t,y)|^2\, dy + \lambda^- \int_{Q\setminus \mathrm{cl}\,\Omega} |Du^-(t,y)|^2\, dy \right) \\
&+ 2\lambda^+ \int_{\partial\Omega} u^+(t,y) \frac{\partial}{\partial\nu_\Omega(y)} u^+(t,y)\, d\sigma_y - 2\lambda^- \int_{\partial\Omega} u^-(t,y) \frac{\partial}{\partial\nu_\Omega(y)} u^-(t,y)\, d\sigma_y \\
=& -2 \left(\lambda^+ \int_\Omega |Du^+(t,y)|^2\, dy + \lambda^- \int_{Q\setminus \mathrm{cl}\,\Omega} |Du^-(t,y)|^2\, dy \right) \\
&+ 2\lambda^+ \int_{\partial\Omega} u^+(t,y) \frac{\partial}{\partial\nu_\Omega(y)} u^+(t,y)\, d\sigma_y - 2\lambda^+ \int_{\partial\Omega} u^-(t,y) \frac{\partial}{\partial\nu_\Omega(y)} u^+(t,y)\, d\sigma_y \\
=& -2 \left(\lambda^+ \int_\Omega |Du^+(t,y)|^2\, dy + \lambda^- \int_{Q\setminus \mathrm{cl}\,\Omega} |Du^-(t,y)|^2\, dy \right) \\
&+ 2\lambda^+ \int_{\partial\Omega} (u^+(t,y) - u^-(t,x)) \frac{\partial}{\partial\nu_\Omega(y)} u^+(t,y)\, d\sigma_y \\
=& -2 \left(\lambda^+ \int_\Omega |Du^+(t,y)|^2\, dy + \lambda^- \int_{Q\setminus \mathrm{cl}\,\Omega} |Du^-(t,y)|^2\, dy \right) \\
&- \frac{2}{\gamma} \int_{\partial\Omega} \left(\lambda^+ \frac{\partial}{\partial\nu_\Omega(y)} u^+(t,y) \right)^2 d\sigma_y \qquad \forall\, t \in]0,T[.
\end{aligned}$$

Hence $\frac{d}{dt} e \leq 0$ in $]0,T[$. Since $e(0) = 0$ and $e \geq 0$ in $[0,T]$, then $e = 0$ for all in $[0,T]$. Then $u^+ = 0$ in $[0,T] \times \mathrm{cl}\,\Omega$, $u^- = 0$ in $[0,T] \times \mathrm{cl}\,Q \setminus \Omega$ and thus the q-periodicity of u^+, u^- implies the validity of the statement. □

Since we plan to solve the problem (3.37) with two space-periodic single layer heat potentials, we need to solve the related boundary integral equation. We do so by means of the following.

Lemma 3.20. *Let $\alpha \in]0,1[$, $T \in]0,+\infty[$, $m \in \mathbb{N}\setminus\{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $J \equiv (J_1, J_2)$ be the operator from $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ to $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ defined by*

$$
\begin{aligned}
J_1[\mu^+,\mu^-] \equiv& \lambda^+\left(\frac{1}{2}\mu^+ + w_{q,*}[\partial_T\Omega,\mu^+]\right) + \gamma(v_q^+[\partial_T\Omega,\mu^+]_{|\partial_T\Omega} - v_q^-[\partial_T\Omega,\mu^-]_{|\partial_T\Omega}), \qquad (3.44)\\
J_2[\mu^+,\mu^-] \equiv& \lambda^-\left(-\frac{1}{2}\mu^- + w_{q,*}[\partial_T\Omega,\mu^-]\right) - \lambda^+\left(\frac{1}{2}\mu^+ + w_{q,*}[\partial_T\Omega,\mu^+]\right),
\end{aligned}
$$

for all $(\mu^+,\mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$. Then J is a linear homeomorphism.

Proof. Let $\tilde{J} \equiv (\tilde{J}_1, \tilde{J}_2)$ be the linear operator from the space $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ to $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ defined by

$$
\begin{aligned}
\tilde{J}_1[\mu^+,\mu^-] \equiv& \frac{\lambda^+}{2}\mu^+,\\
\tilde{J}_2[\mu^+,\mu^-] \equiv& -\frac{\lambda^-}{2}\mu^- - \frac{\lambda^+}{2}\mu^+,
\end{aligned}
$$

for all $(\mu^+,\mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$. Clearly $\tilde{J}$ is a linear homeomorphism.

Moreover, let $\bar{J} \equiv (\bar{J}_1, \bar{J}_2)$ be the linear operator from the space $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ to $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ defined by

$$
\begin{aligned}
\bar{J}_1[\mu^+,\mu^-] \equiv& \lambda^+ w_{q,*}[\partial_T\Omega,\mu^+] + \gamma(v_q^+[\partial_T\Omega,\mu^+]_{|\partial_T\Omega} - v_q^-[\partial_T\Omega,\mu^-]_{|\partial_T\Omega}),\\
\bar{J}_2[\mu^+,\mu^-] \equiv& \lambda^- w_{q,*}[\partial_T\Omega,\mu^-] - \lambda^+ w_{q,*}[\partial_T\Omega,\mu^+],
\end{aligned}
$$

for all $(\mu^+,\mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$. By Corollary 3.11 (ii), the operator $w_{*,q}[\partial_T\Omega,\cdot]$ is compact from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to itself. By Theorem 3.7 (iv), the map $v_q^+[\partial_T\Omega,\cdot]$ is linear and continuous from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and the map $v_q^-[\partial_T\Omega,\cdot]$ is linear and continuous from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$. Then, by the linearity and continuity of the trace operators from $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ and from $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$, and by the compactness of the embedding of $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ into $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$, which is a consequence of Remark 1.10 and of Lemma 2.17, the operators $v_q^+[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and $v_q^-[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ are compact in $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. Then the operator $\bar{J}$ is compact in $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$. Since compact perturbation of linear homeomorphism are Fredholm operators of index 0, we have that

$$J = \tilde{J} + \bar{J}$$

is a Fredholm operator of index 0. Hence, in order to show that J is a linear homeomorphism, it suffices to show that J is injective. Let $(\mu^+,\mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ such that $J[\mu^+,\mu^-] = (0,0)$. By Theorem 3.7 the functions $v_q^+[\partial_T\Omega,\mu^+]$ and $v_q^-[\partial_T\Omega,\mu^-]$ satisfy the assumptions of Proposition 3.19 and then

$$v_q^+[\partial_T\Omega,\mu^+] = 0 \qquad \text{in } [0,T]\times \mathrm{cl}\,\mathbb{S}_q[\Omega]$$

and

$$v_q^-[\partial_T\Omega,\mu^-]=0 \qquad \text{in } [0,T]\times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-.$$

By the continuity of the single layer heat potential we have also that $v_q^+[\partial_T\Omega,\mu^-]=0$ in $[0,T]\times\partial\mathbb{S}_q[\Omega]$ (see Theorem 3.7 (i)). Hence, $v_q^+[\partial_T\Omega,\mu^-]$ solves the Dirichlet problem

$$\begin{cases} \partial_t u-\Delta u=0 & \text{in }]0,T]\times\Omega,\\ u=0 & \text{on } [0,T]\times\partial\Omega,\\ u(0,\cdot)=0 & \text{in } \mathrm{cl}\,\Omega. \end{cases}$$

The uniqueness of the classical Dirichlet problem implies that also

$$v_q^+[\partial_T\Omega,\mu^-]=0 \qquad \text{in } [0,T]\times\mathrm{cl}\,\Omega.$$

Hence

$$\frac{\partial}{\partial\nu_\Omega}v_q^+[\partial_T\Omega,\mu^-]=\frac{\partial}{\partial\nu_\Omega}v_q^-[\partial_T\Omega,\mu^-]=0 \qquad \text{on } [0,T]\times\partial\Omega.$$

The two jump formulas for the normal derivative of the interior and exterior q-periodic single layer potential of Theorem 3.7 (v) implies that

$$\mu^-=\frac{\partial}{\partial\nu_\Omega}v_q^+[\partial_T\Omega,\mu^-]-\frac{\partial}{\partial\nu_\Omega}v_q^-[\partial_T\Omega,\mu^-]=0 \quad \text{on } [0,T]\times\partial\Omega.$$

Using another time the continuity of the single layer heat potential we have that $v_q^-[\partial_T\Omega,\mu^+]=v_q^+[\partial_T\Omega,\mu^+]=0$ in $[0,T]\times\partial\mathbb{S}_q[\Omega]$ (see Theorem 3.7 (i)). Accordingly, $v_q^-[\partial_T\Omega,\mu^+]$ solves the periodic Dirichlet problem

$$\begin{cases} \partial_t u-\Delta u=0 & \text{in }]0,T]\times\mathbb{S}_q[\Omega]^-,\\ u(t,x+qe_i)=u(t,x) & \forall(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-,\ \forall i\in\{1,\dots,n\},\\ u=0 & \text{on } [0,T]\times\partial\Omega,\\ u(0,\cdot)=0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases}$$

Hence, the maximum principle of Proposition 3.12 implies that also

$$v_q^-[\partial_T\Omega,\mu^+]=0 \qquad \text{in } [0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-.$$

Hence

$$\frac{\partial}{\partial\nu_\Omega}v_q^-[\partial_T\Omega,\mu^+]=\frac{\partial}{\partial\nu_\Omega}v_q^+[\partial_T\Omega,\mu^+]=0 \qquad \text{on } [0,T]\times\partial\Omega.$$

Finally, the two jump formulas for the normal derivative of the interior and exterior q-periodic single layer potential of Theorem 3.7 (v) implies that

$$\mu^+=\frac{\partial}{\partial\nu_\Omega}v_q^+[\partial_T\Omega,\mu^+]-\frac{\partial}{\partial\nu_\Omega}v_q^-[\partial_T\Omega,\mu^+]=0 \quad \text{on } [0,T]\times\partial\Omega,$$

and the statement follows. □

Finally we are ready to prove the following result concerning the non-ideal transmission problem (3.37).

Theorem 3.21. *Let* $\alpha \in]0,1[$, $T \in]0,+\infty[$, $m \in \mathbb{N} \setminus \{0\}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$ *such that* $\mathrm{cl}\,\Omega \subseteq Q$. *Let* $f, g \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. *Then problem (3.37) has a unique solution* $(u^+, u^-) \in C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T) \times C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$. *Moreover*

$$u^+ = v_q^+[\partial_T\Omega, \mu^+] \quad \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]_T, \tag{3.45}$$
$$u^- = v_q^-[\partial_T\Omega, \mu^-] \quad \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-,$$

where (μ^+, μ^-) *is the unique solution in* $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ *of the system of integral equations*

$$J[\mu^+, \mu^-] = (f, g) \quad \text{on } \partial_T\Omega. \tag{3.46}$$

Proof. Proposition 3.19 implies that problem (3.37) has at most one solution. Then we only need to show that the pair (u^+, u^-) defined by (3.45) is a solution of problem (3.37). Lemma 3.20 implies that there exists a unique solution (μ^+, μ^-) in $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ of the system of integral equations (3.46). Then by Theorem 3.7 and by the definition (3.44) of the operator J, the functions u^+, u^- defined by (3.45) are q-periodic functions which solve the heat equation and which satisfy all the transmission conditions in (3.37), and thus the pair (u^+, u^-) is a solution of problem (3.37). □

Part II

Second part

CHAPTER 4

Shape analysis of the effective longitudinal permeability of a periodic array of cylinders

This chapter is devoted to the analysis of the behavior of the longitudinal permeability of a periodic array of cylinders upon perturbation of the shape of the cross section of the cylinders, and of the periodicity structure, when a Newtonian fluid is flowing at low Reynolds numbers around the cylinders.

The shape of the cross section of the cylinders is determined by the image of a base domain through a diffeomorphism ϕ and the periodicity cell is a rectangle of sides of length l and $1/l$, where l is a positive parameter. We also assume that the pressure gradient is parallel to the cylinders. Under such assumptions, the velocity field has only one non-zero component which, by the Stokes equations, satisfies a Poisson equation (cf. problem (4.6)). Then, by integrating the longitudinal component of the velocity field, for each pair (l, ϕ), one defines the longitudinal permeability $K_{II}[l, \phi]$ (cf. equality (4.7)). Here, we are interested in studying the behavior of $K_{II}[l, \phi]$ upon the pair (l, ϕ), which we think as a point in a suitable Banach space.

This chapter is organized as follows. Section 4.1 is where we introduce some notation and some preliminaries. In particular we recall the definition of periodic Schauder spaces and we introduce the class of diffeomorphisms that we use in order to model the shape of our domains. In Section 4.2 we introduce the Roumieu spaces, which are normed spaces of analytic functions, and we introduce also their periodic version. Moreover, we state a result due to Preciso [91] about the real analyticity of a superposition operator. In Section 4.3 we introduce the periodic version of the layer potentials associated with the Laplace operator and the periodic version of the volume potentials, and we present some properties of these maps. Section 4.4 is where we explain the problem that we study in the present chapter and where we define the longitudinal permeability. Next, in Section 4.5, we introduce an auxiliary function in order to reduce our Poisson problem (4.6) to a nonhomogeneus Dirichlet problem for the Laplace equation, and we study the regularity of such an auxiliary function in Section 4.6. Finally, in Section 4.7 we prove our main result about the behavior of the longitudinal permeability with respect to the variation of the periodicity structure and of the shape of the cross section of the cylinders. More precisely, we prove that the map

$$(l, \phi) \mapsto K_{II}[l, \phi],$$

is real analytic when l varies in $]0, +\infty[$ and ϕ varies in a suitable class of diffeomorphisms (cf. Theorem 4.23).

Some of the results of this chapter can be found in Musolino, Pukhtaievych and the author [76].

4.1 Notation and preliminaries

Through all this chapter we retain some of the notation of the previous Chapter 3, in the specific case of $n = 2$. In particular we recall that $\mathbb{D}_2^+$ is the space of 2×2 diagonal matrices with entries in $]0, +\infty[$. Moreover, if

$$(q_{11}, q_{22}) \in]0, +\infty[^2,$$

we set the periodicity cell to be the rectangle defined by

$$Q \equiv \Pi_{j=1}^{2}]0, q_{jj}[, \tag{4.1}$$

and we denote by q the 2×2 diagonal matrix

$$q \equiv \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix}, \tag{4.2}$$

and by $m_2(Q)$ the 2–dimensional measure of the periodicity cell Q, and by ν_Q the outward unit normal to ∂Q, where it exists. Clearly,

$$q\mathbb{Z}^2 = \{qz : z \in \mathbb{Z}^2\}$$

is the set of vertices of a periodic subdivision of $\mathbb{R}^2$ corresponding to the periodicity cell Q (see Figure 3.1).

We recall that we say that a subset $\mathbb{D}$ of $\mathbb{R}^2$ is q-periodic provided that

$$x + qe_i \in \mathbb{D} \qquad \forall\, x \in \mathbb{D}, \forall\, i \in \{1, 2\},$$

where $\{e_1, e_2\}$ denotes the canonical basis of $\mathbb{R}^2$. Let $\mathbb{D}$ be a q-periodic subset of $\mathbb{R}^2$. We say that a function f from $\mathbb{D}$ to $\mathbb{C}$ is q-periodic provided that

$$f(x + qe_i) = f(x) \qquad \forall\, x \in \mathbb{D}, \forall\, i \in \{1, 2\}.$$

Let Ω be an open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\,\Omega \subseteq Q$. We recall that we have set

$$\mathbb{S}_q[\Omega] \equiv \bigcup_{z \in \mathbb{Z}^2} (qz + \Omega) = q\mathbb{Z}^2 + \Omega,$$

and then

$$\mathbb{S}_q[\Omega]^- = \mathbb{R}^2 \setminus \mathrm{cl}\,\mathbb{S}_q[\Omega].$$

Clearly, $\mathbb{S}_q[\Omega]$ and $\mathbb{S}_q[\Omega]^-$ are q-periodic subsets of $\mathbb{R}^2$. Let $k \in \mathbb{N}$, $\alpha \in]0, 1]$. We set

$$C_q^{k,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]) \equiv \left\{ u \in C_b^{k,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]) : u \text{ is } q\text{-periodic} \right\},$$

which we regard as a Banach subspace of $C_b^k(\mathrm{cl}\,\mathbb{S}_q[\Omega])$, and

$$C_q^{k,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-) \equiv \left\{ u \in C_b^{k,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-) : u \text{ is } q\text{-periodic} \right\},$$

which we regard as a Banach subspace of $C_b^{k,\beta}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)$.

In order to consider shape perturbations, we have to introduce a class of admissible diffeomorphisms of a fixed base domain. The perturbation of the shape will be made by perturbing the diffeomorphism in such a class. Let $\alpha \in]0,1[$ and let Ω be a bounded open subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$. We denote by

$$\mathcal{A}_{\mathrm{cl}\,\Omega}$$

the set of functions of class $C^1(\mathrm{cl}\,\Omega, \mathbb{R}^2)$ which are injective and whose differential is injective at all points $x \in \mathrm{cl}\,\Omega$. Moreover, we denote by

$$\mathcal{A}_{\partial\Omega}$$

the set of functions of class $C^1(\partial\Omega, \mathbb{R}^2)$ which are injective and whose differential is injective at all points $x \in \partial\Omega$. One can verify that $\mathcal{A}_{\partial\Omega}$ and $\mathcal{A}_{\mathrm{cl}\,\Omega}$ are open in $C^1(\partial\Omega, \mathbb{R}^2)$ and in $C^1(\mathrm{cl}\,\Omega, \mathbb{R}^2)$, respectively (cf., *e.g.*, Lanza de Cristoforis and Rossi [72, Lemma 2.2, p. 197] and Lanza de Cristoforis and Rossi [71, Lemma 2.5, p. 143]). Then we find convenient to set

$$\mathcal{A}^Q_{\partial\Omega} \equiv \{\phi \in \mathcal{A}_{\partial\Omega} : \phi(\partial\Omega) \subseteq Q\},$$
$$\mathcal{A}^Q_{\mathrm{cl}\,\Omega} \equiv \{\Phi \in \mathcal{A}_{\mathrm{cl}\,\Omega} : \Phi(\mathrm{cl}\,\Omega) \subseteq Q\}.$$

Let now suppose, in addition, that the set Ω is connected and such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. In this case, if $\phi \in \mathcal{A}^Q_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^2 \setminus \phi(\partial\Omega)$ has exactly two open connected components, one bounded and one unbounded. We denote by

$$\mathbb{I}[\phi] \tag{4.3}$$

the bounded open connected component of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$, and by

$$\mathbb{E}[\phi] \tag{4.4}$$

the unbounded open connected component of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$. Since $\phi(\partial\Omega) \subseteq Q$, a simple topological argument shows that $Q \setminus \mathrm{cl}\,\mathbb{I}[\phi]$ is also connected.

4.2 The Roumieu spaces

First of all we recall the definition of regular sets in the sense of Whitney. We need such a notion in order to state a regularity results of a superposition operator. For every bounded open connected subsets Ω of $\mathbb{R}^n$, we set

$$c[\Omega] \equiv \sup\left\{\frac{\lambda(x,y)}{|x-y|} : x, y \in \Omega, x \neq y\right\},$$

where

$$\lambda(x,y) \equiv \inf\left\{\int_0^1 |\xi'(s)|\,ds : \xi \in C^1([0,1],\Omega),\ \xi(0) = x,\ \xi(1) = y\right\}.$$

If $c[\Omega] < +\infty$, then we say that Ω is regular in the sense of Whitney. It is of immediate verification that if Ω is a bounded open connected subset of $\mathbb{R}^n$ of class C^1, then Ω is regular in the sense of Whitney.

Next, we turn to introduce the Roumieu spaces. For all bounded open subsets Ω' of $\mathbb{R}^n$ and for all $\rho \in]0,+\infty[$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\,\Omega') \equiv \left\{ u \in C^\infty(\mathrm{cl}\,\Omega') : \sup_{\gamma\in\mathbb{N}^n} \frac{\rho^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma u\|_{C^0(\mathrm{cl}\,\Omega')} < +\infty \right\},$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\,\Omega')} \equiv \sup_{\gamma\in\mathbb{N}^n} \frac{\rho^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma u\|_{C^0(\mathrm{cl}\,\Omega')} \qquad \forall\, u \in C^0_{\omega,\rho}(\mathrm{cl}\,\Omega').$$

As is well known, the Roumieu space $\left(C^0_{\omega,\rho}(\mathrm{cl}\,\Omega'), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\,\Omega')}\right)$ is a Banach space. By definition, a function u belongs to $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega')$ if and only if it can be expanded into a convergent Taylor series around each point of $\mathrm{cl}\,\Omega'$ and the radius of convergence of the Taylor series can estimated from below by means of ρ, uniformly at all points of $\mathrm{cl}\,\Omega'$.

In this chapter, we resort to Roumieu spaces because they are a natural class of functions which generates analytic superposition operators in Schauder spaces. Indeed, the following slight variant of Preciso [91, Proposition 1.1, p. 101] on the real analyticity of a composition operator holds (see also Lanza de Cristoforis and Musolino [66, Proposition 5.2] and the slight variant of the argument of Preciso of the proof of Lanza de Cristoforis [59, Proposition 9, p. 214]).

Theorem 4.1. *Let $\alpha \in]0,1]$, $\rho \in]0,+\infty[$. Let Ω_1, Ω' be two bounded open subsets of $\mathbb{R}^n$. Let Ω' be regular in the sense of Whitney. Then the composition operator T from $C^0_{\omega,\rho}(\mathrm{cl}\,\Omega_1) \times C^{1,\alpha}(\mathrm{cl}\,\Omega', \Omega_1)$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega')$ defined by*

$$T[u,v] \equiv u \circ v, \qquad \forall\,(u,v) \in C^0_{\omega,\rho}(\mathrm{cl}\,\Omega_1) \times C^{1,\alpha}(\mathrm{cl}\,\Omega', \Omega_1),$$

is real analytic.

Let Q and q be as in (4.1) and (4.2), respectively. We now introduce Roumieu spaces of q-periodic function. Let Ω be an open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\,\Omega \subseteq Q$. We set

$$C^0_{q,\omega,\rho}(\mathrm{cl}\,\mathbb{S}_q[\Omega]) \equiv \Big\{ u \in C^\infty(\mathrm{cl}\,\mathbb{S}_q[\Omega]) : \\ u \text{ is } q\text{-periodic and } \sup_{\gamma\in\mathbb{N}^2} \frac{\rho^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma u\|_{C^0_q(\mathrm{cl}\,\mathbb{S}_q[\Omega])} < +\infty \Big\},$$

and

$$\|u\|_{C^0_{q,\omega,\rho}(\mathrm{cl}\,\mathbb{S}_q[\Omega])} \equiv \sup_{\gamma\in\mathbb{N}^2} \frac{\rho^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma u\|_{C^0_q(\mathrm{cl}\,\mathbb{S}_q[\Omega])} \qquad \forall\, u \in C^0_{q,\omega,\rho}(\mathrm{cl}\,\mathbb{S}_q[\Omega]).$$

Similarly, we set

$$C^0_{q,\omega,\rho}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-) \equiv \Big\{ u \in C^\infty(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-) : \\ u \text{ is } q\text{-periodic and } \sup_{\gamma\in\mathbb{N}^2} \frac{\rho^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma u\|_{C^0_q(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)} < +\infty \Big\},$$

and

$$\|u\|_{C^0_{q,\omega,\rho}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)} \equiv \sup_{\gamma\in\mathbb{N}^2} \frac{\rho^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma u\|_{C^0_q(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)} \qquad \forall\, u \in C^0_{q,\omega,\rho}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-).$$

Then we have the following lemma which shows that, when dealing with q-periodic functions in Roumieu spaces on $\mathrm{cl}\,\mathbb{S}_q[A]^-$, it is sufficient to work on a suitable neighborhood of the periodicity cell.

Lemma 4.2. *Let $\rho \in]0,+\infty[$. Let Q and q be as in (4.1) and (4.2), respectively. Let A be an open connected subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\, A$ is connected and such that*

$$\mathrm{cl}\, A \subseteq Q.$$

Let W be a bounded open connected subset of $\mathbb{R}^2$ such that

$$\mathrm{cl}\, Q \subseteq W \quad \text{and} \quad \mathrm{cl}\, W \cap (qz + \mathrm{cl}\, A) = \varnothing \qquad \forall\, z \in \mathbb{Z}^2 \setminus \{0\}.$$

Then the restriction operator from $C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[A]^-)$ onto the subspace

$$C^0_{q,\omega,\rho}(\mathrm{cl}\, W \setminus A) \equiv \Big\{ v \in C^0_{\omega,\rho}(\mathrm{cl}\, W \setminus A) : \\ \exists\, u \in \mathbb{C}^{\mathrm{cl}\, \mathbb{S}_q[A]^-} \text{ such that } u \text{ is } q\text{-periodic}, v = u_{|\mathrm{cl}\, W \setminus A} \Big\},$$

of $C^0_{\omega,\rho}(\mathrm{cl}\, W \setminus A)$ induces a linear homeomorphism

Proof. If $u \in C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[A]^-)$, then by definition of Roumieu spaces its restriction $u_{|\mathrm{cl}\, W \setminus A}$ belongs to $C^0_{q,\omega,\rho}(\mathrm{cl}\, W \setminus A)$. Indeed, by the q-periodicity of u we have that

$$\sup_{\gamma \in \mathbb{N}^n} \frac{\rho^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma u\|_{C^0(\mathrm{cl}\, W \setminus A)} = \sup_{\gamma \in \mathbb{N}^n} \frac{\rho^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma u\|_{C^0(\mathrm{cl}\, Q \setminus A)}.$$

Conversely let $v \in C^0_{q,\omega,\rho}(\mathrm{cl}\, W \setminus A)$, then there exists a unique q-periodic function u from $\mathrm{cl}\, \mathbb{S}_q[A]^-$ to $\mathbb{C}$ such that $v = u_{|\mathrm{cl}\, W \setminus A}$ and clearly $u \in C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[A]^-)$. Then the restriction operator is a bijection from $C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[A]^-)$ to $C^0_{q,\omega,\rho}(\mathrm{cl}\, W \setminus A)$. Since it is clearly linear and continuous, then the Open Mapping Theorem implies the validity of the statement. □

Finally, we prove the following elementary lemma which shows that, possibly taking a smaller ρ in the target space, the differential operators are linear and continuous in periodic Roumieu spaces on $\mathrm{cl}\, \mathbb{S}_q[\Omega]^-$. Corresponding results hold also for classical Roumieu spaces and for periodic Roumieu spaces in $\mathrm{cl}\, \mathbb{S}_q[\Omega]$. However, we only state the result that we exploit in this chapter.

Lemma 4.3. *Let $\rho \in]0,+\infty[$ and $\rho_1 \in]0,\rho[$. Let Q and q be as in (4.1) and (4.2), respectively. Let Ω be an open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\, \Omega \subseteq Q$. Let $\eta \in \mathbb{N}^2$ such that $|\eta|_1 = 1$. If $u \in C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)$, then $D^\eta u \in C^0_{q,\omega,\rho_1}(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)$. Moreover, the operator which takes u to $D^\eta u$ is linear and continuous from $C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)$ to $C^0_{q,\omega,\rho_1}(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)$.*

Proof. Let $u \in C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)$. Then we have

$$\begin{aligned}
&\sup_{\gamma \in \mathbb{N}^2} \frac{\rho_1^{|\gamma|_1}}{|\gamma|_1!} \|D^\gamma D^\eta u\|_{C^0_q(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)} \\
&\quad = \sup_{\gamma \in \mathbb{N}^2} \frac{\rho_1^{|\gamma|_1}}{|\gamma|_1!} \frac{(|\gamma|_1+1)!}{\rho^{|\gamma|_1+1}} \frac{\rho^{|\gamma|_1+1}}{(|\gamma|_1+1)!} \|D^{\gamma+\eta} u\|_{C^0_q(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)} \\
&\quad = \sup_{\gamma \in \mathbb{N}^2} \frac{1}{\rho_1} \left(\frac{\rho_1}{\rho}\right)^{|\gamma|_1+1} (|\gamma|_1+1) \frac{\rho^{|\gamma|_1+1}}{|\gamma|_1+1} \|D^{\gamma+\eta} u\|_{C^0_q(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)} \\
&\quad \leq \frac{1}{\rho_1} \sup_{\gamma \in \mathbb{N}^2} \left\{ \left(\frac{\rho_1}{\rho}\right)^{|\gamma|_1+1} (|\gamma|_1+1) \right\} \sup_{\gamma \in \mathbb{N}^2} \left\{ \frac{\rho^{|\gamma|_1+1}}{|\gamma|_1+1} \|D^{\gamma+\eta} u\|_{C^0_q(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)} \right\}
\end{aligned}$$

$$\leq \frac{1}{\rho_1} \sup_{\gamma \in \mathbb{N}^2} \left\{ \left(\frac{\rho_1}{\rho} \right)^{|\gamma|_1 + 1} (|\gamma|_1 + 1) \right\} \|u\|_{C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[\Omega]^-)}.$$

Since $\rho_1 < \rho$, we have that

$$\sup_{\gamma \in \mathbb{N}^2} \left\{ \left(\frac{\rho_1}{\rho} \right)^{|\gamma|_1 + 1} (|\gamma|_1 + 1) \right\} < +\infty,$$

and, accordingly, the statement follows. □

4.3 Periodic potentials for the Laplace operator

In this section we introduce the periodic layer potentials associated with the Laplace operator and the periodic volume potential associated with the Laplace operator.

