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Modeling, estimation and identification of stochastic systems with latent variables

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Summary

The main topic of this thesis is the analysis of static and dynamic models in which some variables, although directly influencing the behavior of certain observables, are not accessible to measurements. These models find applications in many branches of science and engineering, such as control systems, communications, natural and biological sciences and econometrics. It is well-known that models with unaccessible - or *latent* - variables, usually suffer from a lack of uniqueness of representation. In other words, there are in general many models of the same type describing a given set of observables say, the measurable input-output variables. This is well-known and has been well-studied for a special class of linear models, called *state-space models*. In this thesis we shall focus on two particular classes of stochastic systems with latent variables: the *generalized factor analysis* models and *errors-in-variables* models. For these classes of models there are still some unresolved issues related to non-uniqueness of the representation and clarifying these issues is of paramount importance for their identification. Since mathematical models usually need to be estimated from experimental data, solving the non-uniqueness problem is essential for their use in statistical inference (system identification) from measured data.

Generalized factor analysis models

The first class of models discussed in this thesis constitutes a generalization of the classical factor analysis models. It was first proposed in the 80's by econometricians for the purpose of describing capital asset pricing in large markets. In recent years, it has been extended to the dynamic context, attracting much attention from the econometrics and system identification communities. These models describe observations of infinite cross-sectional dimension. Quite surprisingly, in this generalized context the inherent non-uniqueness of classic factor analysis models does not occur. In our opinion, the reasons for this have not been spelled out clearly in the literature. We shall argue that the uniqueness is due to a different splitting of the variables into a factor and noise component which is based on the concepts of *aggregate sequences* and *idiosyncratic noise*. We describe conditions which are necessary and sufficient for the latent variables of the model to be uniquely recoverable. In particular, for stationary sequences,we show that there is a natural interpretation of generalized factor analysis models in terms of the well-known *Wold decomposition* and show that a stationary sequence admits a (unique) generalized factor analysis decomposition if and only if two rather natural conditions are satisfied.

Part of this work is focused on possible applications of generalized factor analysis

models to various fields of engineering. We present some scenarios in which this new modeling paradigm may be used in order to model the behavior of multi-agents systems consisting of a very large number of interacting random agents. We show that in a description by generalized factor analysis models the latent factor component has an interpretation as a *flocking component* of the ensemble while the idiosyncratic noise component models the local interactions among neighboring agents.

Motivated by the need of modeling the effects of the latent variables on the observables, one chapter is dedicated to the study of the properties of tall linear systems, i.e. systems with more outputs than inputs. In fact, in order to keep the model complexity low, in the context of generalized factor models it is desirable to deal with zero-free models, i.e. linear systems with no invariant zeros. For this reason, after reviewing some recent literature, we study the zero properties of discrete-time linear systems, assuming multirate outputs. In the literature the zero properties of these systems are defined as those of their corresponding time-invariant blocked systems. Hence, the focus is on the zero properties of blocked systems resulting from blocking of linear systems with multirate outputs. In particular, we study the zero properties of tall blocked systems under a generic setting, i.e. for generic parameter matrices. We demonstrate that tall blocked systems generically have no finite nonzero zeros. Moreover, we show when tall blocked multirate systems can generically be zero-free at the origin of the complex plane and at infinity and when they must have zeros at those aforementioned points.

Errors-in-variables models

The second class of systems studied in this thesis are errors-in-variables models. In particular, we discuss identifiability of dynamic single-input-single-output errors-in-variables models with white measurement errors. Although this class of models turns out to be generically identifiable, it has been pointed out that in certain circumstances there may be two errors-in-variables models which are indistinguishable from external input-output experiments. This lack of (global) identifiability may be prejudicial to identification and needs better understanding. The identifiability conditions found in the literature guarantee uniqueness under certain coprimality assumptions on the (rational) transfer function of the ideal "true" system and the spectral density of the noiseless "true" input. Unfortunately these conditions are not testable since they concern precisely the unknowns of the problem which are not available to the experimenter. We provide new identifiability conditions which are instead expressible in terms of the external description of the observable signals, namely their joint power spectral densities.

Motivated by the need of providing a tool for estimating the power spectra densities,

which is a preliminary step for testing the identifiability of errors-in-variables models, we present a new regularized kernel based approach for the estimation of the second order moments of stationary stochastic processes. The correlation functions are assumed to be summable and estimated as the solution of a Tikhonov-type variational problem. The hypothesis space is a reproducing kernel Hilbert space induced by a recently introduced *stable spline kernel*. In this way, the information on the decay to zero of the functions to be reconstructed is incorporated in the estimation process. We show that the overall complexity of the proposed estimator scales linearly with the number of available samples of the processes. An application to the identification of transfer functions in the case of white noise input is also presented. We provide numerical simulations to show that the proposed method compares favorably with respect to standard nonparametric estimation algorithms that exploit an oracle-type tuning of the parameters.

Sommario

L'argomento principale di questa tesi è l'analisi di modelli statici e dinamici in cui alcune variabili non sono accessibili a misurazioni, nonostante esse influenzino l'evoluzione di certe osservazioni. Questi modelli trovano applicazione in molte discipline delle scienze e dell'ingegneria, come ad esempio l'automatica, le telecomunicazioni, le scienze naturali, la biologia e l'econometria e sono stati studiati approfonditamente nel campo dell'identificazione dei modelli. È ben noto che sistemi con variabili inaccessibili - o *latenti*, spesso soffrono di una mancanza di unicità nella rappresentazione. In altre parole, in generale ci sono molti modelli dello stesso tipo che possono descrivere un dato insieme di osservazioni, come ad esempio variabili misurabili di ingresso-uscita. Questo è ben noto, ed è stato studiato a fondo per una classe speciale di modelli lineari, chiamata *modelli a spazio di stato*. In questa tesi ci si focalizza su due classi particolari di sistemi stocastici a variabili latenti: i modelli *generalized factor analysis* e i modelli *errors-in-variables*. Per queste classi di modelli ci sono ancora alcuni problemi irrisolti legati alla non unicità della rappresentazione e chiarificare questi problemi è di importanza fondamentale per la loro identificazione. Poiché solitamente i modelli matematici necessitano ti essere stimati da dati sperimentali, è essenziale risolvere il problema della non unicità per il loro utilizzo nell'inferenza statistica (identificazione di modelli) da dati misurati.

Modelli generalized factor analysis

La prima classe di modelli discussa in questa tesi costituisce una generalizzazione dei modelli ad analisi fattoriale classici. È stata proposta inizialmente negli anni ottanta dagli econometrici allo scopo di descrivere il *capital asset pricing* in grandi mercati. Recentemente, essa è stata estesa al caso dinamico, attirando l'attenzione delle comunità di econometria e identificazione dei sistemi. Lo scopo di questi modelli è la descrizione di osservazioni la cui dimensione trasversale è infinita. Abbastanza sorprendentemente, la non unicità intrinseca dei modelli ad analisi fattoriale classici non si verifica. È nostra opinione che in letteratura le ragioni di tale unicità non sono state spiegate in modo chiaro. Si dimostra che l'unicità è dovuta ad un modo differente di dividere le varaibili nella somma di una componente fattore e una rumore, basandosi sui concetti di *aggregate sequences* e *idiosyncratic noise*. Si descrivono quali condizioni sono necessarie e sufficienti affinché le variabili latenti siano univocamente identificabili. In particolare, per sequenze stazionarie, si dimostra che esiste un'interpretazione naturale dei modelli generalized factor analysis in termini della nota decomposizione di Wold, mostrando che una sequenza stazionaria ammette una (unica) decomposizione in termini di generalized factor analysis

se e solo se sono soddisfatte due condizioni alquanto naturali.

Parte di questo lavoro si focalizza sulla descrizione di possibili applicazione per i modelli generalized factor analysis in diversi campi dell'ingegneria. Si presentano alcuni scenari in cui può essere possibile applicare questo nuovo paradigma per modellizzare il comportamento di sistemi multiagente costituiti da un numero molto frande di agenti interagenti in modo casuale. Si mostra che la componente dei fattori latenti ammette un'interpretazione in termini di componente di *flocking* del comportamento complessivo del gruppo, mentre il rumore *idiosincratico* modellizza in modo naturale le interazioni locali tra agenti vicini tra loro.

Motivati dalla necessità di modellizzare l'effetto delle variabili latenti sulle misurazione, si dedica un capitolo allo studio delle proprietà dei sistemi lineari alti, ossia sistemi con più uscite che ingressi. Infatti, allo scopo di mantenere la complessità del modello bassa, nel contesto dei generalized factor model è desiderabile avere modelli zero-free, cioè sistemi lineari senza zeri invarianti. Per questo motivo, dopo aver rivisitato la letteratura più recente, si studiano le proprietà degli zeri dei sistemi lineari a tempo discreto, assumendo che le uscite di tipo multirate. In particolare, si studiano le proprietà degli zeri di sistemi blocked alti in condizioni generiche, ossia per parametri matriciali generici. Si dimostra che i sistemi blocked alti genericamente non presentano alcuno zero finito. Inoltre, si mostra quando i sistemi blocked alti possono essere genericamente privi di zeri nell'origine del piano complesso e all'infinito e quando invece devono giocoforza presentare degli zeri in tali punti.

Modelli errors-in-variables

La seconda classe di sistemi studiati in questa tesi sono i modelli errors-in-variables. Si discute in particolare l'identificabilità di modelli errors-in-variables (EIV) SISO con errori di misura bianchi. Sebbene questa classe di modelli sia genericamente non identificabile, è stato osservato che in certe circostanze ci possono essere due modelli EIV i quali sono indistinguibili tramite misurazioni esterne di ingresso-uscita. Questa mancanza di identificabilità (globale) può pregiudicare il processo di identificazione e necessita dunque di una maggiore comprensione. Le condizioni sull'identificabiltà trovate in letteratura garantiscono unicità sotto certe ipotesi di coprimalità della funzione di trasferimento (razionale) del sistema ideale "vero" e della densità spettrale dell'ingresso "vero", cioè privo di rumore. Purtroppo, queste condizioni non sono verificabili, poiché riguardano esattamente le incognite del problema, le quali non sono disponibili a chi effettua l'esperimento. Si forniscono nuove condizioni di identificabilità che sono invece esprimibili in termini di descrzione esterna dei segnali misurabili, vale a dire la loro densità spettrale congiunta.

Motivati dalla volontà di fornire uno strumento per la stima di densità spettrali, la quale rappresenta il passo iniziale per il test dell'identificabilità dei modelli EIV, si presenta un nuovo approccio per la stima di momenti del secondo ordine di processi stocastici stazionari, basato su kernel regolarizzatori. Si ipotizza che le funzioni di correlazione siano sommabili; esse sono stimate risolvendo un problema variazionale á lá Tikhonov. Lo spazio delle ipotesi è uno spazio di Hilbert a nucleo riproducente indotto dagli stable spline kernel, recentemente introdotti in letteratura. In questo modo, si incorpora nel procedura di stima l'informazione sulla sulla decadenza a zero delle funzioni da ricostruire. Si dimostra che la complessità computazionale complessiva dello stimatore proposto scala in modo lineare con il numero di campioni disponibili del processo. Si presenta anche un'applicazione all'identificazione di funzioni di trasferimento nel caso in cui l'ingresso sia rumore bianco. Si forniscono delle simulazioni numeriche che mostrano che il metodo proposto fornisce prestazioni analoghe ad algoritmi non parametrici standard di stima, i quali però sfruttano un oracolo per la regolazione dei propri parametri.

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1 Introduction

Mathematical models are nowadays of paramount importance in applied sciences and engineering. Often models cannot be derived by physical deductions and reasoning and there is a need for automatic instruments which can build models starting from observations or measurements of the phenomena of interest. This is precisely the scope of the discipline called *system identification*. It deals with the development of algorithms and methodologies for automatic model building from observed data. In this thesis we shall discuss the statistical approach to system identification, which leads to the construction of stochastic models, namely models in which the input-output quantities are random variables or random processes (depending on whether we are working in a static or dynamic setting). This is the mainstream approach to system identification, which has many well-known advantages and a rich literature.

Any identification procedure requires some prior knowledge or assumptions on the phenomenon to be modeled. For instance, one typically assumes that a linear time-invariant model (either static or dynamic) should be appropriate to describe the system. Other assumptions that need to be considered regard the data and the data acquisition process, especially the presence of noise, its distribution etc.. In particular, when dealing with input-output models, an hypothesis which is very often implicitly made is that the exogenous inputs to the system are accessible via exact measurements (no measurement errors or noise). This is likely a consequence of the uncritical habit in applied statistics of

modeling almost everything by regression models, where by definition the exogenous variables are exactly known. However, in many realistic situations the inputs cannot be considered exactly measurable and one should model the system as if only a noisy version of the signal was available. In fact, sometimes the input signal cannot be measured at all. The "true" hidden input then becomes a *latent variable*, and the underlying model becomes a model with latent variables. This brings up a fundamental difficulty, since there are in general several latent variables models of the same type which can describe equally well the external measurable variables of the system but involve radically different combinations of true inputs plus noise configurations. This fact makes the identification of these models an ill-posed problem. In general models with latent variables are commonly said to suffer from a lack of *identifiability*, meaning that there does not exist a unique model structure in the predefined model class describing the data[1]. For state space models this difficulty has been well-known for a long time. In this case the universally adopted solution is to choose the unique class of models for which the state variable is a function of the past input noise. These are the so-called innovation models; they bear a strict relation to the Kalman steady-state predictor.

The latent variables models which we shall discuss in this thesis are *errors-in-variables (EIV)* and *factor analysis (FA)* models. For these models a canonical choice like that for state space models is not obvious and in the literature somewhat restrictive assumptions on the model class are often made. For EIV models, to single out a unique model class, one often assumes white additive and uncorrelated noises on the true external signals (both inputs and outputs) and often (especially in FA models) the noise variances are assumed to be equal. Clearly these assumptions may not lead to model classes general enough to satisfactorily fit the data.

In this thesis we shall study the identifiability of dynamic EIV models with white additive noise, and a generalization of FA models, called the *generalized factor analysis (GFA)* models, both in a static and dynamic settings. The latter will be considered first.

Generalized factor analysis

As suggested by the name, GFA models are a generalization of the classical FA models which will be surveyed in Section 3.2. Factor analysis models have a long history; they were apparently first introduced by psychologists and successively been studied and applied in various branches of statistics and econometrics. Nowadays they are widely

[1] Usually identifiability is defined as *parameter identifiability*; i.e. one-to-one parametrization of a given model class. In our case the concept has to do instead with the uniqueness of a model class to describe the external data. Once a unique model class is chosen, it can always be parametrized in a one-to-one way, at least locally.

employed for data analysis in various fields such as psychology, econometrics, chemistry, biology, geosciences, etc. The scope of FA is to describe a large set of observations in terms of few common regressors plus additive noise. The regressors are meant to catch the correlation between the observations, while the noise explains their differences.

This class of models may appear to be similar to regression models. However, there is a substantial difference, that is, the regressors, called *latent factors*, are unaccessible to measurement. Even their number is generally unknown. This lack of prior knowledge makes these models difficult to handle because of non-uniqueness, i.e. lack of identifiability. In practice this appears also as parameter unindentifiability since in practical estimation procedures, say maximum likelihood, it always looks like there are too many parameters to estimate. This difficulty is approached by a plethora of *ad hoc* tricks in the literature, none of which seems to be really satisfactory. Actually the fact is that there are generically many (maybe infinitely many) non-equivalent FA representations which can describe the same set of random observables equally well. This is the key identifiability issues for FA models.

In GFA the basic structural assumptions of factor analysis are relaxed, allowing for

1. an infinite number of observables;

2. an additive noise term whose components instead of being uncorrelated (or independent) may allow for a sort of *weak correlation*.

In this thesis, we shall see that, once the above "weak correlation" is properly specified, there is a one-to-one correspondence between the data and their GFA representation, and hence identifiability.

One may argue that the assumption of an infinite number of observables is unrealistic, making GFA models difficult to apply in real scenarios. Actually, we shall present several applications where there is a very large number of random observables which can be modeled as the result of a common, low dimensional, random latent input plus a noise term describing local interactions. Generalized factor analysis can in fact help in capturing certain underlying simple structure in these phenomena. The key point is that, in all these applications, once the noise term is separated out, there is a simple parsimonious model which describes the collective behavior of the ensemble. We shall call this the *flocking* component of the model.

Modeling dynamic factor models: zeros of tall linear systems

In the econometric literature, dynamic versions of factor analysis models have also been introduced. In recent years, we have been witnessing a revival of interest in

these models, motivated on one hand by the need of modeling very large dimensional vector time series. Vector autoregressive moving average (ARMA) models are inadequate for modeling signals of large cross-sectional dimension, because they involve a huge number of parameters to estimate which may sometimes turn out to be larger than the sample size. Likewise the static case, there are identifiability issues for dynamic factor models. For this reason, recently a generalized version of dynamic factor models has been introduced, following the same principles of generalized factor analysis. Still, when modeling the effects of the latent factor on the observations, one can see that the number of parameters to be estimated is very large. More generally, whenever one has to deal with the identification of systems where the number of outputs is much larger than the number of inputs, the variance of the estimates can be high and hence the estimation of the parameters unreliable.

Motivated by these reasons, a part of this thesis is dedicated to the study of tall linear systems, i.e. systems with more outputs than inputs. In particular, we are interested on their zero properties. In fact, if it is a priori known that the system to be identified has no *invariant zeros*, linear autoregressive (AR) models can be adopted to describe the dynamics. In this way, the number of parameters to be estimated decreases considerably, as does the variance of the estimates. Our attention is focused on multirate systems, i.e. systems in which we assume that two output streams are available at different rates.

Errors-in-variables

In this thesis, we shall also discuss dynamic errors-in-variables (EIV) models with *white measurement errors*. Although this model class is rather restricted, it appears to be a natural and tractable generalization of output-error (OE) models. Several identification algorithms have been developed for estimating the system transfer function and the noise variances in these models. This has been possible because it is known that this model class is *generically identifiable*, in the sense that, given a pair of input-output signals, one can *almost always* uniquely associate only one EIV model with them. However, it has been pointed out that in certain circumstances there may actually be *two* EIV models which are indistinguishable from external input-output experiments.

Motivated by this fact, in this thesis we develop novel conditions for testing the identifiability of EIV models with white measurement errors. In particular, we show that identifiability may be not guaranteed if a linear-affine relation between the input and the output spectra holds. The sufficiency of this condition is also addressed. In contrast to the existing literature, our identifiability conditions can be utilized in practical situations, since they rely upon available information, namely the joint input-output spectral density.

Nonparametric spectrum estimation

In system identification, the power spectral density (or *spectrum*) is a statistical description of paramount importance for stationary stochastic processes. Several identification algorithms and identifiability tests, among which the ones employed in stochastic systems with latent variables, rely upon a preliminary estimation of the spectrum of the input and output processes. However, there are many other practical problems of time series analysis where the power spectrum is employed for data analysis, for example signal processing, control systems design, econometrics and mathematical finance.

The problem of estimating a spectrum can be summarized as follows: *Given a finite array of data samples of a stochastic stationary process, estimate its spectrum.*

The development of tools for estimating the spectrum has been object of research since the beginning of the last century, generating many algorithms to solve this problem. The solutions can be roughly grouped into parametric and nonparametric methods. In the first case, the curve obtained is a closed-form function depending on few parameters which determine its shape. In the second case, one gets a curve which is explicitly pointwise defined on the frequency domain.

Usually, in control systems and system identification it is preferable to have parametric solutions, as the goal is to obtain simple models capable to describe the system behavior, which are then studied in order to properly design the system controller. However, in recent years, the nonparametric paradigm has become increasingly important also in system identification and control design algorithms. This is because new nonparametric identification techniques, which perform significantly better than the classic parametric ones, have been developed.

For this reason, it appears reasonable to follow these novel nonparametric techniques for developing new spectrum estimation algorithms which are able to provide better results than other classical methods.

Summary of the thesis and acknowledgements

The topics treated in this thesis can be divided in four parts, corresponding to Chapters 3, 4, 5 and 6. A short summary of some preliminary concepts is made in **Chapter 2**.

In **Chapter 3** we introduce and analyze GFA models. We show that this class of model provides a well-defined description of an infinite collection of random variables by decomposing the observations as the sum of an *idiosyncratic* sequence and an *aggregate* sequence. The first type of sequences arises from a proper definition of *weakly correlated* noise, which we show to correspond to an infinite number of random variables whose

covariance matrix can be interpreted as a bounded linear operator in separable Hilbert spaces. For the latter type of sequences, which explains the effect of the latent factors on the observations, we give necessary and sufficient conditions which ensure the estimability of the latent factors. For stationary sequences, we show also that there is a natural interpretation of generalized factor analysis models in terms of Wold decomposition of stationary processes. A stationary sequence admits a (unique) generalized factor analysis decomposition if and only if two rather natural conditions are satisfied. Furthermore, we present some possible applications in which generalized factor models may help in providing explanations of observed phenomena. The key point is that, in all these applications, the observable variables are the result of local interactions plus a common, simple behavior. We associate with the latter the concept of *flocking*, which will be defined in Chapter 3. We discuss how to extract the flocking component of a random field for a simple class of separable random fields. The content of this chapter is taken from (Bottegal & Picci, 2011), (Picci & Bottegal, 2012), (Bottegal & Picci, 2013a) and (Bottegal & Picci, 2013b).

In **Chapter 4** we focus on exploring the zero properties of tall linear systems. As has been pointed out previously, this type of systems arise when one has to deal with modeling of the actions of the latent factors in a dynamic GFA context. Part of the chapter is dedicated to a review of some recent results on the zeros of tall linear time-invariant systems, and on the zeros of tall blocked linear systems, i.e. systems obtained by grouping inputs and outputs at different time instants. The main result of this part is that these classes of systems are generically zero-free, where by generically we intuitively mean "for almost all the systems". The, we dedicate to analyzing the zero properties of tall multirate linear systems, i.e. systems where two output streams available at different rates are collected. We show that, quite surprisingly, even in a generic setting, there may be situations in which these systems present zeros at the origin of the complex plane or at infinity. We show that the presence of these zeros depends on the dimensions of the state, the input, the output and the rates at which the outputs are available. The contents of this chapter are taken from (Zamani et al., 2012a) and (Zamani et al., 2012b).

In **Chapter 5** we discuss the problem of checking the identifiability of dynamic SISO errors-in-variables models, under the assumption that the measurement errors are white. First, we provide a mathematical formulation of EIV models, according to the literature. Then, we focus on the problem of finding identifiability conditions. Although this class of models has been proven to be generically identifiable, it is well-known that, under certain conditions on the noiseless input and transfer function, there exist two non-equivalent EIV models which describe the relation between the input and the output equally well. In

the literature, identifiability conditions are not testable since they concern precisely the unknowns of the problem which are not available to the experimenter. We provide a new necessary condition for non-identifiability of EIV models. Such a condition is expressible in terms of the external description of the observable signals, namely their joint power spectral densities, and is rather simple and easily checkable. Then, we address the problem of analyzing when this condition is also sufficient, showing that unidentifiability of EIV models is a rather exceptional case. Finally, we provide numerical examples that confirm the theory developed in the chapter, and a simulation that suggests how to build an identifiability test using the stated condition. The contents of this chapter are taken from (Bottegal et al., 2011).

In **Chapter 6** we propose a novel algorithm for spectrum estimation. It relies upon nonparametric kernel based techniques, which exploit prior knowledge on the function to be estimated. Instead of working in the frequency domain, we address this task in the time domain. Thus, we solve the problem of estimating autocorrelation functions, exploiting the prior information on their zero-decaying for large time lags. This is made by searching the solution in a reproducing kernel Hilbert space as hypothesis space, induced by the recently introduced *stable spline kernel*. The natural estimator for this kind of problems arises from the solution of *Tikhonov-type variational problems*. We show that such a problem can be solved with a computational complexity scales linearly with the number of observed process samples. The estimation of the optimal hyperparameters characterizing the problem is made using a cross-validation strategy. Furthermore, this method can be adopted for identifying the transfer function of a linear time-invariant system, still with a lower computational complexity compared to other novel nonparametric kernel based identification techniques. Finally, we present the results of several numerical experiments, which show that our proposed method results comparable to the standard Matlab algorithm "Spatial Spectrum Estimator" (SPA), and often also better than the "Empirical transfer function estimator" (Etfe) algorithm, also available in Matlab, even if these methods are equipped with an oracle that determines the optimal smoothing parameters by exploiting the knowledge of the true correlation function. The contents of this chapter are taken from (Bottegal & Pillonetto, 2012a) and (Bottegal & Pillonetto, 2012b).

After the presentation of these contributions, we end this dissertation with some conclusions in **Chapter 7**.

2

Notation and preliminaries

In this chapter we introduce some concepts which will be used throughout the dissertation. In particular, we recall notions on Hilbert spaces and theory of functionals and operators, for which we refer to (Akhiezer & Glazman, 1961) and (Rudin, 1991), and on stochastic processes and related systems, which are treated in (Doob, 1990), (Rozanov, 1967) and (Lindquist & Picci, 2011).

2.1 Hilbert spaces, functionals, operators

A metric space is called *complete* if every Cauchy sequence of elements in such a space converges to some element of the space. A *Banach space* is a normed space which is a complete metric space with respect to the metric generated by the norm. A *Hilbert space* $\mathcal{H}$ is a inner product space which is a complete metric space with respect to the metric generated by the inner product. Examples of Hilbert spaces, which will be widely used in this thesis, are:

- the space ℓ^2 of the sequencs $\{f(n)\}_{n\in\mathbb{N}}$ such that $\sum_n |f(n)|^2 < \infty$;
- the space $L^2(A)$, where $A \subseteq \mathbb{R}^n$, of the functions $f(t)$, $t \in A$, such that $\int_A |f(t)|^2 dt < \infty$. In this thesis, A will be either the intervals $[0,\, 1]$ or $[-\pi,\, \pi]$.

- the space $\mathcal{H}(\mathbf{y})$, where $\mathbf{y}$ is a finite variance random vector or a stationary stochastic process, given by all the possible linear combinations of the components of $\mathbf{y}$ (see next section).

A Hilbert space is *separable* if and only if it admits a countable orthonormal basis. Those infinite dimensional Hilbert spaces which are separable are therefore isometrically isomorphic to ℓ^2. Other spaces of interest are the Banach space ℓ^1 of the infinite sequences $\{f(n)\}_{n\in\mathbb{N}}$ such that $\sum_n |f(n)| < \infty$, and the Banach space ℓ^∞ of the infinite sequences $\{f(n)\}_{n\in\mathbb{N}}$ such that $|f(n)| < M$, $M \in \mathbb{R}$. We recall that the chain of inclusions $\ell^1 \subset \ell^2 \subset \ell^\infty$ holds.

When f is a infinite sequence, sometimes we shall use the (infinite dimensional) vector notation

$$f := \begin{bmatrix} f(1) \\ f(2) \\ \vdots \end{bmatrix},$$

which naturally extends to the transpose $f^\top := \begin{bmatrix} f(1) & f(2) & \dots \end{bmatrix}$ and the conjugate transpose $f^* := \begin{bmatrix} f^*(1) & f^*(2) & \dots \end{bmatrix}$.

Projections

Another useful concept is the projection of an element on a subspace. We use the symbol $\oplus$ to denote the orthogonal sum between subspaces. Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace and $\mathcal{S}^\perp$ its orthogonal complement (which is closed as well), i.e. the subspace satisfying $\mathcal{S} \oplus \mathcal{S}^\perp = \mathcal{H}$. Then every $f \in \mathcal{H}$ has an unique decomposition

$$f = f_{\mathcal{S}} + f_{\mathcal{S}^\perp},$$

where $f_{\mathcal{S}} \in \mathcal{S}$ and $f_{\mathcal{S}^\perp} \in \mathcal{S}^\perp$. Moreover, $f_{\mathcal{S}}$ and $f_{\mathcal{S}^\perp}$ are such that

$$f_{\mathcal{S}} = \arg\min_{g\in\mathcal{S}} \|f - g\|_{\mathcal{H}} \qquad , \qquad f_{\mathcal{S}^\perp} = \arg\min_{g\in\mathcal{S}^\perp} \|f - g\|_{\mathcal{H}}.$$

Functionals and operators

Let D denote a subset of $\mathcal{H}$. A function L which relates to each element $f \in D$ a definite complex number $L[f]$ is called a *functional* in the space $\mathcal{H}$ with domain D. A functional

is linear if the usual property of linearity holds. A functional is bounded if

$$\sup_{f\in D,\,\|f\|_{\mathcal{H}}\leq 1} |L[f]| < \infty\,.$$

A functional is continuous if and only if it is bounded and linear. For linear functionals, the following theorem holds.

Theorem 2.1.1. *[Riesz's theorem] Each linear functional in the Hilbert space $\mathcal{H}$ can be expressed in the form*

$$L[f] = \langle h,\, f\rangle_{\mathcal{H}}$$

where h is an element of $\mathcal{H}$ which is uniquely determined by the functional L.

Typical examples of functionals are:

- $L[f] = \int_A f(t)dt$, if A is compact and $f \in L^2(A)$;
- $L[f] = f(x)$, $x \in A$, where L is defined over the subset $D \subset L^2(A)$ of the continuous pointwise defined functions;
- $L_i[f] = \int_0^{+\infty} u(t_i - s)f(s)ds$, $t_i \in \mathbb{R}_+$, representing the ouput of a linear time-invariant system, whose impulse response is $f(t)$, evaluated at the instant t_i and driven by an uniformly bounded input $u(t)$.

A function T which relates to each element $f \in D$ a particular element $Tf = g$, $g \in \mathcal{H}$ is called operator in the space $\mathcal{H}$ with domain D. Also for operators the usual concept of linearity holds. A linear operator T is bounded if

$$\sup_{f\in D,\,\|f\|_{\mathcal{H}}\leq 1} \|Tf\|_{\mathcal{H}} < \infty\,;$$

the left member of the inequality is called *norm* of the operator. A bounded linear operator is continuous; conversely, if a linear operator is continuous at some point of D, then it is bounded.

Let $\mathcal{H}$ be a separable Hilbert space and consider an othonormal basis of its; introduce the infinite matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}\,. \tag{2.1}$$

Then the following theorem holds.

Theorem 2.1.2. *In order that* (2.1) *represent a bounded linear operator defined everywhere in $\mathcal{H}$, it is necessary and sufficient that, for some constant M, the inequality*

$$\left|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ji}x_i y_j^*\right| \leq M\sqrt{\sum_{i=1}^{\infty}|x_i|^2}\sqrt{\sum_{j=1}^{\infty}|y_j|^2}$$

holds for any elements $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$.

Given a bounded linear operator T over $\mathcal{H}$, there exists a unique bounded linear operator T^*, called *adjoint* of T, such that $\langle f, Tg\rangle_{\mathcal{H}} = \langle T^*f, g\rangle_{\mathcal{H}}$ for any $f, g \in \mathcal{H}$. If $T = T^*$, then T is *self-adjoint*. If $TT^* = T^*T$, then T is *normal*. For normal operators, Theorem 2.1.2 holds for any $x = y$.

Reproducing kernel Hilbert spaces

In this dissertation, the theory of reproducing kernel Hilbert spaces (RKHS) will be used in Chapter 6. Let $\mathcal{H}$ be a Hilbert space of continuous pointwise well defined functions on a compact set $\mathcal{X}$. Under the assumption that all the point-wise evaluations are bounded linear functionals on $\mathcal{H}$, i.e.

$$\forall x \in \mathcal{X}, \quad \exists C_x > 0 : \quad |g(x)| \leq C_x\|g\|_{\mathcal{H}}, \qquad \forall g \in \mathcal{H}. \tag{2.2}$$

the RKHS family is obtained, as formalized below.

Definition 2.1.3 (RKHS)**.** A reproducing kernel Hilbert space (RKHS) over a non-empty set $\mathcal{X}$ is a Hilbert space of functions $f : \mathcal{X} \to \mathbb{R}$ such that (2.2) holds.

As suggested by the name, the concept of RKHS is strongly linked with that of positive semidefinite kernel (Aronszajn, 1950).

Definition 2.1.4 (Positive semidefinite kernel)**.** Let $\mathcal{X}$ denote a non-empty set. A symmetric function $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is called *positive semidefinite kernel* (or Mercel Kernel) if, for any finite natural number l, it holds

$$\sum_{i=1}^{l}\sum_{j=1}^{l} c_i c_j K(x_i, x_j) \geq 0, \quad \forall (x_i, c_i) \in (\mathcal{X}, \mathbb{R})$$

Theorem 2.1.5. *(Aronszajn, 1950) Let $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a Mercel kernel and $\mathcal{X}$ be a compact set in $\mathbb{R}^{\alpha}$. Then there exists a unique Hilbert space $\mathcal{H}$ which satisfies:*

1. $K(x, \cdot) \in \mathcal{H} \quad \forall\, x \in \mathcal{X}$;

2. $f(x) = \langle K(x, \cdot),\, f(\cdot)\rangle_{\mathcal{H}} \quad \forall\, x \in \mathcal{X},\, \forall f \in \mathcal{H}.$

The functions in the space $\mathcal{H}$ can be expressed as combinations of the kernel evaluated at certain points of the domain, namely

$$f(\cdot) = \sum_{i=1}^{n} a_i K(x_i, \cdot) \quad , a_i \in \mathbb{R} \quad , n \in \mathbb{N} \tag{2.3}$$

and the norm endowed with $\mathcal{H}$ is

$$\|f\|_{\mathcal{H}}^2 = a^\top \bar{K} a \quad , a := [\, a_1, \ldots, a_n \,]^\top \quad , \bar{K} \text{ s.t. } \bar{K}_{\{i,j\}} = K(x_i, x_j)\,. \tag{2.4}$$

The above result shows that a reproducing kernel Hilbert space $\mathcal{H}$ is completely characterized by its Mercel kernel. One can also show that properties such as smoothness and integrability of the kernel directly transfer to every function of the space $\mathcal{H}$.

2.2 Stochastic variables and processes

In this dissertation, boldface symbols will normally denote random arrays, either finite or infinite, and random processes.

Random variables

A real *random variable* $\mathbf{v}$ is a real-valued measurable function defined on some underlying probability space $\{\Omega,\, \mathcal{A},\, P\}$ (P is the probability measure on Ω and $\mathcal{A}$ the σ-algebra of events). The symbol $\mathbb{E}\,[\mathbf{v}] := \int_\Omega \mathbf{v} dP$ denotes mathematical expectation, or mean, of the random variable $\mathbf{v}$. We shall always consider zero-mean random variables. Random variables which have finite second moment, $\mathbb{E}\,[|\mathbf{v}|^2] < \infty$, are commonly called second order random variables.

We shall consider the standard inner-product space of random variables linearly generated by the scalar components $[\mathbf{v}_1, \ldots, \mathbf{v}_n, \ldots]$ of a (possibly infinite) random string $\mathbf{v}$ and denoted by $\mathcal{H}(\mathbf{v}) := \operatorname{span}\{\mathbf{v}_1 \ldots, \mathbf{v}_n, \ldots\}$, equipped with the inner product

$$\langle \mathbf{x},\, \mathbf{y} \rangle := \mathbb{E}\,[\mathbf{x}\mathbf{y}] \quad , \mathbf{x},\, \mathbf{y} \in \mathcal{H}(\mathbf{v})$$

and thus with norm

$$\|\mathbf{x}\|^2 := \operatorname{var}[\mathbf{x}]\,,$$

The *covariance matrix* of a random vector $\mathbf{v} = [\mathbf{v}_1^\top \; \mathbf{v}_2^\top]^\top$ is denoted by

$$\Sigma_{\mathbf{v}} := \mathbb{E}\left[\mathbf{v}_1 \mathbf{v}_2^\top\right] := \begin{bmatrix} \Sigma_{\mathbf{v}_1} & \Sigma_{\mathbf{v}_1\mathbf{v}_2} \\ \Sigma_{\mathbf{v}_2\mathbf{v}_1} & \Sigma_{\mathbf{v}_2} \end{bmatrix},$$

where $\Sigma_{\mathbf{v}_1\mathbf{v}_2} = \Sigma_{\mathbf{v}_2\mathbf{v}_1}^\top$. Given a vector $\mathbf{v}$ and assuming $\Sigma_{\mathbf{v}} > 0$ (i.e. positive definite) the optimal linear estimate of a random variable $\mathbf{x}$ given $\mathbf{v}$ is

$$\hat{\mathbf{x}} := \mathbb{E}\left[\mathbf{x}|\mathbf{v}\right] = \Sigma_{\mathbf{x}\mathbf{v}}\Sigma_{\mathbf{v}}^{-1}\mathbf{v}\,,$$

which corresponds to the orthogonal projection of a random variable $\mathbf{x}$ on the space $\mathcal{H}(\mathbf{v})$, denoted by $\mathbb{E}[\mathbf{x}|\mathcal{H}(\mathbf{v})]$.

We say that two variables $\mathbf{x}$ and $\mathbf{y}$ of a Hilbert space $\mathcal{H}(\mathbf{v})$ are *conditionally orthogonal* given a vector $\mathbf{v}$ if

$$\langle \mathbf{x} - \mathbb{E}\left[\mathbf{x}|\mathcal{H}(\mathbf{v})\right],\, \mathbf{y} - \mathbb{E}\left[\mathbf{y}|\mathcal{H}(\mathbf{v})\right]\rangle = 0\,. \tag{2.5}$$

This can be extended to subspaces, in the sense that the spaces $\mathcal{H}(\mathbf{x})$ and $\mathcal{H}(\mathbf{y})$ are conditionally orthogonal given $\mathcal{H}(\mathbf{v})$ if (2.5) holds for every $\mathbf{x} \in \mathcal{H}(\mathbf{x})$, $\mathbf{y} \in \mathcal{H}(\mathbf{y})$. In this case, $\mathcal{H}(\mathbf{v})$ is a *splitting subspace*.

Random processes

A *stochastic process* $\mathbf{y}$ is an ordered collection of random variables (or vectors) $\mathbf{y} := \{\mathbf{y}(t)\}$, all defined in the same probability space. The time variable t will in general be discrete ($t \in \mathbb{Z}$), but occasionally we shall also deal with continuous-time processes ($t \in \mathbb{R}$). The mean of $\mathbf{y}$ is the signal $\mathbb{E}\left[\mathbf{y}(t)\right]$ and will be assumed identically equal to zero. We shall be interested on the *autocovariance function*

$$\Sigma(t,\, s) := \mathbb{E}\left[\mathbf{y}(t)\mathbf{y}(s)^\top\right],$$

which is a positive semidefinite matrix when $t = s$ ($\Sigma(t,\, t) \geq 0$). Usually we shall deal with stationary processes, for which it holds that

$$\Sigma(t,\, s) = \Sigma(\tau) \quad , \tau := t - s\,.$$

The *spectral density*, or *spectrum*, of $\mathbf{y}$ is the matrix function

$$S(\omega) = \int_{-\infty}^{\infty} \Sigma(\tau) \exp(-j\omega\tau) d\tau$$

for the continuous time case, or

$$S(\omega) := \sum_{\tau=-\infty}^{+\infty} \Sigma(\tau) \exp(-j\omega\tau)\,,$$

for the discrete time processes. In the latter case, it is positive semidefinite, defined of the interval $[-\pi,\, \pi]$ and can be extended analitically on the entire complex plane. If the components of $\mathbf{y}$ are linearly independent, meaning that $\mathcal{H}$ has dimension equal to the size $\mathbf{y}$, then $S(\omega) > 0$ almost everywhere. In this case we say that $\mathbf{y}$ is a full rank process.

We say that the processes $\mathbf{y}(t)$ and $\mathbf{v}(t)$ are uncorrelated if $\mathbb{E}\left[\mathbf{y}(t)\mathbf{v}(s)^\top\right] = 0$ for every $t,\, s$.

When $t \in \mathbb{Z}$, we associate with $\mathbf{y}$ the following separable Hilbert spaces:

- $\mathcal{H}(\mathbf{y}) := \operatorname{span}\{\mathbf{y}(t),\, t \in \mathbb{Z}\}$;
- $\mathcal{H}_t^-(\mathbf{y}) := \operatorname{span}\{\mathbf{y}(s),\, s < t\}$;
- $\mathcal{H}_t^+(\mathbf{y}) := \operatorname{span}\{\mathbf{y}(s),\, s \geq t\}$.

We denote by $\mathcal{H}_t(\mathbf{y})$ the space spanned by the random vector $\mathbf{y}(t)$; when $\mathcal{H}_t(\mathbf{y}) = \mathcal{H}(\mathbf{y})$ we say that $\mathbf{y}$ is *purely deterministic* (PD). Defining the remote past of $\mathbf{y}$ as

$$H_{-\infty}(\mathbf{y}) = \bigcap_{t \leq k} H_t(\mathbf{y})\,,$$

we say that $\mathbf{y}$ is *purely non deterministic* (PND) if and only if $H_{-\infty}(\mathbf{y}) = 0$. Szegö-Kolmogorov theorem states that a full rank process with an absolutely continuous spectrum is PND if and only if its spectral density satisfies

$$\int_{-\pi}^{\pi} \log\det S(\omega) d\omega > -\infty.$$

Conversely, if the integral diverges, the process is PD. Every process $\mathbf{y}$ admits a unique decompisition $\mathbf{y} = \hat{\mathbf{y}} + \check{\mathbf{y}}$, where $\hat{\mathbf{y}}$ is PD and $\check{\mathbf{y}}$ is PND; moreover, $\hat{\mathbf{y}}$ and $\check{\mathbf{y}}$ are uncorrelated. This is known as the *Wold decomposition*.

3

Generalized factor analysis: modeling and applications

3.1 Introduction

Factor analysis has a long history; it has apparently first been introduced by psychologists (Spearman, 1904; Burt, 1909) and successively been studied and applied in various branches of Statistics and Econometrics (Ledermann, 1937, 1939; Bekker & de Leeuw, 1987; Lawley & Maxwell, 1971). With a few exceptions however, (Kalman, 1983; van Schuppen, 1986; Picci, 1987; Picci & Pinzoni, 1986; Deistler & Zinner, 2007; Ning & Georgiou, 2011), little attention has been payed to these models in the control engineering community. Dynamic versions of factor models have also been introduced in the econometric literature, see e.g. (Geweke, 1977; Peña & Box, 1987; Peña & Poncela, 2006; Hu & Chou, 2004) and references therein.

Recently, we have been witnessing a revival of interest in these models, motivated on one hand by the need of modeling very large dimensional vector time series. Vector AR or ARMA models are inadequate for modeling signals of large cross-sectional dimension, because they involve a huge number of parameters to estimate which may sometimes turn out to be larger than the sample size. On the other hand, an interesting generalization of dynamic factor analysis models allowing the cross-sectional dimension of the

observed time series to go to infinity, has been proposed by Chamberlain, Rothschild, Forni, Lippi and collaborators in a series of widely quoted papers (Chamberlain, 1983; Chamberlain & Rothschild, 1983; Forni et al., 2000; Forni & Lippi, 2001). This new modeling paradigm is attracting a considerable attention also in the engineering system identification community (Deistler et al., 2010; Anderson & Deistler, 2008; Deistler & Zinner, 2007; Peña & Poncela, 2006). These models, called *generalized dynamic factor models* are motivated by economic and econometric applications. We shall argue that, with some elaboration, they may be quite useful also in engineering applications.

Contribution of the work

In this chapter, we want to address both theory and applications of generalized factor analysis models. First, we focus on understanding the theory of GFA; in particular, we study what conditions guarantee identifiability of GFA models, i.e. when the decomposition *latent factor plus idiosyncratic noise* is unique. To this end, we introduce the novel concept of aggregate sequence, which incorporates necessary and sufficient conditions for uniqueness of a GFA decomposition. We show that the covariance matrix of idiosyncratic noise can be interpreted as a bounded linear operator on separable Hilbert spaces. These concepts are used to elaborate a method for extracting the latent factors from an infinite set of observations. Then, we focus on stationary GFA, linking the concept of Wold decomposition for stochastic process to the one of GFA decomposition.

After addressing the structure theory of GFA, we present some possible applications in which this decomposition may help in providing understanding of certain underlying phenomena. The key point is that, in all these applications, the observable variables are the result of a common, simple behavior plus local interactions. We shall address this as *flocking* behavior.

Flocking is a commonly observed behavior in gregarious animals by which many equal individuals tend to group and follow, at least approximately, a common path in space. The phenomenon has similarities with many scenarios observed in artificial/technological and biological environments and has been studied quite actively in recent years (Brockett, 2010; Veerman et al., 2005; Olfati-Saber, 2006; Cucker & Smale, 2007). A few examples are described below.

