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CFT partition functions and moduli spaces of canonical curves

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Le teorie di campo conformi (CFT) in due dimensioni rappresentano un ambito di fecondo interscambio tra alcuni degli argomenti più avanzati in fisica teorica e in geometria algebrica. In particolare, lo studio delle funzioni di partizione in teorie conformi appare strettamente legato all'analisi della corrispondenza tra proprietà analitiche e proprietà algebriche delle superfici di Riemann chiuse. In questa tesi, vengono considerati alcuni nuovi aspetti di questa corrispondenza, in particolare quelli che emergono nelle teorie conformi associate a teorie di stringa e superstringa. Più precisamente, i parametri algebrici che determinano la curva canonica associata ad una superficie di Riemann non-iperellittica sono esplicitamente calcolati in termini di funzioni theta di Riemann valutate su punti generici della curva. I metodi proposti vengono inoltre applicati allo studio del locus singolare della funzione theta, anche in relazione all'approccio di Andreotti-Mayer al problema di Schottky, e alla restrizione della misura di Siegel allo spazio dei moduli delle curve canoniche.

Conformal field theories (CFT) represent a framework of fruitful interplay between some of the most advanced topics in theoretical physics and algebraic geometry. In particular, the investigation of the CFT partition functions is closely related to the analysis of the correspondence between analytic and algebraic properties of closed Riemann surfaces. In the present thesis, some new aspects of this correspondence, in particular the ones arising in the CFTs associated to string and superstring theories, are considered. More precisely, the algebraic parameters, determining the canonical curve associated to a non-hyperelliptic Riemann surface, are explicitly computed in terms of Riemann theta functions, evaluated at generic points of the curve. Moreover, the techniques here introduced are applied to the analysis of the singular locus of the theta function, also considered with respect to the Andreotty-Mayer approach to the Schottky problem, and to the restriction of the Siegel's measure to the moduli space of canonical curves.

CONTENTS

INTRODUCTION

Conformal field theories [8, 19, 27] have played an important role in several areas of theoretical physics and mathematics in the last 25 years.

The most famous application has been to string theory, since the classical and quantum theory of excitations of a string is described by a two-dimensional CFT on the world-sheet of the string. Conformal field theories have also been applied in statistical physics: in two dimensions for the Ising model and in three dimensions to describe the critical points of second or higher order phase transitions. Moreover, four dimensional CFT are supposed to play a role in elementary particle physics models. A result by Nahm [50] implies that six is the maximal number of dimensions for a unitary non-trivial conformal field theory; recently, some hints of the existence of such six-dimensional CFT's have been given [63].

Conformal field theories are defined as field theories which are invariant under the group of local conformal transformations, which, roughly speaking, are symmetries preserving the angles but not the lengths. In particular, in two dimensions, the Lie algebra associated to the conformal group is infinite dimensional. The generators if such algebra correspond to an infinite number of conserved charges and this implies that such theories are, in principle, exactly solvable.

The choice of the conformal class for the metric on a two-dimensional manifold is equivalent to the definition of a complex structure on the surface. Therefore, amplitudes in CFT naturally depend on the analytic structure of the surface, i.e. on the sheaf of holomorphic functions defined on the Riemann surface. Such analyticity properties can be made explicit by splitting the CFT vertex operator algebra in its chiral and anti-chiral part. On the other hand, one of the most fascinating aspects of conformal field theories is its relationship with some of the deepest results in algebraic geometry. This is just a facet of a more general correspondence between classes of algebraic varieties, with regular maps and sheaves, and classes of analytic spaces, with holomorphic mappings and sheaves, known as the GAGA principle (from Serre's *Géométrie algébrique et géométrie analytique*, [58]).

Two manifestations of GAGA principle in conformal field theories will play a prominent role in this thesis. As shown in chapter 1, a CFT assigns to each closed surface of genus g a *partition function*, which is a section of a line bundle on the moduli space $\mathcal{M}_g$ of closed Riemann surfaces of genus g. In particular, for the CFTs related to gauge fixed bosonic strings and superstrings (after integration over odd supermoduli), each partition function defines a measure on the corresponding moduli space. For bosonic strings, this is known as the Polyakov measure, and it can be expressed, apart from a factor representing the obstruction to the holomorphic factorization of the theory, as the square modulus of a holomorphic section of $\lambda_1^{-13} \otimes \lambda_2$. Here λ_n is, roughly speaking, the line bundle whose fibre at the point $C \in \mathcal{M}_g$ is the maximal external product of the space of holomorphic n-differentials on the Riemann surface C. Passing from the analytic to the algebraic data associated to Riemann surfaces, it is well-known that $\mathcal{M}_g$ admits a compactification (à la Deligne-Mumford) $\bar{\mathcal{M}}_g$ that is the moduli space of stable curves of genus g. Mumford proved that the line bundle $\lambda_1^{-13} \otimes \lambda_2$ on $\mathcal{M}_g$ admits a unique (up to normalization) non-vanishing holomorphic section, extending to a meromorphic section on $\bar{\mathcal{M}}_g$ with poles at the boundary. Belavin and Knizhnik [7] proved that such a holomorphic section is exactly the chiral factor in the Polyakov measure. The poles at the boundary admit a physical interpretation as the amplitudes corresponding to the propagation of the bosonic string tachyon for Riemann surfaces degenerating to stable curves with nodes.

Another example of this interplay between analytic and algebraic data is provided by CFTs on Riemann surfaces with $\mathbb{Z}_n$-symmetry. Let us consider the simplest case of hyperelliptic surfaces, corresponding to $n = 2$. Any hyperelliptic surface of genus $g \geq 2$ can be described in terms of an affine curve C in $\mathbb{C}^2$, defined by the polynomial equation

$$w^2 = \prod_{i=1}^{2g+2} (z - a_i) \ ,$$

where $(z, w) \in \mathbb{C}^2$. The restriction to C of the projection $(z, w) \mapsto z$ defines a meromorphic function z of degree 2 on the Riemann surface, which shows that any hyperelliptic surface can be represented as a 2-fold branched covering of the Riemann sphere. The pairwise distinct complex numbers $(a_1, \ldots, a_{2g+2})$, corresponding the position of the branching points on the sphere, represent the coordinates of the universal parameter space of hyperelliptic curves; the correspondent moduli space is the quotient of such a parameter space by the 3-parameters group of automorphisms of the sphere. In [42], a procedure has been described to obtain a CFT partition function on a hyperelliptic Riemann surface, from an amplitude in a “double” CFT with $\mathbb{Z}_2$-symmetry on the sphere. Such an amplitude is characterized by the insertion of $2g+2$ “twisted” operators at the branching points $a_1, \ldots, a_{2g+2}$, so that the dependence of the resulting partition function on such algebraic parameters of the hyperelliptic curve is explicit. Recently, such a procedure has been applied to a conjectural CFT [64, 28, 65], representing the holographic dual to a three-dimensional pure gravity theory with negative cosmological constant, to prove that partition functions on hyperelliptic Riemann surfaces can be consistently defined for all genera.

By computing the same partition functions in terms of the analytic data of the theory, one obtains remarkable algebro-geometric identities. More precisely, to each surface one can attach the data of its *Jacobian torus* and the corresponding theta functions (see appendix B). It is possible, in some cases, to compute the same CFT amplitude in terms of theta functions. For a hyperelliptic Riemann surface, by equating the results of the computations, one obtains the classical Thomae formula

$$\theta[\delta](0)^8 = \Big(\frac{\det A}{(2\pi i)^g}\Big)^4 \prod_{k<l} (a_{i_k} - a_{i_l})(a_{j_k} - a_{j_l}) \ .$$

In this formula, δ is an even theta-characteristic associated to a splitting of the set of complex parameters in a disjoint union $\{a_{i_1}, \ldots, a_{i_{g+1}}\} \sqcup \{a_{j_1}, \ldots, a_{j_{g+1}}\}$ and A is a matrix of base change between bases of holomorphic abelian differentials.

The procedure described above extends to the computation of partition functions of n-fold coverings of the sphere. By applying the same reasoning to CFT amplitudes on Riemann surfaces with $\mathbb{Z}_n$-symmetry, with $n > 2$, Bershadsky and Radul [9] derived a generalization of Thomae formula, which has been successively proved using standard algebro-geometric methods by Nakayashiki [51] (see also [20] for further generalizations).

In this thesis, some new methods are described toward an explicit description of this GAGA correspondence in the case of generic non-hyperelliptic Riemann surfaces of genus higher than two. The examples reported above nicely describe the physical motivations for such an analysis. First of all, one of the fundamental problems both in bosonic and in superstring theories is the definition of the measure on the moduli space $\mathcal{M}_g$ for g higher than two. By the Belavin-Knizhnik theorem in the bosonic string case and by analogous arguments for superstrings, this is strictly related to the problem of deriving an explicit formula for the Mumford form. In second instance, generalizations of the techniques holding for theories with $\mathbb{Z}_n$-symmetry would be of great interest for general CFTs.

There are two reasons for considering the space of non-hyperelliptic Riemann surfaces for genus $g \geq 3$. Fist of all, such a space is dense in $\mathcal{M}_g$, so that one can hope to extend most of the results to the whole moduli space by continuity arguments. This should be compared with the case of families of n-fold coverings of the sphere, which are of positive codimension in the moduli space $\mathcal{M}_g$ for g greater than three. In particular, hyperelliptic surfaces enjoy several peculiar properties, which are not shared by general Riemann surfaces. For instance, in the case of the conjectural CFT dual to three-dimensional gravity proposed in [64], the existence of consistent partition functions for all *non-hyperelliptic* Riemann surface would be a considerably stronger signal of the existence of the whole CFT, than just the hyperelliptic case. Another relevant example is provided by the Polyakov measure on the locus of genus 3 non-hyperellipitc Riemann surfaces, whose expression in terms of Riemann period matrices and theta constants has been derived in [6]. Such an expression does not hold in the hyperelliptic case; in fact, it is a non-trivial problem to check that such a formula admits a regular limit as one approaches the hyperelliptic locus.

From a more technical point of view, another advantage in considering non-hyperelliptic Riemann surfaces is that they admit an algebraic description in terms of canonical curves. As explained in chapter 5, a non-hyperelliptic Riemann surface of genus $g \geq 3$ can be embedded as a projective curve (a 1-*canonical* or simply *canonical* curve) in $\mathbb{P}H^0(K_C)^* \cong \mathbb{P}^{g-1}$, where $H^0(K_C)$ is the space of holomorphic abelian differentials on the surface. A similar construction enters in the definition of the Deligne-Mumford moduli space of stable curves, which is based on an n-canonical embedding in $\mathbb{P}H^0(K_C^n)^* \cong \mathbb{P}^{(2n-1)(g-1)-1}$ for $n \geq 3$; the moduli space $\mathcal{M}_g$ is defined by modding the parameter space of the n-canonical curves of genus g by the group $PGL((2n-1)(g-1), \mathbb{C})$ acting on the projective space. In the case of (1-)canonical curves, Petri's theorem [52] assures that the graded ideal $I(C)$ of homogeneous polynomials in $\mathbb{P}_{g-1}$ vanishing on the curve C is generated, with few exceptions, by its degree-2 component $I_2(C)$, i.e. by quadrics (for trigonal curves and smooth plane quintics, also the cubics are needed, see [3]). It follows that the parameter space of the canonical curves can be given in terms of the coefficients of quadrics and cubics; such coefficients play the same role of the parameters $\{a_1, \ldots, a_{2g+2}\}$ for the hyperelliptic curves.

In analogy with the Thomae formula, it should be possible to express the parameters of the quadrics defining the canonical curve in terms of the analytic data of the Jacobian torus associated to the Riemann surface and, more precisely, in terms of its period matrices and of the Riemann theta functions. Such a problem is one of the main subjects of the present thesis, thus it is worth explaining it in some more detail. The pair $(J, \mathcal{L}_\Theta)$ composed by the Jacobian torus J associated to a Riemann surface and the line bundle corresponding to the theta divisor Θ, defines a principally polarized abelian variety (ppav). Torelli's theorem assures that the map

$$i\colon \mathcal{M}_g \to \mathcal{A}_g \ ,$$

where $\mathcal{A}_g$ is the moduli space of g-dimensional ppav's, is an injection. Such an injection is induced by the period-mapping of the Torelli space $\mathcal{T}_g$ into the Siegel upper half-space $\mathfrak{H}_g$, with $\mathcal{M}_g \cong \mathcal{T}_g/Sp(2g,\mathbb{Z})$ and $\mathcal{A}_g \cong \mathfrak{H}_g/Sp(2g,\mathbb{Z})$. The expression of a CFT amplitudes in terms of theta functions is of great interest: for example, the factorization formulae for theta functions, in the limit of degenerating surfaces, are well-known and this allow non-trivial consistency checks among amplitudes for different genera. In particular, several results have been obtained for genus 2 and 3, a recent example being the expressions for the two-loop measure and 4-points amplitudes in type II superstring theory [12, 13, 14, 15, 17, 18]. Note that, whereas for genus 2 and 3, the image of $\mathcal{M}_g$ is dense in $\mathcal{A}_g$, for genus $g \geq 4$ the locus of Jacobian tori is a sublocus (called the *Jacobian locus* and denoted by $\mathcal{J}_g \cong i(\mathcal{M}_g)$) of positive codimension in $\mathcal{A}_g$. The characterization of $\mathcal{J}_g$ in $\mathcal{A}_g$ is the Schottky problem.

Such a problem has been solved by Shiota [59], who proved a conjecture by Novikov, characterizing the elements in the Jacobian locus in terms of the Kadomtsev-Petviashvili (KP) equation for the Riemann theta function. However, this solution is quite implicit and not so useful for CFT and string-theoretical computations.

It is worth mentioning at least another different approach to such a problem, due to Andreotti and Mayer [2]. In their beautiful construction, Andreotti and Mayer proposed to characterize the Jacobian locus in $\mathcal{A}_g$ through the dimension of the singular locus Θ_s of the theta function, i.e. the locus of points in a ppav where the theta function and all its first derivatives vanish. More precisely, they showed that $\mathcal{J}_g$ is an irreducible component of the variety $\mathcal{N}_4 \subset \mathcal{A}_g$, whose points satisfy $\dim \Theta_s \geq g-4$. A crucial point in the Andreotti-Mayer construction is the proof that, if C is a trigonal curve, $I_2(C)$ is generated by relations in the form

$$\sum_{i,j=1}^{g} \theta_{ij}(e) X_i X_j = 0 \ ,$$

as e varies in Θ_s. Here, $(X_1 : \ldots : X_g)$ are the projective coordinates of $\mathbb{P}^{g-1}$ corresponding to a canonically normalized basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(K_C)$ (see appendix B) and θ_{ij} denotes the second derivative of the theta function. Such a result has received many remarkable generalizations, among which at least two deserve citation: Arbarello and Harris [4] proved that the relations above generate I_2 for general curves of genus $g \leq 6$ and that, for all g, they generate all the quadrics of rank not greater than 4; finally, Green [29] proved that such relations generate I_2 for all genera, so that, as a consequence, I_2 can be generated by quadrics of rank not greater than 4 only.

The relationship between the quadrics passing through a curve and the Jacobian locus $\mathcal{J}_g \cong \mathcal{T}_g/Sp(2g,\mathbb{Z})$ can be understood as follows. The cotangent $T^*_C\mathcal{T}_g$ to the Torelli space $\mathcal{T}_g$ at the point representing the curve C is naturally identified, via the Kodaira-Spencer map, with the space of holomorphic quadratic differentials, which, after canonical embedding in $\mathbb{P}^{g-1}$, correspond to the vector space of homogeneous polynomials of degree 2 on the projective curve. Such a correspondence uniquely extends to an identification of the cotangent $T^*_C\mathfrak{H}_g$ to $\mathfrak{H}_g$ at (the Riemann period matrix of) C with the space of homogeneous polynomials of degree 2 on $\mathbb{P}^{g-1}$. Hence, the quadrics defining the canonical curve C correspond to the linear relations defining $T^*_C\mathcal{T}_g$ as a subspace of $T^*_C\mathfrak{H}_g$ (more details are given in chapter 7). Note that, once one fixes an $Sp(2g,\mathbb{Z})$-invariant measure on $\mathfrak{H}_g$, such linear relations for the cotangent spaces enter in the restriction of such a measure to the moduli space $\mathcal{M}_g$. An example, that is relevant for string theory, is given by the Siegel measure, which, for genus 2 and 3, is proportional to the Polyakov measure times the square modulus of a meromorphic modular form. For genus $g \geq 4$ the restriction of the Siegel measure to the moduli space, derived in [44], is described in chapter 7.

The content of this thesis is mainly based on the papers [44, 45, 46]. In the first chapter, we introduce an axiomatic definition of conformal field theories, following Segal's approach [57]. The main point is the definition of the CFT partition function of genus $g \geq 1$ as a section of a tensor power of the determinant line bundle on the moduli space $\mathcal{M}_g$. As shown in section 1.4, for gauge-fixed bosonic strings, this result specializes to the Belavin-Knizhnik theorem, relating the Polyakov measure on the moduli space to the Mumford form.

In chapter 2, upon introducing a powerful notation for symmetric tensor products of vector spaces, two combinatorial lemmas [45] are proved, which are of interest on their own and will be repeatedly applied in the subsequent derivations.

In chapter 3, some useful propositions due to Fay [23, 24] are presented, relating the determinants of holomorphic n-differentials to theta functions and prime forms. In literature, such formulae have been applied to the computation of string-theoretical multiloop amplitudes and, in particular, partition functions [1, 61, 62]. By combining such propositions and the lemmas of chapter 2, an explicit expression for the Mumford form for genus 2 in terms of theta constants is explicitly derived.

Chapters 4, 5 and 6 constitute the core of the thesis. In chapter 4, distinguished bases of holomorphic n-differentials for non-hyperelliptic Riemann surfaces are introduced. Such a definition resembles an analogous construction introduced by Petri [52] to derive his theorem on ideals of canonical curves. The crucial differences rely on the normalization of such bases and in their definition in terms of determinants, which, when combined with the propositions of chapter 3, immediately leads to a proof of Fay's trisecant identity [44]. Such bases are used in chapter 5 to fix some preferred projective coordinates for the canonical curve. This leads to an explicit expression in terms of Riemann theta functions, evaluated at general points of the surface, for the coefficients of a minimal set of quadrics and cubics generating the ideal of the canonical curve C. We notice that, even if the distinguished bases play a key role in the derivation of such formulae, the coefficients can be readily expressed in terms of an arbitrary basis of holomorphic 1-differentials, or, equivalently, of an arbitrary set of projective coordinates on $\mathbb{P}^{g-1}$. A crucial role in the derivation is played by the combinatorial lemmas of chapter 2, which show that each quadric is essentially equivalent to a determinantal relation among Riemann theta functions on the curve [45]. The relationship between the quadrics derived in chapter 5 and the the quadrics related to singular points of the theta function is analyzed in chapter 6. The main tool introduced in this chapter is the section K [46], which encodes the data of the set of generators of the ideal of quadrics introduced in the former chapter. The main results are Theorem 6.3, relating the zeroes of such a section to points on the singular locus of the theta function and Theorem 6.11, which describes the modular properties of K.

In chapter 7, the constructions of the former chapters are applied to derive the volume form on the moduli space of canonical curves, induced by the Siegel measure. Several equivalent expressions are derived. The first one follows from the Wirtinger's Theorem and its derivation heavily relies on the notation introduced in chapter 2. The second formula is given in terms of the distinguished bases defined in chapter 4. Such a formula is the direct consequence of the description of the cotangent to Torelli space $\mathcal{T}_g$ as a subspace of the cotangent to the Siegel upper half-space $\mathfrak{H}_g$. The linear equations defining such a subspace are in one to one correspondence with the quadrics described in chapter 5. Moreover, a remarkable relationship between the Siegel metric at the point representing the curve C and the Bergman metric on C is shown.

Finally, in chapter 8, the distinguished bases, the set of quadrics and the section K are explicitly constructed for a particular family of genus 4 non-hyperelliptic curves. Notice that a generalized Thomae formula has been recently derived in [20] for this family of curves. In this thesis, we give an independent derivation of the prime forms in term of the algebraic parameters of the curve.

1. AXIOMATIC DEFINITION OF CONFORMAL FIELD THEORY

In this chapter, we describe the main steps towards an axiomatic definition of conformal field theory, as proposed by Segal [57]. Segal's approach is based on the path integral formalism of quantum field theory; in facts, the aim of such an approach is to rigorously axiomatize CFT by, simultaneously, keeping clear the geometric intuition of a "sum over stories" which is typical of path integral. (Several authors, however, point out that, in order to fix all the technical subtleties and give a rigorous and complete mathematical treatment of Segal's definition, some concepts, such as the one of a 2-category, are required, which are far from being "intuitive" from a physical point of view - see for example [34, 35, 36, 25]). This should be compared to other different approaches to CFT, whose starting point is the algebra of operators on the Hilbert spaces of states [8, 27].

One of the concepts we are most interested in is the definition of a genus g partition function. We will show that, in a general CFT, this is the section of a line bundle $\mathrm{Det}^{\otimes p} \otimes \overline{\mathrm{Det}}^{\otimes q}$, $p, q \in \mathbb{C}$, $p - q \in \mathbb{Z}$, on the moduli $\mathcal{M}_g$ space of closed Riemann surfaces of genus g. Some remarkable consequences of this result are the Mumford isomorphism and, when applied to the CFT's arising in bosonic string theory, the Belavin-Knizhnik theorem [7], relating the Polyakov string measure to the Mumford form.

1.1 *Segal's approach: motivations and axioms*

In this section, we discuss the motivations and the problems related to an axiomatic definition of a two-dimensional Conformal Field Theory based on the path integral quantization of the classical theory, and then we propose the axioms along the lines described by G. Segal in [57].

1.1.1 *Field theories and cobordisms*

In order to justify the fundamental Segal's axioms, let us first consider the general features expected from the path-integral formulation of a two-dimensional field theory and then specialize to the case of conformal theories.

We will only focus on theories whose main objects are closed oriented 1-manifolds X (generalizations, for example to open string theories, theories is conceptually analogous, but requires solving some technical issue). Note that any such X is just the disjoint union of circles S^1. Such a theory describes the dynamics of a space of fields $\mathcal{F}(X)$ defined on X. Hence, $\mathcal{F}(X)$ represents the space of classical configurations; correspondingly, a state in the quantum theory is given by a vector ray in the Hilbert space $L^2(\mathcal{F}(X))$ of wave-functions $\psi(f)$

on the field space. Denote by $\bar{X}$ the 1-manifold X with reversed orientation. Let Y be a 2-dimensional surface whose boundary is splitted in the disjoint union $\partial Y \cong \bar{X}_1 \sqcup X_2$ of components diffeomorphic to an "outgoing" 1-manifold X_2 and an "incoming" 1-manifold X_1 with reversed orientation. Let us call such a surface Y a *cobordism* from X_1 to X_2. Depending on which kind of theory we are considering (topological, conformal, gravitational,...), one can require that additional structures are defined on such Y; for example, one can require Y to be a Riemannian manifold. The precise definition for a conformal field theory will be given in Definition 1.1.

In the classical field theory, time evolution from a configuration of fields f_1 on X_1 to the fields f_2 on X_2 can be described by a configuration of fields g on the surface Y such that $g_{|X_i} = f_i$, $i = 1, 2$. Such a g must satisfy the classical equations of motion, i.e. must be a stationary point for a bounded-below real functional $S[g]$ (the action) defined of $\mathcal{F}(Y)$. In the quantum theory, one postulates the existence of an integral operator $K : L^2(\mathcal{F}(X_1)) \to L^2(\mathcal{F}(X_2))$, which can be (formally) expressed as

$$(K\psi)(f_2) = \int_{f_1 \in \mathcal{F}(X_1)} K(f_2, f_1)\psi(f_1)[df_1] \ ,$$

where $[df_1]$ is some measure on the space $\mathcal{F}(X)$. In this expression, $K(f_2, f_1)$ is the sum over all the cobordisms Y from X_1 to X_2 of the operators

$$K_Y(f_2, f_1) := \int_{g \in \mathcal{F}(Y)} e^{-S[g]}[dg] \ ,$$

where the integration is over all the fields $g \in \mathcal{F}(Y)$ such that $g_{|X_i} = f_i$, $i = 1, 2$. More precisely, one should sum over a space of "classes" of such cobordisms, where the equivalence relation defining such classes depends on the particular kind of theory we are considering. Here, $S[g]$ is the action and $e^{-S[g]}[dg]$ is assumed to be a well-defined measure on $\mathcal{F}(Y)$.

For each pair of cobordisms Y_1 from X_1 to X_2 and Y_2 from X_2 to X_3, one can define the composition $Y_2 \circ Y_1$ as the cobordism from X_1 to X_3 given by "gluing" together Y_1 and Y_2 along X_3. The precise definition of the process of "gluing" requires fixing some subtleties in the case some additional structures (for example, a metric) are defined on the surfaces. Locality of the theory imposes that, for any such composition $Y_2 \circ Y_1$,

$$K_{Y_2 \circ Y_1}(f_3, f_1) = \int_{f_2 \in \mathcal{F}(X_2)} K_{Y_2}(f_3, f_2) K_{Y_1}(f_2, f_1) \ .$$

Such a construction applies in general to any theory whose basic objects are 1-dimensional closed manifolds. The specialization to certain classes of theories can be given by specifying some additional data and requirements. Let us describe such data in the case of a conformal field theory:

1. *Simmetries of the action.* The characterizing feature of a CFT is that the action $S[g]$ depends on the conformal class of a metric h (with some regularity conditions) defined on the surface Y. In other words, the action is invariant under local conformal transformations, corresponding to local diffeomorphisms and to local Weyl transformations

$$h(\sigma) \mapsto e^{\omega(\sigma)} h(\sigma) \ ,$$

where h is the metric, σ denotes some local coordinates on Y and ω is a real regular function with suitable boundary conditions.

2. *Isomorphism classes of surfaces.* In correspondence with such an invariance of the action, one must consider 2-dimensional manifolds Y with a fixed conformal structure. Equivalently, Y is a Riemann surface, and the "classes" of cobordisms, one should sum over in the path integral, are identified with isomorphism classes of Riemann surfaces.

3. *Composition of cobordisms.* The "gluing" process is naturally defined among classes of diffeomorphic smooth 2-manifolds. However, given the conformal structures on Y_1 and Y_2, there are several inequivalent ways to obtain a conformal structure on $Y_1 \circ Y_2$.

Let us consider a simple example of the problem considered in point 3. Let Y be a Riemann surface with the topology of a cylinder $\mathrm{S}^1 \times [0, 1]$ with one ingoing and one outgoing boundary circle. Any cylinder is conformally equivalent to an annulus $A_r \subset \mathbb{C}$, given by

$$A_r := \{z \in \mathbb{C} \mid r \leq |z| \leq 1\} ,$$

for some suitable $0 < r < 1$. Then, the moduli space of conformal structures on a cylinder is parametrized by a unique real parameter r, $0 < r < 1$.

Let us consider the process of gluing the ingoing and the outgoing boundary together to obtain a 1-manifold with the topology of a torus. This amounts to choose a diffeomorphism

$$f : X_{out} \to X_{in} ,$$

from the outgoing circle $X_{out} := \{|z| = 1\}$ to the ingoing one $X_{in} := \{|z| = r\}$. Different choices of f lead to diffeomorphic 2-manifolds, so that the gluing process is well-defined from the purely topological point of view. Furthermore, by gluing the conformal structures, such a torus can be naturally seen as a Riemann surface. However, it is clear that such a conformal structure depends on the choice of f: for example, the Riemann surface given by $e^{i\alpha}f$, for any $0 < \alpha < 2\pi$, is not isomorphic to the one given by f. It is also clear that there is no canonical way, for general Riemann surfaces, to choose the gluing function f.

This forces us to provide some additional information than just a conformal structure on Y. It turns out that it is sufficient to fix a real-analytic parametrization for the boundary $\partial Y \cong \bar{X}_{in} \sqcup X_{out}$, compatible with the complex structure on Y and with the orientation of X_{in} and X_{out}. That is, for each circle S^1 in the boundary of Y, one should specify a map f from

$$\mathrm{S}^1 \equiv \{z \in \mathbb{C} \mid |z| = 1\} ,$$

to ∂Y, which extends to a holomorphic map $\tilde{f} : A \to Y$, where

$$A := \begin{cases} A_r \equiv \{z \in \mathbb{C} \mid r < |z| < 1\} , & \text{if } X \text{ is outgoing} , \\ A^{\infty}_{1/r} := \{z \in \mathbb{C} \mid 1 < |z| < 1/r\} , & \text{if } X \text{ is incoming} , \end{cases}$$

for some $r \in (0, 1)$. Two Riemann surfaces Y_1 and Y_2 with parametrized boundary are isomorphic if there exists a biholomorphic function $F : Y_1 \to Y_2$ compatible with parametrization, i.e. such that, for each circle S^1 in ∂Y_1 parametrized by f, the parametrization of $F(\mathrm{S}^1)$ is $F \circ f$. This yields the following definition.

Definition 1.1. In a 2-dimensional conformal field theory, a *cobordism* $Y : X_{in} \rightsquigarrow X_{out}$ between the closed 1-manifolds X_{in} and X_{out}, is an isomorphism class of Riemann surfaces, with a real-analytic parametrization $\bar{X}_{in} \sqcup X_{out} \to \partial Y$ of the boundary. On the space of cobordisms an involution and a composition are defined:

Conjugation. For each cobordism $Y : X_1 \rightsquigarrow X_2$, the conjugate cobordism $\bar{Y} : X_2 \rightsquigarrow X_1$ corresponds to the complex conjugated Riemann surface, with the same boundary parametrization.

Composition. The composition $Y \equiv Y_1 \circ Y_2$ (or gluing) of the cobordisms $Y_1 : X_{1,in} \rightsquigarrow X \sqcup X_{1,out}$ and $Y_2 : X \sqcup X_{2,in} \rightsquigarrow X_{2,out}$ is the cobordism $Y : X_{1,in} \sqcup X_{1,in} \rightsquigarrow X_{2,out} \sqcup X_{2,out}$ such that there exist embeddings $\phi_i : Y_i \to Y$, $i = 1, 2$, satisfying

- ϕ_i is bi-holomorphic as a map from Y_i to $\phi_i(Y_i)$ and is compatible with parametrizations on each component of $X_{i,in}$ and $X_{i,out}$, $i = 1, 2$.
- $\phi_1(Y_1) \cup \phi_2(Y_2) = Y$.
- $\phi_i^{-1}(\phi_1(Y_1) \cap \phi_2(Y_2)) = X$.
- for each component of X, with parametrization f_i with respect to the morphism Y_i, $i = 1, 2$, $\phi_1 \circ f_1 = \phi_2 \circ f_2$ as functions on S^1.

Note that the composition of cobordisms depends on the 1-manifold along which the Riemann surfaces are glued (in this sense, the notation $Y_1 \circ Y_2$ is imprecise). The fourth condition in the definition of $Y = Y_1 \circ Y_2$ implies that the function $f : \mathrm{S}^1 \to Y$ given by $f := \phi_1 \circ f_1 = \phi_2 \circ f_2$ extends to a holomorphic function $\tilde{f} : A \to Y$ on an annulus A, with $\mathrm{S}^1 \subset A \subset \mathbb{C}$. It follows that the complex structure on Y is uniquely determined; furthermore, it is independent of the choice of the embeddings ϕ_1, ϕ_2. (This is true if X has only one component; otherwise, there some subtleties related to permutations of the components of X, which can be elegantly solved in the framework of 2-categories [34, 35, 36, 25].) Finally, we observe that Definition 1.1 also makes sense for the composition of two morphisms Y_1 and Y_2 along an empty 1-manifold $X = \varnothing$, with $Y_1 \circ Y_2$ being the disjoint union $Y_1 \sqcup Y_2$. The space of cobordisms is naturally endowed with a topological structure; each connected component is the set of cobordisms with a fixed topology and orientation of the boundary components for the corresponding surface.

Proposition 1.1. *Let $\mathcal{C}_\alpha$ be the space of morphisms with a given topology α. If α has no closed components, then the tangent space at $Y \in \mathcal{C}_\alpha$ is given by*

$$T_Y \mathcal{C}_\alpha := \mathrm{Vect}(\partial Y) / \mathrm{Vect}(Y) \ ,$$

i.e. the space of deformations of the boundary parametrization mod the subspace of deformations that extend holomorphically to Y.

Proof. Proposition follows since, if Y has no closed components, any morphism Y' sufficiently close to Y can be holomorphically embedded in Y (because Y is a Stein manifold). Therefore, each deformation of Y corresponds to an element $\mathrm{Vect}(\partial Y)$. On the other hand, Y and Y' are isomorphic if and only if the deformation of the boundary extends holomorphically to the whole Y. $\square$

Definition 1.1 also describes the category $\mathcal{C}$ whose objects are the cobordisms and the morphisms are the identity, the conjugation $Y \to \bar{Y}$ and the gluing $Y_1 \sqcup Y_2 \to Y_1 \circ Y_2$.

1.1.2 Axiomatic definition and main properties

Let us consider the axiomatization of CFT on lines proposed by Segal. Formally, the basic object in such an axiomatization, is a functor from the category[1] $\mathcal{S}$ whose objects are 1-dimensional manifolds and the morphisms are the cobordisms, to a category of vector spaces with linear operators as morphisms. More precisely, we will consider topological vector spaces with a non-degenerate bilinear form and trace-class operators on such spaces, defined as follows.

Definition 1.2. Let E and F be complete topological spaces with a non-degenerate bilinear form $(\,\cdot\,,\,\cdot\,)$. An operator $A : E \to F$ is *trace-class* if it can be written as

$$A = \sum_{i \in I} \rho_i (e_i,\,\cdot\,) f_i \ ,$$

where I is a countable set of indices, $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ are orthonormal sets in E and F, respectively, and ρ_i, $i \in I$, are complex numbers such that

$$\sum_{i \in I} |\rho_i| < \infty \ .$$

Definition 1.3. A *conformal field theory* is a projective functor from the category $\mathcal{S}$ of closed oriented 1-manifolds and cobordisms to the category of topological vector spaces and trace-class operators, satisfying the following axioms:

1. To each closed smooth oriented 1-manifold X, a CFT associates a locally convex complete topological vector space $\mathcal{H}_X$ with a non-degenerate bilinear form, in such a way that finite disjoint unions correspond to tensor products. More precisely, there exists a canonical multilinear map

$$\mathcal{H}_{X_1 \sqcup \ldots \sqcup X_n} \to \bigotimes \mathcal{H}_{X_i} \ ,$$

which is required to be compatible with permutations of the components.

2. To each cobordism $Y : X \rightsquigarrow X'$, we associate a ray in the space of trace-class linear operators from $\mathcal{H}_X$ to $\mathcal{H}_{X'}$, such that
   (a) under composition $Y_1 \circ Y_2$ of cobordisms along X, the trace with respect to $\mathcal{H}_X$ induces the projective isomorphism $U_{Y_1 \circ Y_2} \cong U_{Y_1} \circ U_{Y_2}$;
   (b) U_Y varies continuously with respect to deformations of Y.
   (c) If $Y' : X_{in} \rightsquigarrow X_{out} \sqcup \bar{X}$ is obtained from $Y : X_{in} \sqcup X \rightsquigarrow X_{out}$ by reversing the orientation of the 1-manifold X, then U_Y is mapped to $U_{Y'}$ through the canonical isomorphism $\mathrm{Hom}(\mathcal{H}_{X_{in}} \otimes \mathcal{H}_X, \mathcal{H}_{X_{out}}) \to \mathrm{Hom}(\mathcal{H}_{X_{in}}, \mathcal{H}_{X_{out}} \otimes \mathcal{H}^*_X)$, where the map $H_X \to H^*_X$ is induced by the bilinear form $(\,\cdot\,,\,\cdot\,)$.

[1] Strictly speaking, this is not a category, because the identity morphism is missing. This issue can be fixed by letting $\mathcal{S}$ being a nuclear ideal in a larger category (see [10]), or admitting degenerate cobordisms, such as annuli with zero width, among the morphisms.

Here, for a complex topological vector space E, we denote by $\bar{E}$ its complex conjugate and by E^* its dual, i.e. the vector space of complex continuous linear functionals on E.

Let us describe the basic properties and the direct consequences of this axioms.

- Any closed oriented 1-manifold X is just a union of circles, so that the vector space $\mathcal{H}_X$ is completely determined by specifying
$$\mathcal{H} := \mathcal{H}_{\mathrm{S}^1} \ .$$
- The first axiom implies that the empty manifold $X \equiv \varnothing$ is associated to $\mathcal{H}_\varnothing = \mathbb{C}$.
- By considering the disjoint union of cobordisms as a composition along the empty set $X = \varnothing$, the second axiom gives the following rule:
$$U_{Y_1 \sqcup Y_2} \cong U_{Y_1} \otimes U_{Y_2} \ .$$
- Any closed Riemann surface Y is a cobordism $Y : \varnothing \rightsquigarrow \varnothing$, so that the corresponding $U_Y : \mathbb{C} \to \mathbb{C}$ defines a continuous section Z_g of a line bundle on the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g, for each $g \geq 0$.
- Axiom 2) describes a functor from the category $\mathcal{C}$ of cobordisms to 1-dimensional vector spaces, together with a natural embedding $E_Y \to \mathcal{H}_{\partial Y}$, where E_Y is the line associated to the cobordism Y.

A complex structure can naturally be defined on each component in the space of morphisms.

Definition 1.4. A conformal field theory is *holomorphic* if for each holomorphic family of cobordisms $\{Y_b\}_{b\in\mathcal{B}}$, parametrized by the complex manifold $\mathcal{B}$, the rays $\{U_{Y_b}\}_{b\in\mathcal{B}}$ form a holomorphic bundle on $\mathcal{B}$.

1.2 Spaces of cobordisms

By a basic result in the theory of Riemann surfaces, each cobordism is the composition of cobordisms corresponding to disjoint unions of disks, cylinders (that can be also understood as disks with one hole, or annuli) and pants (or disks with two holes). Therefore, to completely determine the theory, it is sufficient to consider the linear operators associated to such topologies and their composition.

1.2.1 The space of disks $\mathcal{D}$

Let $\mathcal{D}$ (resp., $\bar{\mathcal{D}}$) denote the set of disks with one outgoing (resp., incoming) parametrized boundary. Any disk is conformally equivalent to the unit disc D. There is a preferred parametrization of the unit disk D, given by the identity map $\mathrm{S}^1 \to \partial D$. Any other element of $\mathcal{D}$ corresponds to a different parametrization of the boundary, i.e. to an element of the group $\mathrm{Diff}_{an}(\mathrm{S}^1)$ of real-analytic diffeomorphisms of the circle. Thus is a Lie group, whose Lie algebra we denote by $\mathrm{Vect}(\mathrm{S}^1)$. A set of generators of the complexification $\mathrm{Vect}_{\mathbb{C}}(\mathrm{S}^1)$ is given by

$L_n := e^{in\theta}\frac{d}{d\theta}$, The map $\mathrm{Diff}_{an}(\mathrm{S}^1) \to \mathcal{D}$ is not Note that the group $PSU(1,1,\mathbb{C})$ of automorphisms of the unit disk, given by

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}} \ , \qquad |a|^2 - |b|^2 = 1 \ ,$$

is a subgroup of $\mathrm{Diff}_{an}(\mathrm{S}^1)$. Two parametrizations that differ only by an element of $PSU(1,1,\mathbb{C})$ should be identified, so that

$$\mathcal{D} \equiv \mathrm{Diff}^+(\mathrm{S}^1)/PSU(1,1,\mathbb{C}) \ .$$

An analogous description holds for $\bar{\mathcal{D}}$, whose preferred element is given by

$$D_\infty := \{z \in \hat{\mathbb{C}} \mid |z| > 1\} \ ,$$

where $\hat{\mathbb{C}}$ is the Riemann sphere.

1.2.2 The semigroup of annuli $\mathcal{A}$

Let $\mathcal{A}$ denote the set of annuli with one incoming and one outgoing parametrized boundary circles. Note that $\mathcal{A}$ has a natural structure of a semigroup under composition. As stated before, any annulus is conformally equivalent to

$$A_r := \{z \in \mathbb{C} \mid r < |z| < 1\} \tag{1.1}$$

for some $r \in (0,1)$. We identify A_r with the element of $\mathcal{A}$ given by the parametrizations $z \mapsto z$ and $z \mapsto rz$ of the outgoing and incoming circle, respectively. Any element of $\mathcal{A}$ is determined by $r \in (0,1)$ and by the parametrizations of its boundary circles and the group of automorphisms of an annulus is given by $U(1)$, the group of rigid rotations, so that $\mathcal{A}$ is homeomorphic to

$$(0,1) \times (\mathrm{Diff}_{an}(\mathrm{S}^1) \times \mathrm{Diff}_{an}(\mathrm{S}^1))/U(1) \ .$$

More explicitly, to each triple (r,ϕ,ψ), we denote by $\phi A_r \psi^{-1} \in \mathcal{A}$ the annulus given by A_r, with the parametrization ϕ and ψ^{-1} of the outgoing and of the incoming boundary circles, respectively.

The complex structure on $\mathcal{A}$ is provided by the following proposition.

Proposition 1.2. *Each element of $\mathcal{A}$ corresponds to a pair of holomorphic functions $f_0 : D \to \mathbb{C}$ and $f_\infty : D_\infty \to \hat{\mathbb{C}}$ such that $f_0(\partial D) \cap f_\infty(\partial D_\infty) = \varnothing$ and*

$$f_0(z) = a_1 z + a_2 z^2 + \dots \ , \qquad z \in D \ , \tag{1.2}$$

$$f_\infty(z) = (z^{-1} + b_2 z^{-2} + \dots)^{-1} \ , \qquad z \in D_\infty \ . \tag{1.3}$$

Proof. The functions f_0 and f_∞ naturally determine an element of $\mathcal{A}$, corresponding to the annulus bounded by the curves $f_0(\partial D)$ and $f_\infty(\partial D_\infty)$ and parametrizations given by f_0 and f_∞ themselves. Conversely, fix $A \in \mathcal{A}$. The composition $D_\infty \circ A \circ D$ with the disks $D \in \mathcal{D}$ and $D_\infty \in \bar{\mathcal{D}}$ can be identified with the Riemann sphere $\hat{\mathbb{C}}$. By Definition 1.1, such a composition of cobordisms corresponds to holomorphic embeddings of D, D_∞ and A in $\hat{\mathbb{C}}$, we denote by f_0, f_∞ and f_A, respectively. Such functions are determined only up to automorphisms of $\hat{\mathbb{C}}$, but the ambiguity can be fixed by requiring that $f_0(0) = 0$ and that $f_\infty(\infty) = 0 = f'_\infty(\infty)$. Then, f_0 and f_∞ satisfy the conditions of the proposition and the corresponding element of $\mathcal{A}$ is identified with A by f_A. □

Note that the multiplicative semigroup

$$\mathbb{C}^\times_{<1} := \{z \in \mathbb{C}^\times \mid |z| < 1\} \ ,$$

is a sub-semigroup of $\mathcal{A}$. In fact, to each $q \in \mathbb{C}^\times_{<1}$, one can associate the annulus A_q given by the functions $f_\infty(z) = z$ and $f_0(z) = qz$, and it is clear that $A_q \circ A_{q'} = A_{qq'}$. Such a definition restricts to Eq.(1.1) for $q = r \in \mathbb{R}$. On the other hand, a semigroup homomorphism

$$\lambda : \mathcal{A} \to \mathbb{C}^\times_{<1} \ ,$$

is canonically defined. Let $\hat{A}$ be the torus obtained by gluing the incoming and the outgoing boundary components and let $\tau(\hat{A}) \in \mathbb{H}$ be its modular parameter; then

$$\lambda(A) := e^{2\pi i \tau(\hat{A})} \ .$$

Note that $\lambda(A_q) = q$.

1.2.3 Hilbert space structure on $\mathcal{H}$ and unitarity

Axiom 2 associates a ray of trace-class linear maps on the locally convex $\mathcal{H} \equiv \mathcal{H}_{\mathrm{S}^1}$ to each $A \in \mathcal{A}$. In particular, it is possible to choose the operators $U_q := U_{A_q}$, $0 < |q| < 1$, in such a way to obtain a *genuine* representation of the semigroup $\mathbb{C}^\times_{<1}$. Let $\check{\mathcal{H}}$ be the union of the images of $\mathcal{H}$ under U_r, $r \in \mathbb{R}$, $0 < r < 1$. In general, the completion of the image of $\check{\mathcal{H}}$ under the natural map $\check{\mathcal{H}} \to \mathcal{H}$ does not coincide with $\mathcal{H}$. However, since any cobordism Y can be written as a composition with some U_r as a component, it is clear that only the subspace of $\mathcal{H}$ corresponding to completion of such an image is relevant for the description of the CFT. Hence, it is natural to add to the axioms a non-degeneracy assumption [56]:

Assumption 1.1. $U_r \to 1$ as $r \to 1$, uniformly on compact subsets of $\mathcal{H}$.

Such an assumption implies that $\check{\mathcal{H}}$, is dense in $\mathcal{H}$. One can also define an injective map with dense image $\mathcal{H} \to \hat{\mathcal{H}}$, where the elements of $\hat{\mathcal{H}}$ are formally defined as $U_r^{-1}\xi$, $0 < r < 1$, for some $\xi \in \mathcal{H}$, with $U_r^{-1}\xi \equiv U_{rs}^{-1}\eta$ in $\hat{\mathcal{H}}$ if $\eta = U_s\xi$. Such a map assigns to each element $\eta \in \mathcal{H}$ the formal element $U_r^{-1}(U_r\eta)$, for some $0 < r < 1$.

The definition readily generalizes to the spaces $\check{\mathcal{H}}_X$ and $\hat{\mathcal{H}}_X$ for each 1-manifold $X = \mathrm{S}^1 \sqcup \ldots \sqcup \mathrm{S}^1$, by replacing each annulus A_r by a disjoint union $A_r \sqcup \ldots \times \sqcup A_r$ of annuli.

Proposition 1.3. *The spaces $\check{\mathcal{H}}_{\bar{X}}$ and $\hat{\mathcal{H}}_X$ are in natural duality.*

Proof. The natural pairing between an element $U_r\xi$ of $\check{\mathcal{H}}_{\bar{X}}$ and $U_s^{-1}\eta$ of $\hat{\mathcal{H}}_X$, $0 < r < s < 1$, is defined by considering the cobordism $Y_{r/s} : X \sqcup \bar{X} \to \varnothing$, so that the corresponding operator is $U_{r/s} : \mathcal{H}_X \otimes \mathcal{H}_{\bar{X}} \to \mathbb{C}$.

$$(U_s^{-1}\eta, U_r\xi) := U_{r/s}(\xi \otimes \eta) \ .$$

By construction, this is well-defined and independent of the choice of r, s. □

With respect to such a duality, the operator $U_Y : \mathcal{H}^*_{X_2} \to \mathcal{H}^*_{X_1}$ corresponding to the cobordism $Y : \bar{X}_2 \rightsquigarrow \bar{X}_1$ is naturally identified with the transpose of $U_Y : \mathcal{H}_{X_1} \to \mathcal{H}_{X_2}$, corresponding to $Y : X_1 \rightsquigarrow X_2$.

Given an operator U on a complex vector space E, we denote by $\bar{U}$ complex conjugate operator on $\bar{E}$ and, if E is a Hilbert space, by $U^\dagger$ the adjoint operator.

Definition 1.5. A conformal field theory is *unitary* is there is given a natural isomorphism $\bar{\check{\mathcal{H}}}_X \to \mathcal{H}_{\bar{X}}$, making $\check{\mathcal{H}}_X$ a pre-Hilbert space with $\mathcal{H}_X$ as its completion, and such that $U_{\bar{Y}} \cong \bar{U}_Y$. Equivalently, a CFT is unitary if $\mathcal{H}_X$ is a Hilbert space and $U_{\bar{Y}} = U_Y^\dagger$ (reflection-positivity).

1.2.4 *Representations of $\mathcal{A}$ and representations of* $\mathrm{Vect}(\mathrm{S}^1)$

By axiom 2, the space $\mathcal{H}$ (and hence also $\mathcal{H}_X$ for all the 1-manidfolds X) carries a projective representation of the semigroup $\mathcal{A}$ and, in particular, of its sub-semigroup $\mathbb{C}^\times_{<1}$. In this section, we will show that such a representation induces a representation of two copies (one holomorphic and one anti-holomorphic) of the algebra $\mathrm{Vect}_\mathbb{C}(\mathrm{S}^1)$, the complexification of the algebra of generators of $\mathrm{Diff}_{an}(\mathrm{S}^1)$.

The relation between the semi-group $\mathcal{A}$ and the Lie group $\mathrm{Diff}_{an}(\mathrm{S}^1)$ (and its Lie algebra $\mathrm{Vect}(\mathrm{S}^1)$) can be understood by noticing that the tangent space at A of $\mathcal{A}$ is isomorphic to

$$T_A\mathcal{A} \cong (\mathrm{Vect}_\mathbb{C}(\mathrm{S}^1) \oplus \mathrm{Vect}_\mathbb{C}(\mathrm{S}^1)) / \mathrm{Vect}(A) \ ,$$

where each $\mathrm{Vect}_\mathbb{C}(\mathrm{S}^1)$ corresponds to the space of deformations of one of the curves $f_0(\partial D)$ and $f_\infty(\partial D_\infty)$, whereas $\mathrm{Vect}(A)$ is the space of deformations of such curves that extend to the whole A. In the limit $A \to \mathrm{S}^1$ of a thin annulus, $\mathrm{Vect}(A) \to \mathrm{Vect}_\mathbb{C}(\mathrm{S}^1)$, so that $(\mathrm{Vect}_\mathbb{C}(\mathrm{S}^1) \oplus \mathrm{Vect}_\mathbb{C}(\mathrm{S}^1)) / \mathrm{Vect}(A) \to \mathrm{Vect}_\mathbb{C}(\mathrm{S}^1)$. In this sense, one can think that the boundary of $\mathcal{A}$ contains some "complexification" of $\mathrm{Diff}_{an}(\mathrm{S}^1)$ and, more generally of the group $\mathrm{Diff}(\mathrm{S}^1)$ of smooth diffoemorphisms of S^1. However, we notice that such a group, rigorously, does not exists: the complexified algebra $\mathrm{Vect}_\mathbb{C}(\mathrm{S}^1)$ is not the Lie algebra of any Lie group.