Let Q and q be as in (4.1) and (4.2), respectively. As is well known, there exists a q-periodic tempered distribution $S_{q,2}$ such that

$$\Delta S_{q,2} = \sum_{z \in \mathbb{Z}^2} \delta_{qz} - \frac{1}{m_2(Q)},$$

where δ_{qz} denotes the Dirac measure with mass in qz. The distribution $S_{q,2}$ is determined up to an additive constant, and we can take

$$S_{q,2}(x) = - \sum_{z \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{m_2(Q) 4\pi^2 |q^{-1}z|^2} e^{2\pi i (q^{-1}z)\cdot x},$$

in the sense of distributions in $\mathbb{R}^2$ (cf., *e.g.*, Ammari and Kang [4, p. 53], Lanza de Cristoforis and Musolino [65, Section 3]). Moreover, $S_{q,2}$ is even, and real analytic in $\mathbb{R}^2 \setminus q\mathbb{Z}^2$, and locally integrable in $\mathbb{R}^2$ (cf., *e.g.*, [65, Section 3]). The tempered distribution $S_{q,2}$ is said to be a $\{0\}$-analog of a q-periodic fundamental solution of the Laplace operator (cf., *e.g.*, [65, p. 84]).

We are now ready to introduce the q-periodic layer potentials associated with the Laplace operator. Let Ω be a bounded open subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$ such that $\mathrm{cl}\,\Omega \subseteq Q$. Let $\mu \in L^\infty(\partial\Omega)$. We set

$$v_q[\partial\Omega, \mu](x) \equiv \int_{\partial\Omega} S_{q,2}(x-y)\mu(y)\, d\sigma_y \qquad \forall\, x \in \mathbb{R}^2,$$

and

$$\begin{aligned} w_q[\partial\Omega, \mu](x) &\equiv \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(y)} S_{q,2}(x-y)\mu(y)\, d\sigma_y \\ &= - \int_{\partial\Omega} \nu_\Omega(y) \cdot DS_{q,2}(x-y)\mu(y)\, d\sigma_y \qquad \forall\, x \in \mathbb{R}^2, \end{aligned}$$

and

$$\begin{aligned} w_{q,*}[\partial\Omega, \mu](x) &\equiv \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(x)} S_{q,2}(x-y)\mu(y)\, d\sigma_y \\ &= \int_{\partial\Omega} \nu_\Omega(x) \cdot DS_{q,2}(x-y)\mu(y)\, d\sigma_y \qquad \forall\, x \in \partial\Omega \end{aligned}$$

The functions $v_q[\partial\Omega,\mu]$ and $w_q[\partial\Omega,\mu]$ are called the q-periodic single and double layer potential with density μ, respectively. The function $w_{q,*}[\partial\Omega,\mu]$ is instead a function related the normal derivative of the q-periodic single layer potential $v_q[\partial\Omega,\mu]$. We now state two known theorems regarding the basic properties of the functions $v_q[\partial\Omega,\mu]$ and $w_q[\partial\Omega,\mu]$. We start with the following assertion concerning the single layer potential. For a proof we refer, *e.g.*, to Lanza de Cristoforis and Musolino [65, Theorem 3.5, p. 87], where the authors consider the more general case of periodic layer potentials associated with a general second order elliptic operator with constant coefficients.

Theorem 4.4. *Let $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$. Let Q and q be as in (4.1) and (4.2), respectively. Let Ω be a bounded connected open subset of $\mathbb{R}^2$ of class $C^{m,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected and such that $\mathrm{cl}\,\Omega \subseteq Q$. Then the following statements hold.*

i) If $\mu \in L^\infty(\partial\Omega)$, then the function $v_q[\partial\Omega,\mu]$ is continuous in $\mathbb{R}^2$, q-periodic, and of class C^∞ in $\mathbb{S}_q[\Omega] \cup \mathbb{S}_q[\Omega]^-$. Moreover,

$$\Delta v_q[\partial\Omega,\mu](x) = -\frac{1}{m_2(Q)}\int_{\partial\Omega}\mu(y)\,d\sigma_y \qquad \forall\, x \in \mathbb{R}^2 \setminus \partial\mathbb{S}_q[\Omega].$$

We set $v_q^+[\partial\Omega,\mu] \equiv v_q[\partial\Omega,\mu]_{|\mathrm{cl}\,\mathbb{S}_q[\Omega]}$ and $v_q^-[\partial\Omega,\mu] \equiv v_q[\partial\Omega,\mu]_{|\mathrm{cl}\,\mathbb{S}_q[\Omega]^-}$

ii) If $\mu \in C^{m-1,\alpha}(\partial\Omega)$, then the function $v_q^+[\partial\Omega_Q,\mu]$ belongs to $C_q^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega])$ and the function $v_q^-[\partial\Omega,\mu]$ belongs to $C_q^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)$. Moreover, the operator from $C^{m-1,\alpha}(\partial\Omega)$ to $C_q^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega])$ which takes μ to $v_q^+[\partial\Omega,\mu]$ is linear and continuous, and the operator from $C^{m-1,\alpha}(\partial\Omega)$ to $C_q^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)$ which takes μ to $v_q^-[\partial\Omega,\mu]$ is linear and continuous

iii) Let $\mu \in C^{m-1,\alpha}(\partial\Omega)$. Then the following jump formula holds.

$$\frac{\partial}{\partial\nu_\Omega(x)} v_q^\pm[\partial\Omega,\mu](x) = \mp\frac{1}{2}\mu(x) + w_{q,*}[\partial\Omega,\mu](t,x) \qquad \forall\, x \in \partial\Omega.$$

Furthermore, we have the corresponding result for the q-periodic double layer potential. For a proof we refer, *e.g*, to Lanza de Cristoforis and Musolino [65, Theorem 3.18, p. 92].

Theorem 4.5. *Under the same assumptions of Theorem 4.4, the following statements hold.*

i) If $\mu \in L^\infty(\partial\Omega)$, then the function $w_q[\partial\Omega,\mu]$ is q-periodic and of class C^∞ in $\mathbb{S}_q[\Omega] \cup \mathbb{S}_q[\Omega]^-$. Moreover,

$$\Delta w_q[\partial\Omega,\mu](x) = 0 \qquad \forall\, x \in \mathbb{R}^2 \setminus \partial\mathbb{S}_q[\Omega].$$

ii) If $\mu \in C^{m,\alpha}(\partial\Omega)$, then the restriction $w_q[\partial\Omega,\mu]_{|\mathbb{S}_q[\Omega]}$ can be extended uniquely to an element $w_q^+[\partial\Omega,\mu]$ of $C_q^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega])$ and the restriction $w_q[\partial\Omega,\mu]_{|\mathbb{S}_q[\Omega]^-}$ can be extended uniquely to an element $w_q^-[\partial\Omega,\mu]$ of $C_q^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)$. Moreover, the following jump formula holds.

$$w_q^\pm[\partial\Omega,\mu](x) = \pm\frac{1}{2}\mu(x) + w_q[\partial\Omega,\mu](t,x) \qquad \forall\, x \in \partial\Omega.$$

iii) The operator from $C^{m,\alpha}(\partial\Omega)$ to $C_q^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega])$ which takes μ to $w_q^+[\partial\Omega,\mu]$ is linear and continuous, and the operator from $C^{m,\alpha}(\partial\Omega)$ to $C_q^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-)$ which takes μ to $w_q^-[\partial\Omega,\mu]$ is linear and continuous

Next we introduce the exterior periodic volume potential. Let Q and q be as in (4.1) and (4.2), respectively. Let A be an open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\, A \subseteq Q$. Let $\varphi \in L^\infty(Q \setminus \mathrm{cl}\, A)$. Then we define the exterior periodic volume potential $\mathcal{P}_q^-[A, \varphi]$ by

$$\mathcal{P}_q^-[A, \varphi](x) \equiv \int_{Q \setminus \mathrm{cl}\, A} S_{q,2}(x - y)\varphi(y)\, dy \qquad \forall\, x \in \mathbb{R}^2.$$

We have the following result on some properties of the exterior periodic volume potential $\mathcal{P}_q^-[A, \varphi]$ that we need in the following sections.

Proposition 4.6. *Let Q and q be as in (4.1) and (4.2), respectively. Let A be an open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\, A \subseteq Q$. Then the following statements hold.*

i) If $\varphi \in L^\infty(Q \setminus \mathrm{cl}\, A)$, then $\mathcal{P}_q^-[A, \varphi]$ is q-periodic and of class $C^1(\mathbb{R}^2)$.

ii) If $\varphi \in C^{0,\alpha}(\mathrm{cl}\, Q \setminus A)$, then $\mathcal{P}_q^-[A, \varphi] \in C^2(Q \setminus \mathrm{cl}\, A)$ and

$$\Delta \mathcal{P}_q^-[A, \varphi](x) = \varphi(x) - \int_{Q \setminus \mathrm{cl} A} \varphi(y)\, dy \qquad \forall\, x \in Q \setminus \mathrm{cl}\, A, \tag{4.5}$$

Proof. Statement i) is a consequence of Dalla Riva, Lanza de Cristoforis and Musolino [22, Proposition 3.6 (v), Proposition 3.16 (iv)]), where the authors consider a volume potential with a general periodic kernel in some classes of weakly singular functions, and of [22, Section 4]), where it is shown that the kernel $S_{q,2}$ belongs to the right class of weakly singular functions.

Statement ii) can be proved following the argument of the proof of Lanza de Cristoforis and Musolino [69, Proposition A.1], and is a consequence of known properties of the classical volume potential (cf., *e.g.*, Gilbarg and Trudinger [41, Lemma 4.2, p. 55]). □

Finally, we have the following proposition regarding the mapping properties in Roumieu spaces of the exterior periodic volume potential.

Proposition 4.7. *Let $\alpha \in]0, 1[$. Let Q and q be as in (4.1) and (4.2), respectively. Let A be a bounded open Lipschitz subset of $\mathbb{R}^2$ such that $\mathrm{cl}\, A \subseteq Q$. Let A_1 be an open subset of $\mathbb{R}^2$ such that*

$$\mathrm{cl}\, A \subseteq A_1 \subseteq \mathrm{cl}\, A_1 \subseteq Q.$$

Then there exists $\rho_0 \in]0, +\infty[$ such that for all $\rho \in]0, \rho_0[$ and for all $\varphi \in C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[A]^-)$, the restriction of $\mathcal{P}_q^-[A, \varphi_{|Q \setminus \mathrm{cl}\, A}]$ to $\mathrm{cl}\, \mathbb{S}_q[A_1]^-$ belongs to the space $C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[A_1]^-)$. Moreover, the map from $C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[A]^-)$ to $C^0_{q,\omega,\rho}(\mathrm{cl}\, \mathbb{S}_q[A_1]^-)$ which takes φ to $\mathcal{P}_q^-[A, \varphi_{|Q \setminus \mathrm{cl}\, A}]_{|\mathrm{cl}\, \mathbb{S}_q[A_1]^-}$ is linear and continuous.

Proof. The statement follows by Dalla Riva, Lanza de Cristoforis and Musolino [22, Theorem 3.40 (ii)], which holds for exterior periodic volume potentials with a general periodic kernel in some classes of weakly singular functions, and by [22, Section 4]), where it is shown that the kernel $S_{q,2}$ belongs to the right class of weakly singular functions. □

4.4 The problem

In order to introduce the mathematical problem, we take $l \in]0, +\infty[$ and we choose

$$(q_{11}, q_{22}) \equiv (l, 1/l).$$

In this case, the periodicity cell is the rectangle

$$Q_l \equiv]0, l[\times]0, 1/l[,$$

and the periodicity matrix is the 2×2 diagonal matrix

$$q_l \equiv \begin{pmatrix} l & 0 \\ 0 & 1/l \end{pmatrix}.$$

We emphasize that we restrict ourself to the case of a periodic structure induced by q_l in order to have that the area $m_2(Q_l)$ of the periodicity cell Q_l is equal to one for all $l \in]0, +\infty[$. This choice helps making the computations simpler and the exposition clearer and it is of course physically meaningful. However, this restriction is not necessary and we could consider a more general periodic structure and a more general perturbation of the periodic structure.

We will often consider the case in which the periodic structure is induced by $l = 1$. Thus, for us it is convenient to set

$$\widetilde{Q} \equiv Q_1 \,, \qquad \tilde{q} \equiv q_1 \,.$$

Then, we fix $\alpha \in]0, 1[$ and Ω to be a bounded open connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Moreover we fix a diffeomorphism

$$\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2),$$

(see Figure 4.1).

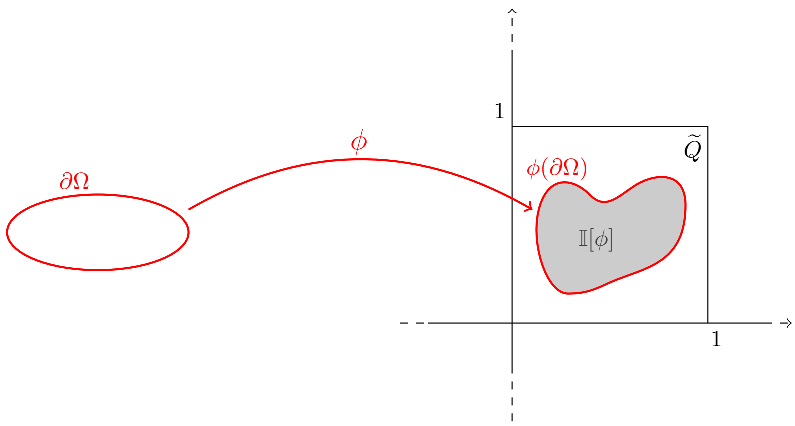

Figure 4.1: *In red the diffeomorphism $\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$ of $\partial\Omega$ onto $\phi(\partial\Omega)$ and in gray the set $\mathbb{I}[\phi]$.*

We recall that $\mathbb{I}[\phi]$ denotes the bounded open connected component of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$, and $\mathbb{E}[\phi]$ denotes the unbounded open connected component of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$ (cf. (4.3), (4.4)). Clearly $\mathrm{cl}\, q_l \mathbb{I}[\phi] \subseteq Q_l$ (see Figure 4.2). Under this assumptions, the set

$$\mathrm{cl}\, \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]] \times \mathbb{R}$$

represents an infinite array of parallel cylinders. Instead, the set

$$\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^- \times \mathbb{R}$$

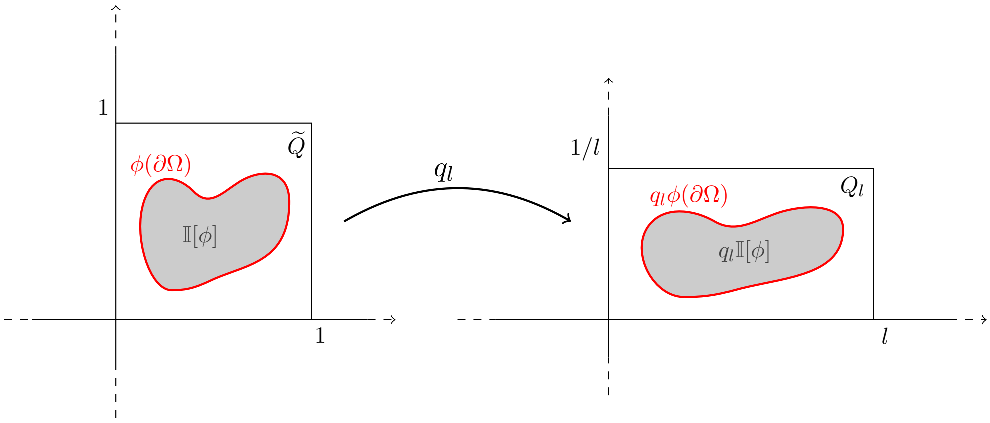

Figure 4.2: *The transformation induced by* q_l*.*

represent the region where, in our model, a Newtonian fluid is flowing at low Reynolds number. Then, we also assume that the driving pressure gradient is constant and parallel to the axis of the cylinders. As a consequence, by a standard argument based on the particular geometry of the problem (cf., *e.g.*, Adler [1, Chapter 4], Sangani and Yao [96], and Mityushev and Adler [85, 86]), one reduces the Stokes system for the fluid flowing in $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \times \mathbb{R}$ to a Poisson equation in $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$ for the non-zero component of the velocity field (see Figure 4.3).

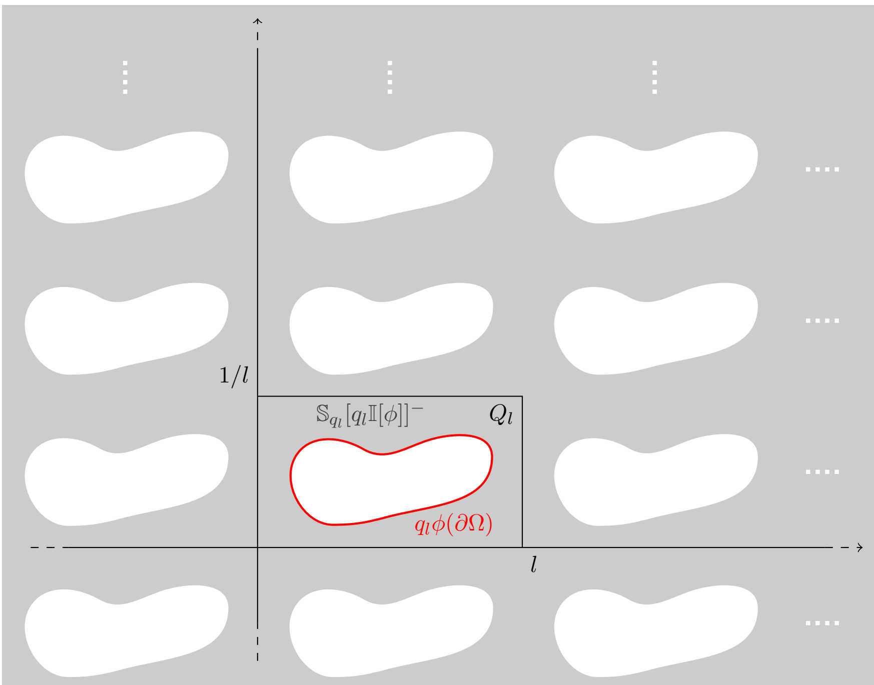

Figure 4.3: *In gray and red the* (l,ϕ)*-dependent sets* $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$ *and* $q_l\phi(\partial\Omega)$*, respectively.*

Since we are working with dimensionless quantities, we may assume that the viscosity of the fluid and the pressure gradient are both set equal to one. For a more complete discussion on spatially periodic structures, we refer to Adler [1, Chapter 4]. Accordingly, we consider the following Dirichlet problem for the Poisson equation:

$$\begin{cases} \Delta u = 1 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \,, \\ u(x+q_l z) = u(x) & \forall\, x \in \mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \,, \forall\, z \in \mathbb{Z}^2 \,, \\ u(x) = 0 & \forall\, x \in \partial\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \,. \end{cases} \tag{4.6}$$

As it is well known, there exists a unique solution in $C^{1,\alpha}_{q_l}(\mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-)$ of problem 4.6 (for a proof we refer, *e.g.*, to Musolino [83, Proposition 2.2, p. 276]). We denote such a solution by

$$u[l,\phi]$$

From the physical point of view, the function $u[l,\phi]$ represents the non-zero component of the velocity field of the fluid flowing in $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \times \mathbb{R}$ (cf. Mityushev and Adler [85, Section 2]). By means of the function $u[l,\phi]$, we can introduce the effective permeability $K_{II}[l,\phi]$ which we define as the integral of the opposite of the flow velocity over the unit cell (cf. Adler [1], Mityushev and Adler [85, Section 3]), *i.e.*,

$$K_{II}[l,\phi] \equiv -\int_{Q_l\setminus q_l\mathbb{I}[\phi]} u[l,\phi](x)\,dx, \tag{4.7}$$

and we pose the following question:

Q) What can be said about the regularity of the map $(l,\phi) \mapsto K_{II}[l,\phi]$? (4.8)

Most of the works in the literature deal with differentiability properties. Here, instead, we are interested into proving higher regularity and we answer the above question (4.8) by showing that $K_{II}[l,\phi]$ depends analytically on (l,ϕ). Such a result is contained in our main Theorem 4.23 and implies, in particular, that if we have a one-parameter analytic family of pairs $(l_\delta,\phi_\delta)_{\delta\in]-\delta_0,\delta_0[}$, then we can deduce the possibility to expand the permeability as a power series, *i.e.*,

$$K_{II}[l_\delta,\phi_\delta] = \sum_{j=0}^{+\infty} c_j\delta^j \tag{4.9}$$

for δ close to zero. Moreover, by the analyticity of the map in (4.8), the coefficients $(c_j)_{j\in\mathbb{N}}$ in (4.9) can be constructively determined by computing the differentials of $K_{II}[\cdot,\cdot]$. Furthermore, another important consequence of our high regularity result is that allows to apply differential calculus in order to find critical *rectangle-shape* pairs (l,ϕ) as a first step to find optimal configurations.

4.5 An auxiliary boundary value problem

In this section we transform our Poisson problem (4.6) into a nonhomogeneus Dirichlet problem for the Laplace equation by means of an auxiliary function.

Since analyticity is a local property, in order to prove the analyticity of the map in (4.8), we can work locally. Therefore, we find convenient to introduce the following lemma, which is an immediate consequence of the fact that the norm in $\mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ is stronger then the uniform norm.

Lemma 4.8. *Let* $\alpha \in]0,1[$. *Let* $\phi_0 \in \mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$. *Let* A_0 *be an open connected Lipschitz subset of* $\mathbb{R}^2$ *such that* $\mathbb{R}^2 \setminus \mathrm{cl}\,A_0$ *is connected and such that* $\mathrm{cl}\,A_0 \subseteq \mathbb{I}[\phi_0]$. *Then* *there exist an open connected subset* A_1 *of* $\mathbb{R}^2$ *such that* $\mathbb{R}^2 \setminus \mathrm{cl}\, A_1$ *is connected, and an open neighborhood* $\mathcal{U}_0$ *of* ϕ_0 *in* $\mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$ *such that*

$$\mathrm{cl}\, A_0 \subseteq A_1 \subseteq \mathrm{cl}\, A_1 \subseteq \mathbb{I}[\phi] \qquad \forall \phi \in \mathcal{U}_0.$$

In order to transform the Dirichlet problem for the Poisson equation (4.6) in a Dirichlet problem for the Laplace equation, we need a function B such that

$$\Delta B = 1.$$

We introduce such a function in the following lemma, which is an immediate consequence of Musolino [82, Thm. 2.1].

Lemma 4.9. *Let* $l \in]0, +\infty[$, $\alpha \in]0, 1[$. *Let* ϕ_0, A_0 *and* $\mathcal{U}_0$ *be as in Lemma 4.8. Let* $p_0 \in A_0$. *Let* $B_{p_0,l}$ *be the function from* $\mathbb{R}^2 \setminus (q_l p_0 + q_l \mathbb{Z}^2)$ *to* $\mathbb{R}$ *defined by*

$$B_{p_0,l}(x) \equiv -S_{q_l,2}(x - q_l p_0) \qquad \forall\, x \in \mathbb{R}^2 \setminus (q_l p_0 + q_l \mathbb{Z}^2).$$

Then

(i) $B_{p_0,l|\mathrm{cl}\, \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-} \in C^{1,\alpha}_{q_l}(\mathrm{cl}\, \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-)$ *for all* $\phi \in \mathcal{U}_0$.

(ii) $\Delta B_{p_0,l} = 1$ *in* $\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-$ *for all* $\phi \in \mathcal{U}_0$.

By means of Lemma 4.9, we can convert problem (4.6) for the Poisson equation into a nonhomogeneous Dirichlet problem for the Laplace equation. Let $l \in]0, +\infty[$. Let ϕ_0, A_0 and $\mathcal{U}_0$ be as in Lemma 4.8. Let $p_0 \in A_0$. Let $\phi \in \mathcal{U}_0$. We note that Lemma 4.9 (i) implies that

$$B_{p_0,l|\partial \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-} \in C^{1,\alpha}(\partial \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-).$$

Accordingly, it is well know that there exists a unique solution in $C^{1,\alpha}_{q_l}(\mathrm{cl}\, \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-)$ of the following auxiliary boundary value problem.

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-, \\ u(x + q_l z) = u(x) & \forall\, x \in \mathrm{cl}\, \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-, \forall\, z \in \mathbb{Z}^2, \\ u(x) = -B_{p_0,l}(x) & \forall\, x \in \partial \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-. \end{cases} \tag{4.10}$$

(cf., *e.g.*, Musolino [83, Proposition 2.2, p. 276] and Proposition 4.16). We denote such a solution of (4.10) by

$$u_\#[l, \phi].$$

Accordingly, one can immediately verify that

$$u[l, \phi] = B_{p_0,l} + u_\#[l, \phi] \qquad \text{in } \mathrm{cl}\, \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-,$$

where $u[l, \phi]$ is the unique solution in $C^{1,\alpha}_{q_l}(\mathrm{cl}\, \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-)$ of problem (4.6). Thus, we can rewrite the longitudinal permeability in the following form.

$$K_{II}[l, \phi] = -\int_{Q_l \setminus q_l \mathbb{I}[\phi]} B_{p_0,l}(x)\, dx - \int_{Q_l \setminus q_l \mathbb{I}[\phi]} u_\#[l, \phi](x)\, dx. \tag{4.11}$$

4.6 Analyticity of the integral of the auxiliary function $B_{p_0,l}$

In this section, we will investigate the analyticity of the first summand in the right hand side of formula (4.11), namely of the map

$$(l,\phi) \mapsto -\int_{Q_l \setminus q_l \mathbb{I}[\phi]} B_{p_0,l}(x)\,dx$$

In order to achieve this objective, we need the following two technical extension results. For a proof we refer to Lanza de Cristoforis and Rossi [72, Section 2] .

Lemma 4.10. *Let $\alpha \in]0,1]$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Let $\beta \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$ be such that $|\beta(x)| = 1$ and $\beta(x) \cdot \nu_\Omega(x) > 1/2$ for all $x \in \partial\Omega$. Then the following statements hold.*

(i) There exists $\delta_\Omega \in]0,+\infty[$ such that for all $\delta \in]0,\delta_\Omega[$ the sets

$$\begin{aligned}
\Omega_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in]-\delta,\delta[\},\\
\Omega^+_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in]-\delta,0[\},\\
\Omega^-_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in]0,\delta[\}
\end{aligned}$$

are connected and of class $C^{1,\alpha}$, and

$$\begin{aligned}
\partial\Omega_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in \{-\delta,\delta\}\},\\
\partial\Omega^+_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in \{-\delta,0\}\},\\
\partial\Omega^-_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in \{0,\delta\}\},
\end{aligned}$$

and

$$\Omega^+_{\beta,\delta} \subseteq \Omega, \qquad \Omega^-_{\beta,\delta} \subseteq \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega.$$

(ii) Let $\delta \in]0,\delta_\Omega[$. If $\Phi \in \mathcal{A}_{\mathrm{cl}\,\Omega_{\beta,\delta}}$, then $\phi \equiv \Phi_{|\partial\Omega} \in \mathcal{A}_{\partial\Omega}$.

(iii) If $\delta \in]0,\delta_\Omega[$, then the set

$$\mathcal{A}'_{\mathrm{cl}\,\Omega_{\beta,\delta}} \equiv \{\Phi \in \mathcal{A}_{\mathrm{cl}\,\Omega_{\beta,\delta}} : \Phi(\Omega^+_{\beta,\delta}) \subseteq \mathbb{I}[\Phi_{|\partial\Omega}]\}$$

is open in $\mathcal{A}_{\mathrm{cl}\,\Omega_{\beta,\delta}}$ and $\Phi(\Omega^-_{\beta,\delta}) \subseteq \mathbb{E}[\Phi_{|\partial\Omega}]$ for all $\Phi \in \mathcal{A}'_{\mathrm{cl}\,\Omega_{\beta,\delta}}$.

(iv) If $\delta \in]0,\delta_\Omega[$ and $\Phi \in \mathcal{A}'_{\mathrm{cl}\,\Omega_{\beta,\delta}} \cap C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta}, \mathbb{R}^2)$, then both $\Phi(\Omega^+_{\beta,\delta})$ and $\Phi(\Omega^-_{\beta,\delta})$ are open sets of class $C^{1,\alpha}$, and

$$\partial\Phi(\Omega^+_{\beta,\delta}) = \Phi(\partial\Omega^+_{\beta,\delta}), \quad \partial\Phi(\Omega^-_{\beta,\delta}) = \Phi(\partial\Omega^-_{\beta,\delta}).$$

Lemma 4.11. *Let $\alpha \in]0,1[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Let $\phi_* \in \mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$. Let β, δ_Ω as in Lemma 4.10. Then the following statements hold.*

(i) There exist $\delta_* \in]0,\delta_\Omega[$ and $\Phi_* \in \mathcal{A}'_{\mathrm{cl}\,\Omega_{\beta,\delta_*}} \cap C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta_*}, \mathbb{R}^2)$ such that $\phi_* = \Phi_{*|\partial\Omega}$.