The mechanism of *formation* of flocks is also called *convergence to consensus* and has been intensely studied in the literature, see e.g. (Fagnani & Zampieri, 2008; Olfati-Saber et al., 2007; Tahbaz-Salehi & Jadbabaie, 2010), and there is now a quite articulated theory addressing the convergence to consensus under a variety of assumptions on the communication strategy among agents etc..

In this chapter we want to address a different issue: given observations of the motion of a large set of equal agents and assuming statistical steady state, decide whether there is a flocking component in the collective motion and estimate its structural characteristics. The reason for doing this is that the very concept of flocking implies an *orderly motion* which must then admit a much simpler mathematical description than the whole ensemble. Once the flocking component (if present) has been separated, the motion of the ensemble splits naturally into flocking plus a random term which describes local random disagreements of the individual agents or the effect of external disturbances. Hence extracting a flocking structure is essentially a parsimonious modeling problem.

The organization of the chapter is as follows. In Section 3.2 we review static finite-dimensional factor analysis; in Section 3.3 we discuss the basic ideas leading to representations of infinite dimensional strings of variables by generalized factor analysis models. The problem of representation by GFA models is discussed in Section 3.4. The restriction to stationary sequences is discussed in Section 3.5; the relation of GFA with the Wold decomposition, the main theme of this section, is believed to be completely original. Then, we formalize the problem of modeling a flocking behavior given set of observations and we present some possible applications of GFA to this type of problems. Also original is the content of Section 3.7 where the extraction of the flocking component for a class of space-time random is finally discussed.

3.2 A review of static factor analysis models

A (static) *factor analysis* model is a representation

$$\mathbf{y} = F\mathbf{x} + \mathbf{e}, \tag{3.1}$$

of N observable random variables $\mathbf{y} = [\,\mathbf{y}_1 \,\ldots\, \mathbf{y}_N\,]^\top$, as linear combinations of q *common factors* $\mathbf{x} = [\,\mathbf{x}_1 \,\ldots\, \mathbf{x}_q\,]^\top$, plus uncorrelated "noise" or "error" terms $\mathbf{e} = [\,\mathbf{e}_1 \,\ldots\, \mathbf{e}_N\,]^\top$. An essential part of the model specification is that the N components of the error $\mathbf{e}$ should be (zero-mean and) mutually uncorrelated random variables, i.e.

$$\mathbb{E}\,[\mathbf{x}\mathbf{e}^\top] = 0\,, \qquad \mathbb{E}\,[\mathbf{e}\mathbf{e}^\top] = \operatorname{diag}\{\sigma_1^2, \ldots, \sigma_N^2\}\,. \tag{3.2}$$

The aim of these models is to provide an "explanation" of the mutual interrelation between the observable variables $\mathbf{y}_i$ in terms of a small number of common factors, in the sense that, setting

$$\hat{\mathbf{y}}_i := f_i^\top \mathbf{x}, \tag{3.3}$$

where $f_i^\top$ is the i-th row of the matrix F, one has exactly

$$\mathbb{E}\left[\mathbf{y}_i \mathbf{y}_j\right] = \mathbb{E}\left[\hat{\mathbf{y}}_i \hat{\mathbf{y}}_j\right], \tag{3.4}$$

for all $i \neq j$. This property is just *conditional orthogonality* (or conditional independence in the Gaussian case) of the family of random variables $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$ *given* $\mathbf{x}$ and is a characteristic property of the factors. It is in fact not difficult to see that $\mathbf{y}$ admits a representation of the type (3.1) if and only if $\mathbf{x}$ renders $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$ pairwise conditionally orthogonal given $\mathbf{x}$, (Picci, 1987; Bartholomew, 1984). We stress that conditional orthogonality given $\mathbf{x}$ is actually *equivalent* to the orthogonality (uncorrelation) of the components of the noise vector $\mathbf{e}$.

Unfortunately these models, although providing a quite natural and useful data compression scheme, in many circumstances, suffer from a serious non-uniqueness problem. In order to clarify this issue we first note that the property of making $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$ conditionally orthogonal is really a property of the subspace of random variables linearly generated by the components of the vector $\hat{\mathbf{y}} := F\mathbf{x}$, denoted $X := \mathcal{H}(\hat{\mathbf{y}})$ and it will hold for any set of generators of X. Any set of generating variables for X can serve as a common factors vector and there is no loss of generality to choose the generating vector $\mathbf{x}$ for X of minimal cardinality (a basis) and normalized, i.e. such that $\mathbb{E}\left[\mathbf{x}\mathbf{x}^\top\right] = I$, which we shall always do in the following. A subspace X making the components of $\mathbf{y}$ conditionally independent is called a *splitting subspace* for $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$. The so-called "true" variables $\hat{\mathbf{y}}_i$ are then just the orthogonal projections $\hat{\mathbf{y}}_i = \mathbb{E}\left[\mathbf{y}_i \mid X\right]$.

We may then call $q = \dim \mathbf{x} = \dim X$ the dimension of the model. Hence a model of dimension q will automatically have $\operatorname{rank} F = q$ as well. Two FA models for the same observable $\mathbf{y}$, whose factors span the same splitting subspace X are *equivalent*. This is a trivial kind of non-uniqueness since two equivalent FA models will have factor vectors related by a real orthogonal transformation matrix.

The serious non-uniqueness comes from the fact that there are in general many (possibly infinitely many) minimal splitting subspaces for a given family of observables $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$. This is by now well known (Picci, 1987; Lindquist & Picci, 2011). Hence there are in general many nonequivalent minimal FA models (with normalized factors) representing a fixed N-tuple of random variables $\mathbf{y}$. For example, one can choose, for each $k \in \{1, \ldots, N\}$, a splitting subspace of the form $X := \operatorname{span}\{\, \mathbf{y}_1 \ \ldots \ \mathbf{y}_{k-1} \ \mathbf{y}_{k+1} \ \ldots \ \mathbf{y}_N \,\}$,

and thereby obtain N "extremal" FA models called *elementary regressions* of the form

$$\begin{cases} \mathbf{y}_1 = [\,1 \,\dots\, 0\,]\,\mathbf{x} + 0 \\ \vdots \\ \mathbf{y}_k = \hat{a}_k^\top \mathbf{x} + \mathbf{e}_k \\ \vdots \\ \mathbf{y}_N = [\,0 \,\dots\, 1\,]\,\mathbf{x} + 0 \end{cases}, \tag{3.5}$$

where $\hat{a}_k^\top = \mathbb{E}\,[\mathbf{y}_k \mathbf{x}^\top](\mathbb{E}\,[\mathbf{x}\mathbf{x}^\top])^{-1}$, which are clearly non equivalent. In this example the factor subspaces are spanned by $N-1$ observable variables. A subspace X contained in the data space $\mathcal{H}(\mathbf{y}) := \operatorname{span}\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ (i.e. generated by linear functionals of $\mathbf{y}$) is called internal. Accordingly, factor analysis models whose factor $\mathbf{x}$ is made of linear functionals of $\mathbf{y}$, are called internal models. Clearly, generically FA models are noninternal.

Note that a factor analysis representation induces a decomposition of the covariance matrix Σ of $\mathbf{y}$ as

$$\Sigma = FF^\top + \operatorname{diag}\{\sigma^2_{\mathbf{e}_1}, \dots, \sigma^2_{\mathbf{e}_N}\} := FF^\top + \Delta \tag{3.6}$$

which can be seen as a special kind of *low rank plus sparse* decomposition of a covariance matrix (Chandrasekaran et al., 2011), a diagonal matrix being, in intuitive terms, as sparse as one could possibly ask for. The following Proposition characterizes noninternal FA models.

Proposition 3.2.1. *All internal factor analysis models are regressions. All nontrivial factor analysis models with $\Delta > 0$ are noninternal.*

The following Theorem describes how to express the latent variables $\mathbf{x}$ starting from $\mathbf{y}$ and from the knowledge of the matrices F and Δ of the structured covariance matrix of the data.

Theorem 3.2.2. *Every normalized latent factors vector for the FA model $\mathbf{y} = F\mathbf{x} + \mathbf{e}$ has the form*

$$\mathbf{x} = F^\top \Sigma^{-1} \mathbf{y} + \mathbf{z}\,,$$

where $\mathbf{z}$ is a q-dimensional zero-mean random vector orthogonal to $\mathcal{H}(\mathbf{y})$ with covariance $I_q - F^\top \Sigma^{-1} F$.

The inherent nonuniqueness of FA models is called "factor indeterminacy", or unindentifiability in the literature and the term is usually referred to parameter unidentifiability as it may appear that there are always "too many" parameters to be estimated. It may

be argued that once a model, in essence, a splitting subspace, is selected, it can always be parametrized in a one-to-one (and hence identifiable) way. Unfortunately, the classification of all possible minimal FA representations and an explicit characterization of minimality are, to a large extent, still an open problem. The difficulty is indeed a serious one.

Since, as we have argued, in essence non-uniqueness is just a consequence of uncorrelation of the noise components, one may try to get uniqueness by giving up or mitigating the requirement of uncorrelation of the components of $\mathbf{e}$. This however tends to make the problem ill-defined as the basic goal of uniquely splitting the external signal into a noiseless component plus "additive noise" is made vacuous, unless some extra assumptions are made on the model and on the very notion of "noise". Quite surprisingly, as we shall see, for models describing an *infinite* number of observables a meaningful weakening of the uncorrelation property can be introduced, so as to guarantee the uniqueness of the decomposition.

3.3 Aggregate and idiosyncratic sequences

In this section we shall review the main ideas of generalized factor analysis, drawing quite heavily on the papers (Chamberlain & Rothschild, 1983; Forni & Lippi, 2001) although with some non-trivial original contributions. We shall restrict for now to the *static case*.

Consider a zero-mean finite variance stochastic process $\mathbf{y} := \{\mathbf{y}(k),\, k \in \mathbb{Z}_+\}$, which we shall normally represent as a random column vector with an infinite number of components. The index k will later have the interpretation of a space variable. Convergence shall always mean convergence in the norm topology of the space $\mathcal{H}(\mathbf{y})$ linearly generated by the components of $\mathbf{y}$. We want to describe the process as a linear combination of a finite number of common random components plus "noise", i.e.

$$\mathbf{y}(k) = \sum_{i=1}^{q} f_i(k)\mathbf{x}_i + \tilde{\mathbf{y}}(k)\,, \qquad k = 1, 2, \ldots \tag{3.7}$$

where the random variables $\mathbf{x}_i\,,\ i = 1, \ldots, q$ are the *common factors* and the deterministic vectors f_i are the *factor loadings*. The $\mathbf{x}_i$ can be taken, without loss of generality, to be orthonormal so as to form a q-dimensional random vector $\mathbf{x}$ with $\mathbb{E}\left[\mathbf{x}\mathbf{x}^\top\right] = I_q$. The $\tilde{\mathbf{y}}(k)$'s are zero mean random variables orthogonal to $\mathbf{x}$. We shall list the linear combinations $\hat{\mathbf{y}}(k) := \sum f_i(k)\mathbf{x}_i$ as the components of an infinite random vector $\hat{\mathbf{y}}$ and likewise for the

noise terms $\tilde{\mathbf{y}}(k)$ so that (3.7) can be written

$$\mathbf{y} = \hat{\mathbf{y}} + \tilde{\mathbf{y}} \tag{3.8}$$

for short. Which specific characteristics qualify the process $\tilde{\mathbf{y}}$ as "noise" is a nontrivial issue which will be one of the main themes of this section and will be made precise later (see the definition of idiosyncratic noise below).

The infinite covariance matrix of the vector $\mathbf{y}$ is formally written as $\Sigma := \mathbb{E}[\mathbf{y}\mathbf{y}^\top]$. We let Σ_n indicate the top-left $n \times n$ block of Σ, equal to the covariance matrix of the first n components of $\mathbf{y}$, the corresponding n-dimensional vector being denoted by $\mathbf{y}^n$. The inequality $\Sigma > 0$ means that all submatrices Σ_n of Σ are positive definite, which we shall always assume in the following.
Letting $\hat{\Sigma} := \mathbb{E}[\hat{\mathbf{y}}\hat{\mathbf{y}}^\top]$ and $\tilde{\Sigma} := \mathbb{E}[\tilde{\mathbf{y}}\tilde{\mathbf{y}}^\top]$, the orthogonality of the noise term and the factor components implies that

$$\Sigma = \hat{\Sigma} + \tilde{\Sigma}\,, \tag{3.9}$$

that is, $\Sigma_n = \hat{\Sigma}_n + \tilde{\Sigma}_n\,,\ \forall n \in \mathbb{N}$. Even imposing $\hat{\Sigma}$ of low rank, this is a priori a highly non unique decomposition. There are situations/examples in which the $\tilde{\Sigma}$ is diagonal as in the static factor analysis case, but these situations are exceptional.

Idiosyncratic sequences

Let $\ell^2(\Sigma)$ denote the Hilbert space of infinite sequences $a := \{a(k),\, k \in \mathbb{Z}_+\}$ such that $\|a\|_\Sigma^2 := a^\top \Sigma a < \infty$.

Definition 3.3.1 (Forni, Lippi)**.** A sequence of elements $\{a_n\}_{n\in\mathbb{N}} \subset \ell^2 \cap \ell^2(\Sigma)$ is an *averaging sequence* (AS) for $\mathbf{y}$, if $\lim_{n\to\infty} \|a_n\|_2 = 0$.
We say that a sequence of random variables $\mathbf{y}$ is **idiosyncratic** if $\lim_{n\to\infty} a_n^\top \mathbf{y} = 0$ for any averaging sequence $a_n \in \ell^2 \cap \ell^2(\Sigma)$.

Whenever the covariance Σ is a bounded operator on ℓ^2 one has $\ell^2(\Sigma) \subset \ell^2$; in this case an AS can be seen just as a sequence of linear functionals in ℓ^2 converging strongly to zero.

Example 3.3.2. The sequence of elements in ℓ^2

$$a_n = \frac{1}{n}[\underbrace{1 \ldots 1}_{n}\, 0 \ldots]^\top \tag{3.10}$$

is an averaging sequence for any Σ. On the other hand, let P_n denote the compression of the n-th power of the left shift operator to the space ℓ^2; i.e. $[P_n a](k) = a(k-n)$ for

$k \geq n$ and zero otherwise. Then $\lim_{n\to\infty} P_n a = 0$ for all $a \in \ell^2$ (Halmos, 1961) so that $\{P_n a\}_{n\in\mathbb{N}}$ is an AS for any $a \in \ell^2$.

Example 3.3.3. Let $\mathbb{1}$ be an infinite column vector of 1's and let $\mathbf{x}$ be a scalar random variable uncorrelated with $\tilde{\mathbf{y}}$, a zero-mean weakly stationary ergodic sequence. Consider the process

$$\mathbf{y} = \mathbb{1}\mathbf{x} + \tilde{\mathbf{y}}$$

and the averaging sequence (3.10). Since $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \tilde{\mathbf{y}}(k) = \mathbb{E}\left[\tilde{\mathbf{y}}(k)\right] = 0$ (limit in L^2) we have $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbf{y}(k) = \mathbf{x}$; hence we can recover the latent factor by averaging. More generally, if $\tilde{\mathbf{y}}$ is idiosyncratic, then $\lim_{n\to\infty} a_n^\top \tilde{\mathbf{y}} = 0$ for any averaging sequence and one could recover $\mathbf{x}$ from AS's such that $\lim_{n\to\infty} a_n^\top \mathbb{1}$ exists and is nonzero.

The following definition is meant to capture the phenomenon described in the previous example.

Definition 3.3.4. Let $\mathbf{z} \in \mathcal{H}(\mathbf{y})$. The random variable $\mathbf{z}$ is an *aggregate (of* $\mathbf{y}$*)* if there exists an AS $\{a_n\}$ such that $\lim_{n\to\infty} a_n^\top \mathbf{y} = \mathbf{z}$. The set of all aggregate random variables in $\mathcal{H}(\mathbf{y})$ is denoted by $\mathcal{G}(\mathbf{y})$.

The space $\mathcal{G}(\mathbf{y})$ is called the *aggregation subspace* of $\mathcal{H}(\mathbf{y})$. The following Lemma characterizes its structure.

Lemma 3.3.5. *(Forni & Lippi, 2001) The aggregation subspace* $\mathcal{G}(\mathbf{y})$ *is closed.*

Clearly, if $\mathbf{y}$ is an idiosyncratic sequence then $\mathcal{G}(\mathbf{y}) = \{0\}$. In general it is possible to define an orthogonal decomposition of the type

$$\mathbf{y} = \mathbb{E}[\mathbf{y} \mid \mathcal{G}(\mathbf{y})] + \mathbf{u}\,, \tag{3.11}$$

where all components $\mathbf{u}(k)$ are uncorrelated with $\mathcal{G}(\mathbf{y})$. The idea behind this decomposition is that, in case $\mathcal{G}(\mathbf{y})$ is finite dimensional, say generated by a q-dimensional random vector $\mathbf{x}$, one may naturally capture a unique decomposition of $\mathbf{y}$ of the type (3.7).

Unfortunately however, in general $\mathcal{G}(\mathbf{y}) = \{0\}$ does not imply that $\mathbf{y}$ is idiosyncratic. See the example below, inspired to a similar one in (Forni & Lippi, 2001).

Example 3.3.6. Consider a sequence $\mathbf{y}$ with $\mathbf{y}(j) \perp \mathbf{y}(h)\ \forall\, j \neq h$ (a possibly non-stationary white noise), and let $\mathbf{z}$ be an aggregate random variable, so that there is an AS $\{a_n\}$ such that

$$\mathbf{z} = \lim_{n\to\infty} a_n^\top \mathbf{y} = \lim_{n\to\infty} \sum_{j=1}^{\infty} a_n(j)\mathbf{y}_j\,. \tag{3.12}$$

Note that, being $\mathbf{z} \in \mathcal{H}(\mathbf{y})$ and $\mathbf{y}$ an orthogonal basis of this space, we can uniquely express $\mathbf{z}$ as

$$\mathbf{z} = \sum_{j=1}^{\infty} b(j)\mathbf{y}(j)\,, \tag{3.13}$$

and, by uniqueness of the representation, it follows that $\lim_{n\to\infty} a_n(j) = b(j)\ \forall j$. On the other hand, being a_n an AS, the limits of $a_n(j)$ must be zero, so that $b(j) = 0$. Hence $\mathbf{z} = 0$. Thus a white noise process has always $\mathcal{G}(\mathbf{y}) = \{0\}$.
However if $\{\mathbf{y}(k)\}$ has unbounded variance, the sequence is not idiosyncratic. For example if $\|\mathbf{y}(k)\|^2 = k$, given the AS

$$d_n = \frac{1}{\sqrt{n}}[\underbrace{0\ \dots\ 0\,1}_{n}\ 0\ \dots]^\top\,, \tag{3.14}$$

we have $\|d_n^\top \mathbf{y}\| = 1\ \forall n$. Hence in this case $\mathbf{y}$ is neither aggregate nor idiosyncratic. On the other hand, when $\|\mathbf{y}(k)\| \le M < \infty$ for all k, we have

$$\|a_n^\top \mathbf{y}\|^2 = \sum_{k=0}^{\infty} a_n(k)^2 \|\mathbf{y}(k)\|^2 \le M^2 \|a_n\|_2^2 \to 0 \tag{3.15}$$

for $n \to \infty$. Hence a white noise process *with a uniformly bounded variance* (has a trivial aggregation subspace and) is idiosyncratic.

The nature of an idiosyncratic sequence is related to certain properties of its covariance matrix. To explain this point, we need to introduce some notations and facts about the eigenvalues of sequences of covariance matrices. Denote by $\lambda_{n,k}(\Sigma)$ the k–th eigenvalue of the $n \times n$ upper left submatrix Σ_n of Σ. The $\lambda_{n,k}(\Sigma)$ are real nonnegative and can be ordered by decreasing magnitude. By Weyl's theorem (Stewart & Sun, 1990, p. 203), see also (Forni & Lippi, 2001, Fact M), the k–th eigenvalue of Σ_n is a non decreasing function of n and hence has a limit, $\lambda_k(\Sigma)$, which may possibly be equal to $+\infty$. Each such limit is called an *eigenvalue of* Σ. These limits however are in general not true eigenvalues, as it is well-known that Σ may not have eigenvalues. For example, a bounded symmetric Toeplitz matrix has a purely continuous spectrum (Hartman & Wintner, 1954). Anyway since Σ is symmetric and positive, its spectrum lies on the positive half line and its elements can also be ordered. Henceforth we shall denote by $\lambda_1(\Sigma)$ the maximal eigenvalue of Σ, as defined above, with the convention that $\lambda_1(\Sigma) = +\infty$ when there are infinite eigenvalues. The following result will be instrumental in understanding the structure of idiosyncratic processes.

Theorem 3.3.7. *If $\lambda_1(\Sigma)$ is finite, then Σ is a bounded operator on ℓ^2.*

Proof. Let $\lambda_1(\Sigma_n)$ be the maximal eigenvalue of Σ_n. Denote the string of the first n elements of an infinite sequence a by a^n. Since

$$\Sigma_n \leq \lambda_1(\Sigma_n) I_n \leq \lambda_1(\Sigma) I_n \tag{3.16}$$

where I_n is the $n \times n$ identity matrix and $\lambda_1(\Sigma) < \infty$ by assumption, it follows that for all sequences $x, y \in \ell^2$

$$x^n \Sigma_n y^n \leq \lambda_1(\Sigma) \|x^n\|_2 \, \|y^n\|_2 \,, \qquad n = 1, 2, \ldots \tag{3.17}$$

Then the result follows from Theorem 2.1.2. □

A strong characterization of idiosyncratic sequences is stated in the following theorem, inspired by (Forni & Lippi, 2001) after some obvious simplifications. For completeness we shall provide a proof.

Theorem 3.3.8. *The sequence* $\mathbf{y}$ *is idiosyncratic if and only if* $\lambda_1(\Sigma)$ *is finite; equivalently, if and only if its covariance matrix defines a bounded operator on* ℓ^2.

Proof. Assume first that $\lim_{n\to\infty} \lambda_{n,1}(\Sigma) = +\infty$. Since $\Sigma_n > 0$ is symmetric it has a spectral representation

$$U_n^\top \Sigma_n U_n = D_n \,, \tag{3.18}$$

where U_n is orthonormal and $D_n = \operatorname{diag}\{\, \lambda_{n,1}(\Sigma), \ldots, \lambda_{n,n}(\Sigma) \,\}$. Consider the first column of U_n, say u_1^n, which is the eigenvector of $\lambda_{n,1}(\Sigma)$ and define the sequence of elements in $\ell^2 \cap \ell^2(\Sigma)$ constructed as

$$a_n := \frac{1}{\sqrt{\lambda_{n,1}(\Sigma)}} \begin{bmatrix} (u_1^n)^\top & 0 & \ldots \end{bmatrix}^\top \,, \qquad n = 1, 2, \ldots \,. \tag{3.19}$$

Since $\lim_{n\to\infty} \lambda_{n,1}(\Sigma) = +\infty$, this is an AS, for which

$$\|a_n^\top \mathbf{y}\|^2 = \frac{1}{\lambda_{n,1}(\Sigma)} (u_1^n)^\top \Sigma_n u_1^n = 1 \tag{3.20}$$

for every n and hence the sequence $\mathbf{y}$ cannot be idiosyncratic.

Conversely, suppose that $\lambda_1(\Sigma) < +\infty$ and again use the diagonalization (3.18). Let a_n be an arbitrary AS and consider the random variable $\mathbf{z} = \lim_{n\to\infty} a_n^\top \mathbf{y} = \lim_{n\to\infty} a_n^{n\top} \mathbf{y}^n$, which has variance

$$\operatorname{var}[\mathbf{z}] = \lim_{n\to\infty} (a_n^n)^\top U_n D_n U_n^\top a_n^n := (d_n^n)^\top D_n \, d_n^n \,, \tag{3.21}$$

where the vector $d_n^n := U_n^\top a_n^n$ is used to form the first n elements of an infinite string, say d_n, whose remaining entries are taken equal to those of a_n; i.e. $d_n(k) = a_n(k)$ for $k > n$. Clearly d_n is an AS.
Since $(d_n^n)^\top D_n\, d_n^n = \sum_{k=1}^n \lambda_{n,k}(\Sigma) d_n(k)^2$, one can write

$$\text{var}\,[\mathbf{z}] = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_{n,k}(\Sigma) d_n(k)^2 \leq \lim_{n\to\infty} \lambda_1(\Sigma) \sum_{k=1}^{n} d_n(k)^2 = \lim_{n\to\infty}\ \lambda_1(\Sigma)\|d_n\|_2^2 = 0$$

which shows that $\mathbf{y}$ is idiosyncratic. □

In particular, since the covariance of a white noise process is diagonal, the covariance of a white noise can be bounded (and therefore $\mathbf{y}$ can be idiosyncratic) only if the variances $\|\mathbf{y}(k)\|^2$ are uniformly bounded. This completes the discussion in Example 3.3.6.

Aggregate sequences

Definition 3.3.9. Let q be a finite integer. A sequence $\mathbf{y}$ is *purely deterministic of rank* q (in short q-PD) if $\mathcal{H}(\mathbf{y})$ has dimension q.

Clearly a q-PD sequence $\mathbf{y}$ can be seen as a (in general non-stationary) purely deterministic process in the classical sense of the term, see e.g. (Cramèr, 1961). Let $\mathbf{x} = \begin{bmatrix}\mathbf{x}_1 & \dots & \mathbf{x}_q\end{bmatrix}^\top$ be an orthonormal basis in $\mathcal{H}(\mathbf{y})$. Obviously $\mathbf{y}$ is a q-PD random sequence if and only if there is $\infty \times q$ matrix $F = \begin{bmatrix} f_1 & f_2 & \dots & f_q\end{bmatrix}$, such that

$$\mathbf{y}(k) = \sum_{i=1}^{q} f_i(k)\, \mathbf{x}_i\,, \qquad k \in \mathbb{Z}_+\,, \tag{3.22}$$

where the columns $f_1,\ f_2,\ \dots f_q$ must be linearly independent, for otherwise the rank of $\mathbf{y}$ would be smaller than q.
We want to relate this concept to the idea of aggregation subspace of $\mathbf{y}$, as defined earlier. In particular we would like to identify $\mathbf{x}$ as an orthonormal basis in $\mathcal{G}(\mathbf{y})$. Quite unfortunately however, there are nontrivial sequences representable in the form (3.22) which are idiosyncratic (or contain idiosyncratic sequences). See the Example below.

Example 3.3.10. Consider a sequence $\mathbf{y}$ whose $k-$th element is

$$\mathbf{y}(k) = \lambda^k \mathbf{x} \quad ,\ |\lambda| < 1, \tag{3.23}$$

where $\mathbf{x}$ is a zero–mean random variable of positive variance σ^2. Clearly, $\mathbf{y}$ is 1-PD, its spanned subspace $\mathcal{H}(\mathbf{y})$ being the one-dimensional space $\mathcal{H}(\mathbf{x})$. The covariance matrix of the first n components of $\mathbf{y}$ is

$$\Sigma_n = \mathbb{E}\left[\mathbf{y}_n \mathbf{y}_n^\top\right] = \sigma^2 \begin{bmatrix} \lambda^2 & \lambda^3 & \dots & \lambda^{n+1} \\ \lambda^3 & \lambda^4 & \dots & \lambda^{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{n+1} & \lambda^{n+2} & \dots & \lambda^{2n} \end{bmatrix} \tag{3.24}$$

Since $\operatorname{rank}(\Sigma_n) = 1$ for every n, we have

$$\lambda_1(\Sigma) = \lim_{n\to\infty} \operatorname{tr}(\Sigma_n) = \lim_{n\to\infty} \sigma^2 \sum_{k=1}^{n} \lambda^{2k} = \frac{\sigma^2\lambda^2}{1-\lambda^2}\,, \tag{3.25}$$

thus, in force of Theorem 3.3.8, $\mathbf{y}$ is idiosyncratic. Hence there are (non-stationary) $q-$PD sequences which are idiosyncratic.

This is a possibility which we clearly must exclude if the decomposition (3.7) has to be unique. The question is which properties need to be satisfied by the functions $f_1,\ f_2,\ \dots f_q$ in order to avoid situations like Example 3.3.10. One necessary condition is easily found: the f_i *cannot be in* ℓ^2 since otherwise any sequence of functionals $\{a_n\}$ in ℓ^2 converging to zero would lead to

$$\lim_{n\to\infty} a_n^\top f_i = 0 \tag{3.26}$$

so that $\lim_{n\to\infty} a_n^\top \mathbf{y} = 0$ as well. This is clearly the problem with Example 3.3.10.

We shall call a sequence $\mathbf{y}$ **q-aggregate** if its covariance matrix has q nonzero eigenvalues, i.e. $\operatorname{rank}\Sigma_n = q,\ \forall n$, and $\lim_{n\to\infty}\lambda_{n,k}(\Sigma) = +\infty$ for $k = 1,\dots,q$. In short, all nonzero eigenvalues of Σ are *infinite*.

The following condition guarantees uniqueness of the decomposition (3.7) when $\hat{\mathbf{y}}$ is q-aggregate and $\tilde{\mathbf{y}}$ is idiosyncratic.

Proposition 3.3.11. *A q-aggregate sequence $\hat{\mathbf{y}}$ can be idiosyncratic only if it is the zero sequence.*

Proof. This follows trivially from Theorem 3.3.8. If $q > 0$ the maximal eigenvalue of the covariance matrix of $\hat{\mathbf{y}}$ is $+\infty$ by definition. □

Of course the question is under what conditions the q eigenvalues of $\hat{\Sigma}$ may tend to infinity. Theorem 3.3.13 below provides an answer.

Definition 3.3.12. Let

$$\tilde{f}_i^n := f_i^n - \Pi[\, f_i^n \,|\, \mathcal{F}_i^n] \tag{3.27}$$

where Π is the orthogonal projection onto the Euclidean space $\mathcal{F}_i^n = \operatorname{span}\{f_j^n,\, j \neq i\}$ of dimension $q-1$.
The vectors f_i, $i = 1, \ldots, q$ in $\mathbb{R}^\infty$ are **strongly linearly independent** if

$$\lim_{n\to\infty} \|\tilde{f}_i^n\|_2 = +\infty \qquad i = 1, \ldots, q\,. \tag{3.28}$$

In a sense, the tails of two strongly linearly independent vectors in $\mathbb{R}^\infty$ cannot get "too close" asymptotically.

Theorem 3.3.13. *Let* $\mathbf{y}$ *be a* $q-$*PD sequence, i.e. let*

$$\mathbf{y}(k) = \sum_{i=1}^{q} f_i(k)\, \mathbf{x}_i\,, \qquad k \in \mathbb{Z}_+\,; \tag{3.29}$$

then $\mathbf{y}$ *is* $q-$*aggregate if and only if, the vectors* f_i, $i = 1, \ldots, q$ *are strongly linearly independent.*

Proof. First we prove the sufficiency of condition (3.28). Let k be a fixed positive constant and let f_1 be such that

$$\lim_{n\to\infty} \|f_1^n - \Pi[f_1^n \,|\, \mathcal{F}_1^n]\|_2 = k^{\frac{1}{2}} < +\infty\,. \tag{3.30}$$

Let

$$\tilde{f}_1^n = f_1^n - \Pi[f_1^n \,|\, \mathcal{F}_1^n] = f_1^n - \alpha_2^n f_2^n - \ldots - \alpha_q^n f_q^n\,; \tag{3.31}$$

whence, defining $\tilde{F}^n := \begin{bmatrix} \tilde{f}_1^n & f_2^n & \ldots & f_q^n \end{bmatrix}$, one can write $\tilde{F}^n = F^n T^n$, with T^n is a full rank matrix of the form

$$T^n = \begin{bmatrix} 1 & 0 \\ -\alpha_n & I_{q-1} \end{bmatrix}, \tag{3.32}$$

where $\alpha_n := \begin{bmatrix} \alpha_2^n & \ldots & \alpha_q^n \end{bmatrix}^\top$. Since $\tilde{f}_1^n \perp f_i^n$, $i \neq 1$, the Gramian matrix of $\tilde{F}^n$ is block diagonal,

$$\tilde{F}^{n\top}\tilde{F}^n = \begin{bmatrix} \|\tilde{f}_1^n\|^2 & 0 \\ 0 & A_n \end{bmatrix}, \tag{3.33}$$

where A_n is a positive definite matrix whose eigenvalues tend to infinity as n increases. Note that the spectrum of $\tilde{F}^{n\top}\tilde{F}^n$ contains the eigenvalue $\|\tilde{f}_1^n\|^2$, which, for $n \to \infty$, converges to $k < +\infty$.

Now, let us compute the trace of both sides of the identity $T^n(\tilde{F}^{n\top}\tilde{F}^n)^{-1}T^{n\top} = (F^{n\top}F^n)^{-1}$ obtaining

$$\begin{aligned}\text{tr}\left[(F^{n\top}F^n)^{-1}\right] &= \text{tr}\left[T^n(\tilde{F}^{n\top}\tilde{F}^n)^{-1}T^{n\top}\right] = \text{tr}\left[T^{n\top}T^n(\tilde{F}^{n\top}\tilde{F}^n)^{-1}\right] \qquad (3.34)\\ &= \text{tr}\begin{bmatrix}1+\|\alpha_n\|^2 & -\alpha_n^\top\\ -\alpha_n & I_{q-1}\end{bmatrix}\begin{bmatrix}k^{-1} & 0\\ 0 & A_n^{-1}\end{bmatrix}\\ &= \text{tr}\begin{bmatrix}k^{-1}(1+\|\alpha_n\|^2) & -\alpha_n^\top A_n^{-1}\\ -\alpha_n k^{-1} & A_n^{-1}\end{bmatrix} = k^{-1}(1+\|\alpha_n\|^2)+\text{tr}\left[A_n^{-1}\right]\end{aligned}$$

(3.35)

Since the eigenvalues of A_n tend to infinity, those of A_n^{-1} tend to zero, while, for every n we have $k^{-1}(1+\|\alpha_n\|^2) > 0$. Thus, one eigenvalue of $(F^{n\top}F^n)^{-1}$ is bounded below by a fixed constant as n tends to infinity. Hence we conclude that one eigenvalue of $F^{n\top}F^n$ remains bounded as n tends to infinity, which is a contradiction.

For the necessity, we define $f_i^{n_1,n_2} := \begin{bmatrix}f_i(n_1) & \dots & f_i(n_2)\end{bmatrix}^\top$ and observe that condition (3.28) implies that

$$\lim_{n\to\infty}\|f_i^{n_1,n} - \Pi[f_i^{n_1,n}\,|\,\mathcal{F}_i^{n_1,n}]\|_2 = +\infty\,, \qquad (3.36)$$

for every index $i = 1, \dots, q$ and natural number n_1. Moreover, by definition of limit, we have that for every $n_1 \in \mathbb{N}$ and $K \in \mathbb{R}_+$ there exists an integer n_2 such that the inequality (with an obvious meaning of the symbols)

$$\|f_i^{n_1,n_2} - \Pi[f_i^{n_1,n_2}\,|\,\mathcal{F}_i^{n_1,n_2}]\|_2^2 \geq K \qquad (3.37)$$

holds for every $i = 1, \dots, q$.

Now, consider the sequence generated by the q-th eigenvalue of the matrix $F^{n\top}F^n$, say $\{\lambda_q^n\,;\, n \in \mathbb{N}\}$. Our goal is to show that for every natural n_1 and arbitrary constant $c > 0$ there exists a natural number n_2 such that $\lambda_q^{n_2} \geq \lambda_q^{n_1} + c$, so that $\lim_{n\to\infty}\lambda_q^n = +\infty$. To this end, fix c and, for a generic n_1, consider the normalized eigenvector of the q-th eigenvalue of the matrix $F^{n_2\top}F^{n_2}$, say $v_q^{n_2}$. Since for every $n_2 > n_1$ it holds that

$$F^{n_2\top}F^{n_2} = F^{n_1\top}F^{n_1} + F^{n_1,n_2\top}F^{n_1,n_2}\,, \qquad (3.38)$$

we can write

$$\lambda_q^{n_2} = v_q^{n_2\top}F^{n_1\top}F^{n_1}v_q^{n_2} + v_q^{n_2\top}F^{n_1,n_2\top}F^{n_1,n_2}v_q^{n_2}\,. \qquad (3.39)$$

Consider the first term on the right side of this identity; expressing $v_q^{n_2}$ as a linear combi-

nation of the eigenvectors of ${F^{n_1}}^\top F^{n_1}$, i.e. $v_q^{n_2} = \alpha_1 v_1^{n_1} + \ldots + \alpha_q v_q^{n_1}$, the orthogonality of these eigenvectors implies that

$$ {v_q^{n_2}}^\top {F^{n_1}}^\top F^{n_1} v_q^{n_2} = \lambda_1^{n_1}\alpha_1^2 + \ldots + \lambda_q^{n_1}\alpha_q^2 \geq \lambda_q^{n_1} \sum_{i=1}^{q} \alpha_i^2 = \lambda_q^{n_1} \,, \tag{3.40} $$

so that

$$ \lambda_q^{n_2} \geq \lambda_q^{n_1} + {v_q^{n_2}}^\top {F^{n_1,n_2}}^\top F^{n_1,n_2} v_q^{n_2} \,. \tag{3.41} $$

Now we have to show that we can always find an integer n_2 such that the quantity

$$ {v_q^{n_2}}^\top {F^{n_1,n_2}}^\top F^{n_1,n_2} v_q^{n_2} $$

can be chosen arbitrarily large, i.e. greater or equal to the previously fixed constant c . To this end, take n_2 such that for every $i = 1, \ldots, q$ the inequality (3.37) holds, with $K = c\sqrt{q}$. Then, there is an index i such that the i-th component of the norm one vector $v_q^{n_2} = \begin{bmatrix} w_1 & \ldots & w_q \end{bmatrix}^\top$, satisfies the inequality $w_i \geq \frac{1}{\sqrt{q}}$. Without loss of generality we may and shall assume that $i = 1$. Let $\alpha_2 \ldots \alpha_q$ be defined as in (3.31) and set

$$ \tilde{f}_1^{n_1,n_2} := f_1^{n_1,n_2} - \alpha_2 f_2^{n_1,n_2} - \ldots - \alpha_q f_q^{n_1,n_2} \,, \tag{3.42} $$

so that we have

$$ {v_q^{n_2}}^\top {F^{n_1,n_2}}^\top F^{n_1,n_2} v_q^{n_2} = {v_q^{n_2}}^\top {T^n}^\top \begin{bmatrix} \|\tilde{f}_1^{n_1,n_2}\|^2 & 0 \\ 0 & A_n \end{bmatrix} T^n v_q^{n_2} \,, \tag{3.43} $$

where T^n has the same structure as in (3.32). Now, observe that

$$ T^n v_q^{n_2} = \begin{bmatrix} w_1 & -\alpha_2 w_1 + w_2 & \ldots & -\alpha_q w_1 + w_q \end{bmatrix}^\top \,, \tag{3.44} $$

which implies that (3.43) is equal to $w_1^2 \|\tilde{f}_1^{n_1,n_2}\|^2 + Q$, where Q is a positive constant. Hence, from (3.42) we have ${v_q^{n_2}}^\top {F^{n_1,n_2}}^\top F^{n_1,n_2} v_q^{n_2} > c$ and hence, recalling (3.41),

$$ \lambda_q^{n_2} \geq \lambda_q^{n_1} + c \,. \tag{3.45} $$

which proves the theorem. □

Example 3.3.14. Consider the following $2-$PD sequence $\mathbf{y}(k) := \sum_{i=1}^{2} f_i(k)\, \mathbf{x}_i$

with

$$f_1(k) = 1 \quad \text{for all } k\,, \qquad f_2(k) = 1 - \left(\frac{1}{2}\right)^k$$

It is not difficult to check that this sequence does not satisfy condition (3.28). We shall show that this sequence is not 2-aggregate. The Gramian matrix of the functions f_1, f_2 restricted to $[1,\, n]$ is

$$F^{n\top} F^n = \begin{bmatrix} \|f_1^n\|_2^2 & \langle f_1^n,\, f_2^n\rangle_2 \\ \langle f_1^n,\, f_2^n\rangle_2 & \|f_2^n\|_2^2 \end{bmatrix}$$

and it can be seen that as $n \to \infty$, the second eigenvalue converges to $\frac{5}{3}$. Hence one eigenvalue of the covariance matrix of $\mathbf{y}$ is finite and the sequence is not 2-aggregate.

3.4 Generalized factor analysis representations: existence and uniqueness

We eventually come to a precise definition of the basic object of our study. The following is the static version of a similar definition of (Forni & Lippi, 2001) for the dynamic setting.

Definition 3.4.1. A random sequence $\mathbf{y}$ is a $q-$**factor sequence** ($q-$FS) if it can be written as an orthogonal sum

$$\mathbf{y}(k) = \sum_{i=1}^{q} f_i(k)\mathbf{x}_i + \tilde{\mathbf{y}}(k)\,, \qquad k = 0, 1, 2, \ldots \tag{3.46}$$

where $\hat{\mathbf{y}} := \sum f_i \mathbf{x}_i$ is a q-aggregate sequence and $\tilde{\mathbf{y}}$ is idiosyncratic and orthogonal to $\mathbf{x}$. The representation (3.46) is called a **generalized factor analysis (GFA) representation of $\mathbf{y}$ with q factors.**

The crucial question is now which random sequences are $q-$FS. A first step is to discuss the problem for covariance matrices.

Definition 3.4.2. The covariance Σ has a GFA decomposition of rank q if it can be decomposed as the sum of a matrix $\tilde{\Sigma}$ which is a bounded operator in ℓ^2 and a $\operatorname{rank}\ q$ perturbation $\hat{\Sigma} = FF^\top$ where $F \in \mathbb{R}^{\infty\times q}$ has strongly linearly independent columns.

Chamberlain and Rothschild (Chamberlain & Rothschild, 1983, Theorem 4) provide a criterion for a GFA decomposition based on separating the bounded from the unbounded eigenvalues of Σ. The criterion has been extended by Forni and Lippi (Forni & Lippi, 2001) to the dynamic case.

Theorem 3.4.3 (Chamberlain-Rothschild)**.** *If and only if for $n \to \infty$, Σ_n has q* **unbounded eigenvalues** *and $\lambda_{q+1}(\Sigma_n)$ stays bounded, then Σ has a GFA decomposition of rank q:*

$$\Sigma = FF^\top + \tilde{\Sigma}\,, \qquad \textit{with} \quad F = \begin{bmatrix} f_1 & \dots & f_q \end{bmatrix}\,, \;\; f_i \in \mathbb{R}^\infty \tag{3.47}$$

The GFA decomposition of Σ is unique.

Note that there may well be sequences (of positive symmetric) Σ_n for which *all eigenvalues* tend to infinity. In this case there is no GFA decomposition. When it applies, the criterion can be seen as a limit of the well-known rule of separating "large" from "small" eigenvalues in Principal Components Analysis (PCA). Let $f_i^n \in \mathbb{R}^n\,;\, i = 1, \dots, q$ be the eigenvectors corresponding to the q (ordered) eigenvalues of Σ_n which increase without bound when $n \to \infty$. We normalize these eigenvectors in such a way that $F_n := \begin{bmatrix} f_1^n & \dots & f_q^n \end{bmatrix}$ yields $\hat{\Sigma}_n = F_n F_n^\top$. Then

$$\lim_{n\to\infty} F_n F_n^\top = FF^\top\,. \tag{3.48}$$

Although the usual orthogonality of the f_i^n in PCA does not make sense in infinite dimensions as the limit eigenvectors do not belong to ℓ^2, one may however interpret the strong linear independence condition as a limit of the orthogonality holding for finite n. Hence we can (asymptotically) get q and F by a limit PCA procedure on the sequence Σ_n.

Trivially, if a random sequence $\mathbf{y}$ admits a GFA representation then its covariance matrix has a GFA decomposition. On the other hand, assume we are given a GFA decomposition $\hat{\Sigma} + \tilde{\Sigma}$ of an infinite covariance Σ. The following Proposition provides a criterion to retrieve the hidden variables in the representation $\mathbf{y} = F\mathbf{x} + \tilde{\mathbf{y}}$.