Let us consider the problem of analytically extending e representation of $\mathrm{Diff}_{an}(\mathrm{S}^1)$. It is useful to first consider to the sub-semigroup $\mathbb{C}^\times_{<1} \subset \mathcal{A}$, that is genuinely a complexification of the subgroup $\mathbb{T} \subset \mathrm{Diff}_{an}(\mathrm{S}^1)$ of rigid rotations. Let the Hilbert space $\mathcal{H}$ carry a representation of $\mathbb{T}$; then, $\mathcal{H}$ splits as a direct sum

$$\mathcal{H} = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_k \ ,$$

where the element $e^{i\theta} \in \mathbb{T}$ acts on $\mathcal{H}_k$ by multiplication by $e^{ik\theta}$.

Definition 1.6. The Hilbert space $\mathcal{H}$ carries a *positive energy* representation of $\mathbb{T}$ if, for some fixed $h \in \mathbb{Z}$, $k < h$ implies $\mathcal{H}_k = 0$.

It is clear that only the positive energy representations of $\mathbb{T}$ can be *holomorphically* extended to a representation of $\mathbb{C}^\times_{<1}$. Analogous considerations hold for the group $\mathrm{Diff}_{an}(\mathrm{S}^1)$ and the semigroup $\mathcal{A}$, as shown in the following proposition. Note that each positive energy representation of $\mathrm{Diff}_{an}(\mathrm{S}^1)$ is necessarily projective. We will restrict to representations of $\mathcal{A}$ such that the action of the subgroup $\mathbb{C}^{\times}_{<1}$ is diagonalizable and induces a (positive energy) representation of $\mathbb{T}$.

Proposition 1.4. *The projective positive energy representations of* $\mathrm{Diff}_{an}(\mathrm{S}^1)$ *are in 1-1 correspondence with projective holomorphic representations of* $\mathcal{A}$. *Moreover, the representation of* $\mathrm{Diff}_{an}(\mathrm{S}^1)$ *is unitary if and only if the representation of* $\mathcal{A}$ *is reflection-positive, i.e.* $U_{\bar{A}} = U_A^\dagger$.

Proof. We will only sketch the basic lines of the proof. Let $\phi A \psi^{-1}$ denote the element of $\mathcal{A}$ corresponding to the annulus A with the parametrizations of the incoming and outgoing circle modified by, respectively, the real-analytic diffeomorphisms ψ and ϕ. If $A \mapsto U_A$ is a projective holomorphic representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$, then we define $\phi \mapsto U_\phi$, $\phi \in \mathrm{Diff}_{an}(\mathrm{S}^1)$, by $U_\phi := U_{\phi A} U_A^{-1}$, which is densely defined in $\mathcal{H}$. More precisely, we define U_ϕ on the dense subspace $\check{\mathcal{H}}$ by $U_\phi(U_{A_s}\xi) := U_{\phi A_s}\xi$, for all $U_{A_s}\xi \in \check{\mathcal{H}}$.

Conversely, let $\phi \mapsto U_\phi$ be a positive-energy representation of $\mathrm{Diff}_{an}(\mathrm{S}^1)$ on $\mathcal{H}$. Then, the representation of the subgroup $\mathbb{T}$ extends in a unique way to a holomorphic representation of $\mathbb{C}^{\times}_{<1}$ (U_{A_q} acts by multiplication by q^k on $\mathcal{H}_k$). Since any element of $\mathcal{A}$ can be written as $\phi A_q \psi^{-1}$, for suitable $0 < |q| < 1$ and $\phi, \psi \in \mathrm{Diff}_{an}(\mathrm{S}^1)$, we set $U_A := U_\phi U_{A_q} U_\psi^{-1}$. It is easy to prove that the map $A \mapsto U_A$ is holomorphic and determines a representation of $\mathcal{A}$ (see [57]).

Finally, if $A = \phi A_q \psi^{-1}$, then $\bar{A} = \psi A_{\bar{q}} \phi^{-1}$; moreover, by diagonalizing U_{A_q}, it is easy to verify that $U_{A_{\bar{q}}} = U_{A_q}^\dagger$, and the proposition follows. □

The proposition above implies that in a *holomorphic* CFT, the space $\mathcal{H}$ is a positive energy holomorphic representation of $\mathrm{Vect}_{\mathbb{C}}(\mathrm{S}^1)$. In a general CFT, the projective representation of $\mathcal{A}$ is not holomorphic, and $\mathcal{H}$ can be split into a direct sum

$$\bigoplus_{(a,\tilde{b})\in\Lambda} \mathcal{H}_{a,\tilde{b}} \ , \tag{1.4}$$

where Λ is a discrete subset of $\mathbb{R}\times\mathbb{R}$, with $(a-\tilde{b}) \in \mathbb{Z}$, and $\mathcal{H}_{a,\tilde{b}}$ are finite-dimensional. The element A_q in the sub-semigroup $\mathbb{C}^{\times}_{<1} \subset \mathcal{A}$ acts on $\mathcal{H}_{a,\tilde{b}}$ by multiplication by $q^a \bar{q}^{\tilde{b}}$. This implies $\mathcal{H}$ is a representation of two copies of the algebra $\mathrm{Vect}_{\mathbb{C}}(\mathrm{S}^1)$, a holomorphic and an anti-holomorphic one. A particular case is given by the *rational conformal field theories* (see, for example, [47]), where the set of indices Λ is finite.

1.2.5 Pants and algebra of operators

A pant is a Riemann surface with the topology of a disk with two holes. It is a basic result that any Riemann surface can be written as a finite union of pants and disks. However, it is more useful in CFT to fix a pant P with two incoming and one outgoing circles, and express any morphism as a composition of morphisms given by disks, annuli and copies of P.

The axioms associate to such a P a ray U_P which gives a map $\mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$. Such a map defines a non-associative algebra on $\mathcal{H}$, which is knows as the *operator product expansion.*

1.3 Conformal anomaly and modular functors

The axioms define a CFT as a *projective* functor from the category of oriented 1-manifolds and (conformal classes with parametrized boundary) cobordisms to the category of vector spaces and trace-class operators. This implies that each vector space in the theory carries a projective representation of the semi-group of annuli $\mathcal{A}$.

In general, a projective representation of a group G can be conveniently seen as a genuine representation of a central extension $\tilde{G}$ of G. In the same spirit, one can look for the definition of an extension of the category of 1-manifolds and cobordisms. The morphisms of an extended category should be pairs (Y, λ) given by a cobordism and a complex number $\lambda \in \mathbb{C}$, satisfying suitable properties under composition.

It is useful to consider more general extension of such a category, in which the morphisms are given by pairs (Y, E_Y), where E_Y is a finite dimensional vector space depending on Y. A correspondence which associates to each cobordism Y a finite-dimensional vector space E_Y must satisfy some compatibility conditions with respect to composition of morphisms. Such conditions yield to the definition of *modular functor*, and will be discussed in section 1.3.2.

The main motivation for considering categories extended by modular functors is the description of the chiral part of the CFT, i.e. of the part depending analytically (or anti-analytically) on the moduli space parameters. In the vertex-operators description of CFT, this corresponds to consider representations of the vertex-operator algebra of meromorphic fields. In Segal's approach the chiral part of CFT can be obtained as a *weakly conformal field theory*, i.e. by applying the axioms of CFT to the extension of the category of 1-manifolds and cobordisms by a modular functor. This will be clarified in section 1.3.4.

1.3.1 Extensions of the semi-group $\mathcal{A}$

A central extension $\hat{G}$ of a topological group G by $\mathbb{C}^\times$ can be given in terms of a short exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \hat{G} \xrightarrow{\pi} G \longrightarrow 1 \quad ,$$

of continuous homomorphisms, such that $\mathbb{C}^\times$ is mapped to the center $Z(\hat{G})$ of $\hat{G}$ and $G \cong \hat{G}/\mathbb{C}^\times$. In particular, in view of the surjection $\pi : \hat{G} \to G$, one can interpret $\hat{G}$ as a principal bundle on the base G with structure group $\mathbb{C}^\times \subseteq Z(\hat{G})$. Equivalently, an extension can be given in terms of the associated line bundle on G. Note that, if L_g and L_h are the fibres at $g, h \in G$, there is a canonical isomorphism $L_{gh} \cong L_g \otimes L_h$ compatible with the action of $\mathbb{C}^\times$.

Proposition 1.5. *Holomorphic extensions of $\mathcal{A}$ by $\mathbb{C}^\times$ correspond to extensions of* $\mathrm{Diff}_{an}(\mathrm{S}^1)$ *by* $\mathbb{C}^\times$.

Proof. The argument is analogous to the one used to prove Proposition 1.4. Fix an extension $A \mapsto L_A$ that associates a line L_A to each annulus $A \in \mathcal{A}$. This determines a line L_ϕ for each $\phi \in \mathrm{Diff}_{an}(\mathrm{S}^1)$, given by $L_\phi := L_{\phi A} \otimes L_A^*$, for an annulus $A \in \mathcal{A}$. The line L_ϕ does not depend on $A \in \mathcal{A}$ and gives an extension of $\mathrm{Diff}_{an}(\mathrm{S}^1)$, since $L_{\psi\phi} = L_{\psi\phi A} \otimes L_A^* \cong L_{\psi\phi A} \otimes L_{\phi A}^* \otimes L_{\phi A} \otimes L_A^* \cong L_\psi \otimes L_\phi$.

Conversely, suppose that we have an extension $\phi \mapsto L_\phi$ of $\mathrm{Diff}_{an}(\mathrm{S}^1)$. Then, we can define an extension of $\mathcal{A}$ by setting $L_{A_q} := \mathbb{C}$ and $L_{\phi A_q \psi} := L_\phi \otimes L_\psi$. $\square$

Consider a projective representation of $\mathrm{Diff}_{an}(\mathrm{S}^1)$, corresponding to a genuine representation of a central extension. This induces a representation of the central extension of the complexified Lie algebra $\mathrm{Vect}_{\mathbb{C}}(\mathrm{S}^1)$ and one can choose the representatives L_n, $n \in \mathbb{Z}$, satisfying the commutation relation

$$[L_n, L_m] = i(m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \ , \tag{1.5}$$

where $c \in \mathbb{R}$ is the *central charge* of the representation.

Proposition 1.6. *The extensions of* $\mathrm{Diff}_{an}(\mathrm{S}^1)$ *by* $\mathbb{C}^\times$ *are classified by* $(c, h) \in \mathbb{C} \times (\mathbb{C}/\mathbb{Z})$*, where* c *is the central charge and* h *is an eigenvalue of* L_0*.*

Proof. We only sketch the main lines of the proof; a complete treatment can be found in [55]. Consider an extension $\hat{G}$ of th topological group G as a principal $\mathbb{C}^\times$-bundle on G. The choice of a splitting $\hat{g}_{\mathbb{C}} \cong g_{\mathbb{C}} \oplus \mathbb{C}$ of the extended Lie algebra $\hat{g}$ into the direct sum of the Lie algebras of G and $\mathbb{C}^\times$, corresponds to a splitting of the tangent space of $\tilde{G}$ at its identity element into a vertical and a horizontal space. Such a splitting extends uniquely as a $\hat{G}$-left-invariant connection on the $\mathbb{C}^\times$-bundle $\hat{G}$. The curvature α of such a connection determines a $\mathbb{C}$-valued 2-form on G. Consider the case $G \equiv \mathrm{Diff}_{an}(\mathrm{S}^1)$, so that $g \equiv \mathrm{Vect}(\mathrm{S}^1)$. Upon suitably choosing the map $g_{\mathbb{C}} \oplus \mathbb{C} \to \hat{g}_{\mathbb{C}}$, the images of the generators of g form a commutator algebra given by (1.5), so that the curvature α corresponds to the central term proportional to the central charge c.

Now, consider two extensions $\pi : \hat{G} \to G$ and $\pi' : \hat{G}' \to G$ with the same central charge. Then, the "difference" extension is the quotient $\hat{G} \times_G \hat{G}'/\mathbb{C}^\times$, where $\hat{G} \times_G \hat{G}'$ is the space of pairs $(g, h) \in \hat{G} \times \hat{G}'$ such that $\pi(g) = \pi'(h)$, and $\mathbb{C}^\times$ acts anti-diagonally $u \to (u, u^{-1})$ on $\hat{G} \times_G \hat{G}'$. Such an extension has $c = 0$, so that the connection is flat and the bundle is determined by a homomorphism $\pi_1(G) \to \mathbb{C}^\times$. But $\pi_1(\mathrm{Diff}_{an}(\mathrm{S}^1)) \cong \pi_1(\mathbb{T}) \cong \mathbb{Z}$, so that all such homomorphisms are given by $\mathbb{Z} \ni n \mapsto e^{2\pi i h n}$ for some $h \in \mathbb{C}/\mathbb{Z}$, which is an eigenvalue of L_0, the generator of $\mathbb{T}$. It follows that the central charge c and the eigenvalue h completely determine the $\mathbb{C}^\times$-bundle and, therefore, the extension of $\mathrm{Diff}_{an}(\mathrm{S}^1)$. □

Under the splitting (1.4), each $\mathcal{H}_{h,\tilde{h}}$ is a representation of two central extensions of $\mathrm{Diff}_{an}(\mathrm{S}^1)$, corresponding to the holomorphic and anti-holomorphic part of $\mathcal{A}$, associated to two pairs (c, h) and $(\tilde{c}, \tilde{h})$. In fact, consistency of the chiral and anti-chiral algebra of operators require that the pair of central charges $c, \tilde{c}$ is the same for all the representations $\mathcal{H}_{h,\tilde{h}}$. The definition of the chiral (or anti-chiral) part of the vertex operator algebra in a CFT requires the concept of modular functor.

1.3.2 Modular functors

Let Φ be a finite set of labels and consider the category $\mathcal{C}_\Phi$ whose objects are (not necessarily connected) Riemann surfaces with *labeled* and parametrized boundary; the labeling is a continuous function $l_Y : \partial Y \to \Phi$ assigning a label in the set Φ to each boundary component of the Riemann surface Y. Two kind of morphisms are defined:

1. a gluing morphism $Y \to \check{Y}$ is defined if $\check{Y}$ is obtained from Y by gluing two circles in ∂Y with the same label;

2. the involution $Y \to \bar{Y}$, is defined for each cobordism Y, where $\bar{Y}$ corresponds to the complex conjugate Riemann surface, with the same boundary parametrization and labeling.

Definition 1.7. A *modular functor* is a functor from the category $\mathcal{C}_\Phi$ to the category of finite dimensional vector spaces and injective linear maps, which associates to each cobordism Y with labeled parametrized boundary a finite dimensional vector space E_Y, satisfying the following axioms:

1. there is a natural isomorphism $E_{Y_1 \sqcup Y_2} \cong E_{Y_1} \otimes E_{Y_2}$.
2. if $\check{Y}$ can be obtained from Y_φ, $\varphi \in \Phi$, by gluing two circles of ∂Y_φ labeled by φ, then there is a natural isomorphism
$$E_{\check{Y}} \cong \bigoplus_{\varphi \in \Phi} E_{Y_\varphi} \ .$$
3. $\dim E_{\mathrm{S}^2} = 1$.
4. For each holomorphic family $\{Y_\alpha\}_{\alpha \in \mathcal{B}}$ of cobordisms, the vector bundle $\pi : E_{\mathcal{B}} \to \mathcal{B}$ with fiber $\pi^{-1}(\alpha) \cong E_{Y_\alpha}$ is a holomorphic vector bundle on the base $\mathcal{B}$.

Axiom 4) implies that, for each fixed topology α, the modular functor E determines a holomorphic vector bundle on space $\mathcal{C}_\alpha$ of Riemann surfaces of topology α, with labeled parametrized boundary. We made the non-degeneracy assumption that for each ϕ, there is a cobordism Y with a boundary component labeled by ϕ such that $E_Y \neq 0$.

Proposition 1.7. *The following properties hold:*

1. *Let D_φ, $\varphi \in \Phi$ be a disk with an outgoing circle labeled by φ. There exists a label, we denote by $1 \in \Phi$, such that $\dim E_{D_\varphi} = 1$ if $\varphi = 1$ and $\dim E_{D_\varphi} = 0$ otherwise.*
2. *Let $A_{\varphi\psi}$ be an annulus with one incoming and one outgoing circle, labeled, respectively, by ϕ and ψ. Then, $\dim E_{A_{\varphi\varphi}} = 1$ and $E_{A_{\varphi\psi}} = 0$ if $\varphi \neq \psi$. In particular, a modular functor determines an extension of $\mathcal{A}$ by $\mathbb{C}^\times$ for each label φ.*
3. *Let $B_{\varphi\psi}$ be an annulus with two outgoing circles labeled by $\varphi, \psi \in \Phi$. There is an involution $\varphi \mapsto \bar{\varphi}$ in Φ such that such that $\dim E_{B_{\varphi\bar{\varphi}}} = 1$ and $E_{B_{\varphi\psi}} = 0$ if $\psi \neq \bar{\varphi}$.*

Proof. Let us first prove 2). The matrix $d_{\varphi\psi} := \dim E_{A_{\varphi\psi}}$ has non-negative integer entries and, by applying axiom 2) in Definition 1.7 to a composition of annuli, it follows that $d^2 = d$. By the non-degeneracy assumption, the only possibility is $d_{\varphi\psi} = \delta_{\varphi\psi}$. To prove 3), notice that the matrices $e_{\varphi\psi} := \dim E_{B_{\varphi\psi}}$ and $\bar{e}_{\varphi\psi} := \dim E_{\bar{B}_{\varphi\psi}}$, where $\bar{B}$ denotes an annulus with two incoming circles, are symmetric and invertible, since, by axiom 2) of Definition 1.7, $e\bar{e} = d \equiv 1$. Note that the functoriality properties with respect to the involution $Y \to \bar{Y}$ give $\dim E_Y = \dim E_{\bar{Y}}$, since the composition of linear maps $E_Y \to E_{\bar{Y}} \to E_Y$ is the identity. In particular, $\dim E_{D_\varphi} = \dim E_{\bar{D}_\varphi}$ and, since S^2 is obtained by gluing disks with opposite boundary orientation, statement 1) follows. □

Property 2) in the proposition above implies that modular functors determine a central extension by $\mathbb{C}^\times$ of the semigroup $\mathcal{A}$ for each label $\phi \in \Phi$, so that it is meaningful to classify E by the central charges associated to its labels. In fact, it can be proved that, in order for the modular functor to be consistently defined, all the labels must correspond to the same central charge.

Proposition 1.8. *Let E be a modular functor with central charge $c = 0$ and denote by E_α the corresponding holomorphic vector bundle on the space $\mathcal{C}_\alpha$ of cobordisms with a fixed topology α. Then, a holomorphic flat connection, compatible with gluing, is canonically defined on E_α, for each topology α with no closed components.*

Proof. Let $Y \in \mathcal{C}_\alpha$ be a Riemann surface with a boundary circle labeled by $\varphi \in \Phi$. The maps $Y \to Y \circ A_{\varphi\varphi}$, with $A_{\varphi\varphi} \in \mathcal{A}$, correspond, through the modular functor, to an action $E_Y \to E_{Y \circ A_{\varphi\varphi}} \cong E_Y$ of $\mathcal{A}$ on the fibre E_Y. In turn, this induces an action of the Lie algebra $\mathrm{Vect}_{\mathbb{C}}(\mathrm{S}^1)$ on the fibre. By considering all the boundary components of Y, a modular functor canonically defines an action of $\mathrm{Vect}_{\mathbb{C}}(\partial Y)$ on the fibre at Y. Recall that, by proposition 1.1 if the topology α has no closed components, the tangent space to $\mathcal{C}_\alpha$ at Y is $\mathrm{Vect}_{\mathbb{C}}(\partial Y)/\mathrm{Vect}_{\mathbb{C}}(Y)$. The action of $\mathrm{Vect}_{\mathbb{C}}(\partial Y)$ on the fibre induces a representation of the subalgebra $\mathrm{Vect}_{\mathbb{C}}(Y)$; however, it can be proven that the only finite dimensional representation of $\mathrm{Vect}(Y)$ is the trivial one (see [57]). Hence, one can canonically define a connection $T\mathcal{M}_\alpha \ni \xi \mapsto D_\xi$ on E_α and this is flat, because it comes from a Lie algebra action of $\mathrm{Vect}(\partial Y)$. Compatibility with gluing follows by construction. □

Proposition 1.9. *The extension of $\mathcal{A}$ associated to the label $1 \in \Phi$ is classified by a pair (c, h), with $h = 0$.*

Proof. Let A_q be an annulus with both boundary circles labeled by $1 \in \Phi$. Then, the relation $A_q \circ D = D$ implies that the action of A_q is trivial. □

1.3.3 The determinant line bundle

Let E, F be Hilbert spaces. An operator $B : E \to F$ is *determinant-class* if $B = 1 + A$, where A is trace-class (see Definition 1.2). Its determinant, defined as

$$\det(1 + A) := \exp[\mathrm{Tr}\log(1 + A)] = \prod_{i=1}^{N}(1 + \rho_i) \ ,$$

is finite.

Definition 1.8. Let E, F be Hilbert spaces. A linear map $T : E \to F$ is a *Fredholm operator* if there exists an operator $P : F \to E$ such that $T \circ P - 1$ and $P \circ T - 1$ are finite rank operators.

It can be proven that a Fredholm operator T has finite-dimensional kernel and cokernel, so that it makes sense to define the *index* of T

$$\mathrm{ind}\, T := \dim\ker T - \dim\mathrm{coker}\, T \ .$$

Definition 1.9. For any Fredholm operator $T : E \to F$, the determinant line Det_T is defined as follows:

- If $\operatorname{ind} T = 0$, then $\det T$ is the line whose points are the equivalence classes of pairs (S, λ), with $S : E \to F$ such that $S - T$ is trace-class and $\lambda \in \mathbb{C}$, where the equivalence relation is
$$(SB, \lambda) \sim (S, \det B\lambda) \ ,$$
where $B : E \to E$ is determinant-class.
- If $\operatorname{ind} T = n \neq 0$, then $\mathrm{Det}_T := \mathrm{Det}_{T'}$, where $T' \equiv T \oplus 0 : E \to F \oplus \mathbb{C}^n$ if $n > 0$ and $T' \equiv T \oplus 0 : E \oplus \mathbb{C}^{-n} \to F$ if $n < 0$.

Definition 1.10. If $T : E \to F$ is Fredholm with $\operatorname{ind} T = 0$, the determinant $\det(T)$ of T is the element $[T, 1]$ of Det_T. If $\operatorname{ind} T = 0$, then $\det(T) = 0 \in \mathrm{Det}_T$.

Proposition 1.10. $\det(T) \neq 0$ *if and only if* T *is invertible.*

Proof. If $\operatorname{ind} T \neq 0$, T is not invertible and $\det(T) = 0$ by definition. If $\operatorname{ind} T = 0$, then there exists an invertible S such that $A := S - T$ is trace-class. The map $\lambda \mapsto (S, \lambda)$ is an isomorphism $\mathbb{C} \to \mathrm{Det}_T$. If T is invertible, than we can choose $S \equiv T$ and $\det(T) \cong 1 \in \mathbb{C}$. If T is not invertible, then $T = SB$, where $B := 1 - S^{-1}A$ is determinant-class and non-invertible, so that $\det(T) = (T, 1) = (SB, 1) \sim (S, \det B) = 0$. □

The following proposition provides an equivalent definition of Det_T

Proposition 1.11. *If* $T : E \to F$ *is Fredholm, with* $\dim\ker T = m$ *and* $\dim\operatorname{coker} T = n$*, then there is canonical isomorphism*
$$\mathrm{Det}_T \cong (\wedge^m \ker T)^* \otimes (\wedge^n \operatorname{coker} T) \ .$$

Proof. One can reduce to the case $\operatorname{ind} T = n - m = 0$. Let $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ be bases for $\ker T^*$ and $\operatorname{coker} T$, respectively. Then, the isomorphism is given by
$$(T + \sum_{i=1}^{n} \alpha_i \otimes \beta_i, 1) \mapsto (\alpha_1 \wedge \ldots \wedge \alpha_n) \otimes (\beta_1 \wedge \ldots \wedge \beta_n) \ .$$
□

Proposition 1.12. *Let* $\{T_x\}_{x\in\mathcal{B}}$ *be a holomorphic family of Fredholm operators* $T_x : E_x \to F_x$*, parametrized by the complex manifold* $\mathcal{B}$*. Then, the lines* Det_{T_x} *form a holomorphic line bundle on* $\mathcal{B}$*.*

Proposition 1.13. *Let*
$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r] \arrow[d, "T'"] & E \arrow[r] \arrow[d, "T"] & E'' \arrow[r] \arrow[d, "T''"] & 0 \\
0 \arrow[r] & F' \arrow[r] & F \arrow[r] & F'' \arrow[r] & 0
\end{tikzcd}$$
where E, E', E'', F, F', F'' *are Hilbert spaces and* T, T', T'' *are Fredholm, be a commutative diagram with exact horizontal arrows. Then,*
$$\mathrm{Det}_T \cong \mathrm{Det}_{T'} \otimes \mathrm{Det}_{T''} \ .$$

To each closed Riemann surface Y, one can associate an operator $\bar{\partial} : \Omega^0(Y) \to \Omega^{0,1}(Y)$, which maps smooth functions to $(0, 1)$-forms. If Y has a boundary, one can define $\bar{\partial}$ as a map on the space $\Omega^0(Y, \partial Y)$ of smooth functions on Y that, when restricted to the boundary, can be expressed as $\sum_{n>0} a_n e^{i\theta n}$ on outgoing circles and $\sum_{n<0} b_n e^{i\theta n}$ on incoming circles. With such a choice, the operator $\bar{\partial} : \Omega^0(Y, \partial Y) \to \Omega^{0,1}(Y, \partial Y)$ is Fredholm and one can define the determinant line

$$\mathrm{Det}_Y := \mathrm{Det}_{\bar{\partial}} \cong (\wedge \ker \bar{\partial})^* \otimes (\wedge \operatorname{coker} \bar{\partial}) \ .$$

By Proposition 1.12, this determines a holomorphic line bundle, called the determinant line bundle, on the moduli space of Riemann surfaces for each fixed topology. It is also obvious that

$$\mathrm{Det}_{Y_1 \sqcup Y_2} \cong \mathrm{Det}_{Y_1} \otimes \mathrm{Det}_{Y_2} \ .$$

(Note, however, that such an isomorphism, in general, is invariant only up to a sign under permutation of terms in the disjoint union; for example, the group of permutations acting on the terms of $\mathrm{S}^2 \sqcup \ldots \sqcup \mathrm{S}^2$, induces the sign representation on $\mathrm{Det}_{\mathrm{S}^2 \sqcup \ldots \sqcup \mathrm{S}^2}$.) Hence, $Y \mapsto \mathrm{Det}_Y$ satisfies the axioms 1), 3) and 4) in Definition 1.7 for a modular functor with one label. Axiom 2) is satisfied thanks to the following proposition, which is proved in [57].

Proposition 1.14. *Let $\check{Y}$ be the Riemann surface obtained from Y by gluing together an incoming and an outgoing circle in ∂Y. Then, there is a canonical isomorphism* $\mathrm{Det}_{\check{Y}} \cong \mathrm{Det}_Y$.

By this proposition and by the above remarks, the following theorem follows.

Theorem 1.15. *Any (even) tensor power* $\mathrm{Det}^{\otimes n}$*, $n \in \mathbb{Z}$, of the determinant line is a one-dimensional modular functor.*

Rigorously, one should restrict to even tensor powers of Det, in order for $\mathrm{Det}^{\otimes n}$ to be invariant under permutations of components in disjoint unions. In the following section, we will see that the tensor powers of the determinant line are essentially the unique one-dimensional modular functors.

1.3.4 CFT from weak conformal field theories

A modular functor E associated to a set Φ of labels determines an extension $\mathcal{S}_E$ of the category $\mathcal{S}$ of oriented 1-manifolds and cobordisms. The objects of $\mathcal{S}_E$ are oriented 1-manifolds labeled by an element of Φ and morphisms are the pairs (Y, η) where Y is cobordism with labeled boundary and $\eta \in E_Y$. Composition is defined between compatibly labeled cobordisms $(Y_1, \eta) \circ (Y_2, \xi) = (Y_1 \circ Y_2, \rho)$, where ρ is the image of $\eta \otimes \xi$ in $E_{Y_1 \circ Y_2}$ through the canonical injections $E_{Y_1} \otimes E_{Y_2} \to E_{Y_1 \sqcup Y_2} \to E_{Y_1 \circ Y_2}$, defined, respectively, by axioms 2) and 3) in Definition 1.7.

Definition 1.11. A *weak conformal field theory* is a functor from the category $\mathcal{S}_E$ to the category of topological vector spaces with trace-class linear maps, satisfying the axioms of Definition 1.3, with the following modifications:

- Isomorphisms in axioms 2a) and 2c) hold genuinely and not just projectively.

- $U_{(Y,\xi)}$ depends holomorphically on the morphism (Y, ξ).

A weakly conformal field theory assigns a topological vector space $\mathcal{H}_\varphi$ to the circle S^1, for each label $\varphi \in \Phi$, and a finite-dimensional subspace E_Y, with a natural injection $E_Y \to \mathcal{H}_{\varphi_1} \otimes \mathcal{H}_{\varphi_n}$, to each labeled cobordism Y with n outgoing (and no incoming) circles labeled by $\varphi_1, \ldots, \varphi_n$. This should be compared with the definition of a CFT, where there is only one label and a cobordism with all outgoing circles is associated to a 1-dimensional space, corresponding to the ray of trace-class operators $U_Y : \mathbb{C} \to \mathcal{H}_{\partial Y}$. Furthermore, for a general CFT, no holomorphicity condition is required on the dependence of such a 1-dimensional space on Y.

The idea behind the definition of a weakly conformal field theory is that it should correspond to the chiral or to the anti-chiral part of a CFT. Hence, one expects to be able to construct a CFT by gluing two weakly conformal field theory. This procedure is not clear in general, but it is described in [57] in case the modular functor satisfies a unitarity condition. We closely follow Segal's definitions.

Definition 1.12. A modular functor E is unitary if there is a positive non-degenerate transformation

$$\bar{E}_Y \otimes E_Y \to |\,\mathrm{Det}_Y\,|^c \ ,$$

for each surface Y with labeled boundary such that the diagram

$$\begin{tikzcd} \bigoplus_\varphi \bar{E}_{Y_\varphi} \otimes E_{Y_\varphi} \arrow[r] \arrow[d] & |\,\mathrm{Det}_Y\,|^c \arrow[d] \\ \bar{E}_Y \otimes E_Y \arrow[r] & |\,\mathrm{Det}_Y\,|^c \end{tikzcd}$$

commutes.

Proposition 1.16. *A pair of weakly conformal field theories corresponding to the same unitary modular functor E with index set Φ defines a conformal field theory based on the space $\bigoplus_\varphi \bar{\mathcal{H}}_\varphi \otimes \mathcal{H}_\varphi$ and the central extension $|\,\mathrm{Det}\,|^c$ of $\mathcal{C}$.*

Proposition 1.17. *Any one-dimensional modular functor is determined by its restriction to the semigroup $\mathcal{A}$. More precisely, given two one-dimensional modular functors E' and E'' with the same restriction to $\mathcal{A}$ and a normalizing isomorphism $E'_D \cong E''_D$, for an arbitrary disk D, there are canonical isomorphisms $E'_Y \cong E''_Y$ for all the cobordisms Y.*

Proof. Axiom 2) of Definition 1.7 and property 2) of Proposition 1.7 imply that $\dim E_{\mathbb{T}}$, for a torus $\mathbb{T}$, equals the number of labels $|\Phi|$. Hence, one-dimensional modular functors have only one label. Let E' and E'' be one-dimensional modular functors with the same restriction to $\mathcal{A}$ and normalized so that $E'_D \cong E''_D$ for a fixed disk D. Then, $E := E' \otimes (E'')^*$ is a modular functor with trivial restriction to $\mathcal{A}$ and $E_D \cong \mathbb{C}$; we have to prove that there are canonical isomorphisms $E_Y \cong \mathbb{C}$ for all Y. Let $\mathcal{M}_\alpha$ be the space of surfaces of a given topology α, not containing any closed component. Since E has vanishing central charge, Proposition 1.8 assures that there is a canonical flat connection on E_α. It follows that E_α corresponds to a one-dimensional representation of $\pi_1(\mathcal{M}_\alpha)$. Such a group is generated by Dehn twists, as follows. Choose a closed curve γ on $Y \in \mathcal{M}_\alpha$ and let A_r, $0 < r < 1$, be an annulus, embedded in Y, containing γ. Then, Y can be obtained by gluing both circles of A_r to a surface Y'. Let Y_t, $t \in [0, 1]$, be the surface obtained by gluing both circles of the annulus $A_{q(t)}$, $q(t) := e^{2\pi i t} r$, to Y'; then, $t \mapsto Y_t$ represents a non-trivial element (with base-point Y) in $\pi_1(\mathcal{M}_\alpha)$. It can be proved that $\pi_1(\mathcal{M}_\alpha)$ is generated by such elements; it follows that the bundle E_α is completely determined by the restriction of E on $\mathcal{A}$. But E is trivial on $\mathcal{A}$, so that we can canonically identify the fibres of E_α for each α. This implies that E_Y depends only on the topology of Y.

On the other hand, if Y is obtained from Y' by removing k disks, the isomorphism $E(D) \cong \mathbb{C}$ induces an isomorphism $E_Y \cong E_{Y'}$. Hence, $E(Y)$ depends only on the genus of Y, $E_Y \cong E_g$. In particular, the line bundle E_α has a canonical flat connection also for surfaces with closed components. Finally, let Y_g be a surface of genus g with two holes; then, one can glue the boundary circles of Y_g to obtain a surface Y_{g+1} of genus $g+1$, and axiom 2) of Definition 1.7 implies that $E_g \cong E_{g+1}$ and that such an isomorphism is compatible with composition of cobordisms. □

The determinant line $Y \mapsto \mathrm{Det}_Y$ is a one-dimensional modular functor and induces an extension of $\mathcal{A}$ classified by $(c, h) = (-2, 0)$.

Corollary 1.18. *The only one-dimensional modular functors are integral tensor powers of the determinant line.*

Corollary 1.19. *The only central extensions of the category $\mathcal{S}$ of 1-manifolds and cobordisms by $\mathbb{C}^\times$ are given by $Y \mapsto (\overline{\mathrm{Det}}_Y)^{\otimes p} \otimes (\mathrm{Det}_Y)^{\otimes q}$, with $p, q \in \mathbb{C}$, $p - q \in \mathbb{Z}$.*

Proof. We will only sketch the main lines of the proof. By Proposition 1.17, such corollaries are equivalent to the claim that the only holomorphic central extensions of $\mathcal{A}$ by $\mathbb{C}^\times$ that extend to a one dimensional modular functor are classified by $(c, 0)$, with c an even integer. The condition $h = 0$ follows by Proposition 1.9. A holomorphic extension of $\mathcal{A}$ with central charge c must correspond to the one given by $\mathrm{Det}^{\frac{c}{2}}$. In order to be a modular functor, a line bundle must be defined on each space of cobordisms $\mathcal{C}_\alpha$ of fixed topology α. In particular, it must determine an element in $H^2(\mathcal{C}_\alpha, \mathbb{Z})$, since the topological classification of line bundles is given by the first Chern class. But it can be proved that $H^2(\mathcal{C}_\alpha, \mathbb{Z}) \cong \mathbb{Z}$ for genus high enough, and the Chern class of Det is a generator. Hence, only integer powers of Det are well-defined. □

1.4 Axiomatic CFT and bosonic string theory

The axiomatic approach to CFT described in the previous sections can be applied to bosonic string theory. In this respect, two problems arise. Fist of all, if the target space of string theory is not a compact manifold, the operators U_Y fail to be trace-class. This is a usual problem in quantum field theories in infinite volume spaces. There are several standard ways to treat this issue, for example by considering a manifold with finite volume V and then taking the limit $V \to \infty$ at the end of the calculations.

A more serious issue concerns to the fact that Segal's axiomatization considers Riemann surfaces with parameterized boundary, whereas no such a parameterization is defined in bosonic string theory. This is strictly related to the problem of restrict the Hilbert space of states to obtain conformal invariant amplitudes. As is well-known, the simple restriction to conformal invariant states is a too strong condition. Instead, it is necessary to restrict to the BRST cohomology. The definition of the BRST cohomology in axiomatic CFT has been developed in [57]; we just notice that a consistency condition for such a definition is that the total central charge of the theory is 0.

In this thesis, we will limit to show how Corollaries 1.18 and 1.19 can be used to derive the partition functions related to closed Riemann surfaces of genus $g \geq 2$. Such partition functions define the bosonic string measure on the moduli space $\mathcal{M}_g$ of genus g.

We recall that bosonic string theory is formally defined by a path integral over the space of embeddings in a flat D-dimensional manifold M (the target spaces) and over the space of world-sheet metrics. The measure is given by a conformal and diffeomorphisms invariant action on the world-sheet. After gauge-fixing, we obtain a CFT with D real fields, corresponding to the target space coordinates, and two copies (holomorphic and anti-holomorphic) bc system of weight 2. The resulting CFT does not admit holomorphic factorization, the obstruction being related to the zero-modes of the operator $\bar{\partial}$. Each amplitude can be written as an integration over the *internal momenta* of a holomorphic times an anti-holomorphic contribution. For a finite volume target space, such an integration is substituted by a discrete sum, which corresponds to the summation in Eq.(1.4). It is straightforward to see that the one-dimensional modular functor giving the central extension of the category $\mathcal{S}$ corresponds to $(\mathrm{Det}_{\bar{\partial}_1})^{\otimes\frac{D}{2}} \otimes \mathrm{Det}^*_{\bar{\partial}_2}$, where $\bar{\partial}_n$ is the derivative operator acting on n-differentials. By Proposition 1.18, such a modular functor must correspond to an integral power of Det. Obviously $\mathrm{Det}_{\bar{\partial}_1} \equiv \mathrm{Det}$; the relation between $\mathrm{Det}_{\bar{\partial}_n}$ and Det is given by Mumford theorem, we state in terms of the dual bundles $\lambda_n := \mathrm{Det}^*_{\bar{\partial}_n}$, for later reference.

Theorem 1.20 (Mumford). *Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g \geq 2$. For each $n > 1$,*

$$\lambda_n \cong \lambda_1^{\otimes c_n} \ ,$$

where $c_n := 6n^2 - 6n + 1$.

Such a theorem can be seen as a direct consequence of Corollary 1.18, once one proves that the central extension of $\mathcal{A}$ induced by $\mathrm{Det}_{\bar{\partial}_n}$ has central charge $-2c_n$. In particular, since $c_2 = 13$, we get $\mathrm{Det}_{\bar{\partial}_2} \cong \mathrm{Det}^{13}$. Hence, the modular functor associated to such a theory is $\mathrm{Det}^{\frac{D}{2}-26}$; however, we recall that consistency conditions in string theory, in particular the requirement of nilpotency of the BRST charge, constrains the central charge of the modular functor to be 0. This fixes the critical dimension $D = 26$.

The genus g partition function (or, equivalently, the string measure on $\mathcal{M}_g$), must be given by integration over the internal momenta of the modulus square of a non-vanishing holomorphic section of $\mathrm{Det}^{13}_{\partial_1}\otimes\mathrm{Det}^{*}_{\partial_2}\equiv\lambda_1^{-13}\otimes\lambda_2$. The only such section is the Mumford form

$$\mu_{g,2}\equiv F_{g,2}[\phi]\,\frac{\phi_1\wedge\ldots\wedge\phi_{3g-3}}{(\omega_1\wedge\ldots\wedge\omega_g)^{13}}\ ,$$

where $\{\phi_1,\ldots,\phi_{3g-3}\}$ is a basis of holomorphic quadratic differentials and $\{\omega_1,\ldots,\omega_g\}$ is the canonical basis of holomorphic abelian differentials (see section B). After integrating over internal momenta, we obtain the Belavin-Knizhnik theorem [7] relating the bosonic string measure $d\mu_g$ to the Mumford form

$$d\mu_g=\frac{|F_{g,2}|^2}{(\det\operatorname{Im}\tau)^{13}}|\phi_1\wedge\ldots\wedge\phi_{3g-3}|^2\ ,$$

where $\operatorname{Im}\tau$ is the Riemann period matrix.

A more extensive treatment of the Mumford isomorphism and the derivation of explicit expressions for the Mumford form for genera 2 and 3, are given in chapter 3.

2. COMBINATORICS OF DETERMINANTS

Determinants of holomorphic quadratic differentials play a crucial role in our construction. In particular, in the following chapters, we will construct bases of $H^0(K_C^2)$ in terms of two-fold products of holomorphic abelian differentials. In this section, we will consider the purely combinatorial problem concerning the determinants of a basis of a two-fold symmetric product of a finite dimensional space of functions. We first introduce a very useful notation for symmetric tensor products of vector space, which we will adopt all along the paper; then we derive two lemmas on determinants which are of interest on their own.

2.1 Identities in symmetric products of vector spaces

Definition 2.1. For each $n \in \mathbb{Z}_{>0}$, set

$$I_n := \{1, \ldots, n\} \ ,$$

and let $\mathcal{P}_n$ denote the group of permutations of n elements.

Let V be a g-dimensional vector space and let

$$M_n := \binom{g+n-1}{n} \ ,$$

be the dimension of the n-fold symmetrized tensor product $\operatorname{Sym}^n V$. We denote by

$$\operatorname{Sym}^n V \ni \eta_1 \cdot \eta_2 \cdots \eta_n := \sum_{s \in \mathcal{P}_n} \eta_{s_1} \otimes \eta_{s_2} \otimes \ldots \otimes \eta_{s_n} \ ,$$

the symmetrized tensor product of an n-tuple $(\eta_1, \ldots, \eta_n)$ of elements of V.

Fix a surjection $m : I_g \times I_g \to I_M$, $M := M_2 = g(g+1)/2$, such that

$$m(i,j) = m(j,i) \ , \tag{2.1}$$

$i, j \in I_g$. Such a surjection corresponds to an isomorphism $\mathbb{C}^M \to \operatorname{Sym}^2 \mathbb{C}^g$ with $\tilde{e}_{m(i,j)} \mapsto e_i \cdot e_j$.

A useful choice for such an isomorphism is considered in the following definition.

Definition 2.2. Let $A : \mathbb{C}^M \to \operatorname{Sym}^2\mathbb{C}^g$, $M \equiv M_2$, be the isomorphism $A(\tilde{e}_i) := e_{\mathtt{1}_i} \cdot e_{\mathtt{2}_i}$, with $\{\tilde{e}_i\}_{i \in I_M}$ the canonical basis of $\mathbb{C}^M$ and

$$(\mathtt{1}_i, \mathtt{2}_i) := \begin{cases} (i,i) \ , & 1 \le i \le g \ , \\ (1, i-g+1) \ , & g+1 \le i \le 2g-1 \ , \\ (2, i-2g+3) \ , & 2g \le i \le 3g-3 \ , \\ \qquad \vdots & \qquad \vdots \\ (g-1, g) \ , & i = g(g+1)/2 \ , \end{cases}$$

so that $\mathtt{1}_i\mathtt{2}_i$ is the i-th element in the M-tuple $(11, 22, \ldots, gg, 12, \ldots, 1g, 23, \ldots)$. Similarly, let $\{\tilde{e}_i\}_{i\in I_{M_3}}$ be the canonical basis of $\mathbb{C}^{M_3}$, and fix an isomorphism $A : \mathbb{C}^{M_3} \to \mathrm{Sym}^3\mathbb{C}^g$, $M_3 := g(g+1)(g+2)/6$, with $A(\tilde{e}_i) := (e_{\mathtt{1}_i}, e_{\mathtt{2}_i}, e_{\mathtt{3}_i})_S$, whose first $6g-8$ elements are

$$(\mathtt{1}_i, \mathtt{2}_i, \mathtt{3}_i) := \begin{cases} (i,i,i) \ , & 1 \le i \le g \ , \\ (1,1,i-g+2) \ , & g+1 \le i \le 2g-2 \ , \\ (2,2,i-2g+4) \ , & 2g-1 \le i \le 3g-4 \ , \\ (1,2,i-3g-4) \ , & 3g-3 \le i \le 4g-4 \ , \\ (1,i-4g+6,i-4g+6) \ , & 4g-3 \le i \le 5g-6 \ , \\ (2,i-5g+8,i-5g+8) \ , & 5g-5 \le i \le 6g-8 \ . \end{cases}$$

As we will see, we do not need the explicit expression of $A(\tilde{e}_i)$ for $6g-8 < i \le M_3$. In general, one can define an isomorphism $A : \mathbb{C}^{M_n} \to \mathrm{Sym}^n\mathbb{C}^g$, with $A(\tilde{e}_i) := (e_{\mathtt{1}_i}, \ldots, e_{\mathtt{n}_i})$, by fixing the n-tuples $(\mathtt{1}_i, \ldots, \mathtt{n}_i)$, $i \in I_{M_n}$, in such a way that $\mathtt{1}_i \le \mathtt{2}_i \le \ldots \le \mathtt{n}_i$.

For each vector $u := {}^t(u_1, \ldots, u_g) \in \mathbb{C}^g$ and matrix $B \in M_g(\mathbb{C})$, set

$$\underbrace{u\cdots u_i}_{n \text{ times}} := \prod_{\mathtt{m}\in\{\mathtt{1},\ldots,\mathtt{n}\}} u_{\mathtt{m}_i} \ , \qquad (\underbrace{B\cdots B}_{n \text{ times}})_{ij} := \sum_{s\in\mathcal{P}_n} \prod_{\mathtt{m}\in\{\mathtt{1},\ldots,\mathtt{n}\}} B_{\mathtt{m}_i s(\mathtt{m})_j} \ ,$$

$i, j \in I_{M_n}$, where the product is the standard one in $\mathbb{C}$. In particular, let us define

$$\chi_i \equiv \chi_i^{(n)} := \prod_{k=1}^{g} \Big(\sum_{\mathtt{m}\in\{\mathtt{1},\ldots,\mathtt{n}\}} \delta_{k\mathtt{m}_i} \Big)! = (\delta\cdots\delta)_{ii} \ ,$$

$i \in I_{M_n}$, (we will not write the superscript (n) when it is clear from the context) where δ denotes the identity matrix, so that, for example,

$$\chi_i^{(2)} = 1 + \delta_{\mathtt{1}_i\mathtt{2}_i} \ , \qquad \chi_i^{(3)} = (1 + \delta_{\mathtt{1}_i\mathtt{2}_i} + \delta_{\mathtt{2}_i\mathtt{3}_i})(1 + \delta_{\mathtt{1}_i\mathtt{3}_i}) \ .$$

Such a single indexing satisfies basic identities, repeatedly used in the following.

Lemma 2.1. *Let V be a vector space and f an arbitrary function $f : I_g^n \to V$, where $I_g^n := I_g \times \ldots \times I_g$ (n times). Then, the following identity holds*

$$\sum_{i_1,\ldots,i_n=1}^{g} f(i_1, \ldots, i_n) = \sum_{i=1}^{M_n} \chi_i^{-1} \sum_{s\in\mathcal{P}_n} f(s(\mathtt{1})_i, \ldots, s(\mathtt{n})_i) \ , \tag{2.2}$$

that, for f completely symmetric, reduces to

$$\sum_{i_1,\ldots,i_n=1}^{g} f(i_1, \ldots, i_n) = n! \sum_{i=1}^{M_n} \chi_i^{-1} f(\mathtt{1}_i, \ldots, \mathtt{n}_i) \ . \tag{2.3}$$

Proof. Use

$$\sum_{i_1,\ldots,i_n=1}^{g} f(i_1, \ldots, i_n) = \sum_{i_n\ge\ldots\ge i_1=1}^{g} \sum_{s\in\mathcal{P}_n} \frac{f(i_{s_1}, \ldots, i_{s_n})}{\prod_{k=1}^{g}(\sum_{m=1}^{n} \delta_{k i_m})!} \ .$$

□

Note that $u^{\otimes n} \equiv u \otimes \ldots \otimes u$ is an element of $\mathrm{Sym}^n \mathbb{C}^g \cong \mathbb{C}^{M_n}$, for each $u \in \mathbb{C}^g$. By (2.2), the following identities are easily verified

$$u^{\otimes n} \cong \sum_{i=1}^{M_n} \chi_i^{-1} u \cdots u_i \tilde{e}_i \ , \quad (Bu)^{\otimes n} \cong \sum_{i,j=1}^{M_n} \chi_i^{-1} \chi_j^{-1} (B \cdots B)_{ij} u \cdots u_j \tilde{e}_i \ ,$$

where $\mathbb{C}^{M_n} \ni \tilde{e}_i \cong e \cdots e_i \in \mathrm{Sym}^n \mathbb{C}^g$, $i \in I_{M_n}$. Furthermore,

$$\sum_{j=1}^{M_n} \chi_j^{-1} (B \cdots B)_{ij} (C \cdots C)_{jk} = ((BC) \cdots (BC))_{ik} \ , \tag{2.4}$$

where B, C are arbitrary $g \times g$ matrices. This identity yields, for any non-singular B

$$\sum_{j=1}^{M_n} \chi_j^{-1} \chi_k^{-1} (B \cdots B)_{ij} (B^{-1} \cdots B^{-1})_{jk} = (\delta \cdots \delta)_{ik} \chi_k^{-1} = \delta_{ik} \ , \tag{2.5}$$

and then

$$\det_{ij}\big((B \cdots B)_{ij} \chi_j^{-1}\big) \det_{ij}\big((B^{-1} \cdots B^{-1})_{ij} \chi_j^{-1}\big) = 1 \ . \tag{2.6}$$

Also observe that

$$\prod_{i=1}^{M_n} u \cdots u_i = \prod_{k=1}^{g} u_k^{\frac{n}{g} M_n} \ , \tag{2.7}$$

where the product and the exponentiation are the standard ones among complex numbers; in particular,

$$\prod_{i=1}^{M} u u_i = \prod_{k=1}^{g} u_k^{g+1} \ .$$

In the following we will denote the minors of $(B \cdots B)$ by

$$|B \cdots B|^{i_1 \ldots i_m}_{j_1 \ldots j_m} := \det_{i \in i_1, \ldots, i_m \, j \in j_1, \ldots, j_m} (B \cdots B)_{ij} \ ,$$

$i_1, \ldots, i_m, j_1, \ldots, j_m \in I_{M_n}$, with $m \in I_{M_n}$.