(ii) Let δ_*, Φ_* be as in (i). Then there exist an open neighborhood $\mathcal{W}_*$ of ϕ_* in $\mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$, and a real analytic extension operator

$$\mathbf{E}_*[\cdot]$$

of $C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta_*}, \mathbb{R}^2)$ which maps $\mathcal{W}_*$ to $\mathcal{A}'_{\mathrm{cl}\,\Omega_{\beta,\delta_*}} \cap C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta_*}, \mathbb{R}^2)$ and such that $\mathbf{E}_*[\phi_*] = \Phi_*$ and $\mathbf{E}_*[\phi]_{|\partial\Omega} = \phi$, for all $\phi \in \mathcal{W}_*$.

We also need the following technical lemma about the real analyticity upon the diffeomorphism ϕ of some maps related to the change of variables in the integrals and to the outer normal field (for a proof we refer to Lanza de Cristoforis and Rossi [71, p. 166], and to Lanza de Cristoforis [59, Proposition 1]).

Lemma 4.12. *Let $\alpha \in]0, 1[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Then the following statements hold.*

(i) For each $\phi \in \mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$, there exists a unique map $\tilde{\sigma}[\phi] \in C^{0,\alpha}(\partial\Omega)$ such that $\tilde{\sigma}[\phi] > 0$ and

$$\int_{\phi(\partial\Omega)} w(s)\, d\sigma_s = \int_{\partial\Omega} w \circ \phi(y) \tilde{\sigma}[\phi](y)\, d\sigma_y, \qquad \forall\, \omega \in L^1(\phi(\partial\Omega)).$$

Moreover, the map $\tilde{\sigma}[\cdot]$ of $\mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic.

(ii) The map of $\mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$ to $C^{0,\alpha}(\partial\Omega, \mathbb{R}^2)$ which takes ϕ to $\nu_{\mathbb{I}[\phi]} \circ \phi$ is real analytic.

We are now ready to prove the following theorem, where we show the analyticity of the map

$$(\phi, G) \mapsto \int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} G\, dx,$$

where ϕ is in a suitable class of diffeomorphisms and G is in a Roumieu space of $\tilde{q}$-periodic functions.

Theorem 4.13. *Let $\alpha \in]0, 1[$. Let $\rho \in]0, +\infty[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Let ϕ_0, A_0 and $\mathcal{U}_0$ be as in Lemma 4.8. Then the map of $\mathcal{U}_0 \times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $\mathbb{R}$ which takes (ϕ, G) to $\int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} G\, dx$ is real analytic.*

Proof. We first note that, if $(\phi, G) \in \mathcal{U}_0 \times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$, equality (4.5) for the Laplace operator applied to the exterior volume potential implies that

$$\int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} G(x)\, dx = \int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} \Delta \mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q} \setminus \mathrm{cl}\, A_0}](x)\, dx + \int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} \int_{\widetilde{Q} \setminus \mathrm{cl}\, A_0} G(y)\, dy\, dx. \tag{4.12}$$

We now consider the two integrals in the right hand side of equality (4.12) separately. We start with the second one. By the Divergence Theorem, we have

$$\begin{aligned}
\int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} \int_{\widetilde{Q} \setminus \mathrm{cl}\, A_0} G(y)\, dy\, dx &= \int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} dx \int_{\widetilde{Q} \setminus \mathrm{cl}\, A_0} G(y)\, dy \\
&= \left(1 - \int_{\mathbb{I}[\phi]} dx\right) \int_{\widetilde{Q} \setminus \mathrm{cl}\, A_0} G(y)\, dy \\
&= \left(1 - \frac{1}{2} \int_{\phi(\partial\Omega)} x \cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x\right) \int_{\widetilde{Q} \setminus \mathrm{cl}\, A_0} G(y)\, dy.
\end{aligned}$$

We note that the map from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $L^1(\widetilde{Q}\setminus \mathrm{cl}\,A_0)$ which takes G to $G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}$ is linear and continuous, and that the map from $L^1(\widetilde{Q}\setminus \mathrm{cl}\,A_0)$ to $\mathbb{R}$ which takes f to $\int_{\widetilde{Q}\setminus \mathrm{cl}\,A_0} f(y)\,dy$ is linear and continuous. Accordingly, the map from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $\mathbb{R}$ which takes G to $\int_{\widetilde{Q}\setminus \mathrm{cl}\,A_0} G(y)\,dy$ is linear and continuous, and thus real analytic. Moreover, by Lemma 4.12 (i), we have that

$$\int_{\phi(\partial\Omega)} x\cdot \nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x = \int_{\partial\Omega} \phi(y)\cdot(\nu_{\mathbb{I}[\phi]}\circ\phi(y))\tilde{\sigma}[\phi](y)\,d\sigma_y.$$

Then, taking into account that the map from $(C^{0,\alpha}(\partial\Omega,\mathbb{R}^2))^2$ to $C^{0,\alpha}(\partial\Omega)$ which takes (f,g) to $f\cdot g$ is bilinear and continuous, that the embedding of $C^{0,\alpha}(\partial\Omega)$ in $L^1(\partial\Omega)$ is linear and continuous, and that the map from $L^1(\partial\Omega)$ to $\mathbb{R}$ which takes h to $\int_{\partial\Omega} h\,d\sigma$ is linear and continuous, Lemma 4.12 implies that the map from $\mathcal{U}_0$ to $\mathbb{R}$ which takes ϕ to $\int_{\phi(\partial\Omega)} x\cdot\nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x$ is real analytic. Accordingly, the map from $\mathcal{U}_0\times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $\mathbb{R}$ which takes the pair (ϕ,G) to $\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]}\int_{\widetilde{Q}\setminus \mathrm{cl}\,A_0} G(y)\,dy\,dx$ is real analytic.

Next, we consider the first integral in the right hand side of equality (4.12). Proposition 4.6 implies that the periodic exterior volume potential $\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}]$ is of class $C^1(\mathbb{R}^2)$. Accordingly, the Divergence Theorem implies that

$$\begin{aligned}
&\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} \Delta\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}](x)\,dx\\
&\quad= \int_{\partial\widetilde{Q}} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}](x))\cdot\nu_{\widetilde{Q}}(x)\,d\sigma_x - \int_{\phi(\partial\Omega)} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}](x))\cdot\nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x\\
&\quad= -\int_{\phi(\partial\Omega)} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}](x))\cdot\nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x.
\end{aligned}$$

Indeed the $\tilde{q}$-periodicity of $\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}]$ (cf. Proposition 4.6 (i)) implies that

$$\int_{\partial\widetilde{Q}} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}](x))\cdot\nu_{\widetilde{Q}}(x)\,d\sigma_x = 0$$

Next, we set

$$\delta_0\equiv\delta_*\,,\quad \mathcal{W}_0\equiv\mathcal{W}_*\,,\quad \mathbf{E}_0\equiv\mathbf{E}_*\,,$$

where δ_*, $\mathcal{W}_*$, $\mathbf{E}_*$, and β are as in Lemma 4.11, with $\phi_*=\phi_0$. Let

$$\mathcal{U}\equiv\mathcal{U}_0\cap\mathcal{W}_0.$$

Let A_1 be as in Lemma 4.8. Then, in particular, we have that

$$\mathrm{cl}\,A_0\subseteq A_1\subseteq\mathrm{cl}\,A_1\subseteq\mathbb{I}[\phi]\subseteq\widetilde{Q}\qquad\forall\,\phi\in\mathcal{U}.$$

Possibly shrinking δ_0 we can assume that

$$\mathrm{cl}\,\mathbf{E}_0[\phi_0](\Omega_{\beta,\delta_0})\subseteq\widetilde{Q}\setminus\mathrm{cl}\,A_1.$$

Moreover, possibly shrinking $\mathcal{U}$ we can assume that

$$\mathrm{cl}\,\mathbf{E}_0[\phi](\Omega_{\beta,\delta_0})\subseteq\widetilde{Q}\setminus\mathrm{cl}\,A_1\qquad\forall\,\phi\in\mathcal{U}.$$

By Corollary 4.7, there exists $\rho'\in]0,\rho[$ such that the map from $C^0_{\tilde{q},\omega,\rho'}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^0_{\tilde{q},\omega,\rho'}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_1]^-)$ which takes F to $\mathcal{P}^-[F_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}]_{|\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_1]^-}$ is linear and continuous. By the linearity and continuity of the embedding of $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ into $C^0_{\tilde{q},\omega,\rho'}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$, the map from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^0_{\tilde{q},\omega,\rho'}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_1]^-)$ which takes G to $\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}]_{|\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_1]^-}$ is linear and continuous. Thus thanks to Lemma 4.3, possibly taking a smaller ρ', we can verify that the map from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^0_{\tilde{q},\omega,\rho'}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_1]^-)$ which takes G to

$$\frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}]_{|\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_1]^-}$$

is linear and continuous and then real analytic, for all $j \in \{1,2\}$. Moreover, we note that the restriction operator from $C^0_{\tilde{q},\omega,\rho'}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_1]^-)$ to $C^0_{\omega,\rho'}(\mathrm{cl}\,\widetilde{Q}\setminus A_1)$ is linear and continuous and then real analytic. Thus, by Lemma 4.11 on the real analyticity of the extension operator $\mathbf{E}_0$ and by Theorem 4.1 on the real analyticity of a superposition operator in Schauder spaces, the map of $\mathcal{U}\times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta_0})$ which takes the pair (ϕ, G) to

$$\frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}]\circ \mathbf{E}_0[\phi]$$

is real analytic, for all $j\in\{1,2\}$. Then we note that

$$\begin{aligned}\int_{\phi(\partial\Omega)} & D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}](x))\cdot \nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x \\ &= \int_{\partial\Omega}\Big(D\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}]\circ \mathbf{E}_0[\phi](x)\Big)\cdot(\nu_{\mathbb{I}[\phi]}\circ\phi(x))\tilde{\sigma}[\phi](x)\,d\sigma_x. \\ &= \sum_{j=1}^{2}\int_{\partial\Omega}\frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}]\circ \mathbf{E}_0[\phi](x)(\nu_{\mathbb{I}[\phi]}\circ\phi(x))_j\tilde{\sigma}[\phi](x)\,d\sigma_x.\end{aligned}$$

By Lemmas 4.11, 4.12, and by the linearity and continuity of the trace operator from $C^{0,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta_0})$ to $C^{0,\alpha}(\partial\Omega)$, and by the linearity and continuity of the embedding of $C^{0,\alpha}(\partial\Omega)$ in $L^1(\partial\Omega)$, and by the linearity and continuity of the map from $L^1(\partial\Omega)$ to $\mathbb{R}$ which takes f to $\int_{\partial\Omega} f\,d\sigma$, we have that the map from $\mathcal{U}\times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\,\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $\mathbb{R}$ which takes the pair (ϕ, G) to $\int_{\phi(\partial\Omega)} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus \mathrm{cl}\,A_0}](x))\cdot\nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x$ real analytic. Thus, the validity of the statement follows. □

We recall $B_{p_0,l}$ is the function defined in Lemma 4.9. We are now ready to analyze the regularity of the map

$$(l,\phi)\mapsto\int_{Q_l\setminus q_l\mathbb{I}[\phi]} B_{p_0,l}(x)\,dx,$$

when l is in $]0,+\infty[$ and ϕ is a suitable class of diffeomorphisms.

Proposition 4.14. *Let ϕ_0, A_0 and $\mathcal{U}_0$ be as in Lemma 4.8. Let $p_0\in A_0$. Then the map from $]0,+\infty[\times\mathcal{U}_0$ to $\mathbb{R}$, which takes the pair (l,ϕ) to $\int_{Q_l\setminus q_l\mathbb{I}[\phi]} B_{p_0,l}(x)\,dx$, is real analytic.*

Proof. Since real analyticity is a local property, we can work locally. Accordingly, we fix

$$l_0\in]0,+\infty[.$$

Let

$$\mathcal{L}_0$$

be a bounded open subset of $]0,+\infty[$ containing l_0. We set

$$\mathcal{Q}_0\equiv\{q_l\in\mathbb{D}^+_2 : l\in\mathcal{L}_0\}.$$

Clearly, $\mathcal{Q}_0$ is a bounded open subset of $\mathbb{D}_2^+(\mathbb{R})$, and

$$\mathrm{cl}\,\mathcal{Q}_0 \subseteq \mathbb{D}_2^+(\mathbb{R}).$$

Now, we note that

$$\int_{Q_l \setminus q_l \mathbb{I}[\phi]} B_{p_0,l}(x)\,dx = \int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} B_{p_0,l}(q_l x)\,dx = -\int_{\widetilde{Q} \setminus \mathbb{I}[\phi]} S_{q_l,2}(q_l(x-p_0))\,dx \tag{4.13}$$

for all $(l,\phi) \in]0,+\infty[\, \times \mathcal{U}_0$. Then we take a bounded open connected subset W of $\mathbb{R}^2$ of class C^∞ such that

$$\mathrm{cl}\,\widetilde{Q} \subseteq W \qquad \text{and} \qquad W \cap (z + \mathrm{cl}\,A_0) = \varnothing \qquad \forall\, z \in \mathbb{Z}^2 \setminus \{0\}.$$

By Lanza de Cristoforis and Musolino [70, Theorem 8], there exists $\rho \in]0,+\infty[$ such that the map from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0 - p_0)$, which takes $\hat{q}$ to the function $S_{\hat{q},2}(\hat{q}\,\cdot)_{|\mathrm{cl}\,W \setminus A_0 - p_0}$, is real analytic. Since the translation operator from $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0 - p_0)$ to $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0)$ which takes f to $f(\cdot - p_0)$ is linear and continuous, then the map from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0)$, which takes $\hat{q}$ to the function $S_{\hat{q},2}(\hat{q}(\cdot - p_0))$, is real analytic. Then, taking into account the real analyticity of the map from $]0,+\infty[$ to $\mathbb{D}_2^+(\mathbb{R})$ which takes l to q_l, we deduce that the map from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0)$, which takes l to $S_{q_l,2}(q_l(\cdot - p_0))$, is real analytic.

Then, due to the Lemma 4.2, we can apply Theorem 4.13 to the last integral in equality (4.13), and the validity of the statement follows. □

4.7 Analyticity of the effective longitudinal permeability

In this section we prove our main result about the real analyticity of the longitudinal permeability. By the previous sections, this aim is reduced to the study of the behavior of the second integral in (4.11), that is the map

$$(l,\phi) \mapsto -\int_{Q_l \setminus q_l \mathbb{I}[\phi]} u_\#[l,\phi](x)\,dx, \tag{4.14}$$

when l is in $]0,+\infty[$ and ϕ is in a suitable class of diffeomorphisms.

In order to achieve this objective, we exploit some of the results of Musolino [82], where the behavior of a (singularly) perturbed Dirichlet problem for the Laplace equation has been studied by means of periodic potentials.

As we shall see, we want to reduce the analysis of the solution $u_\#[l,\phi]$ of our Dirichlet problem (4.10) to that of a related integral equation. To do so, we start with the following result on a boundary integral operators. For a proof, we refer to Musolino [82, Proposition A.3].

Lemma 4.15. *Let $l \in]0,+\infty[$, $\alpha \in]0,1[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Let $\phi \in \mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$. Let $M[\cdot]$ be the map from $C^{1,\alpha}(q_l \partial \mathbb{I}[\phi])$ to itself, defined by*

$$M[\mu] \equiv -\frac{1}{2}\mu + w_{q_l}[q_l \partial \mathbb{I}[\phi], \mu] \qquad \forall\, \mu \in C^{1,\alpha}(q_l \partial \mathbb{I}[\phi]).$$

Then $M[\cdot]$ is a linear homeomorphism from $C^{1,\alpha}(q_l \partial \mathbb{I}[\phi])$ to $C^{1,\alpha}(q_l \partial \mathbb{I}[\phi])$.

Then, we have the following result where we establish a correspondence between the solution of a Dirichlet problem and the solution of an integral equation.

Proposition 4.16. *Let $l \in]0, +\infty[$, $\alpha \in]0, 1[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Let $\phi \in \mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$. Let $\Gamma \in C^{1,\alpha}(q_l \partial\mathbb{I}[\phi])$. Then the following boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \\ u(x + q_l z) = u(x) & \forall\, x \in \mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \forall\, z \in \mathbb{Z}^2, \\ u(x) = \Gamma(x) & \forall\, x \in q_l\partial\mathbb{I}[\phi]. \end{cases} \tag{4.15}$$

has a unique solution u in $C^{1,\alpha}_{q_l}(\mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-)$. Moreover,

$$u(x) = w^-_{q_l}[q_l\partial\mathbb{I}[\phi], \mu](x) \qquad \forall\, x \in \mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \tag{4.16}$$

where μ is the unique solution in $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ of the following integral equation

$$-\frac{1}{2}\mu(x) + w_{q_l}[q_l\partial\mathbb{I}[\phi], \mu](x) = \Gamma(x) \qquad \forall\, x \in q_l\partial\mathbb{I}[\phi] \tag{4.17}$$

Proof. By the Maximum Principle for periodic functions in $\mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$, problem (4.15) has at most one solution (cf. Musolino [82, Prop. A.1]). As a consequence, we only need to prove that the function defined by (4.16) solves problem (4.15). By Lemma 4.15 there exists a unique solution $\mu \in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ of the integral equation (4.17). Then by the properties of the periodic double layer potential the function defined by (4.16) solves problem (4.15) (cf. Theorem 4.5). □

By Propostion 4.16, we know that the solution $u_\#[l, \phi]$ of our Dirichlet problem (4.10) can be written in terms of a double layer potential. As a consequence, in the following two lemmas we study the dependence upon l and ϕ of some integral operators related to the double layer potential. We start with the following result.

Lemma 4.17. *Let $\alpha \in]0, 1[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Let β and δ_Ω be as in Lemma 4.10. Let*

$$\mathcal{A}'^{\widetilde{Q}}_{\mathrm{cl}\,\Omega_{\beta,\delta}} \equiv \mathcal{A}'_{\mathrm{cl}\,\Omega_{\beta,\delta}} \cap \mathcal{A}^{\widetilde{Q}}_{\mathrm{cl}\,\Omega_{\beta,\delta}} \qquad \forall\, \delta \in]0, \delta_\Omega[.$$

Let $\eta \in]0, 1[$. Then there exists $\delta_\eta \in]0, \delta_\Omega[$ such that for all $\delta \in]0, \delta_\eta[$ the map which takes

$$(l, \Phi, \theta) \in]0, +\infty[\times \left(\mathcal{A}'^{\widetilde{Q}}_{\mathrm{cl}\,\Omega_{\beta,\delta}} \cap C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta}, \mathbb{R}^2) \right) \times C^{1,\alpha}(\partial\Omega)$$

to the function $W^+[l, \Phi, \theta]$, which is defined as the continuous extension to $\mathrm{cl}\,\Omega^+_{\beta,\delta}$ of the function

$$-\int_{q_l\Phi(\partial\Omega)} DS_{q_l,2}(q_l\Phi(x) - s) \cdot \nu_{q_l\mathbb{I}[\Phi]}(s)(\theta \circ \Phi^{(-1)} \circ q_l^{-1})(s)\, d\sigma_s \qquad \forall\, x \in \Omega^+_{\beta,\delta},$$

is real analytic from $\mathcal{O}(\eta) \times \mathcal{U}_{\eta,\delta} \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega^+_{\beta,\delta})$, where

$$\mathcal{O}(\eta) \equiv \left\{ l \in]0, +\infty[:\ \max\{l^{-2}, l^2\} < \eta^{-1} \right\},$$

$$\mathcal{U}_{\eta,\delta} \equiv \left\{ \Phi \in \mathcal{A}'^{\widetilde{Q}}_{\mathrm{cl}\,\Omega_{\beta,\delta}} \cap C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta}, \mathbb{R}^2) :\ \sup_{\mathrm{cl}\,\Omega_{\beta,\delta}} |\det(D\Phi)| < \eta^{-1} \right\}.$$

Proof. First of all, let $\delta \in]0, \delta_\Omega[$. Our plan is to follow the proof of Corollary 5.7 of Lanza de Cristoforis and Musolino [65]. To do so, we first need to rewrite the operators W^+, $\frac{\partial}{\partial x_1} W^+$ and $\frac{\partial}{\partial x_2} W^+$ in terms of single layer potentials. Let $R \in]0, +\infty[$ such that

$$R > \sup_{x \in \Omega \cup \Omega_{\beta,\delta}} |x|$$

Let F be a linear and continuous extension operator from $C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathbb{B}_2(0,R))$, such that $F[\theta]_{|\partial\Omega} = \theta$ for all $\theta \in C^{1,\alpha}(\partial\Omega)$ (see, *e.g.*, Troianiello [103, Theorem. 1.3 and Lemma 1.5]). Then, by using [65, equalities (5.8) and (5.9), p. 109], with Φ replaced by $q_l \circ \Phi$, we obtain that

$$W^+[l, \Phi, \theta] = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(V^+\left[l, \Phi, \mathbf{n}_j[q_l \circ \Phi]\theta\right]\right) \left((D(q_l \circ \Phi))^{-1}\right)_{ij} \tag{4.18}$$

and

$$\begin{aligned} &\frac{\partial}{\partial x_k} \left(W^+[l, \Phi, \theta]\right) \\ &= \sum_{r=1}^{2} \frac{\partial (q_l \circ \Phi)_r}{\partial x_k} \sum_{j,t=1}^{2} \frac{\partial}{\partial x_t} \left(V^+\left[l, \Phi, M_{rj}[l, q_l \circ \Phi, \theta]\right]\right) \left((D(q_l \circ \Phi))^{-1}\right)_{ij} \\ &+ \sum_{r=1}^{2} \frac{\partial (q_l \circ \Phi)_r}{\partial x_k} \int_{\partial\Omega} \mathbf{n}_j[q_l \circ \Phi](y)\theta(y)\tilde{\sigma}[q_l \circ \Phi](s)d\sigma_s \end{aligned} \tag{4.19}$$

for all $k \in \{1,2\}$, where

$$\begin{aligned} M_{rj}[l, q_l \circ \Phi, \theta] \equiv\ & |(D(q_l \circ \Phi))^{-t} \cdot \nu_\Omega|^{-1} \\ &\times \Bigg[\left(\sum_{i=1}^{2} \left((D(q_l \circ \Phi))^{-1}\right)_{ir} (\nu_\Omega)_i \right) \left(\sum_{i=1}^{2} \frac{\partial (F[\theta])}{\partial x_i} \left((D(q_l \circ \Phi))^{-1}\right)_{ij} \right) \\ &- \left(\sum_{i=1}^{2} \left((D(q_l \circ \Phi))^{-1}\right)_{ij} (\nu_\Omega)_i \right) \left(\sum_{i=1}^{2} \frac{\partial (F[\theta])}{\partial x_i} \left((D(q_l \circ \Phi))^{-1}\right)_{ir} \right) \Bigg], \end{aligned}$$

and,

$$\mathbf{n}_j[q_l \circ \Phi] \equiv \left(\frac{(D(q_l \circ \Phi))^{-t} \cdot \nu_\Omega}{|(D(q_l \circ \Phi))^{-t} \cdot \nu_\Omega|} \right)_j,$$

and,

$$V^+\left[l, \Phi, \mu\right](\cdot) = \int_{q_l \Phi(\partial\Omega)} S_{q_l,2}(q_l \Phi(\cdot) - s) \left(\mu \circ \Phi^{(-1)} \circ q_l^{-1}\right)(s) d\sigma_s \qquad \forall \mu \in C^{1,\alpha}(\partial\Omega).$$

By the chain rule, we have

$$\begin{aligned} (D(q_l \circ \Phi))_{ij} &= (q_l)_{ii} (D\Phi)_{ij} && \forall\, i,j \in \{1,2\}, \\ ((D(q_l \circ \Phi))^{-1})_{ij} &= \frac{1}{(q_l)_{ii}} ((D\Phi)^{-1})_{ij} && \forall\, i,j \in \{1,2\}, \\ (D(q_l \circ \Phi))^{-t} &= q_l^{-1} \cdot (D\Phi)^{-t}. \end{aligned} \tag{4.20}$$

Next, we consider V^+ and we note that

$$V^+[l,\Phi,\mu](x) = \int_{q_l\Phi(\partial\Omega)} S_{q_l,2}(q_l\Phi(x)-s)\left(\mu\circ\Phi^{(-1)}\circ q_l^{-1}\right)(s)d\sigma_s$$
$$= \int_{\Phi(\partial\Omega)} S_{q_l,2}(q_l(\Phi(x)-s))\left(\mu\circ\Phi^{(-1)}\right)(s)d\sigma_s$$

for all $\mu\in C^{0,\alpha}(\partial\Omega)$ and for all $x\in\Omega^+_{\beta,\delta}$. Then we set

$$\tilde{S}_{\tilde{q},l,2}(x) \equiv S_{q_l,2}(q_l x) \qquad \forall\, x\in\mathbb{R}^2\setminus\mathbb{Z}^2\,. \tag{4.21}$$

We note that the $\tilde{q}$-periodic function $\tilde{S}_{\tilde{q},l,2}$ is a $\tilde{q}$-periodic $\{0\}$-analog of the fundamental solution of the operator

$$\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2}+l^2\frac{\partial^2}{\partial x_2^2}\,.$$

Namely, it is a tempered distribution such that

$$\Big(\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2}+l^2\frac{\partial^2}{\partial x_2^2}\Big)\tilde{S}_{\tilde{q},l,2} = \sum_{z\in\mathbb{Z}^2}\delta_{\tilde{q}z}-1\,,$$

in the sense of distributions (cf. Lanza de Cristoforis and Musolino [65, Section 1]). Then we can write

$$\begin{aligned}\int_{\Phi(\partial\Omega)} & S_{q_l,2}(q_l(\Phi(x)-s))\left(\mu\circ\Phi^{(-1)}\right)(s)d\sigma_s \\ &= \int_{\Phi(\partial\Omega)} \tilde{S}_{\tilde{q},l,2}(\Phi(x)-s)\left(\mu\circ\Phi^{(-1)}\right)(s)d\sigma_s \\ &\equiv \widetilde{V}^+_{\tilde{q}}[l,\Phi,\mu](x) \qquad \forall x\in\Omega^+_{\beta,\delta},\end{aligned} \tag{4.22}$$

for all $(l,\Phi,\mu)\in]0,+\infty[\times\mathcal{U}_{\eta,\delta}\times C^{0,\alpha}(\partial\Omega)$. Now, one can rewrite the operators W^+, $\frac{\partial}{\partial x_1}W^+$ and $\frac{\partial}{\partial x_2}W^+$ using the single layer potential $\widetilde{V}^+_{\tilde{q}}$. More precisely, equalities (4.18), (4.19) together with the three equalities in (4.20) and with equality (4.22) imply that

$$W^+[l,\Phi,\theta] = -\sum_{m,i,j=1}^{2}\frac{\partial}{\partial x_i}\left(\widetilde{V}^+_{\tilde{q}}[l,\Phi,\tilde{\mathbf{n}}_j[l,\Phi]\theta]\right)\frac{1}{(q_l)_{ii}}\left(D\Phi)^{-1}\right)_{im} \tag{4.23}$$

and

$$\frac{\partial}{\partial x_k}\left(W^+[l,\Phi,\theta]\right) = \sum_{r=1}^{2}\frac{\partial\Phi_r}{\partial x_k}(q_l)_{rr}\sum_{m,j,t=1}^{2}\frac{\partial}{\partial x_t}\left(\widetilde{V}^+_{\tilde{q}}\left[l,\Phi,\widetilde{M}_{rj}[l,\Phi,\theta]\right]\right)\frac{1}{(q_l)_{ii}}\left((D\Phi)^{-1}\right)_{im}$$
$$+\sum_{r=1}^{2}\frac{\partial\Phi_r}{\partial x_k}(q_l)_{rr}\int_{\partial\Omega}\tilde{\mathbf{n}}_j[l,\Phi](y)\theta(y)\tilde{\sigma}[q_l\circ\Phi](s)d\sigma_s \tag{4.24}$$

for all $k\in\{1,2\}$, where

$$\widetilde{M}_{rj}[l,\Phi,\theta] = |q_l^{-1}\cdot(D\Phi)^{-t}\cdot\nu_\Omega|^{-1}\times$$

$$\times \Bigg[\left(\sum_{i=1}^{2} \left(D\Phi)^{-1}\right)_{ir} \frac{1}{(q_l)_{ii}} (\nu_\Omega)_i \right) \left(\sum_{i=1}^{2} \frac{\partial (F[\theta])}{\partial x_i} \left(D\Phi)^{-1}\right)_{ij} \frac{1}{(q_l)_{ii}} \right) - \left(\sum_{i=1}^{2} \left(D\Phi)^{-1}\right)_{ij} \frac{1}{(q_l)_{ii}} (\nu_\Omega)_i \right) \left(\sum_{i=1}^{2} \frac{\partial (F[\theta])}{\partial x_i} \left(D\Phi)^{-1}\right)_{ir} \frac{1}{(q_l)_{ii}} \right) \Bigg],$$

and,

$$\widetilde{\mathbf{n}}_j[l,\Phi] = \left(\frac{q_l^{-1} \cdot (D\Phi)^{-t} \cdot \nu_\Omega}{|q_l^{-1} \cdot (D\Phi)^{-t} \cdot \nu_\Omega|} \right)_j .$$

Now we note that

- the map from $]0,+\infty[$ to $\mathbb{D}_2^+$ which takes l to

  $$\mathbf{a}(l) \equiv \begin{pmatrix} l^{-2} & 0 \\ 0 & l^2 \end{pmatrix}$$

  is real analytic.