Proposition 3.4.4. *Assume that its covariance matrix Σ has a GFA decomposition of rank q. Then $\mathbf{y}$ has a GFA representation with q factors where both $\mathbf{x}$ and $\tilde{\mathbf{y}}$ have components in $\mathcal{H}(\mathbf{y})$.*

Proof. By a standard Q-R factorization we can orthogonalize the columns of F_n,

$$\begin{bmatrix} f_1^n & f_2^n & \dots & f_q^n \end{bmatrix} = \begin{bmatrix} g_1^n & g_2^n & \dots & g_q^n \end{bmatrix} \begin{bmatrix} 1 & r_{1,2} & r_{1,3} & \dots & r_{1,q} \\ 0 & 1 & r_{2,3} & \dots & r_{2,q} \\ 0 & 0 & 1 & \ddots & r_{3,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{3.49}$$

which we shall write compactly as

$$F_n = Q_n R_n \tag{3.50}$$

where $Q_n := \begin{bmatrix} g_1^n & g_2^n & \dots & g_q^n \end{bmatrix}$ has orthogonal columns. It is well-known that each g_i^n can be obtained by a sequential Gram-Schmidt orthogonalization procedure as the difference of f_i^n with its projection onto the subspace $\operatorname{span}\{f_j^n,\, j < i\} \subset \mathcal{F}_i^n$. Hence $\|g_i^n\| \geq \|\tilde{f}_i^n\|$ and hence, by assumption, tends to ∞ when $n \to \infty$.
Next, define

$$a_{i,n}^\top := \frac{1}{\|g_i^n\|_2^2} \begin{bmatrix} g_i^n(1) & g_i^n(2) & \dots & g_i^n(n) & 0 & \dots \end{bmatrix} \tag{3.51}$$

where the g_i^n's are as defined above. Since $\|g_i^n\|_2 \to \infty$ with n, we have $\|a_{i,n}\|_2 = 1/\|g_i^n\|_2 \to 0$ as $n \to \infty$. Hence $a_{i,n}$ is an AS.
Note that we can express each f_i^n as

$$f_i^n = g_i^n + \sum_{j=1}^{i-1} r_{j,i} g_j^n \tag{3.52}$$

so that

$$a_{i,n}^\top f_i = \frac{1}{\|g_i^n\|_2^2} \|g_i^n\|_2^2 \;=\; 1 \tag{3.53}$$

for all n large enough and by a similar calculation one can easily check that $a_{i,n}^\top f_j = 0$, for all $j < i$. With these $a_{i,n}$ construct a sequence of $q \times \infty$ matrices

$$A_n := \begin{bmatrix} a_{1,n}^\top \\ \dots \\ a_{q,n}^\top \end{bmatrix} \tag{3.54}$$

which provides an asymptotic left-inverse of F, in the sense that $\lim_{n\to\infty} A_n F = R$, where R is the limit of a sequence of $q \times q$ matrices all of which are upper triangular with ones on the main diagonal. Next, define the random vector $\mathbf{z}_n := A_n \mathbf{y}$ which converges as $n \to \infty$ to a q-dimensional $\mathbf{z}$ whose components must belong to $\mathcal{G}(\mathbf{y})$. These q components form in fact a basis for $\mathcal{G}(\mathbf{y})$ as the covariance $\mathbb{E}\left[\mathbf{z}_n \mathbf{z}_n^\top\right]$ converges to $RR^\top$ which is non singular. From this, one can easily get an orthonormal basis $\mathbf{x}$, in $\mathcal{H}(\hat{\mathbf{y}})$. Hence, since F is known, we can form $\hat{\mathbf{y}} = F\mathbf{x}$ and letting $\tilde{\mathbf{y}} := \mathbf{y} - \hat{\mathbf{y}}$ does yield a GFA representation of $\mathbf{y}$ inducing the given GFA decomposition of Σ. Uniqueness is then guaranteed in force of Proposition 3.3.11. □

This proposition highlights the fact that GFA models are *asymptotically internal* models

with respect to $\mathcal{H}(\mathbf{y})$, since the latent factors $\mathbf{x}$ can be obtained from a linear combination of the observations. Recalling Theorem 3.2.2, this corresponds to having $\mathbf{z} = 0$.

3.5 Stationary sequences and the Wold decomposition

As we have seen, non-stationarity may bring in some pathologies which seem to be difficult to rule out. We consider now the special case in which the sequence $\mathbf{y}$, defined on $\mathbb{N}$, is (weakly) stationary; i.e. $\mathbb{E}[\mathbf{y}(k)\mathbf{y}(j)] = \sigma(k-j)$ for $k,\, j \geq 0$. Let $\mathcal{H}_k(\mathbf{y})$ be the closed linear span of all random variables $\{\mathbf{y}(s)\,;\, s \geq k\}$. Introducing the *remote future subspace* of $\mathbf{y}$:

$$\mathcal{H}_\infty(\mathbf{y}) = \bigcap_{k\geq 0} \mathcal{H}_k(\mathbf{y})\,, \tag{3.55}$$

the sequence of orthogonal wandering subspaces $E_k := \mathcal{H}_k(\mathbf{y}) \ominus \mathcal{H}_{k+1}(\mathbf{y})$ and their orthogonal direct sum

$$\check{\mathcal{H}}(\mathbf{y}) = \bigoplus_{k\geq 0} E_k\,, \tag{3.56}$$

it is well known, see e.g. (Doob, 1990; Rozanov, 1967; Halmos, 1961), that one has the orthogonal decomposition

$$\mathbf{y} = \hat{\mathbf{y}} + \check{\mathbf{y}}\,, \qquad \hat{\mathbf{y}}(k) \in \mathcal{H}_\infty(\mathbf{y}) \quad \check{\mathbf{y}}(k) \in \check{\mathcal{H}}(\mathbf{y}) \tag{3.57}$$

for all $k \in \mathbb{Z}_+$, the component $\hat{\mathbf{y}}$ being the purely deterministic (PD), while $\check{\mathbf{y}}$ the purely non-deterministic (PND) components. The two sequences are orthogonal and uniquely determined. Furthermore, it is well known that $\check{\mathbf{y}}$ has an absolutely continuous spectrum with a spectral density function, say $S_y(\omega)$ satisfying the log-integrability condition $\int \log S_y(\omega)\, d\omega > -\infty$, while the spectral distribution of $\hat{\mathbf{y}}$ is singular with respect to Lebesgue measure (for example consisting only of jumps) possibly together with a spectral density such that $\int \log S_y(\omega)\, d\omega = -\infty$, compare e.g. (Rozanov, 1967).

In this section we want to give an interpretation of the decomposition (3.7) in the light of the Wold decomposition. First we prove the following two lemmas.

Lemma 3.5.1. *Let $\mathbf{y}$ be stationary and assume it has an absolutely continuous spectrum with a bounded spectral density; i.e.*

$$S_y(\omega) \in L^\infty([-\pi,\, \pi])\,. \tag{3.58}$$

Then $\mathbf{y}$ is idiosyncratic. In particular, PND sequences with a bounded spectral density are idiosyncratic sequences.

Proof. By a well known theorem of Szegö (Grenander & Szegö, 1984, p.65) see also (Hartman & Wintner, 1954), Σ is a bounded Toeplitz operator, thus for any AS a_n,

$$\|a_n^\top \mathbf{y}\|^2 = \|a_n\|_\Sigma^2 = a_n^\top \Sigma a_n \leq \|\Sigma\| \, \|a_n\|_2^2 \,. \tag{3.59}$$

and since $\|a_n\|_2^2 \to 0$, $\|a_n^\top \mathbf{y}\|^2 \to 0$, and $\mathbf{y}$ is idiosyncratic. □

Lemma 3.5.2. *Let* $\mathbf{y}$ *be a stationary sequence with a bounded spectral density, then*

$$\mathcal{G}(\mathbf{y}) \subseteq \mathcal{H}_\infty(\mathbf{y}) \,. \tag{3.60}$$

Proof. Assume that $\mathbf{z} \in \mathcal{G}(\mathbf{y})$. Then there exists an AS a_n such that $\mathbf{z} = \lim_n a_n^\top \mathbf{y}$. Applying the Wold decomposition we obtain

$$\mathbf{z} = \lim_{n\to\infty} a_n^\top \mathbf{y} = \lim_{n\to\infty} a_n^\top \hat{\mathbf{y}} + \lim_{n\to\infty} a_n^\top \check{\mathbf{y}} \,. \tag{3.61}$$

By Lemma 3.5.1, the PND part vanishes as n tends to infinity, thus $\mathbf{z} \in \mathcal{H}_\infty(\mathbf{y})$. □

Note that the statement holds in particular for PD processes with a singular spectrum, as in this case $S_y(\omega) \equiv 0$. The converse inclusion, i.e. $\mathcal{H}_\infty(\mathbf{y}) \subseteq \mathcal{G}(\mathbf{y})$, is in general not true. However, for stationary sequences with a *finite dimensional remote future*, we can state the following.

Theorem 3.5.3. *Assume that* $\mathbf{y}$ *is a stationary sequence with a bounded spectral density and that* $\dim \mathcal{H}_\infty(\mathbf{y}) < \infty$. *Then* $\mathcal{H}_\infty(\mathbf{y}) \equiv \mathcal{G}(\mathbf{y})$.

Proof. It is sufficient to show that $\mathcal{H}_\infty(\mathbf{y}) \subseteq \mathcal{G}(\mathbf{y})$.
Let $\dim\, H_\infty(\mathbf{y}) = q$. By assumption $\mathcal{H}_k(\mathbf{y}) \supseteq \mathcal{H}_\infty(\mathbf{y})$ has dimension greater than or equal to q for all $k \geq 0$. It follows that for any k, the random variables $\mathbf{y}(k+1), \ldots, \mathbf{y}(k+q)$ must be linearly independent. For otherwise the $q \times q$ covariance matrix

$$\Sigma_q := \mathbb{E} \begin{bmatrix} \mathbf{y}(k+1) \\ \ldots \\ \mathbf{y}(k+q) \end{bmatrix} \begin{bmatrix} \mathbf{y}(k+1) \\ \ldots \\ \mathbf{y}(k+q) \end{bmatrix}^\top \tag{3.62}$$

would be singular of rank $r < q$ and hence, because of the Toeplitz structure, one would have $\operatorname{rank} \Sigma_n = r < q$ for all $n \geq q$, which implies that one can extract only r linearly independent random variables from an arbitrarily long string of random variables of the process. This in turn would imply $\dim \mathcal{H}_\infty(\mathbf{y}) = r < q$ contrary to our assumption. Therefore

$$\operatorname{span}\{\mathbf{y}(k+1), \ldots, \mathbf{y}(k+q)\} \supseteq \mathcal{H}_\infty(\mathbf{y}) \,, \qquad \text{for all } \; k$$

and for any $\mathbf{z} \in \mathcal{H}_\infty(\mathbf{y})$ there is a nonzero $b_k \in \mathbb{R}^q$ such that

$$\mathbf{z} = b_k^\top \begin{bmatrix} \hat{\mathbf{y}}(k+1) \\ \dots \\ \hat{\mathbf{y}}(k+q) \end{bmatrix}, \tag{3.63}$$

where $\hat{\mathbf{y}}(k+1), \dots, \hat{\mathbf{y}}(k+q)$ are the projections of $\mathbf{y}(k+1), \dots, \mathbf{y}(k+q)$ onto $\mathcal{H}_\infty(\mathbf{y})$. Furthermore, the Euclidean norm $\|b_k\|$ is the same for all k because of stationarity. Hence, choosing $k = 0, q, 2q, \dots, (n-1)q$, one also has

$$\mathbf{z} = \frac{1}{n} [\underbrace{b_0^\top \quad b_1^\top \quad \dots \quad b_{n-1}^\top}_{n} 0 \quad \dots \quad 0] \, \hat{\mathbf{y}} := a_n^\top \, \hat{\mathbf{y}} \tag{3.64}$$

where the sequence $\{a_n, \, n \in \mathbb{N}\}$ is clearly an AS. It follows that

$$a_n^\top \mathbf{y} = \lim_{n\to\infty} a_n^\top \mathbf{y} = \lim_{n\to\infty} a_n^\top \hat{\mathbf{y}} + \lim_{n\to\infty} a_n^\top \check{\mathbf{y}} = \mathbf{z} \,, \tag{3.65}$$

where the last identity is a consequence of Lemma 3.5.1. Therefore $\mathbf{z} \in \mathcal{G}(\mathbf{y})$. □

Hence,

Theorem 3.5.4. *Every stationary sequence with a bounded spectral density and remote future space of dimension q is a $q-$factor sequence. It admits a unique generalized factor analysis representation* (3.46) *where $\hat{\mathbf{y}}$ is the purely deterministic and $\tilde{\mathbf{y}}$ is the purely non-deterministic component of $\mathbf{y}$.*

Note in particular that the spectral density of $\tilde{\mathbf{y}}$ must necessarily satisfy the log-integrability condition.
When $\mathcal{H}_\infty(\mathbf{y})$ is finite dimensional, the PD component of a stationary process has a special structure, namely

$$\hat{\mathbf{y}}(k) = \sum_{i=1}^{\nu} \mathbf{v}_i \cos \omega_i k + \mathbf{w}_i \sin \omega_i k \,, \tag{3.66}$$

where $e^{\pm j\omega_i}$, $i = 1, 2, \dots, q$ are the q eigenvalues of the unitary shift operator of the process (Rozanov, 1967). The ω_i are distinct real frequencies in $[0, \, \pi)$ and $\mathbf{v}_i$ and $\mathbf{w}_i$ are mutually uncorrelated zero-mean random variables with $\mathrm{var}\,[\mathbf{v}_i] = \mathrm{var}\,[\mathbf{w}_i]$ which span the subspace $\mathcal{H}(\hat{\mathbf{y}}) \equiv \mathcal{H}_\infty(\mathbf{y})$.

In the following proposition, we show how to construct AS's that generate a basis in the finite-dimensional remote future space.

Proposition 3.5.5. *The latent factors of a stationary q-factor sequence can be recovered*

using averaging sequences $\{a_{i,n}\}_{n\in\mathbb{N}}$ *of the type*

$$a_{i,n}(k) = \begin{cases} \dfrac{1}{n}\sin\omega_i k & k \le n \\ 0 & k > n \end{cases} \tag{3.67}$$

or

$$a_{i,n}(k) = \begin{cases} \dfrac{1}{n}\cos\omega_i k & k \le n \\ 0 & k > n \end{cases}, \tag{3.68}$$

by letting ω_i *vary on the set of proper frequencies of the signal* (3.66).

Proof. Consider the AS $\{a_n\}_{n\in\mathbb{N}}$ of (3.67), with a fixed frequency $\omega_i = \omega_p$, $p \in \{1, \ldots, \nu\}$ and apply it to the sequence $\mathbf{y}$. While the idiosyncratic (PND) part vanishes asymptotically, the q-aggregate (PD) component (3.66) yields the sequence of random variables

$$\mathbf{z}_n = a_n^\top \hat{\mathbf{y}} = \sum_{k=1}^{n} a_n(k)\hat{\mathbf{y}}(k) = \frac{1}{n}\sum_{k=1}^{n}\Big[\sin\omega_p k \sum_{i=1}^{\nu}(\mathbf{v}_i\cos\omega_i k + \mathbf{w}_i \sin\omega_i k)\Big] \tag{3.69}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\Big[\sum_{i=1, i\neq p}^{\nu}(\mathbf{v}_i \sin\omega_p k\cos\omega_i k + \mathbf{w}_i \sin\omega_p k \sin\omega_i k) + \mathbf{v}_p \sin\omega_p k\cos\omega_p k + \mathbf{w}_p \sin^2\omega_p k\Big]$$

It is well-known and not difficult to check directly, using elementary trigonometric identities such as $\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$ and the formula

$$\left|\frac{1}{n}\sum_{k=1}^{n} e^{j\omega k}\right| = \frac{1}{n}\left|e^{j\omega}\frac{1-e^{j\omega n}}{1-e^{j\omega}}\right| \le \frac{1}{n}\left|\frac{1}{\sin\omega/2}\right|$$

that all time averages of products of $\sin$ and $\cos$ functions in this sum vanish asymptotically except for the $\sin^2$ term, which has the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbf{w}_p\sin^2\omega_p k = \frac{\mathbf{w}_p}{2}, \tag{3.70}$$

which is one of the latent factors. Similarly, the random variables $\mathbf{v}_i$, associated with cosine-type oscillations, can be recovered using averaging sequences of the type (3.68). □

Obviously one can obtain arbitrary linear combinations $\sum_{i=1}^{\nu} c_i\mathbf{v}_i + d_i\mathbf{w}_i$ by properly combining the AS's (3.67) and (3.68).

Discussion

We have shown that there is a natural interpretation of GFA models in terms of the Wold decomposition of stationary sequences. A stationary sequence admits a GFA representation if and only if its spectral density is bounded and the remote future space is finite dimensional. Both conditions are necessary since a PD stationary process has a finite factor representation if and only if its remote future has finite dimension. On the other hand there are stationary processes with a finite dimensional remote future space, whose PND component has an unbounded spectral density. It follows from Szegö's theorem that $\tilde{\Sigma}$ is an unbounded operator and these processes are neither aggregate nor idiosyncratic.

In the classical papers (Chamberlain & Rothschild, 1983; Forni & Lippi, 2001), stationarity with respect to the cross-sectional (space) index is not assumed. However without stationarity, there may be random sequences which fail to satisfy the eigenvalue conditions of Theorem 3.4.3 and do not admit a generalized factor analysis representation. A precise characterization of which class of non-stationary sequences admits a GFA representation seems still to be an open problem.

3.6 Flocking and generalized factor analysis

So far we have presented a rigorous formulation of the problem of modeling an infinite collection of random variables using generalized factor analysis. In this section, we show some possible situation in which GFA decomposition may help in understanding the phenomenon. We shall see that in all these applications there are some global effects which affect every observations, which we shall address as a *flocking behavior*, plus some local interactions. In terms of GFA, the first correspond to the effect of the latent variables, while the latter are modeled through idiosyncratic noise.

Detection of emitters

In this scenario we suppose there is an unknown number, say q, of emitters, each of them broadcasting radio impulse trains at a fixed common frequency. Such impulses are received by a large array of N antennas spread in space. The measurement of each antenna is corrupted by noise, generated by measurement errors or local disturbances, possibly correlated with that of neighboring antennas. The set up can be described mathematically, by indexing each antenna by an integer $i = 1, 2, \ldots, N$ and denoting by $\mathbf{y}_i(t)$ the signal received at time t by antenna i. Then the following model can be used to

describe the received signal

$$\mathbf{y}_i(t) = f_{i1}x_1(t) + \ldots + f_{iq}x_q(t) + \tilde{\mathbf{y}}_i(t)\,, \tag{3.71}$$

where:

- $x_j(t)$ is the signal sent by the j-th emitter at time t;
- f_{ij} is a coefficient related to the distance between j-th emitter and antenna i;
- $\tilde{\mathbf{y}}_i(t)$ is the disturbance affecting antenna i at time t.

The goal is to detect the number of emitters q and possibly estimate the signal components $x_j(t)$ impinging on the antenna array.

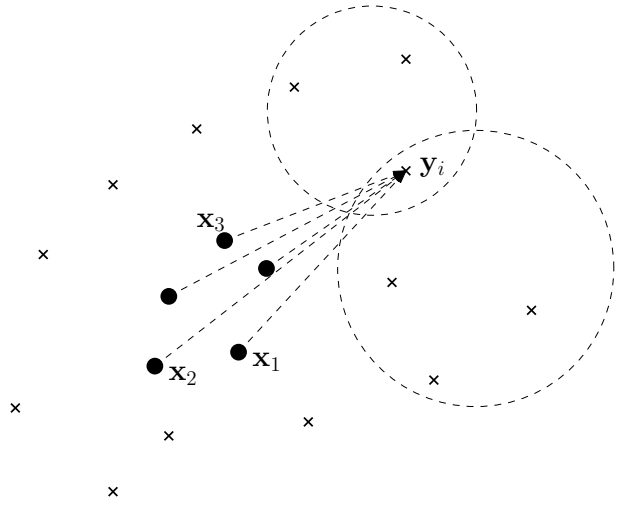


Figure 3.1: Detection of emitters. The signal observed by each antenna $\mathbf{y}_i$ is a combination of the signals sent by the emitters $\mathbf{x}_j$ plus a local noise (dashed circles).

Let $\mathbf{y}(t)$, $\mathbf{x}(t)$, $\tilde{\mathbf{y}}(t)$ denote vector valued quantities in the model (3.71) of respective dimensions N, q and N. The model (3.71) can be compactly written as

$$\mathbf{y}(t) = F\mathbf{x}(t) + \tilde{\mathbf{y}}(t)\,, \tag{3.72}$$

where $\mathbf{y}$ is the N-dimensional random process of observables; $\mathbf{x}(t) = [\mathbf{x}_1(t)\ \ldots\ \mathbf{x}_q(t)\,]^\top$ is the unobservable vector of random signals generated by the emitters; $F = \{f_{ij}\} \in \mathbb{R}^{N\times q}$ is an unknown matrix of coefficients and $\tilde{\mathbf{y}}$ is a N-dimensional random process of disturbances, uncorrelated with $\mathbf{x}$, describing the local disturbance on the i-th antenna.

Note that in the model there are several hidden (non-measurable) variables, including the dimension q. In our setting N is assumed to be very large; ideally we shall assume $N \to \infty$.

We may identify $F\mathbf{x}(t)$ as the flocking component of $\mathbf{y}(t)$. In a primitive statistical formulation all signals in the model are i.i.d. process, and the sample values $\{y(t)\}$ are interpreted as random samples generated by a underlying static model of the form

$$\mathbf{y} = F\mathbf{x} + \tilde{\mathbf{y}}\,. \tag{3.73}$$

One should observe that estimation of this model from observations $\{y(t)\}$ of $\mathbf{y}$, consists first of estimating the model parameters, say F and the covariance matrix of $\tilde{\mathbf{y}}$ but also in constructing the hidden random quantities $\mathbf{x}$ and $\tilde{\mathbf{y}}$. The covariance matrix of $\mathbf{y}$, say $\Sigma \in \mathbb{R}^{N\times N}$ may be obtained from the data by standard procedures.

A problem leading to models of similar structure is automated speaker detection. This is the problem of detecting the speaking persons (emitters) in a noisy environment at any particular time, from signals coming from a large array of N microphones distributed in a room. Here the number of emitters is generally small but could be varying with time. Robustly solving this problem is useful in areas such as surveillance systems, and human-machine interaction.

In the model specification it is customary to assume that the noise vector $\tilde{\mathbf{y}}$ has uncorrelated components. In this case the model (3.73), is a (static) factor analysis model. Statistical inference on these models leads in general to ill-posed problems and to resolve the issue it is often imposed that the variances of the scalar components of $\tilde{\mathbf{y}}$ should all be equal. The problem can then be solved by computing the smallest eigenvalue of the covariance matrix of $\mathbf{y}$, following an old idea (Pisarenko, 1973) which has generated an enormous literature. The assumption of uncorrelated noise and, especially, of equal variances is however rather unrealistic in many instances.

Inference of gene regulatory networks

In systems biology, an important task is the inference of gene regulatory networks in order to understand cell physiology and pathology. Genes are known to interact among each other forming a network, and their expression is directly regulated by few transcription factors (TFs). Typically, TFs and genes are modeled as two distinct networks of interactions which are able also to interact with each other. While methods for measuring the gene expressions using microarray data are extremely popular, there are still problems in understanding the action of TFs and the scientific community is

currently working on computational methods for extraction of the action of the TFs from the available measurements of gene expression. To this end, a simplification of the interaction between genes and TFs is commonly accepted and consists in projecting the TFs network on the "gene space" (Brazhnik et al., 2002).

Denoting by a random variable $\mathbf{y}_i$ the measured expression profile of the i-th gene of the network, usually the model (3.73) is also proposed in this framework. In this case:

- The N dimensional vector $\mathbf{y}$ represents all the gene expressions. The experimenter can usually observe a large amount of genes, and it is reasonable to assume that $N \to \infty$.
- Each component of the random vector $\mathbf{x}$ is associated with a TF. The number q of TF's is a priori unknown; furthermore $N \gg q$.
- The $N \times q$ matrix F models the strength of the TFs effect on each gene.
- The vector $\tilde{\mathbf{y}}$ describes the interaction of connected genes.

Factor analysis models (see Section 3.2) have been considered to deal with this problem, see e.g. (see e.g. (Pournara & Wersnich, 2007), (Sanguinetti et al., 2006), (West, 2003), (Sabatti & James, 2006) or (Lin & Husmeier, 2010) for a survey); in such a case, the vector $\tilde{\mathbf{y}}$ is assumed to have uncorrelated components. However, in the context of gene regulatory networks the latter assumption may be relaxed, since it is well-known that there are interactions among genes that are not determined by TFs. Then, a possible assumption is that $\tilde{\mathbf{y}}$ admits some "weak correlation" among its components.

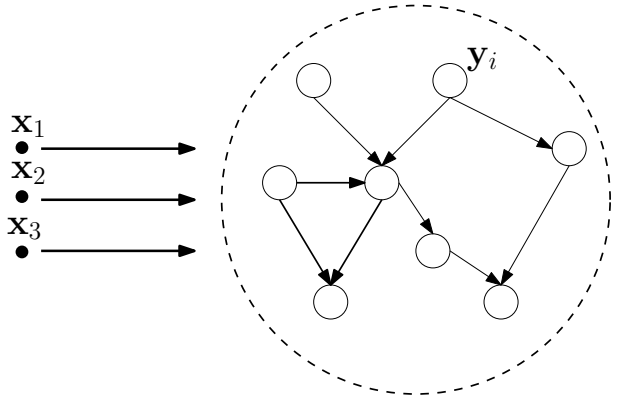


Figure 3.2: Example of a gene network under the action of transcription factors.

Modeling energy consumption

In this example, we may want to model the energy consumption (or production) of a network of N users distributed geographically in a certain area, say a city or a region. The energy consumption $\mathbf{y}_i(t)$ of user i is a random variable which can be seen as the sum of two contributions

$$\mathbf{y}_i(t) = f_i^\top \mathbf{x}(t) + \tilde{\mathbf{y}}_i(t)\,. \tag{3.74}$$

where the term $f_i^\top \mathbf{x}(t)$ represents a linear combination of q hidden variables $\mathbf{x}_i(t)$ which model different factors affecting the energy consumption (or production) of the whole ensemble; say heating or air conditioning consumption related to seasonal climatic variations, energy production related to the current status of the economy etc. The factor vector $\mathbf{x}(t)$ determines the average time pattern of energy consumption/production of each unit, the importance of each scalar factor being determined by a q-ple of constant weight coefficients $f_i(k)$. One may identify the component $F\mathbf{x}(t)$ as the flocking component of the model (3.74). The terms $\tilde{\mathbf{y}}_i(t)$, represent local random fluctuations which model the consumption due to devices that are usually activated randomly, for short periods of time. They are assumed uncorrelated with the process $\mathbf{x}$. The covariance $\mathbb{E}\left[\tilde{\mathbf{y}}_i(t)\tilde{\mathbf{y}}_j(t)\right]$ could be non zero for neighboring users but is reasonable to expect that it decays to zero when $|i-j| \to \infty$. To identify such a model one should start from real data of energy consumption collected from a large amount of units. A possible application for such a model is the forecasting of the average requirement of energy in a certain geographical area.

Dynamic modeling in computer vision

Large-dimensional time series occur often in signal processing applications, typically for example, in computer vision and dynamic image processing. The role of identification in image processing and computer vision has been addressed by several authors. We may refer the reader to the recent survey (Chiuso & Picci, 2008) for more details and references. One starts from a signal $\mathbf{y}(t) := \mathrm{vec}(\mathbf{I}(\cdot, t))$, obtained by vectorizing at each time t, the intensities $\mathbf{I}(\cdot, t)$ at each pixel of an image, into a vector, say $\mathbf{y}(t) \in \mathbb{R}^N$, with a "large" number (typically tens of thousands) of components. We may for instance be interested in modeling (and in identification methodologies thereof) of "dynamic textures" (see (Doretto et al., 2003)), by linear state space models or in extracting classes of models describing rigid motions of objects of a scene. Most of these models involve hidden variables, say the state of linear models of textures, or the displacement-velocity coordinates of the rigid motions of objects in the scene. The purpose is of course to

compress high dimensional data into simple mathematical structures. Note that the number of samples that can be used for identification is very often of the same order (and sometimes smaller) than the data dimensionality. For instance, in dynamic textures modeling, the number of images in the sequences is of the order of a few hundreds while N (which is equal to the number of pixels of the image) is certainly of the order of a few hundreds or thousands (Doretto et al., 2003; Bissacco et al., 2007).

Mathematical formulation of flocking

Let $\mathbf{y}(k,t)$ be a second order finite variance random field depending on a space variable k and on a time variable t. The variable k is indexing a large ensemble of space locations where equal "agents" produce at each time t the measurement, $\mathbf{y}(k,t)$, of a scalar quantity, say the received voltage signal of the k-th antenna or the expression level of the k-th cell in a cell array. We shall assume that k varies on some ordered index set of N elements and let $t \in \mathbb{Z}$ or $\mathbb{Z}_+$, depending on the context. Eventually we shall be interested in problems where $N = \infty$. We shall denote by $\mathbf{y}(t)$ the random (column) vector with components $\{\mathbf{y}(k,t)\,;\, k = 1, 2, \ldots, N\}$. Suitable mathematical assumptions on this process will be specified in due time.

A **(random) flock** is a random field having the multiplicative structure $\hat{\mathbf{y}}(k,t) = \sum_{i=1}^{q} f_i(k)\mathbf{x}_i(t)$, or equivalently,

$$\hat{\mathbf{y}}(t) = \sum_{i=1}^{q} f_i \mathbf{x}_i(t) \tag{3.75}$$

where $f_i = \begin{bmatrix} f_i(1) & f_i(2) & \ldots & f_i(N) \end{bmatrix}^\top$, $i = 1, 2, \ldots, q$ are nonrandom N-vectors and $\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1(t) & \ldots & \mathbf{x}_q(t) \end{bmatrix}^\top$ is a random processes with orthonormal components depending on the time variable only; i.e.

$$\mathbb{E}\left[\mathbf{x}(t)\mathbf{x}(t)^\top\right] = I_q\,, \qquad t \in \mathbb{Z}\,.$$

The idea is that a flock is essentially a deterministic geometric configuration of N points in a q-dimensional space moving rigidly in a random fashion. We want to investigate when a second order random field has a flocking component and study the problem of extracting it from sample measurements of $\mathbf{y}(k,t)$. This means that one should be

searching for decompositions of the type:

$$\mathbf{y}(t) = \sum_{i=1}^{q} f_i \mathbf{x}_i(t) + \tilde{\mathbf{y}}(t) \tag{3.76}$$

where $q \geq 1$ and $\tilde{\mathbf{y}}(t)$ is a "random noise" field which should not contain flocking components. Naturally for the problem to be well-defined one has to specify conditions making this decomposition *unique*.

A generalization of this setting where $\mathbf{y}$ may take vector values is possible, but for the sake of clarity we shall here restrict to scalar-valued processes.

Short and long distance interactions

After having formulated our model of a flocking structure, we suggest an interpretation in terms GFA models. We shall imagine a scenario of an ensemble of infinitely many agents distributed in space generating the random variables $\{\mathbf{y}(k) = \hat{\mathbf{y}}(k) + \tilde{\mathbf{y}}(k)\ ;\ k = 1, 2, \ldots\}$ and interacting in a random fashion.

The idiosyncratic covariances $\tilde{\sigma}(k, j) = \mathbb{E}\left[\tilde{\mathbf{y}}(k)\tilde{\mathbf{y}}(j)\right]$ measure the mutual influence of neighboring units noises $\tilde{\mathbf{y}}(k)$, $\tilde{\mathbf{y}}(j)$. Since $\tilde{\Sigma}$ is a bounded operator in ℓ^2, it is a known fact (Akhiezer & Glazman, 1961, Section 26) that $\tilde{\sigma}(k, j) \to 0$ as $|k - j| \to \infty$ so in a sense the idiosyncratic component $\tilde{\mathbf{y}}$ of a GFA representation models only *short range* interaction among the agents, as $\tilde{\sigma}(k, j)$ is decaying with distance. Agents which are far away from each other essentially do not resent of mutual influence.

On the other hand, $\mathbb{E}\left[\hat{\mathbf{y}}(k)\hat{\mathbf{y}}(j)\right] = \sum_i f_i(k) f_i(j)$ and the elements of the column vectors f_i cannot be in ℓ^2. In particular, as stated in the proposition below, if the variances of the random variables $\mathbf{y}(k)$ are uniformly bounded then $f_i \in \ell^\infty$.

Proposition 3.6.1. *If $\mathbf{y}$ is a $q-$FS and has uniformly bounded variance, then the f_i's are uniformly bounded sequences (i.e. belong to the space ℓ^∞).*

Proof. The statement follows since $\|\hat{\mathbf{y}}(k)\|^2 \leq M^2$, which is the same as

$$\sum_{i=1}^{q} f_i(k)^2 \leq M^2$$

and hence $|f_i(k)| \leq M$ for all k's. □

Hence since the components $f_i(k)$ do not decay with distance, the products $f_i(k) f_i(j)$ generically do not vanish when $|k - j| \to \infty$. Therefore the factor loadings describe "long

range" correlation between the factor components and the $\hat{\mathbf{y}}$ component of $\mathbf{y}$ can be interpreted as variables modeling the *long range interaction* among agents. In this sense $\hat{\mathbf{y}}$ models a *collective behavior* of the ensemble.

3.7 Generalized factor analysis models of random fields

In this section we deal with the question raised in Section 3.6, namely when does a second order random field have a flocking component and how to extract it from sample measurements of $\mathbf{y}(k,t)$. A simple class of random fields for which this question can be answered positively is the class of separable space-time processes

$$\mathbf{y}(k,t) = \mathbf{v}(k)\mathbf{u}(t) \tag{3.77}$$

which are the product of a space, $\mathbf{v}(k)$, and time component, $\mathbf{u}(t)$, both zero mean and with finite variance. This model can be generalized, for example making both $\mathbf{v}(k)$ and $\mathbf{u}(t)$ vector-valued but this would require extending our static theory in the preceding sections to vector-valued processes as well. Although this is quite straightforward involving no new concepts but just more notations, for the sake of clarity we shall restrain to the scalar case.

The model (3.77) needs to be specified probabilistically, as the dynamics of the "time" process $\{\mathbf{u}(t)\}$ may well be space dependent and dually, the distribution of $\mathbf{v}(k)$ may be a priori time-dependent. The following assumption specifies in probabilistic terms the multiplicative structure (3.77) of the random field $\mathbf{y}(k,t)$.

Assumption : The space and time evolutions of $\mathbf{y}(k,t)$ are *multiplicatively uncorrelated* in the sense that

$$\mathbb{E}\left[\mathbf{v}(k_1)\mathbf{v}(k_2) \mid \mathbf{u}(t_1)\mathbf{u}(t_2)\right] = \mathbb{E}_{\mathbf{v}}[\mathbf{v}(k_1)\mathbf{v}(k_2)] \tag{3.78}$$

where the first conditional expectation is made with respect to the conditional probability distribution of $\mathbf{v}$ given the random variables $\mathbf{u}(t_1)$, $\mathbf{u}(t_2)$, while the second expectation is with respect to the marginal distribution of $\mathbf{v}$.

From the multiplicative uncorrelation (3.78) one gets

$$\mathbb{E}\left[\mathbf{v}(k_1)\mathbf{v}(k_2)\mathbf{u}(t_1)\mathbf{u}(t_2)\right] = \mathbb{E}\left[\mathbf{v}(k_1)\mathbf{v}(k_2)\right]\mathbb{E}\left[\mathbf{u}(t_1)\mathbf{u}(t_2)\right] = \sigma_{\mathbf{v}}(k_1,k_2)\,\sigma_{\mathbf{u}}(t_1,t_2) \tag{3.79}$$

where $\sigma_{\mathbf{v}}$ and $\sigma_{\mathbf{u}}$ are the covariance functions of the two processes. Hence the covariance function of the random field inherits the separable structure of the process. If $\mathbf{v}$ and $\mathbf{u}$ are jointly Gaussian, the multiplicative uncorrelation property follows if the two components

are uncorrelated; namely their joint covariance is separable. This is a structure which is often assumed in the literature, see (Li et al., 2008) and references therein. Assume now that the space process has a nontrivial GFA representation with q factors

$$\mathbf{v}(k) = \sum_{i=1}^{q} f_i(k)\mathbf{z}_i + \tilde{\mathbf{v}}(k) \tag{3.80}$$

where $\hat{\mathbf{v}}(k) := \sum_i f_i(k)\mathbf{z}_i$ is the aggregate and $\tilde{\mathbf{v}}(k)$ the idiosyncratic component of $\mathbf{v}(k)$. Then setting $\mathbf{x}_i(t) = \mathbf{z}_i\mathbf{u}(t)$ and $\tilde{\mathbf{y}}(k,t) := \tilde{\mathbf{v}}(k)\mathbf{u}(t)$ one can represent the random field (3.77) by a dynamic GFA model,

$$\mathbf{y}(k,t) = \sum_{i=1}^{q} f_i(k)\mathbf{x}_i(t) + \tilde{\mathbf{y}}(k,t) := \hat{\mathbf{y}}(k,t) + \tilde{\mathbf{y}}(k,t) \tag{3.81}$$

Proposition 3.7.1. *If the processes* $\mathbf{v}$ *and* $\mathbf{u}$ *are multiplicatively uncorrelated then the two terms* $\hat{\mathbf{y}}(k,t)$ *and* $\tilde{\mathbf{y}}(h,s)$ *in the GFA model* (3.81) *are uncorrelated for all* k,h *and* t,s*. Hence a separable random field satisfying the multiplicative uncorrelation property has a flocking component if and only if its space process* $\mathbf{v}$ *has a nontrivial aggregate component.*

Proof. We have

$$\mathbb{E}\left[\hat{\mathbf{y}}(k,t)\tilde{\mathbf{y}}(h,s)\right] = \sum_{i=1}^{q} f_i(k)\,\mathbb{E}\left[\mathbf{z}_i\mathbf{u}(t)\tilde{\mathbf{v}}(h)\mathbf{u}(s)\right] \tag{3.82}$$

where the expectation in the last term can be written as

$$\mathbb{E}\left[\mathbf{z}_i\tilde{\mathbf{v}}(h)\mathbf{u}(t)\mathbf{u}(s)\right] = \mathbb{E}\left[\mathbb{E}_{\mathbf{v}}[\mathbf{z}_i\tilde{\mathbf{v}}(h) \mid \mathbf{u}(t)\mathbf{u}(s)]\,\mathbf{u}(t)\mathbf{u}(s)\right] = \mathbb{E}\left[\mathbb{E}_{\mathbf{v}}[\mathbf{z}_i\tilde{\mathbf{v}}(h)\,]\mathbf{u}(t)\mathbf{u}(s)\right] = 0 \tag{3.83}$$

since the $\mathbf{z}_i$'s are random variables in $\mathcal{H}(\hat{\mathbf{v}})$ and $\tilde{\mathbf{v}}(h)$ is orthogonal to this space. The last statement then follows directly. □

Let now $\mathbf{v}$ be second-order weakly stationary satisfying the conditions of Theorem 3.5.4. Here is probably the simplest nontrivial example of decomposition (3.81).

Example 3.7.2 (Exchangeable space processes)**.** Consider the case of a *(weakly) exchangeable* space process $\mathbf{v}$; i.e. a process whose second order statistics are invariant with respect to all index permutations of locations (k,j). Clearly the covariances $\sigma_{\mathbf{v}}(k,j) = \mathbb{E}\left[\mathbf{v}(k)\mathbf{v}(j)\right]$ must be independent of k, j for $k \neq j$ and $\sigma_{\mathbf{v}}(k,k) = \sigma^2 > 0$

must be independent of k (Aldous, 1985). Letting $\rho := \sigma_{\mathbf{v}}(k,j)$, $k \neq j$, one has

$$\Sigma_{\mathbf{v}} = \begin{bmatrix} \sigma^2 & \rho & \rho & \rho & \dots \\ \rho & \sigma^2 & \rho & \rho & \dots \\ \dots & & \ddots & & \dots \end{bmatrix} \tag{3.84}$$

where $\sigma^2 > |\rho|$ for positive definiteness. Letting f denote an infinite column vector with components all equal to ρ, one can decompose $\Sigma_{\mathbf{v}}$ as

$$\Sigma_{\mathbf{v}} = ff^{\top} + (\sigma^2 - \rho)I \tag{3.85}$$

where here I denotes an infinite identity matrix. This is a factor analysis decomposition of rank $q = 1$ of $\Sigma_{\mathbf{v}}$ with $\tilde{\Sigma}_{\mathbf{v}}$ a diagonal matrix. Hence a weakly exchangeable space process is a 1-factor process with an idiosyncratic component which is actually white. In the GFA representation (3.80) there is just one factor $\mathbf{z}$ and the factor loading vector f does not depend on the space coordinate.

Consider a random field with the multiplicative structure (3.77), then the flocking component

$$\hat{\mathbf{y}}(k,t) = f\mathbf{x}(t)\,, \qquad \mathbf{x}(t) = \mathbf{z}\mathbf{u}(t)$$

describes a constant, space independent, configuration moving randomly in time.

Statistical estimation

Assume that the space component of the random field is stationary and we have a snapshot of the system at certain time t_0; that is we have observations of a "very large" portion of the process $\{\mathbf{y}(k,t_0),\, k = 1,2,\dots,N\}$ at some fixed time t_0. With these sample data we may form the sample covariance estimates

$$\hat{\sigma}_N(h,t_0) := \frac{1}{N}\sum_{k=1}^{N} y(k+h,t_0)y(k,t_0) = \frac{1}{N}\sum_{k=1}^{N} v(k+h)v(k)\,u(t_0)^2\,, \qquad h = 0,1,2,\dots \tag{3.86}$$

which also have the multiplicative structure $\hat{\sigma}_N(h,t_0) = \hat{\sigma}_{\mathbf{v},N}(h)u(t_0)^2$, where $\hat{\sigma}_{\mathbf{v},N}(h)$ is the sample covariance estimate of the $\mathbf{v}$ process based on N data. Now by the assumptions made on the space-process $\mathbf{v}$ the limit $\lim_{N\to\infty}\hat{\sigma}_N(h,t_0)$ exists (although it may be sample dependent for the PD part), so the sample matrix covariance estimate,

which has the form

$$\hat{\Sigma}_N(t_0) := \begin{bmatrix} \hat{\sigma}_N(0,t_0) & \hat{\sigma}_N(1,t_0) & \dots & & \hat{\sigma}_N(N-1,t_0) \\ \hat{\sigma}_N(1,t_0) & \hat{\sigma}_N(0,t_0) & \hat{\sigma}_N(1,t_0) & \dots & \hat{\sigma}_N(N-2,t_0) \\ \dots & \dots & & \dots & \dots \\ \hat{\sigma}_N(N-1,t_0) & \dots & & \hat{\sigma}_N(1,t_0) & \hat{\sigma}_N(0,t_0) \end{bmatrix}$$
$$= u(t_0)^2\, \hat{\Sigma}_{\mathbf{v},N} \tag{3.87}$$

will converge to a limit for $N \to \infty$.

Following (Chamberlain & Rothschild, 1983; Forni & Lippi, 2001) the idea is now to do PCA on the covariance estimate for increasing N and isolate q eigenvalues which tend to grow without bound as $N \to \infty$ while the others stay bounded. The q corresponding eigenvectors will tend as $N \to \infty$ to the q factor loadings $f_1, \dots, f_q$ and therefore provide asymptotically the FA decomposition of the $\Sigma_{\mathbf{v}}$ matrix

$$\Sigma_{\mathbf{v}} = FF^\top + \tilde{\Sigma}_{\mathbf{v}}\,.$$

After F and $\tilde{\Sigma}_{\mathbf{v}}$ are estimated, the stochastic realization procedure of Sect. 3.4 permits to construct the factor vector $\mathbf{z}$ and the idiosyncratic component $\tilde{\mathbf{v}}$ of the GFA representation of $\mathbf{v}$ as in (3.80). The reconstruction of the time varying factor variables $\mathbf{x}_i(t) = \mathbf{z}_i\mathbf{u}(t)$ of $\mathbf{y}$ from the observations $\mathbf{y}(k,t) = \mathbf{v}(k)\mathbf{u}(t)$ can be done, in several equivalent ways, by averaging on the space variable.

4

Zeros of tall linear multirate systems

4.1 Introduction

In the previous chapter it was pointed out that the main feature of factor analysis, both in static and dynamic settings, is the description of the (denoised) observable variables in terms of few latent factors. From a "control systems" point of view, this corresponds to modeling a stochastic system which has a large number of outputs driven by few inputs. Identification of these systems, which are addressed as *tall* in the specific literature, may present some practical issues, since the presence of a large number of outputs implies the presence of a (much) larger number of parameters to be estimated, possibly more than the available data samples. Moreover, having few inputs compared to the number of outputs may compromise the performance of the identification procedure, i.e. increase the variance of the estimates of the system parameters.