Definition 2.3. Fix $g, n \in \mathbb{Z}_{>0}$. Set

$$I_{M_n} \supset I_n^{\mathrm{diag}} := \{ i \in I_{M_n} \mid \mathfrak{1}_i = \mathfrak{2}_i = \ldots = \mathfrak{n}_i \} \ .$$

Fix $l < g$ and $a, a_1, \ldots, a_l \in I_g$ and define the following subsets of I_{M_n}

$$\begin{aligned}
I_n^a &:= \{ i \in I_{M_n} \mid \mathfrak{1}_i = a \vee \mathfrak{2}_i = a \vee \ldots \vee \mathfrak{n}_i = a \} \ , \\
I_n^{a_1 \ldots a_l} &:= \bigcup_{k \in I_l} I_n^{a_k} \ , \\
I_N^{a_1 a_2} &:= I_2^{\mathrm{diag}} \cup I_2^{a_1 a_2} \ , \\
I_{M_n, l} &:= I_n^{1 \ldots l} \ .
\end{aligned}$$

2.2 Combinatorial lemmas

Fix a surjection $m : I_g \times I_g \to I_M$, $M := g(g+1)/2$, such that

$$m(i,j) = m(j,i) \ ,$$

$i, j \in I_g$. Such a surjection corresponds to an isomorphism $\mathbb{C}^M \to \mathrm{Sym}^2\mathbb{C}^g$ with $\tilde{e}_{m(i,j)} \mapsto (e_i \otimes e_j)_S$.

For each morphism $s : I_M \to I_M$ consider the g-tuples $d^k(s)$, $k \in I_{g+1}$, where

$$d^i_j(s) = d^{j+1}_i(s) = s_{m(i,j)} \ , \tag{2.8}$$

$i \le j \in I_g$. Note that if s is a monomorphism, then each g-tuple consists of distinct integers, and each $i \in I_M$ belongs to two distinct g-tuples.

Consider $\mathcal{P}^{g+1}_g \equiv \underbrace{\mathcal{P}_g \times \cdots \times \mathcal{P}_g}_{g+1 \text{ times}}$ and define $\varkappa : \mathcal{P}^{g+1}_g \times I_M \to I_M$, depending on m, by

$$\varkappa_{m(i,j)}(r^1, \ldots, r^{g+1}) = m(r^i_j, r^{j+1}_i) \ , \tag{2.9}$$

$i \le j \in I_g$, where $(r^1, \ldots, r^{g+1}) \in \mathcal{P}^{g+1}_g$. Note that

$$d^i_j(\varkappa(r^1, \ldots, r^{g+1})) = d^{j+1}_i(\varkappa(r^1, \ldots, r^{g+1})) = m(r^i_j, r^{j+1}_i) \ ,$$

$i \le j \in I_g$. Consider the subset of I_M determined by

$$I_{M,n} := \{m(i,j) | i \in I_n, \, j \in I_g\} \ ,$$

$n \in I_g$, with the ordering inherited from I_M, and denote by

$$L := M - (g-n)(g-n+1)/2 \ ,$$

its cardinality. The elements $\varkappa_l(r^1, \ldots, r^{g+1})$, $l \in I_{M,n}$, are independent of r^j_i, with $n+1 \le i, j \le g$, and $\varkappa$ can be generalized to a function $\varkappa : I_{M,n} \times \tilde{\mathcal{P}}^{g,n} \to I_M$, where $\tilde{\mathcal{P}}^{g,n} := \mathcal{P}^n_g \times \mathcal{P}^{g-n+1}_n$, by

$$\varkappa_i(\tilde{r}^1, \ldots, \tilde{r}^{g+1}) := \varkappa_i(r^1, \ldots, r^{g+1}) \ , \tag{2.10}$$

$i \in I_{M,n}$, $(\tilde{r}^1, \ldots, \tilde{r}^{g+1}) \in \tilde{\mathcal{P}}^{g,n}$, where $r^j \in \mathcal{P}_g$, $j \in I_{g+1}$, are permutations satisfying $r^j = \tilde{r}^j$, $j \in I_n$, and $r^j_i = \tilde{r}^j_i$, $i \in I_n$, $n+1 \le j \le g$. Furthermore, if $\{\varkappa_i(\tilde{r}^1, \ldots, \tilde{r}^{g+1})\}_{i \in I_{M,n}}$ consists of distinct elements, then it is a permutation of $I_{M,n}$. By a suitable choice of the surjection

$$m(j,i) = m(i,j) := M - (g-j)(g-j-1)/2 + i \ , \tag{2.11}$$

$j \le i \in I_g$, we obtain $I_{M,n} = I_L$ as an equality between ordered sets.

Consider the maps $s : I \to I$, where I is any ordered subset of I_M; if s is bijective, then it is a permutation of I. We define the function $\epsilon(s)$ to be the sign of the permutation if s is bijective, and zero otherwise.

Let F be a commutative field and S a non-empty set. Fix a set f_i, $i \in I_g$, of F-valued functions on S, and $x_i \in S$, $i \in I_M$. Set

$$ff_{m(i,j)} := f_i f_j \ ,$$

$i, j \in I_g$, and

$$\det f(x_{d^j(s)}) := \det_{ik} f_k(x_{d_i^j(s)}) \ ,$$

$j \in I_{g+1}$, where $x_i \in S$, $i \in I_M$. Furthermore, for any ordered set $I \subseteq I_M$, we denote by

$$\det_I ff(x_1, \dots, x_{\mathrm{Card}(I)}) \ ,$$

the determinant of the matrix $(ff_m(x_i))_{\substack{i \in I_{\mathrm{Card}(I)} \\ m \in I}}$.

Lemma 2.2. *Choose $n \in I_g$ and L points x_i in S, $i \in I_L$. Fix $g - n$ points $p_i \in S$, $n + 1 \le i \le g$ and g F-valued functions f_i on S, $i \in I_g$. The following $g(g - n)$ conditions*

$$f_i(p_j) = \delta_{ij} \ , \tag{2.12}$$

$1 \le i \le j$, $n + 1 \le j \le g$, imply

$$\begin{aligned} &\det_{I_{M,n}} ff(x_1, \dots, x_L) \\ &= \frac{1}{c_{g,n}} \sum_{s \in \mathcal{P}_L} \epsilon(s) \prod_{j=1}^{n} \det f(x_{d^j(s)}) \prod_{k=n+1}^{g+1} \det f(x_{d_1^k(s)}, \dots, x_{d_n^k(s)}, p_{n+1}, \dots, p_g) \end{aligned} \tag{2.13}$$

where

$$c_{g,n} := \sum_{(\tilde r^1, \dots, \tilde r^{g+1}) \in \tilde{\mathcal{P}}^{g,n}} \prod_{k=1}^{g+1} \epsilon(\tilde r^k) \epsilon(\varkappa(\tilde r^1, \dots, \tilde r^{g+1})) \ . \tag{2.14}$$

In particular, for $n = g$

$$c_g \det ff(x_1, \dots, x_M) = \sum_{s \in \mathcal{P}_M} \epsilon(s) \prod_{j=1}^{g+1} \det f(x_{d^j(s)}) \ , \tag{2.15}$$

where

$$c_g := c_{g,g} = \sum_{r^1, \dots, r^{g+1} \in \mathcal{P}_g} \prod_{k=1}^{g} \epsilon(r^k) \, \epsilon(\varkappa(r^1, \dots, r^g)) \ .$$

Proof. It is convenient to fix the surjection m as in (2.11), so that $I_{M,n} = I_L$. Next consider

$$c_{g,n} \det_{I_L} ff(x_1, \dots, x_L) = c_{g,n} \sum_{s \in \mathcal{P}_L} \epsilon(s) ff_1(x_{s_1}) \cdots ff_L(x_{s_L}) \ . \tag{2.16}$$

Restrict the sums in (2.14) to the permutations $(\tilde r^1, \dots, \tilde r^{g+1}) \in \mathcal{P}^{g,n}$, $i \in I_n$, such that $\epsilon(\varkappa(\tilde r^1, \dots, \tilde r^{g+1})) \ne 0$, and set $s' := s \circ \varkappa(\tilde r^1, \dots, \tilde r^{g+1})$, so that

$$ff_1(x_{s_1}) \cdots ff_L(x_{s_L}) = ff_{\varkappa_1}(x_{s'_1}) \cdots ff_{\varkappa_L}(x_{s'_L}) \ ,$$

where $\varkappa_i$ is to be understood as $\varkappa_i(\tilde r^1, \dots, \tilde r^{g+1})$. Note that $\forall \, l \in I_M$, there is a unique pair $i, j \in I_g$, $i \le j$, such that $l = m(i, j)$, and by (2.8) and (2.9) the following identity

$$ff_{\varkappa_l(r^1, \dots, r^{g+1})}(x_{s'_l}) = ff_{m(r_j^i, r_i^{j+1})}(x_{s'_{m(i,j)}}) = f_{r_j^i}(x_{d_j^i(s')}) f_{r_i^{j+1}}(x_{d_i^{j+1}(s')}) \ ,$$

holds $\forall\,(r^1,\ldots,r^{g+1})\in\mathcal{P}_g^{g+1}$. On the other hand, if $l\in I_L$, then $i\le n$ and by Eq.(2.10)

$$ff_1(x_{s_1})\cdots ff_L(x_{s_L})=\prod_{i=1}^{n}f_{\tilde r_1^i}(x_{d_1^i(s')})\cdots f_{\tilde r_g^i}(x_{d_g^i(s')})\prod_{j=n+1}^{g+1}f_{\tilde r_1^j}(x_{d_1^j(s')})\cdots f_{\tilde r_n^j}(x_{d_n^j(s')})\ . \tag{2.17}$$

The condition $f_i(p_j)=\delta_{ij}$, $i\le j$, implies

$$\sum_{\tilde r^j\in\mathcal{P}_n}\epsilon(\tilde r^j)f_{\tilde r_1^j}(x_{d_1^j(s')})\cdots f_{\tilde r_n^j}(x_{d_n^j(s')})=\det f(x_{d_1^j(s')},\ldots,x_{d_n^j(s')},p_{n+1},\ldots,p_g)\ ,$$

$n+1\le j\le g+1$. Hence, Eq.(2.13) follows by replacing the sum over s with the sum over s' in (2.16), and using

$$\epsilon(s)=\epsilon(s')\,\epsilon(\varkappa(\tilde r^1,\ldots,\tilde r^{g+1}))\ .$$

Eq.(2.15) is an immediate consequence of (2.13). $\square$

Remark 2.1. The summation over $\mathcal{P}_M$ in Eq.(2.15) yields a sum over $(g+1)!$ identical terms, corresponding to permutations of the $g+1$ determinants in the product. Such an overcounting can be avoided by summing over the following subset of $\mathcal{P}_M$

$$\mathcal{P}'_M:=\{s\in\mathcal{P}_M,\ s.t.\ s_1=1,\ s_2<s_3<\ldots<s_g,\ s_2<s_i,\ g+1\le i\le 2g-1\}\ ,$$

and by replacing c_g by $c_g/(g+1)!$.

It can be verified that

$$c_{g,1}=g!\ ,\quad c_{g,2}=g!(g-1)!(2g-1)\ ,\quad c_2=6\ ,\quad c_3=360\ ,\quad c_4=302400\ . \tag{2.18}$$

The only non-trivial computation is $c_{g,2}$, which is more interesting case for the following constructions. The computation of its value is reported in section 2.2.1. For $g=2$, $c_g/(g+1)!=1$ and $\mathcal{P}'_{M=3}=\{(1,2,3)\}$, so that

$$\det ff(x_1,x_2,x_3)=\det f(x_1,x_2)\det f(x_1,x_3)\det f(x_2,x_3)\ . \tag{2.19}$$

A crucial point in proving Lemma 2.2 is that if $\varkappa_i(\tilde r^1,\ldots,\tilde r^{g+1})$, $i\in I_{M,n}$, are pairwise distinct elements in I_M, then they belong to $I_{M,n}\subseteq I_M$, with $\varkappa$ a permutation of such an ordered set. For a generic ordered set $I\subseteq I_M$, one should consider $\varkappa$ as a function over $g+1$ permutations $\tilde r^i$, $i\in I_{g+1}$, of suitable ordered subsets of I_g. In particular, $\tilde r^i$ should be a permutation over all the elements $j\in I_g$ such that $m(i,j)\in I$, for $j\ge i$, or $m(i-1,j)\in I$, for $j<i$. However, the condition that the elements $\varkappa_i(\tilde r^1,\ldots,\tilde r^{g+1})$, $i\in I$, are pairwise distinct does not imply, in general, that they belong to I and Lemma 2.2 cannot be generalized to a determinant of products ff_i, $i\in I$. On the other hand, the subsets

$$I:=I_{M,n}\cup\{m(i,j)\}\ , \tag{2.20}$$

satisfy such a condition for $n<i,j\le g$ and yield the following generalization of Lemma 2.2.

Lemma 2.3. *Choose $n \in I_g$ and $L+1$ points x_i in S, $i \in I_L$. Fix $g-n$ points $p_i \in S$, $n+1 \leq i \leq g$ and g F-valued functions f_i on S, $i \in I_g$, satisfying the $g(g-n)$ conditions (2.12). For each fixed pair i, j, $n < i, j \leq g$, the following relation*

$$\begin{aligned}\det{}_I f f(x_1, \ldots, x_{L+1}) = \frac{1}{c'_{g,n}} \qquad\qquad (2.21)\\ \cdot \sum_{s \in \mathcal{P}_{L+1}} \epsilon(s) \det f(x_{d_1^{n+1}(s)}, \ldots, x_{d_{n+1}^{n+1}(s)}, p_{n+1}, \ldots, \check{p}_i, \ldots, p_g)\\ \cdot \det f(x_{d_1^{n+2}(s)}, \ldots, x_{d_{n+1}^{n+2}(s)}, p_{n+1}, \ldots, \check{p}_j, \ldots, p_g)\\ \cdot \prod_{k=1}^{n} \det f(x_{d^k(s)}) \prod_{l=n+3}^{g+1} \det f(x_{d_1^l(s)}, \ldots, x_{d_n^l(s)}, p_{n+1}, \ldots, p_g) \ ,\end{aligned}$$

holds, where

$$c'_{g,n} := \sum_{(\tilde{r}^1, \ldots, \tilde{r}^{g+1}) \in \tilde{\mathcal{P}}^I} \prod_{i=1}^{g+1} \epsilon(\tilde{r}^i) \epsilon(\varkappa(\tilde{r}^1, \ldots, \tilde{r}^{g+1})) \ ,$$

$\tilde{\mathcal{P}}^I := \mathcal{P}_g^n \times \mathcal{P}_{n+1}^2 \times \mathcal{P}_n^{g-n-1}$, and I is defined in (2.19).

Proof. A straightforward generalization of the proof of Lemma 2.2. □

2.2.1 Computation of $c_{g,2}$

Let us choose the definition (2.11) for $m(i,j)$, so that, in particular,

$$m(1,i) = i \ , \quad i \in I_g \ , \qquad m(2,j) = j + g \ , \quad j \in I_g \setminus \{1\} \ .$$

Definition (2.14), for $n = 2$, can be written as

$$c_{g,2} = \sum_{t^2, \ldots, t^g \in \mathcal{P}_2} \sum_{r,s \in \mathcal{P}_g} \prod_{i=2}^{g} \epsilon(t^i) \epsilon(r) \epsilon(s) \epsilon(\varkappa(r, s, t^2, \ldots, t^g)) \ ,$$

where $\varkappa(r, s, t^2, \ldots, t^g)$ is a $(2g-1)$-tuple of elements in I_{2g-1}, whose i-th element is given by applying m to the i-th element of

$$(r(1)s(1), t^2(1)r(2), \ldots, t^g(1)r(g), t^2(2)s(2), \ldots, t^g(2)s(g)) \ . \qquad (2.22)$$

In other words, $\epsilon(\varkappa(r, s, t^2, \ldots, t^g))$ vanishes if the elements in (2.22) are not pairwise distinct; otherwise, (2.22) is necessarily given by a permutation of

$$(11, 12, \ldots, 1g, 22, 23, \ldots, 2g) \ ,$$

and $\epsilon(\varkappa(r, s, t^2, \ldots, t^g))$ is the sign of such a permutation.

Denote by e and p the identity element and the non-trivial permutation of $\mathcal{P}_2$. Fix $t^2, \ldots, t^g \in \mathcal{P}_2$ and let k, $0 \leq k \leq g-1$, be the number of permutations in this set which do not correspond to the identity element $e \in \mathcal{P}_2$. Then, $\prod_2^g \epsilon(t^i) = (-)^k$. We can reorder

$$t^2, \ldots, t^g \to \tilde{t}^2, \ldots, \tilde{t}^g \qquad (2.23)$$

in such a way that

$$\tilde{t}^2, \ldots, \tilde{t}^{g-k} = e \ , \qquad \tilde{t}^{g-k+1}, \ldots, \tilde{t}^g = p \ .$$

For each choice of r, s, one can apply the reordering (2.23) to $r(2), \ldots, r(g)$ and to $s(2), \ldots, s(g)$, while keeping $r(1)$ and $s(1)$ fixed, to obtain two new permutations $\tilde{r}, \tilde{s}$. Then, it is readily verified that

$$\epsilon(r)\epsilon(s)\epsilon(\varkappa(r, s, t^2, \ldots, t^g)) = \epsilon(\tilde{r})\epsilon(\tilde{s})\epsilon(\varkappa(\tilde{r}, \tilde{s}, \tilde{t}^2, \ldots, \tilde{t}^g)) = \epsilon(\tilde{r})\epsilon(\tilde{s})\epsilon(r, s, k) \ ,$$

where

$$\epsilon(r, s, k) := \epsilon(\varkappa(r, s, \underbrace{e, \ldots, e}_{g-1-k \text{ times}}, \underbrace{p, \ldots, p}_{k \text{ times}})) \ .$$

It follows that

$$c_{g,2} = \sum_{k=0}^{g-1} (-)^k \binom{g-1}{k} \sum_{r,s \in \mathcal{P}_g} \epsilon(r)\epsilon(s)\epsilon(r, s, k) \ ,$$

where $\binom{g-1}{k}$ are all the different ways to choose the k non-trivial permutations among $t^2, \ldots, t^g$. Let us consider the dependence of the product $\epsilon(r)\epsilon(s)\epsilon(r, s, k)$ on $r \in \mathcal{P}_g$, for a fixed k. For each fixed $r(1)$, such a product only depends on the splitting of $I_g \setminus \{r_1\}$ into the disjoint union of two subsets, given by

$$I_g \setminus \{r_1\} = \{r_2, \ldots, r_{g-k}\} \sqcup \{r_{g-k+1}, \ldots, r_g\} \ .$$

In particular, two different permutations r and $\tilde{r}$ such that $r_1 = \tilde{r}_1$ and such that

$$\{r_2, \ldots, r_{g-k}\} = \{\tilde{r}_2, \ldots, \tilde{r}_{g-k}\} \ ,$$
$$\{r_{g-k+1}, \ldots, r_g\} = \{r_{g-k+1}, \ldots, r_g\} \ ,$$

give the same contribution to the summation. The conditions above determine an equivalence relation, depending on k, between elements in $\mathcal{P}_g$, each equivalence class corresponding to $k!(g-k-1)!$ elements. The same considerations apply to the permutation s, so that

$$c_{g,2} = \sum_{k=0}^{g-1} (-)^k \binom{g-1}{k} [k!(g-k-1)!]^2 \sum_{r_1,s_1=1}^{g} \sum_{[r],[s] \in \binom{g-1}{k}_{r_1,s_1}} \epsilon(r)\epsilon(s)\epsilon(r, s, k) \ .$$

Here, the notation $\sum_{[r] \in \binom{g-1}{k}_{r_1}}$ means that we are summing the equivalence classes corresponding to a fixed k and r_1. A representative for each class can be chosen by imposing, for example, $r_2 < \ldots < r_{g-k}$ and $r_{g-k+1} < \ldots < r_g$; this will be our ususal choice in the following.

Let us consider the sums over r_1 and s_1. If both r_1 and s_1 are greater than 2, then $\epsilon(r, s, k) = 0$. If $r_1 = s_1$, then $\epsilon(r, s, k) \neq 0$ if and only if $r = s$ as permutations (so that $\epsilon(r)\epsilon(s) = 1$). In this case, it is easy to check that $\epsilon(r, s, k) = (-)^k$. Then,

$$\sum_{i=1}^{2} \sum_{[r],[s] \in \binom{g-1}{k}_{i,i}} \epsilon(r)\epsilon(s)\epsilon(r, s, k) = 2\binom{g-1}{k}(-)^k \ .$$

The other cases for which $\epsilon(r, s, k)$ is non-vanishing are:

– $r_1 = 1$, $s_1 = i > 1$. Let n, m be the integers such that $r_n = i$ and $s_m = 1$. Then, to obtain a non-vanishing $\epsilon(r, s, k)$, a necessary condition is $n, m > g - k$; we can choose representatives r, s of the equivalence classes, in such a way that $r_{g-k+1} = i, s_{g-k+1} = 1$ and $r_2 < \ldots < r_{g-k}$, $r_{g-k+2} < \ldots < r_g$ and analogous ordering for s. Then, $\epsilon(r, s, k) \neq 0$ if and only if $r_l = s_l$ for all $l \notin \{1, g - k + 1\}$ and, in this case, we have $\epsilon(r, s, k) = (-)^k$. It follows that

$$\sum_{i=2}^{g} \sum_{[r],[s]\in\binom{g-1}{k}_{1,i}} \epsilon(r)\epsilon(s)\epsilon(r, s, k) = (g-1)\binom{g-2}{k-1}(-)^k \ .$$

Here, the factor $\binom{g-2}{k-1}$ is the number of different ways to choose $r_{g-k+2} < \ldots < r_g$ in the set $I_g \setminus \{1, i\}$ and the factor $g - 1$ is due to the sum over $s_1 = i$.

– $r_1 = i > 1$, $s_1 = 1$. In this case the conditions for $\epsilon(r, s, k) \neq 0$ are $r_2 = 1$, $s_2 = i$, and $r_l = s_l$ for all $l > 2$, and again $\epsilon(r, s, k) = (-)^k$, so that

$$\sum_{i=2}^{g} \sum_{[r],[s]\in\binom{g-1}{k}_{i,1}} \epsilon(r)\epsilon(s)\epsilon(r, s, k) = (g-1)\binom{g-2}{k}(-)^k \ .$$

The factor $\binom{g-2}{k}$ is to the number of different ways to choose $r_{g-k+1} < \ldots < r_g$ in $I_g \setminus \{1, i\}$.

– $r_1 = 2$, $s_1 = i > 2$. The contribution is

$$\sum_{i=3}^{g} \sum_{[r],[s]\in\binom{g-1}{k}_{2,i}} \epsilon(r)\epsilon(s)\epsilon(r, s, k) = (g-2)\binom{g-2}{k}(-)^k \ ,$$

where the factor $g - 2$ comes from the sum over $s_1 = i > 2$.

– $s_1 = 2$, $r_1 = i > 2$. The contribution is

$$\sum_{i=3}^{g} \sum_{[r],[s]\in\binom{g-1}{k}_{i,2}} \epsilon(r)\epsilon(s)\epsilon(r, s, k) = (g-2)\binom{g-2}{k-1}(-)^k \ .$$

To summarize, we have

$$\begin{aligned} c_{g,2} =& \sum_{k=0}^{g-1} \binom{g-1}{k} [k!(g-k-1)!]^2 \\ & \cdot \left[2\binom{g-1}{k} + (2g-3)\left(\binom{g-2}{k} + \binom{g-2}{k-1} \right) \right] \\ =& \sum_{k=0}^{g-1} (2g-1) \left[\binom{g-1}{k} k!(g-k-1)! \right]^2 \\ =& (2g-1)g[(g-1)!]^2 = (2g-1)g!(g-1)! \ . \end{aligned}$$

2.2.2 Examples of the combinatorial lemmas

We now show some examples of the combinatorial construction described in the last subsection. Set $g = 4$, so that $M = g(g+1)/2 = 10$. Fix a surjection $m : I_4 \times I_4 \to I_{10}$ with $m(i,j) = m(j,i)$, for example by setting $m(i,j) = [m]_{ij}$, with $[m]$ the symmetric matrix

$$[m] = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix} .$$

For each function $s : I_{10} \to I_{10}$, the 4-tuples $d^i(s)$, $i = 1, \ldots, g+1 = 5$, are determined by

$$d^i_j(s) = d^{j+1}_i(s) = s_{m(i,j)} \ ,$$

$i \leq j \in I_g$, so that, with the above choice of m,

$$\begin{aligned} d^1(s) &= (s_1, s_2, s_3, s_4) \ , \\ d^2(s) &= (s_1, s_5, s_6, s_7) \ , \\ d^3(s) &= (s_2, s_5, s_8, s_9) \ , \\ d^4(s) &= (s_3, s_6, s_8, s_{10}) \ , \\ d^5(s) &= (s_4, s_7, s_9, s_{10}) \ . \end{aligned}$$

Let $\mathcal{P}_g$ be the group of permutations of g elements. The function $\varkappa : \mathcal{P}_4^5 \times I_{10} \to I_{10}$ is defined by

$$\varkappa_{m(i,j)}(r^1, \ldots, r^5) = m(r^i_j, r^{j+1}_i) \ , \tag{2.24}$$

$i \leq j \in I_g$, where $(r^1, \ldots, r^5) \in \mathcal{P}_4^5$. For example, fix

$$\begin{aligned} r^1 &= (3,4,1,2) \ , \\ r^2 &= (1,2,4,3) \ , \\ r^3 &= (2,4,1,3) \ , \\ r^4 &= (1,2,3,4) \ , \\ r^5 &= (2,4,1,3) \ . \end{aligned}$$

To determine $\varkappa_1(r^1, \ldots, r^5)$, note that $1 = m(1,1)$, so that, by definition,

$$\varkappa_{m(1,1)}(r^1, \ldots, r^5) = m(r^1_1, r^2_1) = m(3,1) = 3 \ .$$

As a further example note that $2 = m(1,2) = m(2,1)$, so that

$$\varkappa_{m(1,2)}(r^1, \ldots, r^5) = m(r^1_2, r^3_1) = m(4,2) = 7 \ ,$$

(observe that Eq.(2.24), which defines $\varkappa$, holds only for $i \leq j$). The 4-tuples $d^i(\varkappa(r_1, \ldots, r_5))$ are

$$\begin{aligned} d^1(\varkappa) &= (3,7,1,5) \ , \\ d^2(\varkappa) &= (3,7,7,9) \ , \\ d^3(\varkappa) &= (7,7,3,3) \ , \\ d^4(\varkappa) &= (1,7,3,9) \ , \\ d^5(\varkappa) &= (5,9,3,9) \ . \end{aligned}$$

It is readily verified the general relation

$$d_j^i(\varkappa(r^1,\ldots,r^5)) = d_i^{j+1}(\varkappa(r^1,\ldots,r^5)) = m(r_j^i, r_i^{j+1}) \ ,$$

$i \le j \in I_g$. Note that if $\varkappa(r^1,\ldots,r^5) : I_{10} \to I_{10}$, for some fixed $r^1,\ldots,r^5$, is a monomorphism, then it determines a permutation of I_{10}. Hence, we can define the function $\epsilon(\varkappa(r^1,\ldots,r^5))$ to be the sign of the permutation $\varkappa(r^1,\ldots,r^5)$ if it is a monomorphism, and zero otherwise.
Consider the subset

$$I_{M,n} = \{m(i,j) \mid i \in I_n, j \in I_g\} \ ,$$

for some $n \in I_g$. $\varkappa$ can be generalized to a function from $\tilde{\mathcal{P}}^{g,n} \times I_{M,n}$, where $\tilde{\mathcal{P}}^{g,n} := \mathcal{P}_g^n \times \mathcal{P}_n^{g-n+1}$, into I_M. As an example, consider $\varkappa : \tilde{\mathcal{P}}^{4,2} \times I_{10,2} \to I_{10}$, where $I_{10,2} = \{1,2,3,4,5,6,7\}$ (the precise form of $I_{10,2}$ depends on the choice of m). Fix $(\tilde{r}^1,\ldots,\tilde{r}^5) \in \tilde{\mathcal{P}}^{4,2} = \mathcal{P}_4^2 \times \mathcal{P}_2^3$, *e.g.* by

$$\begin{aligned}
\tilde{r}^1 &= (3,4,1,2) \ , \\
\tilde{r}^2 &= (1,2,4,3) \ , \\
\tilde{r}^3 &= (2,1) \ , \\
\tilde{r}^4 &= (1,2) \ , \\
\tilde{r}^5 &= (1,2) \ .
\end{aligned}$$

As a specific case, say $\varkappa_6$, note that $6 = m(2,3) = m(3,2)$ and set

$$\varkappa_{m(2,3)}(\tilde{r}^1,\ldots,\tilde{r}^5) = m(\tilde{r}_3^2, \tilde{r}_2^4) = m(4,2) = 7 \ .$$

For general choices of $\tilde{r}^1,\ldots,\tilde{r}^5$, $\varkappa(\tilde{r}^1,\ldots,\tilde{r}^5) : I_{10,2} \to I_{10}$ may not be a monomorphism. It can be verified that if the image $\varkappa(\tilde{r}^1,\ldots,\tilde{r}^5)(I_{10,2}) \not\subseteq I_{10,2}$, then $\varkappa(\tilde{r}^1,\ldots,\tilde{r}^5)$ is not a monomorphism. Therefore, if $\varkappa(\tilde{r}^1,\ldots,\tilde{r}^5)$ is a monomorphism, then it determines a permutation of $I_{10,2}$. Hence, we can define the function $\epsilon(\varkappa(\tilde{r}^1,\ldots,\tilde{r}^5))$ to be the sign of $\varkappa(\tilde{r}^1,\ldots,\tilde{r}^5)$ if it is a monomorphism, and zero otherwise.

Let us apply Lemma 2.2 to the previous examples. Consider four linearly independent functions $f_1,\ldots,f_4 : \mathbb{C} \to \mathbb{C}$, and set

$$ff_{m(i,j)}(z) := f_i(z)f_j(z) \ .$$

Next, fix $x_1,\ldots,x_{10} \in \mathbb{C}$ and consider

$$\det\begin{pmatrix} ff_1(x_1) & \ldots & ff_{10}(x_1) \\ \vdots & \ddots & \vdots \\ ff_1(x_{10}) & \ldots & ff_{10}(x_{10}) \end{pmatrix} = \det\begin{pmatrix} f_1(x_1)f_1(x_1) & \ldots & f_4(x_1)f_4(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_{10})f_1(x_{10}) & \ldots & f_4(x_{10})f_4(x_{10}) \end{pmatrix}$$

so that $m(i,j)$ determines the column where f_if_j appears. It is easily verified that the above determinant is proportional to

$$\sum_{s\in\mathcal{P}_{10}} \epsilon(s) \det f_i(x_{d_j^1(s)}) \det f_i(x_{d_j^2(s)}) \det f_i(x_{d_j^3(s)}) \det f_i(x_{d_j^4(s)}) \det f_i(x_{d_j^5(s)}) \ . \tag{2.25}$$

This expression, after expanding each determinant, consists of a summation over products of twenty factors $f_i(x_j)$, where each x_k appears twice. After skew-symmetrization of the x_k's, this expression is necessarily proportional to the original determinant.

In Lemma 2.2 it is also considered the more general case of determinants made up of functions ff_i, where i varies in a subset $I_{M,n} \subset I_M$ of $L < M$ elements. For example, let us consider the subset $I_{10,2} = \{1, \ldots, 7\}$ and fix the points $x_1, \ldots, x_7 \in \mathbb{C}$. We are interested in the determinant

$$\det \begin{pmatrix} ff_1(x_1) & \ldots & ff_7(x_1) \\ \vdots & \ddots & \vdots \\ ff_1(x_7) & \ldots & ff_7(x_7) \end{pmatrix} = \det \begin{pmatrix} f_1(x_1)f_1(x_1) & \ldots & f_2(x_1)f_4(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_7)f_1(x_7) & \ldots & f_2(x_7)f_4(x_7) \end{pmatrix} . \tag{2.26}$$

By repeating the above construction, this determinant can be expressed as (a sum over) products of two determinants of 4×4 matrices times three determinants of lower-dimensional 2×2 matrices

$$\sum_{s \in \mathcal{P}_{10}} \epsilon(s) \det_{I_4} f_i(x_{d^1_j(s)}) \det_{I_4} f_i(x_{d^2_j(s)}) \det_{I_2} f_i(x_{d^3_j(s)}) \\ \det_{I_2} f_i(x_{d^4_j(s)}) \det_{I_2} f_i(x_{d^5_j(s)}) ,$$

where $\det_{I_n} f_i(x_j) := \det_{ij \in I_n} f_i(x_j)$. In order to obtain products of five determinants of 4×4 matrices in the form similar to Eq.(2.25), one has to impose some conditions on the functions f_i. In particular, it is sufficient to require that there exist two points, $p_3, p_4 \in \mathbb{C}$, such that

$$\begin{aligned} & f_1(p_i) = f_2(p_i) = 0 , \qquad i = 3, 4 , \\ & f_3(p_4) = f_4(p_3) = 0 , \\ & f_3(p_3) = f_4(p_4) = 1 . \end{aligned}$$

In this case, the following identity

$$\det \begin{pmatrix} f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{pmatrix} = \det \begin{pmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) & f_4(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) & f_4(x_2) \\ f_1(p_3) & f_2(p_3) & f_3(p_3) & f_4(p_3) \\ f_1(p_4) & f_2(p_4) & f_3(p_4) & f_4(p_4) \end{pmatrix} ,$$

holds and the determinants in (2.26) are proportional to

$$\sum_{s \in \mathcal{P}_7} \epsilon(s) \det f_i(x_{d^1_j(s)}) \det f_i(x_{d^2_j(s)}) \det f(x_{d^3_1(s)}, x_{d^3_2(s)}, p_3, p_4) \\ \cdot \det f(x_{d^4_1(s)}, x_{d^4_2(s)}, p_3, p_4) \det f(x_{d^5_1(s)}, x_{d^5_2(s)}, p_3, p_4) , \tag{2.27}$$

where $\det f(z_1, \ldots, z_4) := \det_{ij \in I_4} f_i(z_j)$. Lemma 2.2 generalizes such a result to any g and n. Proportionality of Eqs.(2.26) and (2.27) can be understood as follows. Upon expanding the determinants in (2.27) and using the conditions on f_i, this expression corresponds to a summation of products of the form

$$f_1 f_2 f_3 f_4 \cdot f_1 f_2 f_3 f_4 \cdot f_1 f_2 \cdot f_1 f_2 \cdot f_1 f_2 , \tag{2.28}$$

with the f_i's evaluated at $x_1, \ldots, x_7$ (each x_i appears twice). Such a product can be re-arranged as

$$ff_{i_1}(x_1)ff_{i_2}(x_2)\ldots ff_{i_7}(x_7) \ ,$$

for some $i_1, \ldots, i_7 \in I_{10}$. After skew-symmetrization over the variables x_i, only the products with distinct $i_1, \ldots, i_7$ contribute. But this implies $i_1, \ldots, i_7 \in I_{10,2}$, since the only possibility to construct seven different functions $f_i f_j$ out of the fourteen functions in Eq.(2.28) is

$$f_1^2(x_1)f_1f_2(x_2)f_1f_3(x_3)f_1f_4(x_4)f_2^2(x_5)f_2f_3(x_6)f_2f_4(x_7) \ , \tag{2.29}$$

up to permutations of the x_i's. This is strictly related to the observation that if $\varkappa(\tilde{r}^1, \ldots, \tilde{r}^5)$ is a monomorphism, then it corresponds to a permutation of $I_{10,2}$. The skew-symmetrization of (2.29) with respect to $x_1, \ldots, x_7$ is exactly the determinant we were looking for.

Note that Lemma 2.2 may not be generalized to the case of determinants of matrices with rows $ff_{i_1}, \ldots, ff_{i_L}$, when $I := \{i_1, \ldots, i_L\}$ is a generic subset of I_{10}. One can always define a generalization of the $\varkappa$ function as $\varkappa(\tilde{r}^1, \ldots, \tilde{r}^5) : I \to I_{10}$, with $\tilde{r}^1, \ldots, \tilde{r}^5$ in some suitable subset of $\mathcal{P}_4^5$. However, the necessary condition for the generalization of Lemma 2.2 is that if $\varkappa$ is a monomorphism, then $\varkappa(I) = I$. Such a condition is verified, for example, if $I = I_{10,n}$, as showed before for $I_{10,2}$. The condition still holds when $I = I_{10,n} \cup \{j\}$, for all the elements $j \in I_{10} \backslash I_{10,n}$, which is the content of Lemma 2.3. An example for which the analog of Lemma 2.2 does not exist is for $I = \{1, 5, 8, 10\}$, corresponding to determinants of matrices with rows $f_1^2, f_2^2, f_3^2, f_4^2$. Actually, defining a formula similar to (2.25) in order to obtain terms in the form $f_1^2(x_1)f_2^2(x_2)f_3^2(x_3)f_4^2(x_4)$, some unwanted terms, such as $f_1f_2(x_1)f_2f_3(x_2)f_3f_4(x_3)f_4f_1(x_4)$, do not cancel in the RHS.

3. DETERMINATS OF HOLOMORHPIC DIFFERENTIALS AND THETA FUNCTIONS SURFACES

After reminding some basic facts about theta functions, we investigate the divisor structures of the theta function and its derivatives that will be used in the subsequent chapters.

3.1 Determinants in terms of theta functions

Set

$$S(p_1 + \ldots + p_g) := \frac{\theta(\sum_1^g p_i - y)}{\sigma(y) \prod_1^g E(y, p_i)} \ , \tag{3.1}$$

$y, p_1, \ldots, p_g \in C$.

Lemma 3.1. *For all $p_1, \ldots, p_g \in C$, $S(p_1 + \ldots + p_g)$ is independent of y. For each fixed $d \in C_{g-1}$, consider the map $\pi_d : C \to C_g$, $\pi_d(p) := p + d$. The pull-back $\pi_d^* S$ vanishes identically if and only if d is a special divisor; if d is not special, then $\pi_d^* S$ is the unique (up to a constant) holomorphic $1/2$-differential such that $[(\pi_d^* S) + d]$ is the canonical divisor class.*

Proof. If $p_1 + \ldots + p_g$ is a special divisor, the Riemann Vanishing Theorem implies $S = 0$ identically in y; if $p_1 + \ldots + p_g$ is not special, S is a single-valued meromorphic section in y with no zero and no pole. It follows that, in any case, S is a constant in y. This also shows that $S(p_1 + \ldots + p_g) = 0$ if and only if $p_1 + \ldots + p_g$ is a special divisor. Hence, if $d \in C_{g-1}$ is a special divisor, $S(p + d) = 0$ for all $p \in C$. On the contrary, if d is not special, then $h^0(K_C \otimes \mathcal{O}(-d)) = 1$, and $S(p+d) = 0$ if and only if p is one of the zeros of the (unique, up to a constant) holomorphic section of $H^0(K_C \otimes \mathcal{O}(-d))$, and this concludes the proof. □

Proposition 3.2 (Fay, [23, 24]). *Fix $n \in \mathbb{N}_+$, set $N_n := (2n-1)(g-1) + \delta_{n1}$ and let $\{\phi_i^n\}_{i \in I_{N_n}}$ be arbitrary bases of $H^0(K_C^n)$. There are constants $\kappa[\phi^n]$ depending only on the marking of C and on $\{\phi_i^n\}_{i \in I_{N_n}}$ such that*

$$\kappa[\phi^1] = \frac{\det \phi_i^1(p_j)}{S(\sum_1^g p_i) \prod_1^g \sigma(p_i) \prod_{i<j}^g E(p_i, p_j)} \ , \tag{3.2}$$

and

$$\kappa[\phi^n] = \frac{\det \phi_i^n(p_j)}{\theta(\sum_1^{N_n} p_i) \prod_1^{N_n} \sigma(p_i)^{2n-1} \prod_{i<j}^{N_n} E(p_i, p_j)} \ , \tag{3.3}$$

for $n \geq 2$, for all $y, p_1, \ldots, p_{N_n} \in C$.

Proof. $\kappa[\phi^n]$ is a meromorphic function with empty divisor with respect to $y, p_1, \ldots, p_{N_n}$. □

For each set $\{\phi_i^n\}_{i\in I_{N_n}} \subset H^0(K_C^n)$, consider the Wronskian

$$W[\phi^n](p) := \det \partial_p^{j-1} \phi_i^n(p) \ .$$

If $W[\phi^n](p)$ does not vanish identically, then, for each $\{\phi_i^{n'}\}_{i\in I_{N_n}} \subset H^0(K_C^n)$, we have the constant ratio

$$\frac{\kappa[\phi^n]}{\kappa[\phi^{n'}]} = \frac{\det \phi^{n'}(p_1, \ldots, p_{N_n})}{\det \phi^n(p_1, \ldots, p_{N_n})} = \frac{W[\phi^{n'}](p)}{W[\phi^n](p)} \ , \tag{3.4}$$

for arbitrary $p, p_1, \ldots, p_{N_n} \in C$.

3.2 Relations among higher order theta derivatives and holomorphic differentials

By Riemann Vanishing Theorem it follows that

$$\theta(np + c_{g-n} - y)$$

$n \in I_g$, as a function of y, has a zero of order n at p for all the effective divisors c_{g-n} of degree $g - n$. In particular,

$$\sum_i \theta_i(p + c_{g-2})\omega_i(p) = 0 \ . \tag{3.5}$$

Proposition 3.3. *Fix $x_1, \ldots, x_{g-1} \in C$. The following relations hold*

$$\sum_i \theta_i(x_1 + \ldots + x_{g-1})\omega_i(x_1) = 0 \ ,$$

$$\sum_{i,j} \theta_{ij}(x_1 + \ldots + x_{g-1})\omega_i(x_1)\omega_j(x_2) = 0 \ ,$$

$$\vdots$$

$$\sum_{i_1,\ldots,i_{g-1}} \theta_{i_1\ldots i_{g-1}}(x_1 + \ldots + x_{g-1})\omega_{i_1}(x_1)\cdots\omega_{i_{g-1}}(x_{i_{g-1}}) = 0 \ .$$

Proof. Without loss of generality, we can assume distinct $x_1, \ldots, x_{g-1}$; the general case follows by continuity arguments. The first relation is just Eq.(3.5). Let us assume that the equation

$$\sum_{i_1,\ldots,i_n} \theta_{i_1\ldots i_n}(x_1 + \ldots + x_{g-1})\omega_{i_1}(x_1)\ldots\omega_{i_n}(x_n) = 0 \ ,$$

holds, for all $n \in I_{N-1}$, with $1 < N \le g-1$. Then by taking its derivative with respect to x_{n+1} one obtains the subsequent relation. □

Corollary 3.4. *Fix $p \in C$ and a set of effective divisors c_k, $k \in I_{g-2}$ of degree k. The following relations hold*

$$
\begin{aligned}
&\sum_i \theta_i(p + c_{g-2})\omega_i(p) = 0 \ ,\\
&\sum_{i,j} \theta_{ij}(2p + c_{g-3})\omega_i\omega_j(p) = 0 \ ,\\
&\vdots\\
&\sum_{i_1,\ldots,i_{g-1}} \theta_{i_1\ldots i_{g-1}}((g-1)p)\omega_{i_1}\cdots\omega_{i_{g-1}}(p) = 0 \ .
\end{aligned}
$$

We denote by $\lambda := \{\lambda_1, \ldots, \lambda_l\}$ a partition of length $|\lambda| := l$ of some integer $d > 0$, that is

$$\sum_{i=1}^{l} \lambda_i = d \ , \qquad \lambda_1 \geq \ldots \geq \lambda_l > 0 \ .$$

On the set of the partitions of an integer d, a total order relation can be defined by setting

$$\lambda' > \lambda \qquad \Longleftrightarrow \qquad \exists i,\ 0 < i \leq \min\{|\lambda|, |\lambda'|\},\ \text{ s.t. } \begin{cases} \lambda'_j = \lambda_j, & 1 \leq j < i \ ,\\ \lambda'_i > \lambda_i \ . \end{cases}$$

With respect to such a relation, the minimal and the maximal partitions λ^{min} and λ^{max} of d, are

$$\lambda_1^{min} = \ldots = \lambda_d^{min} = 1 \ , \qquad \lambda_1^{max} = d \ .$$

Also observe that λ^{min} and λ^{max} have, respectively, the maximal and minimal lengths $|\lambda^{min}| = d$, $|\lambda^{max}| = 1$.
For a general holomorphic d differential η, let $\eta(z)$ be its trivialization around a point $p \in C$, with respect to some local coordinate z and let us define

$$\eta^{(0)}(p) := \eta(z) \ , \qquad \eta^{(n)}(p) := \frac{\partial^n \eta}{\partial z^n}(z) \ , \qquad n > 0 \ .$$

Theorem 3.5. *Fix $d \in I_{g-1}$, a point $p \in C$ and a effective divisor c_{g-d} of degree $g - d$. Then, for each partition λ of d, there exists $c(\lambda) \in \mathbb{Z}$ independent of C, p, c_{g-d}, such that*

$$\sum_{i_1,\ldots,i_l}^{g} \theta_{i_1\ldots i_l}((d-1)p + c_{g-d})\omega_{i_1}^{(\lambda_1-1)} \cdots \omega_{i_l}^{(\lambda_l-1)}(p) \tag{3.6}$$

$$= c(\lambda) \sum_{j_1,\ldots,j_d}^{g} \theta_{j_1\ldots j_d}((d-1)p + c_{g-d})\omega_{j_1} \cdots \omega_{j_d}(p) \ , \tag{3.7}$$

where $l := |\lambda|$.

Proof. The theorem is just an identity for $\lambda = \lambda^{min}$, with $c(\lambda^{min}) = 1$. Let us consider a partition $\lambda > \lambda^{min}$ of d, and set $l := |\lambda| < d$ ($|\lambda| = d$ necessarily implies $\lambda = \lambda^{min}$). Fix $c = x_1 + \ldots + x_{g-1}$, with $x_1, \ldots, x_{g-1} \in C$, and apply the derivative operator

$$\mathcal{D}^{(\lambda)} := \Big(\frac{d}{dx_1}\Big)^{\lambda_1} \cdots \Big(\frac{d}{dx_l}\Big)^{\lambda_l} ,$$

to the identity

$$\theta(c - \Delta) = 0 \; . \tag{3.8}$$

Upon taking the limit $x_1, \ldots, x_l \to p$, we obtain a sum, such that each term can be associated to a partition λ' of d and written as

$$\sum_{i_1,\ldots,i_{l'}}^{g} \theta_{i_1\ldots i_{l'}}(lp + c_{g-1-l} - \Delta)\omega_{i_1}^{(\lambda'_1-1)} \cdots \omega_{i_{l'}}^{(\lambda'_{l'}-1)}(p) \; ,$$

with $l' := |\lambda'|$ and $c_{g-1-l} = x_{l+1} + \ldots + x_{g-1}$. The sum is over a set of partitions λ' satisfying $\lambda' \leq \lambda$ and $l' \geq l$, so that λ is the maximal partition appearing. Thus, the sum can be rearranged as

$$\sum_{i_1,\ldots,i_l} \theta_{i_1\ldots i_l}(lp + c_{g-1-l})\omega_{i_1}^{(\lambda_1-1)} \cdots \omega_{i_l}^{(\lambda_l-1)}(p) \tag{3.9}$$

$$= \sum_{\lambda'<\lambda} b(\lambda, \lambda') \sum_{i_1,\ldots,i_{l'}} \theta_{i_1\ldots i_{l'}}(lp + c_{g-1-l})\omega_{i_1}^{(\lambda'_1-1)} \cdots \omega_{i_{l'}}^{(\lambda'_{l'}-1)}(p) \; , \tag{3.10}$$

for some coefficients $b(\lambda, \lambda') \in \mathbb{Z}$. If the only non-vanishing contribution to the RHS corresponds to $\lambda' = \lambda^{min}$, the theorem follows after taking the limit $x_{l+1}, \ldots, x_{d-1} \to p$. Otherwise, for each $\lambda' > \lambda^{min}$, one can obtain a further identity by applying the operator $\mathcal{D}^{(\lambda')}$ to the identity (3.8) and taking the limit $x_1, \ldots, x_{l'} \to p$. This procedure leads to an expression for

$$\sum_{i_1,\ldots,i_{l'}} \theta_{i_1\ldots i_{l'}}(l'p + c_{g-1-l'})\omega_{i_1}^{(\lambda'_1-1)} \cdots \omega_{i_{l'}}^{(\lambda'_{l'}-1)}(p) \; ,$$

analogous to Eq.(3.9), where the RHS is a sum of terms corresponding to partitions $\lambda'' < \lambda'$. This expression can be used to replace the term corresponding to λ' in Eq.(3.9), considered in the limit $x_{l+1}, \ldots, x_{l'} \to p$, with a sum over a set of partitions $\lambda'' < \lambda'$. After a finite number of steps, the RHS of Eq.(3.9) reduces to a term corresponding to λ^{min} times an integer coefficient

$$c(\lambda) := \sum_{\lambda'<\lambda} \sum_{\lambda''<\lambda'} \cdots \sum_{\lambda^{\cdots}} b(\lambda, \lambda')b(\lambda', \lambda'') \cdots b(\lambda^{\cdots}, \lambda^{min}) \; .$$

The arguments of the θ-functions on both sides are

$$l'p - c_{g-1-l'} - \Delta \; ,$$

where l' is the length of the minimal partition $\lambda' > \lambda^{min}$ appearing in any intermediate step of the procedure. Therefore, $l' \leq d - 1$ and the theorem follows. (Actually, with some more effort, it can be proved that the bound $d-1$ cannot be improved). $\square$

Corollary 3.6. *Fix $d \in I_{g-1}$, a point $p \in C$ and an effective divisor c_{g-d-1} of degree $g-d-1$. Then, for each partition λ of d,*

$$\sum_{i_1,\ldots,i_l}^{g} \theta_{i_1\ldots i_l}(dp + c_{g-d-1})\omega_{i_1}^{(\lambda_1-1)}\cdots\omega_{i_l}^{(\lambda_l-1)}(p) = 0 \ ,$$

where $l := |\lambda|$.