Moreover, by Lanza de Cristoforis and Musolino [70, Theorem 7] and by Lanza de Cristoforis and Musolino [65, Section 3]

- the map from $]0,+\infty[\times(\mathbb{R}^2 \setminus \tilde{q}\mathbb{Z}^2)$ to $\mathbb{R}$ which takes the pair (l,x) to $\tilde{S}_{\tilde{q},l,2}(x) = S_{q_l,2}(q_l x)$ is real analytic. Moreover, for all $l \in]0,+\infty[$, the map $\tilde{S}_{\tilde{q},l,2}(\cdot)$ is a $\tilde{q}$-periodic function in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $\left(\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2} + l^2 \frac{\partial^2}{\partial x_2^2}\right)\tilde{S}_{\tilde{q},l,2} = \sum_{z\in\mathbb{Z}^2} \delta_{\tilde{q}z} - 1$ in the sense of distributions.

Accordingly, one can readily verify that the assumptions of Lanza de Cristoforis and Musolino [65, (1.8), pp. 78, 79] are satisfied and thus we can apply the results of [65]. Thus, [65, Proposition 5.6, pp. 105, 106] implies that there exists $\delta_\eta \in]0,\delta_\Omega[$ such that for all $\delta \in]0,\delta_\eta[$ the map $\widetilde{V}^+_{\tilde{q}}[\cdot,\cdot,\cdot]$ is real analytic from

$$\mathcal{O}(\eta) \times \mathcal{U}_{\eta,\delta} \times C^{0,\alpha}(\partial\Omega) \quad \text{to} \quad C^{1,\alpha}(\mathrm{cl}\,\Omega^+_{\beta,\delta}).$$

Then, if $\delta \in]0,\delta_\eta[$, by the real analyticity of the pointwise product in Schauder spaces, and by the real analyticity of the map which takes an invertible matrix with Schauder entries to its inverse, and by the real analyticity of the linear and continuous extension operator $F[\cdot]$ and of the trace operator, and by identities (4.23) and (4.24), we conclude that the operators

$$W^+[\cdot,\cdot,\cdot], \quad \frac{\partial}{\partial x_1} W^+[\cdot,\cdot,\cdot], \quad \frac{\partial}{\partial x_2} W^+[\cdot,\cdot,\cdot]$$

are real analytic from

$$\mathcal{O}(\eta) \times \mathcal{U}_{\eta,\delta} \times C^{1,\alpha}(\partial\Omega) \quad \text{to} \quad C^{0,\alpha}(\mathrm{cl}\,\Omega^+_{\beta,\delta}).$$

Accordingly, the operator $W^+[\cdot,\cdot,\cdot]$ is real analytic from

$$\mathcal{O}(\eta) \times \mathcal{U}_{\eta,\delta} \times C^{1,\alpha}(\partial\Omega) \quad \text{to} \quad C^{1,\alpha}(\mathrm{cl}\,\Omega^+_{\beta,\delta}),$$

and thus the statement follows. □

Then we have the following lemma where we prove the analyticity of the trace of the periodic double layer potential upon the periodicity parameter, the shape, and the density.

Lemma 4.18. *Let* $\alpha \in]0,1[$. *Let* Ω *be a bounded connected subset of* $\mathbb{R}^2$ *of class* $C^{1,\alpha}$ *such that* $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ *is connected. Then the map from* $]0,+\infty[\times(\mathcal{A}_{\partial\Omega}^{\widetilde{Q}} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)) \times C^{1,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\partial\Omega)$ *which takes a triple* (l,ϕ,θ) *to the function*

$$W[l,\phi,\theta](x) \equiv -\int_{q_l\phi(\partial\Omega)} DS_{q_l,2}(q_l\phi(x)-s)\cdot \nu_{q_l\mathbb{I}[\phi]}(q_l s)(\theta\circ\phi^{(-1)}\circ q_l^{-1})(s)\,d\sigma_s \qquad \forall\, x\in\partial\Omega,$$

is real analytic.

Proof. Since analyticity is a local property, it is suffices to show that if

$$(l_*,\phi_*,\theta_*)\in]0,+\infty[\times(\mathcal{A}_{\partial\Omega}^{\widetilde{Q}} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)) \times C^{1,\alpha}(\partial\Omega),$$

then $W[\cdot,\cdot,\cdot]$ is real analytic in a neighborhood of (l_*,ϕ_*,θ_*).

Let β, δ_*, $\mathbf{E}_*$, $\mathcal{W}_*$ be as in Lemma 4.11. Possibly shrinking $\mathcal{W}_*$, we can assume that there exists $\eta\in]0,1[$ such that

$$\sup_{\phi\in\mathcal{W}_*}\ \sup_{x\in \mathrm{cl}\,\Omega^+_{\beta,\delta}} |\det(D\mathbf{E}_*[\phi](x))| < \eta^{-1} \quad \text{and} \quad l_*\in\mathcal{O}[\eta],$$

where $\mathcal{O}[\eta]$ is as in Lemma 4.17. Possibly shrinking δ_* and $\mathcal{W}_*$, we can also assume that

$$\mathbf{E}_*[\phi](\mathrm{cl}\,\Omega_{\beta,\delta_*})\subseteq \widetilde{Q} \qquad \forall \phi\in\mathcal{W}_*.$$

Then we note that the jump formula for the double layer potential implies that

$$W[l,\phi,\theta] = -\frac{1}{2}\theta + W^+\left[l,\mathbf{E}_*[\phi],\theta\right] \qquad \text{on } \partial\Omega, \tag{4.25}$$

for all $(l,\phi,\theta)\in\mathcal{O}[\eta]\times\mathcal{W}_*\times C^{1,\alpha}(\partial\Omega)$, where W^+ is as in Lemma 4.17 for some $\delta\in]0,\min\{\delta_*,\delta_\eta\}[$. Then, by equality (4.25), and by Lemma 4.11 for the real analyticity of the extension operator $\mathbf{E}_*$, and by Proposition 4.17 about the real analyticity of the operator $W^+[\cdot,\cdot,\cdot]$, and by the linearity and continuity of the trace operator from $C^{1,\alpha}(\mathrm{cl}\,\Omega^+_{\beta,\delta})$ to $C^{1,\alpha}(\partial\Omega)$, we have that the operator $W[\cdot,\cdot,\cdot]$ is real analytic from

$$\mathcal{O}[\eta]\times\mathcal{W}_*\times C^{1,\alpha}(\partial\Omega) \quad \text{to} \quad C^{1,\alpha}(\partial\Omega),$$

and, accordingly, the statement follows. □

By Proposition 4.16, the Dirichlet problem (4.10) can be converted into the following integral equation.

$$-\frac{1}{2}\mu(x) + w_{q_l}[q_l\partial\mathbb{I}[\phi],\mu](x) = S_{q_l,2}(x-q_l p_0) \qquad \forall\, x\in q_l\partial\mathbb{I}[\phi]\,.$$

Therefore, the first thing that we have to do in order to study the dependence of the solution $u_\#[l,\phi]$ of problem (4.10) upon (l,ϕ) is to study the dependence upon the same pair of the solution of the integral equation above. On the other hand, since the above integral equation is defined on the (l,ϕ)-dependent domain $q_l\partial\mathbb{I}[\phi]$, in the following lemma we provide a reformulation on the fixed domain $\partial\Omega$.

Lemma 4.19. *Let* $l \in]0,+\infty[$, $\alpha \in]0,1[$. *Let* Ω *be a bounded connected subset of* $\mathbb{R}^2$ *of class* $C^{1,\alpha}$ *such that* $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ *is connected. Let* A_0, ϕ_0 *and* $\mathcal{U}_0$ *be as in Lemma 4.8. Let* $p_0 \in A_0$ *and* $\phi \in \mathcal{U}_0$. *Then the function* $\theta \in C^{1,\alpha}(\partial\Omega)$ *solves the equation*

$$-\frac{1}{2}\theta(t) - \int_{\phi(\partial\Omega)} DS_{q_l,2}(q_l(\phi(t)-s)) \cdot \nu_{q_l\mathbb{I}[\phi]}(q_l s)(\theta \circ \phi^{(-1)})(s)d\sigma_s - S_{q_l,2}(q_l(\phi(t)-p_0)) = 0, \tag{4.26}$$

for all $t \in \partial\Omega$, *if and only if the function* $\mu \in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$, *with* μ *delivered by*

$$\mu(x) = (\theta \circ \phi^{(-1)} \circ q_l^{-1})(x) \qquad \forall x \in q_l\partial\mathbb{I}[\phi], \tag{4.27}$$

solves the equation

$$-\frac{1}{2}\mu(x) + w_{q_l}[q_l\partial\mathbb{I}[\phi],\mu](x) = S_{q_l,2}(x - q_l p_0) \qquad \forall x \in q_l\partial\mathbb{I}[\phi]\,. \tag{4.28}$$

Moreover, equation (4.26) *has a unique solution in* $C^{1,\alpha}(\partial\Omega)$.

Proof. The equivalence of equation (4.26) in the unknown θ and of equation (4.28) in the unknown μ, with μ delivered by (4.27), is a straightforward consequence of the Theorem of change of variables in integrals. Then the existence and uniqueness of a solution of equation (4.26) in $C^{1,\alpha}(\partial\Omega)$, follows from Lemma 4.9 and from Lemma 4.15 applied to equation (4.28), and from the equivalence of equations (4.26), (4.28). □

Now, our aim is to prove the analytic dependence of the function θ which solves equation (4.26) upon (l,ϕ) by exploiting the Implicit Function Theorem for real analytic maps in Banach spaces. To do so, inspired by the previous Lemma 4.19, we introduce the map

$$\Lambda :]0,+\infty[\times\mathcal{U}_0 \times C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$$

by setting

$$\begin{aligned}\Lambda[l,\phi,\theta](t) \equiv -\frac{1}{2}\theta(t) - \int_{\phi(\partial\Omega)} DS_{q_l,2}(q_l(\phi(t)-s))\cdot\nu_{q_l\mathbb{I}[\phi]}(q_l s)(\theta \circ \phi^{(-1)})(s)\,d\sigma_s \\ - S_{q_l,2}(q_l(\phi(t)-p_0)) \qquad \forall t \in \partial\Omega,\end{aligned} \tag{4.29}$$

for all $(l,\phi,\theta) \in]0,+\infty[\times\mathcal{U}_0 \times C^{1,\alpha}(\partial\Omega)$, where $\mathcal{U}_0$ and p_0 are as in Lemma 4.9. In order to apply the Implicit Function Theorem for real analytic maps to the equation

$$\Lambda[l,\phi,\theta] = 0\,,$$

we need to understand the regularity of Λ. The analyticity upon (l,ϕ,θ) of the second term in the right hand side of (4.29) is shown in Lemma 4.18. Accordingly, in order to show the analyticity of the map Λ, it remains to show that the map which takes (l,ϕ) to the function $S_{q_l,2}(q_l(\phi(\cdot)-p_0))$ is real analytic.

Lemma 4.20. *Let* $\alpha \in]0,1[$. *Let* Ω *be a bounded connected subset of* $\mathbb{R}^2$ *of class* $C^{1,\alpha}$ *such that* $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ *is connected. Let* ϕ_0, A_0 *and* $\mathcal{U}_0$ *be as in Lemma 4.8. Let* $p_0 \in A_0$. *Then the map from* $]0,+\infty[\times\mathcal{U}_0$ *to* $C^{1,\alpha}(\partial\Omega)$ *which takes a pair* (l,ϕ) *to the function* $S_{q_l,2}(q_l(\phi(\cdot)-p_0))$ *is real analytic.*

Proof. Since real analyticity is a local property, we can work locally. Accordingly, we fix

$$l_0 \in]0, +\infty[.$$

Let

$$\mathcal{L}_0$$

be a bounded open subset of $]0, +\infty[$ containing l_0. We set

$$\mathcal{Q}_0 \equiv \{q_l \in \mathbb{D}_2^+ : l \in \mathcal{L}_0\}.$$

Clearly, $\mathcal{Q}_0$ is a bounded open subset of $\mathbb{D}_2^+(\mathbb{R})$, and

$$\mathrm{cl}\,\mathcal{Q}_0 \subseteq \mathbb{D}_2^+(\mathbb{R}).$$

We take a bounded open connected subset W of $\mathbb{R}^2$ of class C^∞ such that

$$\mathrm{cl}\,\widetilde{Q} \subseteq W \qquad \text{and} \qquad W \cap (z + \mathrm{cl}\,A_0) = \varnothing \qquad \forall z \in \mathbb{Z}^2 \setminus \{0\}.$$

By Lanza de Cristoforis and Musolino [70, Theorem 8], there exists $\rho \in]0, +\infty[$ such that the map from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0 - p_0)$, which takes $\hat{q}$ to the function $S_{\hat{q},2}(\hat{q}\,\cdot)$, is real analytic. Since the translation operator from $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0 - p_0)$ to $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0)$ which takes f to $f(\cdot - p_0)$ is linear and continuous, then the map from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\mathrm{cl}\,W \setminus A_0)$, which takes $\tilde{q}$ to the function $S_{\tilde{q},2}(\tilde{q}(\cdot - p_0))$, is real analytic. Now we set

$$\delta_0 \equiv \delta_*\,, \quad \mathcal{W}_0 \equiv \mathcal{W}_*\,, \quad \mathbf{E}_0 \equiv \mathbf{E}_*\,,$$

where δ_*, $\mathcal{W}_*$, $\mathbf{E}_*$, and β are as in Lemma 4.11, with $\phi_* = \phi_0$. Let

$$\mathcal{U} \equiv \mathcal{U}_0 \cap \mathcal{W}_0.$$

Let A_1 be as in Lemma 4.8. Then, in particular, we have that

$$\mathrm{cl}\,A_0 \subseteq A_1 \subseteq \mathrm{cl}\,A_1 \subseteq \mathbb{I}[\phi] \subseteq \widetilde{Q} \qquad \forall \phi \in \mathcal{U}.$$

Possibly shrinking δ_0 we can assume that

$$\mathrm{cl}\,\mathbf{E}_0[\phi_0](\Omega_{\beta,\delta_0}) \subseteq \widetilde{Q} \setminus \mathrm{cl}\,A_1.$$

Moreover, possibly shrinking $\mathcal{U}$ we can assume that

$$\mathrm{cl}\,\mathbf{E}_0[\phi](\Omega_{\beta,\delta_0}) \subseteq \widetilde{Q} \setminus \mathrm{cl}\,A_1 \qquad \forall \phi \in \mathcal{U}.$$

Thus, by the real analyticity of the map from $\mathcal{L}_0$ to $\mathcal{Q}_0$ which takes l to q_l, and by Lemma 4.11 on the real analyticity of the extension operator $\mathbf{E}_0$, and by Lemma 4.1 on the real analyticity of a superposition operator in Schauder spaces, we have that the map from $\mathcal{L}_0 \times \mathcal{U}$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta_0})$ which takes (l, ϕ) to $S_{q_l,2}(q_l(\cdot - p_0)) \circ \mathbf{E}_0[\phi]$ is real analytic. Accordingly, the map from $]0, +\infty[\times \mathcal{U}_0$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta_0})$ which takes (l, ϕ) to $S_{q_l,2}(q_l(\cdot - p_0)) \circ \mathbf{E}_0[\phi]$ is real analytic. Finally, the linearity and continuity of the trace operator from $C^{1,\alpha}(\mathrm{cl}\,\Omega_{\beta,\delta_0})$ to $C^{1,\alpha}(\partial\Omega)$ implies the validity the statement. □

We are now ready to show that the solution of the integral equation (4.26) depends analytically on (l, ϕ). The proof is based on the Implicit Function Theorem for real analytic maps in Banach spaces.

Proposition 4.21. *Let $\alpha \in]0,1[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Let ϕ_0, A_0 and $\mathcal{U}_0$ be as in Lemma 4.8. Let $p_0 \in A_0$. Then the following statements hold.*

(i) For each $(l,\phi) \in]0,+\infty[\times\mathcal{U}_0$, there exists a unique θ in $C^{1,\alpha}(\partial\Omega)$ such that

$$\Lambda[l,\phi,\theta] = 0 \qquad \text{on } \partial\Omega,$$

and we denote such a function by $\theta[l,\phi]$.

(ii) The map $\theta[\cdot,\cdot]$ from $]0,+\infty[\times\mathcal{U}_0$ to $C^{1,\alpha}(\partial\Omega)$ which takes (l,ϕ) to $\theta[l,\phi]$ is real analytic.

Proof. Statement (i) is a straightforward consequence of Lemma 4.19.

Next we turn to consider statement (ii). We first observe that by Lemmas 4.18 and 4.20, $\Lambda[\cdot,\cdot,\cdot]$ is a real analytic map from $]0,+\infty[\times\mathcal{U}_0 \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$. Since analyticity is a local property, we fix (l_1,ϕ_1) in $]0,+\infty[\times\mathcal{U}_0$ and we show that $\theta[\cdot,\cdot]$ is real analytic in some neighborhood of (l_1,ϕ_1) in $]0,+\infty[\times\mathcal{U}_0$. By standard calculus in normed spaces, the partial differential

$$\partial_\theta\Lambda[l_1,\phi_1,\theta[l_1,\phi_1]]$$

of Λ at $(l_1,\phi_1,\theta[l_1,\phi_1])$ with respect to the variable θ is delivered by

$$\begin{aligned}\partial_\theta\Lambda[l_1,&\phi_1,\theta[l_1,\phi_1]](\psi)(t)\\ &= -\frac{1}{2}\psi(x) - \int_{\phi(\partial\Omega)} DS_{q_{l_1},2}(q_{l_1}(\phi(t)-s))\cdot\nu_{q_{l_1}\mathbb{I}[\phi]}(q_{l_1}s)(\psi\circ\phi^{(-1)})(s)\,d\sigma_s \quad \forall t\in\partial\Omega,\end{aligned}$$

for all $\psi \in C^{1,\alpha}(\partial\Omega)$. By Lemma 4.15 and by the proof of Lemma 4.19, we deduce that $\partial_\theta\Lambda[l_1,\phi_1,\theta[l_1,\phi_1]]$ is a linear homeomorphism of $C^{1,\alpha}(\partial\Omega)$ onto $C^{1,\alpha}(\partial\Omega)$. Accordingly, we can apply the Implicit Function Theorem for real analytic maps in Banach spaces (see, *e.g.*, Prodi and Ambrosetti [92, Theorem 11.6] and Deimling [29, Theorem 15.3]), and we deduce that $\theta[\cdot,\cdot]$ is real analytic in a neighborhood of (l_1,ϕ_1) in $]0,+\infty[\times\mathcal{U}_0$ to $C^{1,\alpha}(\partial\Omega)$. Thus, the statement follows. □

Now we are ready to consider the second integral in the right hand side of (4.11), that is the map in (4.14).

Theorem 4.22. *Let $\alpha \in]0,1[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Let ϕ_0, A_0 and $\mathcal{U}_0$ be as in Lemma 4.8. Let $p_0 \in A_0$. Then the map from $]0,+\infty[\times\mathcal{U}_0$ to $\mathbb{R}$ which takes the pair (l,ϕ) to $\int_{Q_l\setminus q_l\mathbb{I}[\phi]} u_\#[l,\phi]\,dx$ is real analytic.*

Proof. We first note that by Proposition 4.16, by Lemma 4.19 and by Proposition 4.21 we have that

$$u_\#[l,\phi](x) = w^-_{q_l}[q_l\partial\mathbb{I}[\phi],\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1}](x) \qquad \forall x\in\mathrm{cl}\,\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-,$$

for all $(l,\phi) \in]0,+\infty[\times\mathcal{U}_0$, where $\theta[l,\phi]$ is defined in Proposition 4.21 (i). Accordingly,

$$\int_{Q_l\setminus q_l\mathbb{I}[\phi]} u_\#[l,\phi]\,dx = \int_{Q_l\setminus q_l\mathbb{I}[\phi]} w^-_{q_l}[q_l\partial\mathbb{I}[\phi],\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1}]\,dx, \tag{4.30}$$

for all $(l,\phi) \in]0,+\infty[\times\mathcal{U}_0$. We note that by classical differentiation theorems for integrals depending on a parameter we have that

$$w^-_{q_l}[q_l\partial\mathbb{I}[\phi],\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1}](x)$$

$$
\begin{aligned}
&= \int_{q_l\phi(\partial\Omega)} \frac{\partial}{\partial \nu_{q_l\mathbb{I}[\phi]}(y)} S_{q_l,2}(x-y)(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})(y)\,d\sigma_y \\
&= -\int_{q_l\phi(\partial\Omega)} DS_{q_l,2}(x-y)\cdot \nu_{q_l\mathbb{I}[\phi]}(y)(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})(y)\,d\sigma_y \\
&= -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \int_{q_l\partial\phi(\Omega)} S_{q_l,2}(x-y)(\nu_{q_l\mathbb{I}[\phi]}(y))_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})(y)\,d\sigma_y \\
&= -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} v_{q_l}^-[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](x), \qquad \forall\, x\in \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-,
\end{aligned}
$$

for all $(l,\phi)\in]0,+\infty[\times\mathcal{U}_0$. Then

$$
\begin{aligned}
&\int_{Q_l\setminus q_l\mathbb{I}[\phi]} w_{q_l}^-[q_l\partial\mathbb{I}[\phi],\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1}](x)\,dx \qquad (4.31)\\
&\qquad = -\sum_{j=1}^{2} \int_{Q_l\setminus q_l\mathbb{I}[\phi]} \frac{\partial}{\partial x_j} v_{q_l}^-[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](x)\,dx,
\end{aligned}
$$

for all $(l,\phi)\in]0,+\infty[\times\mathcal{U}_0$. We now fix $j\in\{1,2\}$. Lemma 4.12 (i), The Divergence Theorem, and the continuity in $\mathbb{R}^2$ of the single layer potential (see Theorem 4.4 (i)) imply that

$$
\begin{aligned}
&\int_{Q_l\setminus q_l\mathbb{I}[\phi]} \frac{\partial}{\partial x_j} v_{q_l}^-[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](x)\,dx \qquad (4.32)\\
&= \int_{\partial Q_l} v_{q_l}^-[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](x)(\nu_{Q_l}(x))_j\,d\sigma_x \\
&\quad - \int_{q_l\phi(\partial\Omega)} v_{q_l}^-[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](x)(\nu_{q_l\mathbb{I}[\phi]}(x))_j\,d\sigma_x \\
&= -\int_{q_l\phi(\partial\Omega)} v_{q_l}^-[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](x)(\nu_{q_l\mathbb{I}[\phi]}(x))_j\,d\sigma_x \\
&= -\int_{\partial\Omega} v_{q_l}^-[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](q_l\phi(x))((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)(x)\tilde{\sigma}[\phi](x)\,d\sigma_x \\
&= -\int_{\partial\Omega} v_{q_l}[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](q_l\phi(x))((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)(x)\tilde{\sigma}[\phi](x)\,d\sigma_x,
\end{aligned}
$$

for all $(l,\phi)\in]0,+\infty[\times\mathcal{U}_0$. Indeed, the periodicity of the periodic single layer potential (cf. Theorem 4.4 (i)) implies that

$$
\int_{\partial Q_l} v_{q_l}^-[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](x)(\nu_{Q_l}(x))_j\,d\sigma_x = 0.
$$

Now we note that if $S_{\tilde{q},l,2}$ is the $\tilde{q}$-periodic $\{0\}$-analog of the fundamental solution of the operator

$$
\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2} + l^2\frac{\partial^2}{\partial x_2^2},
$$

defined as in (4.21) (cf. Lanza de Cristoforis and Musolino [65, Section 1]), we have

$$
\begin{aligned}
&v_{q_l}[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](q_l\phi(x)) \\
&\qquad = \int_{q_l\phi(\partial\Omega)} S_{q_l,2}(q_l\phi(x)-y)(\nu_{q_l\mathbb{I}[\phi]}(y))_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})(y)\,d\sigma_y
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\phi(\partial\Omega)} S_{q_l,2}(q_l(\phi(x)-y))(\nu_{q_l\mathbb{I}[\phi]}(q_l y))_j(\theta[l,\phi]\circ\phi^{(-1)})(y)\,d\sigma_y \\
&= \int_{\phi(\partial\Omega)} \tilde{S}_{\tilde{q},l,2}(\phi(x)-y)(\nu_{q_l\mathbb{I}[\phi]}(q_l y))_j(\theta[l,\phi]\circ\phi^{(-1)})(y)\,d\sigma_y \\
&\equiv \tilde{v}_{l,\tilde{q}}[\partial\mathbb{I}[\phi],(((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)\theta[l,\phi])\circ\phi^{(-1)})](\phi(x)) \qquad \forall x\in\partial\Omega,
\end{aligned}
$$

for all $(l,\phi)\in]0,+\infty[\times\mathcal{U}_0$. Here

$$\tilde{v}_{l,\tilde{q}}[\partial\mathbb{I}[\phi],\cdot]$$

is the $\tilde{q}$-periodic single layer potential associated with the analog $\tilde{S}_{\tilde{q},l,2}$ (cf. Lanza de Cristoforis and Musolino [65, Thorem 3.7, pp. 87–89]). Now we note that

- the map from $]0,+\infty[$ to $\mathbb{D}_2^+$ which takes l to
$$\mathbf{a}(l)\equiv\begin{pmatrix} l^{-2} & 0 \\ 0 & l^2\end{pmatrix}$$
is real analytic.

Moreover, by Lanza de Cristoforis and Musolino [70, Theorem 7] and by Lanza de Cristoforis and Musolino [65, Section 3]

- the map from $]0,+\infty[\times(\mathbb{R}^2\setminus\tilde{q}\mathbb{Z}^2)$ to $\mathbb{R}$ which takes the pair (l,x) to $\tilde{S}_{\tilde{q},l,2}(x)=S_{q_l,2}(q_l x)$ is real analytic. Moreover, for all $l\in]0,+\infty[$, the map $\tilde{S}_{\tilde{q},l,2}(\cdot)$ is a $\tilde{q}$-periodic function in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $\left(\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2}+l^2\frac{\partial^2}{\partial x_2^2}\right)\tilde{S}_{\tilde{q},l,2}=\sum_{z\in\mathbb{Z}^2}\delta_{\tilde{q}z}-1$ in the sense of distributions.

Accordingly, one can readily verify that the assumptions of Lanza de Cristoforis and Musolino [65, (1.8), pp. 78, 79] are satisfied and thus we can apply the results of [65]. Moreover, we note that map from $]0,+\infty[\times\mathcal{U}_0$ to $\mathcal{A}_{\partial\Omega}\cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ which takes (l,ϕ) to $q_l\phi$ is real analytic and then Lemma 4.12 (ii) implies that the map from $]0,+\infty[\times\mathcal{U}_0$ to $C^{0,\alpha}(\partial\Omega)$ which takes (l,ϕ) to $(\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi$ is real analytic. Taking into account Proposition 4.21 (ii), Lanza de Cristoforis and Musolino [65, Theorem 5.10 (i)] implies that the map from $]0,+\infty[\times\mathcal{U}_0$ to $C^{1,\alpha}(\partial\Omega)$ which takes (l,ϕ) to

$$\tilde{V}_{\tilde{q}}[l,\phi,((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)\theta[l,\phi])]\equiv\tilde{v}_{l,\tilde{q}}[\partial\mathbb{I}[\phi],(((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)\theta[l,\phi])\circ\phi^{(-1)})]\circ\phi$$

is real analytic. Then Lemma 4.12 (i), the linearity and continuity of the map from $L^1(\partial\Omega)$ to $\mathbb{R}$ which takes f to $\int_{\partial\Omega} f\,d\sigma$, and equality (4.32) imply that the map from $]0,+\infty[\times\mathcal{U}_0$ to $\mathbb{R}$ which takes (l,ϕ) to

$$\int_{Q_l\setminus q_l\mathbb{I}[\phi]}\frac{\partial}{\partial x_j}v^-_{q_l}[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1})](x)\,d\sigma_x,$$

is real analytic. Accordingly, equality (4.31) implies that the map from $]0,+\infty[\times\mathcal{U}_0$ to $\mathbb{R}$ which takes (l,ϕ) to

$$\int_{Q_l\setminus q_l\mathbb{I}[\phi]} w^-_{q_l}[q_l\partial\mathbb{I}[\phi],\theta[l,\phi]\circ\phi^{(-1)}\circ q_l^{-1}](x)\,dx,$$

is real analytic and then, by equality (4.30), we can conclude that the map from $]0,+\infty[\times\mathcal{U}_0$ to $\mathbb{R}$ which takes the pair (l,ϕ) to $\int_{Q_l\setminus q_l\mathbb{I}[\phi]} u_\#[l,\phi]\,dx$ is real analytic. □

Combining Proposition 4.14 and Theorem 4.22 together with the representation formula (4.11) for $K_{II}[l,\phi]$, we can finally deduce our main result regarding the real analyticity of the longitudinal permeability $K_{II}[l,\phi]$ upon (l,ϕ).