For these reasons, it has become necessary a study of the properties of tall systems; in (Anderson & Deistler, 2007), it was proven that model tallness generically implies that the corresponding linear time-invariant dynamic system is zero-free. Hence, in the context of dynamic factor models one can exploit this result to model the dynamic relation between the latent variables and the denoised observations as a singular autoregressive process whose parameters can be easily identified from covariance data using Yule-Walker

equations. Furthermore, from a classical control perspective, controller design is much easier for zero-free systems. This suggests that when one is dealing with a generic system, the controller design can be easier if one can add extra sensors to make the system have more outputs than inputs. This important result regarding zero-freeness does not cover all practical situations; if the outputs are measured at different rates, the system is not time-invariant. For example, in a sensor network there may be devices operating at a certain frequency and communicating with some other sensors working at a lower sample rate. Also, in econometric modeling it is common to deal with variables collected monthly while some other may be obtained quarterly or even annually (Forni et al., 2000), (Schumacher & Breitung, 2006), (Raknerud et al., 2007). These type of systems constitute a special class of the so called periodic systems and are known as multirate systems.

In this chapter we want to extend the results of (Anderson & Deistler, 2007) to multirate systems, addressing the problem of exploring their zero properties under the tallness condition. We shall focus on generic properties, i.e. properties that hold almost surely for the whole class of systems in analysis. Thus, our goal is to establish which characteristics a tall multirate system must have in order to be zero-free.

Brief review of the literature

Discrete-time multirate linear systems have attracted attentions for some decades. The properties of these systems have been studied in such subdisciplines as systems and control (Chen & Francis, 1995), signal processing (Vaidyanathan, 1993), communications (Wang et al., 2005) and econometric modeling (Clements & Galvao, 2008). A technique termed blocking or lifting has been developed in systems and control to deal with periodic linear systems (Chen & Francis, 1995). In systems and control literature, this method was initially introduced to transform linear discrete-time periodic systems to linear time-invariant systems, so that the well-developed tools in linear time-invariant systems can be extended for design and analysis of linear discrete-time periodic systems (Bolzern et al., 1986), (Grasselli & Longhi, 1988), (Bittanti, 1986) and (Bittanti & Colaneri, 2009).

To our best knowledge the pole properties of the blocked systems are well understood (Bittanti & Colaneri, 2009), (Khargonekar et al., 1985); whereas, it is little known about the zero properties of tall blocked systems. For instance, (Bolzern et al., 1986), (Grasselli & Longhi, 1988) have explored the zero properties of blocked systems obtained from blocking of linear periodic systems (a class of systems which includes multi-rate systems). The results show that the blocked system has a finite zero if it is obtained from a time-invariant unblocked system, and the latter has a finite zero, which is a form of sufficiency

condition. However, this reference does not provide a necessary condition for a blocked system obtained this way to have a finite zero. This gap has been covered in (Zamani et al., 2011) and (Chen et al., 2012) where the authors have introduced some additional information about the zero properties of blocked systems obtained from blocking of time-invariant systems.

Contribution of this work

In contrast to the case where the unblocked system is time-invariant, very few results indeed deal with zeros of blocked systems where these systems have been obtained by blocking of a truly multirate system, i.e. one that is periodic in the parameters. Our starting point is a review of the zero properties of blocked systems resulting from blocking of linear time-invariant systems (Chen et al., 2012), (Zamani et al., 2011). The main result of this preliminary study is that tall blocked systems are zero-free if and only if the related (time-invariant) unblocked systems are zero-free. Then, in order to deal with multirate systems, we assume that there exists an underlying system operating at the highest sample rate, which is linear time-invariant. However, because not all the outputs of this underlying system are actually measured at the same rate, we end up with a multirate linear system linking the inputs of the original system to those of its outputs which are measured. Then, the zero properties of such a system are explored. Quite surprisingly, the results obtained from tall blocked time-invariant systems do not extend straightforwardly to the multirate case. Indeed, the analysis of multirate zeros turns out to be quite complicated. For this reason, we consider three cases separately, that is

1. finite nonzero system zeros;

2. system zeros at infinity;

3. system zeros at zero.

First, we focus on the zero properties of tall blocked systems associated with finite nonzero zeros. It is explicitly established that tall blocked systems generically have no finite nonzero zeros. Moreover, in the subsequent section the zero properties of tall blocked systems are examined at zero and infinity. We precisely state the conditions under which tall blocked systems can have a zero at the origin or infinity and when they are zero-free at those aforementioned points. We show that these conditions depend only on the dimensions of the state, the input and the output.

4.2 Zeros theory of tall linear systems

In this section we review some recent results on the zero properties of tall linear time-invariant systems.

We consider a linear time-invariant unblocked system $\sum := \{A,\, B,\, C,\, D\}$ described by the equations

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{4.1}$$

where $t \in \mathbb{Z}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$. Furthermore, we assume that the system is tall, i.e. $p \geq m$. One important assumption that we will consider throughout the whole chapter is that the parameter matrices $\{A,\, B,\, C,\, D\}$ assume generic values. This assumption has some direct consequences which are listed below and which will be exploited in order to obtain the main results of this chapter.

1. The matrices $A,\, B,\, C,\, D$ are full rank.
2. The matrix A has distinct eigenvalues.
3. The system $\sum$ is reachable and observable.

It is easy to check that the subset of the systems which do not satisfy these properties has null measure compared to the set of all the possible systems described by (4.1).

In order to study the zero properties of the system we need to introduce the system matrix

$$M(z) := \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}, \tag{4.2}$$

which is a matrix pencil defined on the complex plane. We define the normal rank of $M(z)$ as follows

$$\operatorname{nrank} M(z) := \max_{z \in \mathbb{C}} \operatorname{rank} M(z) \tag{4.3}$$

or, equivalently, as the value of $\operatorname{rank} M(z)$ for almost every $z \in \mathbb{C}$. Then, the following definition of zeros of a linear systems is available (see e.g. (Kailath, 1980) and (Hespanha, 2009)).

Definition 4.2.1. The finite zeros of the system $\sum$ are defined to be the values of $z \in \mathbb{C}$ for which the rank of the system matrix $M(z)$ falls below its normal rank. Equivalently, $z \in \mathbb{C}$ is a zero of $\sum$ if is a zero of all nonzero minors of order equal to $\operatorname{nrank} M(z)$.

Further, $M(z)$ is said to have an infinite zero when $n + \operatorname{rank} D$ is less than the normal rank of $M(z)$.

The main result of this section is derived from a modification of the results of (Anderson & Deistler, 2007) and is aimed to show that a tall system with generic parameter matrices A, B, C, D is zero-free i.e. its associated system matrix has full-column rank for all $z \in \mathbb{C} \cup \{\infty\}$. First, we need to introduce the following lemma, inspired from results in (Filler, 2010), which provides the zero properties of the system (4.1) when $p = m$ and the parameter matrices of $\sum$ accept generic values. It states that generically the rank reduction of the system matrix at any zero is equal to 1.

Lemma 4.2.2. *The set* $\mathcal{F} = \{\{A,\, B,\, C,\, D\} \,|\, p = m,\, \operatorname{rank} D = m,\, \operatorname{rank} M(z) \geq n + m - 1,\, \forall z \in \mathbb{C}\}$ *is open and dense in the set* $\{\{A,\, B,\, C,\, D\} \,|\, p = m,\, \operatorname{rank} D = m\}$.

Proof. **Dense**: Consider the system matrix $M(z)$ and suppose that there exists a z_0 such that $\operatorname{rank} M(z_0) = n + m - 2$ (note that only the case where rank drops to $n + m - 2$ is discussed here and generalization to $n + m - k, k \geq 2$ is straightforward). Therefore, there exist two linearly independent vectors, say x_1 and x_2, which span the kernel of $M(z_0)$. Let $x_i = [x_{i1}^\top\ x_{i2}^\top]^\top$, $i = 1, 2$ with $x_{i1} \in \mathbb{R}^n$, then x_{11} and x_{21} must be linearly independent otherwise there would exist nonzero scalars a_1 and a_2 such that

$$a_1 x_1 + a_2 x_2 = [0\ \ a_1 x_{12}^\top + a_2 x_{22}^\top]^\top$$

with $D[a_1 x_{12} + a_2 x_{22}] = 0$, which implies $a_1 x_{12} + a_2 x_{22} = 0$. The latter means that x_1 and x_2 are linearly dependent which violates the initial assumption. Now it is easy to verify that $[z_0 I - A + BD^{-1}C][x_{11}\ x_{21}] = 0$, which implies that $A - BD^{-1}C$ has a repeated eigenvalue. By manipulation of an entry of A by an arbitrarily small amount, one can verify that, for any z, the kernel of the system matrix will have dimension at most 1, since $A - BD^{-1}C$ will have nonrepeated eigenvalues.

Open: Set $\mathcal{F}$ being open is equivalent to its complement, call it $\mathcal{F}^C$, being closed. To obtain a contradiction, suppose $\mathcal{F}^C$ is not closed. Then there must exist a sequence $\{A_m,\, B_m,\, C_m,\, D_m\}_{m \in \mathbb{N}} \to \{A_0,\, B_0,\, C_0,\, D_0\}$ where $\{A_0,\, B_0,\, C_0,\, D_0\} \in \mathcal{F}$ and $\{A_m,\, B_m,\, C_m,\, D_m\} \in \mathcal{F}^C$ for all m. Moreover, there exists a $z_m \in \mathbb{C}$ such that $\operatorname{rank}(M_m(z_m)) \leq n + m - 2$, where $M_m(z)$ denotes the system matrix associated with $\{A_m,\, B_m,\, C_m,\, D_m\}$. Consequently, $\sigma_1(\, M_m(z_m)) = \sigma_2(M_m(z_m)) = 0$ where $\sigma_i(F)$ denotes the i-th smallest singular value of F. Now $M_m(z_m) \to M_0(z_0)$ holds as $\{A_m, B_m, C_m, D_m\} \to \{A_0,\, B_0,\, C_0,\, D_0\}$ and $z_m \to z_0$. Hence, $\sigma_2(M_m(z_m)) \to \sigma_2(M_0(z_0))$; however, by assumption $\sigma_2(M_0\ (z_0)) > 0$ which contradicts the fact that $\sigma_2(M_0(z_0)) \to 0$ and the result follows. □

Theorem 4.2.3. *Consider the system $\sum = \{A,\, B,\, C,\, D\}$ with $p > m$. If the entries of $A,\, B,\, C,\, D$ assume generic values, then $\sum$ has no finite or infinite zeros.*

Proof. We observe first that generically the normal rank of $M(z)$ is equal to $n + m$. To see this, take $A = C = 0$ and D as any full column rank matrix, to get a particular $M(z)$ which for any nonzero z has rank $n+m$. Since the normal rank cannot exceed $n+m$ and this rank is attained for a particular choice of $\{A,\, B,\, C,\, D\}$, so $n+m$ must be the normal rank for generic $M(z)$. Observe also that generically D has rank m, and hence the normal rank of M equals $n + \operatorname{rank} D$, which shows that generically $\sum$ has no infinite zero. For the finite zeros, observe that any such zero must be a zero of every minor of dimension $(n + m) \times (n + m)$. Since $M(z)$ has normal rank $n + m$, there must be at least one minor of dimension $(n + m) \times (n + m)$ which is nonzero for almost all values of z. Choose A, B and the first m rows of C, D generically, and consider the associated minor. For each of the finite set of values of z for which the minor is zero, determine the associated kernel which has the dimension at most one based on the result of Lemma 4.2.2. Then a generic $(n + m)$-dimensional vector will not be orthogonal to any single one of these kernels, and since there are a finite number of such kernels, a generic $(n + m)$-dimensional vector will not be orthogonal to any of the kernels considered simultaneously. If the next, i.e $(m + 1)$-th, row of $[C \;\; D]$ is set equal to this vector, then any vector in any of the finite set of kernels of the $(n + m)$-dimensional minors formed using the first m rows of $[C \;\; D]$ will not be orthogonal to the added row of $[C \;\; D]$, which means that the $(m + n + 1)$ row matrix obtained by adjoining the new row of $[C \;\; D]$ must have an empty kernel for any value of z, i.e. there is no zero. Given that $C,\, D$ are actually generic and may have more rows again, the result is now evident. □

Tall blocked linear systems

In the previous section we showed that tall time-invariant unblocked systems are generically zero-free. In this section we study the zero properties of their associated blocked systems. The results of this section enable us to study the zero properties of blocked systems resulted from blocking of linear systems with multirate output in the next section. For some of the theorems proofs are omitted and can be found in (Zamani et al., 2011).

We first introduce an arbitrary integer $N > 1$, called blocking rate and define the following multivariate signals

$$U(t) = \begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+N-1) \end{bmatrix} \quad , \quad Y(t) = \begin{bmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+N-1) \end{bmatrix},$$

where $t = 0, N, 2N, \ldots$. Then, the blocked system $\sum_b$ is given by

$$\begin{aligned} x(t+N) &= A_b x(t) + B_b U(t) \\ Y(t) &= C_b x(t) + D_b U(t)\,. \end{aligned} \tag{4.4}$$

The blocked system, mapping the $U(t)$ sequence to the $Y(t)$ sequence, has a time-invariant state-variable description given by

$$\begin{aligned} A_b &= A^N, \quad B_b = \begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix}, \\ C_b &= \begin{bmatrix} C^\top & A^\top C^\top & \ldots & A^{(N-1)\top} C^\top \end{bmatrix}^\top, \\ D_b &= \begin{bmatrix} D & 0 & \ldots & 0 \\ CB & D & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \ldots & D \end{bmatrix}. \end{aligned} \tag{4.5}$$

We define the operator Z such that $Zx(t) = x(t+N)$, $ZU(t) = U(t+N)$, $ZY(t) = Y(t+N)$, giving to the symbol Z also the meaning of complex value. We denote the transfer function associated with (4.4) as $V(Z) := D_b + C_b(ZI - A_b)^{-1}C_b$ and it is worthwhile remarking that minimality of $\sum$ is equivalent to minimality of $\sum_b$.

Similar to Definition 4.2.1 we have the following definition for the zeros of the system (4.4).

Definition 4.2.4. The finite zeros of the system $\sum_b$ are defined to be the values of $z \in \mathbb{C}$ for which the rank of the following system matrix

$$M_b(Z) = \begin{bmatrix} ZI - A_b & -B_b \\ C_b & D_b \end{bmatrix}. \tag{4.6}$$

falls below its normal rank. Equivalently, $z \in \mathbb{C}$ is a zero of $\sum_b$ if is a zero of all nonzero minors of order equal to $\operatorname{nrank} M_b(z)$.

According to the above definition, the normal rank of the system matrix $M_b(Z)$ plays an important role in the zero properties of its associated blocked system.

Lemma 4.2.5. *Suppose that $p \geq m$. Then the normal rank of $M(z)$ is $n+m$ if and only if the normal rank $M_b(Z)$ is $n+Nm$.*

A consequence of this result is stated in the following lemma.

Lemma 4.2.6. *For a generic choice of matrices $\{A,\, B,\, C,\, D\}$ with $p \geq m$, the system matrix $M_b(Z)$ has normal rank equal to $n+Nm$.*

Proof. In the generic setting and $p \geq m$, matrix D is of full column rank. So, due to the structure of D_b, see (4.5), one can easily conclude that D_b is of full column rank as well. Then the result easily follows. □

The following theorem establishes the relation between the zeros of a unblocked system and its blocked version.

Theorem 4.2.7. *Suppose that the system matrix of the unblocked system $\sum$ has full-column normal rank. Then the following facts hold*

- *if $\sum$ has a finite zero at $z = z_0 \neq 0$, then the system $\sum_b$ has a finite zero at $Z = Z_0 = z_0^N \neq 0$. Conversely, if the system $\sum_b$ has a finite zero at $Z = Z_0 = z_0^N \neq 0$, then the system $\sum$ has a finite zero at one or more of $z = z_0 \neq 0$ or $z = \omega z_0 \neq 0$,..., $z = \omega^{N-1} z_0 \neq 0$, where $\omega = \exp(\frac{2\pi j}{N})$;*
- *the system $\sum_b$ has a zero at $Z_0 = \infty$ if and only if the system $\sum$ has a zero at $z_0 = \infty$;*
- *the system $\sum_b$ has a zero at $Z_0 = 0$ if and only if the system $\sum$ has a zero at $z_0 = 0$.*

We are now ready to state the main result of this section, regarding the zero-freeness of the blocked system $\sum_b$

Theorem 4.2.8. *Consider the system $\sum$ defined by the quadruple $\{A,\, B,\, C,\, D\}$, in which the individual matrices are generic. Then*

- *If $p > m$, the system $M_b(Z)$ has full column rank for all z.*
- *If $p = m$, then the system matrix $M_b(Z)$ can only have finite zeros with one-dimensional kernel.*

Proof. Suppose first that $p > m$. Using the results of Lemma 4.2.6 and Lemma 4.2.5, it can be readily seen that the system matrix of tall unblocked systems generically have full-column normal rank. Furthermore, Theorem 4.2.3 shows that tall unblocked systems are generically zero-free. If the blocked system had its system matrix with less than full column rank for a finite $Z_0 \neq 0$, then according to Theorem 4.2.7, there would be

necessarily a nonzero null vector of the system matrix of the unblocked system for $z_0 \neq 0$ equal to some $N-th$ root of Z_0, which would be a contradiction. If the blocked system had a zero at $Z_0 = \infty$, then based on Theorem 4.2.7 the D matrix of the unblocked system would be less than full column rank which would be a contradiction. Analogously, using the argument in Theorem 4.2.7, one can easily conclude that the blocked system has full column rank system matrix at $Z_0 = 0$.

Now we consider the case $p = m$; since D is generic, it has full column rank. Hence, based on the conclusion of Theorem 4.2.7, both the unblocked system and the blocked system do not have zeros at infinity. In the second part of this proof we use the conclusion of Theorem 4.2.7. Furthermore, one should note that since matrices A, B, C and D assume generic values it can be easily understood that the quadruple $\{A_b,\ B_b,\ C_b,\ D_b\}$ is a minimal realization. Now, based on the fact that D_b is nonsingular, one can conclude that the zeros of the blocked system are the eigenvalues of $A_b - B_b D_b^{-1} C_b$. If the eigenvalues of $A_b - B_b D_b^{-1} C_b$ are distinct, then the associated eigenspace for each eigenvalue is one-dimensional; it is equivalent to saying that the associated kernel of $M_b(Z)$ evaluated at the eigenvalue has dimension one. One should note that the unblocked system has distinct zeros due to the genericity assumption. Furthermore, zeros of the unblocked system generically have distinct magnitudes except for complex conjugate pairs. It is obvious that those zeros of the unblocked system with distinct magnitudes produce distinct blocked zeros. Now, we focus on zeros of the unblocked system with the same magnitudes, i.e. complex conjugate pairs. The only case where the generic unblocked system has distinct zeros but its corresponding blocked system has non-distinct zeros happens when the $N-th$ power of the complex conjugate zeros of the unblocked system coincide. We now show by contradiction that this is generically impossible. In order to illustrate a contradiction, suppose that the unblocked system has a complex conjugate pair, say z_{01} and z_{01}^*. If they produce an identical zero for the blocked system, their $N-th$ powers must be the same. The latter condition implies that the angle between z_{01} and z_{01}^* has to be exactly $2\pi h/N$, where h is an integer, which contradicts the genericity assumption for the unblocked system. Hence, the zeros of the blocked system generically have distinct values and consequently the corresponding kernels of system matrix evaluated at the zeros are one-dimensional. □

4.3 Multirate linear systems: problem statement

In this section, we focus our attention on the analysis of the zeros of blocked multirate systems. We shall see that such a study is non-trivial and needs a deep insight in the

analysis of the normal rank of the system matrix associated with a given multirate system. Quite surprisingly, we shall find that even if the parameter matrices are generic, such systems may have zeros at the origin of the complex plane or at infinity.

After introducing multirate systems, we focus on their finite nonzero zeros, then the possible presence of infinite zeros and zeros at the origin is explored.

We assume that the dynamics of an underlying system operating at the highest sample rate are defined by a linear system of the type

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{4.7}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ the output, and $u(t) \in \mathbb{R}^m$ the input. For this system, $y(t)$ exists for all t, and, separately, can be measured at every time t. However, we are also interested in the situation where $y(t)$ exists for all t, but not every entry is measured for all t. In particular, we consider the case where $y(t)$ has components that are observed at different rates. For simplicity, in this chapter we consider a case where outputs are provided at two rates which we refer to as the fast rate and the slow rate. Without loss of generality we decompose $y(t)$ as

$$y(t) = \begin{bmatrix} y^f(t) \\ y^s(t) \end{bmatrix}$$

where $y^f(t) \in \mathbb{R}^{p_1}$ is observed at all t, the fast part, and $y^s(t) \in \mathbb{R}^{p_2}$ is observed at $k = 0, N, 2N, \cdots$, the slow part. Moreover, $p_1 > 0$, $p_2 > 0$ and $p_1 + p_2 = p$. Accordingly, we decompose C and D as

$$C = \begin{bmatrix} C^f \\ C^s \end{bmatrix}, \; D = \begin{bmatrix} D^f \\ D^s \end{bmatrix}.$$

Thus, the multirate linear system corresponding to what is measured has the following dynamics:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) && t = 0, 1, 2, \ldots \\ y^f(t) &= C^f x(t) + D^f u(t) && t = 0, 1, 2, \ldots \\ y^s(t) &= C^s x(t) + D^s u(t) && t = 0, N, 2N, \ldots \end{aligned} \tag{4.8}$$

We have actually N distinct alternative ways to block the system, depending on how fast signals are grouped with the slow signals. Even though these N different systems

share some common zero properties, their zero properties are not identical in the whole complex plane (see (Bittanti & Colaneri, 2009), pages 173-179). For $\tau \in \{1,\, 2,\, \ldots,\, N\}$ and $k = 0,\, N,\, 2N, \ldots$ define

$$U_\tau(t) := \begin{bmatrix} u(t+\tau) \\ u(t+\tau+1) \\ \vdots \\ u(t+\tau+N-1) \end{bmatrix} \quad , \quad Y_\tau(t) := \begin{bmatrix} y^f(t+\tau) \\ y^f(t+\tau+1) \\ \vdots \\ y^f(t+\tau+N-1) \\ y^s(t+N) \end{bmatrix} \tag{4.9}$$

and $x_\tau(t) := x(t+\tau)$. Then the blocked system $\sum_\tau$ is defined by

$$\begin{aligned} x_\tau(t+N) &= A_\tau x_\tau(t) + B_\tau U_\tau(t) \\ Y_\tau(t) &= C_\tau X_\tau(t) + D_\tau U_\tau(t), \end{aligned} \tag{4.10}$$

where,

$$\begin{aligned} A_\tau &:= A^N, \\ B_\tau &:= \begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & AB & B \end{bmatrix}, \\ C_\tau &:= \begin{bmatrix} C^{f\top} & A^\top C^{f\top} & \ldots & A^{(N-1)\top} C^{f\top} & A^{(N-\tau)^\top} C^{s\top} \end{bmatrix}^\top, \\ D_\tau &:= \begin{bmatrix} D^f & 0 & \ldots & 0 \\ C^f B & D^f & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-2} B & C^f A^{N-3} B & \ldots & D^f \\ C^s A^{N-\tau-1} B & \ldots & D^s & 0_{p_2 \times [m(\tau-1)]} \end{bmatrix}, \end{aligned} \tag{4.11}$$

where $0_{i\times j}$ denotes a zero-entries matrix of size $i \times j$ and when $N - \tau - 1 < 0$, $C^s A^{-1} B$ is replaced by D^s and rest of the terms in the last row are replaced by zero matrices of size $p_2 \times m$.

Reference (Bittanti & Colaneri, 2009) defines a zero of (4.8) at time τ as a zero of its corresponding blocked system $\sum_\tau$ [1]. Hence, in the rest of this section we focus on the zero properties of the blocked system $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$.

Definition 4.3.1. The finite zeros of the system $\sum_\tau$ are defined to be the finite values of

[1] Zeros of the transfer function defined from (4.10) are identical with those defined here, provided the quadruple $\{A_\tau,\, B_\tau,\, C_\tau,\, D_\tau\}$is minimal.

Z for which the rank of the following system matrix falls below its normal rank

$$M_\tau(Z) = \begin{bmatrix} ZI - A_\tau & -B_\tau \\ C_\tau & D_\tau \end{bmatrix}. \tag{4.12}$$

Further, $V_\tau(Z) = C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau$, $\tau \in \{1,\, 2,\, \ldots,\, N\}$, is said to have an infinite zero when $n + \operatorname{rank} D_\tau$, $\tau \in \{1,\, 2,\, \ldots,\, N\}$, is less than the normal rank of $M_\tau(Z)$, $\tau \in \{1,\, 2,\, \ldots,\, N\}$, or equivalently the rank of D_τ, $\tau \in \{1,\, 2,\, \ldots, N\}$, is less than the normal rank of $V_\tau(Z)$, $\tau \in \{1,\, 2,\, \ldots,\, N\}$.

In addition to the above definition the following results from (Chen et al., 2012) and (Colaneri & Longhi, 1995) are useful to the rest of this chapter.

Lemma 4.3.2. *The pair (A, B) is reachable if and only if the pair (A_τ, B_τ), $\forall \tau \in \{1, 2, \ldots, N\}$ is reachable.*

The above lemma studies the reachability property of $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$ and the lemma below explores its transfer function.

Lemma 4.3.3. *The transfer function $V_\tau(Z)$ associated with the blocked system (4.10) has the following property*

$$V_{\tau+1}(Z) = \begin{bmatrix} 0 & I_{p_{1(N-1)}} & 0 \\ ZI_{p_1} & 0 & 0 \\ 0 & 0 & I_{p_2} \end{bmatrix} V_\tau(Z) \begin{bmatrix} 0 & Z^{-1}I_m \\ I_{m(N-1)} & 0 \end{bmatrix}, \tag{4.13}$$

where $\tau \in \{1,\, 2,\, \ldots,\, N\}$.

The result of the above lemma is crucial for the study of the zero properties of $\sum_\tau$, $\forall \tau \in \{1,\, 2,\, \ldots,\, N\}$, for choice of finite nonzero zeros. The latter is the main focus for the remainder of this section. We treat the zero properties of $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$, under genericity and tallness assumptions. Given that $p_1,\, p_2 > 0$ and tallness is defined by $Np_1 + p_2 > Nm$, it proves convenient to consider partition the set of $p_1,\, p_2$ defining tallness into two subsets, as follows

1. $p_1 > m$.
2. $p_1 \leq m$, $Np_1 + p_2 > Nm$.

Such a partitioning is depicted in Figure 4.1. The first case is common, perhaps even overwhelmingly common in econometric modeling but the second case is important from a theoretical point of view, and possibly in other applications. Moreover, our results are able to cover both cases.

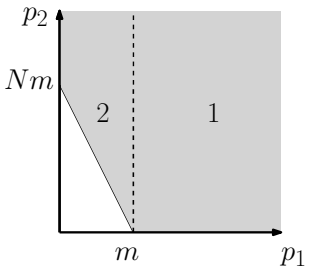


Figure 4.1: Partitioning of the condition of tallness.

4.4 Multirate systems: finite nonzero zeros analysis

Case $p_1 > m$

According to Definition 4.3.1, the normal rank for the system matrix of $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$, plays an important role in the analysis of its zero properties; thus, we state the following result for the normal rank of $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$.

Lemma 4.4.1. *For generic choice of the matrices $\{A, B, C^s, C^f, D^f, D^s\}$, $p_1 \geq m$, the system matrix of $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$, has normal rank equal to $n + Nm$.*

Proof. In a generic setting and with $p_1 \geq m$, the matrix D^f has full column rank. So, due to the structure of D_τ, $\forall \tau \in \{1, 2, \ldots, N\}$, one can easily conclude that D_τ has full-column rank as well. Furthermore,

$$M_\tau(Z) = \begin{bmatrix} ZI - A_\tau & -B_\tau \\ C_\tau & D_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_\tau(ZI - A_\tau)^{-1} & I \end{bmatrix} \begin{bmatrix} ZI - A_\tau & -B_\tau \\ 0 & C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau \end{bmatrix}. \tag{4.14}$$

Now observe that $M_\tau(Z)$ has $n+Nm$ columns so, $n+Nm \geq \operatorname{nrank} M_\tau(Z) = \operatorname{nrank}(ZI - A_\tau) + \operatorname{nrank}(C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau) \geq n + \operatorname{rank}(\lim_{Z\to\infty}[C_\tau(ZI - A_\tau)^{-1}B_\tau + D_\tau]) = n + \operatorname{rank}(D_\tau) = n + Nm$. Hence, the normal rank of $M_\tau(Z)$ equals the number of its columns. □

In the situation where $p_1 > m$, obtaining a result on the absence of finite nonzero zeros is now rather trivial, since the blocked system contains a subsystem obtained by deleting some outputs which is provably zero-free.

Theorem 4.4.2. *For a generic choice of the matrices $\{A, B, C^s, C^f, D^s, D^f\}$, $p_1 > m$, the system matrix of $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$, has full column rank for all finite nonzero Z.*

Proof. Let us define a system matrix $M^f(Z)$ by deleting rows of $M_\tau(Z)$, $\tau \in \{1,\, 2,\, \ldots, N\}$, which contain any entries of C^s. With $p_1 > m$, one can use the results of the previous section and conclude that $M^f(Z)$ is generically of full-column rank for all finite nonzero Z. Then it is immediate that $M_\tau(Z)$, $\tau \in \{1,\, 2,\, \ldots,\, N\}$, will be of full-column rank for all finite nonzero Z. □

Case $p_1 \leq m$, $Np_1 + p_2 > Nm$

In the previous subsection the case $p_1 > m$ was treated where only considering the fast outputs alone generically leads to a zero-free blocked system, and the zero-free property is not disturbed by the presence of the further slow outputs. A different way in which the blocked system will be tall arises when $p_1 \leq m$ and $Np_1 + p_2 > Nm$. The main result of this subsection is to show when $\sum_\tau$, $\forall \tau \in \{1,\, 2,\, \ldots,\, N\}$ with $p_1 \leq m$, $Np_1 + p_2 > Nm$ is again generically zero-free. In order to study the latter case we need to review properties of the Kronecker canonical form of a matrix pencil. Since the system matrix of $\sum_\tau$, $\forall \tau \in \{1,\, 2,\, \ldots,\, N\}$ is actually a matrix pencil, the Kronecker canonical form turns out to be a very useful tool to obtain insight into the zeros of (4.10) and the structure of the kernels associated with those zeros.

The main theorem on the Kronecker canonical form of the matrix pencil is obtained from (Van Dooren, 1979).

Theorem 4.4.3. *(Van Dooren, 1979) Consider a matrix pencil $zR + S$. Then under the equivalence defined using pre- and postmultiplication by nonsingular constant matrices $\widetilde{P}$ and $\widetilde{Q}$, there is a canonical quasidiagonal form:*

$$\widetilde{P}(zR + S)\widetilde{Q} = \operatorname{diag}\left[L_{\epsilon_1}, \ldots, L_{\epsilon_r}, \tilde{L}_{\eta_1}, \ldots, \tilde{L}_{\eta_s}, zN - I, zI - K\right], \tag{4.15}$$

where:

1. *L_μ is the $\mu \times (\mu + 1)$ bidiagonal pencil*

$$\begin{bmatrix} z & -1 & 0 & \ldots & 0 & 0 \\ 0 & z & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & z & -1 \end{bmatrix}. \tag{4.16}$$

2. *$\tilde{L}_\mu$ is the $(\mu+1)\times\mu$ transposed bidiagonal pencil*

$$\begin{bmatrix} -1 & 0 & \dots & 0 & 0 \\ z & -1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & z & -1 \\ 0 & 0 & \dots & 0 & z \end{bmatrix}. \tag{4.17}$$

3. *N is a nilpotent Jordan matrix.*

4. *K is in Jordan canonical form.*

Furthermore, the possibility that $\mu = 0$ exists. The associated L_0 is deemed to have a column but not a row and $\tilde{L}_0$ is deemed to have a row but not a column, see (Van Dooren, 1979). The following corollary can be directly derived easily from the above theorem and provides detail about the vectors in the null space of the Kronecker canonical form. Because the matrices $\tilde{P}$ and $\tilde{Q}$ are nonsingular, it is trivial to translate these properties back to an arbitrary matrix pencil, including a system matrix.

Corollary 4.4.4. *1. For all z except for those which are eigenvalues of K, the kernel of the Kronecker canonical form has dimension equal to the number of matrices L_μ appearing in the form; likewise the co-kernel dimension is determined by the number of matrices $\tilde{L}_\mu$.*

2. *The vector $[1 \;\; z \;\; z^2 \dots z^\mu]^\top$ is the generator of the kernel of L_μ, a set of vectors $[0 \dots 0 \; 1 \; z \; z^2 \dots z^\mu \; 0 \dots 0]^\top$ are generators for the kernel of the whole canonical form which depend continuously on z, provided that z is not an eigenvalue of K; when z is an eigenvalue of K, the vectors form a subset of a set of generators.*

3. *When z equals an eigenvalue of K, the dimension of the kernel jumps by the geometric multiplicity of that eigenvalue, the rank of the pencil drops below the normal rank by that geometric multiplicity, and there is an additional vector or vectors in the kernel apart from those defined in point 2, which are of the form $[0 \; 0 \dots v^\top]^\top$, where v is an eigenvector of K. Such a vector is orthogonal to all vectors in the kernel which are a linear combination of the generators listed in the previous point.*

4. *When z is an eigenvalue, say z_0 of K, the associated kernel of the matrix pencil can be generated by two types of vectors: those which are the limit of the generators defined by adding extra zeros to vectors such as $[1 \;\; z_0 \;\; z_0^2 \dots, z_0^\mu]^\top$ (these being the limits of the generators when $z \neq z_0$ but continuously approaches z_0), and those obtained*

by adjoining zeros to the eigenvector(s) of K with eigenvalue z_0, the latter set being orthogonal to the former set.

According to Definition 4.3.1, the normal rank plays an important role in the zero properties of the blocked system under study. Now a generic tall single-rate system has full-column rank as do blocked versions thereof also, in the earlier work (Zamani et al., 2011) it was made the explicit assumption that a blocked time-invariant version of a multirate system, if tall, has full-column rank. However, it may be for such a system that the normal rank is less than the number of columns when $p_1 < m$. The situation is shown in the example below.

Example 4.4.5. Consider a tall multirate system with $n = 1$, $m = 3$, $N = 2$, $p_1 = 1$, $p_2 = 5$. Let the parameter matrices for the multi-rate system be $A = a$, $B = [b_1\, b_2\, b_3]$, $C^f = c^{f\top}$, $C^s = [c_1^s\, c_2^s\, c_3^s\, c_4^s\, c_5^s]^\top$, $D^f = [d_1^f\, d_2^f\, d_3^f]$ and

$$D^s = \begin{bmatrix} d_{11}^s & d_{12}^s & d_{13}^s \\ \vdots & \vdots & \vdots \\ d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$

We consider $\tau = 1$ and write the associated system matrix as

$$M_1(Z) = \begin{bmatrix} Z-a^2 & -ab_1 & -ab_2 & -ab_3 & -b_1 & -b_2 & -b_3 \\ c^f & d_1^f & d_2^f & d_3^f & 0 & 0 & 0 \\ c^f a & c^f b_1 & c^f b_2 & c^f b_3 & d_1^f & d_2^f & d_3^f \\ c_1^s a & c_1^s b_1 & c_1^s b_2 & c_1^s b_3 & d_{11}^s & d_{12}^s & d_{13}^s \\ c_2^s a & c_2^s b_1 & c_2^s b_2 & c_2^s b_3 & d_{21}^s & d_{22}^s & d_{23}^s \\ c_3^s a & c_3^s b_1 & c_3^s b_2 & c_3^s b_3 & d_{31}^s & d_{32}^s & d_{33}^s \\ c_4^s a & c_4^s b_1 & c_4^s b_2 & c_4^s b_3 & d_{41}^s & d_{42}^s & d_{43}^s \\ c_5^s a & c_5^s b_1 & c_5^s b_2 & c_5^s b_3 & d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$

It is obvious that first two rows are linearly independent. Now consider the rows 3 to 8;

they can be written as a product of matrices $G\Gamma$, with

$$G := \begin{bmatrix} c^f & c^f & c^f & c^f & d_1^f & d_2^f & d_3^f \\ c_1^s & c_1^s & c_1^s & c_1^s & d_{11}^s & d_{12}^s & d_{13}^s \\ c_2^s & c_2^s & c_2^s & c_2^s & d_{21}^s & d_{22}^s & d_{23}^s \\ c_3^s & c_3^s & c_3^s & c_3^s & d_{31}^s & d_{32}^s & d_{33}^s \\ c_4^s & c_4^s & c_4^s & c_4^s & d_{41}^s & d_{42}^s & d_{43}^s \\ c_5^s & c_5^s & c_5^s & c_5^s & d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}$$

and $\Gamma := \operatorname{diag}(a, b_1, b_2, b_3, I_3)$. The matrix G has rank at most 4; hence, with generic parameter matrices the normal rank of $M(Z)$ equals 6 and thus $M(Z)$ cannot attain full-column normal rank.

The following proposition provides a nice connection between the normal ranks of the matrices $M_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$.

Proposition 4.4.6. *If there exists τ, $\tau \in \{1, 2, \ldots, N\}$, such that the normal rank of $M_\tau(Z) = \rho$, then ρ is the normal rank of the system matrix $M_\tau(Z)$ $\forall \tau \in \{1, 2, \ldots, N\}$.*

Proof. Note that there exists a finite value $Z_0 \neq 0$ such that $\operatorname{rank} V_\tau(Z)$ and $\operatorname{rank} V_{\tau+1}(Z)$ are properly defined i.e. both $V_\tau(Z)$ and $V_{\tau+1}(Z)$ have no poles at Z_0. Furthermore, we assume that

$$\max_{Z \in \mathbb{C}} \operatorname{rank} V_\tau(Z) = \operatorname{rank} V_\tau(Z_0)\,. \tag{4.18}$$

Then, by using the equation (4.13) we have $\operatorname{rank} V_\tau(Z_0) = \operatorname{rank} V_{\tau+1}(Z_0)$ and

$$\max_{Z \in \mathbb{C}} \operatorname{rank} V_{\tau+1}(Z) = \operatorname{rank} V_{\tau+1}(Z_0)\,. \tag{4.19}$$

Also, we know that

$$\max_{Z \in \mathbb{C}} \operatorname{rank} V_\tau(Z) = n + \max_{Z \in \mathbb{C}} \operatorname{rank} M_\tau(Z) \tag{4.20}$$

and the conclusion of the proposition becomes immediate. □

In order to state the exact normal rank of the blocked system matrix, we need to introduce the following proposition regarding the rank of the matrix D_1.

Proposition 4.4.7. *Consider the system $\sum_1$, with $p_1 < m$, $Np_1 + p_2 > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Then*

- *if $n < (N-1)(m-p_1)$ the matrix D_1 has rank equal to $(N-1)p_1 + m + n$;*
- *if $n \geq (N-1)(m-p_1)$ the matrix D_1 has full column rank, namely Nm.*

Proof. Assume first $n < (N-1)(m-p_1)$ and write the matrix D_1 as

$$D_1 = \left[\begin{array}{ccccc} D^f & 0 & 0 & \dots & 0 \\ C^f B & D^f & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ C^f A^{N-3} B & \dots & C^f B & D^f & 0 \\ \hline C^f A^{N-2} B & C^f A^{N-3} B & \dots & C^f B & D^f \\ C^s A^{N-2} B & C^s A^{N-3} B & \dots & C^S B & D^s \end{array}\right] := \left[\begin{array}{c} \Delta \\ \hline \Pi \end{array}\right]. \tag{4.21}$$

It is well-known (see e.g. (Zamani et al., 2011)) that, due to genericity of the matrix D^f, Δ is full row rank, namely $(N-1)p_1$. The matrix Π admits the factorization

$$\begin{bmatrix} C^f & \dots & C^f & D^f \\ C^s & \dots & C^s & D^s \end{bmatrix} \Gamma, \tag{4.22}$$

where $\Gamma := \operatorname{diag}\{A^{N-2}B,\, A^{N-3}B,\, \dots,\, B,\, I_m\}$. Now consider the following cases.
1) $n \leq m$. Clearly, Γ is full row rank, hence the rank of Π is equal to the rank of the matrix

$$\bar{\Pi} := \begin{bmatrix} C^f & D^f \\ C^s & D^s \end{bmatrix} \in \mathbb{R}^{[p_1+p_2]\times[n+m]}. \tag{4.23}$$

Under the assumption $n < (N-1)(m-p_1)$ and recalling that the condition of tallness implies that $p_2 > N(m-p_1)$, it holds that $n+m < p_1+p_2$ and hence $\operatorname{rank}(\bar{\Pi}) = n+m$.
2) $n > m$. In this case, Γ is not full row rank. However, one can select $n-m$ rows from each block $A^{N-2}B,\, A^{N-3}B,\, \dots,\, B$ and discard them in order to obtain a matrix, say $\tilde{\Gamma}$, which becomes full row rank, namely Nm. Similarly, one can construct the matrix $\tilde{\Pi}$, which is defined by discarding the corresponding columns from the matrix

$$\begin{bmatrix} C^f & \dots & C^f & D^f \\ C^s & \dots & C^s & D^s \end{bmatrix}.$$

Clearly, if such an operation of discarding is made carefully, the matrix $\tilde{\Pi}$ still contains all the columns of the matrix $\bar{\Pi}$, thus the considerations made for the previous case still hold and $\operatorname{rank}\Pi = n+m$.

Finally, due to genericity of the parameter matrices, one can always find $n+m$ rows of Π which are linearly independent of the rows of Δ. Hence $\operatorname{rank} D_1 = (N-1)p_1 + m + n$.

Assume now $n = (N-1)(m-p_1)$; recalling the arguments used in the previous case one can find again that $\operatorname{rank} D_1 = (N-1)p_1 + m + n$. However, in this case $(N-1)p_1 + m + n = Nm$, thus $\operatorname{rank} D_1 = Nm$. Finally, consider the case $n > (N-1)(m-p_1)$;

clearly, increasing the dimension of the state cannot make the rank of D_1 decrease. Hence, $\operatorname{rank} D_1 = Nm$. □

The following Lemma precisely determines the normal rank of $M_\tau(Z)$; also it provides a sufficient and necessary condition for system matrix $M_\tau(Z)$ to have less than full-column normal rank.

Lemma 4.4.8. *Consider the system $\sum_\tau$, $\tau \in \{1, 2, \ldots, N\}$, with $p_1 < m$, $Np_1 + p_2 > Nm$ and generic values of the defining matrices $\{A,\ B,\ C^f,\ C^s,\ D^f,\ D^s\}$. Then the normal rank of the system matrix $M_\tau(Z)$ is equal to:*

- $(N-1)p_1 + m + 2n$ *if* $n < (N-1)(m-p_1)$*;*
- $n + Nm$ *if* $n \geq (N-1)(m-p_1)$.

Proof. Without loss of generality, we focus on the matrix $M_1(Z)$; every result on its normal rank can be easily extended to any value of $\tau = \{2, \ldots, N\}$ using Proposition 4.4.6.