Proof. A trivial application of Eq.(3.6), with $c_{g-d} := p + c_{g-d-1}$, and Corollary 3.4. □

3.3 Combinatorial lemmas and determinants of holomorphic differentials

Applying Lemmas 2.2 and 2.3 to determinants of symmetric products of holomorphic 1-differentials on an algebraic curve C of genus g leads to combinatorial relations. By Eq.(3.2) and (3.3), such combinatorial relations yield non trivial identities among products of theta functions.

Proposition 3.7. *The following identities*

$$\det\eta\eta(x_1,x_2,x_3) = \det\eta(x_1,x_2)\det\eta(x_1,x_3)\det\eta(x_2,x_3) \ , \qquad g=2 \ , \tag{3.11}$$

$$\det\eta\eta(x_1,\ldots,x_6) = \frac{1}{15}\sum_{s\in\mathcal{P}'_6}\epsilon(s)\prod_{i=1}^{4}\det\eta(x_{d_1^i(s)},x_{d_2^i(s)},x_{d_3^i(s)}) \ , \qquad g=3 \ , \tag{3.12}$$

$$0 = \sum_{s\in\mathcal{P}_M}\epsilon(s)\prod_{i=1}^{g+1}\det\eta(x_{d^i(s)}) \ , \qquad g\geq 4 \ , \tag{3.13}$$

where $\{\eta_i\}_{i\in I_g}$ is an arbitrary basis of $H^0(K_C)$ and x_i, $i\in I_M$, are arbitrary points of C, hold. Furthermore, they are equivalent to

$$\det\eta\eta(x_1,x_2,x_3) = -\kappa[\eta]^3\frac{\prod_{i=1}^3\theta(\sum_{j=1}^3 x_j - 2x_i)\prod_1^3\sigma(x_j)}{\prod_{i<j}E(x_i,x_j)} \ , \tag{3.14}$$

for $g=2$

$$\det\eta\eta(x_1,\ldots,x_6) = \frac{\kappa[\eta]^4}{15}\prod_{i=1}^{6}\sigma(x_i)^2 \sum_{s\in\mathcal{P}'_6}\epsilon(s)\prod_{k=1}^{4}\frac{\theta(\sum_{i=1}^3 x_{d_i^k(s)} - y_{k,s})\prod_{i<j}^3 E(x_{d_i^k(s)},x_{d_j^k(s)})}{\prod_{i=1}^3 E(y_{k,s},x_{d_i^k(s)})\sigma(y_{k,s})} \ , \tag{3.15}$$

for $g = 3$

$$\sum_{s\in\mathcal{P}_M} \epsilon(s) \prod_{k=1}^{g+1} \frac{\theta\big(\sum_{i=1}^{g} x_{d_i^k(s)} - y_{k,s}\big) \prod_{i<j}^{g} E(x_{d_i^k(s)}, x_{d_j^k(s)})}{\prod_{i=1}^{g} E(y_{k,s}, x_{d_i^k(s)})\sigma(y_{k,s})} = 0 \ , \tag{3.16}$$

for $g \geq 4$*, where* $y_{k,s}$*,* $k \in I_{g+1}$*,* $s \in \mathcal{P}_M$*, are arbitrary points of* C*.*

Proof. Eqs.(3.11)-(3.13) follow by applying Lemma 2.2 to $\det \eta\eta(x_1, \ldots, x_M)$ and noting that it vanishes for $g \geq 4$. Eqs.(3.14)-(3.16) then follow by Eq.(3.2). □

In [15] D'Hoker and Phong made the interesting observation that for $g = 2$

$$\det \omega\omega(x_1, x_2, x_3) = \det \omega(x_1, x_2) \det \omega(x_1, x_3) \det \omega(x_2, x_3) \ , \tag{3.17}$$

that proved by first expressing the holomorphic differentials in the explicit form and then using the product form of the Vandermonde determinant. Eq.(3.17) corresponds to (3.11) when the generic basis η_1, η_2 of $H^0(K_C)$ is the canonical one. On the other hand, the way (3.11) has been derived shows that (3.17) is an algebraic identity since it does not need the explicit hyperelliptic expression of ω_1 and ω_2. Eq.(3.17) is the first case of the general formulas, derived in Lemmas 2.2 and 2.3, expressing the determinant of the matrix $ff_i(x_j)$ in terms of a sum of permutations of products of determinants of the matrix $f_i(x_j)$. In particular, by (3.12), for $g = 3$ we have

$$\det \omega\omega(x_1, \ldots, x_6) = \frac{1}{15} \sum_{s\in\mathcal{P}_6'} \epsilon(s) \prod_{i=1}^{4} \det \omega(x_{d^i(s)}) \ .$$

3.4 The Mumford isomorphism

Let $\mathcal{C}_g \xrightarrow{\pi} \mathcal{M}_g$ be the universal curve over $\mathcal{M}_g$ and $L_n = R\pi_*(K^n_{\mathcal{C}_g/\mathcal{M}_g})$ the vector bundle on $\mathcal{M}_g$ of rank $(2n-1)(g-1) + \delta_{n1}$ with fiber $H^0(K_C^n)$ at the point of $\mathcal{M}_g$ representing C. Let $\lambda_n := \det L_n$ be the determinant line bundle. According to Mumford [49]

$$\lambda_n \cong \lambda_1^{\otimes c_n} \ ,$$

where $c_n = 6n^2 - 6n + 1$, which corresponds to (minus) the central charge of the chiral $b - c$ system of conformal weight n [11]. The Mumford form

$$\mu_{g,n} = F_{g,n}[\phi^n] \frac{\phi_1^n \wedge \cdots \wedge \phi_{N_n}^n}{(\omega_1 \wedge \cdots \wedge \omega_g)^{c_n}} \ ,$$

where $\{\phi_i^n\}_{i\in I_{N_n}}$ is a basis of $H^0(K_C^n)$, $n \geq 2$. is the unique, up to a constant, holomorphic section of $\lambda_n \otimes \lambda_1^{-\otimes c_n}$ nowhere vanishing on $\mathcal{M}_g$.

Explicit expressions of the Mumford form were derived in [7, 5, 1, 61] and [24]. In particular, in the following proposition, a modification of the expression derived by Fay [24] is presented.

Proposition 3.8 (Fay [24]). *The Mumford form $\mu_{g,n}$ is given by*

$$\mu_{g,n} = \frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\phi^n]} \frac{\phi_1^n \wedge \cdots \wedge \phi_{N_n}^n}{(\omega_1 \wedge \cdots \wedge \omega_g)^{c_n}} \ . \tag{3.18}$$

Proof. We will only sketch the main lines of the proof; the details can be found in [24]. Let us consider the Teichmüller space $\mathcal{T}_g$ of genus g; each point of $\mathcal{T}_g$ corresponds to a Riemann surface C with marking. For each positive integer n, consider the bundle $\tilde{L}_n$ of rank N_n, whose fiber at the point representing C is $H^0(K_C^n)$. Since $\mathcal{T}_g$ is topologically trivial, the sheaf of sections of such a bundle is freely generated by N_n global holomorphic sections. In particular, a natural choice for $\tilde{L}_1$, at the point representing the marked Riemann surface C, is given by the canonical basis $\{\omega_i\}_{i\in I_g}$. The expression $\mu_{g,n}$ in Eq.(3.18) determines a non-vanishing holomorphic section of the line bundle $\tilde{\lambda}_n \otimes \tilde{\lambda}_1^{-\otimes c_n}$ on $\mathcal{T}_g$, where $\tilde{\lambda}_n := \wedge^{N_n}\tilde{L}_n$, $n > 0$. Now, the moduli space $\mathcal{M}_g$ is the quotient of $\mathcal{T}_g$ by the mapping class group, and it is clear that a section of $\tilde{\lambda}_n \otimes \tilde{\lambda}_1^{-\otimes c_n}$ on $\mathcal{T}_g$ corresponds to a section of $\lambda_n \otimes \lambda_1^{-\otimes c_n}$ on $\mathcal{M}_g$ if and only if it is invariant under a change of marking. Any dependence of the basis $\{\phi_i^n\}_{i\in I_{N_n}}$ on the marking cancels in the ratio $\kappa[\phi^n]/\phi_1^n \wedge \cdots \wedge \phi_{N_n}^n$. Consider the definition (3.2) and (3.3) of $\kappa[\omega]$ and $\kappa[\phi^n]$, respectively. Note that the number of functions σ in the numerator and denominator of $\frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\phi^n]}$ is the same, so that, by Eq.(B.9), they can be replaced by theta functions and prime forms. The transformations of theta functions, prime forms and determinants of the canonical basis $\{\omega_i\}_{i\in I_g}$ under the change of marking are well-known (see Eq.(B.10) and (B.11)), and direct computation shows that, under a modular transformation,

$$\frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\phi^n]}\phi_1^n \wedge \cdots \wedge \phi_{N_n}^n \to (\det(C\tau + D))^{c_n} \frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\phi^n]}\phi_1^n \wedge \cdots \wedge \phi_{N_n}^n \ ,$$

and the proposition follows. □

The normalization of (3.18) is chosen so that the Polyakov bosonic string measure on $\mathcal{M}_g$ is given by (see [24] and [15])

$$d\mu_{Pol} = \frac{|F_{g,2}[\phi]|^2}{(\det \operatorname{Im}\tau)^{13}}|\phi_1 \wedge \ldots \wedge \phi_N|^2 \ .$$

The Mumford form extends as a meromorphic section to the Deligne-Mumford compactification $\bar{\mathcal{M}}_g$ of the moduli space, with prescribed polar singularities at the boundary. In particular, such a form have poles of order $n(n-1)$ in the limit in which the genus g Riemann surface C degenerates a Riemann surface with a node, separating it in lower genera components C_1 and C_2. From the point of view of bosonic string theory, such poles correspond to the divergence due to tachyon states propagating between the Riemann surfaces C_1 and C_2. (For genus 2, the holomorphic section of $\lambda_n \otimes \lambda_1^{\otimes c_n}$ on $\mathcal{M}_2$ is unique only upon prescribing such a behavior on the boundary of $\bar{\mathcal{M}}_2$.)

In [6, 48] it has been shown that

$$F_{2,2}[\omega\omega] = \frac{c_{2,2}}{\Psi_{10}(\tau)} \ , \tag{3.19}$$

with $c_{2,2}$ a complex constant and Ψ_{10} the modular form of weight 10

$$\Psi_{10}(\tau) := \prod_{a,b\, even} \theta \left[{}^a_b \right](0)^2 \ ,$$

where the product is over the 10 even characteristics of $g = 2$. The derivation simply follows by noting that $F_{2,2}$ must be the inverse a modular form of weight 10, with the correct polar singularities at the boundary of $\mathcal{M}_2$. Since the genus 2 modular forms have been completely classified by Igusa [37], this is enough to identify $F_{2,2}$ up to the constant $c_{2,2}$. This can be fixed by requiring that the bosonic string measure correctly factorizes in the degeneration limits. In [15] it has been proved that the correct normalization for the bosonic string measure is given by $c_{2,2} = 1/\pi^{12}$.

For what concerns the higher genus cases, it has been conjectured that [6, 48]

$$F_{3,2}[\omega\omega] = \frac{c_{3,2}}{\Psi_9(\tau)} \ , \tag{3.20}$$

with $\Psi_9(\tau)^2 \equiv \Psi_{18}(\tau)$

$$\Psi_{18}(\tau) := \prod_{a,b\, even} \theta \left[{}^a_b \right](0) \ ,$$

where the product is over the 36 even characteristics of $g = 3$ and $c_{3,2} = 1/2^6\pi^{18}$ [16]. It is clear that such a derivation of the Mumford form can hardly be generalized to higher genus cases, since, due to the Schottky problem, $F_{g,n}$, for $g \geq 4$, is not well defined on the whole Siegel upper half-space $\mathfrak{H}_g$, but only on a $3g-3$-dimensional subspace.

Remarkably, Eq.(3.19) can be directly derived from Eq.(3.18), without references to Igusa classification of modular forms. It is natural to ask whether an analogous computation can be performed for genus 3. This is still an open problem; however, the constructions presented in the following chapters provide a higher genus generalization to most of the steps involved in the genus 2 computation.

The remainder of this section is devoted to the description of such a procedure (a similar derivation is presented in [15]). Let us consider Eq.(3.18) for $n = g = 2$

$$\mu_{2,2} = \frac{\kappa[\omega]^9}{\kappa[\omega\omega]} \frac{d\tau_{11} \wedge d\tau_{12} \wedge d\tau_{22}}{(\omega_1 \wedge \omega_2)^{13}} \ .$$

With respect to the canonical basis of $H^0(K_C)$, Eq.(3.11), which follows by Lemma 2.2, reads

$$\det_{i,j=1,2,3} \omega\omega_i(z_j) = \det \omega(z_1, z_2) \det \omega(z_2, z_3) \det \omega(z_1, z_3) \ ,$$

for all $z_1, z_2, z_3 \in C$, where

$$(\omega\omega_1, \omega\omega_2, \omega\omega_3) \equiv (\omega_1\omega_1, \omega_1\omega_2, \omega_2\omega_2) \ .$$

Then, by Eq.(3.3)

$$\kappa[\omega\omega] = \frac{\det \omega(z_1, z_2) \det \omega(z_2, z_3) \det \omega(z_1, z_3)}{\theta(z_1 + z_2 + z_3) \prod_{i<j}^3 E(z_i, z_j) \prod_{i=1}^3 \sigma^3(z_i)} \ , \tag{3.21}$$

for arbitrary $z_1, z_2, z_3 \in C$. Let us derive some useful identities which hold for genus 2. By Eq.(3.2)

$$\det \omega(p_1, p_2) = (-)^i \kappa[\omega]\sigma(p_i) \sum_{k=1}^{2} \theta_k(p_i)\omega_k(p_{3-i}) \ ,$$

$i = 1, 2$, holds for all $p_1, p_2 \in C$. By integrating both hands along the cycle α_j, $j = 1, 2$, with respect to p_{3-i}, we obtain

$$\omega_j(p_i) = (-)^j \kappa[\omega]\theta_{3-j}(p_i)\sigma(p_i) \ ,$$

and, by taking the determinant of both sides with respect to the indices i and j, we get

$$\det \omega(p_1, p_2) = \kappa[\omega]^2 \sigma(p_1)\sigma(p_2) \det_{ij} \theta_i(p_j) \ . \tag{3.22}$$

By comparing Eq.(3.22) and Eq.(3.2), it follows that

$$\kappa[\omega] = \frac{\theta(p_1 + p_2 - y)E(p_1, p_2)}{E(y, p_1)E(y, p_2)\sigma(y)D(p_1, p_2)} \ , \tag{3.23}$$

where

$$D(p_1, p_2) := \det \theta_i(p_j) \ .$$

By Eqs.(3.21)(3.22)

$$\kappa[\omega\omega] = \frac{\kappa[\omega]^6 D(z_1, z_2)D(z_2, z_3)D(z_1, z_3)}{\theta(z_1 + z_2 + z_3)\prod_{i<j}^{3} E(z_i, z_j)\prod_{i=1}^{3}\sigma(z_i)} \ ,$$

and by (3.23) it follows that

$$\frac{\kappa[\omega]^9}{\kappa[\omega\omega]} = -\frac{\theta(z_1 + z_2 + z_3)\theta(z_1 + z_2 - z_3)\theta(z_1 + z_3 - z_2)\theta(z_2 + z_3 - z_1)}{D(z_1, z_2)^2 D(z_1, z_3)^2 D(z_2, z_3)^2} \ . \tag{3.24}$$

This expression holds for z_1, z_2, z_3 arbitrary points in C. Let us recall that any Riemann surface C of genus 2 is necessarily hyperelliptic, i.e. it can be defined by the equation

$$w^2 = \prod_{i=1}^{6}(z - e_i) \ ,$$

$(z, w) \in \mathbb{C}^2$, with $e_1, \ldots, e_6 \in \mathbb{C}$ distinct complex numbers. Three of such parameters can be fixed (a conventional choice is $e_1 = 0, e_2 = 1, e_3 = \infty$) by a fractional linear transformation on $\hat{\mathbb{C}}$ and the other three correspond to the three complex moduli of the curve. Denote by $p_1, \ldots, p_6 \in C$ the branch points $p_i := (e_i, 0)$, $i = 1, \ldots, 6$. For each $i \in I_6$,

$$(z - e_i)\frac{dz}{w} \ ,$$

is a holomorphic Abelian differential with a double zero at the branch point p_i. Therefore, the divisor $2p_i$ is canonical and $\nu_i \equiv \left[\begin{smallmatrix}\nu_i' \\ \nu_i''\end{smallmatrix}\right] := I(p_i)$ is a (necessarily odd) spin structure. The corresponding holomorphic 1/2-differential ϕ_{ν_i} is given by

$$\phi^2_{\nu_i} = \sum_{j=1}^{2} \theta_j[\nu_i](0)\omega_j = e^{\pi i \nu_i' \tau \nu_i' + 2\pi i \nu_i' \nu_i''} \sum_{j=1}^{2} \theta_j(p_i)\omega_j = N_{\nu_i}(z - e_i)\frac{dz}{w} ,$$

for all $i \in I_6$, where N_{ν_i} is a normalization constant. By evaluating the formula (3.24) at three branch points $z_i \equiv p_i$, we obtain (note the exponential factors in front of the thetas simplify)

$$\frac{\kappa[\omega]^9}{\kappa[\omega\omega]} = \frac{\theta[\nu_1 + \nu_2 + \nu_3]^4}{[\nu_1, \nu_2]^2[\nu_1, \nu_3]^2[\nu_2, \nu_3]^2} , \tag{3.25}$$

where

$$[\nu_1, \nu_2] := \det_{i,j=1,2} \theta_i[\nu_j] .$$

The last tool needed to explicitly compute the Mumford form for genus 2, is a $g = 2$ generalization of the Jacobi's derivative formula which holds for $g = 1$ (see Appendix B.2). This is given by the Rosenhain's formula [54, 32, 33]

$$[\nu_i, \nu_j] = \pm\pi^2 \prod_{\substack{k=1 \\ k\neq i,j}}^{6} \theta[\nu_i + \nu_j + \nu_k] ,$$

where ν_i, ν_j are arbitrary odd spin structures, $i \neq j$. Similar extensions have been proved up to genus 5 [26, 23] and a modified version is conjectured to hold to all genera [40, 41].

By Rosenhain's formula, Eq.(3.25) gives

$$\mu_{2,2} = \frac{1}{\pi^{12}\Psi_{10}} \frac{d\tau_{11} \wedge d\tau_{12} \wedge d\tau_{22}}{(\omega_1 \wedge \omega_2)^{13}} , \tag{3.26}$$

as expected.

For $g = 3$, no such a derivation is known. However, the formula (3.20) can be used to derive a non-trivial expression for the constant $\kappa[\omega]$, considered in the following Proposition. Higher genus generalizations of such an identity are considered in section 6.5.

Proposition 3.9. *For $g = 3$*

$$\kappa[\omega]^5 = \frac{2^{-6}\pi^{-18}}{15\Psi_9(\tau)} \frac{\sum_{s\in\mathcal{P}_6'} \epsilon(s) \prod_{k=1}^{4}[\theta\big(\sum_{i=1}^{3} p_{d_i^k(s)} - y\big) \prod_{i<j}^{3} E(p_{d_i^k(s)}, p_{d_j^k(s)})]}{\theta\big(\sum_1^6 p_i\big) \prod_{i=1}^{6} \sigma(p_i)\sigma(y)^4 \prod_{i=1}^{6} E(y, p_i)^2 \prod_{i<j}^{6} E(p_i, p_j)} . \tag{3.27}$$

Proof. By (3.15)

$$\frac{\det \omega\omega(p_1, \ldots, p_6)}{\kappa[\omega]^4} = \frac{\prod_1^6 \sigma(p_i)^2]}{15\sigma(y)^4 \prod_{i=1}^{6} E(y, p_i)^2} \cdot \sum_{s\in\mathcal{P}_6'} \epsilon(s) \prod_{k=1}^{4}[\theta\big(\sum_{i=1}^{3} p_{d_i^k(s)} - y\big) \prod_{i<j}^{3} E(p_{d_i^k(s)}, p_{d_j^k(s)}) ,$$

and (3.27) follows by the identity $\prod_{i=1}^{6} c(p_i)^{-\frac{3}{2}} = \kappa[\omega]^9 \prod_{i=1}^{6} \sigma(p_i)^3$. □

4. DISTINGUISHED BASES OF HOLOMORPHIC N-DIFFERENTIALS

One of the main tools in genus 2 calculations, both in bosonic string theories and, more generally, in 2-dimensional Conformal Field Theories, is the hyperelliptic representation, i.e. the representation of any Riemann surface C of genus 2 by an algebraic curve defined by the equation

$$w^2 = \prod_i (z - e_i) \ ,$$

$z, w \in \mathbb{C}$, where e_i are distinct points in $\mathbb{C}$ and the product is over 5 or 6 factors. For example, in the explicit calculation of the Mumford form, described in section 3.4, several steps rely on such a representation.

One of the advantages in using the hyperelliptic representation is the possibility explicitly define bases for abelian differentials in terms of of z, w. In turn, this allows to derive explicit expressions for bases of holomorphic n-differentials, for all the integer $n > 1$, in terms of n-fold products of such Abelian differentials. In [15], formula (3.14) was proved with respect to such bases of construction turned out to be a crucial point in the derivation of the Mumford form. Such a problem is greatly simplified by the combinatorial relation between determinant of Abelian differential and determinant of holomorphic n-differentials.

It is natural to ask whether a higher genus realization of such a construction exists. The Max Noether's theorem assures that, for $g > 2$, the natural map

$$\psi : \mathrm{Sym}^n(H^0(K_C)) \to H^0(K_C^n) \ ,$$

is surjective if and only if C is not hyperelliptic. Hence, in the following, we will only consider the case of non-hyperelliptic Riemann surfaces C.

A general procedure to define of a basis of holomorphic n-differentials in terms of a distinguished basis of abelian differentials is provided by the Petri's construction [52]. This has been used to study the ideal of the smooth irreducible algebraic curve, given by the canonical embedding of the Riemann surface C in $\mathbb{P}^{g-1}$. The main result of this approach is Petri's theorem, we recall in section 5.3. The starting point of such a construction is the choice of g distinct points $p_1, \ldots, p_g$ on C in general position (the precise condition is given below); then, one defines a a basis $\{\sigma_1, \ldots, \sigma_g\}$ of abelian differentials, such that $\sigma_i(p_j) = 0$ for all $i \neq j$, $1 \leq i, j \leq g$. Such conditions determine the basis $\{\sigma_1, \ldots, \sigma_g\}$ up to a non-singular diagonal transformation. Such an ambiguity in the normalization was not relevant for the aims of the original construction, but it is a crucial point for the following derivations. In facts, since we are going to look at $\sigma_1, \ldots, \sigma_g$ as sections on a line bundle on the moduli space $\mathcal{M}_g$ (more precisely, they should be considered as sections on the space $\mathcal{M}_{g,g}$ of Riemann surfaces with g

distinguished points), one needs to specify the dependence of the normalization of each σ_i on the moduli.

In this chapter, we provide a suitable refinement of Petri's basis, which addresses such a issue.

4.1 Duality between N_n-tuples of points and bases of $H^0(K_C^n)$

Let C be a canonical curve of genus g and let C_d, $d > 0$, be the set of effective divisors of degree d. Consider the pair (p, λ) given by a point $p \in C$ and an element $\lambda \in \pi^{-1}(p)$ in the fibre of the canonical bundle $\pi : K_C \to C$ at p. Such a pair corresponds to an element of $H^0(K_C)^*$, given by

$$p_\lambda[\eta] := \frac{\varphi(\eta(p))}{\varphi(\lambda)} \ ,$$

for all $\eta \in H^0(K_C)$, where $\varphi : K_U \to U \times \mathbb{C}$ is an arbitrary trivialization of the canonical bundle on a neighborhood U of p. Note that the definition is independent of φ. Similarly, (p, λ) determines an element of $H^0(K_C^n)^*$. Let $\{\phi_i\}_{i\in I_{N_n}}$ be a basis of $H^0(K_C^n)$ and fix $(p_1, \lambda_1), \ldots, (p_{N_n}, \lambda_{N_n})$. A necessary and sufficient condition for $\{p_{1\lambda_1}, \ldots, p_{N_n\lambda_{N_n}}\}$ to be a basis of $H^0(K_C^n)^*$ is that $\det_{i,j\in I_{N_n}} p_i[\phi_j] \neq 0$ (here and in the following, we drop the notation of λ_i when the meaning is clear). Note that such a condition only depends on the points $p_1, \ldots, p_{N_n}$ and is independent of the choice of $\lambda_1, \ldots, \lambda_{N_n}$ and of the basis $\{\phi_i\}_{i\in I_{N_n}}$.

In the following, the notation

$$\phi(p) \equiv p[\phi] := p_\lambda[\phi] \ ,$$

for an arbitrary $\phi \in H^0(K_C^n)$, and

$$\det \phi(p_1, \ldots, p_{N_n}) := \det_{i,j\in I_{N_n}} \phi_i(p_j) \ ,$$

is used, where the choice of λ is understood.

Proposition 4.1. *Fix $n \in \mathbb{N}_+$ and let $p_1, \ldots, p_{N_n}$ be a set of points of C such that*

$$\det \phi^n(p_1, \ldots, p_{N_n}) \neq 0 \ ,$$

with $\{\phi_i^n\}_{i\in I_{N_n}}$ an arbitrary basis of $H^0(K_C^n)$. Choose a class $[\alpha_i]$ of local trivializations around each p_i, $i \in N_n$. Then, $\{\gamma_i^n\}_{i\in I_{N_n}}$, with

$$\gamma_i^n(z) := \frac{\det \phi^n(p_1, \ldots, p_{i-1}, z, p_{i+1}, \ldots, p_{N_n})}{\det \phi^n(p_1, \ldots, p_{N_n})} \ , \tag{4.1}$$

is a basis of $H^0(K_C^n)$ which is independent of the choice of the basis $\{\phi_i^n\}_{i\in I_{N_n}}$ and on the classes of local trivializations up to a non-singular diagonal transformation.

Proof. Since the matrix $\phi_i^n(p_j)$ is non-singular, by

$$\phi_i^n = \sum_{j=1}^{N_n} \phi_i^n(p_j)\gamma_j^n \ , \tag{4.2}$$

$i \in I_{N_n}$, it follows that $\gamma_i^n, \ldots, \gamma_{N_n}^n$ are linearly independent. ☐

Note that the basis $\{\gamma_i^n\}_{i\in I_{N_n}}$ of $H^0(K_C^n)$ and the basis $\{p_{1,\lambda_1},\ldots,p_{N_n,\lambda_{N_n}}\}$ of $H^0(K_C^n)^*$ depend on the choice of $\lambda_1,\ldots,\lambda_{N_n}$ in such a way that the relation

$$\gamma_i^n(p_j) \equiv p_{j\lambda_j}(\gamma_i^n) = \delta_{ij} \ , \tag{4.3}$$

$i,j = 1,\ldots,N_n$, hold for all the choices of λ_i, $i \in I_{N_n}$. In the following, we will refer to $\{\gamma_i^n\}_{i\in I_{N_n}}$ and $\{p_1,\ldots,p_{N_n}\}$ as dual bases, while keeping the choice of $\lambda_1,\ldots,\lambda_{N_n}$ understood.

More generally, the choice of $p_1,\ldots,p_{N_n}$ (and corresponding $\lambda_1,\ldots,\lambda_{N_n}$) also determines a basis of $\mathrm{Sym}^k H^0(K_C^n)$ and of its dual space, for all $k > 0$. In the case of $\mathrm{Sym}^2(H^0(K_C))$, we will denote by $p\cdot q \in \mathbb{P}(\mathrm{Sym}^2(H^0(K_C))^*)$ the element corresponding to the symmetrized pair $((p,\lambda_p),(q,\lambda_q))$, defined by

$$(p\cdot q)\big[\sum_k \eta_k\cdot\rho_k\big] := \sum_k (\eta_k(p)\rho_k(q) + \eta_k(q)\rho_k(p)) \ , \tag{4.4}$$

where $\sum_k \eta_k\cdot\rho_k \in \mathrm{Sym}^2 H^0(K_C)$.

For $n = 1$, for each choice of $p_1,\ldots,p_g \in C$ with $\det \eta_i(p_j) \neq 0$, we set

$$\sigma_i(z) := \gamma_i^1(z) \ , \qquad i \in I_g \ , \tag{4.5}$$

so that

$$p_i[\sigma_j] = \delta_{ij} \ , \qquad (p\cdot p)_k[\sigma\cdot\sigma_l] = \chi_k\delta_{kl} \ , \tag{4.6}$$

$i,j \in I_g$, $k,l \in I_M$, where $(p\cdot p)_k := (\prod p_{1_k}p_{1_k})$ in the notation of section 2.1.

For any pair of bases $\{\phi_i\}_{i\in I_{N_n}}$ and $\{\psi_i\}_{i\in I_{N_n}}$ of $H^0(K_C^n)$, we denote by

$$\big[{\phi \atop \psi}\big] \equiv \big[{\psi \atop \phi}\big]^{-1} \ ,$$

the matrix of basis change

$$\phi_i = \sum_{j\in I_{N_n}} \big[{\phi \atop \psi}\big]_{ij}\psi_j \ ,$$

for all $i \in I_{N_n}$. Then, the proof of proposition 4.1 shows that, for all the bases $\{\phi_i\}_{i\in I_{N_n}}$,

$$\big[{\phi^n \atop \gamma^n}\big]_{ij} = \phi_i^n(p_j) \ .$$

The results of chapter 3 can be used to derive an explicit expression for the matrix $\big[{\sigma \atop \omega}\big]_{ij}$, with $\{\omega_i\}_{i\in I_g}$ the dual basis of the symplectic basis of $H_1(C,\mathbb{Z})$.

Definition 4.1. For each fixed g-tuple $(p_1,\ldots,p_g) \in C^g$ let us define the following effective divisors

$$\mathrm{a} := \sum_{j\in I_g} p_j \ , \qquad \mathrm{a}_i := \mathrm{a} - p_i \ , \qquad b := \mathrm{a} - p_1 - p_2 \ ,$$

$i \in I_g$. Define the subset of C^g

$$\mathcal{A} := \{(p_1,\ldots,p_g) \in C^g \mid \det \eta_i(p_j) = 0\} \ ,$$

with $\{\eta_i\}_{i\in I_g}$ an arbitrary basis of $H^0(K_C)$.

Fix $g+1$ arbitrary points $p_1,\ldots,p_g,z\in C$. By taking the limit $y\to z$ in Eq.(3.2), we obtain

$$\det\eta(z,p_1,\ldots,\check{p}_i,\ldots,p_g)=\kappa[\eta]\sum_{l=1}^{g}\theta_l(\mathrm{a}_i)\omega_l(z)\prod_{\substack{j,k\neq i\\ j<k}}E(p_j,p_k)\prod_{j\neq i}\sigma(p_j)\ , \quad (4.7)$$

for all $i\in I_g$. Note that, by (4.7), the condition $(p_1,\ldots,p_g)\in C^g\setminus\mathcal{A}$ implies

$$\sum_j\theta_j(\mathrm{a}_i)\omega_j(p_i)\neq 0\ , \quad (4.8)$$

for all $i\in I_g$.

Proposition 4.2. *Fix $(p_1,\ldots,p_g)\in C^g\setminus\mathcal{A}$, with $\mathcal{A}$ defined in 4.1. Then*

$$\left[{\omega\atop\sigma}\right]_{ij}=\omega_i(p_j)\ ,\qquad \left[{\sigma\atop\omega}\right]_{ij}=\oint_{\alpha_j}\sigma_i=\frac{\theta_j\left(\mathrm{a}_i\right)}{\sum_k\theta_k\left(\mathrm{a}_i\right)\omega_k(p_i)}\ , \quad (4.9)$$

$i,j\in I_g$, so that

$$\sigma_i(z)=\frac{\sigma(z)}{\sigma(p_i)}\theta(\mathrm{a}+z-y-p_i)\frac{\theta(\mathrm{a}-y)E(z,p_i)}{E(y,p_i)}E(y,z)\prod_{i=1}^{g}\frac{E(z,p_i)}{\prod_{j\neq i}E(p_i,p_j)}\ , \quad (4.10)$$

and

$$\kappa[\sigma]=\frac{\sigma(y)\prod_1^gE(y,p_i)}{\theta(\mathrm{a}-y)\prod_{i<j}^gE(p_i,p_j)\prod_1^g\sigma(p_k)}\ , \quad (4.11)$$

for all $z,y,x_i,y_i\in C$, $i\in I_g$, with a,a_i as in Definition 4.1. Furthermore, fix $p_1,\ldots,p_{N_n}\in C$ such that $\det\phi^n(p_1,\ldots,p_{N_n})\neq 0$, *with $\{\phi_i^n\}_{i\in I_{N_n}}$ an arbitrary basis of $H^0(K_C^n)$. Then,*

$$\gamma_i^n(z)=\sigma(z,p_i)^{2n-1}\frac{\theta\left(\sum_1^{N_n}p_j+z-p_i\right)\prod_{\substack{j=1\\ j\neq i}}^{N_n}E(z,p_j)}{\theta\left(\sum_1^{N_n}p_j\right)\prod_{\substack{j=1\\ j\neq i}}^{N_n}E(p_i,p_j)}\ , \quad (4.12)$$

$i\in I_{N_n}$, and

$$\kappa[\gamma^n]=\frac{1}{\theta\left(\sum_1^{N_n}p_i\right)\prod_1^{N_n}\sigma(p_i)^{2n-1}\prod_{\substack{i,j=1\\ i<j}}^{N_n}E(p_i,p_j)}\ . \quad (4.13)$$

Proof. The first identity of (4.9) follows by (4.2) and (4.5) and the second one by (3.2) and (4.7). Eqs.(4.10)(4.11) follow by (3.2) and by $\det\sigma_i(p_j)=1$, respectively. Similarly, (4.12) follows by (4.1) and (3.3). Eq.(4.13) follows by $\det\gamma_i^n(p_j)=1$. □

Corollary 4.3. *Fix $(p_1,\ldots,p_g)\in C^g\setminus\mathcal{A}$. Then*

$$\sum_{i\in I_g}\frac{\theta_j\left(\mathrm{a}_i\right)}{\sum_l\theta_l\left(\mathrm{a}_i\right)\omega_l(p_i)}\omega_k(p_i)=\delta_{jk}\ ,$$

$j,k\in I_g$.

Proof. Apply (4.9) to the identity $\sum_{j\in I_g}\left[{\omega\atop\sigma}\right]_{ij}\left[{\sigma\atop\omega}\right]_{jk}=\delta_{jk}$. □

4.2 Special loci in C^g and linear independence for holomorphic differentials

There exist natural homomorphisms from $\mathrm{Sym}^n(H^0(K_C))$ to $H^0(K_C^n)$, which, for $n = 2$, we denote by

$$\begin{aligned} \psi\colon\ \mathrm{Sym}^2(H^0(K_C^2)) &\to H^0(K_C^2) \\ \eta \cdot \rho \quad &\mapsto \eta\rho \ . \end{aligned}$$

By Max Noether's Theorem, if C is a Riemann surface of genus two or non-hyperelliptic with $g \geq 3$, then ψ is surjective. Set

$$v_i := \psi(\sigma \cdot \sigma)_i = \sigma_{1_i}\sigma_{2_i} \ , \tag{4.14}$$

$i \in I_M$, so that

$$v_i(p_j) = \begin{cases} \delta_{ij} \ , & i \in I_g \ , \\ 0 \ , & g+1 \leq i \leq M \ , \end{cases} \tag{4.15}$$

$j \in I_g$. By dimensional reasons, it follows that for $g = 2$ and $g = 3$ in the non-hyperelliptic case, the set $\{v_i\}_{i\in I_N}$ is a basis of $H^0(K_C^2)$ if and only if $\{\sigma_i\}_{i\in I_g}$ is a basis of $H^0(K_C)$. On the other hand, for $g \geq 3$ in the hyperelliptic case, there exist holomorphic quadratic differentials which cannot be expressed as linear combinations of products of elements of $H^0(K_C)$, so that $v_1, \ldots, v_N$ are not linearly independent. The other possibilities are considered in the following proposition.

Proposition 4.4. *Fix the points $p_1, \ldots, p_g \in C$, with C non-hyperelliptic of genus $g \geq 4$. If the following conditions are satisfied*

i. $\det \eta_i(p_j) \neq 0$, *with $\{\eta_i\}_{i\in I_g}$ an arbitrary basis of $H^0(K_C)$;*

ii. $b := \sum_{i=3}^g p_i$ *is the greatest common divisor of (σ_1) and (σ_2), with $\{\sigma_i\}_{i\in I_g}$ defined in* (4.5),

then $\{v_i\}_{i\in I_N}$ is a basis of $H^0(K_C^2)$. Conversely, if there exists a set $\{\hat\sigma_i\}_{i\in I_g}$ of holomorphic 1-differentials, such that

a. $i \neq j \Rightarrow \hat\sigma_i(p_j) = 0$, *for all $i, j \in I_g$;*

b. $\{\hat v_i\}_{i\in I_N}$ *is a basis of $H^0(K_C^2)$, with $\hat v_i := \hat\sigma\hat\sigma_i$, $i \in I_N$;*

then i) and ii) hold.

Proof. To prove that *i*) and *ii*) imply that $\{v_i\}_{i\in I_N}$ is a basis of $H^0(K_C^2)$, we first prove that σ_i is the unique 1-differential, up to normalization, vanishing at $c_i := (\sigma_i) - b$, $i = 1, 2$. Any 1-differential $\sigma_i' \in H^0(K_C)$ vanishing at c_i corresponds to an element σ_i'/σ_i of $H^0(\mathcal{O}(b))$, the space of meromorphic functions f on C such that $(f) + b$ is an effective divisor. Suppose that there exists a σ_i' such that σ_i'/σ_i is not a constant, so that $h^0(\mathcal{O}(b)) \geq 2$. By the Riemann-Roch Theorem

$$h^0(K_C \otimes \mathcal{O}(-b)) = h^0(\mathcal{O}(b)) - \deg b - 1 + g \geq 3 \ ,$$

there exist at least 3 linearly independent 1-differentials vanishing at the support of b and, in particular, there exists a linear combination of such differentials vanishing at $p_1,\ldots,p_g$. This implies that $\det\eta_i(p_j)=0$, with $\{\eta_i\}_{i\in I_g}$ an arbitrary basis of $H^0(K_C)$, contradicting the hypotheses. Fix $\zeta_i,\zeta_{1i},\zeta_{2i}\in\mathbb{C}$ in such a way that

$$\sum_{i=3}^g\zeta_i\sigma_i^2+\sum_{i=1}^g\zeta_{2i}\sigma_1\sigma_i+\sum_{i=2}^g\zeta_{1i}\sigma_2\sigma_i=0\ .$$

Evaluating this relation at the point p_j, $3\le j\le g$ yields $\zeta_j=0$. Set

$$t_1:=-\sum_{j=2}^g\zeta_{1j}\sigma_j\ ,\quad t_2:=\sum_{j=1}^g\zeta_{2j}\sigma_j\ , \tag{4.16}$$

so that $\sigma_1t_2=\sigma_2t_1$. Since the supports of c_1 and c_2 are disjoint, t_i must be an element of $H^0(K_C\otimes\mathcal{O}(-c_i))$, $i=1,2$ and then, by the previous remarks, $t_1/\sigma_1=t_2/\sigma_2=\zeta\in\mathbb{C}$. By (4.16)

$$\zeta\sigma_1+\sum_{j=2}^g\zeta_{1j}\sigma_j=0\ ,\quad \zeta\sigma_2-\sum_{k=1}^g\zeta_{2k}\sigma_k=0\ ,$$

and, by linear independence of $\sigma_1,\ldots,\sigma_g$, it follows that $\zeta=\zeta_{1j}=\zeta_{2k}=0$, $2\le j\le g$, $k\in I_g$.

Let us now assume that *a*) and *b*) hold for some set $\{\hat\sigma_i\}_{i\in I_g}$. Then $\{\hat\sigma_i\}_{i\in I_g}$ is a basis of $H^0(K_C)$ if and only if $\det\eta_i(p_j)\neq0$. If $\{\hat\sigma_i\}_{i\in I_g}$ is not a basis of $H^0(K_C)$, the corresponding $\hat v_i$, $i\in I_N$, cannot span a N-dimensional vector space. Then *i*) is satisfied and the basis $\{\hat\sigma_i\}_{i\in I_g}$ corresponds, up to a non-singular diagonal transformation, to the basis $\{\sigma_i\}_{i\in I_g}$, defined in (4.5).

Without loss of generality, to prove *ii*) we can assume that $\hat\sigma_i\equiv\sigma_i$, $i\in I_g$ and then $\hat v_i\equiv v_i$, $i\in I_N$. Suppose there exists $p\in C$ such that $p+b\le(\sigma_i)$, for all $i\in I_2$. If $p\equiv p_1$ or $p\equiv p_2$, then $\sigma_i(p)=0$, for all $i\in I_g$, and therefore $\{\sigma_i\}_{i\in I_g}$ would not be a basis, which contradicts *b*).

Suppose there exists i, $3\le i\le g$, with $p\equiv p_i$. In this case, each v_j, $j\in I_N\setminus\{i\}$, has a double zero in p_i, whereas $v_i(p_i)\neq0$; therefore, an element of $H^0(K_C^2)$ with a single zero in p_i (such as, for example, $\sigma_i\sigma_j$, with $3\le j\le g$, $j\neq i$) cannot be expressed as a linear combination of $v_1,\ldots,v_N$, in contradiction with the assumptions.

Finally, suppose that $p\neq p_i$, for all $i\in I_g$. In this case, there exists at least one σ_i, $3\le i\le g$, with $\sigma_i(p)\neq0$, since, on the contrary, $\{\sigma_i\}_{i\in I_g}$ would not be a basis of $H^0(K_C)$. Suppose that $\sigma_i(p)\neq0$ and $\sigma_j(p)\neq0$ for some $3\le i,j\le g$, $i\neq j$. Then $\sigma_i\sigma_j$ cannot be expressed as a linear combination of v_k, $k\in I_N$. In fact, $\sigma_i\sigma_j(p_k)=0$, for all $k\in I_g$, would imply that $\sigma_i\sigma_j=\sigma_1\rho_1+\sigma_2\rho_2$, for some $\rho_1,\rho_2\in H^0(K_C)$; but this is impossible, since $\sigma_1(p)=0=\sigma_2(p)$, whereas $\sigma_i\sigma_j(p)\neq0$. Therefore, there should exist exactly one $i\in I_g$ with $\sigma_i(p)\neq0$. It follows that $\sigma_j(p)=0=\sigma_j(p_i)$, for all $j\in I_g\setminus\{i\}$; then $h^0(K_C\otimes\mathcal{O}(-p-p_j))\ge g-1$ and, by Riemann-Roch Theorem, there exists a non-constant meromorphic function on C, with only single poles in p and p_j. But this would imply that C is hyperelliptic, in contradiction with the hypotheses. □

The proof that *i*) and *ii*) imply that $\{v_i\}_{i\in I_N}$ is a basis is due to Petri [52] (see also [3]). It can be proved that on a non-hyperelliptic curve there always exists a set of points $\{p_1, \ldots, p_g\}$ satisfying the hypotheses of Proposition 4.4. This is related to the classical result $\dim \Theta_s = g-4$ for non-hyperelliptic surfaces of genus $g \geq 4$, as will be shown in Corollary 4.11.

In view of Theorem 4.4, it is useful to introduce the following subset of $C^g \equiv \underbrace{C \times \ldots \times C}_{g \text{ times}}$.

Definition 4.2. Let $\mathcal{B}$ be the subset of C^g

$$\mathcal{B} := \{(p_1, \ldots, p_g) \in C^g \mid \det \eta_i(p_j) = 0 \vee \gcd((\sigma_1), (\sigma_2)) \neq b\} \ ,$$

for an arbitrary basis $\{\eta_i\}_{i \in I_g}$ of $H^0(K_C)$.

Corollary 4.5. *Fix $(p_1, \ldots, p_g) \in C^g \setminus \mathcal{A}$ such that the greatest common divisor of (σ_1) and (σ_2) be $b + q_1 + \ldots + q_n$, for some $q_1, \ldots, q_n \in C$, $n \geq 1$. Then the dimension r of the vector space generated by $\{v_i\}_{i \in I_N}$ is $r = N - n$.*

Proof. Let us prove that n is the number $(N-r)$ of independent linear relations among $v_1, \ldots, v_N$. Set $d := q_1 + \ldots + q_n$. By $\det \eta_i(p_j) \neq 0$, the quadratic differentials σ_i^2, $i \in I_g$, are linearly independent and independent of $\sigma_1\sigma_2, \sigma_1\sigma_i, \sigma_2\sigma_i$, $i \in I_g \setminus \{1, 2\}$. Therefore, all the independent linear relations have the form

$$\sigma_1 t_2 = \sigma_2 t_1 \ , \tag{4.17}$$

for some $t_1, t_2 \in H^0(K_C)$, with the condition $t_1(p_1) = 0$ in order to exclude the trivial relation $t_i = \sigma_i$, $i = 1, 2$. Consider the effective divisors $\hat{c}_1, \hat{c}_2$ of degree $g - n$ with no common points, defined by $\hat{c}_i := (\sigma_i) - d - b$, $i = 1, 2$. By $\det \eta_i(p_j) \neq 0$, it follows that $h^0(K_C \otimes \mathcal{O}(-b)) = 2$, so that $h^0(K_C \otimes \mathcal{O}(-b - d)) = 2$ too. This implies that σ_1/σ_2 and σ_2/σ_1 are the unique elements of $H^0(\mathcal{O}(\hat{c}_1))$ and $H^0(\mathcal{O}(\hat{c}_2))$, respectively. Then, by Riemann-Roch Theorem, we have $h^0(K_C \otimes \mathcal{O}(-\hat{c}_i)) = n + 1$, $i = 1, 2$. By Eq.(4.17), the divisors of t_1, t_2 satisfy

$$\hat{c}_1 + (t_2) = \hat{c}_2 + (t_1) \ ,$$

so that $t_i \in H^0(K_C \otimes \mathcal{O}(-\hat{c}_i))$. In particular, a basis $\sigma_1, \alpha_1, \ldots, \alpha_n$ of $H^0(K_C \otimes \mathcal{O}(-\hat{c}_1))$ can be chosen in such a way that $\alpha_i(p_1) = 0$, for all $i \in I_n$. Hence, t_1 is a linear combination of $\alpha_1, \ldots, \alpha_n$ and there are at most n linearly independent relations of the form (4.17). This implies $N - r \leq n$.

Let us now prove that such n linearly independent relations exist. By the Riemann-Roch Theorem, since $h^0(K_C \otimes \mathcal{O}(-b - d)) = 2$, we obtain $h^0(\mathcal{O}(b + d)) = n + 1$; a basis for $H^0(\mathcal{O}(b + d))$ is given by $\alpha_1/\sigma_1, \ldots, \alpha_n/\sigma_1$ and the constant function. On the other hand, if $\sigma_2, \beta_1, \ldots, \beta_n$ is a basis for $H^0(K_C \otimes \mathcal{O}(-\hat{c}_2))$, then $\beta_1/\sigma_2, \ldots, \beta_n/\sigma_2$ are n linearly independent elements of $H^0(\mathcal{O}(b + d))$. Hence, there exist n linearly independent relations

$$\frac{\beta_i}{\sigma_2} = \sum_{j=1}^n c_{ij} \frac{\alpha_j}{\sigma_1} + c_{i0} \ ,$$

$i \in I_n$, for some $c_{ij} \in \mathbb{C}$, $0 \leq j \leq n$. By multiplying both sides by $\sigma_1\sigma_2$, we obtain

$$\sigma_1 \beta_i = \sum_{j=1}^n c_{ij} \sigma_2 \alpha_j + c_{i0} \sigma_1 \sigma_2 \ .$$

Therefore, $N - r \geq n$ and the corollary follows. $\square$

Consider the holomorphic 3-differentials (with the notation defined in section 2.1)

$$\varphi_i = \sigma\sigma\sigma_i := \sigma_{1_i}\sigma_{2_i}\sigma_{3_i} \ , \tag{4.18}$$

$i \in I_{M_3}$, with $\{\sigma_i\}_{i\in I_g}$ a basis of $H^0(K_C)$. By the Max Noether's Theorem and dimensional reasons, it follows that the first $N_3 := 5g-5$ of such differentials are a basis of $H^0(K_C^3)$ for $g=3$ in the non-hyperelliptic case, whereas they are not linearly independent for $g \geq 2$ in the hyperelliptic case. The other possibilities are considered in the following proposition.