Theorem 4.23. *Let $\alpha \in]0,1[$. Let Ω be a bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega$ is connected. Then the map from $]0,+\infty[\times\left(\mathcal{A}^{\tilde{Q}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\right)$ to $\mathbb{R}$ which takes a pair (l,ϕ) to $K_{II}[l,\phi]$ is real analytic.*

As already mentioned, one of the consequences of the previous Theorem 4.23 is that if we have a one-parameter analytic family of pairs $(l_\delta,\phi_\delta)_{\delta\in]-\delta_0,\delta_0[}$, then we can deduce the possibility to expand the longitudinal permeability as a power series, *i.e.*,

$$K_{II}[l_\delta,\phi_\delta] = \sum_{j=0}^{+\infty} c_j \delta^j \tag{4.33}$$

for δ close to zero. Once an expansion of this type is shown, for practical applications it is of interest to compute the coefficients $\{c_j\}_{j\in\mathbb{N}}$. Dalla Riva, Musolino, and Rogosin [27] developed a completely constructive method to compute the coefficients for the solution of the Dirichlet problem for the Laplace equation in a planar domain with a small hole. The computation is based on the solutions of systems of integral equations. This type of approach can be exploited also in our case for the longitudinal permeability, in order to obtain an explicit expression for all the coefficients $\{c_j\}_{j\in\mathbb{N}}$ in the series (4.33). This could be the object of future investigations and the present chapter of the Dissertation provides the theoretical background for this aim.

Part III

Appendix

APPENDIX A

Harmonic layer potentials and relation with layer heat potentials

In this appendix we recall the definitions and we collect some know properties of the harmonic layer potentials. Moreover we show some relations between harmonic layer potentials and layer heat potentials.

A.1 Harmonic layer potentials

The fundamental solution of the Laplace equation is well known to be the function S_n from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n} \log|x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n \geq 3, \end{cases} \tag{A.1}$$

where

$$s_n \equiv \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. If $\mu \in L^\infty(\partial\Omega)$, we set

$$\begin{aligned} \tilde{v}[\partial\Omega, \mu](x) &\equiv \int_{\partial\Omega} S_n(x-y)\mu(y)\, d\sigma_y \\ &= \begin{cases} \int_{\partial\Omega} \frac{1}{s_n} \log|x-y|\mu(y)\, d\sigma_y & \text{if } n = 2, \\ \int_{\partial\Omega} \frac{1}{(2-n)s_n} |x-y|^{2-n}\mu(y)\, d\sigma_y & \text{if } n \geq 3, \end{cases} \qquad \forall\, x \in \mathbb{R}^n. \end{aligned}$$

The map $\tilde{v}[\partial\Omega, \mu]$ is the harmonic single layer potential with density μ. We collect in the following statement some known properties of the harmonic single layer potential. For a proof we refer, *e.g.*, to Miranda [79], Wiegner [111], Dondi and Lanza de Cristoforis [32] and references therein.

Theorem A.1. *Let $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.*

i) *Let* $\mu \in C^0(\partial\Omega)$. *Then the function* $\tilde{v}[\partial\Omega,\mu]$ *is continuous in* $\mathbb{R}^n$, *is of class* $C^\infty(\mathbb{R}^n \setminus \partial\Omega)$, *and is harmonic in* $\mathbb{R}^n \setminus \partial\Omega$. *Let* $\tilde{v}^+[\partial\Omega,\mu] \equiv \tilde{v}[\partial\Omega,\mu]_{|\mathrm{cl}\,\Omega}$ *denote the restriciton of* $\tilde{v}[\partial\Omega,\mu]$ *to* $\mathrm{cl}\,\Omega$, *and* $\tilde{v}^-[\partial\Omega,\mu] \equiv \tilde{v}[\partial\Omega,\mu]_{|\mathrm{cl}\,\Omega^-}$ *denote the restriction of* $\tilde{v}[\partial\Omega,\mu]$ *to* $\mathrm{cl}\,\Omega^-$.

ii) *If* $\mu \in C^{m-1,\alpha}(\partial\Omega)$, *then the function* $\tilde{v}^+[\partial\Omega,\mu]$ *belongs to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, *and the function* $\tilde{v}^-[\partial\Omega,\mu]$ *belongs to* $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega^-)$. *Moreover the map which takes* μ *to* $\tilde{v}^+[\partial\Omega,\mu]$ *is continuous from* $C^{m-1,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, *and the map which takes* μ *to* $\tilde{v}^-[\partial\Omega,\mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega}$ *is continuous from* $C^{m-1,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)$ *for all* $R \in]0,+\infty[$ *such that* $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$.

iii) *Let* $r \in \{1,\ldots,n\}$. *If* $\mu \in C^{0,\alpha}(\partial\Omega)$, *then the following jump relations hold.*

$$\frac{\partial}{\partial x_r}\tilde{v}^\pm[\partial\Omega,\mu](x) = \mp\frac{1}{2}\mu(x)\nu_r(x) + \int_{\partial\Omega}\frac{\partial}{\partial x_r}S_n(x-y)\mu(y)\,d\sigma_y \qquad \forall\, x \in \partial\Omega,$$

where the integral in the right hand side exists in the sense of the principal value.

Then we turn to the harmonic double layer potential. Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $\mu \in L^\infty(\partial\Omega)$, we set

$$\begin{aligned}\tilde{w}[\partial\Omega,\mu](x) &\equiv \int_{\partial\Omega}\frac{\partial}{\partial\nu(y)}S_n(x-y)\mu(y)\,d\sigma_y \\ &= -\int_{\partial\Omega}\frac{(x-y)^t\nu(y)}{s_n|x-y|^n}\mu(y)\,d\sigma_y \qquad \forall\, x \in \mathbb{R}^n.\end{aligned}$$

The map $\tilde{w}[\partial\Omega,\mu]$ is the harmonic double layer potential with density μ. We collect in the following statement some known properties of the harmonic double layer potential. For a proof we refer, *e.g.*, to Miranda [79], Wiegner [111], Dondi and Lanza de Cristoforis [32] and references therein.

Theorem A.2. *Let* $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$. *Then the following statements hold.*

i) *Let* $\mu \in C^0(\partial\Omega)$. *The function* $\tilde{w}[\partial\Omega,\mu]$ *is of class* $C^\infty(\mathbb{R}^n\setminus\partial\Omega)$ *and is harmonic in* $\mathbb{R}^n\setminus\partial\Omega$. *Moreover the restriction* $\tilde{w}[\partial\Omega,\mu]_\Omega$ *can be extended uniquely to a continuous function* $\tilde{w}^+[\partial\Omega,\mu]$ *from* $\mathrm{cl}\,\Omega$ *to* $\mathbb{C}$, *and the restriction* $\tilde{w}[\partial\Omega,\mu]_{\Omega^-}$ *can be extended uniquely to a continuous function* $\tilde{w}^-[\partial\Omega,\mu]$ *from* $\mathrm{cl}\,\Omega^-$ *to* $\mathbb{C}$. *Moreover the following jump formula hold.*

$$\tilde{w}^\pm[\partial\Omega,\mu](x) = \pm\frac{1}{2}\mu(x) + \tilde{w}[\partial\Omega,\mu](x) \qquad \forall\, x \in \partial\Omega.$$

ii) *If* $\mu \in C^{m,\alpha}(\partial\Omega)$, *then the function* $\tilde{w}^+[\partial\Omega,\mu]$ *belongs to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, *and the function* $\tilde{w}^-[\partial\Omega,\mu]$ *belongs to* $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega^-)$. *Moreover the map which takes* μ *to* $\tilde{w}^+[\partial\Omega,\mu]$ *is continuous from* $C^{m,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, *and the map which takes* μ *to* $\tilde{w}^-[\partial\Omega,\mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega}$ *is continuous from* $C^{m,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)$ *for all* $R \in]0,+\infty[$ *such that* $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$.

A.2 Relation between harmonic and heat layer potentials

In this section we show that if the density μ is time independent, *i.e.* is a function from $\partial\Omega$ to $\mathbb{C}$, then harmonic layer potentials coincide up to a minus sign with the corresponding layer heat potentials. We start with the following concerning the single layer potential.

Lemma A.3. *Let $T \in]\infty, +\infty]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\mu \in L^\infty(\partial\Omega)$. Then*

$$v[\partial_T\Omega, \mu](t, x) = -(\tilde{v}[\partial\Omega, \mu](x) - \delta_{2,n}\tilde{v}[\partial\Omega, \mu](x_0)) \qquad \forall\, (t, x) \in (\mathbb{R}^n)_T,$$

(for the definition of $v[\partial_T\Omega, \mu]$ see (2.4) and (2.5)).

Proof. We first consider the case $n = 2$. We fix $(t, x) \in (\mathbb{R}^n)_T$. Then

$$\begin{aligned} &v[\partial_T\Omega, \mu](t, x) \\ &= \int_{-\infty}^{+\infty} \int_{\partial\Omega} [\Phi_n(t - \tau, x - y) - \Phi_n(-\tau, x_0 - y)]\mu(y)\, d\sigma_y d\tau \\ &= \lim_{h\to+\infty} \int_{-h}^{+\infty} \int_{\partial\Omega} [\Phi_n(t - \tau, x - y) - \Phi_n(-\tau, x_0 - y)]\mu(y)\, d\sigma_y d\tau \\ &= \lim_{h\to+\infty} \Big\{ \int_{-h}^{t} \int_{\partial\Omega} \Phi_n(t - \tau, x - y)\mu(y)\, d\sigma_y d\tau \\ &\qquad - \int_{-h}^{0} \int_{\partial\Omega} \Phi_n(-\tau, x_0 - y)\mu(y)\, d\sigma_y d\tau \Big\}. \end{aligned}$$

By the changes of variable $t - \tau = \frac{|x-y|^2}{4\xi}$ and $-\tau = \frac{|x_0-y|^2}{4\xi}$ in the first and second integrals in the right hand side of the previous equality, respectively, we have

$$\begin{aligned} &v[\partial_T\Omega, \mu](t, x) \\ &= \lim_{h\to+\infty} \Big\{ \int_{\partial\Omega} \int_{\frac{|x-y|^2}{4(t+h)}}^{+\infty} \frac{1}{4\pi\xi} e^{-\xi}\mu(y)\, d\xi d\sigma_y - \int_{\partial\Omega} \int_{\frac{|x_0-y|^2}{4h}}^{+\infty} \frac{1}{4\pi\xi} e^{-\xi}\mu(y)\, d\xi d\sigma_y \Big\} \\ &= \lim_{h\to+\infty} \int_{\partial\Omega} \int_{\frac{|x-y|^2}{4(t+h)}}^{\frac{|x_0-y|^2}{4h}} \frac{1}{4\pi\xi} e^{-\xi}\mu(y)\, d\xi d\sigma_y. \end{aligned}$$

Let g be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$g(\xi) \equiv \begin{cases} \frac{e^{-\xi}-1}{-\xi} & \text{if } \xi \neq 0, \\ 1 & \text{if } \xi = 0. \end{cases}$$

It is easy to see that g is continuous in $\mathbb{R}$ and that

$$\xi^{-1}e^{-\xi} = \xi^{-1} - g(\xi) \qquad \forall\, \xi \in \mathbb{R} \setminus \{0\}$$

Accordingly, the Dominated Convergence Theorem implies that

$$\begin{aligned} &v[\partial_T\Omega, \mu](t, x) \\ &= \lim_{h\to+\infty} \Big\{ \int_{\partial\Omega} \int_{\frac{|x-y|^2}{4(t+h)}}^{\frac{|x_0-y|^2}{4h}} \frac{1}{4\pi\xi}\, d\xi\, \mu(y) d\sigma_y - \int_{\partial\Omega} \int_{\frac{|x-y|^2}{4(t+h)}}^{\frac{|x_0-y|^2}{4h}} \frac{g(\xi)}{4\pi} d\xi\, \mu(y) d\sigma_y \Big\} \\ &= \lim_{h\to+\infty} \Big\{ \int_{\partial\Omega} \frac{1}{4\pi} \log \left| \frac{|x_0 - y|^2}{4h} \frac{4(t + h)}{|x - y|^2} \right| \mu(y)\, d\sigma_y \\ &\qquad - \int_{\partial\Omega} \int_{\frac{|x-y|^2}{4(t+h)}}^{\frac{|x_0-y|^2}{4h}} \frac{g(\xi)}{4\pi} d\xi\, \mu(y) d\sigma_y \Big\} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega} \frac{1}{4\pi} \log\left(\frac{|x_0 - y|^2}{|x - y|^2}\right) \mu(y)\, d\sigma_y \\
&= \int_{\partial\Omega} \frac{1}{2\pi} \log(|x_0 - y|)\, \mu(y) d\sigma_y - \int_{\partial\Omega} \frac{1}{2\pi} \log(|x - y|)\, \mu(y) d\sigma_y \\
&= -\left(\tilde{v}[\partial\Omega, \mu](x) - \tilde{v}[\partial\Omega, \mu](x_0)\right),
\end{aligned}
$$

which proves the statement in the case $n = 2$.

Next we turn to consider the case $n \geq 3$. We fix $(t, x) \in (\mathbb{R}^n)_T$. By the change of variable $|x - y|^2 u = 4(t - \tau)$ we have that

$$
\begin{aligned}
v[\partial_T\Omega, \mu](t, x) &= \int_{-\infty}^{t} \int_{\partial\Omega} \Phi_n(t - \tau, x - y)\mu(y)\, d\sigma_y d\tau \\
&= \int_{-\infty}^{t} \int_{\partial\Omega} \frac{1}{(4\pi(t - \tau))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-\tau)}} \mu(y)\, d\sigma_y d\tau \\
&= \frac{1}{4\pi^{\frac{n}{2}}} \int_{0}^{+\infty} u^{-\frac{n}{2}} e^{-\frac{1}{u}}\, du \int_{\partial\Omega} \frac{1}{|x - y|^{n-2}} \mu(y)\, d\sigma_y d\tau \\
&= \frac{1}{4\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2} - 1\right) \int_{\partial\Omega} \frac{1}{|x - y|^{n-2}} \mu(y)\, d\sigma_y d\tau \\
&= \frac{1}{2\pi^{\frac{n}{2}}(n - 2)} \Gamma\left(\frac{n}{2}\right) \int_{\partial\Omega} \frac{1}{|x - y|^{n-2}} \mu(y)\, d\sigma_y d\tau \\
&= \frac{1}{(n - 2)s_n} \int_{\partial\Omega} \frac{1}{|x - y|^{n-2}} \mu(y)\, d\sigma_y d\tau \\
&= -\tilde{v}[\partial\Omega, \mu](x),
\end{aligned}
$$

which proves the statement in the case $n \geq 3$. □

Then we turn to consider the case of the double layer potential.

Lemma A.4. *Let $\alpha \in]0, 1[$. Let $T \in]\infty, +\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in L^\infty(\partial\Omega)$. Then*

$$
w[\partial_T\Omega, \mu](t, x) = -\tilde{w}[\partial\Omega, \mu](x) \qquad \forall\, (t, x) \in (\mathbb{R}^n)_T,
$$

(for the definition of $w[\partial_T\Omega, \mu]$ see (2.1)).

Proof. We fix $(t, x) \in (\mathbb{R}^n)_T$. By the change of variable $|x - y|^2 u = 4(t - \tau)$ we have that

$$
\begin{aligned}
w[\partial_T\Omega, \mu](t, x) &= \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial\nu(y)} \Phi_n(t - \tau, x - y)\mu(y)\, d\sigma_y d\tau \\
&= \int_{-\infty}^{t} \int_{\partial\Omega} \frac{(x - y)^t \nu(y)}{2(4\pi)^{\frac{n}{2}} (t - \tau)^{\frac{n}{2}+1}} e^{-\frac{|x-y|^2}{4(t-\tau)}} \mu(y)\, d\sigma_y d\tau \\
&= \frac{1}{2\pi^{\frac{n}{2}}} \int_{0}^{+\infty} u^{-\frac{n}{2}-1} e^{-\frac{1}{u}}\, du \int_{\partial\Omega} \frac{(x - y)^t \nu(y)}{|x - y|^n} \mu(y)\, d\sigma_y \\
&= \frac{1}{2\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right) \int_{\partial\Omega} \frac{(x - y)^t \nu(y)}{|x - y|^n} \mu(y)\, d\sigma_y \\
&= \frac{1}{s_n} \int_{\partial\Omega} \frac{(x - y)^t \nu(y)}{|x - y|^n} \mu(y)\, d\sigma_y \\
&= -\tilde{w}[\partial\Omega, \mu](x).
\end{aligned}
$$

Accordingly, the statement follows. □

Finally we have the following immediate corollary of lemma A.3 and of classical differentiation theorems for integrals depending on a parameter.

Corollary A.5. *Let* $T \in]\infty, +\infty]$*. Let* Ω *be a bounded open Lipschitz subset of* $\mathbb{R}^n$*. Let* $\mu \in L^\infty(\partial\Omega)$*. Then*

$$\int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial}{\partial t} \Phi_n(t-\tau, x-y)\mu(y)\, d\sigma_y d\tau = 0 \qquad \forall\, (t,x) \in (\mathbb{R}^n \setminus \partial\Omega)_T.$$

APPENDIX B

Schauder regularity of heat potentials in $\mathrm{cl}\,\Omega_T$

In this appendix we prove some regularity results for layer heat potentials that we have exploited in the Dissertation which may be considered as *folklore* and for which we could not provide a proper reference. For the definitions of single and double layer heat potentials see (2.4), (2.5) and (2.1).

B.1 Derivatives of layer heat potentials

As a first step, we prove some formulas for time and space derivatives for the single and the double layer heat potentials. We start with the following easy consequence of differentiation under integral sign.

Lemma B.1. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in L^\infty(\partial_T\Omega)$. Then*

$$w[\partial_T\Omega,\mu](t,x) = -\sum_{j=1}^n \frac{\partial}{\partial x_j} v[\partial_T\Omega,\nu_j\mu](t,x) \qquad \forall\,(t,x) \in (\mathbb{R}^n \setminus \partial\Omega)_T. \tag{B.1}$$

Proof. We note that

$$\begin{aligned}\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y) &= -\sum_{j=1}^n \frac{\partial}{\partial x_j}\Phi_n(t-\tau,x-y)\nu_j(y)\\ &= -\sum_{j=1}^n \frac{\partial}{\partial x_j}\Big(\Phi_n(t-\tau,x-y) - \delta_{2,n}\Phi_n(-\tau,x_0-y)\Big)\nu_j(y)\end{aligned}$$

for all $t,\tau \in \overline{]-\infty,T[}$, $x \in \mathbb{R}^n \setminus \partial\Omega$, $y \in \partial\Omega$. Then the statement follows by the definition of layer heat potentials and by classical differentiation theorems for integrals depending on a parameter. □

Next, we prove the following formula which describes the space derivatives of the double layer heat potential in terms of space and time derivatives of the single layer heat potential.

Lemma B.2. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$, $i \in \{1,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in C^{\frac{1}{2};1}(\partial_T\Omega)$. Then*

$$\frac{\partial}{\partial x_i} w[\partial_T\Omega,\mu](t,x) \tag{B.2}$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} v[\partial_T\Omega, M_{ij}[\mu]](t,x) - \frac{\partial}{\partial t} v[\partial_T\Omega, \nu_i\mu](t,x) \qquad \forall\, (t,x) \in (\mathbb{R}^n \setminus \partial\Omega)_T.$$

Proof. First we note that

$$w[\partial_T\Omega, \mu](t,x) = \int_{-\infty}^{t} \int_{\partial\Omega} \sum_{j=1}^{n} \frac{\partial}{\partial y_j} \Phi_n(t-\tau, x-y)\nu_j(y)\mu(\tau,y)\, d\sigma_y d\tau,$$

for all $(t,x) \in (\mathbb{R}^n \setminus \partial\Omega)_T$. Since Φ_n solves the heat equation we have that

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\Big[\sum_{j=1}^{n} \nu_j(y) \frac{\partial}{\partial y_j}\Phi_n(t-\tau, x-y)\Big] \\
&\quad = \sum_{j=1}^{n} \Big[\nu_j(y)\frac{\partial}{\partial y_i} - \nu_i(y)\frac{\partial}{\partial y_j}\Big]\Big(\frac{\partial}{\partial x_j}\Phi_n(t-\tau, x-y)\Big) \\
&\qquad - \nu_i(y)\frac{\partial}{\partial t}\Phi_n(t-\tau, x-y) \\
&\quad = \sum_{j=1}^{n} M_{ji,y}\Big[\frac{\partial}{\partial x_j}\Big(\Phi_n(t-\tau, x-y) - \delta_{2,n}\Phi_n(-\tau, x_0-y)\Big)\Big] \\
&\qquad - \nu_i(y)\frac{\partial}{\partial t}\Big(\Phi_n(t-\tau, x-y) - \delta_{2,n}\Phi_n(-\tau, x_0-y)\Big),
\end{aligned}$$

for all $t, \tau \in \overline{]-\infty, T[}$, $x \in \mathbb{R}^n \setminus \partial\Omega$, $y \in \partial\Omega$. Then classical differentiation theorems for integrals depending on a parameter and the consequence of the Divergence Theorem for the tangential derivatives of Lemma 1.4 imply that

$$\begin{aligned}
&\frac{\partial}{\partial x_i} w[\partial_T\Omega, \mu](t,x) \\
&= \int_{-\infty}^{t}\int_{\partial\Omega} \sum_{j=1}^{n} M_{ji,y}\Big[\frac{\partial}{\partial x_j}\Big(\Phi_n(t-\tau, x-y) - \delta_{2,n}\Phi_n(-\tau, x_0-y)\Big)\Big]\mu(\tau,y)\, d\sigma_y d\tau \\
&\quad - \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Big(\Phi_n(t-\tau, x-y) - \delta_{2,n}\Phi_n(-\tau, x_0-y)\Big)\nu_i(y)\mu(\tau,y)\, d\sigma_y d\tau \\
&= \sum_{j=1}^{n} \int_{-\infty}^{+\infty}\int_{\partial\Omega} \frac{\partial}{\partial x_j}\Big(\Phi_n(t-\tau, x-y) - \delta_{2,n}\Phi_n(-\tau, x_0-y)\Big) M_{ij,y}[\mu](\tau,y)\, d\sigma_y d\tau \\
&\quad - \int_{-\infty}^{+\infty}\int_{\partial\Omega} \frac{\partial}{\partial t}\Big(\Phi_n(t-\tau, x-y) - \delta_{2,n}\Phi_n(-\tau, x_0-y)\Big)\nu_i(y)\mu(\tau,y)\, d\sigma_y d\tau \\
&= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} v[\partial_T\Omega, M_{ij}[\mu]](t,x) - \frac{\partial}{\partial t} v[\partial_T\Omega, \nu_i\mu](t,x),
\end{aligned}$$

for all $(t,x) \in (\mathbb{R}^n \setminus \partial\Omega)_T$. Accordingly, formula (B.2) is proved. □

In the following lemma we turn to consider the space derivatives of the single layer heat potential.

Lemma B.3. *Let* $\alpha \in]0,1[$, $T \in]-\infty, +\infty]$, $i \in \{1, \dots, n\}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Let* $\mu \in C^{\frac{1}{2};1}(\partial_T\Omega)$. *Then*

$$\frac{\partial}{\partial x_i} v[\partial_T\Omega, \mu](t,x) \tag{B.3}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n} v[\partial_T\Omega, M_{ij}[\mu\nu_j]](t,x) - w[\partial_T\Omega, \mu\nu_i](t,x) \\
&\quad + \delta_{2,n}\int_{-\infty}^{0}\int_{\partial\Omega}\sum_{j=1}^{n} M_{ij,y}[\Phi_n(-\tau, x_0 - y)]\nu_j(y)\mu(\tau,y)\,d\sigma_y d\tau \qquad \forall\,(t,x)\in(\mathbb{R}^n\setminus\partial\Omega)_T
\end{aligned}
$$

Proof. Since $\sum_{j=1}^{n}\nu_j^2 = 1$, then

$$
\begin{aligned}
&\frac{\partial}{\partial y_i}\Phi_n(t-\tau, x-y) \\
&= \sum_{j=1}^{n}\nu_j^2(y)\frac{\partial}{\partial y_i}\Phi_n(t-\tau,x-y) \\
&= \sum_{j=1}^{n}\nu_j^2(y)\frac{\partial}{\partial y_i}\Phi_n(t-\tau,x-y) - \sum_{j=1}^{n}\nu_i(y)\nu_j(y)\frac{\partial}{\partial y_j}\Phi_n(t-\tau,x-y) \\
&\quad + \sum_{j=1}^{n}\nu_i(y)\nu_j(y)\frac{\partial}{\partial y_j}\Phi_n(t-\tau,x-y) \\
&= -\sum_{j=1}^{n} M_{ij,y}[\Phi_n(t-\tau,x-y)]\nu_j(y) + \sum_{j=1}^{n}\nu_i(y)\nu_j(y)\frac{\partial}{\partial y_j}\Phi_n(t-\tau,x-y),
\end{aligned}
$$

for all $(t,x)\in(\mathbb{R}^n\setminus\partial\Omega)_T$, $\tau\in\mathbb{R}$, $y\in\partial\Omega$. Then the consequence of the Divergence Theorem for the tangential derivatives of Lemma 1.4 and classical differentiation theorems for integrals depending on a parameter imply that

$$
\begin{aligned}
&\frac{\partial}{\partial x_i} v[\partial_T\Omega,\mu](t,x) \\
&= \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial x_i}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&= -\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial y_i}\Phi_n(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau \\
&= \int_{-\infty}^{+\infty}\int_{\partial\Omega}\sum_{j=1}^{n} M_{ij,y}[\Phi_n(t-\tau,x-y) - \delta_{2,n}\Phi_n(-\tau,x_0-y)]\nu_j(y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\quad + \int_{-\infty}^{0}\int_{\partial\Omega}\sum_{j=1}^{n} M_{ij,y}[\delta_{2,n}\Phi_n(-\tau,x_0-y)]\nu_j(y)\mu(\tau,y)\,d\sigma_y d\tau \\
&\quad - \int_{-\infty}^{t}\int_{\partial\Omega}\sum_{j=1}^{n}\frac{\partial}{\partial y_j}\Phi_n(t-\tau,x-y)\nu_j(y)\nu_i(y)\mu(\tau,y)\,d\sigma_y d\tau \\
&= -\sum_{j=1}^{n} v[\partial_T\Omega, M_{ij}[\mu\nu_j]](t,x) - w[\partial_T\Omega,\mu\nu_i](t,x) \\
&\quad + \delta_{2,n}\int_{-\infty}^{0}\int_{\partial\Omega}\sum_{j=1}^{n} M_{ij,y}[\Phi_n(-\tau,x_0-y)]\nu_j(y)\mu(\tau,y)\,d\sigma_y d\tau,
\end{aligned}
$$

for all $(t,x)\in(\mathbb{R}^n\setminus\partial\Omega)_T$, and then the statement follows. □

Finally, we consider the time derivatives of the single and the double layer heat potentials. We can prove the following easy consequence of integration by parts.

Lemma B.4. *Let $T \in]-\infty, +\infty]$, $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{2,\alpha}$. Let $\mu \in C^{1;2}(\partial_T\Omega)$. Then the following statements hold.*

(i)

$$\frac{\partial}{\partial t}v[\partial_T\Omega,\mu](t,x) = v[\partial_T\Omega,\partial_t\mu](t,x) - \delta_{2,n}\int_{-\infty}^{0}\int_{\partial\Omega}\frac{\partial}{\partial\tau}\Phi_n(-\tau,x_0-y)\mu(\tau,y)d\sigma_y\,d\tau \qquad \forall\,(t,x)\in(\mathbb{R}^n\setminus\partial\Omega)_T. \tag{B.4}$$

(ii)

$$\frac{\partial}{\partial t}w[\partial_T\Omega,\mu](t,x) = w[\partial_T\Omega,\partial_t\mu](t,x) \qquad \forall\,(t,x)\in(\mathbb{R}^n\setminus\partial\Omega)_T. \tag{B.5}$$

Proof. We first consider statement (i). By classical differentiation theorems for integrals depending on a parameter and by integration by parts we have that

$$\begin{aligned}
&\frac{\partial}{\partial t}v[\partial_T\Omega,\mu](t,x)\\
&= \int_{-\infty}^{+\infty}\int_{\partial\Omega}\frac{\partial}{\partial t}\Big(\Phi(t-\tau,x-y)-\delta_{2,n}\Phi_n(-\tau,x_0-y)\Big)\mu(\tau,y)d\sigma_y\,d\tau\\
&= -\int_{-\infty}^{+\infty}\int_{\partial\Omega}\frac{\partial}{\partial\tau}\Big(\Phi(t-\tau,x-y)-\delta_{2,n}\Phi_n(-\tau,x_0-y)\Big)\mu(\tau,y)d\sigma_y\,d\tau\\
&\quad -\delta_{2,n}\int_{-\infty}^{0}\int_{\partial\Omega}\frac{\partial}{\partial\tau}\Phi_n(-\tau,x_0-y)\mu(\tau,y)d\sigma_y\,d\tau\\
&= v[\partial_T\Omega,\partial_t\mu](t,x)-\delta_{2,n}\int_{-\infty}^{0}\int_{\partial\Omega}\frac{\partial}{\partial\tau}\Phi_n(-\tau,x_0-y)\mu(\tau,y)d\sigma_y\,d\tau,
\end{aligned}$$

for all $(t,x)\in(\mathbb{R}^n\setminus\partial\Omega)_T$, which proves statement (i).