Consider the matrix D_1 and define $r := \operatorname{rank} D_1$; note that the condition of tallness of the system implies $r \leq Nm$. Define the full row rank matrix $\bar{D}_1 \in \mathbb{R}^{r \times Nm}$, obtained by discarding a proper number of linearly dependent rows of D_1. Similarly, define $\bar{C}_1$ discarding the corresponding rows from C_1. Without loss of generality assume A diagonal. This hypothesis is not limiting; in fact, under a generic setting, A has n distinct eigenvalues and so it is diagonalizable. If one considers a change of basis T such that $T^{-1}AT$ is diagonal, then the other parameter matrices $T^{-1}B$, CT, D are still in a generic setting. Define $\bar{M}_1(Z)$ as follows

$$\bar{M}_1(Z) = \begin{bmatrix} Z - a_1^N & 0 & & \ldots & 0 & -b_1^\top \\ 0 & Z - a_2^N & 0 & \ldots & 0 & -b_2^\top \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & Z - a_n^N & -b_n^\top \\ \bar{c}_{1,1} & \bar{c}_{1,2} & \ldots & \ldots & \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix}, \tag{4.24}$$

where the a_i's represent the diagonal elements of A, $b_i^\top$ is the i-th row of B_1 and $\bar{c}_{i,1}$ is the i-th column of $\bar{C}_1$. Consider the submatrix $\begin{bmatrix} \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix}$. Since $\bar{D}_1$ is full row rank, also this matrix is full row rank. Consider the equation

$$v^\top \begin{bmatrix} \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix} = \begin{bmatrix} Z - a_n^N & -b_n^\top \end{bmatrix}, \tag{4.25}$$

in which v and Z are yet to be specified and which can be rewritten as

$$\begin{cases} v^\top \bar{c}_{1,n} &= Z - a_n^N \\ v^\top \bar{D}_1 &= -b_n^\top \end{cases} . \tag{4.26}$$

Since $\bar{D}_1$ is full row rank there exists at most one vector $\bar{v}^\top$ satisfying the second relation. Clearly, if one were to insert such a vector in the first relation, there could exist only one value $Z_n \in \mathbb{C}$ such that this equation is satisfied. Choose $Z \neq Z_n$ and consider the submatrix

$$\begin{bmatrix} 0 & Z - a_n^N & -b_n^\top \\ \bar{c}_{1,n-1} & \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix} , \tag{4.27}$$

which is clearly full row rank, namely $r+1$. Write the equation

$$v^\top \begin{bmatrix} 0 & Z - a_n^N & -b_n^\top \\ \bar{c}_{1,n-1} & \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix} = \begin{bmatrix} Z - a_{n-1}^N & 0 & -b_{n-1}^\top \end{bmatrix} , \tag{4.28}$$

which in turn can be rewritten as

$$\begin{cases} v^\top \begin{bmatrix} 0 \\ \bar{c}_{1,n-1} \end{bmatrix} &= Z - a_{n-1}^N \\ v^\top \begin{bmatrix} Z - a_n^N & -b_n^\top \\ \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix} &= \begin{bmatrix} 0 & -b_{n-1}^\top \end{bmatrix} \end{cases} . \tag{4.29}$$

Again, the second relation admits at most one solution, which is compatible with the first equation for only one value $Z_{n-1} \in \mathbb{C}$. Hence, choosing $Z \notin \{Z_n,\, Z_{n-1}\}$ one can build the matrix

$$\begin{bmatrix} 0 & Z - a_{n-1}^N & 0 & -b_{n-1}^\top \\ 0 & 0 & Z - a_n^N & -b_n^\top \\ \bar{c}_{1,n-2} & \bar{c}_{1,n-1} & \bar{c}_{1,n} & \bar{D}_1 \end{bmatrix} , \tag{4.30}$$

which is full row rank, namely $r+2$, and repeat the previous steps until all the rows containing the a_i^N's and the $b_i^\top$'s, $i \in \{1, \ldots, n\}$, are considered. This procedure ends after n iterations, when all the rows of the matrix $\bar{M}_1(Z)$ are included; clearly the rank turns out to be $r+n$. Since $\bar{M}_1(Z)$ is a submatrix of $M_1(Z)$, the normal rank of $M_1(Z)$ is greater than or equal to $r+n$. Now consider the following cases.

1) $n \geq (N-1)(m-p_1)$. Recalling Proposition 4.4.7, $r = Nm$; hence $\operatorname{nrank} \bar{M}_1(Z) = n + Nm$ and $M_1(Z)$ is full normal rank.

2) $n < (N-1)(m-p_1)$. In this case, from Proposition 4.4.7 we have $r = (N-1)p_1 + m + n$,

hence $\operatorname{nrank} M_1(Z) \geq \operatorname{nrank} \bar{M}_1(Z) = (N-1)p_1 + m + 2n$. Now, consider the submatrix formed by the first $n + (N-1)p_1$ rows of $M_1(Z)$. Such a submatrix is full normal rank, since it can be seen also as a submatrix of the system matrix

$$\begin{bmatrix} ZI_n - A^N & -A^{N-1}B & \dots & -B \\ C^f & D^f & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-1} & C^f A^{N-2}B & \dots & D^f \end{bmatrix}, \tag{4.31}$$

which is the system matrix of a blocked fat system with generic parameter matrices. From (Zamani et al., 2011), it is well-known that (4.31) is full normal rank. Now consider the remaining rows of $M_1(Z)$, i.e. the matrix

$$\Pi = \begin{bmatrix} C^f A^{N-1} & C^f A^{N-2}B & \dots & C^f B & D^f \\ C^s A^{N-1} & C^s A^{N-2}B & \dots & C^s B & D^s \end{bmatrix}$$

which can be factorized as

$$\Pi = \begin{bmatrix} C^f & \dots & C^f & D^f \\ C^s & \dots & C^s & D^s \end{bmatrix} \bar{\Gamma},$$

where $\bar{\Gamma} := \operatorname{diag}(A^{N-1}, A^{N-2}B, \dots, B, I_m)$. For generic choice of matrices C^s, D^s, C^f, D^f, the matrix Π has rank equal to $\alpha := \min\{p_1 + p_2, m + n\}$. Then $\operatorname{nrank} M(Z) \leq n + (N-1)p_1 + \alpha$. However, since for the condition of tallness $p_2 > N(m - p_1)$ and by assumption $n < (N-1)(m - p_1)$, we have $\alpha = n + m$ hence $\operatorname{nrank} M(Z) \leq (N-1)p_1 + m + 2n$. Combining this bound with the overbound found previously, we conclude that $\operatorname{nrank} M(Z) = (N-1)p_1 + m + 2n$. □

In the rest of this subsection, we explore the zero properties of $M_\tau(Z)$, $\forall \tau \in \{1, 2, \dots, N\}$. To achieve this, we first focus on the particular case of $M_1(Z)$. Later, we introduce the main result for the zero properties of $M_\tau(Z)$, $\forall \tau \in \{1,\, 2,\, \dots,\, N\}$.

First we need to introduce some parameters. To this end, we argue first that the first $n + Np_1$ rows of $M_1(Z)$ are linearly independent. For the submatrix formed by these rows is the system matrix of the blocked system obtained by blocking the fast system defined by $\{A,\, B,\, C^f,\, D^f\}$, and accordingly has full-row normal rank, since the unblocked system is generic and square or fat under the condition $p_1 \leq m$. Now define the square submatrix of $M_1(Z)$:

$$N(Z) := \begin{bmatrix} ZI - A_1 & -B_1 \\ \mathcal{C}_1 & \mathcal{D}_1 \end{bmatrix}, \tag{4.32}$$

such that $\operatorname{nrank} N(Z) = \operatorname{nrank} M_1(Z)$, by including the first $n + Np_1$ rows of $M_1(Z)$ and followed by appropriate other rows of $M_1(Z)$ to meet the normal rank and squareness requirements. Hence there exists a permutation matrix P such that

$$PM_1(Z) = \begin{bmatrix} N(Z) \\ \mathcal{C}_2 \quad \mathcal{D}_2 \end{bmatrix} \tag{4.33}$$

where $\mathcal{C}_2$ and $\mathcal{D}_2$ capture those rows of C_1 and D_1 that are not included in $\mathcal{C}_1$ and $\mathcal{D}_1$, respectively.

The zero properties of $N(Z)$ are studied in the following proposition.

Proposition 4.4.9. *Let the matrix $N(Z)$ be the submatrix of $M_1(Z)$ formed via the procedure described. Then for generic values of the matrices A, B, etc. with $p_1 \leq m$ and $Np_1 + p_2 > Nm$, for any finite Z_0 for which the matrix $N(Z_0)$ has less rank than its normal rank, its rank is one less than its normal rank.*

Proof. We distinguish two cases, $p_1 = m,\ p_1 < m$. In case $p_1 = m$, then $N(Z)$ is the system matrix for the system obtained by blocking the original system with slow outputs discarded. As such, the blocked system zeros are precisely the N-th powers of the unblocked system zeros (Zamani et al., 2011). For generic coefficient matrices, the unblocked system will have n distinct zeros; then the blocked system will have the same property. Further, the unblocked system will generically have a nonsingular direct feedthrough matrix, as will then the blocked system, so that $\mathcal{D}_1$ can be assumed to be nonsingular. It follows then that the zeros of the system with system matrix $N(Z)$ are identical with the eigenvalues of $\mathcal{A} - \mathcal{B}\mathcal{D}_1^{-1}\mathcal{C}_1$, which are then distinct, and since this matrix is $n \times n$, the eigenvector associated with each zero will be uniquely defined to within a scaling constant. It follows easily that there is a unique vector (to within scaling) in the kernel of $N(Z_0)$ where Z_0 is the zero of the blocked system.

We turn therefore to the case $p_1 < m$. We study the co-kernel of $N(Z_0)$. Let $Z_1, Z_2, \ldots,$ be a sequence of complex numbers such that (a) $Z_i \to Z_0$ and (b) $\operatorname{rank} N(Z_i)$ equals the normal rank of $N(Z)$. From what has been described earlier using the Kronecker canonical form, we know that the sequence of co-kernels of $N(Z_i)$ converges, say to $\mathcal{K}$, with any vector in this limit also in the co-kernel of $N(Z_0)$. In addition, since $N(Z_0)$ has lower rank than the normal rank of $N(Z)$, the co-kernel, call it $\bar{\mathcal{K}}$, will be strictly greater than $\mathcal{K}$. Suppose its dimension is at least two more than that of $\mathcal{K}$. We shall show this

situation is nongeneric.

Select two vectors w_1, w_2 which are in $\bar{\mathcal{K}}$ and which are orthogonal to $\mathcal{K}$. Then it is evident that there are two vectors call them v_1, v_2, constructed from linear combinations of w_1, w_2, which belong to $\bar{\mathcal{K}}$, which are still orthogonal to $\mathcal{K}$, and which for some pair $r < s$ have 1 and 0 in the r-th entry and 0 and 1 in the s-th entry respectively. Choose v_1, v_2 so that firstly, s is maximal, and secondly, for that s then r is maximal. It is not difficult to see that this means that v_1 has zero entries beyond the r-th and v_2 has zero entries beyond the s-th.

Now again we must consider two cases. Suppose firstly that s obeys $n + Np_1 + 1 \leq s \leq n + Nm$; in forming the product $v_2^\top N(Z_0)$, the s-th entry of v_2 will be multiplying entries of $N(Z_0)$ defined using C^s, A, B, D^s. Consider an entry in the s-th row of $N(Z_0)$ and in the last m columns. Such an entry is an entry of D^s, and is independent of all other entries in $N(Z_0)$. Suppose this entry of D^s is continuously perturbed by a small amount. Then clearly v_1 remains in the co-kernel of $N(Z_0)$ but v_2 cannot.

The particular values of Z for which $N(Z)$ has rank less than its normal rank, i.e. the zeros of $N(Z)$, will depend continuously on the perturbation.

Accordingly, with a small enough perturbation, those not equal before perturbation to Z_0 will never change to Z_0, and it is therefore guaranteed that with a small enough nonzero perturbation, the co-kernel of $N(Z_0)$ is reduced by one in dimension, though never to zero. If the original (before perturbation) co-kernel $\bar{\mathcal{K}}$ had dimension greater than two in excess of the dimension of $\mathcal{K}$, and the excess after perturbation is still greater than one, the argument can be repeated. Eventually, the co-kernel of $N(Z_0)$ will have an excess dimension over $\mathcal{K}$ of 1, i.e. $N(Z_0)$ will have rank one less than the normal rank of $N(Z)$.

Now suppose that s obeys $s \leq n + Np_1$. Then the last $N(m - p_1)$ entries of each of v_1, v_2 are zero. Remove these entries to define two linearly independent vectors $\tilde{v}_1, \tilde{v}_2$ of length $n + Np_1$, which evidently satisfy

$$\tilde{v}_i^\top \begin{bmatrix} ZI_n - A^N & -A^{N-1}B & \dots & -B \\ C^f & D^f & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^f A^{N-1} & C^f A^{N-2} B & \dots & D^f \end{bmatrix} = 0, \;\; i = 1, 2. \tag{4.34}$$

The above equation contains a fat system matrix, corresponding to a blocked version of a fat time-invariant unblocked system. It can be concluded easily form the results provided in (Zamani et al., 2011) that for generic values of the underlying matrices, there can be no Z_0 for which an equation such as (4.34) can even hold for a single nonzero $\tilde{v}_i$,

let alone two linearly independent ones. This ends the proof. □

The result of the previous proposition, although restricted to $\tau = 1$, enables us to establish the following main result applicable for any τ.

Theorem 4.4.10. *Consider the system $\sum_\tau$, $\forall\tau \in \{1, 2, \ldots, N\}$, with $p_1 \leq m$, and $Np_1 + p_2 > Nm$. Then for generic values of the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ the system matrix $M_\tau(Z)$, $\forall\tau \in \{1, 2, \ldots, N\}$, has rank equal to its normal rank for all finite nonzero values of Z_0, and accordingly $\sum_\tau$ has no finite nonzero zero.*

Proof. We first focus on the case $\tau = 1$. Now, apart from the $p_2 - N(m - p_1)$ rows of the C^s, D^s which do not enter the matrix $N(Z)$ defined by (4.32), choose generic values for the defining matrices, so that the conclusions of the preceding proposition are valid.

Let $Z_a, Z_b, \ldots$ be the finite set of Z for which $N(Z)$ has less rank than its normal rank (the set may have less than n elements, but never has more), and let $w_a, w_b, \ldots$ be vectors which are in the corresponding kernels (*not co-kernels*) and orthogonal to the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \to Z_a, Z_b, \ldots$ etc. Now, due to the facts that $M_1(Z)$ and $N(Z)$ have the same normal rank and any existing vector in the kernel of $M_1(Z)$ is in the kernel of $N(Z)$ one can conclude that the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \to Z_a, Z_b, \ldots$ etc, coincides with the subspace in the kernel obtained from the limit of the kernel of $M_1(Z)$ as $Z \to$ zeros of $M_1(Z)$.

Now, to obtain a contradiction, we suppose that the system matrix $M_1(Z)$ is such that, for $Z_0 \neq 0$, $M_1(Z_0)$ has rank less than its normal rank, i.e. the dimension of its kernel increases. Since the kernel of $M_1(Z_0)$ is a subspace of the kernel of $N(Z_0)$, Z_0 must coincide to one of the values of $Z_a, Z_b, \ldots$ and the rank of $M_1(Z_0)$ must be only one less than its normal rank; moreover, there must exist an associated nonzero w_1 unique up to a scaler multiplier, in the kernel of $M_1(Z_0)$ which is orthogonal to the limit of the kernel of $M_1(Z)$ as $Z \to Z_0$. Then w_1 is necessarily in the kernel of $N(Z_0)$, orthogonal to the limit of the kernel of $N(Z)$ as $Z \to Z_0$ and thus w_1 in fact must coincide to within a nonzero multiplier with one of the vectors $w_a, w_b, \ldots$.

Write this w_1 as

$$w_1 = \begin{bmatrix} x_1 \\ u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \tag{4.35}$$

and suppose the input sequence $u(i) = u_i$ is applied for $i = 1, 2 \ldots, N$ to the original system, starting in initial state x_1 at time 1. Let $y^f(1), y^f(2), \ldots$ denote the corresponding fast outputs and $y^s(N)$ the slow output at time N. Break this up into two subvectors, $y^{s1}(N), y^{s2}(N)$, where $y^{s1}(N)$ is associated with those rows of C^s, D^s which are included in $\mathcal{C}_1$, $\mathcal{D}_1$ and $y^{s2}(N)$ is related with the remaining rows of C^s and D^s . We have

$$\begin{aligned} N(Z_0)w_1 &= \begin{bmatrix} Z_0 I_n - A^N & -A^{N-1}B & -A^{N-2}B & \ldots & -B \\ C^f & D^f & 0 & \ldots & 0 \\ C^f A & C^f B & D^f & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C^f A^{N-1} & C^f A^{N-2} B & C^f A^{N-3} B & \ldots & D^f \\ C^{s1} A^{N-1} & C^{s1} A^{N-2} B & C^{s1} A^{N-3} B & \ldots & D^{s1} \end{bmatrix} w_1 \\ &= \begin{bmatrix} Z_0 x_1 - x(N+1) \\ y^f(1) \\ y^f(2) \\ \vdots \\ y^f(N) \\ y^{s1}(N) \end{bmatrix} = 0. \end{aligned} \tag{4.36}$$

Now it must be true that $x_1 \neq 0$. For otherwise, we would have $N(Z)w_1 = 0$ for all Z, which would violate assumptions. Since also $Z_0 \neq 0$, there must hold $x(N+1) \neq 0$. Hence there cannot hold both $x(N) = 0$ and $u(N) = 0$. Consequently, we can always find C^{s2}, D^{s2} such that $y^{s2}(N) = C^{s2}x(N) + D^{s2}u(N) \neq 0$, i.e. the slow output value is necessarily nonzero, no matter whether $w_1 = w_a, w_b$, etc. Equivalently, the equation $[\mathcal{C}_2 \;\; \mathcal{D}_2]w_1 = 0$ cannot hold. Hence, if $M_1(Z)$ defines a system with a finite zero and it is nonzero, this is a nongeneric situation. Hence, $M_1(Z)$ generically has rank equal to its normal rank for all finite nonzero Z. Now, we show that the latter property holds for all $M_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$. First, note that the pair (A, B) is generically reachable so, according to Lemma 4.3.2 the pair (A_τ, B_τ), $\forall \tau \in \{1, 2, \ldots, N\}$, is also reachable. Consider $Z_\zeta \in \mathbb{C} - \{0, \infty\}$, if Z_ζ does not coincide with the eigenvalues of A_τ then

$$\operatorname{rank} M_\tau(Z_\zeta) = n + \operatorname{rank} V_\tau(Z_\zeta). \tag{4.37}$$

Hence, using the result of Lemma 4.3.3, it is immediate that

$$\operatorname{rank} M_\tau(Z_\zeta) = \operatorname{rank} M_{\tau+1}(Z_\zeta)\,. \tag{4.38}$$

If Z_ζ does coincide with an eigenvalue of A_τ then $\operatorname{rank} V_\tau(Z_\zeta)$ is ill-defined. However, since zeros of $M_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$, are invariant under state feedback and pair (A_τ, B_τ) is reachable, one can easily find a state feedback to replace that eigenvalue (Zhou et al., 1996) and then (4.37) is a well-defined equation and $\operatorname{rank} M_\tau(Z_\zeta) = \operatorname{rank} M_{\tau+1}(Z_\zeta)$. Thus, we can conclude that all $M_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$ generically have no finite nonzero zeros. This ends the proof. □

4.5 Multirate systems: zeros at the origin and infinity

So far we have explored the presence of zeros in tall blocked systems with generic parameter matrices for finite nonzero values of the complex variable Z. We have shown that these systems generically have no finite nonzero zeros. However, in order to complete the analysis investigating the cases $Z = 0$ and $Z = \infty$, we need to tackle this problem using a different approach. We shall see that the result of Theorem 4.4.10 does not hold for this two particular points. As in the previous section, it is convenient to break up our examination of tall systems into separate cases based on the relation between p_1 and m.

We begin our analysis stating the following result, which relates the zeros of the system $\sum_\tau(Z)$ at infinity to the zeros of the system $\sum_{N-\tau+1}$ at the origin and conversely.

Lemma 4.5.1. *Consider the family of systems $\sum_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$, where the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ accept generic values. Then the following fact holds: $\sum_\tau$ has ν zeros at $Z = 0$ and μ zeros at $Z = \infty$ if and only if $\sum_{N-\tau+1}$ has μ zeros at $Z = 0$ and ν zeros at $Z = \infty$.*

Proof. Consider a reversed time description of the system (4.8), namely

$$\begin{aligned}
x(t-1) &= A^{-1}x(t) - A^{-1}Bu(t-1) && t = 1, 2, \ldots \\
y^f(t-1) &= C^f x(t-1) + D^f u(t-1) && t = 1, 2, \ldots \\
&= C^f A^{-1}x(t) + (D^f - C^f A^{-1}B)u(t-1) \\
y^s(t-1) &= C^s x(t-1) + D^s u(t-1) && t = 1, N+1, \ldots \\
&= C^s A^{-1}x(t) + (D^s - C^s A^{-1}B)u(t-1) && (4.39)
\end{aligned}$$

and define the following matrices

$$\begin{aligned}
\tilde{A} &:= A^{-1} & \tilde{B} &:= -A^{-1}B \\
\tilde{C}^f &:= C^f A^{-1} & \tilde{D}^f &:= D^f - C^f A^{-1}B \\
\tilde{C}^s &:= C^s A^{-1} & \tilde{D}^s &:= D^s - C^s A^{-1}B
\end{aligned} \qquad (4.40)$$

which are still in a generic setting since the genericity of $\{A,\, B,\, C^f,\, D^f,\, C^s,\, D^s\}$ is assumed. Note that the matrix A^{-1} is well-defined, since A is generically full rank. Recall the blocking procedure introduced in (4.9) for a given value of τ; we can obtain the blocked time-invariant system associated with the system (4.39) as

$$\begin{aligned} x_\tau(t-N) &= \tilde{A}_\tau x_\tau(t) + \tilde{B}_\tau U_\tau(t-N) \\ Y_\tau(t-N) &= \tilde{C}_\tau x_\tau(t) + \tilde{D}_\tau U_\tau(t-N), \end{aligned} \tag{4.41}$$

where $t = 0, N, 2N, \ldots$, and

$$\begin{aligned} \tilde{A}_\tau &:= \tilde{A}^N, \\ \tilde{B}_\tau &:= \left[\begin{array}{cccc} \tilde{B} & \tilde{A}\tilde{B} & \ldots & \tilde{A}^{N-2}\tilde{B} \quad \tilde{A}^{N-1}\tilde{B} \end{array}\right], \\ \tilde{C}_\tau &:= \left[\begin{array}{cccc} \tilde{A}^{(N-1)\top}\tilde{C}^{f\top} & \ldots & \tilde{C}^{f\top} & \tilde{A}^{(\tau-1)\top}\tilde{C}^{s\top} \end{array}\right]^\top, \\ \tilde{D}_\tau &:= \left[\begin{array}{cccc} \tilde{D}^f & \ldots & \tilde{C}^f\tilde{A}^{N-3}\tilde{B} & \tilde{C}^f\tilde{A}^{N-2}\tilde{B} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \tilde{D}^f & \tilde{C}^f\tilde{B} \\ 0 & \ldots & 0 & \tilde{D}^f \\ 0_{p_s\times(N-\tau-1)} & \tilde{D}^s & \ldots & \tilde{C}^s\tilde{A}^{\tau-2}\tilde{B} \end{array}\right]. \end{aligned} \tag{4.42}$$

Furthermore, when $\tau - 2 < 0$, $\tilde{C}^s\tilde{A}^{-1}\tilde{B}$ is replaced by $\tilde{D}^s$. Now let us introduce the N-step backward operator ζ, such that $\zeta x(t) = x(t-N)$; the transfer function $\tilde{V}_\tau(\zeta) := \tilde{C}_\tau(\zeta I - \tilde{A}_\tau)^{-1}\tilde{B}_\tau + \tilde{D}_\tau$, associated with the blocked system (4.41) is readily available. It can be easily checked through simple computations that this transfer function is connected to the transfer function $V_\tau(Z)$ associated with the system $\sum_\tau$ through the equalities below

$$\tilde{V}_\tau(0) = \lim_{Z\to\infty} V_\tau(Z) \qquad \lim_{\zeta\to\infty} \tilde{V}_\tau(\zeta) = V_\tau(0). \tag{4.43}$$

Define the system matrix associated with the system (4.41) as

$$\tilde{M}_\tau(\zeta) := \left[\begin{array}{cc} \zeta I - \tilde{A}_\tau & -\tilde{B}_\tau \\ \tilde{C}_\tau & \tilde{D}_\tau \end{array}\right]. \tag{4.44}$$

For our purpose in this chapter, we define the following equalities

$$\begin{aligned} \operatorname{rank} M_\tau(\infty) &:= n + \operatorname{rank} D_\tau \\ \operatorname{rank} \tilde{M}_\tau(\infty) &:= n + \operatorname{rank} \tilde{D}_\tau\,, \end{aligned} \tag{4.45}$$

thus using the equation (4.43) one can write

$$\begin{aligned} \operatorname{rank} M_\tau(\infty) &= \operatorname{rank} \tilde{M}_\tau(0) \\ \operatorname{rank} \tilde{M}_\tau(\infty) &= \operatorname{rank} M_\tau(0)\,. \end{aligned} \tag{4.46}$$

Note that the above equalities are well-defined, since the matrices A_τ and $\tilde{A}_\tau$ generically do not have eigenvalues at the origin. Now, observe that, after some row and column reordering $\tilde{M}_\tau(\zeta)$ has the same structure as $M_{N-\tau+1}(Z)$. Since the parameter matrices $\{A,\, B,\, C^f,\, D^f,\, C^s,\, D^s\}$ assume generic values, we have the following equalities

$$\begin{aligned} \operatorname{rank} \tilde{M}_\tau(\infty) &= \operatorname{rank} M_{N-\tau+1}(\infty) \\ \operatorname{rank} \tilde{M}_\tau(0) &= \operatorname{rank} M_{N-\tau+1}(0)\,. \end{aligned} \tag{4.47}$$

Moreover, with the help of Proposition 4.4.6, we have

$$\operatorname{nrank} M_{N-\tau+1}(Z) = \operatorname{nrank} M_\tau(Z). \tag{4.48}$$

Finally, by combining equations (4.46) and (4.47) we obtain

$$\operatorname{rank} M_\tau(\infty) = \operatorname{rank} M_{N-\tau+1}(0) \tag{4.49}$$

and

$$\operatorname{rank} M_\tau(0) = \operatorname{rank} M_{N-\tau+1}(\infty)\,. \tag{4.50}$$

Thus, by using the equations (4.48), (4.49), (4.50) the conclusion of lemma readily follows. □

Case $p_1 > m$

As for the finite nonzero complex values, the case $p_1 > m$ is rather trivial.

Theorem 4.5.2. *For a generic choice of the matrices $\{A,\, B,\, C^s,\, C^f,\, D^s,\, D^f\}$, $p_1 > m$, the system matrix of $\sum_\tau$, $\forall \tau \in \{1,\, 2, \ldots,\, N\}$, has full column rank at $Z = 0$.*

Proof. It was shown in Section 4.2 that $M^f(0)$, where the system matrix $M^f(0)$ can be formed by deleting rows of $M(0)$ which are related to C^s, has full-column rank at $Z = 0$ for generic parameter matrices A, B, etc. Then, it becomes immediate that $M_\tau(0)$, $\forall \tau \in \{1,\, 2,\, \ldots,\, N\}$ has full-column rank. □

An analogous result holds for the zeros at infinity.

Theorem 4.5.3. *For a generic choice of the matrices* $\{A,\ B,\ C^s,\ C^f,\ D^s,\ D^f\}$*,* $p_1 > m$*, the system* $\sum_\tau$*,* $\tau \in \{1,\ 2, \dots\ N\}$*, has no zero at* $Z = \infty$*.*

Proof. Since $M_\tau(Z)\,,\ \forall \tau \in \{1,\ \dots,\ N\}$ has no zeros at zero, then, using Lemma 4.5.1, it follows that $M_\tau(Z)\,,\ \forall \tau \in \{1,\ \dots,\ N\}$ has no zeros at infinity. □

Case $p_1 \leq m$**,** $Np_1 + p_2 > Nm$

In this subsection, in order to explore zero properties of the blocked system $\sum_\tau$ at $Z = 0$ and $Z = \infty$, two cases are considered. We first focus on the case where $p_1 < m$, $Np_1 + p_2 > Nm$. The following theorem treats zeros at infinity.

Theorem 4.5.4. *Consider the system* $\sum_\tau$*,* $\forall \tau \in \{1, \dots, N\}$*, with* $p_1 < m$ *and* $Np_1 + p_2 > Nm$*. Assume that the defining matrices* $\{A,\ B,\ C^f,\ D^f,\ C^s,\ D^s\}$ *accept generic values and the system matrix* $M_\tau(Z)$*, has full column normal rank. Then* $M_\tau(Z)$ *has zeros at* $Z = \infty$ *with multiplicity equal to* $(\tau - 1)(m - p_1)$*.*

Proof. By using Definition 4.3.1 and assumptions provided in the theorem statement, one can conclude that the system $\sum_\tau$, $\{1,\ 2\ \dots,\ N\}$, has a zero at infinity if and only if D_τ has rank less than full column rank. Assume that $1 \leq i \leq N$ let $\tau = i$. Now, since D^f matrix is fat, there exists a nonsingular matrix with the proper size

$$J_l = \begin{bmatrix} \Xi & & & \\ & \ddots & & \\ & & \Xi & 0 \\ 0 & 0 & 0 & I_{p_2} \end{bmatrix} \tag{4.51}$$

where Ξ is a nonsingular constant matrix such that $\Xi D^f = \begin{bmatrix} I_{p_1 \times p_1} & X_{(p_1)\times(m-p_1)} \end{bmatrix} = \Theta$

and X could be a nonzero matrix. Then one has

$$J_l D_i = J_l \begin{bmatrix} D^f & & & & & \\ C^f B & D^f & & & & \\ \vdots & & \ddots & & & \\ C^f A^{N-3} B & & & & & \\ C^f A^{N-2} B & C^f A^{N-3} B & \dots & & & D^f \\ C^s A^{i-2} B & C^s A^{i-3} B & \dots & D^s & \dots & 0_{p_2 \times (i-1)m} \end{bmatrix} \tag{4.52}$$

$$= \begin{bmatrix} \Theta & & & & & \\ \Xi C^f B & \Theta & & & & \\ \vdots & & \ddots & & & \\ \Xi C^f A^{N-3} B & & & & & \\ \Xi C^f A^{N-2} B & \Xi C^f A^{N-3} B & \dots & & & \Theta \\ C^s A^{i-2} B & C^s A^{i-3} B & \dots & D^s & \dots & 0 \end{bmatrix} \tag{4.53}$$

Due to the genericity of matrices C^s, B, the equality $\operatorname{rank}(C^s B) = \min\{\operatorname{rank} C^s, \operatorname{rank} B\}$ holds. Furthermore, since matrix A assumes generic values, it is nonsingular and has distinct eigenvalues. It is easy to see that in the above matrix those block columns which contain $I_{p_1 \times p_1}$ are linearly independent from each other and also do not linearly depend on those block columns which contain X and D^s. Now due to genericity of matrices A, B, C^s, one can verify that other block columns which contain X have to be also linearly independent from each other. Finally, it is obvious that the block column which has D^s, is linearly independent from the block columns which contain X. Thus, the number of dependent columns is precisely $(i-1)(m-p_1)$.Thus, the multiplicity of zeros of $\sum_\tau$ at infinity is $(i-1)(m-p_1)$. This ends the proof. □

The case of zeros at infinity is considered in the above theorem and the following corollary studies the zeros at the origin.

Corollary 4.5.5. *Consider the system $\sum_\tau$, $\forall \tau \in \{1, 2, \dots, N\}$, with $p_1 < m$ and $Np_1 + p_2 > Nm$. Assume that the defining matrices $\{A, B, C^f, D^f, C^s, D^s\}$ accept generic values and the system matrix $M_\tau(Z)$, has full column normal rank. Then $M_\tau(Z)$ has zeros at $Z = 0$ with multiplicity equal to $(N-\tau)(m-p_1)$.*

Proof. Using the results of Theorem 4.5.4 and Lemma 4.4.8 the claimed statement readily follows. □

In the above results we treated the case where the normal rank of $M_\tau(Z)$ is equal to its number of columns. Now, we study the scenario where $M_\tau(Z)$ has less than full-column

rank.

Theorem 4.5.6. *Consider the system $\sum_\tau$, $\tau \in \{1, 2, \ldots, N\}$, with $p_1 < m$, $Np_1 + p_2 > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Suppose that $M_\tau(Z)$ has less than full-column normal rank. Then $M_\tau(Z)$ has zeros at $Z = \infty$ with multiplicity equal to* $\max\{0, n - (N - \tau)(m - p_1)\}$.

Proof. Under the assumption made in the theorem statement the normal rank of $M_\tau(Z)$ is precisely $2n + m + (N - 1)p_1$ (See Lemma 4.4.8). Moreover, with help of Definition 4.3.1 the multiplicity of zeros at infinity for the system matrix $M_\tau(Z)$ is equal to

$$\operatorname{nrank} M_\tau(Z) - n - \operatorname{rank} D_\tau = n + m + (N - 1)p_1 - \operatorname{rank} D_\tau \,.$$

Using the same argument provided in the previous proposition one can easily observe that $\operatorname{rank} D_\tau = (N - \tau + 1)m + (\tau - 1)p_1$. Hence, the multiplicity of zeros for the system matrix $M_\tau(Z)$ at infinity is $\max\{0, n - (N - \tau)(m - p_1)\}$. □

Corollary 4.5.7. *Consider the system $\sum_\tau$, $\tau \in \{1, 2, \ldots, N\}$, with $p_1 < m$, $Np_1 + p_2 > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Suppose that $M_\tau(Z)$ has less than full-column normal rank. Then $M_\tau(Z)$ has zeros at $Z = 0$ with multiplicity equal to* $\max\{0, n - (\tau - 1)(m - p_1)\}$.

Proof. Using the results of Theorem 4.5.6 and Lemma 4.4.8 the claim of this corollary is immediate. □

Remark 4.5.8. The above results reveal that, if the matrices A, B, C^f, D^f, C^s, D^s assume generic values with $p_1 < m$ and $Np_1 + p_2 > Nm$, then when $\tau = 1$ all zeros are at the origin and no zero are at infinity; on the other hand, when $\tau = N$ all zeros are at infinity and there are no zeros at the origin.

So far we have examined the zero properties of blocked systems for the choice $p_1 < m$. Now, we consider the case $p_1 = m$. We first give attention to the zero properties of the blocked system $\sum_\tau$ at $Z = 0$ and then at $Z = \infty$. Let us consider a submatrix of $M_\tau(Z)$, $\tau \in \{1, 2, \ldots, N\}$, that is

$$M^f(Z) := \begin{bmatrix} ZI - \mathcal{A} & -\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}. \tag{4.54}$$

where $\mathcal{C}$ denotes the first Np_1 rows of the matrix C_τ, $\tau \in \{1, 2, \ldots, N\}$, and $\mathcal{D}$ is the $Np_1 \times Nm$ square matrix defined using the top Np_1 rows of D_τ, $\tau \in \{1, 2, \ldots, N\}$. Finally, $\mathcal{A} = A_1$ and $\mathcal{B} = B_1$.

Proposition 4.5.9. *For a generic choice of the matrices* $\{A,\, B,\, C^f,\, D^f\}$*,* $p_1 = m$*, the system matrix of* $M^f(0)$ *has full rank.*

Proof. We use a contradiction to prove the theorem statement. Assume that the system matrix $M^f(0)$ has rank less than full rank; then using the decomposition originally provided in (Zamani et al., 2011), we can write

$$\begin{aligned}\begin{bmatrix} -\mathcal{A} & -\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = &\begin{bmatrix} -A & 0 & \dots & 0 & -B \\ 0 & -I_m & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \\ & & & & \\ 0 & \dots & 0 & -I_m & 0 \\ C^f & 0 & \dots & 0 & D^f \end{bmatrix} \dots \\ &\begin{bmatrix} -A & 0 & -B & 0 & \dots & 0 \\ 0 & -I_m & 0 & 0 & \dots & 0 \\ C^f & 0 & D^f & & & \\ 0 & \dots & 0 & -I_m & & \\ \vdots & & & & \ddots & \\ 0 & & & & & -I_m \end{bmatrix} \\ &\begin{bmatrix} -A & -B & 0 & \dots & 0 \\ C^f & D^f & & & \\ 0 & & -I_m & & \\ \vdots & & & \ddots & \\ 0 & & & & -I_m \end{bmatrix}.\end{aligned} \tag{4.55}$$

With each matrix in the product being square, then the matrix

$$\begin{bmatrix} -A & -B \\ C^f & D^f \end{bmatrix}$$

attains rank less than full rank for a set of generic matrices A, B, C^f and D^f, which is a contradiction. □

Theorem 4.5.10. *For a generic choice of the matrices* $\{A,\, B,\, C^f,\, C^s,\, D^f,\, D^s\}$ *with* $p_1 = m$ *the system* $\sum_\tau$*,* $\tau \in \{1,\, 2,\, \dots,\, N\}$*, has no zero at* $Z = 0$

Proof. With the help of the above proposition, the result is immediate. □

To complete this part of the analysis we provide the following result for infinite zeros.

Theorem 4.5.11. *Consider the system* $\sum_\tau$, $\forall \tau \in \{1, \ldots, N\}$, *with* $p_1 = m$. *Then for generic values of the defining matrices* $\{A, B, C^f, D^f, C^s, D^s\}$ *the system matrix* $M_\tau(Z)$, $\tau \in \{1, 2 \ldots, N\}$ *always has no zero at* $Z = \infty$.

Proof. The proof is immediate by recalling Lemma 4.5.1. □

Several theorems and propositions have been introduced in this chapter about the zero properties of the system $\sum_\tau$ given a generic underlying multirate system. Accordingly, we summarize results from Theorem 4.4.2 to Theorem 4.5.11 in the table below.

Zero \ Region	$p_1 \geq m$	$p_1 < m$, $Np_1 + p_2 > Nm$
Finite nonzero zeros	No	No
Zeros at zero	No	Zeros can be at these points depending on τ and n.
Zeros at infinity	No	

Table 4.1: Summarizing Results from Theorem 4.4.2 to Theorem 4.5.11

4.6 Examples and simulations

In order to support the results obtained on the zeros of multirate systems, some examples and simulations are reported below.

Example 4.6.1. Consider a tall multi-rate system with $n = 1$, $m = 3$, $N = 2$, $p_1 = 1$, $p_2 = 5$. Let the parameter matrices for the multi-rate system be $A = a$, $B = [b_1\, b_2\, b_3]$, $C = [c^{f\top} C^{s\top}]^\top$, $C^s = [c_1^s\ c_2^s\ c_3^s\ c_4^s\ c_5^s]^\top$, $D^f = [d_1^f\ d_2^f\ d_3^f]$ and

$$D^s = \begin{bmatrix} d_{11}^s & d_{12}^s & d_{13}^s \\ \vdots & \vdots & \vdots \\ d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$

First, consider $\tau = 1$ and write the associated system matrix as

$$M_1(Z) = \begin{bmatrix} Z - a^2 & -ab_1 & -ab_2 & -ab_3 & -b_1 & -b_2 & -b_3 \\ c^f & d_1^f & d_2^f & d_3^f & 0 & 0 & 0 \\ c^f a & c^f b_1 & c^f b_2 & c^f b_3 & d_1^f & d_2^f & d_3^f \\ c_1^s a & c_1^s b_1 & c_1^s b_2 & c_1^s b_3 & d_{11}^s & d_{12}^s & d_{13}^s \\ c_2^s a & c_2^s b_1 & c_2^s b_2 & c_2^s b_3 & d_{21}^s & d_{22}^s & d_{23}^s \\ c_3^s a & c_3^s b_1 & c_3^s b_2 & c_3^s b_3 & d_{31}^s & d_{32}^s & d_{33}^s \\ c_4^s a & c_4^s b_1 & c_4^s b_2 & c_4^s b_3 & d_{41}^s & d_{42}^s & d_{43}^s \\ c_5^s a & c_5^s b_1 & c_5^s b_2 & c_5^s b_3 & d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$

It is clear that the first two rows are linearly independent. Now, consider rows 3 to 8; they can be written as

$$\begin{bmatrix} c^f & c^f & c^f & c^f & d_1^f & d_2^f & d_3^f \\ c_1^s & c_1^s & c_1^s & c_1^s & d_{11}^s & d_{12}^s & d_{13}^s \\ c_2^s & c_2^s & c_2^s & c_2^s & d_{21}^s & d_{22}^s & d_{23}^s \\ c_3^s & c_3^s & c_3^s & c_3^s & d_{31}^s & d_{32}^s & d_{33}^s \\ c_4^s & c_4^s & c_4^s & c_4^s & d_{41}^s & d_{42}^s & d_{43}^s \\ c_5^s & c_5^s & c_5^s & c_5^s & d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix} \operatorname{diag}(a, b_1, b_2, b_3, I_3) := G \operatorname{diag}(a, b_1, b_2, b_3, I_3).$$

The matrix G has rank at most 4; hence, with generic parameter matrices the normal rank of $M(Z)$ equals 6. Furthermore, it is easy to observe that the system matrix has a zero at $Z = 0$. However, for $\tau = 2$ we can write the system matrix $M_2(Z)$ as

$$M_2(Z) = \begin{bmatrix} Z - a^2 & -ab_1 & -ab_2 & -ab_3 & -b_1 & -b_2 & -b_3 \\ c^f & d_1^f & d_2^f & d_3^f & 0 & 0 & 0 \\ c^f a & c^f b_1 & c^f b_2 & c^f b_3 & d_1^f & d_2^f & d_3^f \\ c_1^s & d_{11}^s & d_{12}^s & d_{13}^s & 0 & 0 & 0 \\ c_2^s & d_{21}^s & d_{22}^s & d_{23}^s & 0 & 0 & 0 \\ c_3^s & d_{31}^s & d_{32}^s & d_{33}^s & 0 & 0 & 0 \\ c_4^s & d_{41}^s & d_{42}^s & d_{43}^s & 0 & 0 & 0 \\ c_5^s & d_{51}^s & d_{52}^s & d_{53}^s & 0 & 0 & 0 \end{bmatrix}.$$

Observe that the normal rank of the system matrix is still 6 and the matrix D_2 (with its nonzero entries assuming generic values) has rank 4; hence, the only zero of the system matrix is now at infinity.

In the following we present some numerical simulations. In each experiment, the

parameter matrices were generated using random numbers. To keep numerical stability, we generated diagonal matrices A, with random diagonal elements picked in the interval $[0.5,\ 1]$. This choice is motivated by the fact that, if N is large, then the elements of A^N may diverge or converge to zero exponentially fast. The normal rank of $M(Z)$ is computed as $\operatorname{rank} M(Z_0)$, $Z_0 \neq 0$.

Example 4.6.2. Consider the following system, with

$$n = 3 \qquad m = 3 \qquad p_1 = 1 \qquad p_2 = 10 \qquad N = 4$$

and $\tau = \{1, \ldots, 4\}$. The corresponding matrices, for $\tau = 4$ are

$$A = \begin{bmatrix} 0.9865 & 0 & 0 \\ 0 & 0.8245 & 0 \\ 0 & 0 & 0.9002 \end{bmatrix} \qquad B = \begin{bmatrix} 0.4538 & 0.0835 & 0.3909 \\ 0.4324 & 0.1332 & 0.8314 \\ 0.8253 & 0.1734 & 0.8034 \end{bmatrix}$$

$$C = \begin{bmatrix} C^f \\ C^s \end{bmatrix} = \begin{bmatrix} 0.1814 & 1.1978 & 1.5806 \\ 2.0840 & 0.9906 & 4.7137 \\ 3.2843 & 2.4484 & 2.0887 \\ 3.1399 & 1.6975 & 4.9153 \\ 1.4599 & 4.7582 & 1.5073 \\ 2.1583 & 4.6017 & 3.5055 \\ 0.0774 & 0.2634 & 3.3317 \\ 4.9203 & 3.6893 & 2.6956 \\ 0.8358 & 1.3456 & 3.4905 \\ 0.5311 & 2.1142 & 3.3326 \\ 1.8620 & 2.7394 & 0.8907 \end{bmatrix} \qquad D = \begin{bmatrix} D^f \\ D^s \end{bmatrix} = \begin{bmatrix} 1.0241 & 7.9926 & 1.3690 \\ 0.1304 & 2.5791 & 1.0072 \\ 2.2448 & 1.5051 & 1.1618 \\ 3.5275 & 0.7637 & 2.4684 \\ 2.6767 & 1.7130 & 1.0611 \\ 0.7617 & 1.9281 & 3.2975 \\ 1.4757 & 0.4824 & 3.9307 \\ 1.8429 & 2.3580 & 2.9210 \\ 3.9266 & 0.9048 & 1.3755 \\ 0.6256 & 1.5385 & 2.3363 \\ 3.4221 & 2.3319 & 0.4311 \end{bmatrix}$$

The normal rank of $M(Z)$ turns out to be 12, while the number of columns is 15. For different values of τ we obtain the following zeros.

value of τ \ # of zeros	0	∞
1	3	0
2	1	0
3	0	1
4	0	3

In this example, $n < (N-1)(m-p_1) = 6$, hence the system matrix is not full normal rank.