Proposition 4.6. *Fix the points $p_1, \ldots, p_g \in C$, with C non-hyperelliptic of genus $g \geq 4$. If the following conditions are satisfied for a fixed $i \in I_g \setminus \{1,2\}$:*

i. $\det \eta_j(p_k) \neq 0$, *with $\{\eta_j\}_{j\in I_g}$ an arbitrary basis of $H^0(K_C)$;*

ii. $b := \sum_{j=3}^{g} p_j$ *is the greatest common divisor* (gcd) *of (σ_1) and (σ_2), with $\{\sigma_j\}_{j\in I_g}$ defined in (4.5);*

iii. p_k *is a single zero for σ_1, for all $k \neq i$, $3 \leq k \leq g$;*

then the set $\{\varphi_j\}_{j\in I_{N_3-1}} \cup \{\varphi_{i+5g-8}\}$ is a basis of $H^0(K_C^3)$. In particular, if i), ii) and

iii'. $p_3, \ldots, p_g$ are single zeros for σ_1,

are satisfied, then, for each i, $3 \leq i \leq g$, the set $\{\varphi_j\}_{j\in I_{N_3-1}} \cup \{\varphi_{i+5g-8}\}$ is a basis of $H^0(K_C^3)$. Conversely, if for some fixed $i \in I_g \setminus \{1,2\}$ there exists a set $\{\hat\sigma_j\}_{j\in I_g}$ of holomorphic 1*-differentials, such that*

a. $j \neq k \Rightarrow \hat\sigma_j(p_k) = 0$, for all $j,k \in I_g$;

b. $\{\hat\varphi_j\}_{j\in I_{N_3-1}} \cup \{\hat\varphi_{i+5g-8}\}$ is a basis of $H^0(K_C^3)$, with $\hat\varphi_j := \hat\sigma\hat\sigma\hat\sigma_j$, $j \in I_{M_3}$;

then i), ii) and iii) hold.

Proof. We first prove that if *i*), *ii*) and *iii*) hold for a fixed i, $3 \leq i \leq g$, then $\{\varphi_j\}_{j\in I_{N_3-1}} \cup \{\varphi_{i+5g-8}\}$ is a basis of $H^0(K_C^3)$. To this end it is sufficient to prove that the equation

$$\sum_{j=3}^{g} (\zeta_j \sigma_j^3 + \zeta_{1j}\sigma_1\sigma_j^2 + \zeta_{12j}\sigma_1\sigma_2\sigma_j) + \sigma_1^2\mu + \sigma_2^2\nu + \zeta_{2i}\sigma_2\sigma_i^2 = 0 \ ,$$

is satisfied if and only if $\zeta_j, \zeta_{1j}, \zeta_{2i}, \zeta_{12j} \in \mathbb{C}$, $3 \leq j \leq g$, and $\mu, \nu \in H^0(K_C)$ all vanish identically (no non-trivial solution). Evaluating such an equation at $p_j \in C$, $3 \leq j \leq g$, gives $\zeta_j = 0$. Furthermore, note that, by condition *iii*), for each $j \neq i$, $3 \leq j \leq g$, $\sigma_1\sigma_j^2$ is the unique 3-differential with a single zero in p_j, so that $\zeta_{1j} = 0$. We are left with

$$\zeta_{1i}\sigma_1\sigma_i^2 + \zeta_{2i}\sigma_2\sigma_i^2 + \sigma_1^2\mu + \sigma_2^2\nu + \sum_{j=3}^{g} \zeta_{12j}\sigma_1\sigma_2\sigma_j = 0 \ . \tag{4.19}$$

By Riemann-Roch Theorem, for each k, $3 \leq k \leq g$, $h^0(K_C \otimes \mathcal{O}(-b-p_k)) \geq 1$; the condition *ii*) implies that $h^0(K_C \otimes \mathcal{O}(-b-p_k)) \leq 1$, so that, in particular, there exists a unique (up to a constant) non-vanishing β in $H^0(K_C\otimes\mathcal{O}(-b-p_i))$. Furthermore,

$$H^0(K_C\otimes\mathcal{O}(-b))\neq\bigcup_{k=3}^{g}H^0(K_C\otimes\mathcal{O}(-b-p_k))\ ,$$

because the LHS is a 2-dimensional space and the RHS is a finite union of 1-dimensional subspaces; then, there exists $\alpha\in H^0(K_C\otimes\mathcal{O}(-b))$ such that $p_3,\ldots,p_g$ are single zeros for α. Note that α and β span $H^0(K_C\otimes\mathcal{O}(-b))$ and α^2, β^2 and $\alpha\beta$ span $H^0(K_C^2\otimes\mathcal{O}(-2b))$. Hence, the existence of non-trivial $\zeta_{1i},\zeta_{2i},\zeta_{12j},\nu,\mu$ satisfying Eq.(4.19) is equivalent to the existence of non-trivial $\nu',\mu'\in H^0(K_C)$ and $\zeta_\alpha,\zeta_\beta,\zeta_{\alpha\beta j}\in\mathbb{C}$ satisfying

$$\zeta_\alpha\alpha\sigma_i^2+\zeta_\beta\beta\sigma_i^2+\alpha^2\mu'+\beta^2\nu'+\sum_{j=3}^{g}\zeta_{\alpha\beta j}\alpha\beta\sigma_j=0\ .$$

Note that $\alpha\sigma_i^2$ is the unique 3-differential with a single zero in p_i, so that $\zeta_\alpha=0$. Condition *ii*) implies that b is the greatest common divisor of (α) and (β). Then $\alpha\neq0$ on the support of c_β, where $c_\beta:=(\beta)-b-p_i$. Hence, $\mu'\in H^0(K_C\otimes\mathcal{O}(-c_\beta))$, which, by Riemann-Roch Theorem, is a 1-dimensional space, so that $\mu'=\zeta'_\mu\beta$, for some $\zeta'_\mu\in\mathbb{C}$. Since, by construction, $\beta\neq0$, we have

$$\zeta_\beta\sigma_i^2+\zeta'_\mu\alpha^2+\beta\nu'+\sum_{j=3}^{g}\zeta_{\alpha\beta j}\alpha\sigma_j=0\ .$$

By evaluating such an equation at p_i gives $\zeta_\beta=0$. Furthermore, since $\beta\neq0$ on the support of c_α, where $c_\alpha:=(\alpha)-b$, it follows that $\nu'=\zeta'_\nu\alpha$, for some $\zeta'_\nu\in\mathbb{C}$. Since $\alpha\neq0$

$$\zeta'_\mu\alpha+\zeta'_\nu\beta+\sum_{j=3}^{g}\zeta_{\alpha\beta j}\sigma_j=0\ ,$$

which implies that $\zeta'_\mu=\zeta'_\nu=\zeta_{\alpha\beta j}=0$, for all $3\leq j\leq g$.

Conversely, suppose that *a*) and *b*) hold for some fixed i, with $3\leq i\leq g$, and for some set $\{\hat\sigma_j\}_{j\in I_g}$. If $\det\eta_j(p_k)=0$, then $\{\hat\sigma_j\}_{j\in I_g}$ is not a basis of $H^0(K_C)$ and $\{\hat\varphi_j\}_{j\in I_{N_3-1}}$ cannot span a (N_3-1)-dimensional vector space. Then *i*) is satisfied and the basis $\{\hat\sigma_j\}_{j\in I_g}$ corresponds, up to a non-singular diagonal transformation, to the basis $\{\sigma_j\}_{j\in I_g}$, defined in (4.5).

Without loss of generality, we can prove *ii*) and *iii*) for $\hat\sigma_j\equiv\sigma_j$, $j\in I_g$ and then $\hat\phi_j\equiv\phi_j$, $j\in I_{M_3}$. Since the 3-differentials σ_1v_j, $j\in I_N$, are distinct elements of a basis of $H^0(K_C^3)$, then v_j, $j\in I_N$, are linearly independent elements of $H^0(K_C^2)$ and, by Proposition 4.4, also condition *ii*) is satisfied.

Finally, assume that there exists $k\neq i$, $3\leq k\leq g$, such that σ_1 has a double zero in p_k. Then, apart from $\varphi_k\equiv\sigma_k^3$, which satisfies $\varphi_k(p_k)\neq0$, all the other 3-differentials of the basis have a double zero in p_k. Therefore, an element of $H^0(K_C^3)$ with a single zero in p_k cannot be a linear combination of the elements of such a basis, which is absurd. (An example of a holomorphic 3-differential with a single zero in p_k is $\sigma_2\sigma_k^2$, since, by condition *ii*), σ_2 cannot have a double zero in p_k). □

4.2.1 Determinants of distinguished bases and Fay's identity

In this sections, the combinatorial lemmas 2.2 and 2.3 are applied to the computation of determinants of the distinguished bases introduced in (4.14). For $n < g$, a necessary condition for Eq.(2.13) to hold is the existence of the points p_i, $3 \le i \le g$, satisfying Eq.(2.12); in particular, Lemmas 2.2 and 2.3 can be applied to the basis $\{\sigma_i\}_{i\in I_g}$, of $H^0(K_C)$, defined in Eq.(4.5).

Theorem 4.7. *Fix the points $p_1, \ldots, p_g \in C$, and $\hat\sigma_i \in H^0(K_C)$, $i \in I_g$, in such a way that $\hat\sigma_i(p_j) = 0$, for all $i \neq j \in I_g$. Define $\hat v_i \in H^0(K_C^2)$, $i \in I_N$, by*

$$\hat v_i := \psi(\hat\sigma \cdot \hat\sigma)_i = \hat\sigma_{1_i}\hat\sigma_{2_i} \ ,$$

and let $\{\eta_i\}_{i\in I_g}$ be an arbitrary basis of $H^0(K_C)$. Then, the following identity

$$\begin{aligned}\det \hat v(p_3, \ldots, p_g, x_1, \ldots, x_{2g-1}) = &\Big(\frac{\hat A_1 \hat A_2}{\det \eta_i(p_j)}\Big)^{g+1} \prod_{i=3}^{g} \hat A_i^4 \\ &\cdot \frac{(-)^{g+1}}{c_{g,2}} \sum_{s\in\mathcal{P}_{2g-1}} \epsilon(s) \det \eta(x_{d^1(s)}) \det \eta(x_{d^2(s)}) \\ &\cdot \prod_{i=3}^{g+1} \det \eta(x_{d_1^i(s)}, x_{d_2^i(s)}, p_3, \ldots, p_g) \ ,\end{aligned} \tag{4.20}$$

holds for all $x_1, \ldots, x_{2g-1} \in C$, where, according to (2.18), $c_{g,2} = g!(g-1)!(2g-1)$, *and for each $i \in I_g$*

$$\hat A_i := \hat\sigma_i(p_i) \ ,$$

is a 1*-differential in p_i.*

Proof. Assume that $p_1, \ldots, p_g$ satisfy the hypotheses of Proposition 4.1, so that $\{\hat\sigma_i\}_{i\in I_g}$ is a basis of $H^0(K_C)$ and $\hat\sigma_i(p_i) \neq 0$, for all $i \in I_g$. Since the points $p_1, \ldots, p_g$ satisfying such a condition are a dense set in C^g, it suffices to prove Eq.(4.20) in this case and then conclude by continuity arguments. A relation analogous to (4.15) holds

$$v_i(p_j) = \begin{cases} \hat A_j^2 \,\delta_{ij} \ , & i \in I_g \ , \\ 0 \ , & g+1 \le i \le M \ , \end{cases}$$

$j \in I_g$, so that

$$\det \hat v(p_3, \ldots, p_g, x_1, \ldots, x_{2g-1}) = (-)^{g+1} \prod_{i=3}^{g} \hat\sigma_i(p_i)^2 \det_{I_{M,2}} \hat\sigma\hat\sigma(x_1, \ldots, x_{2g-1}) \ .$$

By Lemma 2.2 for $n = 2$, $\det_{I_{M,2}} \hat\sigma\hat\sigma(x_1, \ldots, x_{2g-1})$ is equal to the RHS of (2.13) divided by $\prod_{i=3}^{g} \hat A_i^{g-1}$. Eq.(4.20) then follows by the identity

$$\det \hat\sigma_i(z_j) = \frac{\det \eta_i(z_j)}{\det \eta_i(p_j)} \det \hat\sigma_i(p_j) = \frac{\det \eta_i(z_j)}{\det \eta_i(p_j)} \prod_{i=1}^{g} \hat A_i \ .$$

∎

Remark 4.1. If $\det \eta_i(p_j) \neq 0$, then Theorem 4.7 holds for $\hat\sigma_i \equiv \sigma_i$, so that $\hat\sigma_i(p_i) = 1$, $i \in I_g$, and $\hat v_i \equiv v_i$, $i \in I_N$.

Corollary 4.8. *Let* $\mathrm{b} := \sum_{i=3}^g p_i$ *be a fixed divisor of* C *and define* $\hat v_i$, $i \in I_N$, *as in Theorem 4.7. Then for all* $x_1, \ldots, x_N \in C$

$$\det \hat v(x_1, \ldots, x_N) = -\frac{F}{c_{g,2}} \frac{\theta\big(\sum_1^N x_i\big) \prod_{i=2g}^N (\sigma(x_i)^3 \prod_{j=1}^{i-1} E(x_j, x_i))}{\theta\big(\sum_1^{2g-1} x_i + \mathrm{b}\big) \prod_{i=3}^g \prod_{j=1}^{2g-1} E(p_i, x_j)} \prod_{i=1}^{2g-1} \sigma(x_i)^2 \tag{4.21}$$

$$\cdot \sum_{s \in \mathcal{P}_{2g-1}} \epsilon(s) S\big(\sum_{i=1}^g x_{s_i}\big) S\big(\sum_{i=g}^{2g-1} x_{s_i}\big) \prod_{\substack{i,j=1\\ i<j}}^g E(x_{s_i}, x_{s_j}) \prod_{\substack{i,j=g\\ i<j}}^{2g-1} E(x_{s_i}, x_{s_j})$$

$$\cdot \prod_{k=1}^{g-1} \Big(S(x_{s_k} + x_{s_{k+g}} + \mathrm{b}) E(x_{s_k}, x_{s_{k+g}}) \prod_{i=3}^g E(x_{s_k}, p_i) E(x_{s_{k+g}}, p_i) \Big) ,$$

where $F \equiv F(p_1, \ldots, p_g)$ *is*

$$F := \Big(\frac{\hat\sigma_1(p_1) \hat\sigma_2(p_2)}{S(\mathrm{a}) \sigma(p_1) \sigma(p_2) E(p_1, p_2)} \Big)^{g+1}$$
$$\prod_{i=3}^g \frac{\hat\sigma_i(p_i)^4}{\sigma(p_i)^5 (E(p_1, p_i) E(p_2, p_i))^{g+1} \prod_{j>i}^g E(p_i, p_j)^3} .$$

Proof. Apply Eq.(4.20) to

$$\det \hat v(x_1, \ldots, x_N) = \frac{\det \rho(x_1, \ldots, x_N) \det \hat v(p_3, \ldots, p_g, x_1, \ldots, x_{2g-1})}{\det \rho(p_3, \ldots, p_g, x_1, \ldots, x_{2g-1})} ,$$

with $\{\rho_i\}_{i \in I_N}$ an arbitrary basis of $H^0(K_C^2)$. Eq.(4.21) then follows by Eqs.(3.2) and (3.3). □

In this section, we will use the bases introduced in section 4 to derive a combinatorial proof of the Fay's trisecant identity.

Theorem 4.9. *The following are equivalent*

a) *Proposition* (3.2) *holds;*

b) *The Fay's trisecant identity [22]*

$$\frac{\theta(w + \sum_1^m (x_i - y_i)) \prod_{i<j} E(x_i, x_j) E(y_i, y_j)}{\theta(w) \prod_{i,j} E(x_i, y_j)} = \pm \det_{ij} \frac{\theta(w + x_i - y_j)}{\theta(w) E(x_i, y_j)} , \tag{4.22}$$

$m \geq 2$, *holds for all* $x_1, \ldots, x_m, y_1, \ldots, y_m \in C$, $w \in J_0(C)$.

Proof. ($a \Rightarrow b$) Fix $x_1, \ldots, x_m, y_1, \ldots, y_m \in C$ and $w \in J_0(C)$, with $\theta(w) \neq 0$. Choose $y_1, \ldots, y_m$ distinct, otherwise the identity is trivial. Set $p_i := y_i$, $i \in I_m$, and fix $n \in \mathbb{N}_+$, with $d := N_n - m \geq g$, and $p_{m+1}, \ldots, p_{N_n} \in C$, in such a way that

$$I(\sum_1^{N_n} p_i) = w .$$

By Jacobi Inversion Theorem, such a choice is always possible. Note that the set of divisors $p_{m+1}+\ldots+p_{N_n}$, such that $p_i = p_j$ for some $i \neq j \in I_{N_n}$, is the set of points of a subvariety in the space of positive divisors of degree d. Then the image of such a variety under the Jacobi map, which is analytic, corresponds to a proper subvariety W of $J_0(C)$. Hence, the conditions $\theta(w) \neq 0$ and

$$w - I(\sum_1^m y_i) \in J_0(C) \setminus W \ ,$$

are satisfied for w a dense subset in $J_0(C)$. It is therefore sufficient to prove Eq.(4.22) on such a subset and the theorem follows by continuity arguments.

Let us then choose the points $p_{m+1},\ldots,p_{N_n}$ to be pairwise distinct and distinct from $y_1,\ldots,y_m$ and fix a basis $\{\phi_i^n\}_{i\in I_{N_n}}$ of $H^0(K_C^n)$. Since $p_1,\ldots,p_{N_n}$ are pairwise distinct and

$$\theta(\sum_1^{N_n} p_i) = \theta(w) \neq 0 \ ,$$

it follows by Eq.(3.3) that $\det \phi_i^n(p_j) \neq 0$. Therefore, by Proposition 4.1, one can define the basis $\{\gamma_i^n\}_{i\in I_{N_n}}$ of $H^0(K_C^n)$ with the property $\gamma_i^n(p_j) = \delta_{ij}$, $i,j \in I_{N_n}$. On the other hand, note that

$$\det \gamma^n(x_1,\ldots,x_m,p_{m+1},\ldots,p_{N_n}) = \det_{ij\in I_m} \gamma_i^n(x_j) \ ,$$

can be expressed either by means of Eq.(4.12)

$$\prod_{i=1}^m \sigma(x_i,y_i)^{2n-1} \prod_{j=m+1}^{N_n} \frac{E(x_i,p_j)}{E(y_i,p_j)} \frac{\prod_{i,j=1}^m E(x_i,y_j)}{\prod_{\substack{i,j=1\\ i\neq j}}^m E(y_i,y_j)} \det_{ij} \frac{\theta(w+x_i-y_j)}{\theta(w)E(x_i,y_j)} \ ,$$

or by means of (3.3) and (4.13)

$$\prod_{i=1}^m \sigma(x_i,y_i)^{2n-1} \prod_{j=m+1}^{N_n} \frac{E(x_i,p_j)}{E(y_i,p_j)} \frac{\theta\big(w+\sum_1^m(x_i-y_i)\big)\prod_{i<j}^m E(x_i,x_j)}{\theta(w)\prod_{\substack{i,j=1\\ i<j}}^m E(y_i,y_j)} \ .$$

Eq.(4.22) then follows by observing that

$$\prod_{\substack{i,j=1\\ i\neq j}}^m E(y_i,y_j) = (-)^{m(m-1)/2} \prod_{\substack{i,j=1\\ i<j}}^m E(y_i,y_j)^2 \ . \qquad (4.23)$$

($b \Rightarrow a$) Fix $p_1,\ldots,p_{N_n} \in C$, $n \geq 2$, in such a way that the hypothesis of Proposition 4.1 is satisfied. Let $\{\gamma_i^n\}_{i\in I_{N_n}}$ be the corresponding basis of $H^0(K_C^n)$ satisfying (4.3). $\det \gamma_i^n(z_j)$ can be evaluated, for arbitrary $z_1,\ldots,z_{N_n} \in C$, by expressing $\gamma_i^n(z_j)$ by means of (4.12). In particular, by using (4.22) with $m = N_n$, $x_i = z_i$, $y_i = p_i$, $i \in I_{N_n}$, and $w = I(\sum_1^{N_n} p_i)$, after a computation analogous to the previous one, (3.3) follows, with $\kappa[\gamma^n]$ given by Eq.(4.13). Therefore, (3.3) holds for an arbitrary basis $\{\phi_i^n\}_{i\in I_{N_n}}$ of $H^0(K_C^n)$, with $\kappa[\phi^n] = \kappa[\gamma^n]\det\phi_i^n(p_j)$. The same result holds for (3.2) by using (4.22) with $w = I(\sum_1^g p_i - y)$. □

4.3 The function H and the characterization of the $\mathcal{B}$ locus

Proposition 4.1 shows that $\det \eta_i(p_j) \neq 0$, for an arbitrary basis $\{\eta_i\}_{i\in I_g}$ of $H^0_C(K)$, is a necessary and sufficient condition on the points $p_1, \ldots, p_g$ for the existence of a basis of holomorphic 1-differentials $\{\hat{\sigma}_i\}_{i\in I_g}$, such that $i \neq j \Rightarrow \sigma_i(p_j) = 0$, $i, j \in I_g$. By Eq.(3.2) and (3.1) it follows that the subset $\mathcal{A} \subset C^g$, for which such a condition is not satisfied, corresponds to the set of solutions of the equation

$$S(a)\prod_{i<j}^{g} E(p_i, p_j) = 0 \ .$$

It is more difficult to characterize the locus $\mathcal{B} \subset C^g$, whose elements are the g-tuples of points $p_1, \ldots, p_g$ which do not satisfy the conditions of Proposition 4.4. The following theorems show that such a locus can be characterized as the set of solutions of the equation $H = 0$ for a suitable function $H(p_1, \ldots, p_g)$.

Theorem 4.10. *Fix $g-2$ distinct points $p_3, \ldots, p_g \in C$ such that*

$$\{I(p+b)|p \in C\} \cap \Theta_s = \varnothing \ , \tag{4.24}$$

$b := \sum_3^g p_i$. Then, for each $p_2 \in C \setminus \{p_3, \ldots, p_g\}$, there exists a finite set of points S, depending on b and p_2, with $\{p_2, \ldots, p_g\} \subset S \subset C$, such that, for all $p_1 \in C \setminus S$, the holomorphic 1-differentials $\{\sigma_i\}_{i\in I_g}$, associated to the points $p_1, \ldots, p_g$ by Proposition 4.1, is a basis of $H^0(K_C)$ and the corresponding quadratic differentials $\{v_i\}_{i\in I_N}$ is a basis of $H^0(K_C^2)$. Conversely, if for some fixed $g-2$ arbitrary points $p_3, \ldots, p_g \in C$, there exist $p_1, p_2 \in C$ such that the associated $\{\sigma_i\}_{i\in I_g}$ and $\{v_i\}_{i\in I_N}$ are bases of $H^0(K_C)$ and $H^0(K_C^2)$, then (4.24) holds.

Proof. Eq.(4.24) implies that $h^0(K_C \otimes \mathcal{O}(-b-p)) = 1$, for all $p \in C$. Hence, $h^0(K_C \otimes \mathcal{O}(-b)) = 2$ and, for each pair of linearly independent elements σ_1, σ_2 of $H^0(K_C \otimes \mathcal{O}(-b))$, the supports of $(\sigma_1) - b$ and $(\sigma_2) - b$ are disjoint. Fix $p_2 \in C \setminus \{p_3, \ldots, p_g\}$ and let σ_1 be a non-vanishing element of $H^0(K_C \otimes \mathcal{O}(-b-p_2))$. Define the finite set S as the support of (σ_1) or, equivalently, as the union of $\{p_2, \ldots, p_g\}$ and the set of zeros of $S(x + p_2 + b)$. Then, for all $p_1 \in C \setminus S$, fix $\sigma_2 \in H^0(K_C \otimes \mathcal{O}(-b-p_1))$ so that σ_1 and σ_2 are linearly independent. Then $p_1, \ldots, p_g$ satisfy the conditions *i*) and *ii*) of Proposition 4.4, and $\{v_i\}_{i\in I_N}$, as defined in (4.14), is a basis of $H^0(K_C^2)$. Conversely, if $I(p + b) \in \Theta_s$ for some $p \in C$, then, for each pair $\sigma_1, \sigma_2 \in H^0(K_C \otimes \mathcal{O}(-b))$, their greatest common divisor satisfies $\gcd(\sigma_1, \sigma_2) \geq p + b$ and the condition *ii*) of Proposition 4.4 does not hold. □

The classical result that the dimension of Θ_s is $g-4$ for a non-hyperelliptic Riemann surface of genus $g \geq 4$, immediately gives the following corollary by simple dimensional considerations.

Corollary 4.11. *In a non-hyperelliptic Riemann surface C of genus $g \geq 4$, there always exist g points $p_1, \ldots, p_g \in C$ such that the corresponding $\{v_i\}_{i\in I_N}$ is a basis of $H^0(K_C^2)$.*

Proof. By Theorem 4.10, it is sufficient to prove that there exists $b \in C_{g-2}$ satisfying the condition (4.24). Suppose, by absurd, that this is not true. Then $W_{g-2} = I(C_{g-2})$ is a subset of $\Theta_s \ominus W_1 := \{e - I(p) \mid e \in \Theta_s, p \in C\}$. The corollary then follows by observing that W_{g-2} has dimension $g-2$, whereas the dimension of each component of $\Theta_s \ominus W_1$ is less than $\dim \Theta_s + \dim W_1 = g-3$. □

Theorem 4.12. *Fix $p_1 \dots, p_g \in C$. The function $H \equiv H(p_1, \dots, p_g)$*

$$H := \frac{S(\mathrm{a})^{5g-7} E(p_1,p_2)^{g+1}}{\theta\big(\mathrm{b} + \sum_1^{2g-1} x_i\big) \prod_{i=1}^{2g-1} \sigma(x_i)} \prod_{i=3}^{g} \frac{E(p_1,p_i)^4 E(p_2,p_i)^4 \prod_{j>i}^{g} E(p_i,p_j)^5}{\sigma(p_i)}$$
$$\cdot \sum_{s \in \mathcal{P}_{2g-1}} \frac{S\big(\sum_{i=1}^{g} x_{s_i}\big) S\big(\sum_{i=g}^{2g-1} x_{s_i}\big)}{\prod_{i=3}^{g} E(x_{s_g}, p_i)} \prod_{i=1}^{g-1} \frac{S(x_{s_i} + x_{s_{i+g}} + \mathrm{b})}{\prod_{\substack{j=1 \\ j \neq i}}^{g-1} E(x_{s_i}, x_{s_{j+g}})} , \qquad (4.25)$$

is independent of the points $x_1, \dots, x_{2g-1} \in C$. Furthermore, the set $\{v_i\}_{i \in I_N}$, defined as in (4.14)*, is a basis of $H^0(K_C^2)$ if and only if $H \neq 0$.*

Proof. Consider the holomorphic 1-differentials

$$\hat{\sigma}_i(z) := A_i^{-1} \sigma(z) S(\mathrm{a}_i + z) \prod_{\substack{j=1 \\ j \neq i}}^{g} E(z, p_j) = A_i^{-1} \sum_{j=1}^{g} \theta_j(\mathrm{a}_i) \omega_j(z) ,$$

$i \in I_g$, with a_i as in Definition 4.1 and $A_1, \dots, A_g$ non-vanishing constants. If the points $p_1, \dots, p_g$ satisfy the hypotheses of Proposition 4.1, then $\{\hat{\sigma}_i\}_{i \in I_g}$ corresponds, up to a non-singular diagonal transformation, to the basis defined in (4.5). Let $\{\rho_i\}_{i \in I_N}$ be an arbitrary basis of $H^0(K_C^2)$. By (3.3) the following identity

$$\det \rho(p_3, \dots, p_g, x_1, \dots, x_{2g-1}) = \kappa[\rho] \epsilon(s) \prod_{\substack{i,j=1 \\ i<j}}^{2g-1} E(x_{s_i}, x_{s_j}) \prod_{i=3}^{g} \sigma(p_i)^3 \prod_{i=1}^{2g-1} \sigma(x_i)^3$$
$$\theta\Big(\sum_1^{2g-1} x_i + \mathrm{b}\Big) \prod_{\substack{i,j=3 \\ i<j}}^{g} E(p_i, p_j) \prod_{i=3}^{g} \prod_{j=1}^{2g-1} E(p_i, x_j) ,$$

holds for all $s \in \mathcal{P}_{2g-1}$. Together with Eq.(4.21) and the above expression for $\hat{\sigma}_i$, it implies that

$$H = \kappa[\rho] c_{g,2} (A_1 A_2)^{g+1} \prod_{i=3}^{g} A_i^4 \frac{\det \hat{v}(p_3, \dots, p_g, x_1, \dots, x_{2g-1})}{\det \rho(p_3, \dots, p_g, x_1, \dots, x_{2g-1})} . \qquad (4.26)$$

Hence, H is independent of $x_1, \dots, x_{2g-1}$, and $H \neq 0$ if and only if $\{\hat{v}_i\}_{i \in I_N}$ is a basis of $H^0(K_C^2)$. On the other hand the vector $(\hat{v}_1, \dots, \hat{v}_N)$ corresponds, up to a non-singular diagonal transformation, to $(v_1, \dots, v_N)$, with v_i, $i \in I_N$, defined in (4.14). □

Remark 4.2. By (4.26)

$$\kappa[\hat{v}] = \frac{H(p_1, \ldots, p_g)}{c_{g,2}(A_1 A_2)^{g+1} \prod_{i=3}^{g} A_i^4} \ .$$

Furthermore, if $(p_1, \ldots, p_g) \notin \mathcal{A}$, then one can choose

$$A_i = \sigma(p_i) S(\mathrm{a}) \prod_{\substack{j=1 \\ j \neq i}}^{g} E(p_i, p_j) = \sum_{j=1}^{g} \theta_j(\mathrm{a}_i) \omega_j(p_i) \ ,$$

to obtain $\hat{\sigma}_i \equiv \sigma_i$, $i \in I_g$, and

$$\begin{aligned} \kappa[v] &= \frac{H(p_1, \ldots, p_g)}{c_{g,2} \prod_{i=1}^{2} \big(\sum_{j=1}^{g} \theta_j(\mathrm{a}_i) \omega_j(p_i)\big)^{g+1} \prod_{i=3}^{g} \big(\sum_{j=1}^{g} \theta_j(\mathrm{a}_i) \omega_j(p_i)\big)^4} \\ &= \frac{H(p_1, \ldots, p_g)}{c_{g,2} S(\mathrm{a})^{6g-6} \prod_{i=1}^{2} \big(\sigma(p_i) \prod_{\substack{j=1 \\ j \neq i}}^{g} E(p_i, p_j)\big)^{g+1} \prod_{i=3}^{g} \big(\sigma(p_i) \prod_{\substack{j=1 \\ j \neq i}}^{g} E(p_i, p_j)\big)^4} . \end{aligned} \tag{4.27}$$

Observe that $\mathcal{A} \subset \mathcal{B}$. Theorem 4.10 shows that if $(p_1, \ldots, p_g) \notin \mathcal{A}$, a necessary and sufficient condition for $(p_1, \ldots, p_g)$ to be in $\mathcal{B}$ is that there exists $p \in C$ such that $I(b + p) \in \Theta_s$. Hence, $\mathcal{B}$ is the union of $\mathcal{A}$ together with the pull-back of a divisor in C^{g-2} by the projection $C^g \to C^{g-2}$ which “forgets” the first pair of points: $(p_1, \ldots, p_g) \to (p_3, \ldots, p_g)$. Such a divisor is characterized by the equation $K = 0$, where K is defined in the following chapter.

5. THE IDEAL OF A CANONICAL CURVE

Denote by $\tilde{\phi}^n : H^0(K_C^n) \to \mathbb{C}^{N_n}$ the isomorphism $\tilde{\phi}^n(\phi_i^n) = e_i$, with $\{e_i\}_{i \in I_{N_n}}$ the canonical basis of $\mathbb{C}^{N_n}$. The isomorphism $\tilde{\eta}$ induces an isomorphism $\tilde{\eta} \cdot \tilde{\eta}$: $\mathrm{Sym}^2(H^0(K_C^2)) \to \mathrm{Sym}^2\mathbb{C}^g$. The natural map $\psi : \mathrm{Sym}^2(H^0(K_C^2)) \to H^0(K_C^2)$ is surjective if C is canonical.

The choice of a basis $\{\eta_i\}_{i \in I_g}$ of $H^0(K_C)$ determines an embedding of the curve C in $\mathbb{P}_{g-1}$ by $p \mapsto (\eta_1(p), \ldots, \eta_g(p))$, so that the elements of $\{\eta_i\}_{i \in I_g}$ correspond to a set of homogeneous coordinates $X_1, \ldots, X_g$ on $\mathbb{P}_{g-1}$. Each holomorphic n-differential corresponds to a homogeneous n-degree polynomial in $\mathbb{P}_{g-1}$ by

$$\phi^n := \sum_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n} \eta_{i_1} \cdots \eta_{i_n} \mapsto \sum_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n} X_{i_1} \cdots X_{i_n} \ ,$$

where $X_1, \ldots, X_g$ are homogeneous coordinates on $\mathbb{P}_{g-1}$. A basis of $H^0(K_C^n)$ corresponds to a basis of the homogeneous polynomials of degree n in $\mathbb{P}_{g-1}$ that are not zero when restricted to C. The curve C is identified with the ideal of all the polynomials in $\mathbb{P}_{g-1}$ vanishing at C. Enriques-Babbage and Petri's Theorems state that, with few exceptions, such an ideal is generated by quadrics

$$\sum_{j=1}^{M} C_{ij}^{\eta} XX_j = 0 \ ,$$

$N + 1 \leq i \leq M$, where $XX_j := X_{1_j} X_{2_j}$. Here, $\{C_i^\eta\}_{N<i\leq M}$, with $C_i^\eta := (C_{i1}^\eta, \ldots, C_{iM}^\eta)$, is a set of linearly independent elements of $\mathbb{P}(\mathrm{Sym}^2 \mathbb{C}^g) \cong \mathbb{P}_M$, each one defining a quadric. The isomorphism $\tilde{\eta} \cdot \tilde{\eta}$ induces the identification $\mathbb{P}(\mathrm{Sym}^2(H^0(K_C))) \cong \mathbb{P}_M$, under which each quadric corresponds to an element of $\ker \psi \subset \mathrm{Sym}^2(H^0(K_C))$

$$\ker \psi \ni u_i := \sum_{j=1}^{M} C_{ij}^{\eta} \eta \cdot \eta_j \ ,$$

$N + 1 \leq i \leq M$, or, equivalently, to a relation among holomorphic quadratic differentials

$$\psi(u_i) \equiv \sum_{j=1}^{M} C_{ij}^{\eta} \eta \eta_j = 0 \ .$$

Canonical curves that are not cut out by such quadrics are trigonal or isomorphic to smooth plane quintic. In these cases, Petri's Theorem assures that the ideal is generated by the quadrics above together with a suitable set of cubics.

This section is devoted to the study of such relations among quadratic and cubic differentials.

5.1 Relations among holomorphic quadratic differentials

In the following we derive the matrix form of the map $\tilde{v} \circ \psi \circ (\tilde{\sigma} \cdot \tilde{\sigma})^{-1}$, with respect to the basis $\{\sigma_i\}_{i \in I_g}$ constructed in the previous subsection. This will lead to the explicit expression of $\ker \psi$. Set

$$\tilde{\psi}_{ij} := \frac{\kappa[v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_N]}{\kappa[v]} \ . \tag{5.1}$$

$i \in I_N$, $j \in I_M$.

Lemma 5.1. *$v_1, \ldots, v_M$ satisfy the following $(g-2)(g-3)/2$ linearly independent relations*

$$v_i = \sum_{j=1}^{N} \tilde{\psi}_{ji} v_j = \sum_{j=g+1}^{N} \tilde{\psi}_{ji} v_j \ , \tag{5.2}$$

$i = N+1, \ldots, M$.

Proof. The first equality trivially follows by the Cramer rule. The identities (4.15) imply $\tilde{\psi}_{ji} = 0$ for $j \in I_g$ and $i = N+1, \ldots, M$, and the lemma follows. $\square$

Eq.(5.2) implies that the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\psi"] \arrow[d, "\tilde{\sigma}\cdot\tilde{\sigma}"'] & H^0(K_C^2) \arrow[d, "\tilde{v}"] \\ \mathbb{C}^M \arrow[r, "\tilde{\psi}"] & \mathbb{C}^N \end{tikzcd}$$

where $\tilde{\psi} : \mathbb{C}^M \to \mathbb{C}^N$ is the homomorphism with matrix elements $\tilde{\psi}_{ij}$ and $\mathrm{Sym}^2\mathbb{C}^g$ is isomorphic to $\mathbb{C}^M$ through A, introduced in Definition 2.2, commutes.

Let $\iota : \mathbb{C}^N \to \mathbb{C}^M$ be the injection $\iota(e_i) = \tilde{e}_i$, $i \in I_N$. The matrix elements of the map $\iota \circ \tilde{\psi} : \mathbb{C}^M \to \mathbb{C}^M$ are

$$(\iota \circ \tilde{\psi})_{ij} = \begin{cases} \tilde{\psi}_{ij} \ , & 1 \le i \le N \ , \\ 0 \ , & N+1 \le i \le M \ , \end{cases}$$

$j \in I_M$. Noting that $(\iota \circ \tilde{\psi})_{ij} = \delta_{ij}$, for all $i, j \in I_N$, we obtain

$$\sum_{i=1}^{M} (\iota \circ \tilde{\psi})_{ji} (\iota \circ \tilde{\psi})_{ik} = \sum_{i=1}^{N} (\iota \circ \tilde{\psi})_{ji} \tilde{\psi}_{ik} = (\iota \circ \tilde{\psi})_{jk} \ ,$$

$j, k \in I_M$. Hence, $\iota \circ \tilde{\psi}$ is a projection of rank N and, since ι is an injection,

$$\ker \tilde{\psi} = \ker \iota \circ \tilde{\psi} = (\mathrm{id} - \iota \circ \tilde{\psi})(\mathbb{C}^M) \ . \tag{5.3}$$

Lemma 5.2. *The set $\{\tilde{u}_{N+1}, \ldots, \tilde{u}_M\}$, $\tilde{u}_i := \tilde{e}_i - \sum_{j=1}^{N} \tilde{e}_j \tilde{\psi}_{ji}$, $N+1 \le i \le M$, is a basis of $\ker \tilde{\psi}$.*

Proof. Since $(\mathrm{id} - \iota \circ \tilde{\psi})(\tilde{e}_i) = 0$, $i \in I_N$, by (5.3), the $M - N$ vectors $\tilde{u}_i = (\mathrm{id} - \iota \circ \tilde{\psi})(\tilde{e}_i)$, $N < i \le M$, are a set of generators for $\ker \psi$ and, since $\dim \ker \psi = M - N$, the lemma follows. □

Set $\eta\eta_i := \psi(\eta \cdot \eta)_i$, $i \in I_g$, and let X^η be the automorphism on $\mathbb{C}^M$ in the commutative diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\mathrm{id}} & \mathcal{S} \\ {\scriptstyle \tilde{\sigma}\cdot\tilde{\sigma}}\downarrow & & \downarrow{\scriptstyle \tilde{\eta}\cdot\tilde{\eta}} \\ \mathbb{C}^M & \xrightarrow{X^\eta} & \mathbb{C}^M \end{array}$$

whose matrix elements are

$$X^\eta_{ji} = \chi_j^{-1}\left(\begin{bmatrix}\sigma\\\eta\end{bmatrix}\begin{bmatrix}\sigma\\\eta\end{bmatrix}\right)_{ij} = \left(\begin{bmatrix}\sigma\\\eta\end{bmatrix}_{1_i 1_j}\begin{bmatrix}\sigma\\\eta\end{bmatrix}_{2_i 2_j} + \begin{bmatrix}\sigma\\\eta\end{bmatrix}_{1_i 2_j}\begin{bmatrix}\sigma\\\eta\end{bmatrix}_{2_i 1_j}\right)(1+\delta_{1_j 2_j})^{-1} \ , \quad (5.4)$$

$i, j \in I_M$, so that

$$v_i = \sum_{j=1}^{M} X^\eta_{ji}\, \eta\eta_j \ , \quad (5.5)$$

$i \in I_M$. Since $\eta\eta_i$, $i \in I_M$, are linearly dependent, the matrix X^η_{ij} is not univocally determined by Eq.(5.5). More precisely, an endomorphism $X^{\eta'} \in \mathrm{End}(\mathbb{C}^M)$ satisfies Eq.(5.5) if and only if the diagram

$$\begin{array}{ccc} \mathbb{C}^M & \xrightarrow{\tilde{\psi}} & \mathbb{C}^N \\ {\scriptstyle X^{\eta'}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{C}^M & \xrightarrow{B^\eta} & \mathbb{C}^N \end{array}$$

where $B^\eta := \tilde{\psi} \circ (X^\eta)^{-1}$, commutes or, equivalently, if and only if

$$(X^{\eta'} - X^\eta)(\mathbb{C}^M) \subseteq X^\eta(\ker \tilde{\psi}) \ . \quad (5.6)$$

Next theorem provides an explicit expression for such a homomorphisms. Consider the following determinants of the d-dimensional submatrices of X^η

$$|X^\eta|^{j_1\ldots j_d}_{i_1\ldots i_d} := \det \begin{pmatrix} X^\eta_{i_1 j_1} & \ldots & X^\eta_{i_1 j_d} \\ \vdots & \ddots & \vdots \\ X^\eta_{i_d j_1} & \ldots & X^\eta_{i_d j_d} \end{pmatrix} \ ,$$

$i_1, \ldots, i_d, j_1, \ldots, j_d \in I_M$, $d \in I_M$.

Theorem 5.3.

$$\sum_{j=1}^{M} C^\eta_{ij}\eta\eta_j = 0 \ , \quad (5.7)$$

$N + 1 \le i \le M$, *where*

$$C^\eta_{ij} := \sum_{k_1,\ldots,k_N=1}^{M} |X^\eta|^{1\ \ldots\ N i}_{k_1\ldots k_N j}\, \frac{\kappa[\eta\eta_{k_1}, \ldots, \eta\eta_{k_N}]}{\kappa[v]} \ , \quad (5.8)$$

are $M-N$ independent linear relations among holomorphic quadratic differentials. Furthermore, for all $p \in C$

$$W[v](p) = \sum_{i_1,\ldots,i_N=1}^{M} |X^\eta|_{i_1 \ldots i_N}^{1 \ \ldots \ N} W[\eta\eta_{k_1}, \ldots, \eta\eta_{k_N}](p) \ . \tag{5.9}$$

Proof. By (5.2) and (5.5)

$$\sum_{j=1}^{M} (X_{ji}^\eta - \sum_{k=1}^{N} \tilde{\psi}_{ki} X_{jk}^\eta)\eta\eta_j = 0 \ ,$$

for all $N+1 \leq i \leq M$, and by (5.1)

$$\sum_{j=1}^{M} \left[\sum_{k=1}^{N} (-)^k \frac{\kappa[v_i, v_1, \ldots, \check{v}_k, \ldots, v_N]}{\kappa[v]} X_{jk}^\eta + X_{ji}^\eta \right] \eta\eta_j = 0 \ .$$

By (5.5)

$$\frac{\kappa[v_{i_1}, \ldots, v_{i_N}]}{\kappa[v]} = \sum_{k_1,\ldots,k_N=1}^{M} |X^\eta|_{k_1 \ldots k_N}^{i_1 \ldots i_N} \frac{\kappa[\eta\eta_{k_1}, \ldots, \eta\eta_{k_N}]}{\kappa[v]} \ ,$$

$i_1, \ldots, i_N \in I_M$, and we get (5.7) with

$$C_{ij}^\eta = \sum_{k_1,\ldots,k_N=1}^{M} \left[\sum_{l=1}^{N} (-)^l X_{jl}^\eta \ |X^\eta|_{k_1 \ldots\ldots k_N}^{i\,1 \ldots \check{l} \ldots N} + X_{ji}^\eta \ |X^\eta|_{k_1 \ldots k_N}^{1 \ \ldots \ N} \right] \frac{\kappa[\eta\eta_{k_1}, \ldots, \eta\eta_{k_N}]}{\kappa[v]} \ ,$$

which is equivalent to (5.8) by the identity

$$\sum_{l=1}^{N} (-)^l X_{jl}^\eta \ |X^\eta|_{k_1 \ldots\ldots k_N}^{i\,1 \ldots \check{l} \ldots N} + X_{ji}^\eta \ |X^\eta|_{k_1 \ldots k_N}^{1 \ \ldots \ N} = |X^\eta|_{j k_1 \ldots k_N}^{i 1 \ \ldots \ N} \ .$$

Eq.(5.9) follows by (5.5). □

The homomorphisms $(X^{\eta'} - X^\eta) \in \mathrm{End}(\mathbb{C}^M)$, satisfying (5.6), are the elements of a $M(M-N)$ dimensional vector space, spanned by

$$(X^{\eta'} - X^\eta)_{ij} = \sum_{k=N+1}^{M} \Lambda_{jk} C_{ki}^\eta \ ,$$

$i, j \in I_M$, with Λ_{jk} an arbitrary $M \times (M-N)$ matrix. An obvious generalization of (5.2) yields

$$\eta\eta_i = \sum_{j=1}^{N} v_j B_{ji}^\eta \ , \tag{5.10}$$

$i \in I_M$, implying that $B_{ij}^\eta = \kappa[v_1, \ldots, v_{j-1}, \eta\eta_i, v_{j+1}, \ldots, v_N]/\kappa[v]$, are the matrix elements of the homomorphism $B^\eta = \tilde{\psi} \circ (X^\eta)^{-1}$. Such coefficients can be expanded as

$$B_{ij}^\eta = \sum_{k_1,\ldots,k_{N-1}=1}^{M} (-)^{j+1} |X^\eta|_{k_1 \ldots \ k_{N-1}}^{1 \ldots \check{\jmath} \ldots N} \frac{\kappa[\eta\eta_i, \eta\eta_{k_1}, \ldots, \eta\eta_{k_{N-1}}]}{\kappa[v]} \ . \tag{5.11}$$

Define $C^{\eta,ij}_{kl}$, $3 \le i < j \le g$, $k, l \in I_g$, by

$$C^{\eta, 1_m 2_m}_{1_n 2_n} := C^{\eta}_{mn} \ ,$$

$m, n \in I_M$, $m > N$.

The following result is a direct consequence of the Petri-like approach. The bound $r \le 6$ for the rank of quadrics is not sharp, however: M. Green proved that the ideal of quadrics of a canonical curve is generated by elements of rank 4 [29].

Theorem 5.4. *All the relations among holomorphic quadratic differentials have rank $r \le 6$.*

Proof. The statement is trivial for $g \le 6$, so let us assume $g \ge 7$. Each relation can be written as

$$\begin{aligned} 0 =& \sigma_i\sigma_j + C^{\sigma,ij}_{12}\sigma_1\sigma_2 + C^{\sigma,ij}_{1i}\sigma_1\sigma_i + C^{\sigma,ij}_{1j}\sigma_1\sigma_j + C^{\sigma,ij}_{2i}\sigma_2\sigma_i + C^{\sigma,ij}_{2j}\sigma_2\sigma_j \\ &+ \sum_{k\neq 1,2,i,j} C^{\sigma,ij}_{1k}\sigma_1\sigma_k + \sum_{k\neq 1,2,i,j} C^{\sigma,ij}_{2k}\sigma_2\sigma_k \ , \end{aligned}$$

where $3 \le i < j \le g$ and $C^{\eta,1_i 2_i}_{1_j 2_j} := C^{\eta}_{ij}$. Set $\eta_1 \equiv \sigma_1$, $\eta_2 \equiv \sigma_2$, $\eta_3 \equiv \sigma_i$, $\eta_4 \equiv \sigma_j$, $\eta_5 \equiv \sum_{k\neq 1,2,i,j} C^{\sigma,ij}_{1k}\sigma_k$, $\eta_6 \equiv \sum_{k\neq 1,2,i,j} C^{\sigma,ij}_{2k}\sigma_k$. Then the relations can be written as

$$\sum_{k<l}^{6} C^{\eta,ij}_{kl}\eta_k\eta_l = 0 \ ,$$

for suitable $C^{\eta,ij}_{kl}$, and the theorem follows. □

5.1.1 Consistency conditions on the quadrics coefficients

In the construction in chapter 4, the points p_1 and p_2 play a special role with respect to $p_3, \ldots, p_g$. Relations among holomorphic quadratic differentials can be obtained by replacing p_1 and p_2 with p_a and p_b, $a, b \in I_g$, $a < b$, $(a, b) \neq (1, 2)$. In the following of this section, we will consider the relationships between the coefficients C^σ obtained in section 4 and the analogous coefficients obtained upon replacing $(1, 2)$ by (a, b).

Proposition 5.5. *There exist g distinct points $p_1, \ldots, p_g \in C$ such that*

$$K(p_1, \ldots, \check{p}_i, \ldots, \check{p}_j, \ldots, p_g) \neq 0 \ ,$$

for all $i, j \in I_g$, $i \neq j$.