Next we consider statement (ii). Another time by classical differentiation theorems for integrals depending on a parameter and by integration by parts we have

$$\begin{aligned}
&\frac{\partial}{\partial t}w[\partial_T\Omega,\mu](t,x)\\
&= \int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial t}\frac{\partial}{\partial\nu(y)}\Phi(t-\tau,x-y)\mu(\tau,y)d\sigma_y\,d\tau\\
&= -\int_{-\infty}^{t}\int_{\partial\Omega}\frac{\partial}{\partial\tau}\frac{\partial}{\partial\nu(y)}\Phi(t-\tau,x-y)\mu(\tau,y)d\sigma_y\,d\tau\\
&= w[\partial_T\Omega,\partial_t\mu](t,x)
\end{aligned}$$

for all $(t,x)\in(\mathbb{R}^n\setminus\partial\Omega)_T$, which proves statement (ii). □

B.2 Regularity of layer heat potentials

Ladyzheskaia, Solonnikov and Ural'tseva in [58, Chapter 4.2] prove the Schauder regularity of the layer heat potentials $v[\partial_T\Omega,\mu]$ and $w[\partial_T\Omega,\mu]$ with respect to the Schauder regularity of the density function μ when the domain Ω is the half-space

$$\mathbb{R}^n_+ = \mathbb{R}^n \cap \{x_n > 0\}.$$

In this Section we will adapt the treatment of [58] to the case in which the domain Ω is a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. We plan to prove first the result for Schauder spaces of lower order, and then deduce the general case by induction, by exploiting the formulas for the derivatives we have stated in the previous section.

The results contained in this section can be considered as *mathematical folklore*, that is, they are known results, however, to the best of our knowledge, it is not possible to find a complete exposition in the literature. In this direction we mention, in addition to the already cited Ladyzheskaia, Solonnikov and Ural'tseva [58], also the work of Baderko [8, Theorem 3.4] where the author proves that the single layer heat potential is bounded from $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ to $C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\Omega_T)$ under the assumption that Ω is of class $C^{1,\alpha}$, and the series of works by Kamynin [50], [51], [52], [53] where the author proves results in the same spirit but under stronger regularity assumption on Ω.

Let $\alpha \in]0,1[$, $m \in \mathbb{N}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. The definition of sets of class $C^{m,\alpha}$ implies that for all $P \in \partial\Omega$ there exists an open neighborhood W_P of P in $\mathbb{R}^n$, and a diffeomorphism $\phi_P \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n, \mathbb{R}^n)$ of $\mathbb{B}_n$ onto W_P such that $\phi_P(0) = P$, $\phi_P(\{x \in \mathbb{B}_n : x_n = 0\}) = W_P \cap \partial\Omega$, and $\phi_P(\{x \in \mathbb{B}_n : x_n < 0\}) = W_P \cap \Omega$. We set

$$V_P \equiv W_P \cap \partial\Omega,$$

which is an open set in $\partial\Omega$. By the compactness of $\partial\Omega$ there exist $r \in \mathbb{N}\setminus\{0\}$ and $P_1, \ldots, P_r \in \partial\Omega$ such that

$$\partial\Omega = \bigcup_{j=1}^{r} V_{P_j}.$$

Let $\{\eta_i\}_{j=1,\ldots,r}$ be a partition of unity of $\partial\Omega$ of class $C^{m,\alpha}$ subordinate to the open covering $\{V_{P_j}\}_{j=1,\ldots,r}$ of $\partial\Omega$. We note that if $\mu \in C^{\frac{h+\beta}{2};h+\beta}(\partial_T\Omega)$, where $\beta \in]0,1[$, $h \in \mathbb{N}$, and $h+\beta \leq m+\alpha$, then

$$\mu(t,x) = \sum_{j=1}^{r} \eta_j(x)\mu(t,x) \qquad \forall\,(t,x) \in \partial_T\Omega,$$

and

(i) $\eta_j\mu \in C^{\frac{h+\beta}{2};h+\beta}(\partial_T\Omega)$ for all $j = 1, \ldots, r$,

(ii) $\mathrm{supp}(\eta_j\mu) \subseteq (V_{P_j})_T$ for all $j = 1, \ldots, r$.

Then, if we want to estimate the Schauder norm of a layer heat potential with density μ it suffices to estimate the Schauder norm of the same layer heat potential with density $\eta_j\mu$, which has support in $(V_{P_j})_T$, for all $j = 1, \ldots, n$. Accordingly, without loss of generality we can suppose that all the density functions have support in the infinite cylinder with basis equal to the domain V_P of a local parametrization around a point $P \in \partial\Omega$. More precisely, in all the proofs we can suppose that all density functions μ are such that

$$\mathrm{supp}\,(\mu) \subseteq (V_P)_T.$$

Up to rotation and translation of the domain we can further assume that P is the origin, *i.e.*, $P = 0$, and that there exists $\psi_P \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_{n-1})$ such that

$$\phi_P(x_1, \ldots, x_{n-1}, 0) = (x_1, \ldots, x_{n-1}, \psi_P(x_1, \ldots, x_{n-1})) \qquad \forall\,(x_1, \ldots, x_n) \in \mathrm{cl}\,\mathbb{B}_{n-1}.$$

From now on if $x = (\bar{x}, x_n) \in \mathbb{R}^n$, $\bar{x}$ denotes the vector of the first $n-1$ components of x. Moreover, for $m \in \mathbb{N} \setminus \{0\}$, we set

$$J(y) \equiv \sqrt{1 + |D\psi_P(y)|^2} \qquad \forall\, y \in \mathrm{cl}\,\mathbb{B}_{n-1}.$$

We note that clearly

$$C_J \equiv \sup_{y \in \mathrm{cl}\,\mathbb{B}_{n-1}} J(y) < +\infty.$$

We are ready to start considering the Schauder regularity of layer heat potentials. We start with the following lemma on the time regularity of the single layer heat potential.

Lemma B.5. *Let $\alpha \in]0,1[$, $T \in]-\infty, +\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) Let $n \geq 3$. Then the map from $C_b^{\frac{\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\mathrm{cl}\,\Omega)\right)$ which takes μ to $v^+[\partial_T\Omega, \mu]$ is linear and continuous. Moreover, the map from the space $C_b^{\frac{\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\mathrm{cl}\,\Omega^-)\right)$ which takes μ to $v^-[\partial_T\Omega, \mu]$ is linear and continuous.

(ii) Let $n = 2$. Then the map from $C_b^{\frac{\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\mathrm{cl}\,\Omega)\right)$ which takes μ to $v^+[\partial_T\Omega, \mu]$ is linear and continuous. Moreover, the map from the space $C_b^{\frac{\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\mathrm{cl}\,\Omega^-)\right)$ which takes μ to $v^-[\partial_T\Omega, \mu]$ is linear and continuous.

Proof. First we consider statements (i) and (ii) for the interior single layer heat potential $v^+[\partial_T\Omega, \cdot]$ together. Let $\mu \in C_b^{\frac{\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$. As already pointed out at the beginning of this section, we can assume that $\mathrm{supp}\,(\mu) \subseteq (V_P)_T$.

If $n \geq 3$, Lemma 1.15 (i) implies that the kernel Φ_n satisfies the assumptions of Lemma 1.23, which implies that the map $v^+[\partial_T\Omega, \cdot]$ is bounded from $C_b^{\frac{\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C_b^0(\mathrm{cl}\,\Omega_T)$.

Now let $n \geq 2$. We note that

$$v^+[\partial_T\Omega, \mu](t,x) = v^+[\partial_T\Omega, \mu - \mu(t^*, \cdot)](t,x) + v^+[\partial_T\Omega, \mu(t^*, \cdot)](t,x) \tag{B.6}$$

for all $(t^*, x), (t,x) \in \Omega_T$. Now we fix $(t', x), (t'', x) \in \Omega_T$, with $t' < t''$. The equality (B.6) implies that

$$v^+[\partial_T\Omega, \mu](t'', x) = v^+[\partial_T\Omega, \mu - \mu(t', \cdot)](t'', x) + v^+[\partial_T\Omega, \mu(t', \cdot)](t'', x),$$

and

$$v^+[\partial_T\Omega, \mu](t', x) = v^+[\partial_T\Omega, \mu - \mu(t', \cdot)](t', x) + v^+[\partial_T\Omega, \mu(t', \cdot)](t', x).$$

We now note that Lemma A.3 implies that

$$v^+[\partial_T\Omega, \mu(t', \cdot)](t'', x) = v^+[\partial_T\Omega, \mu(t', \cdot)](t', x) = -(\tilde{v}[\partial\Omega, \mu(t', \cdot)](x) - \delta_{2,n}\tilde{v}[\partial\Omega, \mu(t', \cdot)](x_0)).$$

Then

$$\begin{aligned} & v^+[\partial_T\Omega, \mu](t'', x) - v^+[\partial_T\Omega, \mu](t', x) \\ &\quad = v^+[\partial_T\Omega, \mu - \mu(t', \cdot)](t'', x) - v^+[\partial_T\Omega, \mu - \mu(t', \cdot)](t', x) \end{aligned} \tag{B.7}$$

$$
\begin{aligned}
&= \int_{t''-2|t''-t'|}^{t''} \int_{\partial\Omega} \Phi_n(t''-\tau, x-y)(\mu(\tau,y)-\mu(t'',y))\, d\sigma_y d\tau \\
&\quad - \int_{t''-2|t''-t'|}^{t'} \int_{\partial\Omega} \Phi_n(t'-\tau, x-y)(\mu(\tau,y)-\mu(t',y))\, d\sigma_y d\tau \\
&\quad + \int_{-\infty}^{t''-2|t''-t'|} \int_{\partial\Omega} (\Phi_n(t''-\tau, x-y)-\Phi_n(t'-\tau,x-y))(\mu(\tau,y)-\mu(t',y))\, d\sigma_y d\tau \\
&\quad + \int_{\partial\Omega} (\mu(t'',y)-\mu(t',y)) \int_0^{2|t''-t'|} \Phi_n(\tau, x-y)\, d\tau d\sigma_y.
\end{aligned}
$$

First we consider the first term in the right hand side of (B.7).

$$
\begin{aligned}
&\Big| \int_{t''-2|t''-t'|}^{t''} \int_{\partial\Omega} \Phi_n(t''-\tau, x-y)(\mu(\tau,y)-\mu(t'',y))\, d\sigma_y d\tau \Big| \\
&\leq \int_{t''-2|t''-t'|}^{t''} \int_{\partial\Omega} \Phi_n(t''-\tau, x-y)|\mu(\tau,y)-\mu(t'',y)|\, d\sigma_y d\tau \\
&= \int_{t'-2|t''-t'|}^{t''} \int_{V_P} \Phi_n(t''-\tau, x-y)|\mu(\tau,y)-\mu(t'',y)|\, d\sigma_y d\tau \\
&\leq \frac{\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{(4\pi)^{\frac{n}{2}}} \int_{t''-2|t''-t'|}^{t''} \int_{V_P} (t''-\tau)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t''-\tau)}} (t''-\tau)^{\frac{\alpha}{2}}\, d\sigma_y d\tau \\
&= \frac{\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{(4\pi)^{\frac{n}{2}}} \int_{t''-2|t''-t'|}^{t''} (t''-\tau)^{\frac{\alpha-n}{2}} \int_{\mathbb{B}_{n-1}} e^{-\frac{|\bar{x}-\bar{y}|^2}{4(t''-\tau)}} e^{-\frac{(x_n-\psi_P(\bar{y}))^2}{4(t''-\tau)}} J(\bar{y})\, d\bar{y} d\tau \\
&\leq \frac{C_J\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{(4\pi)^{\frac{n}{2}}} \int_{t''-2|t''-t'|}^{t''} (t''-\tau)^{\frac{\alpha-1}{2}} \left\{ \frac{1}{(t''-\tau)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|\bar{x}-\bar{y}|^2}{4(t''-\tau)}}\, d\bar{y} \right\} d\tau.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Big| \int_{t''-2|t''-t'|}^{t''} \int_{\partial\Omega} \Phi_n(t''-\tau, x-y)(\mu(\tau,y)-\mu(t'',y))\, d\sigma_y d\tau \Big| \\
&\qquad \leq \frac{C_J\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{\sqrt{4\pi}} \int_{t''-2|t''-t'|}^{t''} (t''-\tau)^{\frac{\alpha-1}{2}}\, d\tau \\
&\qquad = \frac{2^{\frac{\alpha+3}{2}} C_J\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{(\alpha+1)\sqrt{4\pi}} (t''-t')^{\frac{1+\alpha}{2}}.
\end{aligned}
$$

The second term in the right hand side of (B.7) can be estimated in the same way, by switching the roles of t'' and t'. Next we consider the third term in the right han side of (B.7). Let $\tau \in]-\infty, t''-2|t'-t''|[$ and $y \in \partial\Omega$. Then Lemma 1.15 (iii) implies that there exists a constant $\tilde{C}'_{0,16,\mathrm{cl}\,\Omega} \in]0,+\infty[$ such that

$$
|\Phi_n(t''-\tau, x-y) - \Phi_n(t'-\tau, x-y)| \leq \tilde{C}'_{0,16,\mathrm{cl}\,\Omega}(t''-t') \frac{e^{-\frac{|x-y|^2}{16(t'-\tau)}}}{(t'-\tau)^{\frac{n}{2}+1}},
$$

for all $\tau < t''-2|t'-t''|, y \in \partial\Omega$. Then

$$
\Big| \int_{-\infty}^{t''-2|t''-t'|} \int_{\partial\Omega} (\Phi_n(t''-\tau, x-y)-\Phi_n(t'-\tau,x-y))(\mu(\tau,y)-\mu(t',y))\, d\sigma_y d\tau \Big|
$$

$$\leq \tilde{C}'_{0,16,\mathrm{cl}\,\Omega}\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t')\int_{-\infty}^{t''-2|t''-t'|}\int_{V_P}(t'-\tau)^{\frac{\alpha-n}{2}-1}e^{-\frac{|x-y|^2}{16(t'-\tau)}}\,d\sigma_y d\tau$$

$$\leq (16\pi)^{\frac{n-1}{2}}\tilde{C}'_{0,16,\mathrm{cl}\,\Omega}C_J\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t')\int_{-\infty}^{t''-2|t''-t'|}(t'-\tau)^{\frac{\alpha-3}{2}}\,d\tau$$

$$=\frac{2(16\pi)^{\frac{n-1}{2}}\tilde{C}'_{0,16,\mathrm{cl}\,\Omega}C_J\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{(1-\alpha)}(t''-t')^{\frac{1+\alpha}{2}}.$$

Finally we consider the fourth term in the right hand side of (B.7).

$$\left|\int_{\partial\Omega}(\mu(t'',y)-\mu(t',y))\int_0^{2|t''-t'|}\Phi_n(\tau,x-y)d\tau d\sigma_y\right|$$

$$\leq \|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t'')^{\frac{\alpha}{2}}\int_0^{2|t''-t'|}\int_{V_P}\Phi_n(\tau,x-y)\,d\sigma_y d\tau$$

$$=\frac{\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t')^{\frac{\alpha}{2}}}{(4\pi)^{\frac{n}{2}}}\int_0^{2|t''-t'|}\int_{V_P}\tau^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4\tau}}\,d\sigma_y d\tau$$

$$\leq \frac{C_J\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t')^{\frac{\alpha}{2}}}{(4\pi)^{\frac{n}{2}}}\int_0^{2|t''-t'|}\tau^{-\frac{1}{2}}\left\{\tau^{-\frac{n-1}{2}}\int_{\mathbb{R}^{n-1}}e^{-\frac{|\bar{x}-\bar{y}|^2}{4\tau}}\,d\bar{y}\right\}d\tau$$

$$=\frac{C_J\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t')^{\frac{\alpha}{2}}}{\sqrt{4\pi}}\int_0^{2|t''-t'|}\tau^{-\frac{1}{2}}\,d\tau$$

$$=\frac{2\sqrt{2}C_J\|\mu\|_{C_b^{\frac{\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{\sqrt{4\pi}}(t''-t')^{\frac{1+\alpha}{2}}.$$

Thus we can conclude that statements (i) and (ii) hold for $v^+[\partial_T\Omega,\cdot]$.

The proof of statement (i) and (ii) for $v^-[\partial_T\Omega,\cdot]$ follows exactly the same lines. □

Next we consider the following lemma about the regularity of the map $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$.

Lemma B.6. *Let* $\alpha\in]0,1[$, $T\in]-\infty,+\infty]$, $i\in\{1,\dots,n\}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$. *Let* $R\in]0,+\infty[$ *such that* $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. *Then the map from* $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ *to* $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$ *which takes* μ *to* $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\mu]$ *is linear and continuous. Moreover, the map from* $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ *to* $C^{\frac{\alpha}{2};\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T)$, *which takes* μ *to* $\frac{\partial}{\partial x_i}v^-[\partial_T\Omega,\mu]$, *is linear and continuous.*

Proof. We first prove the statement for $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$. Let $\mu\in C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. As already pointed out at the beginning of this section, we can assume that $\mathrm{supp}\,(\mu)\subseteq(V_P)_T$. Clearly

$$\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\mu](t,x)=\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\mu-\mu(t,\cdot)](t,x)+\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\mu(t,\cdot)](t,x), \tag{B.8}$$

for all $(t,x)\in\Omega_T$. Lemma A.3 implies that

$$\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\mu(t,\cdot)](t,x)=-\frac{\partial}{\partial x_i}\tilde{v}^+[\partial\Omega,\mu(t,\cdot)](x),$$

for all $(t,x)\in\Omega_T$. Then, the classical regularity properties of the harmonic single layer potential of Theorem A.1 imply that

$$\left\|\frac{\partial}{\partial x_i}\tilde{v}[\partial\Omega,\mu(t,\cdot)]\right\|_{C^{0,\alpha}(\mathrm{cl}\,\Omega)}\leq\|\mu(t,\cdot)\|_{C^{0,\alpha}(\partial\Omega)}\left\|\frac{\partial}{\partial x_i}\tilde{v}[\partial\Omega,\cdot]\right\|_{\mathcal{L}(C^{0,\alpha}(\partial\Omega);C^{0,\alpha}(\mathrm{cl}\,\Omega))},$$

for all $t \in \overline{]-\infty, T[}$. Taking the supremum over t in $\overline{]-\infty, T[}$, we have

$$\left\| \frac{\partial}{\partial x_i} \tilde{v}^+[\partial\Omega, \mu] \right\|_{B\left(\overline{]-\infty,T[};C^{0,\alpha}(\mathrm{cl}\,\Omega)\right)} \leq \|\mu\|_{C^{\frac{\alpha}{2};\alpha}(\partial\Omega)} \left\| \frac{\partial}{\partial x_i} \tilde{v}^+[\partial\Omega, \cdot] \right\|_{\mathcal{L}(C^{0,\alpha}(\partial\Omega);C^{0,\alpha}(\mathrm{cl}\,\Omega))} .$$

Thus, to estimate the L^∞-norm and the space Hölder quotient of $\frac{\partial}{\partial x_i} v^+[\partial_T\Omega, \mu]$, we only need to consider the first term in right hand side of equality (B.8). The time Hölder continuity of μ and Lemma 1.16 (i) imply that there exists a constant $C_{0,1,\mathbb{R}^n} \in]0, +\infty[$ such that

$$\left| \frac{\partial}{\partial x_i} \Phi_n(t-\tau, x-y)(\mu(\tau, y) - \mu(t, y)) \right| \leq C_{0,1,\mathbb{R}^n} \|\mu\|_{C^{\frac{\alpha}{2};\alpha}(\partial\Omega)} \frac{|x-y|}{(t-\tau)^{\frac{n}{2}+1-\frac{\alpha}{2}}} e^{-\frac{|x-y|^2}{4(t-\tau)}},$$

for all $(t,x) \in \mathrm{cl}\,\Omega_T$, $(\tau, y) \in \partial_T\Omega$, $\tau < t$. Accordingly, Lemma 1.23 implies that the map which takes μ to the function

$$\frac{\partial}{\partial x_i} v^+[\partial_T\Omega, \mu - \mu(t,\cdot)](t,x) \qquad \forall\, (t,x) \in \mathrm{cl}\,\Omega_T, \tag{B.9}$$

is bounded from $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

Next we consider the space Hölder quotient of the function in (B.9). We fix $x', x'' \in \Omega$, $t \in \overline{]-\infty, T[}$. Classical differentiation theorems for integrals depending on a parameter imply that

$$\begin{aligned} &\frac{\partial}{\partial x_i} v^+[\partial_T\Omega, \mu - \mu(t,\cdot)](t,x') - \frac{\partial}{\partial x_i} v^+[\partial_T\Omega, \mu - \mu(t,\cdot)](t,x'') \qquad \text{(B.10)} \\ &= \int_0^{+\infty} \int_{\partial\Omega} \left[\frac{\partial}{\partial x_i} \Phi_n(\tau, x'-y) - \frac{\partial}{\partial x_i} \Phi_n(\tau, x''-y) \right] [\mu(t-\tau, y) - \mu(t,y)]\, d\sigma_y d\tau \\ &= \int_0^{|x'-x''|^2} \int_{V_P} \frac{\partial}{\partial x_i} \Phi_n(\tau, x'-y)[\mu(t-\tau, y) - \mu(t,y)]\, d\sigma_y d\tau \\ &\quad - \int_0^{|x'-x''|^2} \int_{V_P} \frac{\partial}{\partial x_i} \Phi_n(\tau, x''-y)[\mu(t-\tau, y) - \mu(t,y)]\, d\sigma_y d\tau \\ &\quad + \int_{|x'-x''|^2}^{+\infty} \int_{V_P} \left[\frac{\partial}{\partial x_i} \Phi_n(\tau, x'-y) - \frac{\partial}{\partial x_i} \Phi_n(\tau, x''-y) \right] [\mu(t-\tau, y) - \mu(t,y)]\, d\sigma_y d\tau \end{aligned}$$

First we consider the first integral in the right hand side of (B.10). By the estimate (1.4) for the derivatives of the fundamental solution Φ_n, there exists a constant $K_{e_i,0} \in]0, +\infty[$ such that

$$\begin{aligned} &\left| \int_0^{|x'-x''|^2} \int_{V_P} \frac{\partial}{\partial x_i} \Phi_n(\tau, x'-y)[\mu(t-\tau, y) - \mu(t,y)]\, d\sigma_y d\tau \right| \\ &\qquad \leq \int_0^{|x'-x''|^2} \int_{V_P} \left| \frac{\partial}{\partial x_i} \Phi_n(\tau, x'-y) \right| |\mu(t-\tau, y) - \mu(t,y)|\, d\sigma_y d\tau \\ &\qquad \leq K_{e_i,0} \|\mu\|_{C^{\frac{\alpha}{2},\alpha}(\partial_T\Omega)} \int_0^{|x'-x''|^2} \int_{V_P} \tau^{-\frac{n-\alpha+1}{2}} e^{-\frac{|x'-y|^2}{8\tau}}\, d\sigma_y d\tau \\ &\qquad \leq K_{e_i,0} C_J \|\mu\|_{C^{\frac{\alpha}{2},\alpha}(\partial_T\Omega)} (8\pi)^{\frac{n-1}{2}} \int_0^{|x'-x''|^2} \tau^{\frac{\alpha}{2}-1}\, d\tau \\ &\qquad = \frac{2K_{e_i,0} C_J \|\mu\|_{C^{\frac{\alpha}{2},\alpha}(\partial_T\Omega)} (8\pi)^{\frac{n-1}{2}}}{\alpha} |x'-x''|^\alpha. \end{aligned}$$

The second integral in the right hand side of (B.10) can be estimated in the same way, by switching x' with x''. Next we consider the last term in the right hand side of (B.10). By the Fundamental Theorem of Calculus and by inequality (1.4) for the derivatives of the fundamental solution of the heat equation there exists a constant $K_{e_i+e_j,0} \in]0,+\infty[$ such that

$$\begin{aligned}
&\left| \int_{|x'-x''|^2}^{+\infty} \int_{V_P} \left[\frac{\partial}{\partial x_i}\Phi_n(\tau,x'-y) - \frac{\partial}{\partial x_i}\Phi_n(\tau,x''-y)\right] [\mu(t-\tau,y)-\mu(t,y)]\, d\sigma_y d\tau \right| \\
&\quad \le \|\mu\|_{C^{\frac{\alpha}{2},\alpha}(\partial_T\Omega)} \int_{|x'-x''|^2}^{+\infty} \int_{V_P} \left| \frac{\partial}{\partial x_i}\Phi_n(\tau,x'-y) - \frac{\partial}{\partial x_i}\Phi_n(\tau,x''-y)\right| \tau^{\frac{\alpha}{2}}\, d\sigma_y d\tau \\
&\quad \le \|\mu\|_{C^{\frac{\alpha}{2},\alpha}(\partial_T\Omega)} \sum_{j=1}^n |x'_j - x''_j| \\
&\quad\quad \times \int_0^1 \int_{|x'-x''|^2}^{+\infty} \tau^{\frac{\alpha}{2}} \int_{V_P} \left| \frac{\partial^2}{\partial x_j \partial x_i} \Phi_n(\tau, \lambda x' + (1-\lambda)x'' - y)\right| d\sigma_y d\tau d\lambda \\
&\quad \le \|\mu\|_{C^{\frac{\alpha}{2},\alpha}(\partial_T\Omega)} \sum_{j=1}^n K_{e_i+e_j,0} |x'_j - x''_j| \\
&\quad\quad \times \int_0^1 \int_{|x'-x''|^2}^{+\infty} \tau^{\frac{\alpha-3}{2}} \int_{V_P} \frac{e^{-\frac{|\lambda x' + (1-\lambda)x''-y|^2}{8\tau}}}{\tau^{\frac{n-1}{2}}}\, d\sigma_y d\tau d\lambda \\
&\quad \le n \sum_{j=1}^n K_{e_i+e_j,0} C_J \|\mu\|_{C^{\frac{\alpha}{2},\alpha}(\partial_T\Omega)} (8\pi)^{\frac{n-1}{2}} |x'-x''| \int_{|x'-x''|^2}^{+\infty} \tau^{\frac{\alpha-3}{2}} d\tau \\
&\quad = \frac{2n \sum_{j=1}^n K_{e_i+e_j,0} C_J \|\mu\|_{C^{\frac{\alpha}{2},\alpha}(\partial_T\Omega)} (8\pi)^{\frac{n-1}{2}}}{1-\alpha} |x'-x''|^\alpha,
\end{aligned}$$

for all $j \in \{1,\dots,n\}$. Thus, the map from $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ to $B(\overline{]-\infty,T[}, C^{0,\alpha}(\mathrm{cl}\,\Omega))$ which takes μ to $\frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu]$ is linear and continuous.

Now we turn to consider the time Hölder quotient of $\frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu]$. We fix $x \in \Omega$, $t',t'' \in \overline{]-\infty,T[}$. We can assume that $|t''-t'|<1$. Lemma B.5 and interpolation estimates (cf., *e.g.*, von Wahl [108, p. 255] and Agmon, Douglis and Niremberg [2, p. 657]) imply that there exist two constants $d_1, d_2 \in]0,+\infty[$ such that

$$\begin{aligned}
&\left\| \frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu](t',\cdot) - \frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu](t'',\cdot) \right\|_{C^0(\mathrm{cl}\,\Omega)} \\
&\qquad \le d_1 \left| \frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu](t',\cdot) - \frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu](t'',\cdot) : \mathrm{cl}\,\Omega \right|_\alpha^{\frac{1}{1+\alpha}} \\
&\qquad\quad \times \|v^+[\partial_T\Omega,\mu](t',\cdot) - v^+[\partial_T\Omega,\mu](t'',\cdot)\|_{C^0(\mathrm{cl}\,\Omega)}^{\frac{\alpha}{1+\alpha}} \\
&\qquad\quad + d_2 \|v^+[\partial_T\Omega,\mu](t',\cdot) - v^+[\partial_T\Omega,\mu](t'',\cdot)\|_{C^0(\mathrm{cl}\,\Omega)}
\end{aligned}$$

Then Lemma B.5 implies that there exists $c_1 \in]0,+\infty[$ such that

$$\begin{aligned}
&\left\| \frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu](t',\cdot) - \frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu](t'',\cdot) \right\|_{C^0(\mathrm{cl}\,\Omega)} \\
&\qquad \le 2^{\frac{1}{1+\alpha}} c_1 d_1 \left\| \frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu] \right\|_{B\left(\overline{]-\infty,T[}, C^{0,\alpha}(\mathrm{cl}\,\Omega)\right)}^{\frac{1}{1+\alpha}} |t'-t''|^{\frac{\alpha}{2}} + c_1 d_2 |t'-t''|^{\frac{1+\alpha}{2}}
\end{aligned}$$

$$\leq \max \left\{ 2^{\frac{1}{1+\alpha}} c_1 d_1 \left\| \frac{\partial}{\partial x_i} v^+[\partial_T \Omega, \mu] \right\|_{B\left(\overline{]-\infty,T[},C^{0,\alpha}(\mathrm{cl}\,\Omega)\right)}^{\frac{1}{1+\alpha}}, c_1 d_2 \right\} |t'-t''|^{\frac{\alpha}{2}}.$$

Accordingly, the map from the space $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ to $C_b^{\frac{\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\mathrm{cl}\,\Omega)\right)$ which takes μ to $\frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\mu]$ is linear and continuous. Thus the statement for $\frac{\partial}{\partial x_i} v^+[\partial_T\Omega,\cdot]$ holds true.

The statement for $\frac{\partial}{\partial x_i} v^-[\partial_T\Omega,\cdot]$ can be proved following exactly the same lines. □

Next we consider the following lemma about the regularity of the map $\frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu]$.