Example 4.6.3. In this example, we consider several systems where we keep the same dimensions considered in the previous example and increase n.

Case n = 4

value of τ \ # of zeros	0	∞
1	4	0
2	2	0
3	0	2
4	0	4

The normal rank of $M(Z)$ turns out to be 14, while the number of columns is 16.

Case n = 5

value of τ \ # of zeros	0	∞
1	5	0
2	3	1
3	1	3
4	0	5

The normal rank of $M(Z)$ turns out to be 16, while the number of columns is 17.

Case n = 6

value of τ \ # of zeros	0	∞
1	6	0
2	4	2
3	2	4
4	0	6

The normal rank of $M(Z)$ turns out to be 18, while the number of columns is 18. The matrix is full normal rank, as $n = (N-1)(m-p_1)$.

Case n = 7.

value of τ \ # of zeros	0	∞
1	6	0
2	4	2
3	2	4
4	0	6

Note that now $n > (N-1)(m-p_1)$, but the maximum number of zeros is $(N-1)(m-p_1) = 6$. Moreover, the position of the zeros is the same of the case $n = 6$. For $n > 7$, the position of the zeros remain the same.

Example 4.6.4. We report another set of systems, where $m = 4$, $p_1 = 2$, $p_2 = 22$, $N = 6$ and n increases from 8 to 12.

Case n = 8.

value of τ \ # of zeros	0	∞
1	8	0
2	6	0
3	4	2
4	2	4
5	0	6
6	0	8

The normal rank of $M(Z)$ turns out to be 30, while the number of columns is 32.

Case n = 9.

value of τ \ # of zeros	0	∞
1	9	0
2	7	1
3	5	3
4	3	5
5	1	7
6	0	9

The normal rank of $M(Z)$ turns out to be 32, while the number of columns is 33.

Case n = 10.

value of τ \ # of zeros	0	∞
1	10	0
2	8	2
3	6	4
4	4	6
5	2	8
6	0	10

The normal rank of $M(Z)$ turns out to be 34, as the number of columns, since $n = (N-1)(m-p_1)$. The cases $n = 11$ and $n = 12$ have the same properties of the case $n = 10$.

Example 4.6.5. In this example we consider a system with parameters dimensions

$$n = 5 \qquad m = 5 \qquad p_1 = 3 \qquad p_2 = 24 \qquad N = 8\,.$$

The resulting zeros at the origin and infinity are reported below

value of τ \ # of zeros	0	∞
1	5	0
2	3	0
3	1	0
4	0	0
5	0	0
6	0	1
7	0	3
8	0	5

This example suggests that, if n is suitably smaller than the threshold $(N-1)(m-p_1)$, then there may be some values of τ (always different from 1 and N) for which the system $\sum_\tau$ is completely zero-free.

5

Identifiability of errors-in-variables models

5.1 Introduction

The identification of errors-in-variables (EIV) models is a "classical" subject which has been studied in the statistical literature since the beginning of the last century and has generated many papers, among which (Gini, 1921), (Frisch, 1934), (Madansky, 1959), (Kalman, 1982), initially dealing with static EIV models only. In the following years attention has been shifting more towards *dynamic* EIV models and their use in system identification, (Söderström, 1981), (Anderson, 1985) since these models provide a more realistic description of the situation encountered in many practical instances where the input signal may also be affected by "noise" or by random errors of various kinds. This in contrast to the use of standard ARMAX or Box-Jenkins models, where the input signal is invariably supposed to be measured exactly by the data acquisition device.

Several techniques have been proposed for the analysis and the identification of dynamic EIV models; see e.g. (Fernando & Nicholson, 1985), (Stoica et al., 1995), (Beghelli et al., 1990), (Zheng & Feng, 1992), (Tugnait, 1992), (Zheng, 1999), (Song & Chen, 2008). Yet, a main difficulty with EIV models is that they are generally non-identifiable. This is by now well-known, in particular for dynamic EIV models, and many papers have appeared dealing with identifiability of general dynamic EIV model structure

such as (Anderson & Deistler, 1984), (Solo, 1986), (Picci & Pinzoni, 1986), (Deistler & Anderson, 1989), (Schachermayer & Deistler, 1998), (Scherrer & Deistler, 1998) and (Aguero & Goodwin, 2008). In order to overcome this structural difficulty, dynamic EIV models with *white measurement errors* have recently been considered and identification of these models is now a rather active research subject. Although the model class is rather restricted because of the assumption of white measurement errors, it appear to be a natural and tractable generalization of ARMAX or output-error (OE) models, where the standard identification techniques may generalize naturally. Indeed this restricted model class turns out to be "generically identifiable" (where the attribute "generic" can here be given an intuitive meaning of "almost always"). Yet, it has been pointed out by (Stoica & Nehorai, 1987) and (Picci et al., 1993) that in certain circumstances there may be *two* EIV models which are indistinguishable from external input-output experiments. This lack of (global) identifiability, although it should hopefully almost never be encountered in practice, is a fact which needs better understanding. For several reasons, the first being obviously the desire of guaranteeing the well-posedness of the parameter estimation phase in all circumstances. The second is that in statistical estimation it is desirable to know when the estimation problem is near to an ill-conditioning situation. For it is a general fact in system identification that the variance of parameter estimates is related to certain indices measuring the degree of identifiability (for example the condition number of the Fisher matrix) and being "close" to non-identifiability may lead to poor or unreliable estimates. Further, we note that a motivation for our work is also the fact that the study of identification algorithms of dynamic EIV models with white measurement errors has greatly advanced in recent years, and the proposed techniques seem to be quite mature to become standard tools in applications, see e.g. (Söderström, 2007), (Söderström et al., 2002), (Söderström et al., 2003), (Chen & Yang, 2005), (Guidorzi & Diversi, 2009), (Diversi & Guidorzi, 2009) and (Söderström et al., 2009).

For these, besides other more "theoretical" reasons, we believe that a better understanding of identifiability of EIV models with white measurement errors is necessary. In particular we need better understanding of when we may be *close* to a situation where two different models may describe the data equally well.

Brief review of the literature

Several papers are concerned with the study of the identifiability of EIV models. Their contributions to the literature can be divided in two categories. The first one addresses the problem of EIV models with colored measurement errors and the main results are focused on describing the classes of equivalent EIV models given a statistical input-output

description (see (Schachermayer & Deistler, 1998), (Scherrer & Deistler, 1998) and (Aguero & Goodwin, 2008)).

The second category includes EIV models with white measurement errors (also known as Frisch scheme). Here, the identifiability conditions found in the literature, e.g. in (Anderson & Deistler, 1984), (Stoica & Nehorai, 1987) (Castaldi & Soverini, 1996) guarantee uniqueness of the EIV description under certain coprimality assumptions on the (rational) transfer function of the "true" system and the spectral density of the "true" input. Unfortunately these conditions are *not testable* since they concern precisely the unknowns of the problem which are not available to the experimenter. Ideally, identifiability conditions should instead be expressible in terms of the "external" description of the observable signals; namely their joint power spectral densities.

Contribution of this work

In this chapter we take precisely the latter point of view. We shall provide conditions on the spectral densities of the external (measurable) signals under which a SISO EIV structure with white measurement noises is *non-identifiable*. Our conditions state that a necessary and "almost" sufficient condition for non-identifiability is the existence of a linear affine relation between the spectra of the two external signals. The "almost" sufficiency of this condition has to do with the nonlinear constraint of positivity of the variances of the additive noises. We provide conditions on the parameters of the affine relation under which the condition is actually also sufficient. Furthermore, we provide some numerical example which illustrate our conditions for non-identifiability and we describe a possible application in a simulated real scenario.

It is worth remarking that, in contrast to the existing literature, the identifiability conditions which will be described in this chapter do not require to assume any fixed order polynomial structure for the input and output spectra.

5.2 Background on dynamic errors-in-variables models

Consider a pair of real scalar second-order stationary zero-mean discrete-time stochastic processes $(\mathbf{y},\, \mathbf{u})$, whose joint spectral density is a rational matrix function

$$S(z) = \begin{bmatrix} S_{\mathbf{y}}(z) & S_{\mathbf{yu}}(z) \\ S_{\mathbf{uy}}(z) & S_{\mathbf{u}}(z) \end{bmatrix} \quad , \, z \in \mathbb{C}\,, \tag{5.1}$$

which will be assumed positive definite almost everywhere on the unit circle $\{|z| = 1\}$. Recall that by Hermitian symmetry of the spectrum we have $S_{\mathbf{yu}}(z) = S_{\mathbf{uy}}(z^{-1})$.

The background motivation for EIV models is to describe the pair $(\mathbf{y},\ \mathbf{u})$ as "measurements corrupted by additive noise" of two "true" (unobservable) stochastic processes denoted $\mathbf{x}$ and $\mathbf{z}$, which are related by some time-invariant linear relation described by a rational transfer function $G(z)$, $z \in \mathbb{C}$. For the moment we shall just require that G should be such that the following stochastic integral is well defined:

$$\mathbf{z}(t) = \int_{-\pi}^{\pi} e^{j\theta t} G(e^{j\theta})\, d\hat{\mathbf{x}}(e^{j\theta})\,, \tag{5.2}$$

where $\hat{\mathbf{x}}(e^{j\theta})$ is the spectral representative of $\mathbf{x}$, (Rozanov, 1967, p. 34). This relation is customarily written in symbolic form as $\mathbf{z}(t) = G(z)\mathbf{x}(t)$. We shall therefore make no assumptions on $G(z)$ like causality, stability or other. The model can then be described by the equations (see Figure 5.1):

$$\left\{ \begin{array}{rcl} \mathbf{y}(t) & = & G(z)\mathbf{x}(t) + \mathbf{e}_{\mathbf{y}}(t) \\ \mathbf{u}(t) & = & \mathbf{x}(t) + \mathbf{e}_{\mathbf{u}}(t) \end{array} \right. . \tag{5.3}$$

where the processes $\mathbf{e}_{\mathbf{u}}(t)$ and $\mathbf{e}_{\mathbf{y}}(t)$ called "measurement noises" are mutually uncorrelated and uncorrelated also with the process $\mathbf{x}(t)$.

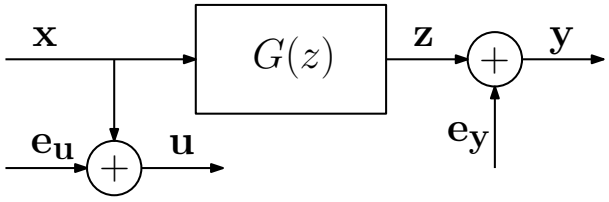


Figure 5.1: Scheme of EIV model.

Note that even in the case when $G(z)$ is assumed causal, the "causal" appearance of (5.3) is actually misleading. According to the standard notions of causality in the literature (Granger, 1963; Caines, 1988) it is in fact generally not true that $\mathbf{y}(t)$ is *caused* by $\mathbf{u}(t)$, as it is easy to check that in general there is feedback from one variable to the other. The pair of processes $\mathbf{y}(t)$ and $\mathbf{u}(t)$ is feedback-free if and only if the condition

$$\mathbb{E}\left[\mathbf{y}(s)|\mathcal{H}(\mathbf{u})\right] = \mathbb{E}\left[\mathbf{y}(s)|\mathcal{H}_t^-(\mathbf{u})\right] \tag{5.4}$$

is satisfied (Gevers & Anderson, 1981). Clearly, in the EIV case, writing

$$\mathbf{y}(t) = G(z)\mathbf{u}(t) - G(z)\mathbf{e}_{\mathbf{u}}(t) + \mathbf{e}_{\mathbf{y}}(t)\,, \tag{5.5}$$

it is easy to check that condition (5.6) generically does not hold. Hence, from an external point of view, the pair $(\mathbf{y}(t),\, \mathbf{u}(t))$, although generated by the model (5.3), could be described equally well by the feedback model (Figure 5.2)

$$\begin{cases} \mathbf{y}(t) &= F(z)\mathbf{u}(t) + L(z)\mathbf{e}_1(t) \\ \mathbf{u}(t) &= H(z)\mathbf{y}(t) + K(z)\mathbf{e}_2(t) \end{cases}, \tag{5.6}$$

where $F(z)$ is strictly causal, $H(z)$, $L(z)$ and $K(z)$ are causal and $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$ are white noise such that $\mathbb{E}\left[\mathbf{e}_1(t)\mathbf{e}_2(s)\right] = 0$, for all $t,\, s \in \mathbb{Z}$. We shall nevertheless agree to call $\mathbf{u}$

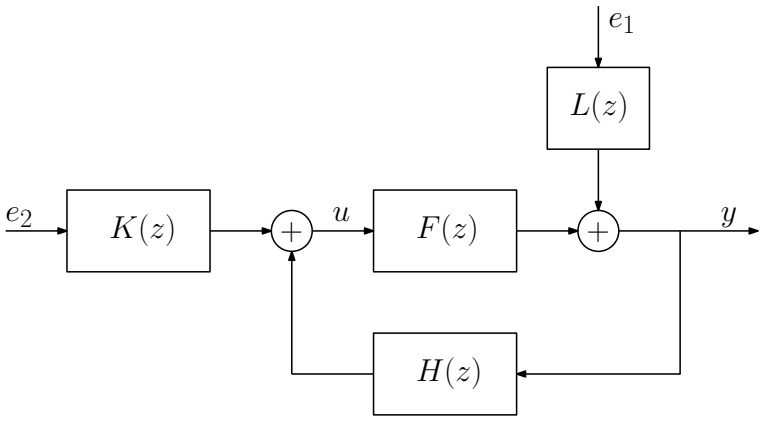


Figure 5.2: Scheme of a feedback model.

the *input* and $\mathbf{y}$ the *output* processes.

As it is well-known (Anderson & Deistler, 1984), any joint spectral density matrix $S(z)$ admits decompositions of the form, $S(z) = \hat{S}(z) + \tilde{S}(z)$ where the "true" spectrum $\hat{S}(z)$ has rank one almost everywhere on the unit circle, that is

$$S_{\mathbf{z}}(z)S_{\mathbf{x}}(z) = S_{\mathbf{yu}}(z)S_{\mathbf{uy}}(z) \qquad \forall z : |z| = 1 \tag{5.7}$$

and $\tilde{S}(z)$ is a diagonal spectral density. It is then easy to see that by defining new variables $\mathbf{x}$, $\mathbf{z}$, $\mathbf{e_u}$, $\mathbf{e_y}$ such that

$$\hat{S}(z) = \begin{bmatrix} S_{\mathbf{z}}(z) & S_{\mathbf{zx}}(z) \\ S_{\mathbf{xz}}(z) & S_{\mathbf{x}}(z) \end{bmatrix}, \quad \tilde{S}(z) = \begin{bmatrix} S_{\mathbf{e_y}}(z) & 0 \\ 0 & S_{\mathbf{e_u}}(z) \end{bmatrix}, \tag{5.8}$$

where $S_{\mathbf{zx}}(z) := S_{\mathbf{yu}}(z)$, one has indeed a representation of the form (5.3) with $G(z) := S_{\mathbf{zx}}(z)/S_{\mathbf{x}}(z)$. Hence all joint spectra (5.1) admit EIV representations.

In this dissertation however we shall only consider *EIV models with white measurement*

errors, where

$$S_{\mathbf{e_y}}(z) = \sigma^2_{\mathbf{y}} \quad , \quad S_{\mathbf{e_u}}(z) = \sigma^2_{\mathbf{u}} \tag{5.9}$$

and refer to them simply as "EIV models" hereafter. Clearly not all joint spectra admit EIV representations of this kind.

The family of EIV models with white measurement errors

For brevity, we shall say that an EIV model (5.3) is a *realization* of the joint spectrum of the $(\mathbf{y},\, \mathbf{u})$ processes it represents. This joint spectrum is in a sense an "external" description of the $(\mathbf{y},\, \mathbf{u})$ processes which is uniquely attached to them, while specifying an EIV description requires the introduction of additional nonobservable variables - the *latent* variables - so that there are in general many EIV realizations of the same spectrum. A basic identifiability question of EIV models that has been studied in the literature and we shall also address in this chapter is how many different EIV models can realize the same rational joint spectrum (5.1).

A first observation to be made is that, given the joint spectrum, the family of EIV models realizing it can be parametrized in terms of the two variances $(\sigma^2_{\mathbf{y}},\, \sigma^2_{\mathbf{u}})$, subjected to a *non-negativity* plus a *rank one condition* which we illustrate below. Letting

$$\begin{aligned} R_{\mathbf{y}}(z) &:= S_{\mathbf{y}}(z) - \frac{S_{\mathbf{yu}}(z) S_{\mathbf{uy}}(z)}{S_{\mathbf{u}}(z)} \\ R_{\mathbf{u}}(z) &:= S_{\mathbf{u}}(z) - \frac{S_{\mathbf{uy}}(z) S_{\mathbf{yu}}(z)}{S_{\mathbf{y}}(z)} \,, \end{aligned} \tag{5.10}$$

the *non-negativity constraint* (see e.g. (Anderson, 1985)) is

$$\begin{aligned} 0 \leq \sigma^2_{\mathbf{y}} \leq \bar{R}_{\mathbf{y}} &:= \min\{R_{\mathbf{y}}(z),\, z : |z| = 1\} \\ 0 \leq \sigma^2_{\mathbf{u}} \leq \bar{R}_{\mathbf{u}} &:= \min\{R_{\mathbf{u}}(z),\, z : |z| = 1\} \,. \end{aligned} \tag{5.11}$$

Given $(\sigma^2_{\mathbf{y}},\, \sigma^2_{\mathbf{u}})$, satisfying (5.11), let $S_{\mathbf{z}}(z) := S_{\mathbf{y}}(z) - \sigma^2_{\mathbf{y}}$ and $S_{\mathbf{x}}(z) := S_{\mathbf{u}}(z) - \sigma^2_{\mathbf{u}}$; then $S_{\mathbf{z}}(z)$ and $S_{\mathbf{x}}(z)$ are bona-fide spectral densities since they certainly satisfy the non-negativity constraints, $S_{\mathbf{y}}(z) - \sigma^2_{\mathbf{y}} \geq 0$ and $S_{\mathbf{u}}(z) - \sigma^2_{\mathbf{u}} \geq 0$ on the unit circle $\{\, z : |z| = 1\}$. The *rank one constraint* comes from rewriting (5.7) as

$$(S_{\mathbf{y}}(z) - \sigma^2_{\mathbf{y}})(S_{\mathbf{u}}(z) - \sigma^2_{\mathbf{u}}) = S_{\mathbf{yu}}(z) S_{\mathbf{uy}}(z) \quad \{\, z : |z| = 1\} \,. \tag{5.12}$$

It follows from a well-known result in the literature (see e.g. (Anderson, 1985) and (Picci & Pinzoni, 1986)) that if the noise variances $\sigma^2_{\mathbf{y}},\, \sigma^2_{\mathbf{u}}$ satisfy these two constraints then they are valid noise variances of an EIV model realizing the given spectrum.

Since the very definition of an EIV model entails that the cross spectral density of $\mathbf{z}$ and $\mathbf{x}$ must coincide with that of $\mathbf{y}$ and $\mathbf{u}$, we can obtain $G(z)$ from

$$G(z) = \frac{S_{\mathbf{zx}}(z)}{S_{\mathbf{x}}(z)} = \frac{S_{\mathbf{yu}}(z)}{S_{\mathbf{u}}(z) - \sigma^2_{\mathbf{u}}},$$

the reciprocal formula providing the symmetric representation of $\mathbf{x}$ in terms of $\mathbf{z}$. Our problem then reduces to investigating how many pairs $(\sigma^2_{\mathbf{y}}, \sigma^2_{\mathbf{u}})$ can lead to EIV realizations of a given joint spectral density. To avoid trivial pathological cases of non uniqueness, from now on we shall assume that $S_{\mathbf{yu}}(z)S_{\mathbf{uy}}(z)$ is not identically zero and that neither $\mathbf{y}$ nor $\mathbf{u}$ are white noise processes.
The following result (Stoica & Nehorai, 1987; Picci et al., 1993) lies at the background of our investigations.

Theorem 5.2.1. *There are at most two pairs of noise variances $(\sigma^2_{\mathbf{y}}, \sigma^2_{\mathbf{u}})$ which satisfy condition (5.12). Equivalently, there are at most two EIV models (with white measurement errors) which are compatible with the joint spectrum* (5.1).

For ease of reference we also recall here the following obvious fact.

Lemma 5.2.2. *For every variance pair $(\sigma^2_{\mathbf{y}}, \sigma^2_{\mathbf{u}})$ satisfying the rank one condition (5.12), one of the two variance values uniquely determines the value of the other.*

5.3 Conditions for non-identifiability

Conditions under which two EIV models exist both describing the same joint spectrum (non-identifiability), have been described in (Stoica & Nehorai, 1987). However these conditions are given in terms of the unknown signal spectra and transfer function and are not testable. It is therefore important to characterize this occurrence in terms of the available "external" spectral data.
A preliminary condition is given in the following Lemma. An explicit necessary condition will follow and be given in Theorem 5.3.2 below.

Lemma 5.3.1. *If the "true" spectra $S_{\mathbf{z}}(z)$ and $S_{\mathbf{x}}(z)$ have common zeros then the model is identifiable.*

Proof. Assume that the "true" spectra $S_{\mathbf{z}}(z)$ and $S_{\mathbf{x}}(z)$ are parameterized by the noise variance pair $(\sigma'^2_{\mathbf{y}}, \sigma'^2_{\mathbf{u}})$. Suppose now that there exists another pair $(\sigma''^2_{\mathbf{y}}, \sigma''^2_{\mathbf{u}})$ leading

to a valid EIV model. We shall show that $(\sigma_{\mathbf{y}}^{\prime 2}, \sigma_{\mathbf{u}}^{\prime 2}) = (\sigma_{\mathbf{y}}^{\prime\prime 2}, \sigma_{\mathbf{u}}^{\prime\prime 2})$. Let, without loss of generality, there be one common zero, z_0. Then it must be

$$\begin{aligned} 0 &= (S_{\mathbf{y}}(z_0) - \sigma_{\mathbf{y}}^{\prime 2})(S_{\mathbf{u}}(z_0) - \sigma_{\mathbf{u}}^{\prime 2}) = \\ &= S_{\mathbf{yu}}(z_0) S_{\mathbf{uy}}(z_0) = (S_{\mathbf{y}}(z_0) - \sigma_{\mathbf{y}}^{\prime\prime 2})(S_{\mathbf{u}}(z_0) - \sigma_{\mathbf{u}}^{\prime\prime 2}) \,, \end{aligned}$$

which implies that $(S_{\mathbf{y}}(z_0) - \sigma_{\mathbf{y}}^{\prime\prime 2}) = 0$ or $(S_{\mathbf{u}}(z_0) - \sigma_{\mathbf{u}}^{\prime\prime 2}) = 0$. In the first case we must have $(S_{\mathbf{y}}(z_0) - \sigma_{\mathbf{y}}^{\prime\prime 2}) = 0 = (S_{\mathbf{y}}(z_0) - \sigma_{\mathbf{y}}^{\prime 2})$, and so $\sigma_{\mathbf{y}}^{\prime} = \sigma_{\mathbf{y}}^{\prime\prime}$. Recalling Lemma 5.2.2 it also follows that $\sigma_{\mathbf{u}}^{\prime} = \sigma_{\mathbf{u}}^{\prime\prime}$. Similarly $(S_{\mathbf{u}}(z_0) - \sigma_{\mathbf{u}}^{\prime\prime 2}) = 0$ implies the claim. □

We can now state our characterization of non identifiability.

Theorem 5.3.2. *If there are two EIV models realizing the same joint spectrum, then there are constants $L > 0$ and K such that the following* ***linear-affine relation*** *holds*

$$S_{\mathbf{y}}(z) = L S_{\mathbf{u}}(z) + K \,. \tag{5.13}$$

Assume the pair $(\sigma_{\mathbf{y}}^{\prime 2},\ \sigma_{\mathbf{u}}^{\prime 2})$ parametrizes an EIV realization with true signal spectra $S_{\mathbf{z}}(z),\ S_{\mathbf{x}}(z)$. Then if there is another model realizing the same joint spectrum, it must have the following structure:

$$S_{\mathbf{y}}(z) = L S_{\mathbf{x}}(z) + \sigma_{\mathbf{y}}^{\prime\prime 2} \,, \quad S_{\mathbf{u}}(z) = L^{-1} S_{\mathbf{z}}(z) + \sigma_{\mathbf{u}}^{\prime\prime 2} \,, \tag{5.14}$$

so that one model is obtained by switching and renormalizing the true spectra of the other.

Proof. Assume there are two distinct variance pairs $(\sigma_{\mathbf{y}}^{\prime 2}, \sigma_{\mathbf{u}}^{\prime 2})$ and $(\sigma_{\mathbf{y}}^{\prime\prime 2}, \sigma_{\mathbf{u}}^{\prime\prime 2})$ describing two EIV realizations of the same joint spectrum. From equation (5.12) it must hold that

$$\begin{aligned} (S_{\mathbf{y}}(z) - \sigma_{\mathbf{y}}^{\prime 2})(S_{\mathbf{u}}(z) - \sigma_{\mathbf{u}}^{\prime 2}) &= S_{\mathbf{yu}}(z) S_{\mathbf{uy}}(z) \\ (S_{\mathbf{y}}(z) - \sigma_{\mathbf{y}}^{\prime\prime 2})(S_{\mathbf{u}}(z) - \sigma_{\mathbf{u}}^{\prime\prime 2}) &= S_{\mathbf{yu}}(z) S_{\mathbf{uy}}(z) \,. \end{aligned} \tag{5.15}$$

are simultaneously true. Subtracting the second equation from the first we obtain

$$(\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{u}}^{\prime 2}) S_{\mathbf{y}}(z) = (\sigma_{\mathbf{y}}^{\prime 2} - \sigma_{\mathbf{y}}^{\prime\prime 2}) S_{\mathbf{u}}(z) + (\sigma_{\mathbf{y}}^{\prime\prime 2}\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{y}}^{\prime 2}\sigma_{\mathbf{u}}^{\prime 2}) \,. \tag{5.16}$$

Being $(\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{u}}^{\prime 2}) \neq 0$ we can rewrite (5.16) as

$$S_{\mathbf{y}}(z) = \frac{\sigma_{\mathbf{y}}^{\prime 2} - \sigma_{\mathbf{y}}^{\prime\prime 2}}{\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{u}}^{\prime 2}} S_{\mathbf{u}}(z) + \frac{\sigma_{\mathbf{y}}^{\prime\prime 2}\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{y}}^{\prime 2}\sigma_{\mathbf{u}}^{\prime 2}}{\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{u}}^{\prime 2}} \,, \tag{5.17}$$

which, denoting

$$L = \frac{\sigma_{\mathbf{y}}^{\prime 2} - \sigma_{\mathbf{y}}^{\prime\prime 2}}{\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{u}}^{\prime 2}} \quad , \quad K = \frac{\sigma_{\mathbf{y}}^{\prime\prime 2}\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{y}}^{\prime 2}\sigma_{\mathbf{u}}^{\prime 2}}{\sigma_{\mathbf{u}}^{\prime\prime 2} - \sigma_{\mathbf{u}}^{\prime 2}} \,, \tag{5.18}$$

leads to (5.13). From (5.16) we can also obtain

$$S_{\mathbf{y}}(z) = \frac{\sigma_{\mathbf{y}}'^2 - \sigma_{\mathbf{y}}''^2}{\sigma_{\mathbf{u}}''^2 - \sigma_{\mathbf{u}}'^2}(S_{\mathbf{u}}(z) - \sigma_{\mathbf{u}}'^2) + \sigma_{\mathbf{y}}''^2 = LS_{\mathbf{x}}(z) + \sigma_{\mathbf{y}}''^2$$

and $S_{\mathbf{u}}(z) = L^{-1}S_{\mathbf{z}}(z) + \sigma_{\mathbf{u}}''^2$. Finally, that L is always positive follows since, as pointed out in (Anderson & Deistler, 1984), the admissible variance pairs lay on a hyperbola. Hence whenever $\sigma_{\mathbf{y}}''^2 > \sigma_{\mathbf{y}}'^2$, necessarily $\sigma_{\mathbf{u}}''^2 < \sigma_{\mathbf{u}}'^2$, and conversely. One can then see that for any variance pair determining two (non-identifiable) EIV models one has $L > 0$. □

Sufficiency of the linear-affine relation

Theorem 5.3.2 provides a nice and clean necessary condition for non-identifiability of EIV models. In this section we shall take up the question of assessing when the linear-affine relation (5.13) is also sufficient for non-identifiability. Naturally, we shall have to assume that the joint spectrum (5.1) admits EIV realizations. We shall look for conditions depending on the parameters K and L alone and not on the model variances. We shall first study a particular case.

EIV models with an All-Pass transfer function

Consider EIV models with an all-pass transfer function, namely

$$G(z) = \sqrt{L}\frac{\prod_i (z - z_i)}{\prod_i (z - \bar{z}_i^{-1})} \quad , \quad L > 0\,, \tag{5.19}$$

where the zeros z_i may be repeated. It is easy to check that in this case the linear affine relation is satisfied. In fact, from $G(z)G^*(z) = L$, the true spectra must satisfy

$$S_{\mathbf{z}}(z) = G(z)G^*(z)S_{\mathbf{x}}(z) = LS_{\mathbf{x}}(z) \tag{5.20}$$

and since $S_{\mathbf{y}}(z) = S_{\mathbf{z}}(z) + \sigma_{\mathbf{y}}^2$, summing $\sigma_{\mathbf{y}}^2$ to both members one gets $S_{\mathbf{y}}(z) = LS_{\mathbf{x}}(z) + \sigma_{\mathbf{y}}^2$. Further recalling that $S_{\mathbf{x}}(z) = S_{\mathbf{u}}(z) - \sigma_{\mathbf{u}}^2$, we arrive at

$$S_{\mathbf{y}}(z) = LS_{\mathbf{u}}(z) - L\sigma_{\mathbf{u}}^2 + \sigma_{\mathbf{y}}^2 = LS_{\mathbf{u}}(z) + K\,. \tag{5.21}$$

Hence EIV models with an all-pass transfer function satisfy the linear-affine relation. However it is easy to check that they are identifiable. In fact, since $S_{\mathbf{z}}(z) = G(z)G^*(z)S_{\mathbf{x}}(z) = LS_{\mathbf{x}}(z)$, the "true" spectra have the same zeros. From Lemma 5.3.1, we have identifiability. These models are however quite special; in a sense they correspond to a limit

situation, as explained in the following remark.

Remark 5.3.3. As it follows from equation (5.14), Theorem 5.3.2, for non-identifiable EIV models, the spectrum $S_{\mathbf{y}}(z)$ can be written $S_{\mathbf{y}}(z) = LS_{\mathbf{x}}(z) + \sigma_{\mathbf{y}}^{\prime\prime 2}$, but for all-pass transfer functions one also has $S_{\mathbf{z}}(z) = LS_{\mathbf{x}}(z)$ and therefore $\sigma_{\mathbf{y}}^{\prime\prime 2} = \sigma_{\mathbf{y}}^{\prime 2}$. For this reason even if formally there are two EIV realizations with the same all-pass transfer function, the *two realizations actually coincide.*

Checking non-identifiability

Let us consider a joint spectrum satisfying the linear-affine relation (5.13) admitting an EIV realization corresponding to a variance pair $(\sigma_{\mathbf{y}}^{\prime 2}, \sigma_{\mathbf{u}}^{\prime 2})$. By defining $V_{\mathbf{y}} := K + L\sigma_{\mathbf{u}}^{\prime 2}$ and substituting $S_{\mathbf{u}}(z) = S_{\mathbf{x}}(z) + \sigma_{\mathbf{u}}^{\prime 2}$ into (5.13), one gets a candidate alternative model

$$S_{\mathbf{y}}(z) = LS_{\mathbf{x}}(z) + V_{\mathbf{y}}\,, \tag{5.22}$$

which would prove non-identifiability just in case $V_{\mathbf{y}}$ turns out to be a valid variance $\sigma_{\mathbf{y}}^{\prime\prime 2}$. A similar argument leads to a candidate companion equation

$$S_{\mathbf{u}}(z) = L^{-1}S_{\mathbf{z}}(z) + V_{\mathbf{u}} \quad , \quad V_{\mathbf{u}} := L^{-1}(\,\sigma_{\mathbf{y}}^{\prime 2} - K)\,. \tag{5.23}$$

Hence the question of proving existence of a second valid EIV model reduces to discussing what range of parameters L and K guarantee that $V_{\mathbf{y}}$ in (5.22) is a valid output noise variance, that is, such that the corresponding variance parameters $\sigma_{\mathbf{y}}^2$ and $\sigma_{\mathbf{u}}^2$ satisfy the positivity condition (5.11). It is actually easy to show that if one of the two variances, say $\sigma_{\mathbf{y}}^2$, satisfies the inequality $\sigma_{\mathbf{y}}^2 \in [0\,, \bar{R}_{\mathbf{y}}]$, then the other inequality is automatically satisfied. For this reason we shall henceforth concentrate on $\sigma_{\mathbf{y}}^2$. We may distinguish three different situations:

- Either $V_{\mathbf{y}} < 0$ or $V_{\mathbf{y}} > \bar{R}_{\mathbf{y}}$: in this case $V_{\mathbf{y}}$ cannot be interpreted as a noise variance and (5.22) cannot give rise to a second EIV model. The given model is **identifiable**.
- $V_{\mathbf{y}} = \sigma_{\mathbf{y}}^{\prime 2}$: by lemma 5.2.2 it must also hold that $V_{\mathbf{u}}^2 = \sigma_{\mathbf{u}}^{\prime 2}$. This is the case in which the two EIV models coincide and in fact $G(z)$ is all-pass with gain L. The model is **identifiable**.
- $0 \leq V_{\mathbf{y}} \leq \bar{R}_{\mathbf{y}}\,,\ V_{\mathbf{y}} \neq \sigma_{\mathbf{y}}^{\prime 2}$: in this case $V_{\mathbf{y}}$ can be interpreted as output noise variance; i.e. $V_{\mathbf{y}} = \sigma_{\mathbf{y}}^{\prime\prime 2}$. The decomposition

$$\begin{array}{lll} S_{\mathbf{y}}(z) = LS_{\mathbf{x}}(z) + \sigma_{\mathbf{y}}^{\prime\prime 2} & , & \sigma_{\mathbf{y}}^{\prime\prime 2} = K + L\sigma_{\mathbf{u}}^{\prime 2} \\ S_{\mathbf{u}}(z) = L^{-1}S_{\mathbf{z}}(z) + \sigma_{\mathbf{u}}^{\prime\prime 2} & , & \sigma_{\mathbf{u}}^{\prime\prime 2} = L^{-1}(\,\sigma_{\mathbf{y}}^{\prime 2} - K) \end{array} \tag{5.24}$$

is another valid EIV realization of the given joint spectrum. Therefore the model is **non-identifiable**.

Figure 5.3 provides a graphical description of the three situations.

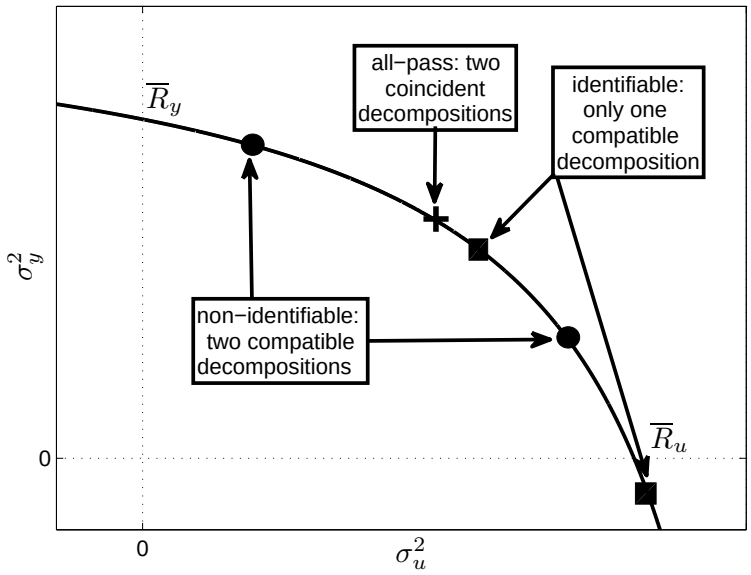


Figure 5.3: Graphical interpretation of identifiability of EIV models.

Checking non-identifiability in case of an unknown true model

Obviously when one of the variance parameters $\sigma^2_{\mathbf{y}}, \sigma^2_{\mathbf{u}}$ of an EIV model is known, it is trivial to check if $\tilde{K}$ lies in the feasible interval $[0,\ \bar{R}_{\mathbf{y}}]$. There may be situations however where it may be a priori known (perhaps from prior physical or engineering knowledge about the system) that there is a true EIV model generating the data but its variance parameters are unknown. In this case we may like to check *a priori* if $V_{\mathbf{y}}$ belongs to the feasible interval just on the basis of the parameters L and K of the linear-affine relation (which can be estimated from frequency domain data say by linear regression). Since $V_{\mathbf{y}}$ is a function also of the unknown value $\sigma'^2_{\mathbf{u}}$, we will in general be able to obtain conditions providing only partial answers. Use of these conditions is illustrated in the examples of Section 5.4.

Lemma 5.3.4. *Assume the linear-affine relation* (5.13) *holds and let* $\bar{R}_{\mathbf{y}}$ *and* $\bar{R}_{\mathbf{u}}$ *be the*

upper limits for the noise variances as defined in (5.11)*. Then:*

$$|K| \geq |\bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}| \quad \textit{and} \quad \operatorname{sgn} K = \operatorname{sgn}(\bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}})\,. \tag{5.25}$$

Moreover, if $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$*,* $K \neq 0$*, then either* $\sigma'^2_{\mathbf{y}} = \bar{R}_{\mathbf{y}}$*, or* $\sigma'^2_{\mathbf{u}} = \bar{R}_{\mathbf{u}}$*.*

Proof. Define the function $\alpha(z)$ by

$$\alpha(z) = \frac{S_{\mathbf{x}}(z)S_{\mathbf{z}}(z)}{S_{\mathbf{u}}(z)S_{\mathbf{y}}(z)}\,. \tag{5.26}$$

Note that $\alpha(z) : [-\pi, \pi] \to [0, 1[$ being the ratio of two nonnegative functions ($S_{\mathbf{x}}(z)$ and $S_{\mathbf{z}}(z)$) and the product of two strictly positive functions ($S_{\mathbf{y}}(z)$ and $S_{\mathbf{u}}(z)$) which is always greater than the numerator. Recalling (5.7) and (5.11), we can rewrite the quantity $\bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$ as

$$\min_{z:|z|=1} \{(1-\alpha(z))S_{\mathbf{y}}(z)\} - L \min_{z:|z|=1} \{(1-\alpha(z))S_{\mathbf{u}}(z)\} \tag{5.27}$$

and, by substituting $S_{\mathbf{y}}(z) = LS_{\mathbf{u}}(z) + K$

$$\min_{z:|z|=1} \{(1-\alpha(z))LS_{\mathbf{u}}(z) + (1-\alpha(z))K\} - \min_{z:|z|=1} \{(1-\alpha(z))LS_{\mathbf{u}}(z)\}\,. \tag{5.28}$$

Now let us consider the case $K > 0$; we shall show that the inequality $\bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}} > K$ cannot be satisfied. Since $\min\{f(x) + C\} = \min\{f(x)\} + C$, we can extract the constant K from the first term of (5.28) and rewrite the inequality as

$$\min_{z:|z|=1} \{(1-\alpha(z))LS_{\mathbf{u}}(z) - \alpha(z)K\} - \min_{z:|z|=1} \{(1-\alpha(z))LS_{\mathbf{u}}(z)\} > 0\,,$$

which cannot hold since the first term is always not greater than the second. Next, to show that $\bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}} > 0$, we shall use again (5.28), getting an inequality which is always true given that $K > 0$. Analogous considerations can be done for $K < 0$. For the case $K = 0$, recalling (5.28) it easily turns out that $\bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}} = 0$.
Finally, let $K \neq 0$ and $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$. This equality can be written

$$\min_{z:|z|=1} \{(1-\alpha(z))LS_{\mathbf{u}}(z) - \alpha(z)K\} - \min_{z:|z|=1} \{(1-\alpha(z))LS_{\mathbf{u}}(z)\} = 0\,.$$

Since $K \neq 0$ by assumption, this equality can only be satisfied if $\alpha(z)$ vanishes on the unit circle, namely if there are z_0, $|z_0| = 1$, such that $\alpha(z_0) = 0$. Since $\alpha(z)$ can vanish on the unit circle if and only if at least one of the two spectra in the numerator vanish, there

must be a z_0, $|z_0| = 1$, such that either $S_{\mathbf{x}}(z_0) = 0$ or $S_{\mathbf{z}}(z_0) = 0$, or both. This amounts to saying that either $\sigma'^2_{\mathbf{u}} = \bar{R}_{\mathbf{u}}$ or $\sigma'^2_{\mathbf{y}} = \bar{R}_{\mathbf{y}}$, or both equalities must be true. □

Remark 5.3.5. In the proof we have excluded the presence of zeros on the unit circle of either $S_{\mathbf{y}}(z)$ or $S_{\mathbf{u}}(z)$. The presence of such zeros would in fact imply either that $\sigma^2_{\mathbf{y}} = 0$ (i.e. $S_{\mathbf{z}}(z) = S_{\mathbf{y}}(z)$) or $\sigma^2_{\mathbf{u}} = 0$ (and $S_{\mathbf{x}}(z) = S_{\mathbf{u}}(z)$). In both cases identifiability analysis would be superfluous.

Theorem 5.3.6. *Assume the joint spectrum* (5.1) *admits EIV realizations and that the linear-affine relation* (5.13) *is satisfied. Assume also that there is no all-pass relation between the true processes. Then,*

- *if $K > \bar{R}_{\mathbf{y}}$ or $K < -L\bar{R}_{\mathbf{u}}$ we have identifiability;*
- *if $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$ we have non-identifiability.*

The situation is described by the figure below.

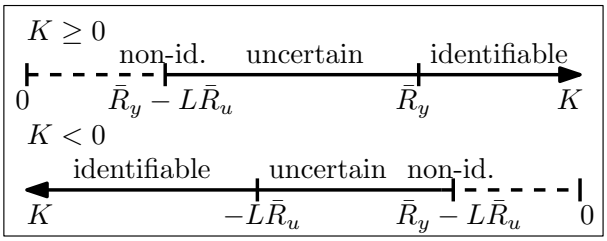


Figure 5.4: Identifiability for various values of K.

Proof. As noted earlier, for identifiability $V_{\mathbf{y}}$ must be such that

$$K + L\sigma'^2_{\mathbf{u}} < 0 \;\vee\; K + L\sigma'^2_{\mathbf{u}} > \bar{R}_{\mathbf{y}} \quad \forall\, \sigma'^2_{\mathbf{u}} \in [0,\, \bar{R}_{\mathbf{u}}]\,. \tag{5.29}$$

As $\sigma'^2_{\mathbf{u}} \leq \bar{R}_{\mathbf{u}}$, the first condition is certainly satisfied if $K < -L\bar{R}_{\mathbf{u}}$, while the second, since $L\sigma'^2_{\mathbf{u}} \geq 0$, is surely true whenever $K > \bar{R}_{\mathbf{y}}$.

When $-L\bar{R}_{\mathbf{u}} \leq K \leq \bar{R}_{\mathbf{y}}$ we may discuss three possible subcases.

First case: $K > 0$. Because of Lemma 5.3.4, the only admissible values for K are in the interval $[\bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}},\, \bar{R}_{\mathbf{y}}]$, with $\bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}} > 0$. For $K \neq \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$ we cannot say anything about identifiability. If $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$, by Lemma 5.3.4 we have either $\sigma'^2_{\mathbf{y}} = \bar{R}_{\mathbf{y}}$ or $\sigma'^2_{\mathbf{u}} = \bar{R}_{\mathbf{u}}$. If $\sigma'^2_{\mathbf{u}} = \bar{R}_{\mathbf{u}}$, being $V_{\mathbf{y}} = K + L\sigma'^2_{\mathbf{u}}$, with $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$, we obtain $V_{\mathbf{y}} = \bar{R}_{\mathbf{y}}$, which is an admissible value for $\sigma''^2_{\mathbf{y}}$. Hence we have non-identifiability. A similar argument can be used in case $\sigma'^2_{\mathbf{y}} = \bar{R}_{\mathbf{y}}$, in which case we obtain $\sigma''^2_{\mathbf{u}} = \bar{R}_{\mathbf{u}}$.