Proof. Consider the function in C^g

$$F(p_1, \ldots, p_g) := \prod_{i<j} K(p_1, \ldots, \check{p}_i, \ldots, \check{p}_j, \ldots, p_g) \ ,$$

and set $Z := \{(p_1, \ldots, p_g) \in C^g \mid F(p_1, \ldots, p_g) = 0\}$. Note that

$$Z = \bigcup_{i<j} \{(p_1, \ldots, p_g) \in \mathcal{C}^g \mid K(p_1, \ldots, \check{p}_i, \ldots, \check{p}_j, \ldots, p_g) = 0\} \ ,$$

so that it is a finite union of varieties of codimension 1 in C^g and, in particular, $Z \neq C^g$. Suppose that $C^g \setminus (\bigcup_{i<j} \Pi_{ij}) \subseteq Z$, where $\Pi_{ij} := \{(p_1, \dots, p_g) \in C^g \mid p_i = p_j\}$, $1 \leq i < j \leq g$. Since $C^g \setminus (\bigcup_{i<j} \Pi_{ij})$ is dense in C^g, it would follow that $Z \equiv C^g$, which is absurd. Hence, there exist pairwise distinct $p_1, \dots, p_g \in C$ such that $F(p_1, \dots, p_g) \neq 0$. $\square$

By Proposition 5.5 and Proposition 4.4, one can choose the points $p_1, \dots, p_g$ in such a way that

$$\{v_i^{(ab)}\}_{i \in I_N} := \{\sigma_i^2\}_{i \in I_g} \cup \{\sigma_a \sigma_b\} \cup \{\sigma_a \sigma_i, \sigma_b \sigma_i\}_{i \in I_g \setminus \{a,b\}} ,$$

is a basis of $H^0(K_C^2)$. Furthermore, one can obtain $M - N$ independent linear relations

$$\sum_{1 \leq k \leq l \leq g} (ab)^{ij}_{kl} \sigma_k \sigma_l = 0 , \tag{5.12}$$

where $i, j \in I_g \setminus \{a, b\}$, $i \neq j$. The coefficients $(ab)^{ij}_{kl}$ are defined by setting $(ab)^{ij}_{ij} := 1$,

$$(ab)^{ij}_{kl} := \frac{\kappa[v_1^{(ab)}, \dots, \check{\sigma}_k \check{\sigma}_l, \sigma_i \sigma_j, \dots, v_N^{(ab)}]}{\kappa[v_1^{(ab)}, \dots, v_N^{(ab)}]} , \tag{5.13}$$

if $k \neq l$ and $\sigma_k \sigma_l \in \{v_i^{(ab)}\}_{i \in I_N}$, and $(ab)^{ij}_{kl} := 0$ for all the other $(k, l) \in I_g \times I_g$. In this notation, the coefficients C^{σ}_{ij} defined in (5.8), with $N < i \leq M$, $j \in I_M$, correspond to $(12)^{1_i 2_i}_{1_j 2_j}$. Eqs.(5.12) and (5.13) can be derived by a trivial generalization of the same construction considered in section 2 in the particular case $a = 1$, $b = 2$.

Proposition 5.6. *The coefficients $(ab)^{ij}_{kl}$ satisfy the following consistency conditions*

$$(ij)^{ab}_{kl} = \sum_{m \leq n} (ij)^{ab}_{mn} (ab)^{mn}_{kl} = \sum_{m \leq n} (ij)^{ab}_{mn} (ai)^{mn}_{kl} = \sum_{m \leq n} (ij)^{ab}_{mn} (aj)^{mn}_{kl} \tag{5.14}$$

$$= \sum_{m \leq n} (ij)^{ab}_{mn} (bi)^{mn}_{kl} = \sum_{m \leq n} (ij)^{ab}_{mn} (bj)^{mn}_{kl} , \tag{5.15}$$

for all $i, j, a, b \in I_g$ pairwise distinct, and for all $k, l \in I_g$.

Proof. Choose $i, j, a, b \in I_g$, with $a < b < i < j$, and consider the relations $\sum_{k \leq l} (ij)^{ab}_{kl} \sigma_k \sigma_l = 0$ and $\sum_{k \leq l} (ab)^{ij}_{kl} \sigma_k \sigma_l = 0$, that is

$$0 = (ij)^{ab}_{ij} \sigma_i \sigma_j + \sigma_a \sigma_b + (ij)^{ab}_{ai} \sigma_a \sigma_i + (ij)^{ab}_{aj} \sigma_a \sigma_j + (ij)^{ab}_{bi} \sigma_b \sigma_i \tag{5.16}$$

$$+ (ij)^{ab}_{bj} \sigma_b \sigma_j + \sum_{k \neq a,b,i,j} (ij)^{ab}_{ik} \sigma_i \sigma_k + \sum_{k \neq a,b,i,j} (ij)^{ab}_{jk} \sigma_j \sigma_k , \tag{5.17}$$

$$0 = \sigma_i \sigma_j + (ab)^{ij}_{ab} \sigma_a \sigma_b + (ab)^{ij}_{ai} \sigma_a \sigma_i + (ab)^{ij}_{aj} \sigma_a \sigma_j + (ab)^{ij}_{bi} \sigma_b \sigma_i \tag{5.18}$$

$$+ (ab)^{ij}_{bj} \sigma_b \sigma_j + \sum_{k \neq a,b,i,j} (ab)^{ij}_{ak} \sigma_a \sigma_k + \sum_{k \neq a,b,i,j} (ab)^{ij}_{bk} \sigma_b \sigma_k . \tag{5.19}$$

Replace the differentials $\sigma_i\sigma_k$ and $\sigma_j\sigma_k$, $k \neq i, j, a, b$, in Eq.(5.16) by

$$\sigma_i\sigma_k = -\sum_{\substack{m\leq n\\(m,n)\neq(i,k)}} (ab)^{ij}_{mn}\sigma_m\sigma_n \ , \quad k \neq i,j,a,b \ ,$$

and the analogous expression for $\sigma_j\sigma_k$. Then multiply Eq.(5.18) by $(ij)^{ab}_{ij}$ and consider the difference between (5.16) and (5.18). We obtain

$$0 = \Big((ij)^{ab}_{ab} - \sum_{m\leq n}(ij)^{ab}_{mn}(ab)^{mn}_{ab}\Big)\sigma_a\sigma_b + \sum_{k\neq a,b}\Big((ij)^{ab}_{ak} - \sum_{m\leq n}(ij)^{ab}_{mn}(ab)^{mn}_{ak}\Big)\sigma_a\sigma_k \tag{5.20}$$

$$+ \sum_{k\neq a,b}\Big((ij)^{ab}_{bk} - \sum_{m\leq n}(ij)^{ab}_{mn}(ab)^{mn}_{bk}\Big)\sigma_b\sigma_k \ . \tag{5.21}$$

Since the holomorphic quadratic differentials appearing in Eq.(5.20) are linearly independent, it follows that each coefficient vanishes, yielding the first identity in (5.14), in the cases in which at least one between k and l is equal to a or b. On the other hand, in the case $k, l \neq a, b$, the only non-vanishing term in the sum $\sum_{m\leq n}(ij)^{ab}_{mn}(ab)^{mn}_{kl}$ is $(ij)^{ab}_{kl}(ab)^{kl}_{kl} = (ij)^{ab}_{kl}$, and the first identity in (5.14) follows. The other identities can be proved by applying the analogous procedure to the relation $\sum_{k\leq l}(ij)^{ab}_{kl}\sigma_k\sigma_l = 0$ and one of the relations $\sum_{k\leq l}(ai)^{bj}_{kl}\sigma_k\sigma_l = 0$, $\sum_{k\leq l}(bi)^{aj}_{kl}\sigma_k\sigma_l = 0$, and so on. □

5.2 A correspondence between quadrics and θ-identities

Theorem 5.7. *Fix $(p_1,\ldots,p_g) \in C^g\backslash\mathcal{B}$ with $\mathcal{B}$ defined in Definition 4.2. Then, the associated holomorphic quadratic differentials v_i, $i \in I_M$, satisfy*

$$v_i = \sum_{j=1}^{M} X^{\omega}_{ji}\,\omega\omega_j \ , \tag{5.22}$$

$i \in I_N$, where

$$X^{\omega}_{ij} = \frac{\theta_{\mathbf{1}_j}(\mathrm{a}_{\mathbf{1}_i})\theta_{\mathbf{2}_j}(\mathrm{a}_{\mathbf{2}_i}) + \theta_{\mathbf{1}_j}(\mathrm{a}_{\mathbf{2}_i})\theta_{\mathbf{2}_j}(\mathrm{a}_{\mathbf{1}_i})}{(1+\delta_{\mathbf{1}_j\mathbf{2}_j})\sum_{l,m}\theta_l(\mathrm{a}_{\mathbf{1}_i})\theta_m(\mathrm{a}_{\mathbf{2}_i})\omega_l(p_{\mathbf{1}_i})\omega_m(p_{\mathbf{2}_i})} \ , \tag{5.23}$$

$i, j \in I_M$, with a_i as in Definition 4.1, correspond to the coefficients defined in (5.4) *for $\eta_i \equiv \omega_i$, $i \in I_g$. Furthermore, the $M-N$ independent linear relations*

$$\sum_{j=1}^{M} C^{\omega}_{ij}\omega\omega_j = 0 \ , \tag{5.24}$$

$N+1 \leq i \leq M$, hold, where

$$C^{\omega}_{ij} = \sum_{k_1,\ldots,k_N=1}^{M} |X^{\omega}|^{1\ \ldots\ Ni}_{k_1\ldots k_N j}\,\frac{\kappa[\omega\omega_{k_1},\ldots,\omega\omega_{k_N}]}{\kappa[v]} \ . \tag{5.25}$$

correspond to the coefficients defined in (5.8).

Proof. Eq.(4.9) implies that Eq.(5.23) is equivalent to (5.4), and the theorem follows by Theorem 5.3. □

Remark 5.1. Choose $p_1, \ldots, p_g$ as in Corollary 4.5, with $n = 1$ and set $q := q_1$. Then, there exists a non-trivial relation

$$a\sigma_1 t_2 + b\sigma_2 t_1 + c\sigma_1\sigma_2 = 0 \ ,$$

where $a, b, c \in \mathbb{C}$. Without loss of generality, we can assume that $t_1(p_1) = 0$ and $t_2(p_2) = 0$. Set

$$(\sigma_1) = p_2 + p_3 + \ldots + p_g + q + \sum_{i=1}^{g-2} r_i \ ,$$

and

$$(\sigma_2) = p_1 + p_3 + \ldots + p_g + q + \sum_{i=1}^{g-2} s_i \ ,$$

for some $r_i, s_i \in C$, $i \in I_{g-2}$. Then, $(t_1) > p_1 + \sum_{i=1}^{g-2} r_i$ and $(t_2) > p_2 + \sum_{i=1}^{g-2} s_i$, so that

$$\begin{aligned} t_1 &\sim -\frac{\theta(p_1 + \sum_i r_i + z - y)}{\sigma(y)E(y,z)E(y,p_1)\prod_i E(y,r_i)}\sigma(z)E(z,p_1)\prod_i E(z,r_i) \\ &\sim -\frac{\theta(p_2 + b + q + y - p_1 - z)}{\sigma(y)E(y,z)E(y,p_1)\prod_i E(y,r_i)}\sigma(z)E(z,p_1)\prod_{i=3}^{g} E(z,r_i) \\ &= \sum_{i=1}^{g} \theta_{\Delta,i}(p_2 + b + q - p_1)\omega_i(z) \ , \end{aligned}$$

where, in the second line, we used $I(\sum_i r_i) = I(-p_2 - b - q)$ in $J_0(C)$. An analogous calculation yields

$$t_2 \sim \sum_{i=1}^{g} \theta_i(p_1 + b + q - p_2)\omega_i(z) \ .$$

(By the symbol $\sim$, we denote the equality up to a factor independent of z; such a factor is not meaningful, since it can be compensated by a redefinition of the constants a, b.)

Theorem 5.8. *Let C be a canonical curve of genus $g \geq 4$ and $\{\omega_i\}_{i\in I_g}$ the canonically normalized basis of $H^0(K_C)$, and fix the points $(p_1, \ldots, p_g) \in C^g \setminus \mathcal{B}$. Then, the following $(g-2)(g-3)/2$ independent relations*

$$\begin{aligned} \sum_{s\in\mathcal{P}_{2g}} \epsilon(s) \det\omega(x_{s_1}, \ldots, x_{s_g}) \det\omega(x_{s_g}, \ldots, x_{s_{2g-1}}) \qquad (5.26) \\ \cdot \det\omega(x_{s_1}, x_{s_{g+1}}, x_{s_{2g}}, p_3, \ldots, \check{p}_i, \ldots, p_g) \\ \cdot \det\omega(x_{s_2}, x_{s_{g+2}}, x_{s_{2g}}, p_3, \ldots, \check{p}_j, \ldots, p_g) \\ \cdot \prod_{k=3}^{g-1} \det\omega(x_{s_k}, x_{s_{k+g}}, p_3, \ldots, p_g) = 0 \ , \end{aligned}$$

$3 \leq i < j \leq g$, hold for all $x_k \in C$, $1 \leq k \leq 2g$.

Proof. Fix i, j, $3 \le i < j \le g$, and choose p_1, p_2 in such a way that $\{\sigma_i\}_{i\in I_g}$ is a basis of $H^0(K_C)$. Observe that, due to Eq.(5.2), $\det_I \sigma\sigma(x_1, \ldots, x_{2g}) = 0$, for all $x_1, \ldots, x_{2g} \in C$, where $I := I_{M,2} \cup \{m(i,j)\}$. Applying Lemma 2.3, with $n = 2$, such an identity corresponds to Eq.(5.26) with the canonical basis $\{\omega_i\}_{i\in I_g}$ of $H^0(K_C)$ replaced by $\{\sigma_i\}_{i\in I_g}$. Eq.(5.26) is then obtained by simply changing the base. $\square$

The relations of Theorem 5.8 can be directly expressed in terms of theta functions. (The conditions on the points $p_3, \ldots, p_g$ in Theorem 5.9 and Corollary 5.10 can be safely replaced by one of the equivalent conditions *iv*), *v*), *vi*), and *vii*) of Theorem 6.2.)

Theorem 5.9. *Fix $p_3, \ldots, p_g \in C$ in such a way that $(p_1, p_2, p_3, \ldots, p_g) \notin \mathcal{B}$ for some $p_1, p_2 \in C$. The following $(g-2)(g-3)/2$ independent relations*

$$\begin{aligned} V_{i_1 i_2}(p_3, \ldots, p_g, x_1, \ldots, x_{2g}) :=& \qquad\qquad\qquad\qquad\qquad\qquad (5.27)\\ \sum_{s\in\mathcal{P}_{2g}} \epsilon(s)\Bigg\{ &\prod_{k=1}^{2} \frac{S(\hat{x}_k + \hat{x}_{g+k} + \hat{x}_{2g} + \mathrm{b}_{i_k}) E(\hat{x}_k, \hat{x}_{2g}) E(\hat{x}_{k+g}, \hat{x}_{2g})}{E(\hat{x}_k, p_{i_k}) E(\hat{x}_{k+g}, p_{i_k}) E(\hat{x}_{2g}, p_{i_k})} \\ &\cdot \prod_{k=1}^{g-1} \big(E(\hat{x}_k, \hat{x}_{k+g}) \prod_{j=3}^{g} E(\hat{x}_k, p_j) E(\hat{x}_{k+g}, p_j)\big) \\ &\cdot S\big(\sum_{k=1}^{g} \hat{x}_k\big) \prod_{\substack{k,j=1\\k<j}}^{g} E(\hat{x}_k, \hat{x}_j) S\big(\sum_{k=g}^{2g-1} \hat{x}_k\big) \prod_{\substack{k,j=g\\k<j}}^{2g-1} E(\hat{x}_k, \hat{x}_j) \\ &\cdot \prod_{k=3}^{g-1} S(\hat{x}_k + \hat{x}_{k+g} + \mathrm{b}) \prod_{j=3}^{g} E(\hat{x}_{2g}, p_j)^2 \Bigg\} = 0 \ , \end{aligned}$$

$3 \le i_1 < i_2 \le g$, where $\hat{x}_i := x_{s_i}$, $i \in I_{2g}$, $\mathrm{b}_i := \mathrm{b} - p_i$, $3 \le i \le g$, hold for all $x_i \in C$, $i \in I_{2g}$.

Proof. By (3.2) $V_{ij}(p_3, \ldots, p_g, x_1, \ldots, x_{2g})$ is equivalent to (5.26). $\square$

Remark 5.2. Note that $V_{ii} \neq 0$ for $i = 3, \ldots, g$, since for $i = j$ the LHS of (5.26) is proportional to a determinant of $2g$ linearly independent holomorphic quadratic differentials on C, evaluated at general points $x_i \in C$, $i \in I_{2g}$.

By a limiting procedure we derive the original Petri's relations, now written in terms of the canonical basis $\{\omega_i\}_{i\in I_g}$ of $H^0(K_C)$ and with the coefficients expressed in terms of theta functions.

Corollary 5.10. *Fix $p_3, \ldots, p_g \in C$ in such a way that $(p_1, p_2, p_3, \ldots, p_g) \notin \mathcal{B}$ for some $p_1, p_2 \in C$. The following $(g-2)(g-3)/2$ linearly independent relations*

$$\sum_{j=1}^{M} C^{\omega}_{ij} \omega\omega_j(z) := \frac{\kappa[\sigma]}{\kappa[v]}^{g+1} F(p, x) \frac{V_{1_i 2_i}(p_3, \ldots, p_g, x_1, \ldots, x_{2g-1}, z)}{\theta\big(\sum_1^{2g-1} x_j + \mathrm{b}\big)} = 0 \ , \tag{5.28}$$

$N+1 \leq i \leq M$, where

$$F(p,x) := c'_{g,2} \frac{\prod_{\substack{j,k=3\\ j<k}}^{g} E(p_j,p_k)^{g-4} \prod_{\substack{j=3\\ j\neq 1_i}}^{g} E(p_{1_i},p_j) \prod_{\substack{j=3\\ j\neq 2_i}}^{g} E(p_{2_i},p_j)}{\prod_{j=1}^{2g-1}(\sigma(x_j)\prod_{k=3}^{g} E(x_j,p_k)\prod_{k=j+1}^{2g-1} E(x_j,x_k))} \ ,$$

hold for all $z \in C$. Furthermore, C^{ω}_{ij} are independent of $p_1, p_2, x_1, \ldots, x_{2g-1} \in C$ and correspond to the coefficients defined in (5.8) *(with $\eta_i \equiv \omega_i$, $i \in I_g$) or, equivalently, in* (5.25).

Proof. Consider the identity

$$\frac{\det_I \sigma\sigma(x_1,\ldots,x_{2g-1},z)}{\det v(p_3,\ldots,p_g,x_1,\ldots,x_{2g-1})} = 0 \ , \tag{5.29}$$

$I := I_{M,2} \cup \{i\}$, $N+1 \leq i \leq M$. Upon applying Lemma 2.3, with $n = 2$, and Eq.(3.2) to the numerator and Eq.(3.3) to the denominator of (5.29), Eq.(5.28) follows by a trivial computation. On the other hand, for arbitrary points $z, y_1, \ldots, y_{g-1} \in C$,

$$S(y_1+\ldots+y_{g-1}+z) = \frac{\sum_{i=1}^{g} \theta_i(y_1+\ldots+y_{g-1})\omega_i(z)}{\sigma(z)\prod_1^{g-1} E(z,y_i)} \ .$$

Consider $V_{1_i 2_i}(p_3,\ldots,p_g,x_1,\ldots,x_{g-1},z)$ and replace each term in of the form $S(d_{g-1}+z)$ by its expression above, for any effective divisor d_{g-1} of degree $g-1$. The dependence on z only enters through $\omega_i\omega_j(z)$ and the relations (5.28) can be written in the form of Eq.(5.7).

To prove that C^{ω}_{ij} are the coefficients in (5.8), with $\eta_i \equiv \omega_i$, $i \in I_g$, first consider the identity

$$\frac{\kappa[\omega\omega_{k_1},\ldots,\omega\omega_{k_N}]}{\kappa[v]} = \frac{\det_{i\in\{k_1,\ldots,k_N\}} \omega\omega_i(p_3,\ldots,p_g,x_1,\ldots,x_{2g-1})}{\det v(p_3,\ldots,p_g,x_1,\ldots,x_{2g-1})} \ ,$$

then recall that

$$v_i := \sigma\sigma_i = \sum_{j=1}^{M} X^{\omega}_{ji}\omega\omega_j \ ,$$

$i \in I_M$, so that one obtains

$$\sum_{k_1,\ldots,k_N,j=1}^{M} |X^{\omega}|^{1\ \ldots\ N i}_{k_1\ldots k_N j} \frac{\kappa[\omega\omega_{k_1},\ldots,\omega\omega_{k_N}]}{\kappa[v]}\omega\omega_j(z) = \frac{\det_I \sigma\sigma(x_1,\ldots,x_{2g-1},z)}{\det v(p_3,\ldots,p_g,x_1,\ldots,x_{2g-1})} \ ,$$

as an algebraic identity (in the sense that it holds as an identity in $\mathrm{Sym}^2(H^0_C(K))$ after replacing $\sigma_i\sigma_j \to \sigma_i\cdot\sigma_j$ and $\omega_i\omega_j \to \omega_i\cdot\omega_j$, $i,j \in I_g$). Hence, the coefficients of $\omega\omega_j(z)$ on the LHS, given by (5.8) or, equivalently, by (5.25) and the ones on the RHS, given by (5.28), are the same.

Eq.(5.8) explicitly shows that the coefficients C^{ω}_{ij} are independent of x_i for all $i = 1,\ldots,2g-1$. By (5.28) it follows that they may depend on p_1 and p_2

only through the term $\kappa[\sigma]^{g+1}/\kappa[v]$. The dependence of $\kappa[\sigma]$ and $\kappa[v]$ on p_1 and p_2 is due to the dependence of the basis $\{\sigma_i\}_{i\in I_g}$ and $\{v_i\}_{i\in I_N}$ on the choice of $p_1,\ldots,p_g \in C$. On the other hand, Eq.(6.4) implies that $\kappa[\sigma]^{g+1}/\kappa[v]$ is independent of p_1, p_2 and the proof of the corollary is complete. □

5.3 Relations among holomorphic cubic differentials

According to Petri's Theorem, in the most general case the ideal of a canonical curve C is generated by its ideals of quadrics together with the ideal of cubics. As discussed in the introduction of this section, such cubics correspond to linear relations among holomorphic 3-differentials on C; a generalization of the previous construction is necessary in order to explicitly determine such relations.

Fix $p_1,\ldots,p_g \in C$ satisfying the conditions *i*), *ii*) and *iii*) of Proposition 4.6 with respect to some fixed i, $3\le i\le g$, and let $\{\varphi_j\}_{j\in I_{N_3-1}}\cup\{\varphi_{i+5g-8}\}$ be the corresponding basis of $H^0(K_C^3)$. The kernel of the canonical epimorphism from $\mathrm{Sym}^3 H^0(K_C)$ onto $H^0(K_C^3)$ has dimension $(g-3)(g^2+6g-10)/6$, and each element corresponds to a linear combination of the following relations

$$\sigma_j\sigma_k\sigma_l = \sum_{m\in I_{N_3-1}} B_{jkl,m}\varphi_m + B_{jkl,i+5g-8}\sigma_2\sigma_i^2 \ , \tag{5.30}$$

$3\le j,k,l\le g$, $j\ne k$, and

$$\sigma_2\sigma_j^2 = \sum_{m\in I_{N_3-1}} B_{2jj,m}\varphi_m + B_{2jj,i+5g-8}\sigma_2\sigma_i^2 \ , \tag{5.31}$$

$3\le j\le g$, $j\ne i$, where $B_{jkl,m}, B_{2jj,m}\in\mathbb{C}$, are suitable coefficients. On the other hand, a trivial computation shows that the relations (5.30) are generated by (5.31) and by the relations among holomorphic quadratic differentials,

$$\sum_{j=1}^{M} C^{\sigma}_{kj}\sigma\sigma_j = 0 \ , \tag{5.32}$$

$k = N+1,\ldots,M$. Therefore, relations among holomorphic 3-differentials, modulo relations among holomorphic quadratic differentials, provide at most $g-3$ independent conditions on products of elements of $H^0(K_C)$.

The relations (5.31) can be restated in terms of an arbitrary basis $\{\eta_j\}_{j\in I_g}$ of $H^0(K_C)$. Let Y^η be the automorphism of $\mathbb{C}^{M_3}$, determined by

$$Y^\eta_{kj} := \chi_k^{-1}([\eta]^{-1}[\eta]^{-1}[\eta]^{-1})_{jk} \ , \tag{5.33}$$

$j,k\in I_{M_3}$, so that

$$\varphi_j = \sum_{k=1}^{M_3} Y^\eta_{kj}\eta\eta\eta_k \ ,$$

$j\in I_{M_3}$. Consider the following determinants of d-dimensional submatrices of Y^η

$$|Y^\eta|^{j_1\ldots j_d}_{i_1\ldots i_d} := \det\begin{pmatrix} Y^\eta_{i_1j_1} & \ldots & Y^\eta_{i_1j_d} \\ \vdots & \ddots & \vdots \\ Y^\eta_{i_dj_1} & \ldots & Y^\eta_{i_dj_d}\end{pmatrix} \ ,$$

$i_1,\ldots,i_d,j_1,\ldots,j_d\in I_{M_3}$, $d\in I_{M_3}$.

Proposition 5.11.

$$\sum_{j=1}^{M_3} D^\eta_{kj}\eta\eta\eta_j = 0 \ , \tag{5.34}$$

$N_3 \leq k \leq N_3 + g - 3$, $k \neq i$ where

$$D^\eta_{kj} := \sum_{k_1,\ldots,k_{N_3}=1}^{M_3} |Y^\eta|^{1\ldots(N_3-1)\, i\, k}_{k_1\ \ldots\ \ k_{N_3}\ j} \ \frac{\kappa[\eta\eta\eta_{k_1},\ldots,\eta\eta\eta_{k_{N_3}}]}{\kappa[\varphi]} \ , \tag{5.35}$$

$j \in I_{M_3}$, are $g-3$ independent linear relations among holomorphic 3-differentials.

Proof. Without loss of generality, we can assume $i = N_3$; such an assumption can always be satisfied after a re-ordering of the points $p_3, \ldots, p_g$. Fix $N_3 + 1$ arbitrary points $x_1, \ldots, x_{N_3}, x_{N_3+1} \equiv z \in C$ and consider the singular matrix $[\varphi_l(x_m)]_{\substack{l\in I \\ m\in I_{N_3+1}}}$ with $I := I_{N_3} \cup \{k\}$, with $N_3 < k \leq N_3 + g - 3$. By expressing the determinant with respect to the column $(\varphi_l(z))_{l\in I}$, the identity $\det \varphi_l(x_m) = 0$, $l \in I$, $m \in I_{N_3+1}$, yields

$$\sum_{m=1}^{M_3}\left[\sum_{l=1}^{N_3}(-)^{l+1}\frac{\kappa[\varphi_1,\ldots,\check{\varphi}_l,\ldots,\varphi_{N_3},\varphi_k]}{\kappa[\varphi]}Y^\eta_{ml} - Y^\eta_{mk}\right]\eta\eta\eta_m = 0 \ .$$

The proposition follows by combinatorial identities analogous to the proof of Theorem 5.3. $\square$

Whereas for $g = 4$ the relations (5.34) are independent of the relation among holomorphic quadratic differentials, for $g \geq 5$, (5.34) are generated by (5.32) in all but some particular curves. Set $\tilde\psi_{1_i 2_i, 1_j 2_j} := \tilde\psi_{ij}$ and $C^\sigma_{1_i 2_i, 1_j 2_j} := C^\sigma_{ij}$, $N+1 \leq i \leq M$, $j \in I_M$. Consider the 3-differentials $\sigma_i\sigma_j\sigma_k$ with $3 \leq i < j < k \leq g$ ($g \geq 5$). By Eq.(5.32) and by $C^\sigma_{ij} = \tilde\psi_{ij} - \delta_{ij}$, $N+1 \leq i \leq M$, $j \in I_M$,

$$\sigma_i\sigma_j\sigma_k = \sum_{m=1}^{2}\sum_{n=3}^{g}\tilde\psi_{ij,mn}\sigma_m\sigma_n\sigma_k + \tilde\psi_{ij,12}\sigma_1\sigma_2\sigma_j \ ,$$

so that

$$\begin{aligned}
&\sum_{m,p=1}^{2}\sum_{q=3}^{g}\Big(\sum_{\substack{n=3\\ n\neq j}}^{g}\tilde\psi_{ik,mn}\tilde\psi_{nj,pq}\Big)\sigma_m\sigma_p\sigma_q + \tilde\psi_{ik,12}\sigma_1\sigma_2\sigma_k + \sum_{m=1}^{2}\tilde\psi_{ik,mj}\sigma_m\sigma_j^2 \\
= &\sum_{m,p=1}^{2}\sum_{q=3}^{g}\Big(\sum_{\substack{n=3\\ n\neq i}}^{g}\tilde\psi_{jk,mn}\tilde\psi_{ni,pq}\Big)\sigma_m\sigma_p\sigma_q + \tilde\psi_{jk,12}\sigma_1\sigma_2\sigma_k + \sum_{m=1}^{2}\tilde\psi_{jk,mi}\sigma_m\sigma_i^2 \ .
\end{aligned}$$

The above equation yields

$$\begin{aligned}
C^\sigma_{ik,2j}\sigma_2\sigma_j^2 = &\sum_{m,p=1}^{2}\sum_{q=3}^{g}\Big(\sum_{\substack{n=3\\ n\neq i}}^{g}C^\sigma_{jk,mn}C^\sigma_{ni,pq} - \sum_{\substack{n=3\\ n\neq j}}^{g}C^\sigma_{ik,mn}C^\sigma_{nj,pq}\Big)\sigma_m\sigma_p\sigma_q \\
&+ C^\sigma_{jk,12}\sigma_1\sigma_2\sigma_k - C^\sigma_{ik,12}\sigma_1\sigma_2\sigma_k + C^\sigma_{jk,1i}\sigma_1\sigma_i^2 \\
&- C^\sigma_{ik,1j}\sigma_1\sigma_j^2 + C^\sigma_{jk,2i}\sigma_2\sigma_i^2 \ .
\end{aligned}$$

If $C^{\sigma}_{ik,2j} \neq 0$ for some k, the above identity shows that the relation (5.31) is generated by Eqs.(5.32). On the other hand, it can be proved [3] that if $C^{\sigma}_{ik,2j} = 0$ for all $3 \leq k \leq g$, $k \neq i, j$, the relation (5.31) is independent of the relations among holomorphic quadratic differentials. This case occurs if and only if the curve C is trigonal or a smooth quintic.

Proposition 5.12. *Fix g points $p_1, \ldots, p_g \in C$ satisfying the conditions of theorem 4.6. The coefficients Y^{ω}_{ij}, defined in Eq.(5.33) with $\eta \equiv \omega$, are given by*

$$Y^{\omega}_{ij} = \frac{(1 + \delta_{\mathfrak{1}_j \mathfrak{2}_j} + \delta_{\mathfrak{2}_j \mathfrak{3}_j})(1 + \delta_{\mathfrak{1}_j \mathfrak{3}_j})}{6 \prod_{\mathfrak{m} \in \{\mathfrak{1},\mathfrak{2},\mathfrak{3}\}} \sum_l \theta_l(a_{\mathfrak{m}_i}) \omega_l(p_{\mathfrak{m}_i})} \sum_{s \in \mathcal{P}_3} \Big(\prod_{\mathfrak{m} \in \{\mathfrak{1},\mathfrak{2},\mathfrak{3}\}} \theta_{s(\mathfrak{m})_j}(a_{\mathfrak{m}_i}) \Big) , \qquad (5.36)$$

$i, j \in I_M$.

Proof. The proposition follows immediately by the definition (5.33) and by Eq.(4.9). ☐

6. THE SECTION K

6.1 Definition and fundamental properties

Definition 6.1. For all $p_3, \dots, p_g, x_1, \dots, x_{2g-1} \in C$, set

$$K(p_3, \dots, p_g) := \frac{1}{\theta\big(b + \sum_1^{2g-1} x_i\big) \prod_1^{2g-1} \sigma(x_i) \prod_{i=3}^g \sigma(p_i)} \cdot \sum_{s \in \mathcal{P}_{2g-1}} \frac{S\big(\sum_{i=1}^g x_{s_i}\big) S\big(\sum_{i=g}^{2g-1} x_{s_i}\big)}{\prod_{i=3}^g E(x_{s_g}, p_i)} \prod_{i=1}^{g-1} \frac{S(x_{s_i} + x_{s_{i+g}} + b)}{\prod_{\substack{j=1 \\ j \neq i}}^{g-1} E(x_{s_i}, x_{s_{j+g}})} \ . \tag{6.1}$$

Theorem 6.1. *For all $p_3, \dots, p_g \in C$, the following properties hold:*

a. *$K \equiv K(p_3, \dots, p_g)$ is independent of $x_1, \dots, x_{2g-1} \in C$.*

b. *For any $p_1, \dots, p_g \in C$ such that $\det \eta_i(p_j) \neq 0$, the set $\{v_i\}_{i \in I_N}$, defined in (4.14), is a basis of $H^0(K_C^2)$ if and only if $K \neq 0$.*

c.

$$S(p_1 + p_2 + b) = 0 \ , \ \forall p_1, p_2 \in C \qquad \Longrightarrow \qquad K = 0 \ . \tag{6.2}$$

d. *If $p_3, \dots, p_g$ are pairwise distinct and $K \neq 0$, then there exist $p_1, p_2 \in C$ such that $H \neq 0$.*

Proof. – *a.* The ratio

$$\frac{H}{K} = S(\mathrm{a})^{5g-7} E(p_1, p_2)^{g+1} \prod_{i=3}^g (E(p_1, p_i) E(p_2, p_i))^4 \prod_{\substack{i,j=3 \\ i<j}}^g E(p_i, p_j)^5 \ , \tag{6.3}$$

is independent of $x_1, \dots, x_{2g-1}$, so that *a*) follows by Theorem 4.12 or, equivalently, noticing that by Eqs.(4.11)(4.25)(4.27) and (6.3)

$$K(p_3, \dots, p_g) := (-)^{g+1} c_{g,2} \frac{\kappa[v]}{\kappa[\sigma]^{g+1}} \prod_{\substack{i,j=3 \\ i<j}}^g E(p_i, p_j)^{2-g} \prod_{i=3}^g \sigma(p_i)^{3-g} \ . \tag{6.4}$$

– *b.* By (3.2) and (6.3) the condition $\det \eta_i(p_j) \neq 0$ implies $H/K \neq 0$. In this case $K \neq 0$ if and only if $H \neq 0$, and *b*) follows by Theorem 4.12.

– *c.* If $S(p_1 + p_2 + b) = 0$, for all $p_1, p_2 \in C$, then the numerators in each term of the sum in (6.1) vanish for all $x_1, \dots, x_{2g-1} \in C$. Since K is independent of $x_1, \dots, x_{2g-1}$, it follows that the proof of point *c*) is equivalent to prove that there exist $x_1, \dots, x_{2g-1} \in C$ such that the denominators in (6.1) do not vanish. On the other hand, the possible zeros of such denominators are the ones corresponding of the zeros of the primes forms, which are avoided by simply choosing $p_3,\ldots,p_g,x_1,\ldots,x_{2g-1}$ pairwise distinct, and the ones of $\theta(b+\sum_1^{2g-1}x_i)$. Fix an arbitrary $y\in C$ and set $w:=I(b+\sum_{g+1}^{2g-1}x_i+y)$. Then

$$\theta\big(b+\sum_1^{2g-1}x_i\big)=\theta\big(w+\sum_1^g x_i-y\big)\ ,$$

and, by the Jacobi Inversion Theorem, by varying the points $x_1,\ldots,x_g\in C$ one can span the whole Jacobian variety. Then, one can always choose $x_1,\ldots,x_{2g-1}$ pairwise distinct and distinct from $p_3,\ldots,p_g$ in such a way that $\theta\big(w+\sum_1^g x_i-y\big)\neq 0$, so that the denominator does not vanish and *c*) follows.

–d. Since $K\neq 0$, by *c*) there exist $p_1,p_2\in C$ such that $S(p_1+p_2+b)\neq 0$. By continuity arguments, it follows that there exist some neighborhoods $U_i\subset C$ of p_i, $i=1,2$, such that $S(x_1+x_2+b)\neq 0$ for all $(x_1,x_2)\in U_1\times U_2$. Hence, we can choose p_1,p_2 so that $S(p_1+p_2+b)\neq 0$ and $p_1,\ldots,p_g$ are pairwise distinct. Then, by Eq.(6.3), $H/K\neq 0$ and, since $K\neq 0$, we conclude that $H\neq 0$.

□

In view of Eq.(6.4), it is useful to define

$$\begin{aligned}k(p_3,\ldots,p_g):=&K(p_3,\ldots,p_g)\prod_{\substack{i,j=3\\ i<j}}^{g}E(p_i,p_j)^{g-2}\prod_{i=3}^{g}\sigma(p_i)^{g-3}\\ =&(-)^{g+1}c_{g,2}\frac{\kappa[v]}{\kappa[\sigma]^{g+1}}\ ,\end{aligned}\tag{6.5}$$

which is a holomorphic $(g-3)$-differential in each of its $g-2$ arguments.

Theorem 6.2. *Fix $p_1,\ldots,p_g\in C$, with C non-hyperelliptic of genus $g\geq 4$ and let $\{\hat\sigma_i\}_{i\in I_g}$ be a set of non-vanishing holomorphic 1-differentials such that $i\neq j\ \Rightarrow\ \hat\sigma_i(p_j)=0$, for all $i,j\in I_g$. The following statements are equivalent*

i) *The conditions*

  i′) $(p_1,\ldots,p_g)\notin\mathcal{A}$;

  i″) $b:=\sum_{i=3}^g p_i$ *is the greatest common divisor of* (σ_1) *and* (σ_2);

  are satisfied;

ii) $H(p_1,\ldots,p_g)\neq 0$, *where H is defined in Eq.*(4.25);

iii) $\{\hat v_i\}_{i\in I_N}$ *is a basis of* $H^0(K_C^2)$, *with* $\hat v_i:=\hat\sigma\hat\sigma_i$, $i\in I_M$.

More generally, fix $p_3,\ldots,p_g\in C$. The following statements are equivalent:

iv) $p_3,\ldots,p_g$ *are pairwise distinct and* $\{I(p+b)|p\in C\}\cap\Theta_s=\varnothing$;

v) $p_3,\ldots,p_g$ *are pairwise distinct and* $K(p_3,\ldots,p_g)\neq 0$, *where K is defined in Eq.*(6.1);

vi) *There exist* $p_1, p_2 \in C$ *such that* $p_1, \ldots, p_g$ *satisfy i), ii) and iii);*

vii) *For all* $p \in C$, $S(x+p+b)$ *does not vanish identically as a function of* x*; furthermore, for each* $p_2 \in C \setminus \{p_3, \ldots, p_g\}$*, the points* $p_1, \ldots, p_g$ *satisfy i), ii) and iii) if and only if* p_1 *is distinct from* $p_2, \ldots, p_g$ *and from the* $g-1$ *zeros of* $S(x+p_2+b)$.

Proof. *i)* $\Leftrightarrow$ *iii)* is proved in Proposition 4.4 (in the direction *i)* $\Rightarrow$ *iii)*, only the case of normalized 1-differentials $\sigma_i(p_i) = 1$, for all $i \in I_g$ is considered; however, by the hypothesis *i′)*, the general case can be reduced to this choice by a non-singular diagonal transformation on $\{\hat{\sigma}_i\}_{i \in I_g}$);

ii) $\Leftrightarrow$ *iii)* is proved in Theorem 4.12;

vii) $\Rightarrow$ *vi)* is obvious;

iv) $\Leftrightarrow$ *vii)* follows by first noting that $S(x+p+b)$ identically vanishes as a function of x if and only if $I(p+b) \in \Theta_s$, and then by Theorem 4.10; in particular, in such a theorem it is proved that for each fixed $p_2 \in C \setminus \{p_3, \ldots, p_g\}$, the points $p_1, \ldots, p_g$ satisfy *i)* if and only if the conditions $p_1 \notin \{p_2, \ldots, p_g\}$, $S(p_1+p_2+b) \neq 0$ and *iv)* hold;

vi) $\Rightarrow$ *iv)* also follows by Theorem 4.10, where it is proved that if *iv)* does not hold, then *i″)* cannot be satisfied;

v) $\Leftrightarrow$ *vi)* follows by Corollary 6.1, where it is proved that *i′)* and *v)* are equivalent to *ii)* and that if *v)* holds, then there exist $p_1, p_2 \in C$ such that $p_1, \ldots, p_g$ satisfy *ii)*.

□

6.2 Zeros of K and the singular locus Θ_s

The function $K(p_3, \ldots, p_g)$ defined in Eq.(6.1), whose zero divisor is characterized in the theorem above, is the fundamental tool in the proof of the following theorem. Such a result heavily relies on the properties of Θ_s in the case the sublying ppav is the Jacobian torus of a canonical curve. By the Riemann Singularity Theorem,

$$\Theta_s = W^1_{g-1} \equiv I(C^1_{g-1}) \ ,$$

where $C^1_{g-1} \subset C_{g-1}$ is the subvariety of codimension 2 in C_{g-1}, whose elements are the effective divisors of degree $g-1$ with index of specialty greater than 1. Note that each effective divisor $d \in C_{g-3}$ of degree $g-3$ canonically determines an embedding

$$\begin{aligned} \pi_d \colon C_2 &\hookrightarrow C_{g-1} \ , \\ c &\mapsto c + d \ , \end{aligned}$$

of C_2 as a subvariety of dimension 2 in C_{g-1}. Hence, by a simple dimensional counting, we expect the intersection $C^1_{g-1} \cap \pi_d(C_2)$ to have (in general) dimension 0. The following theorem shows that, in the general case in which such an intersection does not contain any component of dimension greater than 0, $C^1_{g-1} \cap \pi_d(C_2)$ corresponds (set-theoretically) to a set of $g(g-3)/2$ points; furthermore, a remarkable relation of such a set of points with the canonical divisor is given.

Since the restriction of the Abel-Jacobi map to C_2 is an injection (because C is non-hyperelliptic), such points are in one to one correspondence with the points in the intersection $\Theta_s \cap I(\pi_d(C_2))$.

Theorem 6.3. *Let C be non-hyperelliptic of genus $g \geq 4$ and fix an effective divisor $d \in C_{g-3}$ of degree $g-3$. Then, either:*

a. For each point $p \in C$, there exists a point $q \in C$ such that

$$I(p+q+d) \in \Theta_s \ ;$$

or:

b. There exist $k := g(g-3)/2$ effective divisors $c_1, \ldots, c_k \in C_2$ of degree 2, such that

$$e_i := I(c_i + d) \in \Theta_s \ , \qquad \forall i \in I_k \ . \tag{6.6}$$

Moreover, in this case

$$\sum_{i=1}^{k} c_i + (g-2)d = (g-3)K_C \ .$$

Proof. Set $d := \sum_{i=4}^{g} p_i$ and consider $K(z, p_4, \ldots, p_g)$ as a function of z. It vanishes at $z \equiv p$ if and only if there exists a point $q \in C$ such that $I(p+q+d) \in \Theta_s$. Then, $K = 0$ for all $z \in C$ if and only if statement *a*) holds.
Now, assume that $K(z, p_4, \ldots, p_g)$ does not vanish identically and consider

$$\phi(z) := K(z, p_4, \ldots, p_g) \prod_{i=4}^{g} E(z, p_i)^{g-2} \sigma(z)^{g-3} \ . \tag{6.7}$$

By (6.4), ϕ is a holomorphic $(g-3)$-differential on C. Therefore, the divisor e of $K(z, p_4, \ldots, p_g)$ is effective (K has no poles) of degree $g(g-3)$ and $e + (g-2)d$ is the divisor of a $(g-3)$-differential. It only remains to prove that e is the sum of all the effective divisors of degree 2 satisfying Eq.(6.6). By the equivalence of *iv*) and *v*) in Theorem 6.2, if $c := q_1 + q_2$ satisfies Eq.(6.6), then q_1 and q_2 are both zeros of K. By construction, $K(z, p_4, \ldots, p_g)$ can be written as

$$K(z, p_4, \ldots, p_g) = F(z, p_4, \ldots, p_g, x_1, \ldots, x_{2g-1}) \det \varphi_i(x_j) \ ,$$

with $\{\varphi_1, \ldots, \varphi_{2g-1}\}$ a set of generators (depending on $z, p_4, \ldots, p_g$) of $H^0(K_C^2 \otimes \mathcal{O}(-z-d))$ and $x_1, \ldots, x_{2g-1}$ arbitrary points in C; F is such that, by Corollary 6.1, K does not depend on $x_1, \ldots, x_{2g-1}$. It is easy to verify that K vanishes only if $\det \varphi_i(x_j) = 0$ for all $x_1, \ldots, x_{2g-1} \in C$; the multiplicity of such a zero is $2g-1-r$, where $r := h^0(K_C^2 \otimes \mathcal{O}(-z-d))$. The space $H^0(K_C^2 \otimes \mathcal{O}(-z-d))$ is generated by elements $\sigma_1\eta$, $\sigma_2\rho$, as η, ρ vary in $H^0(K_C)$; here, σ_1, σ_2 is a basis for the 2-dimensional space $H^0(K_C \otimes \mathcal{O}(-z-d))$ (note that if there exists $q \in C$ such that $h^0(K_C \otimes \mathcal{O}(-q-d)) > 2$, then $K(z, p_4, \ldots, p_g)$ identically vanishes). Proposition 4.4 shows that $K(z, p_4, \ldots, p_g) \neq 0$, that is $r = 2g-1$, if and only if $h^0(K_C \otimes \mathcal{O}(-q-z-d)) = 1$ for all $q \in C$. Let q_1 be a zero of K and denote by n the maximal integer for which there exist $n-1$ points $q_2, \ldots, q_n \in C$ such that $h^0(K_C \otimes \mathcal{O}(-q_1 - \ldots - q_n - d) = 2$. By the considerations above, since q_1

is a zero, $n \geq 2$; furthermore, $q_2, \ldots, q_n$ are zeros of K too. Corollary 4.5 shows that

$$r \equiv h^0(K_C^2 \otimes \mathcal{O}(-q_1 - d)) = h^0(K_C^2 \otimes \mathcal{O}(-q_1 - \ldots - q_n - d)) = 2g - n \ ,$$

so that the multiplicity of each q_i, $i \in I_n$, is $2g - 1 - r = n - 1$. Now, consider a zero q'_1 of $K(z, p_4, \ldots, p_g)$, distinct from $q_1, \ldots, q_n$; by the same construction, if q'_1 has multiplicity $n' - 1$, with $n' \geq 2$, then it is an element of a set of n' (possibly coincident) zeroes $\{q'_1, \ldots, q'_n\}$ with the same multiplicity. By repeating this procedure, we obtain a finite number l of disjoint sets of zeroes; for each $i \in I_l$, the i-th set contains $n_i \geq 2$ zeroes, we denote by $q^i_1, \ldots, q^i_{n_i}$, each one with multiplicity $n_i - 1$. Therefore, we have

$$e = \sum_{i=1}^{l} \sum_{j=1}^{n_i} (n_i - 1) q^i_j = \sum_{i=1}^{l} \sum_{\substack{j,k=1 \\ j<k}}^{n_i} (q^i_j + q^i_k) \ ,$$

and, since $h^0(K_C \otimes \mathcal{O}(-q^i_j - q^i_k - d)) = 2$, each $c := q^i_j + q^i_k$ satisfies Eq.(6.6); conversely, it follows immediately that if an element of C_2 satisfies Eq.(6.6), then it is the sum of a pair of zeroes of $K(z, p_4, \ldots, p_g)$ in the same set. $\square$

Theorem 6.4. *There exists a holomorphic section A on $\Theta_s \times \ldots \times \Theta_s \equiv \Theta_s^{M-N}$, completely anti-symmetric in its $M - N$ arguments and such that, for all $p_3, \ldots, p_g \in C$,*

$$A(e_{N+1}, \ldots, e_M) = \frac{\det_{i,j=N+1,\ldots,M} \left(\sum_{k,l=1}^{g} \theta_{kl}(e_i) \omega_k(p_{1_j}) \omega_l(p_{2_j}) \right)}{k(p_3, \ldots, p_g)} \ . \tag{6.8}$$

Furthermore, the quadrics

$$\sum_{ij=1}^{g} \theta_{ij}(e_k) X^i X^j \ , \tag{6.9}$$

$k = N + 1, \ldots, M$, generate the ideal I_2 of quadrics of the canonical curve C if and only if $A(e_{N+1}, \ldots, e_M) \neq 0$.