Lemma B.7. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let Ω be bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $R>0$ such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Then the map from $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C^{\frac{\alpha}{2},\alpha}(\mathrm{cl}\Omega_T)$ which takes μ to $\frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu]$ is linear and continuous. Moreover, the map from $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C^{\frac{\alpha}{2},\alpha}((\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T)$ which takes μ to $\frac{\partial}{\partial t} v^-[\partial_T\Omega,\mu]$ is linear and continuous.*

Proof. We first consider the statement for $\frac{\partial}{\partial t} v^+[\partial_T\Omega,\cdot]$. Let $\mu \in C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$. As already pointed out at the beginning of this section, we can assume that $\mathrm{supp}\,(\mu) \subseteq (V_P)_T$. Lemma A.5 implies that

$$\begin{aligned}\frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu](t,x) &= \frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu-\mu(t^*,\cdot)](t,x) + \frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu(t^*,\cdot)](t,x) \\ &= \frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu-\mu(t^*,\cdot)](t,x) \\ &= \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)(\mu(\tau,y)-\mu(t^*,y))\,d\sigma_y d\tau,\end{aligned} \tag{B.11}$$

for all $t,t^* \in \overline{]-\infty,T[}$, $x\in\Omega$. In particular, by taking $t^*=t$ in the equality (B.11), we have that

$$\frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu](t,x) = \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)(\mu(\tau,y)-\mu(t,y))\,d\sigma_y d\tau, \tag{B.12}$$

for all $(t,x)\in\Omega_T$. The time Hölder continuity of μ and Lemma 1.17 (i) imply that there exists a constant $C_{1,0,\mathrm{cl}\,\Omega} \in]0,+\infty[$ such that

$$\left|\frac{\partial}{\partial t}\Phi_n(t-\tau,x-y)(\mu(\tau,y)-\mu(t,y))\right| \leq C_{1,0,\mathrm{cl}\,\Omega}\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} \frac{1}{(t-\tau)^{\frac{n+1-\alpha}{2}}} e^{-\frac{|x-y|^2}{8(t-\tau)}},$$

for all $(t,x)\in \mathrm{cl}\,\Omega_T$, $(\tau,y)\in\partial_T\Omega$, $\tau<t$. Accordingly, Lemma 1.23 implies that the map which takes μ to $\frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu]$ is bounded from $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[}, C^0(\partial\Omega)\right)$ to $C_b^0(\mathrm{cl}\,\Omega_T)$.

Next we consider the space Hölder quotient. We fix $x',x''\in\Omega$, $t\in\overline{]-\infty,T[}$. The equality (B.12) implies that

$$\begin{aligned}&\frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu](t,x') - \frac{\partial}{\partial t} v^+[\partial_T\Omega,\mu](t,x'') \\ &\quad= \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x'-y)(\mu(\tau,y)-\mu(t,y))\,d\sigma_y d\tau \\ &\quad\quad- \int_{-\infty}^{t}\int_{\partial\Omega} \frac{\partial}{\partial t}\Phi_n(t-\tau,x''-y)(\mu(\tau,y)-\mu(t,y))\,d\sigma_y d\tau\end{aligned} \tag{B.13}$$

$$
\begin{aligned}
&= \int_0^{|x'-x''|^2} \int_{V_P} \frac{\partial}{\partial \tau}\Phi_n(\tau, x'-y)[\mu(t-\tau,y)-\mu(t,y)]\,d\sigma_y d\tau \\
&\quad - \int_0^{|x'-x''|^2} \int_{V_P} \frac{\partial}{\partial \tau}\Phi_n(\tau, x''-y)[\mu(t-\tau,y)-\mu(t,y)]\,d\sigma_y d\tau \\
&\quad + \int_{|x'-x''|^2}^{+\infty} \int_{V_P} \left[\frac{\partial}{\partial \tau}\Phi_n(\tau, x'-y) - \frac{\partial}{\partial \tau}\Phi_n(\tau, x''-y)\right] \\
&\qquad \times [\mu(t-\tau,y)-\mu(t,y)]\,d\sigma_y d\tau.
\end{aligned}
$$

Now we consider the first term in the right hand side of equality (B.13). By inequality (1.4) on the derivatives of the fundamental solution of the heat equation there exists a constant $K_{0,1} \in]0,+\infty[$ such that

$$
\begin{aligned}
&\left| \int_0^{|x'-x''|^2} \int_{V_P} \frac{\partial}{\partial \tau}\Phi_n(\tau, x'-y)[\mu(t-\tau,y)-\mu(t,y)]\,d\sigma_y d\tau \right| \\
&\qquad \le \int_0^{|x'-x''|^2} \int_{V_P} \left|\frac{\partial}{\partial \tau}\Phi_n(\tau, x'-y)\right| |\mu(t-\tau,y)-\mu(t,y)|\,d\sigma_y d\tau \\
&\qquad \le K_{0,1}\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} \int_0^{|x'-x''|^2} \int_{V_P} \tau^{-\frac{n}{2}-1} e^{-\frac{|x'-y|^2}{8\tau}} \tau^{\frac{1+\alpha}{2}}\,d\sigma_y d\tau \\
&\qquad = K_{0,1} C_J \|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} (8\pi)^{\frac{n-1}{2}} \int_0^{|x'-x''|^2} \tau^{\frac{\alpha}{2}-1}\,d\tau \\
&\qquad = \frac{2K_{0,1} C_J \|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} (8\pi)^{\frac{n-1}{2}}}{\alpha} |x'-x''|^{\alpha}.
\end{aligned}
$$

The second integral in the right hand side of equality (B.13) can be estimated in the same way by switching the role of x' and x''. Next we consider the third integral in the right hand side of equality (B.13). By the Fundamental Theorem of Calculus and by inequality (1.4) on the derivatives of the fundamental solution of the heat equation there exists $K_{e_j,1} \in]0,+\infty[$ such that

$$
\begin{aligned}
&\left| \int_{|x'-x''|^2}^{+\infty} \int_{V_P} \left[\frac{\partial}{\partial \tau}\Phi_n(\tau, x'-y) - \frac{\partial}{\partial \tau}\Phi_n(\tau, x''-y)\right] [\mu(t-\tau,y)-\mu(t,y)]\,d\sigma_y d\tau \right| \\
&\quad \le \int_{|x'-x''|^2}^{+\infty} \int_{V_P} \left|\frac{\partial}{\partial \tau}\Phi_n(\tau, x'-y) - \frac{\partial}{\partial \tau}\Phi_n(\tau, x''-y)\right| |\mu(t-\tau,y)-\mu(t,y)|\,d\sigma_y d\tau \\
&\quad \le \|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} \sum_{j=1}^{n} |x'_j - x''_j| \\
&\qquad \times \int_0^1 \int_{|x'-x''|^2}^{+\infty} \tau^{\frac{1+\alpha}{2}} \int_{V_P} \left|\frac{\partial^2}{\partial x_j \partial \tau}\Phi_n(\tau, \lambda x' + (1-\lambda)x'' - y)\right| d\sigma_y d\tau d\lambda \\
&\quad \le C_J \|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} \sum_{j=1}^{n} K_{e_j,1}|x'_j - x''_j| \int_0^1 \int_{|x'-x''|^2}^{+\infty} \tau^{\frac{\alpha-3}{2}} \\
&\qquad \times \tau^{-\frac{n-1}{2}} \int_{V_P} e^{-\frac{|\lambda x' + (1-\lambda)x'' - y|^2}{8\tau}}\,d\sigma_y d\tau d\lambda \\
&\quad = C_J \|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))} (8\pi)^{\frac{n-1}{2}} \sum_{j=1}^{n} K_{e_j,1}|x'_j - x''_j| \int_{|x'-x''|^2}^{+\infty} \tau^{\frac{\alpha-3}{2}}\,d\tau
\end{aligned}
$$

$$\leq \frac{2C_J\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(8\pi)^{\frac{n-1}{2}}}{1-\alpha}\sum_{j=1}^{n}K_{e_j,1}|x'_j-x''_j||x'-x''|^{\alpha-1}$$
$$\leq \frac{2nC_J\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(8\pi)^{\frac{n-1}{2}}\sum_{j=1}^{n}K_{e_j,1}}{1-\alpha}|x'-x''|^{\alpha}.$$

Next we turn to consider the time Hölder quotient of $\frac{\partial}{\partial t}v[\partial_T\Omega,\mu]$. We fix $(t',x),(t'',x)\in\Omega_T$, $t'<t''$. Equality (B.11) with $t^*=t'$ implies

$$\frac{\partial}{\partial t}v^+[\partial_T\Omega,\mu](t'',x)=\int_{-\infty}^{t''}\int_{V_P}\frac{\partial}{\partial t}\Phi_n(t''-\tau,x-y)(\mu(\tau,y)-\mu(t',y))\,d\sigma_y d\tau,$$

and

$$\frac{\partial}{\partial t}v^+[\partial_T\Omega,\mu](t',x)=\int_{-\infty}^{t'}\int_{V_P}\frac{\partial}{\partial t}\Phi_n(t'-\tau,x-y)(\mu(\tau,y)-\mu(t',y))\,d\sigma_y d\tau.$$

Accordingly, we have

$$\begin{aligned}\frac{\partial}{\partial t}v^+[\partial_T\Omega,&\mu](t'',x)-\frac{\partial}{\partial t}v^+[\partial_T\Omega,\mu](t',x) \qquad \text{(B.14)}\\
=&\int_{-\infty}^{t''}\int_{V_P}\frac{\partial}{\partial t}\Phi_n(t''-\tau,x-y)(\mu(\tau,y)-\mu(t',y))\,d\sigma_y d\tau\\
&-\int_{-\infty}^{t'}\int_{V_P}\frac{\partial}{\partial t}\Phi_n(t'-\tau,x-y)(\mu(\tau,y)-\mu(t',y))\,d\sigma_y d\tau\\
=&\int_{t''-2|t''-t'|}^{t''}\int_{V_P}\frac{\partial}{\partial t}\Phi_n(t''-\tau,x-y)(\mu(\tau,y)-\mu(t'',y))\,d\sigma_y d\tau\\
&-\int_{t''-2|t''-t'|}^{t'}\int_{V_P}\frac{\partial}{\partial t}\Phi_n(t'-\tau,x-y)(\mu(\tau,y)-\mu(t',y))\,d\sigma_y d\tau\\
&+\int_{-\infty}^{t''-2|t''-t'|}\int_{V_P}\left(\frac{\partial}{\partial t}\Phi_n(t''-\tau,x-y)-\frac{\partial}{\partial t}\Phi_n(t'-\tau,x-y)\right)\\
&\quad\times(\mu(\tau,y)-\mu(t',y))\,d\sigma_y d\tau\\
&+\int_{V_P}[\mu(t'',y)-\mu(t',y)]\int_0^{2(t''-t')}\frac{\partial}{\partial\tau}\Phi_n(\tau,x-y)\,d\tau d\sigma_y.\end{aligned}$$

First we consider the first term in the right hand side of (B.14). By inequality (1.4) on the derivatives of the fundamental solution of the heat equation there exists $K_{0,1}>0$ such that

$$\begin{aligned}&\left|\int_{t''-2|t''-t'|}^{t''}\int_{V_P}\frac{\partial}{\partial t}\Phi_n(t''-\tau,x-y)(\mu(\tau,y)-\mu(t'',y))\,d\sigma_y d\tau\right|\\
&\leq K_{0,1}\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}\int_{t''-2|t''-t'|}^{t''}\int_{V_P}(t''-\tau)^{-\frac{n}{2}-1}e^{\frac{|x-y|^2}{8(t''-\tau)}}(t''-\tau)^{\frac{1+\alpha}{2}}\,d\sigma_y d\tau\\
&\leq K_{0,1}C_J\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(8\pi)^{\frac{n-1}{2}}\int_{t''-2|t''-t'|}^{t''}(t''-\tau)^{\frac{\alpha-2}{2}}\,d\tau\\
&=\frac{2K_{0,1}C_J\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(8\pi)^{\frac{n-1}{2}}}{\alpha}(t''-t')^{\frac{\alpha}{2}}.\end{aligned}$$

The second integral in the right hand side of (B.14) can be estimated in the same way by swapping the role of t' and t''. Next we consider the third term in the right hand side of (B.14). Lemma 1.17 (iii) implies that there exists a constant $\tilde{C}'_{0,1,16,\mathrm{cl}\,\Omega} \in]0,+\infty[$ such that

$$\left|\frac{\partial}{\partial t}\Phi_n(t''-\tau,x-y)-\frac{\partial}{\partial t}\Phi_n(t'-\tau,x-y)\right| \leq C'_{1,0,16,\mathrm{cl}\,\Omega}\frac{|t'-t''|e^{-\frac{|x-y|}{16|t'-\tau|}}}{|t'-\tau|^{\frac{n}{2}+2}},$$

for all $y \in \partial\Omega$, $\tau \in]-\infty, t''-2|t'-t''|[$. Accordingly,

$$\begin{aligned}
&\left|\int_{-\infty}^{t''-2|t''-t'|}\int_{V_P}\left(\frac{\partial}{\partial t}\Phi_n(t''-\tau,x-y)-\frac{\partial}{\partial t}\Phi_n(t'-\tau,x-y)\right)(\mu(\tau,y)-\mu(t',y))\,d\sigma_y d\tau\right| \\
&\quad\leq \tilde{C}'_{0,1,16,\mathrm{cl}\,\Omega}\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t') \\
&\qquad\times\int_{-\infty}^{t''-2|t''-t'|}\int_{V_P}(t'-\tau)^{-\frac{n}{2}-2}e^{-\frac{|x-y|^2}{16(t'-\tau)}}(t'-\tau)^{\frac{1+\alpha}{2}}\,d\sigma_y d\tau \\
&\quad= (16\pi)^{\frac{n-1}{2}}\tilde{C}'_{0,1,16,\mathrm{cl}\,\Omega}C_J\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t')\int_{-\infty}^{t''-2|t''-t'|}(t'-\tau)^{\frac{\alpha}{2}-2}\,d\tau \\
&\quad= \frac{2^{\frac{\alpha}{2}}(16\pi)^{\frac{n-1}{2}}\tilde{C}'_{0,1,16,\mathrm{cl}\,\Omega}C_J\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{2-\alpha}(t''-t')^{\frac{\alpha}{2}}.
\end{aligned}$$

Next we consider the last term in the right hand side of (B.14).

$$\begin{aligned}
&\left|\int_{V_P}[\mu(t'',y)-\mu(t',y)]\int_0^{2(t''-t')}\frac{\partial}{\partial\tau}\Phi_n(\tau,x-y)\,d\tau d\sigma_y\right| \\
&\quad= \left|\int_{V_P}[\mu(t'',y)-\mu(t',y)]\Phi_n(2(t''-t'),x-y)\,d\sigma_y\right| \\
&\quad\leq \|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}(t''-t')^{\frac{1+\alpha}{2}}\int_{V_P}\Phi_n(2(t''-t'),x-y)\,d\sigma_y \\
&\quad\leq \frac{\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{(8\pi)^{\frac{n}{2}}}(t''-t')^{\frac{\alpha}{2}}\int_{V_P}(t''-t')^{-\frac{n-1}{2}}e^{-\frac{|x-y|^2}{8(t''-t')}}\,d\sigma_y \\
&\quad= \frac{C_J\|\mu\|_{C_b^{\frac{1+\alpha}{2}}(\overline{]-\infty,T[},C^0(\partial\Omega))}}{\sqrt{8\pi}}(t''-t')^{\frac{\alpha}{2}}.
\end{aligned}$$

By the above computations, the statement for $\frac{\partial}{\partial t}v^+[\partial_T\Omega,\cdot]$ holds true.

The statement for $\frac{\partial}{\partial t}v^-[\partial_T\Omega,\cdot]$ can be proved following exactly the same lines. □

Next we consider the following lemma about the regularity of the double layer heat potential $w[\partial_T\Omega,\cdot]$.

Lemma B.8. *Let $\alpha \in]0,1[$, $T \in]-\infty,+\infty]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $R \in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Then the map from $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\mathrm{cl}\,\Omega)\right)$ which takes μ to $w^+[\partial_T\Omega,\mu]$ is linear and continuous. Moreover, the map from $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)\right)$ which takes μ to $w^-[\partial_T\Omega,\mu]$ is linear and continuous.*

Proof. Let $\mu \in C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$. With the same change of variable of Lemma A.4 one can easily see that

$$\int_{-\infty}^{t}\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right|d\sigma_y d\tau=\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu(y)}S_n(x-y)\right|d\sigma_y \qquad \forall\,(t,x)\in\Omega_T.$$

Moreover, by Folland [36, Lemma 3.20] there exists a constant $C\in]0,+\infty[$ such that

$$\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu(y)}S_n(x-y)\right|d\sigma_y\leq C \qquad \forall\,x\in\Omega.$$

Accordingly,

$$\begin{aligned}|w[\partial_T\Omega,\mu](t,x)| &\leq \|\mu\|_{C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)}\int_{-\infty}^{t}\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu(y)}\Phi_n(t-\tau,x-y)\right|d\sigma_y d\tau\\ &=\|\mu\|_{C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)}\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu(y)}S_n(x-y)\right|d\sigma_y\\ &\leq C\|\mu\|_{C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)},\end{aligned}$$

for all $(t,x)\in\Omega_T$. Thus, $w^+[\partial_T\Omega,\cdot]$ is bounded from $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ to $C_b^0(\mathrm{cl}\,\Omega_T)$.

Next we turn to consider the time Hölder quotient. We fix $t',t''\in\overline{]-\infty,T[}$, $x\in\Omega$. Then

$$\begin{aligned}&|w[\partial_T\Omega,\mu](t',x)-w[\partial_T\Omega,\mu](t'',x)|\\ &\quad\leq\int_0^{+\infty}\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu(y)}\Phi_n(\tau,x-y)\right||\mu(t'-\tau,x)-\mu(t''-\tau,x)|\,d\sigma_y d\tau\\ &\quad\leq\|\mu\|_{C^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)}|t'-t''|^{\frac{1+\alpha}{2}}\int_0^{+\infty}\int_{\partial\Omega}\left|\frac{\partial}{\partial\nu(y)}\Phi_n(\tau,x-y)\right|d\sigma_y d\tau\\ &\quad\leq C\|\mu\|_{C^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)}|t'-t''|^{\frac{1+\alpha}{2}}.\end{aligned}$$

This implies that the statement for $w^+[\partial_T\Omega,\cdot]$ holds true.

The statement for $w^-[\partial_T\Omega,\cdot]$ can be proved following exactly the same lines. □

We are now ready to state the following result about the regularity of the single and double layer potential in $\mathrm{cl}\,\Omega_T$ and in $\mathrm{cl}\,\Omega_T^-$ with respect the regularity of the density function.

Theorem B.9. *Let $\alpha\in]0,1[$, $T\in]-\infty,+\infty]$, $m\in\mathbb{N}\setminus\{0\}$, $i\in\{0,\dots,n\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $R\in]0,+\infty[$ be such that $\mathrm{cl}\,\Omega\subseteq\mathbb{B}_n(0,R)$. Then the following statements hold.*

(i) The map from the space $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$, which takes μ to $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\mu]$, is linear and continuous. Moreover, the map from $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$, which takes μ to $\frac{\partial}{\partial t}v^+[\partial_T\Omega,\mu]$, is linear and continuous.

If $n\geq 3$, then the map from $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$, which takes μ to $v^+[\partial_T\Omega,\mu]$, is linear and continuous.

The same statements hold for $v^-[\partial_T\Omega,\cdot]$, $\frac{\partial}{\partial x_i}v^-[\partial_T\Omega,\cdot]$, and $\frac{\partial}{\partial t}v^-[\partial_T\Omega,\cdot]$, replacing $\mathrm{cl}\,\Omega_T$ with $(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T$.

(ii) *The map from* $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ *to* $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$*, which takes* μ *to* $w^+[\partial_T\Omega,\mu]$*, is linear and continuous.*

The same statement holds for $w^-[\partial_T\Omega,\cdot]$*, replacing* $\mathrm{cl}\,\Omega_T$ *with* $(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)_T$*.*

Proof. We first prove statements (i) and (ii) for $v^+[\partial_T\Omega,\cdot]$ and $w^+[\partial_T\Omega,\cdot]$. We proceed by induction on m. The case $m=1$ for $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$ follows by Lemma B.6, and for $\frac{\partial}{\partial t}v^+[\partial_T\Omega,\cdot]$ follows by Lemma B.7. Next we consider the case $m=1$ for $v^+[\partial_T\Omega,\cdot]$. It suffices to prove that

(j) $v^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ to $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\mathrm{cl}\,\Omega)\right)$.

(jj) $\frac{\partial}{\partial x_l}v^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$ for all $l\in\{1,\dots,n\}$.

Statement (j) follows by Lemma B.5 (i) and statement (jj) follows by B.6. Next we consider the case $m=1$ for $w^+[\partial_T\Omega,\cdot]$. It suffices to prove that

(j^i) $w^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ to $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\mathrm{cl}\,\Omega)\right)$.

(jj^i) $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$ for all $l\in\{1,\dots,n\}$.

Statement (j^i) follows by Lemma B.8. Statement (jj^i) follows by formula (B.2) of Lemma B.2 and by Lemmas B.6, B.7. Indeed, the membership of $\nu_i\in C_b^0(\partial\Omega)$, implies that the map from $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$ to $C_b^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\partial\Omega)\right)$ which takes μ to $\nu_i\mu$ is linear and continuous, for all $i\in\{1,\dots,n\}$.

Next we consider the case $m=2$. First we consider the statement for $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$. It suffices to prove that

(j^{ii}) $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ to $C_b^0(\mathrm{cl}\,\Omega_T)$.

(jj^{ii}) $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ to $C^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\mathrm{cl}\,\Omega)\right)$.

(jjj^{ii}) $\frac{\partial^2}{\partial x_l\partial x_i}v^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$ for all $l\in\{1,\dots,n\}$.

Statement (j^{ii}) follows by the case $m=1$ and by the continuity of the embedding of $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ into $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. Then we consider statement (jj^{ii}). By formula (B.3) of Lemma B.3 we have that if $\mu\in C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ then

$$
\begin{aligned}
&\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\mu](t,x) \qquad\qquad (\mathrm{B.15})\\
&\quad=-\sum_{j=1}^{n}v^+[\partial_T\Omega,M_{ij}[\mu\nu_j]](t,x)-w^+[\partial_T\Omega,\mu\nu_i](t,x)\\
&\qquad+\delta_{2,n}\int_{-\infty}^{0}\int_{\partial\Omega}\sum_{j=1}^{n}M_{ij,y}[\Phi_n(-\tau,x_0-y)]\nu_j(y)\mu(\tau,y)\,d\sigma_y d\tau,
\end{aligned}
$$

for all $(t,x)\in\Omega_T$. We point out that the second term in the right hand side of formula (B.15) is a constant. Then statement (jj^{ii}) follows by equality (B.15), by Lemma B.5 and by the case $m = 1$ for $w^+[\partial_T\Omega, \cdot]$. Next we consider statement (jjj^{ii}). By equality (B.15) we have if $\mu \in C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ then

$$\frac{\partial^2}{\partial x_l \partial x_i} v^+[\partial_T\Omega, \mu](t,x) \\ = -\sum_{j=1}^{n} \frac{\partial}{\partial x_l} v^+[\partial_T\Omega, M_{ij}[\mu\nu_j]](t,x) - \frac{\partial}{\partial x_l} w^+[\partial_T\Omega, \mu\nu_i](t,x), \tag{B.16}$$

for all $l \in \{1, \dots, n\}$ and for all $(t,x) \in \Omega_T$. Then statement (jjj^{ii}) follows by equality (B.16) and by case $m = 1$ for $\frac{\partial}{\partial x_l} v^+[\partial_T\Omega, \cdot]$ and $w^+[\partial_T\Omega, \cdot]$. Next we consider $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$. It suffices to prove that

(j^{iii}) $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(jj^{iii}) $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^{\frac{1+\alpha}{2}}\left(\overline{]-\infty, T[}, C^0(\mathrm{cl}\,\Omega)\right)$.

(jjj^{iii}) $\frac{\partial^2}{\partial x_l \partial t} v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$ for all $l \in \{1, \dots, n\}$.

Statement (j^{iii}) follows by the case $m = 1$ and by the continuity of the embedding of $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ into $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$. Next we consider statement (jj^{iii}). By formula (B.4) of Lemma B.4 we have that if $\mu \in C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ then

$$\frac{\partial}{\partial t} v^+[\partial_T\Omega, \mu](t,x) = v^+[\partial_T\Omega, \partial_t\mu](t,x) \\ - \delta_{2,n} \int_{-\infty}^{0} \int_{\partial\Omega} \frac{\partial}{\partial \tau} \Phi_n(-\tau, x_0 - y) \nu_k(y) \mu(\tau, y) d\sigma_y \, d\tau, \tag{B.17}$$

for all $(t,x) \in \Omega_T$. We point out that the second term in the right hand side of formula (B.17) is a constant. Then statement (jj^{iii}) follows by equality (B.17) and by Lemma B.5. Next we consider statement (jjj^{iii}). By formula (B.4) we have that if $\mu \in C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ then

$$\frac{\partial^2}{\partial x_l \partial t} v^+[\partial_T\Omega, \mu](t,x) = \frac{\partial}{\partial x_l} v^+[\partial_T\Omega, \partial_t\mu](t,x), \tag{B.18}$$

for all $l \in \{1, \dots, n\}$ and for all $(t,x) \in \Omega_T$. Then statement (jjj^{iii}) follows by equality (B.18) and by the case $m = 1$ for $\frac{\partial}{\partial x_l} v^+[\partial_T\Omega, \cdot]$. Next we consider $v^+[\partial_T\Omega, \cdot]$. It suffices to prove that

(j^{iv}) $v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(jj^{iv}) $\frac{\partial}{\partial x_l} v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ to $C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\Omega_T)$ and $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$, for all $l \in \{1, \dots, n\}$.

Statement (j^{iv}) follows by the case $m = 1$ and by the continuity of the embedding of $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$ into $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. Statement (jj^{iv}) follows by the case $m = 2$ for $\frac{\partial}{\partial x_l} v^+[\partial_T\Omega, \cdot]$ and by the case $m = 1$ for $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$, respectively. Next we consider $w^+[\partial_T\Omega, \cdot]$. It suffices to prove that

(j^{v}) $w^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(jj^{v}) $\frac{\partial}{\partial x_l} w^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^{\frac{1+\alpha}{2};1+\alpha}(\mathrm{cl}\,\Omega_T)$ and $\frac{\partial}{\partial t} w^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\mathrm{cl}\,\Omega_T)$, for all $l \in \{1, \dots, n\}$.

Statement (j^v) follows the case $m = 1$ and by the continuity of the embedding of $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ into $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$. Next we consider statement (jj^v) for $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$. It suffices to prove that

(k^v) $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(kk^v) $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^{\frac{1+\alpha}{2}}\left(\overline{]-\infty,T[},C^0(\mathrm{cl}\,\Omega)\right)$.

(kkk^v) $\frac{\partial^2}{\partial x_k\partial x_l}w^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ to $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$, for all $k\in\{1,\dots,n\}$.

Statement (k^v) holds by the case $m = 1$ and by the continuity of the embedding of $C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ into $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$. Then we consider statement (kk^v). By formula (B.2) of Lemma B.2 and formula (B.4) of Lemma B.4 we have that if $\mu\in C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ then

$$\begin{aligned}\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\mu](t,x) =&\sum_{j=1}^n\frac{\partial}{\partial x_j}v^+[\partial_T\Omega,M_{lj}[\mu]](t,x)-\frac{\partial}{\partial t}v^+[\partial_T\Omega,\nu_l\mu](t,x) \\ =&\sum_{j=1}^n\frac{\partial}{\partial x_j}v^+[\partial_T\Omega,M_{lj}[\mu]](t,x)-v^+[\partial_T\Omega,\nu_l\partial_t\mu](t,x)\\ &+\delta_{2,n}\int_{-\infty}^0\int_{\partial\Omega}\frac{\partial}{\partial\tau}\Phi_n(-\tau,x_0-y)\nu_l(y)\mu(\tau,y)d\sigma_y\,d\tau,\end{aligned} \tag{B.19}$$

for all $(t,x)\in\Omega_T$. We point out that the second term in the right hand side of formula (B.19) is a constant. Then statement (kk^v) follows by equality (B.19), by the case $m = 2$ for $\frac{\partial}{\partial x_j}v^+[\partial_T\Omega,\cdot]$, and by Lemma B.5. Next we consider statement (kkk^v). By formula (B.19) we have that if $\mu\in C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ then

$$\begin{aligned}\frac{\partial^2}{\partial x_k\partial x_l}w^+[\partial_T\Omega,\mu](t,x) =&\sum_{j=1}^n\frac{\partial^2}{\partial x_k\partial x_j}v^+[\partial_T\Omega,M_{lj}[\mu]](t,x)\\ &-\frac{\partial}{\partial x_k}v^+[\partial_T\Omega,\nu_l\partial_t\mu](t,x),\end{aligned} \tag{B.20}$$

for all $k\in\{1,\dots,n\}$ and for all $(t,x)\in\Omega_T$. Then statement (kkk^v) follows by equality (B.20), and by the cases $m = 1$ and $m = 2$ for $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$. Thus we have proved statement (jj^v) for $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$. Next we consider statement (jj^v) for $\frac{\partial}{\partial t}w^+[\partial_T\Omega,\cdot]$. By formula (B.5) of Lemma B.4 and formula (B.1) of Lemma B.1 we have that if $\mu\in C^{\frac{2+\alpha}{2};2+\alpha}(\partial_T\Omega)$ then

$$\frac{\partial}{\partial t}w^+[\partial_T\Omega,\mu](t,x)=w^+[\partial_T\Omega,\partial_t\mu](t,x)=-\sum_{j=1}^n\frac{\partial}{\partial x_j}v^+[\partial_T\Omega,\nu_j\partial_t\mu](t,x), \tag{B.21}$$

for all $(t,x)\in\Omega_T$. Then the statement follows by equality (B.21) and by the case $m = 1$ for $\frac{\partial}{\partial x_j}v^+[\partial_T\Omega,\cdot]$.