Second case: $K < 0$. Then according to Lemma 5.3.4, the admissible values for K are in the interval $[-L\bar{R}_{\mathbf{u}},\ \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}]$. By a similar analysis as in the previous case ($K > 0$), when $K \neq \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$ we cannot say anything about identifiability, while for $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$ we have non identifiability, since either $\sigma_{\mathbf{y}}^{\prime\prime 2} = \bar{R}_{\mathbf{y}}$ or $\sigma_{\mathbf{u}}^{\prime\prime 2} = \bar{R}_{\mathbf{u}}$.

Third case: $K = 0$. In this case by Lemma 5.3.4, one has $\bar{R}_{\mathbf{y}} = L\bar{R}_{\mathbf{u}}$. Since in this case $V_{\mathbf{y}} = L\sigma_{\mathbf{u}}^{\prime 2}$, given that $L\sigma_{\mathbf{u}}^{\prime 2} \leq L\bar{R}_{\mathbf{u}}$, we see that $V_{\mathbf{y}} \leq \bar{R}_{\mathbf{y}}$ and hence $V_{\mathbf{y}}$ can be interpreted as the output error variance. Therefore in this case we have non identifiability. □

5.4 Numerical Experiments

To illustrate the results of this chapter we shall discuss some examples.

Example 1: a non-identifiable model

Consider the following power spectra:

$$S_{\mathbf{y}}(z) = \frac{0.11z^2 - 4.864z - 14.57 - 4.864z^{-1} + 0.11z^{-2}}{z^2 - 0.138z - 2.83 - 0.138z^{-1} + z^{-2}}$$
$$S_{\mathbf{u}}(z) = \frac{0.01z^2 - 0.971z - 2.88 - 0.971z^{-1} + 0.01z^{-2}}{z^2 - 0.138z - 2.83 - 0.138z^{-1} + z^{-2}}$$
$$S_{\mathbf{yu}}(z) = \frac{0.033z^4 + 0.026z^3 + 0.005z^2 - 0.0002z}{z^4 + 0.2z^3 - 0.83z^2 - 0.084z + 0.176}$$

also represented in Figure 5.5 (solid line). In this case the input and output spectra satisfy the linear-affine relation (5.13) with $L = 5$ e $K = 0.06$. In order to check identifiability we use Theorem 5.3.6. This requires a preliminary computation of $\bar{R}_{\mathbf{y}}$ and $\bar{R}_{\mathbf{u}}$. For this example we find $\bar{R}_{\mathbf{y}} = 2.6521$, $\bar{R}_{\mathbf{u}} = 0.5184$ and in this case we have exactly $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$. In force of Theorem 5.3.6, the model is **non-identifiable**. As a check we may use the geometric method proposed in (Beghelli et al., 1997), interpreting (5.12) as the intersection of an infinite family of hyperbolas in the plane $\{\sigma_{\mathbf{u}}^2,\ \sigma_{\mathbf{y}}^2\}$. The intersection of all these branches in the plane is a point in the plane $\{\sigma_{\mathbf{u}}^2,\ \sigma_{\mathbf{y}}^2\}$ corresponding to the error variance pairs of candidate EIV models. Points of intersection, lying outside of the positive orthant do not correspond to valid EIV models. As we can see from Figure 5.6 there are two nonnegative intersections $(\sigma_{\mathbf{u}}^{\prime 2},\ \sigma_{\mathbf{y}}^{\prime 2}) = (0.3,\ 2.65)$, $(\sigma_{\mathbf{u}}^{\prime\prime 2},\ \sigma_{\mathbf{y}}^{\prime\prime 2}) = (0.52,\ 1.56)$ and we may check that $\sigma_{\mathbf{y}}^{\prime 2} = \bar{R}_{\mathbf{y}}$ and $\sigma_{\mathbf{u}}^{\prime\prime 2} = \bar{R}_{\mathbf{u}}$. According to the analysis made in this chapter, when $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$, at least one of the true spectra must have zeros on the unit circle *for both EIV models*. This can be seen very clearly from Figure 5.5.

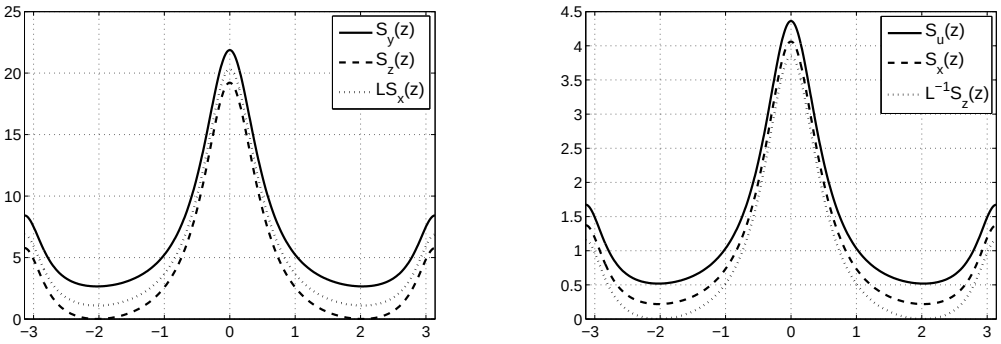


Figure 5.5: Input-output and true spectra for model 1 (dashed line) and model 2 (dotted line), Example 1.

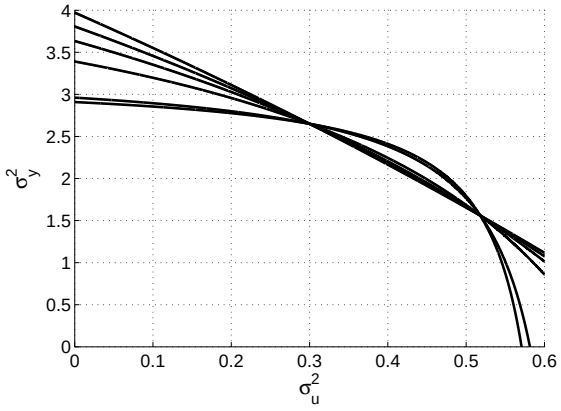


Figure 5.6: Intersections of hyperbolas for Example 1.

Example 2: an identifiable model satisfying the linear-affine relation

Assume the input, output and cross spectra are described by

$$S_{\mathbf{y}}(z) = \frac{-0.2z^3 + 30.9z^2 - 71.5z - 387.2 - 71.3z^{-1} + 30.9z^{-2} - 0.2z^{-3}}{z^3 - 6.35z^2 + 14.13z - 34.014 + 14.13z^{-1} - 6.35z^{-2} + z^{-3}}$$
$$S_{\mathbf{u}}(z) = \frac{z^3 - 10.1z^2 - 24.04z - 252.9 - 24.04z^{-1} - 10.1z^{-2} + z^{-3}}{z^3 - 6.35z^2 + 14.13z - 34.014 + 14.13z^{-1} - 6.35z^{-2} + z^{-3}}$$
$$S_{\mathbf{yu}}(z) = \frac{9.29z^6 + 4.24z^5 - 0.95z^4 - 0.32z^3 - 0.037z^2 + 0.003z + 0.0001}{z^6 - 0.82z^5 + 0.59z^4 - 0.25z^3 + 0.074z^2 - 0.015z + 0.001}$$

and shown in Figure 5.7. We see from the picture that there may be a linear affine relation between the two spectra. By imposing a relation of the type (5.13), we find $L = 1.8$, $K = -2$. In this case the necessary condition of Theorem 5.3.2 is satisfied.

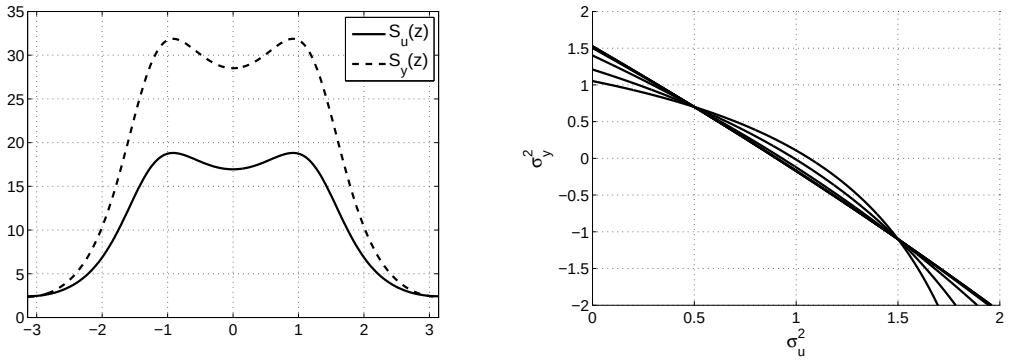


Figure 5.7: Input-output spectra and intersection of hyperbolas, Example 2.

However non-identifiability is not guaranteed. According to Theorem 5.3.6, since here $K < 0$, we need to check if $K < -L\bar{R}_{\mathbf{u}}$, in which case the model would be identifiable. Computing $\bar{R}_{\mathbf{u}}$ one gets $\bar{R}_{\mathbf{u}} = 0.902$, and so $-L\bar{R}_{\mathbf{u}} = -1.624$. Hence we have $K < -L\bar{R}_{\mathbf{u}}$, and the model is **identifiable**. We may in fact check that we have two possible variance pairs but only one of them, $(\sigma'^2_{\mathbf{u}}, \sigma'^2_{\mathbf{y}}) = (0.5,\, 0.7)$ is positive. The other, $(\sigma''^2_{\mathbf{u}}, \sigma''^2_{\mathbf{y}}) = (1.5,\, -1.1)$, is not feasible having a negative component. Note that this happens because $\sigma''^2_{\mathbf{u}} > \bar{R}_{\mathbf{u}}$ (the theoretical upper limit).

Example 3: an all-pass transfer function

Assume an all-pass transfer function

$$G(z) = \sqrt{2}\frac{(z - 1.25)(z + 4)}{(z - 0.8)(z + 0.25)}\,, \tag{5.30}$$

with gain $L = G(z)G^*(z) = 2$ and noise variances $(\sigma^2_{\mathbf{u}}\,,\, \sigma^2_{\mathbf{y}}) = (0.2\,,\, 0.6)$. Then we must have $S_{\mathbf{z}}(z) = 2\,S_{\mathbf{x}}(z)$. Summing to this equation the quantity $\sigma'^2_{\mathbf{y}} + L\sigma'^2_{\mathbf{u}}$ and letting $K = \sigma'^2_{\mathbf{y}} - L\sigma'^2_{\mathbf{u}} = 0.2$ the linear-affine relation between the observable input and output spectra is obtained. We get the following spectra, whose profile is drawn in Figure 5.8 (left), in which the linear-affine relation is clearly visible.

$$\begin{aligned}
S_{\mathbf{y}}(z) &= \frac{0.4z^2 - 2.895z + 8.376 - 2.895z^{-1} + 0.4z^{-2}}{z^2 - 0.74z + 3.18 - 0.74z^{-1} + z^{-2}} \\
S_{\mathbf{u}}(z) &= \frac{0.1z^2 - 1.373z + 3.87 - 1.373z^{-1} + 0.1z^{-2}}{z^2 - 0.74z + 3.18 - 0.74z^{-1} + z^{-2}} \\
S_{\mathbf{yu}}(z) &= \frac{0.447z^4 - 0.599z^3 - 0.055z^2 + 0.124z + 0.009}{z^4 + 1.5z^3 - 1.97z^2 - 1.03z - 0.74}
\end{aligned}$$

As explained in Sect. 5.3, all-pass models are always identifiable. They however

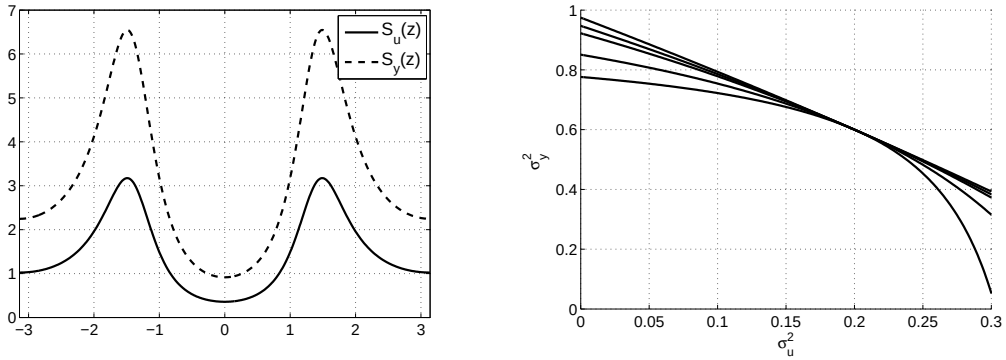


Figure 5.8: Input-output spectra and hyperbolas intersection for Example 3.

constitute a limit case, in which there are two coincident EIV representations. Figure 5.8 (right) confirms this fact. We see that in this case all the hyperbolas are tangent in the unique point corresponding to the (unique) EIV realization of the model. By elementary geometry we know that a point of tangency among conics must always be a *double contact point*.

A simulation experiment

This numerical example simulates an experimental setup. A vector time series realization of the bivariate process $[\,\mathbf{y}(t)\ \mathbf{u}(t)\,]^\top$ is generated from an EIV model with correlated non-white additive noise errors obtained as a filtered linear combinations of two uncorrelated white noises $\mathbf{w}_1$ and $\mathbf{w}_2$ of unit variance, $\sigma^2_{\mathbf{w}_i} = 1$, according to the scheme

$$\begin{aligned}\mathbf{e}_\mathbf{y}(t) &= \sqrt{\sigma^2_\mathbf{y}}F(z)(c_{y,1}\mathbf{w}_1(t) + c_{y,2}\mathbf{w}_2(t))\\ \mathbf{e}_\mathbf{u}(t) &= \sqrt{\sigma^2_\mathbf{u}}F(z)(c_{u,1}\mathbf{w}_1(t) + c_{u,2}\mathbf{w}_2(t))\,,\end{aligned}$$

where $c^2_{i,1} + c^2_{i,2} = 1$ and $F(z)$ is a linear FIR filter, whose spectral profile is plotted in Figure 5.9. The filter introduces a sort of realistic attenuation of the noise spectra at high frequencies.

From the sample time series of $\mathbf{y}(t)$, $\mathbf{u}(t)$ the power spectra of the simulated system are estimated by a standard non parametric method (Welch). These spectra will be called the "true" or "rough" spectra hereafter. The frequency plots of these spectra are the solid lines drawn in Figures 5.10 and 5.11, below: Naturally these true spectra do not comply with the Frisch scheme. One may produce a Frisch scheme approximation

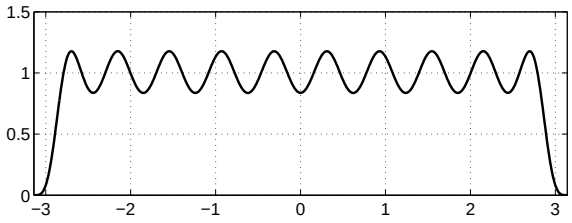


Figure 5.9: Spectral profile of the noise filter.

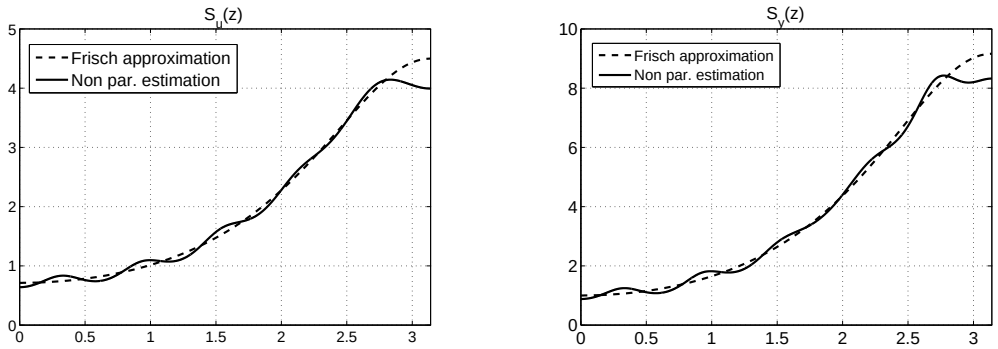


Figure 5.10: Input-output estimated spectra and their approximations.

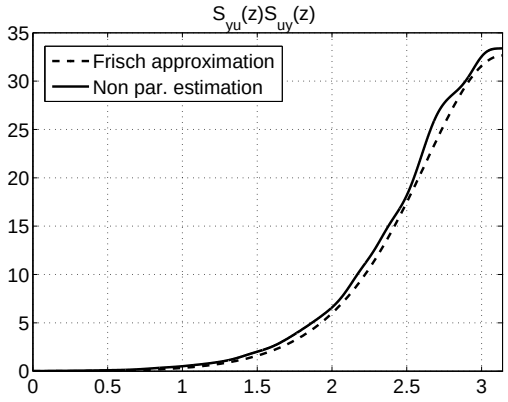


Figure 5.11: Estimated cross–spectrum and its approximation.

which is “best” according to some chosen identification/approximation procedure, see for example (Söderström, 2007). We come up with an estimated joint spectrum of the Frisch

type described by [1]

$$\begin{aligned}\hat{S}_{\mathbf{y}}(z) &= \frac{-2.333z + 10.3 - 2.333z^{-1}}{z + 3.633 + z^{-1}} \\ \hat{S}_{\mathbf{u}}(z) &= \frac{-0.25z + 1.705 - 0.25z^{-1}}{0.3z + 1.09 + 0.3z^{-1}} \\ \hat{S}_{\mathbf{yu}}(z) &= \frac{-1.333z^2 + 4.667z - 3.333}{z^2 + 3.633z + 1}\end{aligned} \tag{5.31}$$

The plots of the approximate spectra are the dashed lines in Figures 5.10 and 5.11.

At this point we may want to check for possible non-identifiability. The existence of a linear-affine relation between the two approximate output spectra (Theorem 5.3.2) is tested by fitting a linear regression of $\hat{S}_{\mathbf{y}}(e^{j\theta})$ versus $\hat{S}_{\mathbf{u}}(e^{j\theta})$ for various frequencies. The regression line is the dashed line shown in Figure 5.12 below. In this case we can

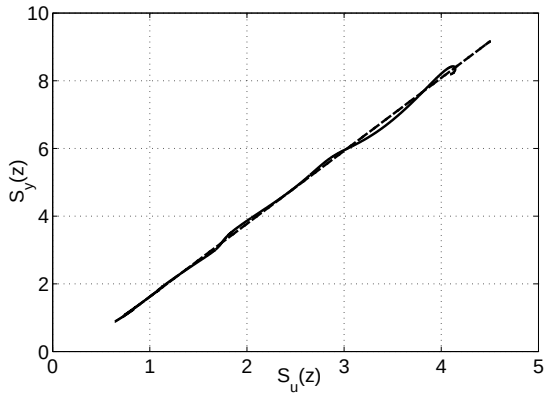


Figure 5.12: Regression line on Frisch spectra vs rough spectra.

see that the linear regression is quite accurate. We find $L = 2.1556$ and $K = -0.537$. Hence we conclude that there may be another Frisch scheme model compatible with the joint spectra (5.31). One may argue that at this stage it may be simpler to use a specific algorithm to get estimates of $\sigma^2_{\mathbf{y}}$, $\sigma^2_{\mathbf{u}}$ from the given spectra (5.31), and thereby check directly for non-uniqueness. Alternatively one may check if the quantities $V_{\mathbf{y}}$, $V_{\mathbf{u}}$ are positive and are therefore interpretable as true noise variances. Experimental procedures of this kind may however be very imprecise and turn out estimates of the model variances which are affected by noise and ultimately provide wrong answers. For this reason we shall instead attempt to use the a priori criteria of Theorem 5.3.6 of Section 5.3, which

[1] As we do not want to be tied up with any specific EIV identification procedure (each of which may give different estimates) we won't even mention which method was used in the experiment.

only depends on the model spectra (5.31) and not on specific variance estimates. We find

$$\bar{R}_{\mathbf{y}} \simeq 1 \quad \bar{R}_{\mathbf{u}} = 0.713\,; \tag{5.32}$$

and we see that $K = \bar{R}_{\mathbf{y}} - L\bar{R}_{\mathbf{u}}$, with good approximation, whence we can conclude, on the basis of Proposition 5.3.6, that the Frisch model describing the data (5.31) is *non-unique* and we have non identifiability. This can also be checked graphically by intersecting hyperbolas in the plane $\{\sigma^2_{\mathbf{y}}, \sigma^2_{\mathbf{u}}\}$ corresponding to different frequencies, see Figure 5.13 below.

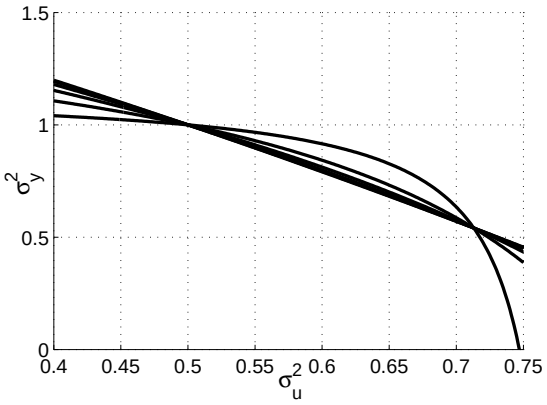


Figure 5.13: Hyperbolas intersection for the spectra (5.31).

Discussion

Although testing for non-identifiability on the rough (true) spectra does not make sense since these spectra are in general not realizable by EIV (Frisch) models, still one may want to see how these tests perform on the rough data in order to get a feeling for the sensitivity of the procedure.

Running a linear regression of the rough spectrum $S_{\mathbf{y}}(z)$ on $S_{\mathbf{u}}(z)$, we obtain slightly different values of L and K, namely $L = 2.1572$, $K = -0.5706$, and the straight line gives an average error of fit of the linear approximation $\mathbf{Y} := LS_{\mathbf{u}}(z) + K$ versus the measured output spectrum,

$$e = \frac{1}{\sqrt{N}}\|\mathbf{Y} - S_{\mathbf{y}}(z)\|_2 = 0.0463\,, \tag{5.33}$$

($N = 4097$ is the sample size) which indicates that a linear-affine relation is a good approximation. Hence in this case we may conclude that there is a warning for possible presence of two compatible models. Checking for actual non-identifiability cannot

however be done on rough spectra and requires fitting a realizable spectrum to the data. The hyperbola intersection test is inconclusive due to sharp differences between rough and approximate spectra for certain frequencies, see Figure 5.10 and also the computation of the bounds $\bar{R}_{\mathbf{y}}$ and $\bar{R}_{\mathbf{u}}$ on the rough spectra may easily become meaningless. This may happen either because of approximation errors, or also because of noise correlation. In our case we get the values $\bar{R}_{\mathbf{y}} = -0.1048$ and $\bar{R}_{\mathbf{u}} = -0.0516$, which are negative, and therefore meaningless.

6

Nonparametric kernel-based spectrum estimation

6.1 Introduction

In the previous chapter we introduced new methods to verify the identifiability of errors-in-variables models. Such methods require the knowledge of the power spectra of the input-output joint process. Hence, a reliable tool for spectrum estimation becomes of paramount importance. However, in many other practical problems of time series analysis, the power spectrum is one of the most intuitive and effective statistical descriptions of the data. It finds applications e.g. in signal processing, control systems design, econometrics and mathematical finance (Jenkins & Watts, 1968). For this reason, methods for spectrum estimation have been studied since the beginning of the last century (Schuster, 1898), still representing a rather active research area (Byrnes et al., 2000; Ramponi et al., 2009; Beran & Heiler, 2009; Rosen & Stoffer, 2007).

In this chapter we propose a new approach to spectrum estimation that exploits novel nonparametric techniques. Our interest is focused on the estimation of regular spectra, defined as square-summable spectral functions. The latter are suitable to represent the second order statistical description of most of the processes that belong to the purely nondeterministic (PND) class.

Review of the literature

In the literature, two main types of approaches to the problem can be found. The first one is concerned with parametric methods, where one adopts autoregressive models involving a finite number of coefficients that need to be estimated from data. Classical examples are the so-called covariance algorithm (Akaike, 1969) and the Burg spectrum estimation technique (Stoica & Moses, 1997). Other methods are based on the solutions of the so-called Yule-Walker equations (Friedlander & Porat, 1984) or on the prediction error minimization (PEM) paradigm, modeling the time series as ARMA processes. Generally, parametric techniques may yield satisfactory estimates, but in most of the cases the estimation process requires the solution of a nonlinear optimization problem possibly subject to local minima. Moreover, this approach requires model order selection and this can be an issue. In fact, complexity criteria such as AIC and BIC (see (Akaike, 1974) and (Schwarz, 1978) respectively) may return unsatisfactory results (Pillonetto & De Nicolao, 2010).

The second family of approaches to spectrum estimation relies upon nonparametric paradigms. The basic scheme consists of refining a rough estimate of the spectrum (usually the periodogram) through techniques based on windowed smoothing. Two standard nonparametric estimators are the Empirical transfer function estimator (Etfe) and the Spatial Spectrum Estimator (SPA), see e.g. (Ljung, 1999) for details. These algorithms provide a spectrum estimate with few computational effort even if they require the user to choose a smoothing parameter. This point is important since such parameter has a major effect on the quality of the final estimate, having to establish the right trade-off between adherence to experimental data and smoothness of the spectrum. Its tuning can be considered as the counterpart of the model order selection step that is encountered in the parametric context. Other spline-based nonparametric techniques have been also developed in (Wahba & Wold, 1975; Wahba, 1980); in this case, the optimal smoothing parameter of the periodogram can be estimated using approximations of the integrated mean squared error or cross validation.

Contribution of this work

As mentioned above, in this chapter we propose a novel method for spectrum estimation that falls inside the nonparametric family. Our method can be adopted to estimate the power spectrum of both continuous and discrete time processes.

The proposed approach is distinct from Etfe and SPA since, instead of performing smoothing in the frequency domain, it regularizes the empirical estimate of the correlation

functions. In particular, the estimate is obtained as the solution of a Tikhonov-type regularization problem, adopting as hypothesis space a reproducing kernel Hilbert space (RKHS) induced by the so-called stable spline kernel. This particular kernel has been recently proposed in (Pillonetto & De Nicolao, 2010) to identify impulse responses of linear time-invariant systems and includes information regarding the exponential stability of the function to reconstruct. The main contribution of this chapter is to show that the stable spline kernel can be used to define an effective spectrum estimator whose computational complexity scales linearly with the number of observed process samples. As shown later on, this result can be obtained exploiting the connection between regularization in RKHS and Bayes estimation of Gaussian processes, also making use of Kalman smoothing concepts. Moreover, the resulting estimator will be specialized to an application regarding the identification of time-invariant linear dynamical systems fed with white noise as input. Numerical experiments show that our novel approach, equipped with cross validation to estimate the kernel parameters from data, yields results comparable to SPA, and often also better than Etfe, equipped with an oracle that determines the optimal smoothing parameters by exploiting the knowledge of the true covariance.

The chapter is organized as follows. After this introduction, Section 6.2 describes the problem statement. In Section 6.3 some concepts of regularization in RKHS and nonparametric estimation are introduced, while in Section 6.4 we introduce the new kernel-based algorithm for spectrum estimation. Section 6.5 provides numerical experiments to test the performance of our approach.

6.2 Framework and problem formulation

Let us introduce the problem of estimating the second order moments of multivariate stochastic process, defined either on a continuous or discrete time domain. Without loss of generality, we consider a bivariate, stationary, zero-mean, PND stochastic process denoted by

$$\mathbf{v}(t) = \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{bmatrix}, \quad t \in \mathcal{I} \tag{6.1}$$

where $\mathbf{y}(t)$ and $\mathbf{u}(t)$ are scalar processes and $\mathcal{I}$ corresponds to $\mathbb{R}$ or $\mathbb{Z}$, depending on whether $\mathbf{v}(t)$ is continuous or discrete.

In view of the stationarity assumptions, the second order statistical description of the

process $\mathbf{v}(t)$ is completely determined by the autocorrelation function, defined as

$$\Sigma(\tau) := \mathbb{E}\left[\mathbf{v}(t)\mathbf{v}^\top(t+\tau)\right], \quad \tau \in \mathcal{I} \tag{6.2}$$

$$= \begin{bmatrix} f_{\mathbf{y}}(\tau) & f_{\mathbf{yu}}(\tau) \\ f_{\mathbf{uy}}(\tau) & f_{\mathbf{u}}(\tau) \end{bmatrix} \tag{6.3}$$

where $f_{\mathbf{y}}(\tau) := \mathbb{E}[\mathbf{y}(t)\mathbf{y}(t+\tau)]$, $f_{\mathbf{u}}(\tau) := \mathbb{E}[\mathbf{u}(t)\mathbf{u}(t+\tau)]$ denote the autocorrelation functions of the processes $\mathbf{y}(t)$, $\mathbf{u}(t)$ respectively, while $f_{\mathbf{yu}}(\tau) := \mathbb{E}[\mathbf{y}(t)\mathbf{u}(t+\tau)]$ is the cross-correlation with $f_{\mathbf{uy}}(\tau) = f_{\mathbf{yu}}(-\tau)$. Note that, since $\Sigma(\tau) = \Sigma^\top(-\tau)$, the autocorrelation function is completely described by its values taken in the domain $\tau \geq 0$.

An equivalent description is obtained by means of the spectral density function. In the discrete time case, the latter is defined as

$$S(\omega) := \sum_{\tau=-\infty}^{+\infty} \Sigma(\tau)\exp(-j\omega\tau), \quad -\pi \leq \omega \leq \pi \tag{6.4}$$

$$= \begin{bmatrix} S_{\mathbf{y}}(\omega) & S_{\mathbf{yu}}(\omega) \\ S_{\mathbf{uy}}(\omega) & S_{\mathbf{u}}(\omega) \end{bmatrix}, \tag{6.5}$$

which, in order to be a well-defined spectral density function, needs to be Hermitian positive semi-definite matrix for every z on the unit circle. Moreover, it is required that all its entries are square summable, i.e. $S_{\mathbf{y}}$, $S_{\mathbf{u}}$, $S_{\mathbf{yu}} \in L^2[-\pi, \pi]$. Using Parseval's theorem, this is equivalent to say that $f_{\mathbf{y}}$, $f_{\mathbf{u}}$, $f_{\mathbf{yu}} \in \ell^2$. In the continuous time case, similar considerations yield the condition $f_{\mathbf{y}}$, $f_{\mathbf{u}}$, $f_{\mathbf{yu}} \in L^2$. In this chapter we make a slightly different assumption, i.e. the autocorrelation functions are bounded and summable, exhibiting an exponential decay to zero. The first two hypotheses are not so limitative, since non-bounded or non-summable autocorrelation functions correspond to non-continuous spectral densities whose estimation with regularization techniques is a rather difficult issue. As for the last point, most of practical applications involves spectral densities modeled by rational processes, whose autocovariance functions have exponential dynamics.

System identification as spectrum estimation

A particular and significant application of spectrum estimation is the frequency-based identification of the transfer function of a linear time-invariant dynamic system. In fact, assume that the components of $\mathbf{v}(t)$ admit a representation in terms of an output error

model (Fig. 6.1), i.e.

$$\mathbf{y}(t) = G(z)\mathbf{u}(t) + \mathbf{e}(t)\,, \tag{6.6}$$

where $G(z)$ is a stable causal transfer function and $\mathbf{e}(t)$ is white noise independent of $\mathbf{u}(t)$. Then, one can recover $G(z)$ using the relation

$$G(\omega) = \frac{S_{\mathbf{yu}}(\omega)}{S_{\mathbf{u}}(\omega)}\,. \tag{6.7}$$

In particular, assuming that $\mathbf{u}(t)$ is white noise of unit variance, the problem of identifying $G(z)$ is equivalent to the estimation of the cross-spectrum $S_{\mathbf{yu}}(\omega)$.

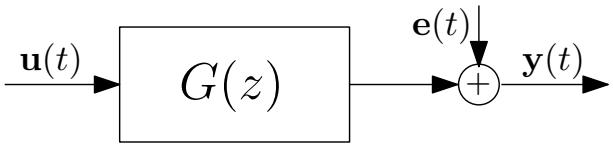


Figure 6.1: Block scheme of the linear dynamic system.

Problem statement

We assume to collect N equispaced observations of the process $\mathbf{v}(t)$, namely $v(1), \ldots, v(N)$, from which some empirical statistics can be computed (for simplicity, we assume N even). In particular, for $i = 1, \ldots, p$, we define

$$\mathcal{Z}_i := \frac{1}{N-\tau_i}\sum_{j=1}^{N-\tau_i} v(j)v^T(j+\tau_i-1) := \begin{bmatrix} \mathcal{Z}_i^{\mathbf{y}} & \mathcal{Z}_i^{\mathbf{yu}} \\ \mathcal{Z}_i^{\mathbf{uy}} & \mathcal{Z}_i^{\mathbf{u}} \end{bmatrix}, \tag{6.8}$$

which, considering different values of $\tau_i > 0$, is the causal part of the empirical autocorrelation function, for p time lags. The vector whose i-th entry is $\mathcal{Z}_i$ is denoted by $\mathcal{Z}$. For our purposes it is convenient to define also the quantities

$$\mathcal{T}_i := \frac{1}{N/2-\tau_i}\sum_{j=1}^{N/2-\tau_i} v(j)v^T(j+\tau_i-1) \tag{6.9}$$

$$\mathcal{V}_i := \frac{1}{N/2-\tau_i}\sum_{j=N/2+1}^{N-\tau_i} v(j)v^T(j+\tau_i-1)\,, \tag{6.10}$$

which are still versions of the empirical autocorrelation function, obtained separately by exploiting the first and second half of the data set (a notation similar to (6.8) will be used

for these quantities). These two vectors will define, respectively, the training and the validation sets, both having size equal to p. Then, we are interested in the development of estimators that take $\mathcal{Z}$, $\mathcal{T}$ and $\mathcal{V}$ as inputs and return an estimate

$$\hat{\Sigma}(\tau) = \begin{bmatrix} \hat{f}_{\mathbf{y}}(\tau) & \hat{f}_{\mathbf{yu}}(\tau) \\ \hat{f}_{\mathbf{uy}}(\tau) & \hat{f}_{\mathbf{u}}(\tau) \end{bmatrix} \quad , \tau \geq 0 \tag{6.11}$$

of the autocorrelation function $\Sigma(\tau)$ (or, equivalently, the spectrum via Fourier transform).

6.3 Regularization in spaces induced by stable spline kernels

Regularization in RKHS

Let us consider the problem of reconstructing an unknown function $f : \mathcal{X} \mapsto \mathbb{R}$ from a finite set of data z_i, each given by noisy version of a linear functional $L_i[f]$. This ill-posed problem has been widely investigated in the literature of inverse problems, opening the way to regularization techniques (Tikhonov & Arsenin, 1977). Such approaches represent an alternative paradigm to parametric estimation, where, in place of constraining the unknown function to a specific parametric structure, f is searched over a possibly infinite-dimensional functional space $\mathcal{H}$. The key ingredient to avoid overfitting and ill-posedness is the introduction of a regularizer Ω in the objective functional:

$$\min_{f \in \mathcal{H}} \left(\sum_{i=1}^{l} (z_i - L_i[f])^2 + \gamma \Omega(f) \right) \tag{6.12}$$

The positive parameter γ balances the error term $(z_i - L_i[f])^2$ and the regularizer $\Omega(f)$. A meaningful analysis of this approach is possible when $\mathcal{H}$ is a Hilbert space subject to the basic requirement that every function in the space be point-wise well defined everywhere on its domain, such as a RKHS. When a RKHS is adopted as hypothesis space, a natural regularizer is the squared norm, i.e. $\Omega(f) = \|f\|_{\mathcal{H}}^2$. Under mild assumptions on L_i, such a choice makes problem (6.12) well-posed.

Before introducing stable splines kernels, we discuss how to obtain a closed form for the solution to the problem (6.12) when a RKHS is adopted as hypothesis space, namely solve

$$\arg\min_{f \in \mathcal{H}} \left(\sum_{i=1}^{l} (z_i - L_i[f])^2 + \gamma \|f\|_{\mathcal{H}}^2 \right) . \tag{6.13}$$

For the moment, we assume that the hyperparameter γ is given; later, we discuss how to estimate it. The second property of Theorem 2.1.5 is also known as *reproducing property*

and it has some important consequences on the representation of linear functionals. Let $L : \mathcal{H} \to \mathbb{R}$ be a bounded and linear functional. Then, in force Theorem 2.1.1 there exists an element $h \in \mathcal{H}$ such that $\langle h,\, f\rangle_{\mathcal{H}} = L[f]$. Applying the reproducing property one has

$$h(s) = \langle h,\, K_s\rangle_{\mathcal{H}} = L[K_s]\,, \tag{6.14}$$

where $K_s := K(s,\, \cdot)$. Hence, from the reproducing kernel we can obtain any bounded and linear functional representer by applying L to K. Moreover, given two bounded linear functionals L_i and L_j, the inner product of their representers satisfies

$$\langle h_i,\, h_j\rangle_{\mathcal{H}} = L_i[h_j] = L_i\left[L_j[K(\cdot,\, \cdot)]\right] = L_j\left[L_i[K(\cdot,\, \cdot)]\right]\,. \tag{6.15}$$

Using this equality, the optimal solution to (6.13) can be written

$$\hat{f} = \sum_{i=1}^{l} \hat{c}_i h_i = \sum_{i=1}^{l} \hat{c}_i L_i[K]\,, \tag{6.16}$$

where h_i is the representer of L_i. The constant values $\hat{c}_i$ are obtained by substituting the expression of $\hat{f}$ into (6.13), yielding a problem of optimization in $\mathbb{R}^l$, namely

$$\begin{aligned}\hat{f} &= \arg\min_{\hat{c}\in\mathbb{R}^l} \left(\sum_{i=1}^{l} \left(z_i L_i \left[\sum_{i=1}^{l} \hat{c}_i L_i[K] \right] \right)^2 + \gamma \left\| \sum_{i=1}^{l} \hat{c}_i L_i[K] \right\|^2 \right) \\ &= \arg\min_{\hat{c}\in\mathbb{R}^l} \left(\|z - \Phi\hat{c}\|^2 + \gamma \hat{c}^\top \Phi \hat{c} \right)\,,\end{aligned} \tag{6.17}$$

where $z := [z_1,\, \ldots,\, z_l]^T$ and Φ is a $n \times n$ matrix such that

$$\Phi_{\{i,\,j\}} = L_i\left[L_j[K(\cdot,\, \cdot)]\right]\,. \tag{6.18}$$

It is well-known that the solution to this problem is

$$\hat{c} = (\Phi - \gamma I)^{-1} z\,, \tag{6.19}$$

hence the solution to (6.13) is

$$\hat{f} = \sum_{i=1}^{l} \hat{c}_i L_i[K] \quad ,\, \hat{c} = (\Sigma - \gamma I)^{-1} z\,. \tag{6.20}$$

This procedure is known in the literature as regularization network (Poggio & Girosi, 1990).

In the next subsection, we review a class of kernels that incorporate information on the exponential decay to zero of f.

Stable spline kernels

The class of the so called *stable spline kernels* introduced in (Pillonetto & De Nicolao, 2010; Pillonetto et al., 2010) induces hypothesis spaces containing smooth and exponentially stable functions. For $(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, it is defined by

$$K(x_1,\, x_2) := \int_0^{+\infty} G_m(e^{-\beta x_1}, e^{-\beta u}) G_m(e^{-\beta x_2}, e^{-\beta u}) \beta e^{-\beta u} du\,, \tag{6.21}$$

$$G_m(r,u) = \frac{(r-u)_+^{m-1}}{(m-1)!}, \qquad (u)_+ := \begin{cases} u & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In (6.21), the parameter $\beta > 0$ regulates how fast the functions in the associated RKHS decay to zero while m is a positive integer that indicates the kernel order.

In order to derive the RKHS associated with the class of kernels (6.21), we introduce a lemma, which will be also instrumental to the proof of the main result of this section.

First we need to set the following notation; given a function h, $h^{(k)}$ denotes its k-th derivative. We also define the notation $h_\beta^{(k)}$ where $h_\beta^{(0)}(t) := h(t)$ while, for $k \in \mathbb{N}$:

$$h_\beta^{(1)}(t) := \frac{e^{\beta t} h^{(1)}(t)}{\beta}, \qquad h_\beta^{(k+1)} := \left(h_\beta^{(k)}\right)_\beta^{(1)}\,. \tag{6.22}$$

In addition, here we use the symbol L^2 to indicate the classical Lebesgue space of squared integrable functions on $[0,1]$ while, for $\beta > 0$, L^2_β is the space of square integrable functions on $\mathbb{R}_+$ with the norm $\|\cdot\|_\beta$ defined by

$$\|h\|_\beta^2 = \int_0^\infty h^2(t) \beta e^{-\beta t} dt\,. \tag{6.23}$$

The spline kernel of order m (Wahba, 1990) is

$$W(s,t) := \int_0^1 G_m(s,u) G_m(t,u) du \tag{6.24}$$

where we still have

$$G_m(r,u) = \frac{(r-u)_+^{m-1}}{(m-1)!}, \qquad (u)_+ := \begin{cases} u & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, it is well known that the RKHS of functions on $[0,1]$ associated with W is (see e.g. (Wahba, 1990))

$$\mathcal{H}_W = \{h : h(0) = 0, \dots, h^{(m-1)} = 0; h, \dots, h^{(m-1)} \text{abs. continuous}; h^{(m)} \in L^2\} \quad (6.25)$$

with norm

$$\|h\|^2_{\mathcal{H}_W} = \int_0^1 \left(h^{(m)}(t)\right)^2 dt \quad (6.26)$$

In the rest of the chapter, for ease of notation, $\mathcal{H}$ will denote the RKHS associated with the stable spline kernel K. Having set the notation, the following lemma holds.

Lemma 6.3.1. *Let $\mathcal{H}$ be the RKHS associated with the stable spline kernel K defined in (6.21). Then, for any $m \in \mathbb{N}$ the transformation $\tau = e^{-\beta t}$ establishes a Hilbert space isomorphism between $\mathcal{H}$ on $\mathbb{R}_+$ and $\mathcal{H}_W$ on $[0,1]$. In particular if $h \in \mathcal{H}$, $g \in \mathcal{H}_W$ and $h(t) = g(e^{-\beta t})$, one has*

$$\|g\|_{\mathcal{H}_W} = \|g^{(m)}\|_{L^2} = \|h\|_{\mathcal{H}} = \|h_\beta^{(m)}\|_\beta \quad (6.27)$$

Proof. From Mercer theorem and RKHS theory, see e.g. (Cucker & Smale, 2001), we know that there exist functions φ_j and positive scalars υ_j such that

$$\upsilon_j \varphi_j(t) = \int_0^1 W(t,u)\varphi_j(u)du, \quad j = 1, 2, \dots$$

It also comes that the RKHS associated with W admits the following representation

$$\mathcal{H}_W = \left\{ g \in L^2 \quad | \quad g = \sum_{j=1}^{\infty} a_j \varphi_j, \quad \sum_{j=1}^{\infty} \frac{a_j^2}{\upsilon_j} < \infty \right\} \quad (6.28)$$

where

$$\|g\|^2_{\mathcal{H}_W} = \|g^{(m)}\|^2_{L^2} = \sum_{j=1}^{\infty} \frac{a_j^2}{\upsilon_j}\,. \quad (6.29)$$

Now, to gain insight about the structure of the RKHS associated with the stable spline

kernel, from the definition of K and simple calculations one obtains

$$\begin{aligned}
&\int_0^\infty K(t,u)\varphi_j(e^{-\beta u})\beta e^{-\beta u}du = \\
&\int_0^\infty W(e^{-\beta t}, e^{-\beta u})\varphi_j(e^{-\beta u})\beta e^{-\beta u}du = \\
&\int_0^1 W(e^{-\beta t}, u)\varphi_j(u)du = \upsilon_j\varphi_j(e^{-\beta t})
\end{aligned}$$

The above expression thus reveals that

- the eigenfunctions $\phi_j(t)$ of K (calculated using the measure on $\mathbb{R}_+$ induced by the density $\beta e^{-\beta u}$) are obtained from those of W (computed using the classical Lebesgue measure on $[0,1]$) after an exponential transformation, i.e. $\phi_j(t) = \varphi_j(e^{-\beta t})$;
- the eigenvalues of K (calculated using the density $\beta e^{-\beta t}$ on $\mathbb{R}_+$) coincide with those of W (calculated using the Lebesgue measure on $[0,1]$).