Proof. Let us first prove that the ratio on the right hand side of Eq.(6.8) does not depend on $p_3, \ldots, p_g \in C$. The numerator of such a ratio is the determinant of a $(M - N) \times (M - N)$ matrix W, which can be expressed as the product $W = UV$ of a $(M - N) \times M$ matrix U and a $M \times (M - N)$ matrix V, with entries

$$U_{ij} := \theta_{1_{j+N} 2_{j+N}}(e_{i+N}) \ , \qquad V_{jk} := \chi_j^{-1} (p \cdot p)_{k+N} [\omega \omega_j] \ ,$$

$i, k = 1, \ldots, M - N$, $j \in I_M$, where, by definition (4.4),

$$(p \cdot p)_k [\omega \omega_j] = \omega_{1_j}(p_{1_k}) \omega_{2_j}(p_{2_k}) + \omega_{1_j}(p_{2_k}) \omega_{2_j}(p_{1_k}) \ .$$

The determinant $\det W$ is a holomorphic $(g - 3)$-differential in each p_i, $i = 3, \ldots, g$; furthermore, it is symmetric (for g even) or anti-symmetric (for g odd), with respect to permutations of such arguments. Fix $p_4, \ldots, p_g \in C$ and consider the divisor of $\det W$ with respect to p_3. Define a local trivialization of K_C and a local coordinate z on an open neighborhood of p_4; with respect to such a trivialization, in the limit $p_3 \to p_4$, $\det W$ can be seen as a holomorphic function in $(p_3, p_4) \in U \times U$. Such a determinant is invariant under the replacement, in the matrix W, of the column

$$\begin{pmatrix} \sum_{ij} \theta_{ij}(e_{N+1})\omega_i(p_3)\omega_j(p_k) \\ \vdots \\ \sum_{ij} \theta_{ij}(e_M)\omega_i(p_3)\omega_j(p_k)) \end{pmatrix} ,$$

by the column

$$\begin{pmatrix} \sum_{ij} \theta_{ij}(e_{N+1})(\omega_i(p_3) - \omega_i(p_4))\omega_j(p_k) \\ \vdots \\ \sum_{ij} \theta_{ij}(e_M)\omega_i(p_3)(\omega_i(p_3) - \omega_i(p_4))\omega_j(p_k) \end{pmatrix} ,$$

for all $k = 5, \ldots, g$. It follows that each of these columns is of order $z(p_3) - z(p_4)$ in the limit $p_3 \to p_4$. Then, consider the element $\sum_{ij} \theta_{ij}(e_k)\omega_i(p_3)\omega_j(p_4)$, for each $k = N+1, \ldots, M$. Such a function vanishes at $p_3 = p_4$ due to the relations (B.6); moreover, since it is symmetric with respect to the exchange $p_3 \leftrightarrow p_4$, the first non-vanishing contribution in the limit $p_3 \to p_4$ must be of order $(z(p_3) - z(p_4))^2$. It follows that $\det W$ has a zero of order $g-2$ at $p_3 = p_4$ and, by symmetry arguments, at $p_3 = p_i$ for all $i = 4, \ldots, g$.

Fix $p_3, \ldots, p_g \in C$ and suppose that there exists a point $p \in C$ such that $\tilde{e} := I(p + p_3 + \ldots + p_g)$ is in Θ_s. Each point of Θ_s is associated to a relation among holomorphic quadratic differentials by Eq.(B.6). On the other hand, since at most $M - N$ such relations can be linearly independent, there exist some coefficients $\tilde{c}, c_{N+1}, \ldots, c_M \in \mathbb{C}$ such that

$$\tilde{c}\theta_{ij}(\tilde{e}) + \sum_{k=N+1}^{M} c_k \theta_{ij}(e_k) = 0 , \tag{6.10}$$

for all $i, j = 1, \ldots, g$. By Proposition 3.3,

$$\sum_{i,j=1}^{g} \theta_{ij}(\tilde{e})\omega_i(p_k)\omega_j(p_l) = 0 ,$$

which, by (6.10), implies

$$\sum_{n=N+1}^{M} c_n \sum_{i,j=1}^{g} \theta_{ij}(e_n)\omega_i(p_k)\omega_j(p_l) = 0 ,$$

for all $3 \le k, l \le g$, so that the rows of W are linearly dependent and $\det W = 0$.

Hence, $\det W$, considered as a holomorphic $(g-3)$-differential in p_3, has a zero of order $g-2$ at each p_i, $i = 4, \ldots, g$, and vanishes if there exists $p \in C$, such that $I(p + p_3 + \ldots + p_g) \in \Theta_s$; it follows that the right hand side of (6.8) is a meromorphic function of p_3 with no poles, and then is a constant. By the same arguments, it does not depend on p_i, for all $i = 3, \ldots, g$.

The condition that the quadrics (6.9) generate the ideal I_2 of degree 2 of C is equivalent to the matrix U having its maximal rank. Therefore, if such quadrics does not generate I_2, then $A(e_{N+1},\ldots,e_M) = 0$. Conversely, since there always exist $p_3,\ldots,p_g \in C$ such that V has rank $M-N$, it follows that if $A(e_{N+1},\ldots,e_M) \neq 0$ then the matrix U has maximal rank. □

Green [29] proved that the ideal I_2 of degree 2 is generated by quadrics in the form (6.9). Together with Theorem 6.4, this implies that $A(e_{N+1},\ldots,e_M)$ does not vanish identically. Consider the coefficients C^ω_{ij}, $i = N+1,\ldots,M$, $j \in I_M$, given by Corollary 5.10 for some suitable $p_3,\ldots,p_g \in C$. The corresponding quadrics (5.28) generate I_2; it follows that each $\theta_{ij}(e_k)$, $k = N+1,\ldots,M$, can be expressed as a linear combination

$$\theta_{\mathbf{1}_j\mathbf{2}_j}(e_k) = \sum_{i=N+1}^{M} c_{ki} C^\omega_{ij} \ ,$$

for all $j \in I_M$, $i = N+1,\ldots,M$, for some complex coefficients c_{ki}. By Eq.(6.8), this implies that $k(z_3,\ldots,z_g)$, for arbitrary $z_3,\ldots,z_g \in C$, is proportional to

$$k(z_3,\ldots,z_g) \sim \det_{i,j=N+1,\ldots,M} \sum_{k=1}^{M} [C^\omega_{ik}(\omega_{\mathbf{1}_k}(z_{\mathbf{1}_j})\omega_{\mathbf{2}_k}(z_{\mathbf{2}_j}) + \omega_{\mathbf{1}_k}(z_{\mathbf{2}_j})\omega_{\mathbf{2}_k}(z_{\mathbf{1}_j}))] \ .$$

The multiplicative constant does not depend on $z_3,\ldots,z_g$; hence, by setting $z_i \equiv p_i$, $i = 3,\ldots,g$, and noting that

$$\sum_{j=1}^{M} C^\omega_{ij}\omega \cdot \omega_j = \sum_{j=1}^{M} C^\sigma_{ij}\sigma \cdot \sigma_j \ ,$$

and

$$(p \cdot p)_k[\sum_{j=1}^{M} C^\sigma_{ij}\sigma \cdot \sigma_j] = \delta_{ik} \ ,$$

$i, k = N+1,\ldots,M$, we obtain

$$k(z_3,\ldots,z_g) = k(p_3,\ldots,p_g) \det_{i,j=N+1,\ldots,M} (z \cdot z)_j[\sum_{k=1}^{M} C^\omega_{ik}\omega \cdot \omega_k] \ .$$

In particular,

$$k(z, p_4,\ldots,p_g) = k(p_3,\ldots,p_g) \det_{i,j=4,\ldots,g} [\sum_{k=1}^{g} C^\sigma_{3i,jk}\sigma_k(z)] \ .$$

6.3 Quadrics from double points on Θ_s

Choose $p_3,\ldots,p_g \in C$ pairwise distinct and such that $K(p_3,\ldots,p_g) \neq 0$. Let $C_2 \ni c := u + v$, $u, v \in C$, be an effective divisor of degree 2, such that u is distinct from $p_3,\ldots,p_g$ and $\sum_{i=3}^{g} p_i + c$ is special. Then there exists $x \in C$ such that $(x, u, p_3,\ldots,p_g) \in C^g \setminus \mathcal{A}$ (or, otherwise, $K(p_3,\ldots,p_g)$ would vanish); let $\{\sigma_i\}_{i\in I_g}$ be the basis of $H^0(K_C)$ associated to $x, u, p_3,\ldots,p_g$ by Proposition 4.1.

Let $A(c) \subset I_g \setminus \{1,2\}$ be the set

$$A(c) := \{i \in I_g \setminus \{1,2\} \mid \sigma_i(v) \neq 0\} \ ,$$

and $\bar{A}(c) := \{3,\ldots,g\} \setminus A(c)$ its complement.

Lemma 6.5. *The set $A(c)$ is independent of x, provided that $(x,u,p_3,\dots,p_g)\in C^g\setminus\mathcal{A}$. Furthermore, for each subset $A'\subseteq I_g\setminus\{1,2\}$, the divisor $\sum_{i\in A'}p_i+c$ is exceptional if and only if $A(c)\subseteq A'$, and $A(c)$ is the unique set satisfying such a property.*

Proof. An effective divisor d, with $\deg d\le g$, is exceptional if and only if $h^0(K_C\otimes\mathcal{O}(-d))>g-\deg D$. Consider the divisor $d:=\sum_{i\in A(c)}p_i+c$ of degree $\deg d=a+2$, where a is the cardinality of $A(c)$. Since $H^0(K_C\otimes\mathcal{O}(-d))$ is generated by σ_1 and by the elements of $\{\sigma_i\}_{i\in\bar{A}(c)}$,

$$h^0(K_C\otimes\mathcal{O}(-d))=g-1-a>g-2-a=g-\deg d\ ,$$

and d is exceptional. It follows that if $A(c)\subseteq A'\subseteq\{3,\dots,g\}$, then $\sum_{i\in A'}p_i+c\ge d$ is special.

Conversely, set $d:=\sum_{i\in A'}p_i$ and suppose that $d+c$ is exceptional. Note that, since $d+u$ is not exceptional,

$$h^0(K_C\otimes\mathcal{O}(-d-u))=g-\deg d-1\le h^0(K_C\otimes\mathcal{O}(-d-c))\ ,$$

and by $H^0(K_C\otimes\mathcal{O}(-d-c))\subseteq H^0(K_C\otimes\mathcal{O}(-d-u))$, it follows that $H^0(K_C\otimes\mathcal{O}(-d-c))=H^0(K_C\otimes\mathcal{O}(-d-u))$; in other words, each element of $H^0(K_C\otimes\mathcal{O}(-d-u))$ also vanishes at v. Now, $H^0(K_C\otimes\mathcal{O}(-d-u))$ is generated by σ_1 and by the elements of $\{\sigma_i\}_{i\in\bar{A}'}$, where $\bar{A}':=\{3,\dots,g\}\setminus A'$. Then, $\sigma_i(v)=0$ for all $i\in\bar{A}'$, so that $\bar{A}'\subseteq\bar{A}(c)$ and then $A(c)\subseteq A'$.

Uniqueness follows by noting that if $\tilde{A}$ satisfies the same property, then $\tilde{A}\subseteq A(c)$ (because $\sum_{i\in A(c)}p_i+c$ is special) and $A(c)\subseteq\tilde{A}$ (because $\tilde{A}\subseteq\tilde{A}$ implies that $\sum_{i\in\tilde{A}}p_i+c$ is special).

Finally, by defining $A(c)$ as the unique set satisfying such a property, it follows that $A(c)$ is independent of x. $\square$

Lemma 6.6. *Suppose that $\bar{A}(c)\ne\varnothing$ and fix $i\in\bar{A}(c)$ and $j\ne i$, $3\le j\le g$. Let $k+1$, with $k\ge0$, be the order of the zero of σ_1 in p_j. Then, the holomorphic 1-differential*

$$\lambda_i^{(c)}(z):=\sum_{a,b\in I_g}\theta_{ab}(c+\sum_{l\ne i}p_l)\omega_a(p_i)\omega_b(z)\ ,$$

has a zero of order $n\ge k$ in $z=p_j$, and $n>k$ if and only if $j\in\bar{A}(c)$.

Proof. Define the points $\tilde{x}_1,\dots,\tilde{x}_{g-2-k}$ by

$$(\sigma_1)=\sum_{l=3}^{g}p_l+u+v+kp_j+\sum_{l=1}^{g-2-k}\tilde{x}_l\ ,$$

so that $I(\sum_{l=3}^g p_l+u+v+kp_j+\sum_{l=1}^{g-2-k}\tilde{x}_l)=b+\tau a$, for some $a,b\in\mathbb{Z}^g$. Consider the identities

$$\begin{aligned}&\sum_{l\in I_g}\theta_l(u+v+\sum_{m=3}^{g}p_m-w)\omega_l(z)\\&\sim\frac{\theta(\sum_m\tilde{x}_m+kp_j+w+z-y)E(z,p_j)^kE(z,w)\prod_l E(z,\tilde{x}_l)\sigma(z)}{E(y,z)E(y,w)E(y,p_j)^k\prod_l E(y,\tilde{x}_l)\sigma(y)}\\&\sim\frac{\theta(y+u+v+\sum_m p_m-w-z)E(z,p_j)^kE(z,w)\prod_l E(z,\tilde{x}_l)\sigma(z)}{E(y,z)E(y,w)E(y,p_j)^k\prod_l E(y,\tilde{x}_l)\sigma(y)}\ ,\end{aligned}$$

where $\sim$ denotes equality up to nowhere vanishing factors, which hold for arbitrary $w, y \in C$. Dividing by $E(p_i, w)$ and taking the limit $w \to p_i$ one obtains

$$\lambda_i^{(c)}(z) = \frac{e^{-2\pi i^t a I(z-y)} E(z,p_j)^k E(p_i,z) \prod_m E(z,\tilde{x}_m)\sigma(z)}{E(y,z)E(y,p_i)E(y,p_j)^k \prod_m E(y,\tilde{x}_m)\sigma(y)} \sum_{l\in I_g} \theta_l(y+u+v+\sum_{m\neq i} p_m - z)\omega_l(p_i) \ ,$$

where we recovered the right phase. Since the right hand side does not depend on y, the factor $E(z,p_j)^k$ cannot be compensated by any factor in the denominator and the 1-differential has a zero of order at least k in $z = p_j$. Furthermore, such a zero if of order strictly greater than k if and only if

$$\sum_{l\in I_g} \theta_l(y+u+v+\sum_{m\in A'} p_m)\omega_l(p_i) = 0 \ ,$$

for all $y \in C$, with $A' := \{3,\ldots,g\} \setminus \{i,j\}$. In particular, for $y \equiv x$, this implies that the holomorphic 1-differential

$$\sum_{l\in I_g} \theta_l(x+u+v+\sum_{m\in A'} p_m - \Delta)\omega_l(z) \ ,$$

vanishes at p_i. Therefore, such a differential vanishes at x, u, v and p_l, for all $l \neq j$, $3 \le l \le g$; hence, it is proportional to σ_j, which is the generator of $H^0(K_C \otimes \mathcal{O}(-u-x-\sum_{l\neq j} p_l))$, and it must be $\sigma_j(v) = 0$, so that $j \in \bar{A}(c)$. Conversely, if $j \in \bar{A}(c)$, then $A(c) \subseteq A'$ and, by Lemma 6.5, $y+u+v+\sum_{l\in A'} p_l$ is a special divisor for all $y \in C$. Then, for each $y \in C$, there exist $q_1, \ldots, q_{g-2} \in C$ such that $I(y+u+v+\sum_{l\in A'} p_l) = I(p_i + \sum_l q_l)$, so that

$$\sum_{l\in I_g} \theta_l(y+u+v+\sum_{m\neq i,j} p_m)\omega_l(p_i) = \sum_{l\in I_g} \theta_l(p_i + \sum_m q_m)\omega_l(p_i) = 0 \ ,$$

for all $y \in C$, and the lemma follows. $\square$

Set

$$\Lambda_{jk}^{(i)}(c) := \sum_{a,b\in I_g} \theta_{ab}(c + \sum_{l\neq i} p_l)\omega_a(p_j)\omega_b(p_k) \ , \tag{6.11}$$

$i, j, k \in I_g \setminus \{1,2\}$. Note that, if $i \in \bar{A}(c)$, then $\Lambda_{jk}^{(i)}(c) = 0$ for $j = k$ and for $j, k \neq i$, and

$$\Lambda_{ij}^{(i)}(c) = \lambda_i^{(c)}(p_j) \ ,$$

$j \neq i$.

Theorem 6.7. *Choose $p_3, \ldots, p_g \in C$, $C_2 \ni c := u+v$ and $x \in C$ as above. Suppose $\bar{A}(c) \neq \varnothing$ and fix $i \in \bar{A}(c)$. If u is a single zero for $K(\,\cdot\,, p_3, \ldots, \check{p}_i, \ldots, p_g)$, then the holomorphic quadratic differentials $\sigma\sigma_k$, $k \in I_N^{1i}$ (see Definition 2.3 for notation), satisfy a unique linear relation*

$$\sum_{k\in I_N^{1i}} \tilde{C}_k^{\sigma(i)}(c)\sigma\sigma_k = 0 \ ,$$

where

$$\tilde{C}_k^{\sigma(i)}(c) = \sum_{\substack{j\in I_2^i\\ j>N}} \Lambda_j^{(i)}(c) C_{jk}^\sigma \ ,$$

$k \in I_N^{1i}$*, with* $\Lambda_j^{(i)}(c) := \Lambda_{\imath_j \jmath_j}^{(i)}(c)$*,* $j \in I_M$*, defined in Eq.*(6.11).

Proof. By Theorem 6.3 and Corollary 4.5, since u is a single zero for the function $K(\,\cdot\,, p_3, \ldots, \check{p}_i, \ldots, p_g)$, then $\sigma\sigma_k$, $k \in I_N^{1i}$, span a $(N-1)$-dimensional vector space in $H^0(K_C^2)$, and then satisfy a relation

$$\sum_{k\in I_N^{1i}} \tilde{C}_k^{\sigma(i)}(c)\sigma\sigma_k = 0 \ .$$

Such a relation determines, up to normalization, an element

$$\ker\psi \ni \phi := \sum_{k\in I_N^{1i}} \tilde{C}_k^{\sigma(i)} \sigma\cdot\sigma_k \ ,$$

where $\psi : \operatorname{Sym}^2 H^0(K_C) \to H^0(K_C^2)$; by Theorem 5.3, $\ker\psi$ is spanned by $\{\sum_{k=1}^M C_{ik}^\sigma \sigma\cdot\sigma_k\}_{N<i\le M}$, so that

$$\sum_{k\in I_N^{1i}} \tilde{C}_k^{\sigma(i)}(c)\sigma\cdot\sigma_k = \sum_{j=N+1}^{M} L_j^{(i)}(c) \sum_{l\in I_M} C_{jl}^\sigma \sigma\cdot\sigma_l \ , \tag{6.12}$$

for some complex coefficients $L_j^{(i)}(c)$, $N < j \le M$. Note that, for all j, k, with $N < j, k \le M$, $C_{jk}^\sigma = \delta_{jk}$. Then, by applying $(p\cdot p)_j$ (see Eq.(4.4)), $j = N+1, \ldots, M$, to both sides of (6.12), and by using Eq.(4.6), we obtain

$$L_j^{(i)} = \begin{cases} \tilde{C}_j^{\sigma(i)}(c) \ , & \text{for } j \in I_N^{1i} \ , \\ 0 \ , & \text{for } j \notin I_N^{1i} \ , \end{cases} \qquad N < j \le M \ .$$

Observe that if $j \in I_N^{1i}$ and $j > N$, then $j \in I_2^i$ (see Def. 2.3), that is, at least one between $\imath_j$ and $\jmath_j$ is equal to i; furthermore, the condition $j > N$ implies $\imath_j \neq \jmath_j$ and $\imath_j, \jmath_j \neq 1, 2$. Therefore, it remains to prove that $L_j^{(i)}(c) \equiv C_j^{\sigma(i)} = \Lambda_{\imath_j\jmath_j}^{(i)}(c)$ for all $j \in I_2^i$, $j > N$, with respect to a suitable normalization of ϕ.

The vector ϕ can be expressed as

$$\phi \equiv \sum_{k\in I_N^{1i}} \tilde{C}_k^{\sigma(i)}(c)\sigma\cdot\sigma_k = \sigma_1\cdot\eta + \sigma_i\cdot\rho + c\sigma_1\cdot\sigma_i \ , \tag{6.13}$$

for some $\eta, \rho \in H^0(K_C)$, $c \in \mathbb{C}$, so that the relation $\psi(\phi) = 0$ corresponds to

$$\sigma_1\eta + \sigma_i\rho + c\sigma_1\sigma_i = 0 \ . \tag{6.14}$$

Note that, by the redefinition $\eta \to \eta + \alpha\sigma_i$, $c \to c - \alpha$, for a suitable $\alpha \in \mathbb{C}$, we can assume $\eta(p_i) = 0$. Applying $p_i \cdot p_j$, $3 \le j \le g$, $j \neq i$, to both sides of (6.13), it follows that

$$L_{ij}^{(i)}(c) = p_i \cdot p_j[\phi] = \rho(p_j) \ ,$$

where $L^{(i)}_{1_k 2_k}(c) = L^{(i)}_{2_k 1_k}(c) := L^{(i)}_k(c)$, $N < k \leq M$. Define $d \in C_{g-2}$ in such a way that

$$(\sigma_1) = b + c + d \ ,$$

and observe that, by (6.14), $\rho \in H^0(K_C \otimes \mathcal{O}(-d))$ (since u is a single zero for $K(\,\cdot\,, p_3, \ldots, \check{p}_i, \ldots, p_g)$, it follows that the gcd of (σ_1) and (σ_i) is $c + \sum_{k \neq i} p_k$). Furthermore, ρ cannot be a multiple of σ_1, since, in this case, the only possibility for Eq.(6.14) to hold would be $\phi = 0$. Finally, $L^{(i)}_{ij}(c)$ is invariant under the redefinition $\rho \to \rho + a\sigma_1$, since $\sigma_1(p_j) = 0$ for all $j = 3, \ldots, g$. Then, we can fix an arbitrary $y \in C \setminus \mathrm{supp}(\sigma_1)$ and assume that ρ is an element of the 1-dimensional space $H^0(K_C \otimes \mathcal{O}(-d - y))$. By using the relation $I(b + c - y) = -I(d + y)$, such an element can be expressed as follows

$$\rho(z) = \frac{a(y)}{A} \frac{\sum_{k \in I_g} \theta_k(b + c - y)\omega_k(z)}{E(y, p_i)} \ , \tag{6.15}$$

where the normalizing constant A can be arbitrarily fixed, and a is a function such that

$$L^{(i)}_{ij} = \frac{a(y)}{A} \frac{\sum_{k \in I_g} \theta_k(b + c - y)\omega_k(p_j)}{E(y, p_i)} \ , \tag{6.16}$$

$3 \leq j \leq g$, $j \neq i$, is independent of y. In other words, we assume that, under the change

$$\begin{aligned} &y \to \tilde{y} \ , \qquad y, \tilde{y} \in C \setminus \mathrm{supp}(\sigma_1) \\ &\rho \to \tilde{\rho} \ , \end{aligned}$$

$\rho(p_i)$ is equal to $\tilde{\rho}(p_i)$; this property, together with the fact that $\tilde{\rho} \in H^0(K^C \otimes \mathcal{O}(-d))$, which is generated by σ_1 and ρ, implies that

$$\tilde{\rho} = \rho + f(y, \tilde{y})\sigma_1 \ , \tag{6.17}$$

for some function f. Though Eq.(6.15) only holds for $y \in C \setminus \mathrm{supp}(\sigma_1)$, the RHS of Eq.(6.16) is a constant and can be continued to all $y \in C$ and, in particular, in the limit $y \to p_i$.

It is now sufficient to prove that $a(p_i) := \lim_{y \to p_i} a(y)$ is finite and non-vanishing (by Eq.(6.16) such a limit necessarily exists); in fact, in this case, after fixing the normalization $A \equiv a(p_i)$, we obtain

$$L^{(i)}_{ij} = \lim_{y \to p_i} \frac{\sum_{k \in I_g} \theta_k(b + c - y)\omega_k(p_j)}{E(y, p_i)} = \Lambda^{(i)}_{ij}(c) \ .$$

Then, to conclude, it remains to prove that $\lim_{y \to p_i} a(y) \neq 0, \infty$. Since $L^{(i)}_{ij}$ and $\Lambda^{(i)}_{ij}$ are finite, $\lim_{y \to p_i} a(y) = 0$ would imply that $L^{(i)}_{ij} = 0$ for all j and then that Eq.(6.13) is trivial, which is absurd.

In order to prove that $\lim_{y \to p_i} a(y) \neq \infty$, let us choose $j \neq i$, $3 \leq j \leq g$, in such a way that, at the point p_j, σ_1 has a zero of order $k + 1$ and $\lambda^{(c)}_i(z)$ has a zero of order k, for some $k \geq 0$. Suppose, by absurd, that such a j does not exist. Then, by Lemma 6.6, $\sigma_l(v) = 0$, for all $l \in I_g \setminus \{2\}$. On the other hand, such differentials also vanish at u, so that $h^0(K_C \otimes \mathcal{O}(-u - v)) = g - 1$. By the Riemann-Roch Theorem, this would imply that $h^0(\mathcal{O}(u + v)) = 1$ and then C would be hyperelliptic, counter the hypotheses.

As discussed above, the hypotheses of the theorem imply that the greater common divisor of (σ_1) and (σ_i) is $c + \sum_{m\neq i} p_m$; in particular, if $k > 0$, then p_j is a single zero for σ_i. Hence, by Eq.(6.14), $\rho(z)$ has a zero of order at least k in p_j. By expanding $\rho(z)$ in the limit $z \to p_j$, we obtain

$$\rho(z) \sim \beta\zeta^k d\zeta + o(\zeta^k) \ ,$$

with respect to some coordinates $\zeta(z)$ centered in p_j. Here, β does not depend on y, since, by Eq.(6.17), $\rho(z)$ depends on y only through a term proportional to $\sigma_1(z)$, which is of order ζ^{k+1}.

By using Eq.(6.15), in the limit $z \to p_j$ we have

$$\frac{\sum_{a\in I_g} \theta_a(u + v + \sum_{m=3}^{g} p_m - y)\omega_a(z)}{E(p_i, y)} \sim \frac{A\beta}{a(y)}\zeta^k d\zeta + o(\zeta^k) \ .$$

In the limit $y \to p_i$, the LHS gives $\lambda_i^{(c)}(z)$, which, by Lemma 6.6, has a zero of order exactly k in $z = p_j$. Therefore,

$$\lim_{y\to p_i} \frac{A\beta}{a(y)} \neq 0 \ ,$$

that concludes the proof. □

A classical result known by Riemann is the relation

$$\sum_{a,b\in I_g} \theta_{ab}(e)\omega_a\omega_b = 0 \ ,$$

which holds for an arbitrary $e \in \Theta_s$. The connection of such a relation to the ones considered in this paper is given by the following lemma.

Lemma 6.8. *Choose $p_1, \ldots, p_g$ satisfying conditions i), ii) or iii) of Theorem 6.2. Then, for all $e \in \Theta_s$, the relation*

$$\sum_{a,b\in I_g} \theta_{ab}(e)\omega_a\omega_b = 0 \ ,$$

is equivalent to

$$\sum_{i=N+1}^{M} A_i(e) \sum_{j\in I_M} C^\sigma_{ij}\sigma\sigma_j = 0 \ ,$$

where

$$A_i(e) := \sum_{a,b\in I_g} \theta_{ab}(e)\omega_a(p_{1_i})\omega_b(p_{2_i}) \ ,$$

$i \in I_M$.

Proof. Two relations are equivalent if they correspond to the same vector in $\ker\psi$, up to normalization. Since $\ker\psi$ is spanned by $\{\sum_{k=1}^{M} C^\sigma_{ik}\sigma\cdot\sigma_k\}_{N<i\leq M}$, then

$$\sum_{a,b\in I_g} \theta_{ab}(e)\omega_a\cdot\omega_b = \sum_{i=N+1}^{M} A_i(e) \sum_{j\in I_M} C^\sigma_{ij}\sigma\cdot\sigma_j \ ,$$

for some complex coefficients $A_i(e)$, $i \in I_M$. By applying $p\cdot p_i$, $i = N+1, \ldots, M$, to both sides of this equation, and using $C^\sigma_{ij} = \delta_{ij}$, for $N < i, j \leq M$, we conclude. □

Theorem 6.9. *Choose $p_3, \ldots, p_g \in C$, $C_2 \ni c := u+v$ and $x \in C$ as above. Suppose $\bar{A}(c) \neq \varnothing$ and fix $i \in \bar{A}(c)$. If u is a single zero for $K(\,\cdot\,, p_3, \ldots, \check{p}_i, \ldots, p_g)$, then the linear relation*

$$\sum_{k \in I_N^{1i}} \tilde{C}_k^{\sigma(i)}(c)\sigma\sigma_k = 0 \ ,$$

is equivalent to

$$\sum_{a,b \in I_g} \theta_{ab}(c + \sum_{j \neq i} p_j)\omega_a\omega_b = 0 \ .$$

Proof. By construction, $I(c + \sum_{j\neq i} p_j) \in \Theta_s$. Then, use Theorem 6.7 and Lemma 6.8, and note that

$$A_k(I(c + \sum_{j \neq i} p_j)) = \Lambda_k^{(i)}(c) \ ,$$

$k = N+1, \ldots, M$. □

Theorem 6.10. *If C is a trigonal curve, then there exist $2g-4$ pairwise distinct points $p_3, \ldots, p_g, u_3, \ldots, u_g \in C$ such that the following conditions are satisfied*

i. $K(p_3, \ldots, p_g) \neq 0$

ii. $K(u_j, p_3, \ldots, \check{p}_i, \ldots, p_g) = 0$ *if and only if $j \neq i$, for all $i, j \in I_g \setminus \{1,2\}$.*

Furthermore, if, for each $i \in I_g \setminus \{1,2\}$, the points u_j, $j \in I_g \setminus \{1,2,i\}$, are single zeros for $K(\,\cdot\,, p_3, \ldots, \check{p}_i, \ldots, p_g)$, then the following statements hold:

a. For each $3 \leq j \leq g$, there exists a unique $v_j \in C$ such that

$$I(c_j + \sum_{k \neq i} p_k - \Delta) \in \Theta_s \ ,$$

for all $i \neq j$, $3 \leq i \leq j$, where $c_j := u_j + v_j$, $3 \leq j \leq g$;

b. The relations

$$\sum_{k \in I_N^{1i}} \tilde{C}_k^{\sigma(i)}(c_j)\sigma\sigma_k = 0 \ ,$$

$3 \leq i < j \leq g$, considered in Lemma 6.7, are linearly independent and then generate the ideal I_2 of quadrics in $\mathbb{P}_{g-1}$ containing the curve C.

Proof. Since C is trigonal, there exists a unique (up to a fractional linear transformation) meromorphic function f with three poles. Hence, for each $p \in C$, $f^{-1}(f(p))$ consists of three (possibly coincident) points; note that, trivially, the sum of such three points (counting multiplicity) corresponds to the unique effective divisor of degree three which is special and containing p in its support.

Fix $x_4, \ldots, x_g \in C$, and consider the function

$$F_{x_4,\ldots,x_g}(p) := \prod_{x \in f^{-1}(f(p)))} K(x, x_4, \ldots, x_g) \ , \qquad p \in C \ .$$

Denote by $[K]_{x_4,\ldots,x_g} \subseteq C$ and $[F]_{x_4,\ldots,x_g}$ the sets of zeros of $K(\,\cdot\,, x_4, \ldots, x_g)$ and $F_{x_4,\ldots,x_g}$, respectively. Then, one of the following alternatives holds: if $K(\,\cdot\,, x_4, \ldots, x_g)$ is not identically vanishing, then both $[K]_{x_4,\ldots,x_g}$ and

$$[F]_{x_4,\ldots,x_g} = \bigcup_{x \in [K]_{x_4,\ldots,x_g}} f^{-1}(f(x)) \ ,$$

are finite sets; otherwise, both $[K]_{x_4,\ldots,x_g}$ and $[F]_{x_4,\ldots,x_g}$ coincide with C.

For each n, $1 \le n \le g-2$, let $N^{(n)}_{x_{n+3},\ldots,x_g} \subseteq C^n$ denote the set of n-tuples $(p_3,\ldots,p_{n+2})$ such that

$$F^{(n)}_{x_{n+3},\ldots,x_g}(p_3,\ldots,p_{n+2}) := \prod_{i=1}^{n} F_{p_3,\ldots,\check{p}_i,\ldots,p_{n+2},x_{n+3},\ldots,x_g}(p_i) \ ,$$

is not zero. Note that $F^{(1)} \equiv F$ and $N^{(1)} = C \setminus [F]$.

Now, assume that, for some m, $1 \le m < g-2$, the set $N^{(n)}$ is dense in C^n for all $n \le m$. The set $[F^{(m+1)}]_{x_{m+4},\ldots,x_g}$ of zeros of

$$\begin{aligned} F^{(m+1)}_{x_{m+4},\ldots,x_g}(p_3,\ldots,p_{m+2},p) = F_{p_3,\ldots,p_{m+2},x_{m+4},\ldots,x_g}(p) \\ \prod_{i=1}^{m} F_{p_3,\ldots,\check{p}_i,\ldots,p_{m+2},p,x_{m+4},\ldots,x_g}(p_i) \ , \end{aligned}$$

as a function of p, is given by

$$\begin{aligned} [F^{(m+1)}]_{x_{m+4},\ldots,x_g} = \bigcup_{i=1}^{m}\Big(\bigcup_{x\in f^{-1}(f(p_i))} [K]_{p_3,\ldots,\check{p}_i,\ldots,p_{m+2},x,x_{m+4},\ldots,x_g}\Big) \\ \cup [F]_{p_3,\ldots,p_{m+2},x_{m+4},\ldots,x_g} \ . \end{aligned}$$

If $(p_3,\ldots,p_{m+2}) \in N^{(m)}$, then the functions

$$K(\,\cdot\,,p_3,\ldots,p_{m+2},x_{m+4},\ldots,x_g) \ ,$$

and

$$K(\,\cdot\,,p_3,\ldots,\check{p}_i,\ldots,p_{m+2},x,x_{m+4},\ldots,x_g) \ ,$$

for each $i = 1,\ldots,m$, and $x \in f^{-1}(f(p_i))$, vanish identically on C (for example, x_{m+3} is not a zero). Hence, $[F^{(m+1)}]_{x_{m+4},\ldots,x_g} \subseteq C$ is a finite set and, therefore, $N^{(m+1)}_{x_{m+4},\ldots,x_g}$ is dense in C^{m+1}. We proved that if $K(\,\cdot\,,x_4,\ldots,x_g)$ does not identically vanish for some $x_4,\ldots,x_g \in C$, then $N^n_{x_{n+3},\ldots,x_g}$ is dense in C^n for all n, $1 \le n \le g-2$. It follows that N^{g-2}, which does not depend on $x_4,\ldots,x_g$, is dense in C^{g-2}. Also note that the subset of C^{g-2} for which

$$\bigcup_{i=3}^{g} f^{-1}(f(p_i)) \ ,$$

consists of pairwise distinct points is dense C_{g-2}. Hence, its intersection with $N^{(g-2)}$ is not empty.

Let us choose $(p_3,\ldots,p_g)$ in such an intersection and fix $u_i \in f^{-1}(f(p_i))$, $u_i \ne p_i$, for all $i \in I_g \setminus \{1,2\}$. Then, the points $p_3,\ldots,p_g,u_3,\ldots,u_g$ are pairwise distinct and satisfy the condition

$$K(u_i,p_3,\ldots,\check{p}_j,\ldots,p_g) = 0 \ \Leftrightarrow \ i \ne j \ ,$$

for all $i,j \in I_g \setminus \{1,2\}$. Furthermore, if u_i, $i \in I_g \setminus \{1,2\}$, is a single zero of $K(\,\cdot\,,p_3,\ldots,\check{p}_j,\ldots,p_g)$, for all $j \in I_g \setminus \{1,2,i\}$, then there exists a unique point v_{ij} such that $v_{ij} + u_i + \sum_{k\ne j} p_k$ is special. Such a point satisfies necessarily

$f^{-1}(f(p_i)) = \{p_i, u_i, v_{ij}\}$, so that it is independent of j, and the statement $a.$ follows.

Finally, note that $A(c_j) = p_j$ and $\bar{A}(c_j) = \{p_i \mid 3 \le i \le g, i \ne j\}$. Hence, by Theorem 6.7, for each k, $N < k \le M$ the coefficients $C_l^{\sigma(\mathfrak{1}_k)}(c_{\mathfrak{2}_k})$, $l \in I_M$, are given by

$$C_l^{\sigma(\mathfrak{1}_k)}(c_{\mathfrak{2}_k}) = \Lambda_k^{(\mathfrak{1}_k)}(c_{\mathfrak{2}_k}) C_{kl}^\sigma \ ,$$

where $\Lambda_k^{(\mathfrak{1}_k)}(c_{\mathfrak{2}_k}) \ne 0$. Linear independence of the $\tilde{C}^{\sigma(i)}(c_j)$'s, $3 \le i < j \le g$, follows by linear independence of the C_k^σ's. □

6.4 The case of genus 4

Consider the case of a non-hyperelliptic curve C of genus 4. The identity (6.4) reduces to

$$K(p_3, p_4) := -c_{4,2} \frac{\kappa[v]}{\kappa[\sigma]^5 E(p_3, p_4)^2 \sigma(p_3)\sigma(p_4)} \ ,$$

where $c_{4,2} = 1008$, and can be used to express Eq.(5.7) in terms of the function K. For $g = 4$ Eq.(5.7) reduces to a unique relation

$$\sum_{i=1}^{10} C_i^\sigma \sigma\sigma_i = 0 \ .$$

It can be derived from the identity

$$\frac{\det_{i,j\in I_{10}} \sigma\sigma_i(x_j)}{\det_{i,j\in I_9} v_i(x_j)} = 0 \ ,$$

by expanding the determinant at the numerator with respect to the column corresponding to $x_{10} \equiv z$. One obtains

$$\sum_{i\in I_{10}} (-)^i \frac{\det_{\substack{j\in I_{10}\setminus\{i\} \\ k\in I_9}} \sigma\sigma_j(x_k)}{\det_{j,k\in I_9} v_j(x_k)} \sigma\sigma_i(z) = 0 \ ,$$

where the ratios of determinants do not depend on $x_1, \ldots, x_9$ and correspond to

$$\frac{\det_{\substack{j\in I_{10}\setminus\{i\} \\ k\in I_9}} \sigma\sigma_j(x_k)}{\det_{j,k\in I_9} v_j(x_k)} = \frac{\kappa[\sigma\sigma_1, \ldots, \check{\sigma\sigma_i}, \ldots, \sigma\sigma_{10}]}{\kappa[v]} \ .$$

Now, note that for $\mathfrak{1}_i = \mathfrak{2}_i$, $\kappa[\sigma\sigma_1, \ldots, \check{\sigma\sigma_i}, \ldots, \sigma\sigma_{10}] = 0$. This can be checked by observing that all the elements in $\{\sigma\sigma_j\}_{j\in I_{10}\setminus\{i\}}$ vanish at p_i, so that it cannot be a basis of $H^0(K_C^2)$. Hence, we can restrict the summation over all the $i \in I_{10}$ with $\mathfrak{1}_i \ne \mathfrak{2}_i$. By a re-labeling of the points $p_1, \ldots, p_4$, the relation between $\kappa[v]$ and K at genus four is

$$K(p_{\mathfrak{1}_i}, p_{\mathfrak{2}_i}) = (-)^{i+1} c_{4,2} \frac{\kappa[\sigma\sigma_1, \ldots, \check{\sigma\sigma_i}, \ldots, \sigma\sigma_{10}]}{\kappa[\sigma]^5 E(p_{\mathfrak{1}_i}, p_{\mathfrak{2}_i})^2 \sigma(p_{\mathfrak{1}_i})\sigma(p_{\mathfrak{2}_i})} \ ,$$

for all i, $5 \le i \le 10$. Hence,

$$C_i^\sigma = \frac{K(p_{\mathfrak{1}_i}, p_{\mathfrak{2}_i}) E(p_{\mathfrak{1}_i}, p_{\mathfrak{2}_i})^2 \sigma(p_{\mathfrak{1}_i})\sigma(p_{\mathfrak{2}_i})}{K(p_3, p_4) E(p_3, p_4)^2 \sigma(p_3)\sigma(p_4)} = \frac{k(p_{\mathfrak{1}_i}, p_{\mathfrak{2}_i})}{k(p_3, p_4)} \ , \qquad (6.18)$$

$5 \leq i \leq 10$, with k defined in Eq.(6.5), whereas $C_i^\sigma = 0$ for $i \leq 4$. Since $\sigma\sigma_i = \sum_{j=1}^{10} X_{ji}\omega\omega_j$, it follows that

$$C_i^\omega = \sum_{j=5}^{10} X_{ij}^\omega C_j^\sigma \ ,$$

$i \in I_{10}$, and we obtain

$$C_i^\omega = \chi_i^{-1} \sum_{k,l=1}^{4} \frac{k(p_k,p_l)}{k(p_3,p_4)} \frac{\theta_{1_i}(\mathrm{a}_k)\theta_{2_i}(\mathrm{a}_l)}{\sum_{m,n}\theta_m(\mathrm{a}_k)\theta_n(\mathrm{a}_l)\omega_m(p_k)\omega_n(p_l)} \ , \tag{6.19}$$

$i \in I_{10}$. Note that $\hat{C}_i^\omega := k(p_3,p_4)C_i^\omega$ is symmetric under any permutation of $p_1,\ldots,p_4$. On the other hand, Corollary 5.10 shows that C_i^ω, and therefore also $\hat{C}_i^\omega$, are independent of p_1,p_2. We conclude that $\hat{C}_i^\omega$, whose explicit form is

$$\begin{aligned}\hat{C}_i^\omega = &-\frac{\chi_i^{-1}}{S(\mathrm{a})^2\prod_1^7\sigma(x_i)} \\ &\cdot \sum_{\substack{k,l=1\\k\neq l}}^{4} \Bigg[\frac{\theta_{1_i}(\mathrm{a}_k)\theta_{2_i}(\mathrm{a}_l)}{\theta\big(p_k+p_l+\sum_1^7 x_i\big)\sigma(p_k)\sigma(p_l)\prod_{i\neq k,l}\big(E(p_k,p_i)E(p_l,p_i)\big)} \\ &\qquad \cdot \sum_{s\in\mathcal{P}_7} \frac{S(\sum_{i=1}^4 x_{s_i})S(\sum_{i=4}^7 x_{s_i})}{E(x_{s_g},p_k)E(x_{s_g},p_l)} \prod_{i=1}^{3} \frac{S(x_{s_i}+x_{s_{i+4}}+p_k+p_l)}{\prod_{\substack{j=1\\j\neq i}}^3 E(x_{s_i},x_{s_{j+4}})}\Bigg] \ ,\end{aligned}$$

does not depend on $p_1,\ldots,p_4$, for all $i \in I_{10}$.

Note that, at genus 4, the equivalent relations

$$\sum_{i\in I_M} C_i^\omega \omega\omega_i = 0,$$

and

$$\sum_{i\in I_M} \hat{C}_i^\omega \omega\omega_i = 0 \ ,$$

must be proportional to Eq.(B.6), with e one of the two points in Θ_s; in other words, C_i^ω and $\hat{C}_i^\omega$ must be proportional to $\chi_i^{-1}\theta_{1_i 2_i}(e)$. Such a proportionality is immediately derived by noting that Eq. (6.8), for genus 4, gives

$$k(p,q) \equiv K(p,q)E(p,q)^2\sigma(p)\sigma(q) = A(e)^{-1}\sum_{a,b\in I_g}\theta_{ab}(e)\omega_a(p)\omega_b(q) \ , \tag{6.20}$$

for all $p,q \in C$, where e one of the two points of Θ_s and $A(e)$ is defined in Eq.(6.8). By Eq.(6.18) and Eq.(6.20), it immediately follows that

$$C_i^\sigma = \frac{\sum_{a,b\in I_g}\theta_{ab}(e)\omega_a(p_{1_i})\omega_b(p_{2_i})}{\sum_{a,b\in I_g}\theta_{ab}(e)\omega_a(p_3)\omega_b(p_4)} \ , \tag{6.21}$$

for all $i \in I_M$, and by Eq.(6.19)

$$C_i^\omega = \chi_i^{-1}\frac{\theta_{1_i 2_i}(e)}{\sum_{a,b\in I_g}\theta_{ab}(e)\omega_a(p_3)\omega_b(p_4)} \ , \tag{6.22}$$

for all $i \in I_M$. Finally, we have

$$\hat{C}_i^{\omega} = \frac{\theta_{1_i 2_i}(e)}{A(e)\chi_i} ,$$

for all $i \in I_M$.

6.5 Modular properties of $K(p_3, \ldots, p_g)$

For each $n \in \mathbb{Z}_{>0}$, let us consider the rank N_n vector bundle L_n on $\mathcal{M}_g$, defined in section 3.4, whose fiber at the point corresponding to a curve C is $H^0(K_C^n)$. A general section $s \in L_n^m$, $i > 1$, admits the local expression on an open set $U \subset \mathcal{M}_g$

$$s(p) = \sum_{i_1,\ldots,i_m \in I_{N_n}} s_{i_1\ldots i_m}(p)\phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_m} , \qquad p \in U \subset \mathcal{M}_g , \quad (6.23)$$

with respect to a set $\{\phi_i\}_{i \in I_{N_n}}$ of linearly independent local sections of L_n on U.

For each non-hyperelliptic C of genus $g \geq 3$, $k(p_3, \ldots, p_g)$ as defined in (6.5), is a holomorphic $(g-3)$-differential in each variable, and is symmetric (for g even) or anti-symmetric (for g odd) in its $g-2$ arguments. Hence,

$$k := \sum_{i_1,\ldots,i_{g-2} \in I_{N_{g-3}}} k_{i_1\ldots i_{g-2}} \phi_{i_1} \otimes \cdots \otimes \phi_{i_{g-2}} , \qquad (6.24)$$

can be naturally seen as an element of E_g, where

$$E_g := \begin{cases} \mathrm{Sym}^{g-2} H^0(K_C^{g-3}) , & g \text{ even} , \\ \bigwedge^{g-2} H^0(K_C^{g-3}) , & g \text{ odd} , \end{cases}$$

for a fixed basis $\{\phi_i\}_{i \in I_{N_{g-3}}}$ of $H^0(K_C^{g-3})$. The definition can be extended in a continuous way to hyperelliptic curves, by setting $k_{i_1\ldots i_{g-2}} \equiv 0$ in this case. At genus $g = 3$, $k(p_3)$ is a holomorphic function on C and therefore is a constant. Furthermore, Eq.(6.5) also makes sense at genus $g = 2$; in this case, k is again a constant. For $g > 3$, let us define $\mathbb{E}_g$ by

$$\mathbb{E}_g := \begin{cases} \mathrm{Sym}^{g-2} L_{g-3} , & g \text{ even} , \\ \bigwedge^{g-2} L_{g-3} , & g \text{ odd} . \end{cases}$$

In view of Eqs.(6.23) and (6.24), it is natural to seek for a section $k \in \mathbb{E}_g$ such that, at the point $p_C \in \mathcal{M}_g$ corresponding to the curve C, it satisfies

$$E_g \ni k(p_3, \ldots, p_g) \cong k(p_C) \in (\mathbb{E}_g)_{|p_C} ,$$

under the identification $(\mathbb{E}_g)_{|p_C} \cong E_g$. On the other hand, $k(p_3, \ldots, p_g)$ is not modular invariant, and then it does not correspond to a well-defined element of E_g for each $p_C \in \mathcal{M}_g$. The correct statement is given by the following theorem.

Theorem 6.11.

$$k := \kappa[\omega]^{g-8} k \otimes (\omega_1 \wedge \ldots \wedge \omega_g)^{12-g} ,$$

is a holomorphic section of λ_1^{12-g} *for* $g = 2, 3$ *and of* $\mathbb{E}_g \otimes \lambda_1^{12-g}$ *for* $g > 3$*, which vanishes only in the hyperelliptic locus for* $g \geq 3$.

Proof. Let us derive the modular properties of

$$\kappa[\omega]^{g-8}k(p_3,\dots,p_g) \ .$$

Eq.(6.5) and the identity $\kappa[\sigma]=\kappa[\omega]/\det\omega_i(p_j)$ yield

$$\kappa[\omega]^{g-8}k(p_3,\dots,p_g)=(-)^{g+1}c_{g,2}\frac{\kappa[v]}{\kappa[\omega]^9}(\det\omega_i(p_j))^{g+1} \ .$$

By Eq.(3.18), it follows that $\kappa[v]/\kappa[\omega]^9$ has a simple modular transformation

$$\frac{\kappa[v]}{\kappa[\omega]^9}\to\frac{\kappa[v]}{\kappa[\omega]^9}(\det(C\tau+D))^{-13} \ , \qquad \begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}(2g,\mathbb{Z}) \ ,$$

and, by using the modular transformation $\det\omega_i(p_j)\to\det\omega_i(p_j)\det(C\tau+D)$, we obtain

$$\kappa[\omega]^{g-8}k\to\kappa[\omega]^{g-8}k(\det(C\tau+D))^{g-12} \ .$$

Hence, $\kappa[\omega]^{g-8}k\otimes(\omega_1\wedge\ldots\wedge\omega_g)^{12-g}$ is modular invariant and determines a section of $\mathrm{Sym}^{g-2}E_{g-3}\otimes\lambda_1^{12-g}$ on $\mathcal{M}_g$. Since $\kappa[\omega]\neq 0$ for all C, $k=0$ at the point corresponding to the C if and only if $k(p_3,\dots,p_g)=0$ for all $p_3,\dots,p_g\in C$, or, equivalently, if and only if C is hyperelliptic. □

For $g=2$ the section k corresponds to

$$k=\kappa[\omega]^6k(\omega_1\wedge\omega_2)^{10} \ ,$$

and for $g=3$

$$k=\kappa[\omega]^5k(\omega_1\wedge\omega_2\wedge\omega_3)^9 \ .$$

Note that, for $g=2,3$, Eqs.(3.11) and (3.12) lead to the following relations

$$\frac{\kappa[v]}{\kappa[\sigma]^{g+1}}=\frac{\kappa[\omega\omega]}{\kappa[\omega]^{g+1}} \ ,$$

and, together with (3.19) and (3.20), we obtain the identification

$$\begin{aligned} k&=6\pi^{12}\kappa[\omega]^6\Psi_{10} \ , && g=2 \ ,\\ k&=15\cdot 2^6\kappa[\omega]^5\pi^{18}\Psi_9 \ , && g=3 \ ,\end{aligned}$$

recovering the results of Proposition 3.9.

Let C be a non-hyperelliptic curve of genus $g=4$. In this case, $k(p_3,p_4)$ is a holomorphic 1-differential in both p_3 and p_4, symmetric in its arguments. Then,

$$k^{(4)}:=\frac{\det k(p_i,p_j)}{(\det\omega_i(p_j))^2}=\det k_{ij} \ ,$$

is a meromorphic function on C in each p_i, $i\in I_4$.