Now we prove that if statements (i) and (ii) hold for all $m'\le m$ and $m\ge 2$, the they hold for $m+1$. First we consider the statement for $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$. It suffices to prove that

(j^{vi}) $\frac{\partial}{\partial x_i}v^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(jj^{vi}) $\frac{\partial^2}{\partial x_l \partial x_i} v^+[\partial_T\Omega, \cdot]$ is continuous from the space $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$ for all $l \in \{1,\ldots,n\}$, and $\frac{\partial^2}{\partial t \partial x_i} v^+[\partial_T\Omega, \cdot]$ is continuous from the space $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-2+\alpha}{2};m-2+\alpha}(\mathrm{cl}\,\Omega_T)$.

Statement (j^{vi}) follows by the case $m = 1$ and by the continuity embedding of $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ into $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. Statement (jj^{vi}) for $\frac{\partial^2}{\partial x_l \partial x_i} v^+[\partial_T\Omega, \cdot]$ follows by equality (B.16) and by the inductive assumptions. Next we consider statement (jj^{vi}) for $\frac{\partial^2}{\partial t \partial x_i} v^+[\partial_T\Omega, \cdot]$. By formula (B.4) of Lemma B.4 we have that if $\mu \in C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ then

$$\frac{\partial^2}{\partial t \partial x_i} v^+[\partial_T\Omega, \mu](t,x) = \frac{\partial}{\partial x_i} v^+[\partial_T\Omega, \partial_t\mu](t,x), \tag{B.22}$$

for all $(t,x) \in \Omega_T$. Then statement (jj^{vi}) for $\frac{\partial^2}{\partial t \partial x_i} v^+[\partial_T\Omega, \cdot]$ follows by formula (B.22) and by the inductive assumptions. Next we consider $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$. It suffices to prove that

(j^{vii}) $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(jj^{vii}) $\frac{\partial^2}{\partial x_l \partial t} v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$ and $\frac{\partial^2}{\partial t^2} v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-2+\alpha}{2};m-2+\alpha}(\mathrm{cl}\,\Omega_T)$, for all $l \in \{1,\ldots,n\}$.

Statement (j^{vii}) follows by the case $m = 1$ and by the continuity of the embedding of $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ into $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$. Statement (jj^{vii}) for $\frac{\partial^2}{\partial x_l \partial t} v^+[\partial_T\Omega, \cdot]$ follows by formula (B.22) and by the inductive assumptions. Next we consider statement (jj^{vii}) for $\frac{\partial^2}{\partial t^2} v^+[\partial_T\Omega, \cdot]$. By formula (B.4) of Lemma B.4 we have that if $\mu \in C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ then

$$\frac{\partial^2}{\partial t^2} v^+[\partial_T\Omega, \mu](t,x) = \frac{\partial}{\partial t} v^+[\partial_T\Omega, \partial_t\mu](t,x), \tag{B.23}$$

for all $(t,x) \in \Omega_T$. Then statement (jj^{vii}) for $\frac{\partial^2}{\partial t^2} v^+[\partial_T\Omega, \cdot]$ follows by (B.23) and by the inductive assumptions. Next we consider $v^+[\partial_T\Omega, \cdot]$. It suffices to prove that

(j^{viii}) $v^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(jj^{viii}) $\frac{\partial}{\partial x_l} v^+[\partial_T\Omega, \cdot]$ is continuous from the space $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$ for all $l \in \{1,\ldots,n\}$, and $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$ is continuous from the space $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$.

Statement (j^{viii}) follows by the case $m = 1$ and by the continuity embedding of $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ into $C^{\frac{\alpha}{2};\alpha}(\partial_T\Omega)$. Statement (jj^{viii}) follows by case m for $\frac{\partial}{\partial t} v^+[\partial_T\Omega, \cdot]$ and by case $m+1$ for $\frac{\partial}{\partial x_l} v^+[\partial_T\Omega, \cdot]$. Finally we consider $w^+[\partial_T\Omega, \cdot]$. It suffices to prove that

(j^{ix}) $w^+[\partial_T\Omega, \cdot]$ is continuous from $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(jj^{ix}) $\frac{\partial}{\partial x_l} w^+[\partial_T\Omega, \cdot]$ is continuous from the space $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\Omega_T)$ for all $l \in \{1,\ldots,n\}$, and $\frac{\partial}{\partial t} w^+[\partial_T\Omega, \cdot]$ is continuous from the space $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$.

Statement (j^{ix}) follows by the case $m=1$ and by the continuity of the embedding of $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ into $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$. Next we consider statement (jj^{ix}) for $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$. It suffices to prove that

(k^{ix}) $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^0_b(\mathrm{cl}\,\Omega_T)$.

(kk^{ix}) $\frac{\partial^2}{\partial x_k\partial x_l}w^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-1+\alpha}{2};m-1+\alpha}(\mathrm{cl}\,\Omega_T)$ for all $k\in\{1,\dots,n\}$, and $\frac{\partial^2}{\partial t\partial x_l}w^+[\partial_T\Omega,\cdot]$ is continuous from $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ to $C^{\frac{m-2+\alpha}{2};m-2+\alpha}(\mathrm{cl}\,\Omega_T)$.

Statement (k^{ix}) holds by the case $m=1$ and by the continuity of the embedding of $C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ into $C^{\frac{1+\alpha}{2};1+\alpha}(\partial_T\Omega)$. Statement ($\mathrm{kk}^{ix}$) for $\frac{\partial^2}{\partial x_k\partial x_l}w^+[\partial_T\Omega,\cdot]$ follows by equality (B.20), and by case m for $\frac{\partial}{\partial x_k}v^+[\partial_T\Omega,\cdot]$ and by case $m+1$ for $\frac{\partial}{\partial x_j}v^+[\partial_T\Omega,\cdot]$. Then we consider statement (kk^{ix}) for $\frac{\partial^2}{\partial t\partial x_l}w^+[\partial_T\Omega,\cdot]$. By formula (B.5) of Lemma B.4 we have that if $\mu\in C^{\frac{m+1+\alpha}{2};m+1+\alpha}(\partial_T\Omega)$ then

$$\frac{\partial^2}{\partial t\partial x_l}w^+[\partial_T\Omega,\mu](t,x)=\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\partial_t\mu](t,x), \tag{B.24}$$

for all $(t,x)\in\Omega_T$. Then statement (kk^{ix}) for $\frac{\partial^2}{\partial t\partial x_l}w^+[\partial_T\Omega,\cdot]$ follows by equality (B.24) and by case m for $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$. Thus we have proved statement (jj^{ix}) for $\frac{\partial}{\partial x_l}w^+[\partial_T\Omega,\cdot]$.

The corresponding statements for $v^-[\partial_T\Omega,\cdot]$, $\frac{\partial}{\partial x_i}v^-[\partial_T\Omega,\cdot]$, $\frac{\partial}{\partial t}v^-[\partial_T\Omega,\cdot]$ and $w^-[\partial_T\Omega,\cdot]$ can be proved in the same way. $\square$

Bibliography

[1] P. M. Adler, *Porous media: geometry and transports*, Butterworth/Heinemann, 1992.

[2] S. Agmon, A. Douglis, and L. Niremberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, *Comm. Pure Appl. Math.*, **12** 1959, 623–727.

[3] H. Ammari, H. Kang, and K. Kim, Polarization tensors and effective properties of anisotropic composite materials. *J. Differential Equations* **215** (2005), no. 2, 401–428.

[4] H. Ammari and H. Kang, *Polarization and moment tensors. With applications to inverse problems and effective medium theory.* Applied Mathematical Sciences, 162. Springer, New York, 2007. x+312 pp.

[5] H. Ammari, H. Kang, and M. Lim, Effective parameters of elastic composites. *Indiana Univ. Math. J.* **55** (2006), no. 3, 903–922.

[6] H. Ammari, H. Kang, and K. Touibi, Boundary layer techniques for deriving the effective properties of composite materials. *Asymptot. Anal.* **41** (2005), no. 2, 119–140 .

[7] D. N. Arnold and P. J. Noon, Coercivity of the single layer heat potential. China-US Seminar on Boundary Integral and Boundary Element Methods in Physics and Engineering (Xi'an, 1987–88). *J. Comput. Math.* **7** (1989), no. 2, 100–104.

[8] E. A. Baderko, Parabolic problems and boundary integral equations. *Math. Methods Appl. Sci.* **20** (1997), no. 5, 449–459.

[9] S. Bernstein, S. Ebert, and R. Sören Kraußhar, On the diffusion equation and diffusion wavelets on flat cylinders and the n-torus. *Math. Methods Appl. Sci.* **34** (2011), no. 4, 428–441.

[10] V. Bonnaillie-Noël, M. Dalla Riva, M. Dambrine, and P. Musolino, A Dirichlet problem for the Laplace operator in a domain with a small hole close to the boundary. *J. Math. Pures Appl.* (9) **116** (2018), 211–267.

[11] R. M. Brown, The method of layer potentials for the heat equation in Lipschitz cylinders. *Amer. J. Math.* **111** (1989), no. 2, 339–379.

[12] M. R. Brown, The initial-Neumann problem for the heat equation in Lipschitz cylinders. *Trans. Amer. Math. Soc.* **320** (1990), no. 1, 1–52.

[13] L. P. Castro and E. Pesetskaya, A transmission problem with imperfect contact for an unbounded multiply connected domain. *Math. Methods Appl. Sci.*, **33** (2010), 517–526.

[14] L.P. Castro, E. Pesetskaya, and S.V. Rogosin, Effective conductivity of a composite material with non-ideal contact conditions, *Complex Var. Elliptic Equ.*, **54** (2009), no. 12, 1085–1100.

[15] A. Cialdea, Appunti di teoria del potenziale. *Corso di Analisi Superiore*, A.A. 2001-02, *Dipartimento di Matematica, Università degli Studi della Basilicata*, 2000; **http://cialdea.altervista.org/dispense/potenziale.pdf**

[16] D. Colton and R. Kress, *Integral equation methods in scattering theory.* Reprint of the 1983 original. Classics in Applied Mathematics, 72. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013. xvi+271 pp.

[17] M. Costabel, Boundary integral operators for the heat equation. *Integral Equations Operator Theory*, **13** (1990), no. 4, 498–552.

[18] M. Costabel, M. Dalla Riva, M. Dauge, and P. Musolino, Converging expansions for Lipschitz self-similar perforations of a plane sector. *Integral Equations Operator Theory*, **88** (2017), no. 3, 401–449.

[19] M. Costabel, K. Onishi, and W. L. Wendland, A boundary element collocation method for the Neumann problem of the heat equation. *Inverse and ill-posed problems* (Sankt Wolfgang, 1986), 369–384, Notes Rep. Math. Sci. Engrg., **4**, Academic Press, Boston, MA, 1987.

[20] M. Dalla Riva, Stokes flow in a singularly perturbed exterior domain, *Complex Var. Elliptic Equ.* **58** (2013), no. 2, 231–257.

[21] M. Dalla Riva and M. Lanza de Cristoforis, A singularly perturbed nonlinear traction boundary value problem for linearized elastostatics. A functional analytic approach. *Analysis (Munich)* 30 (2010), no. 1, 67–92.

[22] M. Dalla Riva, M. Lanza de Cristoforis, and P. Musolino, Mapping properties of weakly singular periodic volume potentials in Roumieu classes. *Submitted* (2018), *https://arxiv.org/abs/1705.06487*.

[23] M. Dalla Riva and P. Musolino, A singularly perturbed nonideal transmission problem and application to the effective conductivity of a periodic composite. *SIAM J. Appl. Math.*, **73** (2013), no. 1, 24–46.

[24] M. Dalla Riva and P. Musolino, A singularly perturbed nonlinear traction problem in a periodically perforated domain: a functional analytic approach. *Math. Methods Appl. Sci.* **37** (2014), no. 1, 106–122.

[25] M. Dalla Riva and P. Musolino, A mixed problem for the Laplace operator in a domain with moderately close holes. *Comm. Partial Differential Equations*, **41** (2016), no. 5, 812–837.

[26] M. Dalla Riva and P. Musolino, The Dirichlet problem in a planar domain with two moderately close holes. *J. Differential Equations*, **263** (2017), no. 5, 2567–2605.

[27] M. Dalla Riva, P. Musolino, and S. Rogosin Series expansions for the solution of the Dirichlet problem in a planar domain with a small hole. *Asymptot. Anal.*, 92(3-4):339–361, 2015.

[28] M. Dalla Riva, P. Musolino, and R. Pukhtaievych. Series expansion for the effective conductivity of a periodic dilute composite with thermal resistance at the two-phase interface. *Asymptot. Anal.*, to appear.

[29] K. Deimling, *Nonlinear Functional Analysis.* Springer-Verlag, Berlin, 1985, xiv+450 pp.

[30] P. Donato and E. C. Jose, Asymptotic behavior of the approximate controls for parabolic equations with interfacial contact resistance, *ESAIM: Control Optim. Calc. Var.*, **21** (2015), no. 1, 138–164.

[31] P. Donato and E. C. Jose, Approximate Controllability of a Parabolic System with Imperfect Interfaces, *Philippine Journal of Science*, 144 (2), 2015, 187–196.

[32] F. Dondi and M. Lanza de Cristoforis, Regularizing properties of the double layer potential of second order elliptic differential operators. *Mem. Differ. Equ. Math. Phys.* **71** (2017), 69–110.

[33] L. C. Evans. *Partial differential equations.* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998. xviii+662 pp.

[34] E. B. Fabes and N. M. Rivière, Dirichlet and Neumann problems for the heat equation in C^1-cylinders. *Harmonic analysis in Euclidean spaces* (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 2, pp. 179–196, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979.

[35] E. B. Fabes, M. Jodeit, and N. M. Rivière, Potential techniques for boundary value problems on C^1-domains. *Acta Math.* 141 (1978), no. 3-4, 165–186.

[36] G. B. Folland, *Introduction to partial differential equations.* Second edition. Princeton University Press, Princeton, NJ, 1995. xii+324 pp.

[37] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications.* Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999. xvi+386 pp.

[38] A. Friedman, *Partial differential equations of parabolic type.* Partial differential equations of parabolic type. Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964 xiv+347 pp.

[39] M. Gevrey, Sur les équations aux dérivées partielle du type parabolique. *Journal de Mathématiques Pures et Appliquées.* **9** (1913), 305–471.

[40] M. Gevrey, Sur les équations aux dérivées partielle du type parabolique (suite). *Journal de Mathématiques Pures et Appliquées.* **10** (1914), 105–148.

[41] D. Gilbarg and N. S.Trudinger, *Elliptic partial differential equations of second order.* Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp.

[42] M. Hamina, An approximation method for the hypersingular heat operator equation. Proceedings of the 8th International Congress on Computational and Applied Mathematics, ICCAM-98 (Leuven). *J. Comput. Appl. Math.* **115** (2000), no. 1-2, 229–243.

[43] H. Hasimoto, On the periodic fundamental solutions of the Stokes' equations and their application to viscous flow past a cubic array of spheres. *J. Fluid Mech.* **5** 1959 317–328.

[44] A. Henrot and M. Pierre, *Variation et optimisation de formes. Une analyse géométrique.* Mathématiques & Applications 48. Springer, Berlin, 2005. xii+334 pp.

[45] S. Hofmann, Parabolic singular integrals of Calderòn-type, rough operators, and caloric layer potentials. *Duke Math. J.* **90** (1997), no. 2, 209–259.

[46] S. Hofmann and J. L. Lewis, L^2 solvability and representation by caloric layer potentials in time-varying domains. *Ann. of Math.* (2) **144** (1996), no. 2, 349–420.

[47] S. Hofmann J. L. Lewis, and M. Mitrea, Spectral properties of parabolic layer potentials and transmission boundary problems in nonsmooth domains. *Illinois J. Math.* **47** (2003), no. 4, 1345–1361.

[48] S. Hofmann, M. Mitrea, and M. Taylor, Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains. *Int. Math. Res. Not. IMRN* 2010, no. 14, 2567–2865.

[49] C. G. Hsiao, J. Saranen, Boundary integral solution of the two-dimensional heat equation. *Math. Methods Appl. Sci.* 16 (1993), no. 2, 87–114.

[50] L. I. Kamynin, On the smoothness of thermal potentials. (Russian) *Differencial'nye Uravnenija* **1** (1965), 799–839.

[51] L. I. Kamynin, On the smoothness of thermal potentials. II. Thermal potentials on the surface of type $L^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}$. (Russian) *Differencial'nye Uravnenija* **2** 1966 647–687.

[52] L. I. Kamynin, On the smoothness of thermal potentials. V. Thermal potentials U, V and W on surfaces of type $\Pi^{m+1,\alpha,\alpha/2}_{2m+1,1,(1+\alpha)/2}$ and $\Pi^{m+1,1,(1+\alpha)/2}_{2m+3,1,\alpha,\alpha/2}$. (Russian) *Differensial'nye Uravnenija* **4** 1968 347–365.

[53] L. I. Kamynin, On the smoothness of thermal potentials. V. Thermal potentials U, V and W on surfaces of type $L^{m+1,\alpha,\alpha/2}_{2m+1,1,(1+\alpha)/2}$ and $L^{m+1,1,(1+\alpha)/2}_{2m+3,\alpha,\alpha/2}$. II. (Russian) *Differencial'nye Uravnenija* **4** 1968 881–895.

[54] D. Kapanadze, G. Mishuris, and E. Pesetskaya, Exact solution of a nonlinear heat conduction problem in a doubly periodic 2D composite material, *Arch. Mech. (Arch. Mech. Stos.)* **67** (2015), 157–178.

[55] D. Kapanadze, G. Mishuris, and E. Pesetskaya, Improved algorithm for analytical solution of the heat conduction problem in doubly periodic 2D composite materials, *Complex Var. Elliptic Equ.*, **60** (2015), 1–23.

[56] A. N. Konënkov, Smoothness of the double-layer heat potential in Zygmund spaces. (Russian) *Differ. Uravn.* **43** (2007), no. 8, 1106–1115, 1152; translation in *Differ. Equ.* **43** (2007), no. 8, 113–1141.

[57] R. Kress, *Linear integral equations.* Applied Mathematical Sciences, 82. Springer-Verlag, Berlin, 1989. xii+299 pp.

[58] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type.* (Russian) Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1968 xi+648 pp.

[59] M. Lanza de Cristoforis, Perturbation problems in potential theory, a functional analytic approach, *J. Appl. Funct. Anal.* **2** (2007), no. 3, 197–222.

[60] M. Lanza de Cristoforis, Asymptotic behavior of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole: a functional analytic approach. *Complex Var. Elliptic Equ.* **52** (2007), no. 10-11, 945–977.

[61] M. Lanza de Cristoforis, Asymptotic behavior of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach, *Analysis (Munich)* **28** (2008), no. 1, 63–93.

[62] M. Lanza de Cristoforis, *Asymptotic behaviour of the solutions of a non-linear transmission problem for the Laplace operator in a domain with a small hole. A functional analytic approach.* Complex Var. Elliptic Equ. 55 (2010), no. 1-3, 269–303.

[63] M. Lanza de Cristoforis and P. Luzzini, Time dependent boundary norms for kernels and regularizing properties of the double layer heat potential. *Eurasian Math. J.* **8** (2017), no. 1, 76–118.

[64] M. Lanza de Cristoforis, P. Luzzini, Tangential derivatives and higher order regularizing properties of the double layer heat potential. *Submitted* (2018).

[65] M. Lanza de Cristoforis, P. Musolino, A perturbation result for periodic layer potentials of general second order differential operators with constant coefficients. *Far East J. Math. Sci.* **52** (2011), no. 1, 75–120.

[66] M. Lanza de Cristoforis and P. Musolino, A real analyticity result for a nonlinear integral operator, *J. Integral Equations Appl.* **25** (2013), no. 1, 21–46.

[67] M. Lanza de Cristoforis and P. Musolino, A singularly perturbed nonlinear Robin problem in a periodically perforated domain: a functional analytic approach. *Complex Var. Elliptic Equ.* **58** (2013), no. 4, 511-536.

[68] M. Lanza de Cristoforis and P. Musolino, A quasi-linear heat transmission problem in a periodic two-phase dilute composite. A functional analytic approach. *Comm. Pure Appl. Anal.* **13** (2014), no. 6, 2509-2542.

[69] M. Lanza de Cristoforis and P. Musolino, A singularly perturbed Neumann problem for the Poisson equation in a periodically perforated domain. A functional analytic approach, *ZAMM Z. Angew. Math. Mech.*, **96** (2016), no. 2, 253–272.

[70] M. Lanza de Cristoforis and P. Musolino, Analytic dependence of a periodic analog of a fundamental solution upon the periodicity parameters, Ann. Mat. Pura Appl. (4) **197** (2018), no. 4, 1089–1116.

[71] M. Lanza de Cristoforis and L. Rossi, Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density, *J. Integral Equations Appl.*, **16** (2004), no. 2, 137–174.

[72] M. Lanza de Cristoforis and L. Rossi, Real analytic dependence of simple and double layer potentials for the Helmholtz equation upon perturbation of the support and of the density. *Analytic methods of analysis and differential equations: AMADE 2006*, 193–220, Camb. Sci. Publ., Cambridge, 2008.

[73] J. L. Lewis and M. A. M. Murray, The method of layer potentials for the heat equation in time-varying domains. *Mem. Amer. Math. Soc.* **114** (1995), no. 545.

[74] R. Lipton, Heat conduction in fine scale mixtures with interfacial contact resistance, *SIAM J. Appl. Math.*, **58** (1998), no. 1, pp. 55–72.

[75] P. Luzzini, Regularizing properties of space-periodic layer heat potentials and applications to boundary value problems in periodic domains. *Submitted* (2018).

[76] P. Luzzini, P. Musolino, R. Pukhtaievych, Shape analyticity of the longitudinal permeability of a periodic array of cylinders. *Draft* (2018).

[77] V. Maz'ya and T. Shaposhnikova, Higher regularity in the layer potential theory for Lipschitz domains. *Indiana Univ. Math. J.* **54** (2005), no. 1, 99–142.

[78] E. A. Jr. McIntyre, Boundary integral solutions of the heat equation. *Math. Comp.* **46** (1986), no. 173, 71–79, S1–S14.

[79] C. Miranda, Sulle proprietà di regolarità di certe trasformazioni integrali, *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez I (8)* **7** (1965), 303–336.

[80] G. Miranda, Integral equation solution of the first initial-boundary value problem for the heat equation in domains with non-smooth boundary. *Comm. Pure Appl Math.* **23** 1970, 757–765.

[81] M. Mitrea, The almighty double layer: recent perspectives. Invited presentation at the 13th International Conference on Integral methods in Science and Engineering, July 21–25, 2014.

[82] P. Musolino, A singularly perturbed Dirichlet problem for the Laplace operator in a periodically perforated domain. A functional analytic approach. *Math. Methods Appl. Sci.* **35** (2012) no. 3, 334–349.

[83] P. Musolino, A singularly perturbed Dirichlet problem for the Poisson equation in a periodically perforated domain. A functional analytic approach, *Advances in harmonic analysis and operator theory*, 269–289, Oper. Theory Adv. Appl., 229, Birkhäuser/Springer Basel AG, Basel, 2013.

[84] P. Musolino and V. Mityushev, Asymptotic behavior of the longitudinal permeability of a periodic array of thin cylinders. *Electron. J. Differential Equations*, 2015, No. 290, 20 pp.

[85] V. Mityushev and P.M. Adler, Longitudinal permeability of spatially periodic rectangular arrays of circular cylinders. I. A single cylinder in the unit cell, *ZAMM Z. Angew. Math. Mech.*, **82** (2002), no. 5, 335–345.

[86] V. Mityushev and P.M. Adler, Longitudinal permeability of spatially periodic rectangular arrays of circular cylinders. II. An arbitrary distribution of cylinders inside the unit cell, *Z. Angew. Math. Phys.*, **53** (2002), no. 3, 486–517.

[87] P. J. Noon, *The single layer heat potential and Galerkin boundary element methods for the heat equation.* Thesis (Ph.D.)-University of Maryland, College Park. 1988. 113 pp.

[88] A. A. Novotny and J. Sokołowski, *Topological derivatives in shape optimization.* Interaction of Mechanics and Mathematics. Springer, Heidelberg, 2013. xxii+412 pp.

[89] M. A. Pinsky, *Introduction to Fourier analysis and wavelets.* Reprint of the 2002 original. Graduate Studies in Mathematics, 102. American Mathematical Society, Providence, RI, 2009. xx+376 pp.

[90] W. Pogorzelski. Sur la solution de l'équation intégrale dans le problème de Fourier. (French) *Ann. Soc. Polo. Math.*, **24** (1951), 56–74 (1952).

[91] L. Preciso, Regularity of the composition and of the inversion operator and perturbation analysis of the conformal sewing problem in Roumieu type spaces. In *Proc. Inst. Math.* (2000), vol. 5, Nat. Acad. Sci. Belarus, pp. 99–104.

[92] G. Prodi and A. Ambrosetti. *Analisi non lineare.* Editrice Tecnico Scientifica, Pisa, 1973

[93] R. Pukhtaievych, Asymptotic behavior of the solution of singularly perturbed transmission problems in a periodic domain, *Math. Methods Appl. Sci.*, **41** (2018), no. 9, 3392–3413.

[94] S. Rogosin, M. Dubatovskaya, and E. Pesetskaya, Eisenstein sums and functions and their application at the study of heat conduction in composites, *Šiauliai Math. Semin.*, **4**(12) (2009), 167–187.

[95] A.S. Sangani and C. Yao, Transport properties in random arrays of cylinders. I. Thermal conduction, *Phys. Fluids*, **31** (1988), 2426–2434.

[96] A.S. Sangani and C. Yao, Transport properties in random arrays of cylinders. II. Viscous flow, *Phys. Fluids*, **31** (1988), 2435–2444.

[97] J. Schauder, Potentialtheoretische Untersuchungen. *Math. Z.*, 33 (1931), no. 1, 602-640.

[98] J. Schauder, Bemerkung zu meiner Arbeit "Potentialtheoretische Untersuchungen I (Anhang)". *Math. Z.* 35 (1932), no. 1, 536–538.

[99] J. Schmid, Longitudinal laminar flow in an array of circular cylinders, *Mass Transfer*, **9** (1966), 925–937.

[100] V. A. Shcherbina, Boundary value problems with a triple periodic solution for the Laplace equation in $\mathbb{R}^3$. (Russian), *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* **No. 45** (1986), 132–139; translation in J. Soviet Math., **48**, no. 4, (1990) 484–490.

[101] J. Sokołowski and J.P. Zolésio, *Introduction to shape optimization. Shape sensitivity analysis.* Springer Series in Computational Mathematics, 16. Springer-Verlag, Berlin, 1992. ii+250 pp.

[102] E. T. Swartz and R. O. Pohl, Thermal boundary resistance, *Rev. Mod. Phys.*, 61 (1989), pp. 605–668.

[103] G.M. Troianiello, *Elliptic Differential Equations and Obstacle Problems.* The Univesrity Series in Mathematics. Plenum Press, New York, 1987. xiv+353 pp.

[104] Van Tun. Theory of the heat potential. I. Level curves of heat potentials and the inverse problem in the theory of the heat potential. (Russian) *Ž. Vyčisl. Mat. i Mat. Fiz.* **4** 1964 660–670.

[105] Van Tun. Theory of the heat potential. II. Smoothness of contour heat potentials. (Russian) *Ž. Vyčisl. Mat. i Mat. Fiz.* **5** 1965 474–487.

[106] Van Tun. A theory of the thermal potential. III. Smoothness of a plane thermal potential. (Russian) *Ž. Vyčisl. Mat. i Mat. Fiz.* **5** 1965 658–666.

[107] G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.* **59** (1984), no. 3, 572-611.

[108] W. von Wahl, Gebrochene Potenzen eines elliptischen operators und parabolische differentialgleichungen in Räumen Hölderstetiger funktionen, *Nachr. Ak. d. Wiss Göttingen, II Mathem. Physik. Klasse*, (1972) 231-258.

[109] W. von Wahl, Abschätzungen für das Neumann-Problem und die Helmholtz-Zerlegung von L^p. (German) *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* 1990, no. 2, 29 pp.

[110] N. A. Watson, *Introduction to heat potential theory.* Mathematical Survays and Monographs, 182. American Mathematical Society, Providence, RI, 2012. xiv+266 pp.

[111] M. Wiegner, Schauder estimates for boundary layer potentials, *Math. Methods Appl. Sci.*, **16** (1993), no. 12, 877-894.