Since the representation of an RKHS is independent of any (non degenerate Borel) measure adopted to calculate the eigenfunctions, we conclude that the RKHS induced by the stable spline kernel K is

$$\mathcal{H} = \left\{ h \in L^2_\beta \quad | \quad h = \sum_{j=1}^\infty a_j\phi_j, \quad \sum_{j=1}^\infty \frac{a_j^2}{\upsilon_j} < \infty \right\} \tag{6.30}$$

and that

$$\|h\|^2_{\mathcal{H}} = \sum_{j=1}^\infty \frac{a_j^2}{\upsilon_j} \tag{6.31}$$

The expressions (6.28, 6.29) and (6.30, 6.31), together with the definitions of ϕ_j and φ_j, prove the isomorphism of $\mathcal{H}_W$ and $\mathcal{H}$ as regulated by the axis transformation $\tau = e^{-\beta t}$. This, combined with (6.26), the definition of $h_\beta^{(m)}$ in (6.22) and other simple integral calculations, eventually proves (6.27). □

Typically, the parameter m is chosen to be 1 or 2. When $m = 1$ one obtains the kernel

$$e^{-\beta \max(x_1,x_2)} \tag{6.32}$$

that, in force of Lemma 6.3.1, induces a RKHS with norm

$$\|f\|^2_{\mathcal{H}} = \int_{\mathbb{R}_+} \left(f^{(1)}(x)\right)^2 \frac{e^{\beta x}}{\beta} dx \tag{6.33}$$

Notice that, beyond the energy of the first-order derivative of g, the norm includes also a weight proportional to $e^{\beta x}$ that ensures exponential BIBO stability. A class of more regular functions can be obtained using the stable spline kernel of order $m = 2$. Originally introduced in (Pillonetto & De Nicolao, 2010), it is given by

$$\frac{e^{-\beta(x_1+x_2)}e^{-\beta\max(x_1,x_2)}}{2} - \frac{e^{-3\beta\max(x_1,x_2)}}{6} \tag{6.34}$$

Still using Lemma 6.3.1, one obtains that the corresponding norm is

$$\|f\|_{\mathcal{H}}^2 = \int_{\mathbb{R}_+} \left(f^{(2)}(x) + \beta f^{(1)}(x)\right)^2 \frac{e^{3\beta x}}{\beta^3} dx \tag{6.35}$$

and again forces stability and smoothness, now introducing derivatives up to the second order.

Computational complexity of the stable spline estimator

Now, we discuss the computational complexity of the estimator (6.12), when stable spline kernels are adopted, considering two different situations connected to the nature of L_i. The first scenario is related to the identification of linear and time-invariant dynamic systems and has been already discussed in (Pillonetto & De Nicolao, 2010). Here, f is thought of as the unknown impulse response and the estimator (6.12) becomes

$$\min_{f\in\mathcal{H}} \left(\sum_{i=1}^{l}(z_i - L_i[f])^2 + \gamma\|f\|_{\mathcal{H}}^2\right) \tag{6.36}$$

with $\|f\|_{\mathcal{H}}^2$ typically given by (6.33) or (6.35), and

$$L_i[f] = \int_0^{+\infty} u(t_i - s)f(s)ds$$

where u is the known system input. In particular, the use of the stable spline estimator (6.36) involves the following two computational steps, see (Pillonetto & De Nicolao, 2010) for details.

1. First, the unknown regularization parameter γ and the kernel parameter β are estimated from data by optimizing a suitable cost function, i.e.

$$(\hat{\gamma}, \hat{\beta}) = \arg\min_{\gamma,\beta\geq 0} J(\gamma, \beta) \tag{6.37}$$

The procedure suggested in (Pillonetto & De Nicolao, 2010) resorts to the Bayesian interpretation of (6.36) and sets J to the minus log of a marginal likelihood. Another possible choice of J may lead to cross validation. In any case, the pointwise evaluation of the objective requires $O(l^3)$ operations.

2. The estimates $\hat{\gamma}$ and $\hat{\beta}$ are plugged into (6.36). A closed form solution for the estimate of f becomes so available: it admits a structure by means of the previously described regularization network, i.e. it is sum of l basis functions given by the stable spline kernel filtered by u. The expansion coefficients are the solution of a system of l linear equations, hence requiring $O(l^3)$ operations.

As also discussed in the next section, the second scenario we consider is intimately related to the perspective for spectrum estimation taken in this chapter. It corresponds to a particular instance of the first problem where each linear functional becomes a pointwise evaluator[1], i.e. $L_i[f] = f(\tau_i)$ with $\{\tau_i\}$ denoting the sampling locations. The estimator (6.36) thus simplifies to

$$\min_{f \in \mathcal{H}} \left(\sum_{i=1}^{l} (z_i - f(\tau_i))^2 + \gamma \|f\|^2_{\mathcal{H}} \right) . \tag{6.38}$$

The following proposition, which represents the main result of this section, then holds.

Proposition 6.3.2. *Let $\hat{f}$ denote the solution of (6.38) when a stable spline kernel of order m is adopted, e.g. (6.32) for $m = 1$ or (6.34) for $m = 2$. Then, for known γ and β, the estimates $\{\hat{f}(\tau_i)\}_{i=1}^{l}$ can be computed with $O(lm^3)$ operations. In addition, once the estimates $\{\hat{f}(\tau_i)\}_{i=1}^{l}$ become known, $\hat{f}(\tau)$ can be computed with $O(m^3)$ operations for every τ.*

Proof. Let

$$\hat{f} = \arg\min_{f \in \mathcal{H}} \left(\sum_{i=1}^{l} (z_i - f(\tau_i))^2 + \gamma \|f\|^2_{\mathcal{H}} \right) \tag{6.39}$$

and

$$\hat{g} = \arg\min_{g \in \mathcal{H}_W} \left(\sum_{i=1}^{l} (y_i - g(t_i))^2 + \gamma \|g\|^2_{\mathcal{H}_W} \right) \tag{6.40}$$

where, for $i = 1, \ldots, l$:

$$y_{l-i+1} = z_i \quad \text{and} \quad t_i = e^{-\beta \tau_i} .$$

[1] This is equivalent to consider the first scenario assuming that the system input is an impulse.

Using Lemma 6.3.1, one obtains

$$\hat{f}(\tau) = \hat{g}(e^{-\beta\tau}), \quad \tau \in \mathbb{R}_+ \tag{6.41}$$

so that we can just focus on how to solve efficiently (6.40). For this purpose, we exploit the isometry between RKHS and Gaussian processes, as e.g. described in (Wahba, 1990). In particular, let e_i be the m-dimensional row vector with $i-th$ component equal to 1 and all the other ones equal to zero while, by definition, the components of e_0 are all set to zero. Define also the following correspondences

$$A = \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{m-1} \end{bmatrix}, \quad B = \begin{bmatrix} e_1^\top \end{bmatrix}, \quad C = \begin{bmatrix} e_m \end{bmatrix} \tag{6.42}$$

Then, define $\mathbf{V}(t,s) = \mathbb{E}\left[\mathbf{x}(t)\mathbf{x}^\top(s)\right]$ where $\mathbf{x}(t) \in \mathbb{R}^m$ is a continuous-time stochastic process whose statistics are defined by the following state-space stochastic model:

$$\begin{cases} \dot{\mathbf{x}}(t)dt &= A\mathbf{x}(t)dt + \gamma^{-1}Bd\zeta(t), \qquad 0 \le t \le 1 \\ \mathbf{x}(0) &= 0 \end{cases} \tag{6.43}$$

where ζ is Brownian motion. Since the last component of $\mathbf{x}(t)$ i the m-fold integration of white noise, it is straightforward to check that the covariance of $C\mathbf{x}(t)$ coincides with the kernel $\gamma^{-1}W_m$. It comes that the optimizer of (6.40) is the Bayes estimate of $C\mathbf{x}(t)$ conditional on the measurements $\mathbf{y}_i = C\mathbf{x}_i + \nu_i$, where $i = 1, \ldots, l$ and the ν_i are Gaussian random variables of unit variance, mutually independent and independent of ζ.

Now, we first consider the problem of obtaining the estimate of $C\mathbf{x}(t)$ at the sampling instants t_i. For this purpose, after simple computations one obtains that the sampled version of (6.43) at the sampling instants t_i, complemented with the measurements model, is

$$\begin{cases} \mathbf{x}_0 &= 0 \\ \mathbf{x}_{i+1} &= F_i\mathbf{x}_i + \gamma^{-1}\omega_i, \qquad i = 0, 1, \ldots, l \\ \mathbf{y}_i &= C\mathbf{x}_i + \nu_i, \qquad i = 1, \ldots, l \end{cases} \tag{6.44}$$

where, letting $\Delta_i = t_{i+1} - t_i$,

- $\{\omega_i\}$ are independent zero-mean Gaussian noises, with $m \times m$ covariance matrix

Q_i whose (k,j)-entry is

$$Q_i(k,j) = \frac{\Delta_k^{k+j-1}}{(k-1)!(j-1)!(k+j-1)} \tag{6.45}$$

- F_i is an $m \times m$ lower-triangular Toeplitz matrix whose $(k,1)$ entry is

$$F_i(k,1) = \frac{\Delta_i^{k-1}}{(k-1)!} \tag{6.46}$$

- $\{\nu_i\}$ are mutually independent zero-mean Gaussian noises of unit variance.

- C is defined as in (6.42).

Now, starting from (6.44), the classical Kalman smoothing filter can be used to obtain the minimum variance estimates of the states $\{\mathbf{x}_i\}_{i=1}^l$, denoted by $\{\hat{\mathbf{x}}_i\}_{i=1}^l$, with a number of operations linear in l (Anderson & Moore, 1979). The first part of Proposition 6.3.2 is then proved just recalling that $\hat{\mathbf{x}}_i = \hat{g}(t_i) = \hat{f}(\tau_i)$, where $\tau_i = e^{-\beta t_i}$. For what regards the second part of Proposition 6.3.2, we need to compute the state estimates outside the sampling instants t_i. One has

$$\begin{aligned}
\hat{\mathbf{x}}(t) &:= \mathbb{E}\left[\mathbf{x}(t) \,|\, \{\mathbf{y}_i\}_{i=1}^l\right] \\
&= \mathbb{E}\left[\mathbb{E}[\mathbf{x}(t) \,|\, \{\mathbf{y}_i\}_{i=1}^l,\, \{\mathbf{x}_i\}_{i=1}^l] \,|\, \{\mathbf{y}_i\}_{i=1}^l\right] \\
&= \mathbb{E}\left[\mathbb{E}[\mathbf{x}(t) \,|\, \{\mathbf{x}_i\}_{i=1}^l] \,|\, \{\mathbf{y}_i\}_{i=1}^l\right]
\end{aligned}$$

Then, using the above result and recalling again that $C\hat{\mathbf{x}}(e^{-\beta\tau}) = \hat{g}(e^{-\beta\tau}) = \hat{f}(\tau)$, one obtains that for every $\tau \geq 0$:

$$\hat{f}(\tau) = \begin{cases}
C\mathbf{V}(\mathbf{x}(e^{-\beta\tau}), \mathbf{x}_i^{i+1})\mathbf{V}\left(\mathbf{x}_i^{i+1}\right)^{-1} \begin{bmatrix} \hat{\mathbf{x}}_i \\ \hat{\mathbf{x}}_{i+1} \end{bmatrix} & \\
t_1 \leq e^{-\beta\tau} \leq t_l & i \text{ s.t.} \quad t_i \leq e^{-\beta\tau} \leq t_{i+1} \\
C\mathbf{V}(\mathbf{x}(e^{-\beta\tau}), \mathbf{x}_1)\mathbf{V}\left(\mathbf{x}_1\right)^{-1} \hat{\mathbf{x}}_1 & e^{-\beta\tau} \leq t_1 \\
C\mathbf{V}(\mathbf{x}(e^{-\beta\tau}), \mathbf{x}_n)\mathbf{V}\left(\mathbf{x}_n\right)^{-1} \hat{\mathbf{x}}_n & e^{-\beta\tau} \geq t_l
\end{cases}$$

where $\mathbf{x}_i^{i+1} := [\mathbf{x}_i^\top \quad \mathbf{x}_{i+1}^\top]^\top$ and $\mathbf{V}(a) := \mathbb{E}\,[aa^\top]$ for every random column vector a. This concludes the proof. □

6.4 Description of the algorithms

Spectrum estimation

In this section, we describe the proposed algorithm for spectrum estimation. The starting point is to consider each component of $\Sigma(\tau)$ separately, so that our original problem is reformulated as the estimation of four scalar functions. Furthermore, we consider the empirical correlations (6.8), namely $\mathcal{Z}_i$, as sampled noisy versions of the correlation function $\Sigma(\tau)$. In this way, the functions $f_{\mathbf{y}}$, $f_{\mathbf{u}}$, $f_{\mathbf{yu}}$, $f_{\mathbf{uy}}$ can be estimated independently by solving problems of the type (6.38), with $l = p$, while the noisy measurements z_i are the corresponding entries of $\mathcal{Z}_i$.

Estimation of the parameters γ and β

In this work, a cross validation strategy for the estimation of the hyperparameters is adopted. This choice appears reasonable and convenient, since we can set up an estimation scheme that still utilizes the empirical moments and relies on the solution of problems of the type (6.38). Hence, recalling the result of Proposition 6.3.2, such a cross validation scheme has a computational complexity that scales linearly with p. In the following, we report the cross validation procedure for the estimation of $f_{\mathbf{y}}$.

1. Define suitable grids of candidate values γ and β. Since both γ and β are positive real numbers, a logarithmic scale for the grids can be adopted.

2. For each point of the grids (γ_i, β_j), solve the problem (6.38) using the training set as available measurements, i.e. setting $z_i = \mathcal{T}_i^{\mathbf{y}}$ (training step).

3. For every solution $\hat{f}_{\mathbf{y}}(\gamma_i, \beta_j)$ to the problem (6.38) obtained at the previous step, evaluate the cost function

$$J_{\mathbf{y}}(\gamma_i, \beta_j) := \|\mathcal{V}^{\mathbf{y}} - \hat{f}_{\mathbf{y}}(\gamma_i, \beta_j)\|_2 \,, \tag{6.47}$$

   which indicates the predictive capability of the estimated covariance functions to the validation data set, and choose the minimizer (validation step).

The same scheme can be applied to the estimation of $f_{\mathbf{u}}$, $f_{\mathbf{yu}}$, $f_{\mathbf{uy}}$, with the proper measurements (i.e. $z_i = \mathcal{T}_i^{\mathbf{u}}$, $\mathcal{T}_i^{\mathbf{yu}}$, $\mathcal{T}_i^{\mathbf{uy}}$ respectively) and cost functions.

The correlation function estimation algorithm can be summarized by the following steps.

Algorithm 1 Correlation function estimation

Input: $\{v(t)\}_{t=1}^{N}$
Output $\hat{\Sigma}(\tau)$

1. Fix p
2. Compute the data sets $\mathcal{T}$, $\mathcal{V}$, $\mathcal{Z}$
3. For each function $f_{\mathbf{y}}$, $f_{\mathbf{u}}$, $f_{\mathbf{yu}}$, $f_{\mathbf{uy}}$
   (a) Define the log-spaced grids of the parameters γ and β
   (b) Solve the minimization problem (6.38) for each value in the grids (training step)
   (c) Choose the values of γ and β that minimize (6.47) (validation step)
   (d) Solve the minimization problem (6.38) with the selected values of γ and β using the measurements $\mathcal{Z}$

As can be seen, Algorithm 1 relies on the solution of several problems of the type (6.38), all requiring a computational complexity that scales linearly with the parameter p. The input of the algorithm are the empirical moments $\mathcal{T}$, $\mathcal{V}$, $\mathcal{Z}$, which need to be computed only once.

Figure 6.2 shows a block scheme version of Algorithm 1.

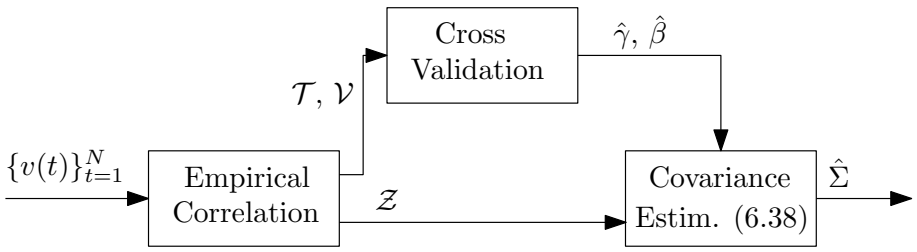


Figure 6.2: Block scheme of Algorithm 1.

Remark 6.4.1. Actually, optimization in (6.38) should be constrained to the functions in the space $\mathcal{H}$ that define covariance matrices. This constraint should be added also when using other approaches. One simple approach to deal with this problem is to project the estimate onto a subset of functions that define true covariances. However, in the experiments shown later on it has never been necessary to perform this further step, i.e. all the estimates define true spectra.

System identification with white noise as input

As already discussed in Section 6.2, an interesting problem is the identification of a transfer function when the input is a white noise. It is strictly related to the estimation of the cross-correlation function of the processes $\mathbf{y}(t)$ and $\mathbf{u}(t)$ (interpreted as output and input respectively).

A successful nonparametric approach for the identification of linear time-invariant dynamic systems relies on the solution of the problem (6.36), which first requires the estimation of parameters γ and β, then the solution of a linear system. Both these steps take $O(l^3)$ operations, where in this case l is the number of collected observations, i.e. N. However, approaching this problem from a spectrum estimation perspective, also in this case we can exploit the cross validation scheme described in Section 6.4. Thus, compared to (Pillonetto & De Nicolao, 2010), where the minimization problem (6.37) is defined by a marginal likelihood strategy that takes $O(N^3)$ operations, here we use a procedure that estimates the hyperparameters with computational complexity $O(p)$.

In the following, we report the proposed algorithm for the identification of systems with white noise as input.

Algorithm 2 System identification

Input: $\{v(t)\}_{t=1}^{N}$
Output $\hat{f}_{\mathbf{yu}}(\tau)$

1. Fix p
2. Compute $\mathcal{T}^{\mathbf{yu}}$, $\mathcal{V}^{\mathbf{yu}}$ and $\mathcal{Z}^{\mathbf{yu}}$
3. Define the log-spaced grids of the parameters γ and β
4. Solve the minimization problem (6.38) for each value in the grids (training step)
5. Choose the values of γ and β that minimize (6.47) (validation step)
6. Solve the minimization problem (6.36) with the selected values of γ and β using the measurements $\{v(t)\}_{t=1}^{N}$

Notice that Algorithm 2 differs from Algorithm 1 only in the last step, where the estimator (6.36) is used. This takes a number of operations that scales with $O(N^3)$. However, this problem has to be solved only one time as final step, involving the inversion of one matrix only (see (Pillonetto & De Nicolao, 2010), Sec. 3.9, eq. (9)).

Figure 6.3 shows a block scheme interpretation of Algorithm 2.

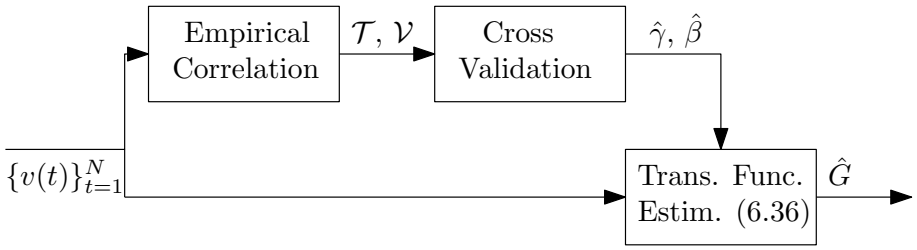


Figure 6.3: Block scheme of Algorithm 2.

6.5 Numerical Experiments

In this section, we report some results regarding the performance of the algorithms proposed in this chapter.

Estimation of autocovariance functions

First, we tested our algorithm for the estimation of autocovariance functions, which was tested by means of 6 Monte Carlo experiments of 100 runs. For the sake of simplicity, a scalar and rational process $\mathbf{v}(t)$ was considered; at every run, N samples of such a process were generated filtering a white noise with a stable minimum phase filter $W_i(z)$, i.e. $\mathbf{v}(t) = W_i(z)\mathbf{e}(t)$, so that the spectrum turns out to be $S(\omega) = W_i(e^{j\omega})W_i(e^{-j\omega})$. Three different scenarios were considered:

1. $W_1(z) = \dfrac{z^{-1}}{1 - 0.3z^{-1}}$;

2. $W_2(z) = \dfrac{2z^{-5}}{1 - 1.9z^{-1} + 2.5z^{-2} - 2.25z^{-3} + 1.49z^{-4} - 0.41z^{-5}}$;

3. At any Monte Carlo run, $W_{\text{rand}}(z)$ has 10 zeros randomly picked in the circle $|z| \leq 0.98$ and 10 poles randomly picked in the circle $|z| \leq 0.95$.

The spectrum profiles of the first two scenarios are shown in Figure 6.4. For each of the 3 scenarios, we considered two different situations where the number of available samples is either $N = 500$ or $N = 2000$. In all the experiments, the parameter p was set to 500. The latter choice follows the empirical rule $p = N/4$, which appeared to be appropriate after several simulations. Hence, there is a total of 6 different Monte Carlo experiments whose features are summarized in Table 6.1.

Exp.#	Data set size (N)	Spectrum
1	500	$W_1(z)$
2	2000	$W_1(z)$
3	500	$W_2(z)$
4	2000	$W_2(z)$
5	500	$W_{\text{rand}}(z)$
6	2000	$W_{\text{rand}}(z)$

Table 6.1: Features of the 6 Monte Carlo experiments.

Three different spectrum estimators are used whose performance is evaluated at any run computing the relative mean squared error, i.e.

$$E_i(\%) = 100\frac{\|S - \hat{S}_i\|_2}{\|S\|_2}\,, \tag{6.48}$$

where $\hat{S}_i$ represents the spectrum estimate obtained in the i-th run. The three adopted estimators are listed below.

- *Stable Spline estimator*: this is the nonparametric approach proposed in this chapter with parameters γ and β determined by cross validation according to Algorithm 1. In particular, the choice of the parameter γ has been made on the logarithmic grid

$$\Gamma = \{\gamma :\, \gamma = 10^{\nu_\gamma}\}\,, \tag{6.49}$$

  where ν_γ is an array of 20 equispaced real numbers in the interval $[-3,\, 1]$. Similarly, the following set has been defined for the parameter β:

$$\Delta = \{\beta :\, \beta = -\log(\nu_\beta)\}\,, \tag{6.50}$$

  where again ν_β is an array of equispaced numbers from 0.04 to 0.99 with a sampling of 0.05.

- *ETFE+Oracle*: this is the classical method Etfe available in Matlab (Ljung, 1999). This estimator needs the user to choose a smoothing parameter that greatly influences its performance. Since there is no standard method to automatically select this parameter, at each run it has been selected by an oracle that knows the true spectrum and chooses the value minimizing the relative mean squared error E_i. This represents an ideal tuning not obtainable in real applications.

- *SPA+Oracle*: the same as above except that the SPA method available in Matlab is now used.

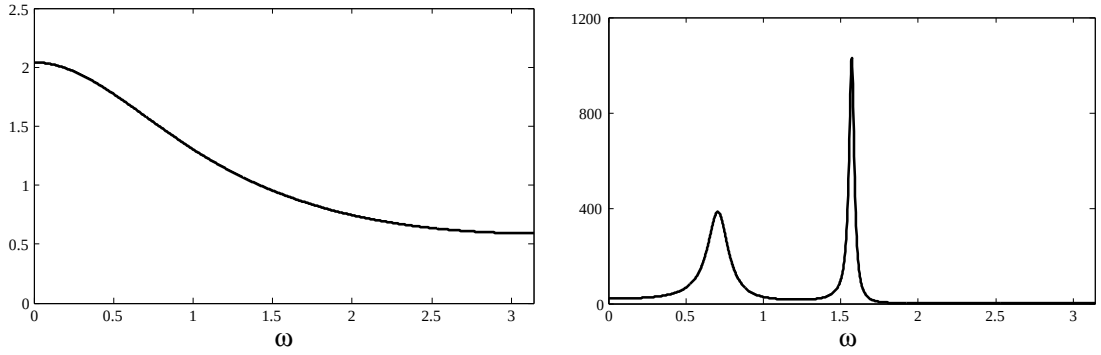


Figure 6.4: Spectra profiles generated by W_1 and W_2.

Figures 6.5-6.7 show the box plots of the 100 reconstruction errors obtained by each estimator after the 6 Monte Carlo experiments. Remarkably, in most of the experiments the performance of the proposed Stable Spline based algorithm with cross validation is comparable or also better than SPA and Etfe with oracle.

Estimation of transfer functions

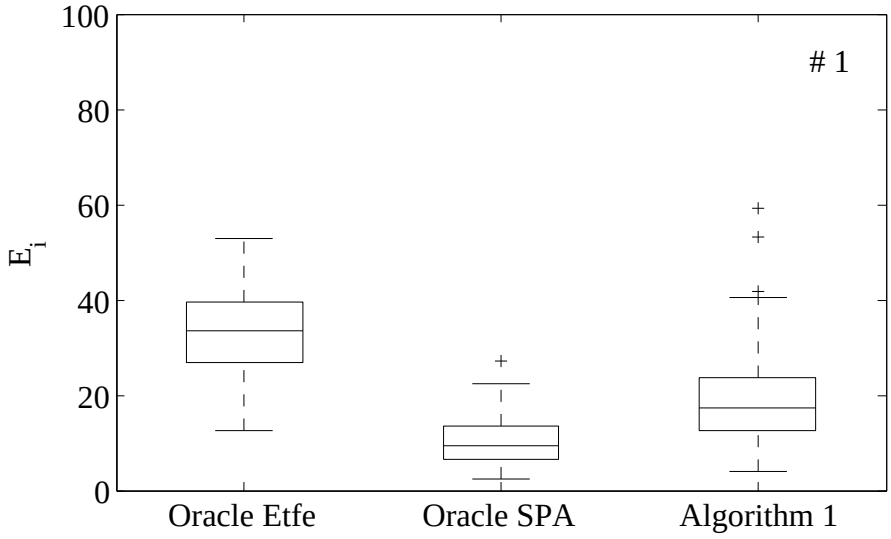
A second set of experiments was performed to test the algorithm for identification of transfer functions with white noise as input. For such purpose, 4 Monte Carlo experiments of 100 runs each were performed. We considered a scenario in which at any Monte Carlo run, $G_{\text{rand}}(z)$ has 10 zeros randomly picked in the circle $|z| \leq 0.95$ and 10 poles randomly picked in the circle $|z| \leq 0.95$, with a random gain in the interval $[5,\ 10]$. The variance of the measurement noise was chosen such that the SNR, defined as $\|G(z)\mathbf{u}(t)/\mathbf{e}(t)\|_2$, took value in the set $\{1,\ 5,\ 10,\ 100\}$. We assumed $N = 2000$, while the parameter p was set to be equal to 500. Again, the grids on which γ and β are evaluated were defined as in (6.49) and (6.50) respectively.

We compared the two stable spline algorithms with the Oracle-SPA estimator described in the previous section, and with the empirical estimator that returns as estimates just (6.8). The performance is evaluated computing at every Monte Carlo run the relative mean squared error

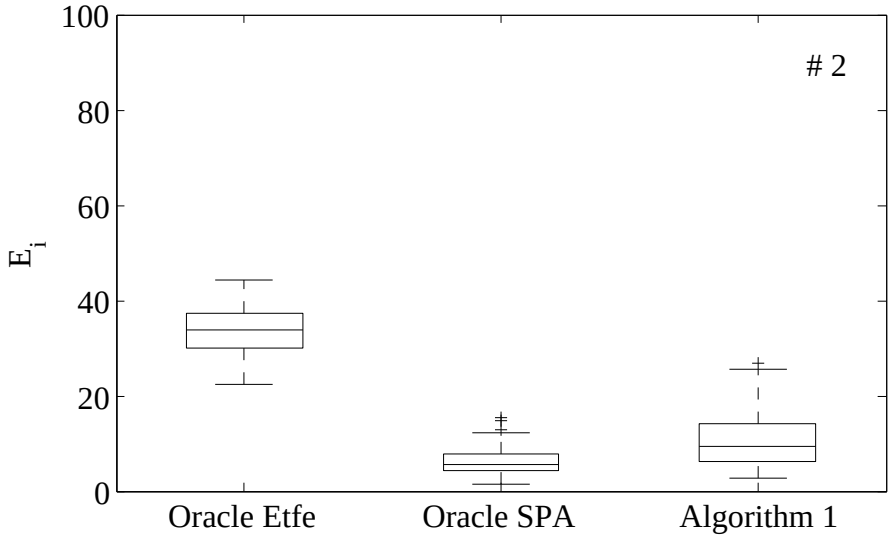
$$E_i(\%) = 100\frac{\|G - \hat{G}_i\|_2}{\|G\|_2}\,, \tag{6.51}$$

where $\hat{G}_i$ represents the transfer function estimate obtained in the i-th run.

Figures 6.8-6.9 show the box plots of the 100 reconstruction errors obtained by each estimator after the 4 Monte Carlo experiments. One can see that the performance of Algorithm 2 is comparable to the SPA algorithm tuned by an oracle. Moreover, when the noise variance increases, the estimation given by the proposed approach is often better than any possible estimate obtainable with the classic non parametric approaches.

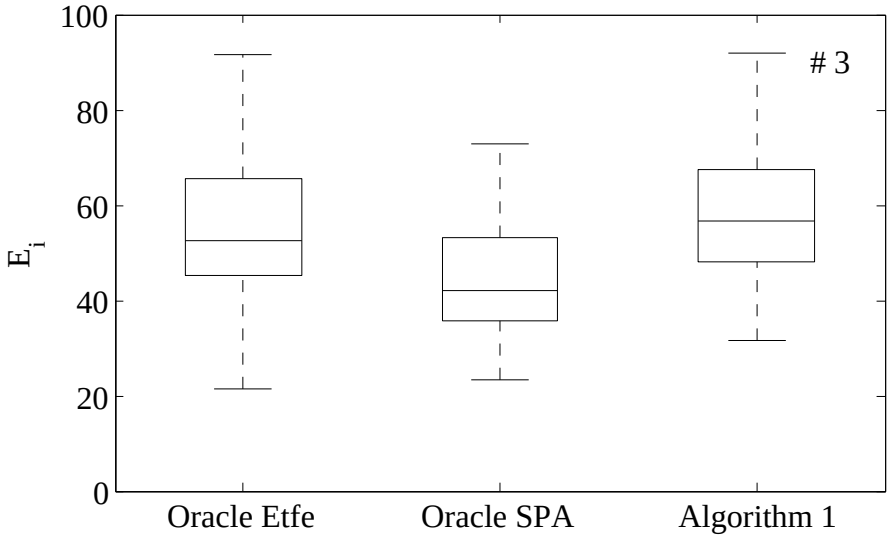


(a) Experiment 1

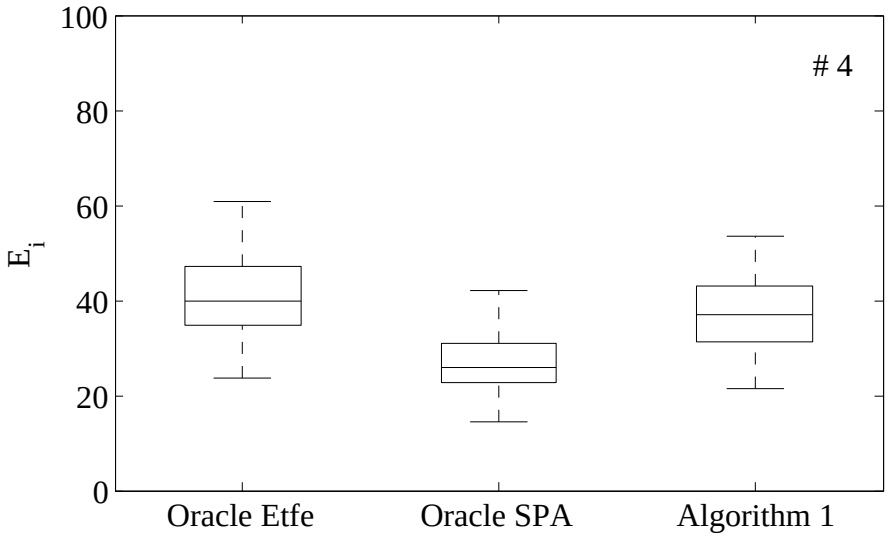


(b) Experiment 2

Figure 6.5: Results of simulations for the estimation of autocovariance functions.

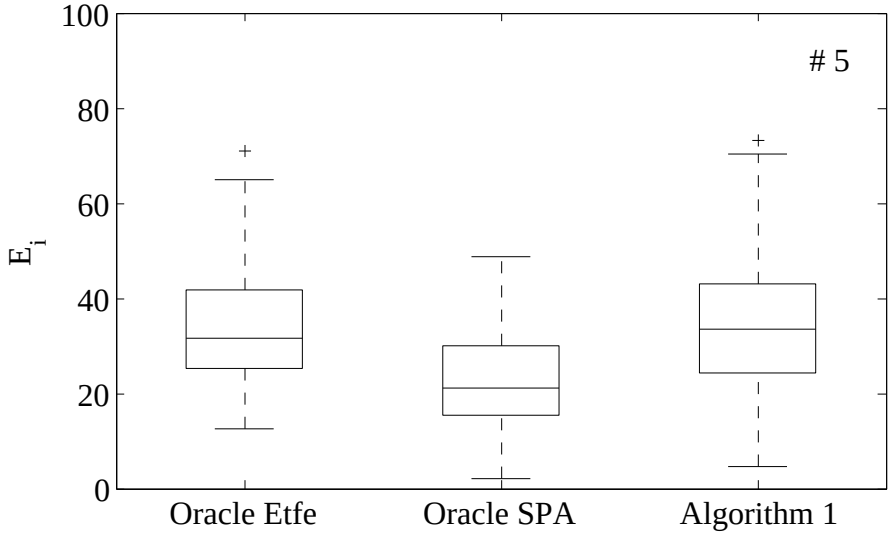


(a) Experiment 3

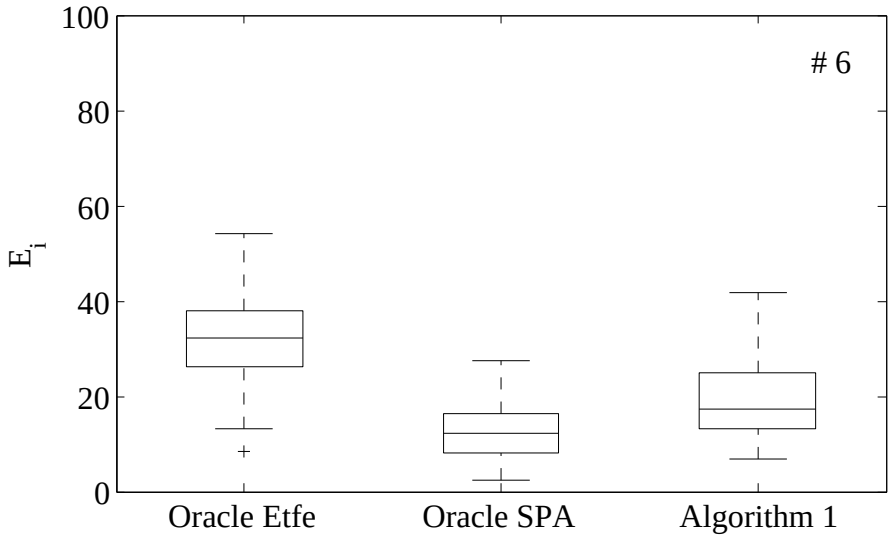


(b) Experiment 4

Figure 6.6: Results of simulations for the estimation of autocovariance functions.

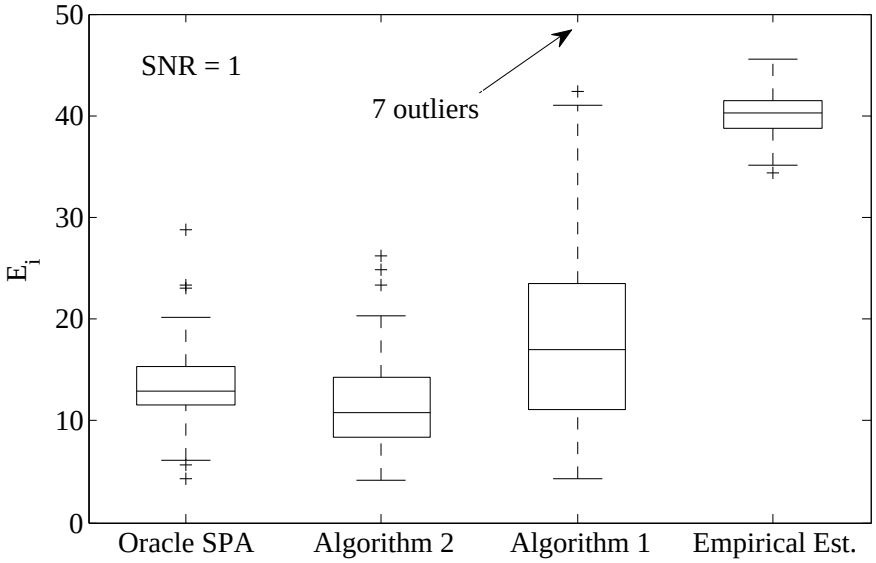


(a) Experiment 5

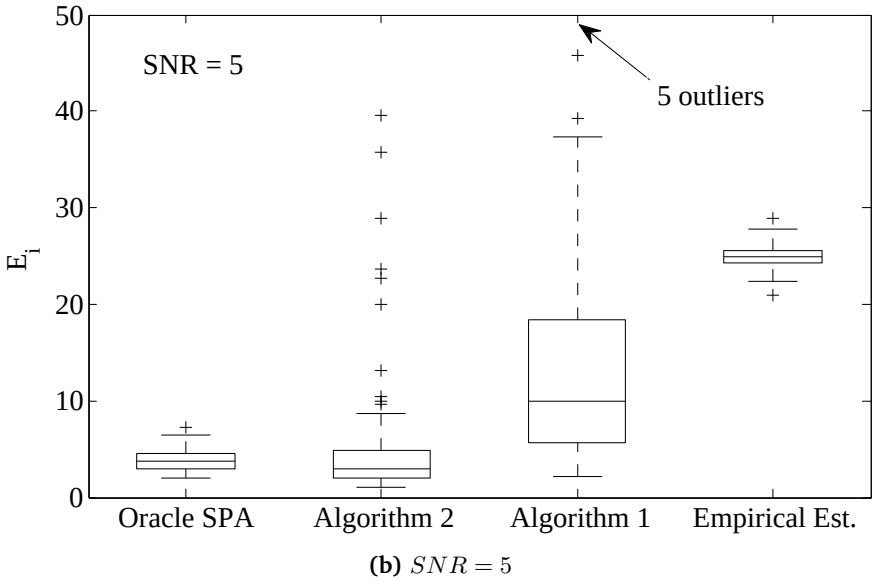


(b) Experiment 6

Figure 6.7: Results of simulations for the estimation of autocovariance functions.

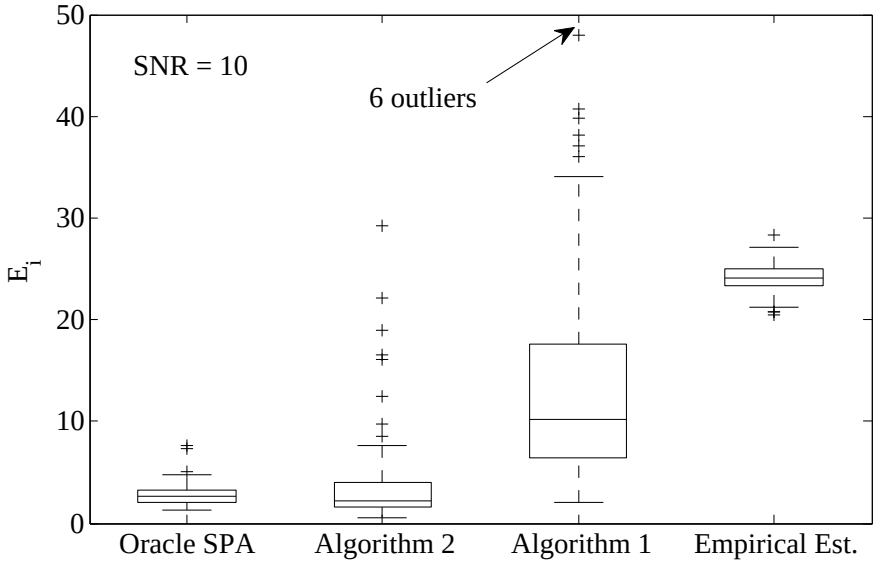


(a) $SNR = 1$

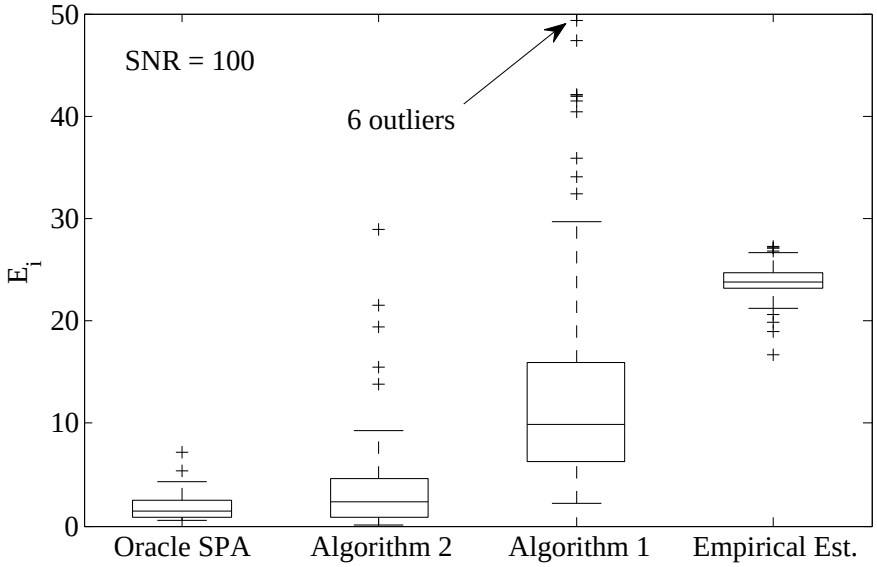


(b) $SNR = 5$

Figure 6.8: Results of the simulations for the identification of transfer functions.



(a) $SNR = 10$



(b) $SNR = 100$

Figure 6.9: Results of the simulations for the identification of transfer functions.

7

Possible extensions and future works

In this dissertation we have discussed some aspects of the analysis of stochastic systems with latent variables. In particular, we have focused on generalized factor analysis, both in static and dynamic settings, and errors-in-variables models. Furthermore, we have proposed a new nonparametric kernel-based spectrum estimation algorithm. In the following, we provide some indications of possible extensions and generalizations of the results presented in this thesis.

In **Chapter 3** we have introduced and analyzed generalized factor analysis. There are several open problems regarding flocking and generalized factor analysis; for example, it could be interesting to understand what conditions make this modeling paradigm applicable to time-varying systems, e.g. flocks in which the agents gradually separate from the group. From a theoretical point of view, some of the results of Section 3.3 do not have a counterpart for the dynamic version of GFA; we believe that this can be done quite straightforwardly. Moreover, we argue that, introducing some further assumptions, the connection between GFA and the Wold decomposition of Section 3.5 holds also for nonstationary sequences. Other extensions could regard the application of convex semidefinite programming to the decomposition of the covariance matrix of the observations, instead of using PCA method.

In **Chapter 4** we have addressed the problem of studying the generic properties of the zeros of tall blocked multirate systems. As part of future work, we intend to extend

the results obtained for multirate linear systems to a general case in which the output streams are available at coprime rates.

A natural future extension of the work on errors-in-variables models, described in **Chapter 5**, would be to discuss EIV identifiability in the multivariable case, in order to derive testable conditions also for this case. One should expect that identifiability is no longer generically guaranteed. We argue that the set of models compatible with a given input-output joint spectrum is dense, i.e. there are infinitely many models that can describe those data equally well.

Future developments regarding nonparametric spectrum estimation, introduced in **Chapter 6**, could concern the refinement of the estimates, e.g. when the process is known to be Gaussian, exploiting more accurate statistical models for the empirical autocovariance samples. If some more information on the process is available, it is possible to derive some results on the convergence rate of the estimates of the autocorrelation function. An interesting application of nonparametric kernel-based techniques could regard the development of a new algorithm for the identification of errors-in-variables models.

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