Proposition 6.12. *The function $k^{(4)}$ is a constant on C that depends only on the choice of the marking. Furthermore, $k^{(4)}=0$ if and only if C is hyperelliptic or if C is non-hyperelliptic and admits a (necessarily even) singular spin structure.*

Proof. Let us suppose that, for a suitable choice of $p_1,\ldots,p_4\in C$, $\{k(p_i,z)\}_{i\in I_4}$ is a basis of $H^0(K_C)$. Then, the determinant

$$\frac{\det k(p_i,z_j)}{\det\omega_i(z_j)}\ ,$$

does not depend on the points $z_1,\ldots,z_4\in C$. Hence, the ratio

$$\frac{\det k(p_i,z_j)}{\det\omega_i(z_j)\det\omega_i(p_j)}\ ,$$

is a non-vanishing constant on C. In particular, by taking $p_i=z_i$, it follows that such a constant is $k^{(4)}$. On the contrary, if for all $p_1,\ldots,p_4\in C$, the holomorphic 1-differentials $k(p_i,z)$, $i\in I_4$, are linearly dependent, then $k^{(4)}$ vanishes identically.

Such a construction shows that $k^{(4)}$ vanishes if and only if $k(p_i,z)$, $i\in I_4$, are linearly dependent for all $p_1,\ldots,p_4\in C$. If C is hyperelliptic, then $k(p_i,p_j)=0$ for all $p_i,p_j\in C$ and $k^{(4)}=0$. Assume that C admits a singular spin structure α and let L_α be the corresponding holomorphic line bundle with $L_\alpha^2\cong K_C$. This implies that Θ_s consists of a unique point of order 2 in the Jacobian torus. For each $p\in C$, the holomorphic 1-differential $k(p,z)$ is the square of an element of $H^0(L_\alpha)$; by varying $p\in C$, such 1-differentials span the image of the map $\varphi:\mathrm{Sym}^2H^0(L_\alpha)\to H^0(K_C)$. If α is even, then $h^0(L_\alpha)=2$ and $\mathrm{Sym}^2H^0(L_\alpha)$ has dimension three, so that φ cannot be surjective and $k^{(4)}=0$. If α is odd, then $h^0(L_\alpha)=3$ so that, for each point $p\in C$, $h^0(L_\alpha\otimes\mathcal{O}(-p))\geq 2$; if h_1,h_2 span $H^0(L_\alpha\otimes\mathcal{O}(-p))$, then h_1/h_2 is a non-constant meromorphic function with 2 poles and C is hyperelliptic.

Suppose that C is non-hyperelliptic and does not admit a singular spin structure. Then, Θ_s consists of 2 distinct points, e and $-e$. Let us first observe that if there exist two points $p,q\in C$ such that $I(p-q)=2e$, then they are unique. For, if $I(\tilde p-\tilde q)=2e=I(p-q)$, then $p+\tilde q-\tilde p-q$ is the divisor of a meromorphic function on C. But, since C is non-hyperelliptic, the unique meromorphic function with less that 3 poles are the constants and, since $p\neq q$ (because $2e\neq 0$ in $J_0(C)$), it follows that $\tilde p=p$ and $\tilde q=q$.

Also, observe that $K(z,z)$ is not identically vanishing as a function of z; since C is compact, $K(z,z)$ has only a finite number of zeros. Fix a point $p_1\in C$ and define $x_1,x_2,y_1,y_2\in C$ by

$$I(p_1+x_1+x_2)=e\ ,\qquad I(p_1+y_1+y_2)=-e\ .$$

Then the divisor of $k(p_1,z)$ with respect to z is $2p_1+x_1+x_2+y_1+y_2$. Observe that at least one between x_1 and x_2 is distinct from y_1 and y_2, since otherwise we would have $e=-e$. We choose p_1 in such a way that p_1,x_1,x_2,y_1,y_2 are distinct from the zeros of $K(z,z)$ and from the points p,q such that $I(p-q)=2e$ (if they exist). Note that such a condition can always be fulfilled, since it is equivalent to require that p_1 is distinct from the zeros of $k(p,\cdot)$, $k(q,\cdot)$ and $k(w,\cdot)$ for each w such that $K(w,w)=0$. Then, the points for which such a condition is not satisfied is a finite set.

Set $p_2:=x_1$ and $p_3:=y_1$. The divisor of $k(p_3,z)$ is $(k(p_3,z))=2p_3+p_1+y_2+z_1+z_2$, where z_1,z_2 satisfy

$$I(p_3+z_1+z_2)=e\ .$$

Since the condition on the choice of p_1 implies $K(p_3, p_3) \neq 0$, it follows that z_1 and z_2 are distinct from p_3. Set $p_4 := z_1$, so that

$$\det k(p_i, p_j) = k(p_1, p_4)^2 k(p_2, p_3)^2 .$$

The identities

$$I(p_1 + p_2 + x_2 - p_3 - p_4 - z_2) = 0 , \qquad I(p_4 + z_2 - p_1 - y_2) = 2e ,$$

imply that p_4 and z_2 are distinct from p_1, p_2, x_2 (for example, if $p_4 = x_2$, then $p_1 + p_2 - p_3 - z_2$ is the divisor of a meromorphic function and C is hyperelliptic) and from y_2 (if $p_4 = y_2$, then $I(z_2 - p_1) = 2e$, counter the requirement that p_1 is distinct from q and p). Therefore, $k(p_1, p_4)k(p_2, p_3) \neq 0$ and then $k^{(4)} \neq 0$. □

By Propositions 6.12 and 6.11, it follows that, for $g = 4$,

$$k^{(4)} := \kappa[\omega]^{-16} \det k_{ij} (\omega_1 \wedge \cdots \wedge \omega_4)^{34} ,$$

is a holomorphic section of λ_1^{34} vanishing only on the hyperelliptic locus, with a zero of order $4[(3g - 3) - (2g - 1)] = 8$, and on the locus of Riemann surfaces with an even singular spin structure, with a zero of order 1. By Eq.(6.20), the following relation holds

$$k^{(4)} = A^4 \det_{ij \in I_4} \theta_{ij}(e) ,$$

where the constant A depends on the moduli. Recently, it has been shown that the Hessian $\det_{ij\in I_4} \theta_{ij}(e)$ plays a key role in the analysis of the Andreotti-Mayer loci at genus 4 and in the corresponding applications to the Schottky problem [21][31]. Whereas no natural generalization of such a Hessian exists at genus $g > 4$, the section $k^{(4)}$ is the $g = 4$ representative of a set of sections $k^{(g)}$ of a tensor power of λ_1 on $\mathcal{M}_g$, defined for each even $g \geq 4$.

Definition 6.2. Let C be a curve of even genus $g \geq 4$. Fix $N_{g-3} = h^0(K_C^{g-3})$ points $p_1, \ldots, p_{N_{g-3}} \in C$ and let $\{\phi_i\}_{i \in I_{N_{g-3}}}$ be a basis of $H^0(K_C^{g-3})$. Set

$$k^{(g)} := \frac{\kappa[\phi]^{g-2} \sum_{s^1,\ldots,s^{g-2}} \prod_{i=1}^{g-2} \epsilon(s^i) \prod_{j=1}^{N_{g-3}} k(p_{s^1_j}, \ldots, p_{s^{g-2}_j})}{N_{g-3}! \kappa[\omega]^{(2g-7)^2(g-2)+(8-g)N_{g-3}} \big(\det \phi(p_1, \ldots, p_{N_{g-3}})\big)^{g-2}} (\omega_1 \wedge \cdots \wedge \omega_g)^{d_g}$$

where $d_g := (12 - g)N_{g-3} + (g - 2)[6(g - 3)(g - 4) + 1]$ and the sum in the numerator runs over $g - 2$ permutations $s^1, \ldots, s^{g-2} \in \mathcal{P}_{N_{g-3}}$.

Proposition 6.13. *For all the even $g \geq 4$, $k^{(g)}$ does not depend on the points $p_1, \ldots, p_{N_{g-3}} \in C$ and on the basis $\{\phi_i\}_{i \in I_{N_{g-3}}}$ of $H^0(K_C^{g-3})$ and determines a section of $\lambda_1^{d_g}$ on $\mathcal{M}_g$.*

Proof. Choose $(g - 2)N_{g-3}$ points $p^i_1, \ldots, p^i_{N_{g-3}} \in C$, $i \in I_{g-2}$ and note that

$$\sum_{s^1,\ldots,s^{g-2} \in \mathcal{P}_{N_{g-3}}} \prod_{i=1}^{g-2} \epsilon(s^i) \prod_{j=1}^{N_{g-3}} k(p^i_{s^1_j}, \ldots, p^i_{s^{g-2}_j}) , \qquad (6.25)$$

is a product of $g-3$ differentials in each p_j^i, $i \in I_{g-2}, j \in I_{N_{g-3}}$. Such a product is completely anti-symmetric with respect to the permutations of each N_{g-3}-tuple $(p_1^i, \ldots, p_{N_{g-3}}^i)$, for all $i \in I_{g-2}$, so that it must be proportional to the determinant $\det \phi(p_1^i, \ldots, p_{N_{g-3}}^i)$. Therefore, the ratio of Eq.(6.25) and

$$\prod_{i \in I_{g-2}} \det \phi(p_1^i, \ldots, p_{N_{g-3}}^i) \ ,$$

does not depend on the points $p_1^i, \ldots, p_{N_{g-3}}^i \in C$, $i \in I_{g-2}$; in particular, by choosing, for each $j \in I_{N_{g-3}}$, $p_j^1 \equiv p_j^2 \equiv \ldots \equiv p_j^{g-2} \equiv p_j$, where $p_1, \ldots, p_{N_{g-3}}$ are the points in the definition 6.2, it follows that $k^{(g)}$ is a constant as a function of $C^{N_{g-3}}$. The proposition follows trivially by Theorem (6.11) and by the expression (3.18) of the Mumford form, with $n = g-3$. $\square$

Definition 6.2 and Proposition 6.13 make sense also at odd genera; however, simple algebraic considerations show that, in this case, $k^{(g)}$ is identically null on $\mathcal{M}_g$. In general, there exist some non-trivial generalizations of $k^{(4)}$ at odd genus, but they are not as simple as the ones at even g. An example at genus $g = 5$ is

$$\begin{aligned}\frac{(\omega_1 \wedge \cdots \wedge \omega_5)^{164} \kappa[\phi]^4}{\kappa[\omega]^{84} \big(\det \phi(p_1, \ldots, p_{12})\big)^4} \sum\nolimits_{i,j,k,l \in \mathcal{P}_{12}} & \epsilon(i)\epsilon(j)\epsilon(k)\epsilon(l) k(p_{i_1}, p_{i_2}, p_{i_3}) \\ & \cdot k(p_{i_4}, p_{i_5}, p_{j_1}) k(p_{i_6}, p_{i_7}, p_{k_1}) k(p_{i_8}, p_{i_9}, p_{l_1}) \\ & \cdot k(p_{i_{10}}, p_{j_2}, p_{j_3}) k(p_{i_{11}}, p_{k_2}, p_{k_3}) k(p_{i_{12}}, p_{l_2}, p_{l_3}) \\ & \cdot k(p_{j_4}, p_{j_5}, p_{j_6}) k(p_{j_7}, p_{j_8}, p_{k_4}) k(p_{j_9}, p_{j_{10}}, p_{l_4}) \\ & \cdot k(p_{j_{11}}, p_{k_5}, p_{k_6}) k(p_{j_{12}}, p_{l_5}, p_{l_6}) k(p_{k_7}, p_{k_8}, p_{k_9}) \\ & \cdot k(p_{k_{10}}, p_{k_{11}}, p_{l_7}) k(p_{k_{12}}, p_{l_8}, p_{l_9}) k(p_{l_{10}}, p_{l_{11}}, p_{l_{12}}),\end{aligned}$$

which does not depend on the points $p_1, \ldots, p_{12} \in C$ and corresponds to a section of λ_1^{164} on $\mathcal{M}_5$.

7. SIEGEL'S INDUCED MEASURE ON THE MODULI SPACE

In this section we derive the explicit expression of the metric $ds^2_{|\hat{\mathcal{M}}_g}$ on the moduli space $\hat{\mathcal{M}}_g$ of genus g canonical curves induced by the Siegel metric. This was previously known only for the trivial cases $g = 2$ and $g = 3$. By Wirtinger Theorem, the explicit expression for the volume form on $\hat{\mathcal{M}}_g$ is also obtained. A remarkable property of $ds^2_{|\hat{\mathcal{M}}_g}$ is that it is given by the Kodaira-Spencer map of the square of the Bergman reproducing kernel (times $4\pi^2$). This is one of the basic properties of the Bergman reproducing kernel derived in this section. Such an approach will led to a notable relation satisfied by the determinant of powers of the Bergman reproducing kernel. The results are a natural consequence of the present approach, which also uses, as for the present derivation of $ds^2_{|\hat{\mathcal{M}}_g}$, the isomorphisms introduced in section 2.1.

The Torelli space $\mathcal{T}_g$ of smooth algebraic curves of genus g can be embedded in $\mathfrak{H}_g$ by the period mapping, which assigns to a curve C, with a fixed basis of $H_1(C, \mathbb{Z})$, representing a point in $\mathcal{T}_g$, the corresponding period matrix. The period mapping has maximal rank $3g-3$ on the subspace $\hat{\mathcal{T}}_g$ of non-hyperelliptic curves and therefore a metric on $\mathfrak{H}_g$ induces the pull-back metric on $\hat{\mathcal{T}}_g$. It is therefore natural to consider the Siegel metric on $\mathfrak{H}_g$ [60]

$$ds^2 := \mathrm{Tr}\,(Y^{-1}dZY^{-1}d\bar{Z}) \ , \tag{7.1}$$

where $Y := \mathrm{Im}\, Z$, $Z \in \mathfrak{H}_g$. Such a metric is $\mathrm{Sp}(2g, \mathbb{R})$ invariant, and since $\hat{\mathcal{M}}_g \cong \hat{\mathcal{T}}_g/\Gamma_g$, it also induces a metric on $\hat{\mathcal{M}}_g$. The Siegel volume form is [60]

$$d\nu = \frac{i^M}{2^g} \frac{\bigwedge_{i\leq j}^{g}(\mathrm{d}Z_{ij} \wedge \mathrm{d}\bar{Z}_{ij})}{(\det Y)^{g+1}} \ . \tag{7.2}$$

The explicit expression of the volume forms on $\hat{\mathcal{M}}_g$ induced by the Siegel metric, which coincides with (7.2) for $g = 2$ and $g = 3$ non-hyperelliptic curves, is given in Theorem 7.7. It is simply written in terms of the Riemann period matrix τ_{ij} and of the basis $\{d\tau_{ij}\}$ of $T^*\hat{\mathcal{T}}_g$.

The Laplacian associated to the Siegel's symplectic metric were derived, ten years after Siegel's paper [60], by H. Maass [43]

$$\Delta = 4\,\mathrm{Tr}\left(Y\, {}^t\!\left(Y\frac{\partial}{\partial\bar{Z}}\right)\frac{\partial}{\partial Z}\right) \ . \tag{7.3}$$

As we will see, as a byproduct of the present approach, and of the formalism developed in section 2.1 in particular, both (7.2) and (7.3) are straightforwardly derived.

7.1 Derivation of the volume form and the Laplacian on $\mathfrak{H}_g$

Proposition 7.1. *The Siegel metric* (7.1) *can be equivalently expressed in the form*

$$ds^2 = \sum_{i,j=1}^{M} g_{ij}^S dZ_i d\bar{Z}_j \ , \tag{7.4}$$

where

$$g_{ij}^S(Z,\bar{Z}) := 2\frac{Y^{-1}_{\mathtt{1}_i\mathtt{1}_j}Y^{-1}_{\mathtt{2}_i\mathtt{2}_j} + Y^{-1}_{\mathtt{1}_i\mathtt{2}_j}Y^{-1}_{\mathtt{2}_i\mathtt{1}_j}}{(1+\delta_{\mathtt{1}_i\mathtt{2}_i})(1+\delta_{\mathtt{1}_j\mathtt{2}_j})} = 2\chi_i^{-1}\chi_j^{-1}(Y^{-1}Y^{-1})_{ij} \ , \tag{7.5}$$

$i,j \in I_M$.

Proof. For $n=2$ the identity (2.3) reads

$$\sum_{i,j=1}^{g} f(i,j) = \sum_{k=1}^{M}(2-\delta_{\mathtt{1}_k\mathtt{2}_k})f(\mathtt{1}_k,\mathtt{2}_k) \ ,$$

where we used the identity

$$2-\delta_{ij} = \frac{2}{1+\delta_{ij}} \ .$$

Hence

$$\begin{aligned} ds^2 &= \sum_{i,j,k,l=1}^{g} Y_{ij}^{-1}dZ_{jk}Y_{kl}^{-1}d\bar{Z}_{li} \qquad (7.6)\\ &= \sum_{i,j=1}^{g} d\bar{Z}_{ji}\sum_{m=1}^{M}\frac{Y^{-1}_{i\mathtt{1}_m}Y^{-1}_{j\mathtt{2}_m}+Y^{-1}_{i\mathtt{2}_m}Y^{-1}_{j\mathtt{1}_m}}{1+\delta_{\mathtt{1}_m\mathtt{2}_m}}dZ_{\mathtt{1}_m\mathtt{2}_m}\\ &= \sum_{m,n=1}^{M}(2-\delta_{\mathtt{1}_n\mathtt{2}_n})d\bar{Z}_{\mathtt{1}_n\mathtt{2}_n}\frac{Y^{-1}_{\mathtt{1}_n\mathtt{1}_m}Y^{-1}_{\mathtt{2}_n\mathtt{2}_m}+Y^{-1}_{\mathtt{1}_n\mathtt{2}_m}Y^{-1}_{\mathtt{2}_n\mathtt{1}_m}}{1+\delta_{\mathtt{1}_m\mathtt{2}_m}}dZ_{\mathtt{1}_m\mathtt{2}_m}\\ &= \sum_{m,n=1}^{M} 2\chi_m^{-1}\chi_n^{-1}(Y^{-1}Y^{-1})_{nm}dZ_m d\bar{Z}_n \ . \end{aligned}$$

$\square$

Let

$$\omega := \frac{i}{2}\sum_{i,j=1}^{M} g_{ij}^S dZ_i\wedge d\bar{Z}_j \ , \tag{7.7}$$

be the $(1,1)$-form associated to the Siegel metric on $\mathfrak{H}_g$, so that the volume form on $\mathfrak{H}_g$ is

$$\frac{1}{M!}\omega^M = \left(\frac{i}{2}\right)^M \det g_{ij}^S \bigwedge_{i\le j}^{g}(\mathrm{d}Z_{ij}\wedge \mathrm{d}\bar{Z}_{ij}) \ .$$

Proposition 7.2.

$$\det g^S_{ij} = \frac{2^{M-g}}{(\det Y)^{g+1}} \ .$$

Proof. Since Y is symmetric and positive-definite, we have

$$PY^{-1}P^{-1} = \mathrm{diag}(\lambda_1, \dots, \lambda_g) \equiv D \ ,$$

for some non-singular $g \times g$ matrix P and some positive $\lambda_1, \dots, \lambda_g$. By (7.5) and (2.6)

$$\begin{aligned}\det g^S_{ij} =& 2^M \det{}_{ij} \frac{(Y^{-1}Y^{-1})_{ij}}{\chi_i \chi_j} \\ =& 2^M \det{}_{ij} \frac{(PP)_{ij}}{\chi_j} \det{}_{ij} \frac{(P^{-1}P^{-1})_{ij}}{\chi_j} \det{}_{ij} \frac{(Y^{-1}Y^{-1})_{ij}}{\chi_i \chi_j},\end{aligned}$$

and by (2.4)

$$\det g^S_{ij} = 2^M \det{}_{ij} \frac{(DD)_{ij}}{\chi_i \chi_j} = 2^M \det{}_{ij} \frac{\lambda \lambda_i (\delta\delta)_{ij}}{\chi_i \chi_j} \ .$$

The proposition then follows observing that $(\delta\delta)_{ij} = \chi_j \delta_{ij}$ and that (2.7) yields

$$\det g^S_{ij} = 2^M \prod_{i=1}^{M} \lambda \lambda_i \chi_i^{-1} = 2^{M-g} \Big(\prod_{k=1}^{g} \lambda_k \Big)^{g+1} \ .$$

□

Proposition 7.3. *The Laplace-Beltrami operator acting on functions on $\mathfrak{H}_g$ is*

$$\Delta = \frac{1}{2} \sum_{i,j=1}^{M} (YY)_{ij} \frac{\partial}{\partial Z_i} \frac{\partial}{\partial \bar{Z}_j} \ .$$

Proof. Just use the definition of Δ and note that $g^{S\,ij} = (YY)_{ij}/2$. □

7.2 The Siegel metric on the moduli space

The following theorem provides a modular invariant basis of the fiber of $T^*\hat{\mathcal{I}}_g$ at the point representing C.

Theorem 7.4. *If $p_3, \dots, p_g \in C$ are $g-2$ pairwise distinct points such that $K(p_3, \dots, p_g) \neq 0$, then*

$$\Xi_i := \sum_{j=1}^{M} X^\omega_{ji} d\tau_j \ , \tag{7.8}$$

$i \in I_N$, with X^ω_{ij}, $i, j \in I_M$, defined in Eq.(5.23), is a modular invariant basis of the fiber of $T^*\hat{\mathcal{I}}_g$ at the point representing C.

Proof. Consider the Kodaira-Spencer map k identifying the space of quadratic differentials on C with the fiber of the cotangent bundle of $\mathcal{M}_g$ at the point representing C. Next, consider a Beltrami differential $\mu \in \Gamma(\bar{K}_C \otimes K_C^{-1})$ (see [11] for explicit constructions) and recall that it defines a tangent vector at C of $\mathcal{T}_g$. The derivative of the period map $\tau_{ij} : \mathcal{T}_g \to \mathbb{C}$ at C in the direction of μ is given by Rauch's formula

$$d_C\tau_{ij}(\mu) = \int_C \mu\omega_i\omega_j \ .$$

It follows that

$$k(\omega_j\omega_k) = \frac{1}{2\pi i} d\tau_{jk} \ ,$$

$j, k \in I_g$, so that, by (5.22),

$$k(v_j) = \frac{1}{2\pi i}\sum_{k=1}^{M} X^{\omega}_{kj} d\tau_k \ , \tag{7.9}$$

$j \in I_N$, where

$$d\tau_i := d\tau_{1_i 2_i} \ ,$$

$i \in I_M$. It follows that the differentials

$$\Xi_j := 2\pi i\, k(v_j) \ , \tag{7.10}$$

$j \in I_N$, are linearly independent. Furthermore, since by construction the basis $\{v_i\}_{i\in I_N}$ is independent of the choice of a symplectic basis of $H_1(C, \mathbb{Z})$, such differentials are modular invariant, i.e.

$$\Xi_i \ \mapsto \ \tilde{\Xi}_i = \Xi_i \ , \tag{7.11}$$

$i \in I_N$, under (B.3). □

Let $ds^2_{|\hat{\mathcal{M}}_g}$ be the metric on $\hat{\mathcal{M}}_g$ induced by the Siegel metric. Set

$$g^{\tau}_{ij} := g^{S}_{ij}(\tau, \bar{\tau}) = 2\chi_i^{-1}\chi_j^{-1}(\operatorname{Im}\tau^{-1}\operatorname{Im}\tau^{-1})_{ij} \ . \tag{7.12}$$

Corollary 7.5.

$$ds^2_{|\hat{\mathcal{M}}_g} = \sum_{i,j=1}^{N} g^{\Xi}_{ij}\Xi_i\bar{\Xi}_j \ , \tag{7.13}$$

where

$$g^{\Xi}_{ij} := \sum_{k,l=1}^{M} g^{\tau}_{kl} B^{\omega}_{ik}\bar{B}^{\omega}_{jl} \ , \tag{7.14}$$

and B^ω is the matrix defined in (5.11) *with $\eta_i \equiv \omega_i$, $i \in I_g$. Furthermore, the volume form on $\hat{\mathcal{M}}_g$ induced by the Siegel metric is*

$$d\nu_{|\hat{\mathcal{M}}_g} = \left(\frac{i}{2}\right)^N \det g^{\Xi}\, dw \wedge d\bar{w} \ , \tag{7.15}$$

where

$$dw := \sum_{i_N > \ldots > i_1 = 1}^{M} X^{\omega}{}^{1\ \ldots\ N}_{i_1 \ldots i_N} d\tau_{i_1} \wedge \cdots \wedge d\tau_{i_N} \ . \tag{7.16}$$

Proof. By (7.4) and (7.5)

$$ds^2_{|\hat{\mathcal{M}}_g} = \sum_{k,l=1}^{M} g^{\tau}_{ij} d\tau_i d\bar{\tau}_j \ . \tag{7.17}$$

Furthermore, by applying the Kodaira-Spencer map to both sides of Eq.(5.10), one obtains

$$d\tau_i = \sum_{j=1}^{N} B^{\omega}_{ji}\, \Xi_j \ , \tag{7.18}$$

$i \in I_M$, and (7.13) follows. On the other hand, by (7.13)

$$d\nu_{|\hat{\mathcal{M}}_g} = \left(\frac{i}{2}\right)^N \det g^{\Xi} \ \textstyle\bigwedge_1^N (\Xi_i \wedge \bar{\Xi}_i) \ , \tag{7.19}$$

and by Theorem 7.4 the proof is completed. □

Applying the Kodaira-Spencer map to (5.24) yields the linear relations satisfied by $d\tau_i$, $i \in I_M$.

Corollary 7.6. *The $(g-2)(g-3)/2$ linear relations*

$$\sum_{j=1}^{M} C^{\omega}_{ij} d\tau_j = 0 \ , \tag{7.20}$$

$N+1 \le i \le M$, where the matrices C^{ω} are defined in (5.25)*, hold.*

Set $\operatorname{Im}\tau := \operatorname{Im}\tau$ and consider the Bergman reproducing kernel

$$B(z,\bar{w}) := \sum_{i,j=1}^{g} \omega_i(z)(\operatorname{Im}\tau)^{-1}_{ij}\bar{\omega}_j(w) \ ,$$

for all $z, w \in C$., and Set $\mathcal{K}(\phi\bar{\psi}) := k(\phi)\bar{k}(\psi)$, $\bar{k}(\bar{\psi}) = \overline{k(\psi)}$, for all $\phi, \psi \in H^0(K_C^2)$, where k is the Kodaira-Spencer map.

Theorem 7.7.

$$ds^2_{|\hat{\mathcal{M}}_g} = 4\pi^2 \mathcal{K}(B^2) \ . \tag{7.21}$$

Furthermore, the volume form on $\hat{\mathcal{M}}_g$ induced by the Siegel metric is

$$d\nu_{|\hat{\mathcal{M}}_g} = i^N \sum_{\substack{i_N > \ldots > i_1 = 1 \\ j_N > \ldots > j_1 = 1}}^{M} \frac{\left|\operatorname{Im}\tau^{-1}\operatorname{Im}\tau^{-1}\right|^{i_1\ldots i_N}_{j_1\ldots j_N}}{\prod_{k=1}^{N}(1+\delta_{1_{i_k}2_{i_k}})(1+\delta_{1_{j_k}2_{j_k}})} \bigwedge_{l=1}^{N} (d\tau_{i_l} \wedge d\bar{\tau}_{j_l}) \ . \tag{7.22}$$

Proof. Eq.(7.21) is an immediate consequence of Proposition 7.1 and of the application of the Kodaira-Spencer map to the identity

$$\sum_{i,j=1}^{M} \omega\omega_i(z) g^{\tau}_{ij} \bar{\omega}\bar{\omega}_j(w) = B^2(z,\bar{w}) \ . \tag{7.23}$$

Consider the $(1,1)$-form ω defined in Eq.(7.7). By Wirtinger's Theorem [30], the volume form on a d-dimensional complex submanifold S is

$$\frac{1}{d!}\omega^d \ ,$$

so that the volume of S is expressed as the integral over S of a globally defined differential form on $\mathfrak{H}_g$. Note that

$$\begin{aligned} d\nu_{|\hat{\mathcal{M}}_g} =& \frac{i^N}{2^N N!}\sum_{\substack{i_1,\ldots,i_N=1\\ j_1,\ldots,j_N=1}}^{M}\prod_{k=1}^{N} g^\tau_{i_k j_k}\bigwedge_{l=1}^{N}(d\tau_{i_l}\wedge d\bar{\tau}_{j_l}) \\ =& \frac{i^N}{2^N N!}\sum_{\substack{i_N>\ldots>i_1=1\\ j_N>\ldots>j_1=1}}^{M}\sum_{r,s\in\mathcal{P}_N}\epsilon(r)\epsilon(s)\prod_{k=1}^{N} g^\tau_{i_{r(k)}j_{s(k)}}\bigwedge_{l=1}^{N}(d\tau_{i_l}\wedge d\bar{\tau}_{j_l}) \ , \end{aligned}$$

and Eq.(7.22) follows by the identity

$$\sum_{r,s\in\mathcal{P}_N}\epsilon(r)\epsilon(s)\prod_{k=1}^{N} g^\tau_{i_{r(ks)}j_{s(k)}} = N!\,|g^\tau|^{i_1\ldots i_N}_{j_1\ldots j_N} \ .$$

□

Fix the points $z_1,\ldots,z_N\in C$ satisfying the conditions of Proposition 4.1. The basis $\{\gamma_i\}_{i\in I_N}$ of $H^0(K_C^2)$, with $\gamma_i\equiv\gamma_i^2$, $i\in I_N$, defined by Eq.(4.1) in the case $n=2$, satisfies the relations

$$\omega\omega_i=\sum_{j=1}^{N}\omega\omega_i(z_j)\gamma_j \ , \qquad v_i=\sum_{j=1}^{N}v_i(z_j)\gamma_j \ ,$$

$i\in I_M$. Set $\Gamma_i:=(2\pi i)^{-1}k(\gamma_i)$ and $[v]_{ij}:=v_i(z_j)$, $i,j\in I_N$.

Corollary 7.8. *Fix the points $z_1,\ldots,z_N\in C$ in such a way that $\det\phi_i(z_j)\neq 0$, for any arbitrary basis $\{\phi_i\}_{i\in I_N}$ of $H^0_C(K^2)$. The metric on $\hat{\mathcal{M}}_g$ induced by the Siegel metric is*

$$ds^2_{|\hat{\mathcal{M}}_g}=\sum_{i,j=1}^{N}B^2(z_i,\bar{z}_j)\Gamma_i\bar{\Gamma}_j \ , \qquad (7.24)$$

and the volume form is

$$d\nu_{|\hat{\mathcal{M}}_g}=\left(\frac{i}{2}\right)^N\det B^2(z_i,\bar{z}_j)\ \textstyle\bigwedge_1^N(\Gamma_i\wedge\bar{\Gamma}_i)=\left(\frac{i}{2}\right)^N\frac{\det B^2(z_i,\bar{z}_j)}{|\det v_i(z_j)|^2}\,dw\wedge d\bar{w} \ , \qquad (7.25)$$

where $\{v_i\}_{i\in I_N}$ is the basis of $H^0(K_C^2)$ defined in Proposition 4.4 and dw is defined in Eq.7.16.

Proof. Eq.(7.24), and therefore the first equality in Eq.(7.25), follows substituting

$$d\tau_i=\sum_{j=1}^{N}\omega\omega_i(z_j)\Gamma_j \ ,$$

$i \in I_M$, in (7.17) and then using the identity (7.23). Next, note that comparing (7.24) and (7.13), and by $\Xi_i = \sum_{j=1}^N [v]_{ij}\Gamma_j$, $i \in I_N$, yields

$$\sum_{k,l=1}^N [v]_{ki} g^{\Xi}_{kl} [\bar{v}]_{lj} = B^2(z_i, \bar{z}_j) \ ,$$

which also follows by the definition (7.14) of g^Ξ and by Eq.(5.10), with $\eta_i \equiv \omega_i$, $i \in I_g$, and Eq.(7.23). Hence

$$\det\, g^{\Xi} = \frac{\det B^2(z_i, \bar{z}_j)}{|\det v_i(z_j)|^2} \ , \tag{7.26}$$

which also follows by $\det \gamma_i(z_j) = 1$ and

$$\Xi_1 \wedge \cdots \wedge \Xi_N = \det\, v_i(z_j)\Gamma_1 \wedge \cdots \wedge \Gamma_N \ ,$$

and the second equality in Eq.(7.25) follows. □

7.3 Determinants of powers of the Bergman reproducing kernel

Corollary 7.8, in particular Eq.(7.25), implies that the ratio

$$\frac{\det B^2(z_i, \bar{z}_j)}{|\det v_i(z_j)|^2} \ ,$$

does not depend on z_i, $i \in I_N$, and therefore $\det B^2(z_i, \bar{z}_j)$ factorizes into a product of a holomorphic times an antiholomorphic function of $z_1, \ldots, z_N$. This is a special case of a more general theorem.

Theorem 7.9. *Fix $n \in \mathbb{N}_+$ and set*

$$B_A(z, \bar{w}) := \sum_{i,j=1}^g \omega_i(z) A_{ij} \bar{\omega}_j(w) \ ,$$

where A is a complex $g \times g$ matrix. Then, for all $z_i, w_i \in C$, $i \in I_{N_n}$,

$$\det B^n_A(z_i, \bar{w}_j) = \left|\kappa[\phi^n]\right|^{-2} \det \phi^n(z_1, \ldots, z_{N_n}) \det \bar{\phi}^n(w_1, \ldots, w_{N_n}) K_n(A) \ , \tag{7.27}$$

where $\{\phi^n_i\}_{i \in I_{N_n}}$ is an arbitrary basis of $H^0(K^n_C)$ and

$$K_n(A) = \sum_{\substack{i_{N_n} > \ldots > i_1 = 1 \\ j_{N_n} > \ldots > j_1 = 1}}^{M_n} \kappa[\omega \cdots \omega_{i_1}, \ldots, \omega \cdots \omega_{i_{N_n}}] \cdot \frac{|A \ldots A|^{i_1 \ldots i_{N_n}}_{j_1 \ldots j_{N_n}}}{\prod_{k=1}^{N_n} \chi_{i_k} \chi_{j_k}} \bar{\kappa}[\omega \cdots \omega_{j_1}, \ldots, \omega \cdots \omega_{j_{N_n}}] \ . \tag{7.28}$$

Furthermore, for $n \geq 2$

$$\det B^n_A(z_i, \bar{z}_j) = \Big|\theta_\Delta\big(\sum_1^{N_n} z_i\big) \prod_{i<j}^{N_n} E(z_i, z_j) \prod_1^{N_n} \sigma(z_i)^{2n-1}\Big|^2 K_n(A) \ . \tag{7.29}$$

Proof. Observe that

$$B_A^n(z_i,\bar w_j) = \sum_{\substack{k_1,\ldots,k_n=1\\ l_1,\ldots,l_n=1}}^{g} \omega_{k_1}(z_i)\cdots\omega_{k_n}(z_i)A_{k_1l_1}\cdots A_{k_nl_n}\bar\omega_{l_1}(w_j)\cdots\bar\omega_{l_n}(w_j)$$
$$= \sum_{k,l=1}^{M_n} \omega\cdots\omega_k(z_i)\frac{(A\cdots A)_{kl}}{\chi_k\chi_l}\bar\omega\cdots\bar\omega_l(w_j)\ ,$$

with the notation of section 2.1. Then

$$\det B_A^n(z_i,\bar w_j) = \sum_{\substack{k_1,\ldots,k_{N_n}=1\\ l_1,\ldots,l_n=1}}^{M_n}\ \sum_{s\in\mathcal{P}_{N_n}} \epsilon(s)\prod_{i=1}^{N_n}\omega\cdots\omega_{k_i}(z_i)\bar\omega\cdots\bar\omega_{l_i}(w_{s_i})\frac{(A\cdots A)_{k_il_i}}{\chi_{k_i}\chi_{l_i}},$$

and by defining $m_{s_i} := l_i$, $i\in I_{M_n}$, $\det B_A^n(z_i,\bar w_j)$ becomes

$$\sum_{\substack{k_1,\ldots,k_{N_n}=1\\ m_1,\ldots,m_{N_n}=1}}^{M_n} \frac{|A\ldots A|^{k_1\ldots k_{N_n}}_{m_1\ldots m_{N_n}}}{\prod_{i=1}^{N_n}\chi_{k_i}\chi_{m_i}}\prod_{i=1}^{N_n}\omega\cdots\omega_{k_i}(z_i)\bar\omega\cdots\bar\omega_{m_i}(w_i)$$
$$= \sum_{\substack{k_{N_n}>\ldots>k_1=1\\ m_{N_n}>\ldots>m_1=1}}^{M_n} \frac{|A\ldots A|^{k_1\ldots k_{N_n}}_{m_1\ldots m_{N_n}}}{\prod_{i=1}^{N_n}\chi_{k_i}\chi_{m_i}}\sum_{r,s\in\mathcal{P}_{N_n}}\epsilon(r)\epsilon(s)\prod_{i=1}^{N_n}\omega\cdots\omega_{k_{r_i}}(z_i)\bar\omega\cdots\bar\omega_{m_{s_i}}(w_i)$$
$$= \sum_{\substack{k_{N_n}>\ldots>k_1=1\\ m_{N_n}>\ldots>m_1=1}}^{M_n} \frac{|A\ldots A|^{k_1\ldots k_{N_n}}_{m_1\ldots m_{N_n}}}{\prod_{i=1}^{N_n}\chi_{k_i}\chi_{m_i}}\det_{\substack{i=k_1,\ldots,k_{N_n}\\ j=1,\ldots,N_n}}\omega\cdots\omega_i(z_j)\det_{\substack{i=m_1,\ldots,m_{N_n}\\ j=1,\ldots,N_n}}\bar\omega\cdots\bar\omega_i(w_j).$$

By Eq.(3.4), for an arbitrary basis $\{\phi_i^n\}_{i\in I_{N_n}}$ of $H^0(K_C^n)$

$$\det_{\substack{i\in\{k_1,\ldots,k_{N_n}\}\\ j\in I_{N_n}}}\omega\cdots\omega_i(z_j) = \det\phi^n(z_1,\ldots,z_{N_n})\frac{\kappa[\omega\cdots\omega_{k_1},\ldots,\omega\cdots\omega_{k_{N_n}}]}{\kappa[\phi^n]}\ ,$$

leading to (7.27). Eq.(7.29) then follows by Eq.(3.3). □

8. A GENUS 4 EXAMPLE: A 3-FOLD COVERING OF THE SPHERE

In this chapter, the objects defined in the previous chapters, in particular the distinguished basis $\{\sigma_i\}_{i\in I_g}$ of $H^0(K_C)$, are explicitly obtained for a family of non-hyperelliptic curves of genus 4, in terms of the algenraic parameters of the family.

8.1 Definition and main properties

Let C be the non-hyperelliptic curve of genus 4 defined by

$$w^3 = z(z-1)(z-\lambda_1)^2(z-\lambda_2)^2(z-\lambda_3)^2 \ ,$$

$(z,w)\in \mathbb{P}_1\times\mathbb{P}_1$ and let

$$\begin{aligned} q_0 &= z^{-1}(0) \ , & q_1 &= z^{-1}(1) \ , \\ q_\infty &= z^{-1}(\infty) \ , & p_i &= z^{-1}(\lambda_i) \ , \quad i\in I_3 \ , \end{aligned}$$

be the branching points on C, all with branching number 1. Since

$$(z) = 3q_0 - 3q_\infty \ , \qquad (w) = q_0 + q_1 + 2p_1 + 2p_2 + 2p_3 - 8q_\infty \ ,$$
$$(dz) = 2q_0 + 2q_1 + 2p_1 + 2p_2 + 2p_3 - 4q_\infty \ ,$$

a basis of $H^0(K_C)$ is given by

$$\begin{aligned} \varphi_1 &:= \frac{dz}{w} \ , & \varphi_3 &:= \frac{\left[\prod_{i=1}^3 (z-\lambda_i)\right]dz}{w^2} \ , \\ \varphi_2 &:= \frac{z\,dz}{w} \ , & \varphi_4 &:= \frac{z\left[\prod_{i=1}^3 (z-\lambda_i)\right]dz}{w^2} \ . \end{aligned}$$

Note that φ_1,φ_2 generate $U := H^0(K_C\otimes\mathcal{O}(-q_0-q_1-q_\infty))$, whereas φ_3,φ_4 generate $V := H^0(K_C\otimes\mathcal{O}(-p_1-p_2-p_3))$. Consider the automorphism ϕ of C given by $\phi(z,w) := (z,\zeta w)$, where $\zeta := e^{2\pi i/3}$; then, the pull-back ϕ^* is an automorphism of $H^0(K_C)$ and U and V are the eigenspaces corresponding to the eigenvalues, respectively, ζ^2 and ζ.

The Riemann surface C is a 3-fold covering of the sphere. Let us define the j-th sheet, $j=0,1,2$, as the one containing the line $\operatorname{Im} z = 0$, $\arg w = j(2\pi i/3)$. Let us fix a basis of $H_1(C,\mathbb{Z})$ as in the following figure.

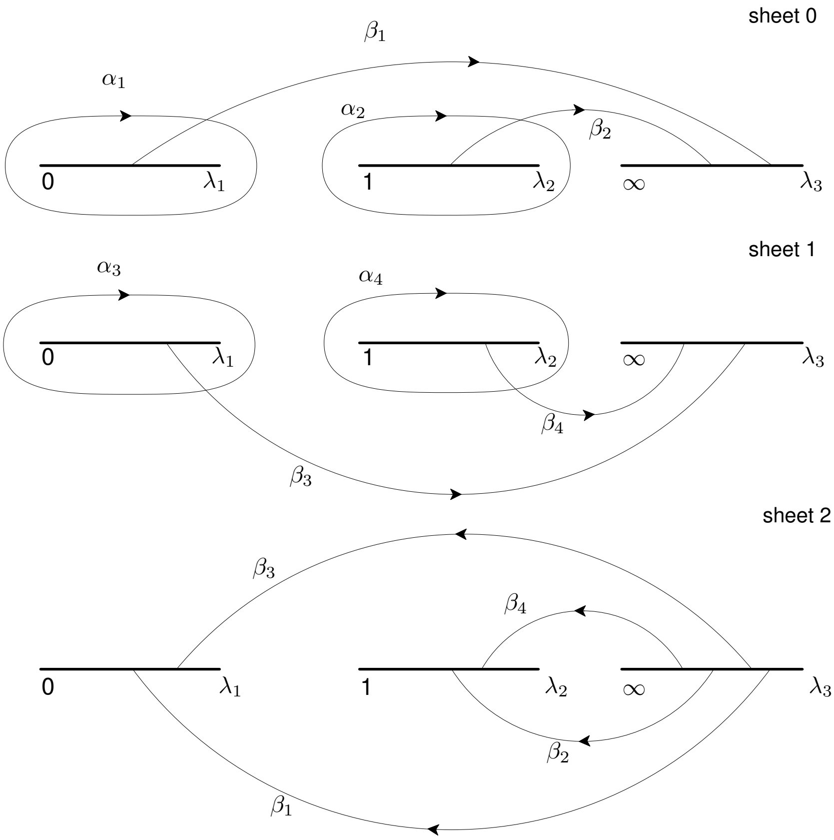

Fig. 1.

$$\int_{\alpha_1} \eta = \int_0^{\lambda_1} \eta - \int_0^{\lambda_1} \phi^* \eta \ , \qquad \int_{\alpha_2} \eta = \int_1^{\lambda_2} \eta - \int_1^{\lambda_2} \phi^* \eta \ ,$$
$$\int_{\alpha_3} \eta = \int_0^{\lambda_1} \phi^* \eta - \int_0^{\lambda_1} \phi^{*2} \eta \ , \qquad \int_{\alpha_4} \eta = \int_1^{\lambda_2} \phi^* \eta - \int_1^{\lambda_2} \phi^{*2} \eta \ ,$$
$$\int_{\beta_1} \eta = \int_{\lambda_1}^{\infty} \eta - \int_{\lambda_1}^{\infty} \phi^{*2} \eta \ , \qquad \int_{\beta_2} \eta = \int_{\lambda_2}^{\infty} \eta - \int_{\lambda_2}^{\infty} \phi^{*2} \eta \ ,$$
$$\int_{\beta_3} \eta = \int_{\lambda_1}^{\infty} \phi^* \eta - \int_{\lambda_1}^{\infty} \phi^{*2} \eta \ , \qquad \int_{\beta_4} \eta = \int_{\lambda_2}^{\infty} \phi^* \eta - \int_{\lambda_2}^{\infty} \phi^{*2} \eta \ .$$

Let $u_1, u_2 \in U$ and $v_1, v_2 \in V$ the holomorphic 1-differentials satisfying

$$\int_0^{\lambda_1} u_1 = \frac{1}{3} = \int_0^{\lambda_1} v_1 \ , \qquad \int_1^{\lambda_2} u_1 = 0 = \int_1^{\lambda_2} v_1 \ ,$$
$$\int_0^{\lambda_1} u_2 = 0 = \int_0^{\lambda_1} v_2 \ , \qquad \int_1^{\lambda_2} u_2 = \frac{1}{3} = \int_1^{\lambda_2} v_2 \ ,$$

where the integration is above the cuts in the 0-sheet. Consistency requires

$$\int_{\lambda_3}^{\infty} u_1 = \int_{\lambda_3}^{\infty} u_1 = \int_{\lambda_3}^{\infty} u_1 = \int_{\lambda_3}^{\infty} u_1 = \frac{1}{3} \ ,$$

where integration is along the cut in the 0-sheet. Then, it can be easily verified that

$$\omega_1 := u_1 + v_1 \ , \quad \omega_2 := u_2 + v_2 \ , \quad \omega_3 := -\zeta^2 u_1 - \zeta v_1 \ , \quad \omega_4 := -\zeta^2 u_2 - \zeta v_2 \ , \tag{8.1}$$

is the canonical basis of $H^0(K_C)$ associated to our choice of basis of $H_1(C,\mathbb{Z})$. Furthermore, a lengthy but straightforward calculation yields

$$\tau = \begin{pmatrix} 2a & 2c & a & c \\ 2c & 2b & c & b \\ a & c & 2a & 2c \\ c & b & 2c & 2b \end{pmatrix} ,$$

where

$$\begin{array}{llll}
\int_{\lambda_1}^{\infty} \omega_1 = a \ , & \int_{\lambda_1}^{\infty} \omega_2 = c \ , & \int_{\lambda_1}^{\infty} \omega_3 = 0 \ , & \int_{\lambda_1}^{\infty} \omega_4 = 0 \ , \\
\int_{\lambda_2}^{\infty} \omega_1 = c \ , & \int_{\lambda_2}^{\infty} \omega_2 = b \ , & \int_{\lambda_2}^{\infty} \omega_3 = 0 \ , & \int_{\lambda_2}^{\infty} \omega_4 = 0 \ , \\
\int_{\lambda_3}^{\infty} \omega_1 = \frac{2}{3} \ , & \int_{\lambda_3}^{\infty} \omega_2 = \frac{2}{3} \ , & \int_{\lambda_3}^{\infty} \omega_3 = \frac{1}{3} \ , & \int_{\lambda_3}^{\infty} \omega_4 = \frac{1}{3} \ , \\
\int_{0}^{\infty} \omega_1 = \frac{2}{3} + a \ , & \int_{0}^{\infty} \omega_2 = c \ , & \int_{0}^{\infty} \omega_3 = \frac{1}{3} \ , & \int_{0}^{\infty} \omega_4 = 0 \ , \\
\int_{1}^{\infty} \omega_1 = c \ , & \int_{1}^{\infty} \omega_2 = \frac{2}{3} + b \ , & \int_{1}^{\infty} \omega_3 = 0 \ , & \int_{1}^{\infty} \omega_4 = \frac{1}{3} \ ,
\end{array}$$

where the path of integration is along the cuts and along the arcs representing part of the β-cycles in the 0-sheet (see the figure above).

The vector of Riemann constants with base point q_∞ can be computed to be

$$K^{q_\infty} = \begin{pmatrix} 1/2 + a - 1/6 + c \\ 1/2 + b - 1/6 + c \\ 1/2 + a + 1/6 + c \\ 1/2 + b + 1/6 + c \end{pmatrix} = \frac{1}{3} \left[\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} + \tau \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right] .$$

Note that $\varphi_1, \varphi_2 \in H^0(K_C \otimes \mathcal{O}(-q_0 - q_1 - q_\infty))$ and $\varphi_1, \varphi_3 \in H^0(K_C \otimes \mathcal{O}(-3q_\infty))$, so that $I(q_0 + q_1 + q_\infty)$ and $I(3q_\infty)$ are in W_3^1. Furthermore, $I(q_0 + q_1 - 2q_\infty)$ is not a period, so that Θ_s consists of two distinct points e_1, e_2

$$e_1 = I(q_0 + q_1 + q_\infty - \Delta) = -K^{q_\infty} \ , \qquad e_2 = I(3q_\infty - \Delta) = K^{q_\infty} \ .$$

By Proposition 6.12, $k_4 \neq 0$ for such a curve.

The points p_1, p_2, q_0, q_1 satisfy the condition of Proposition 4.1 with $n = 1$, and the corresponding basis of $H^0(K_C)$ is

$$\begin{aligned}
\sigma_{p_1} :=& \frac{\sqrt[3]{\lambda_1(\lambda_1-1)(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2}}{3(\lambda_1-\lambda_2)} \frac{(z-\lambda_2)dz}{wd\zeta_{\lambda_1}} \ , \\
\sigma_{p_2} :=& \frac{\sqrt[3]{\lambda_2(\lambda_2-1)(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_1)^2}}{3(\lambda_2-\lambda_1)} \frac{(z-\lambda_1)dz}{wd\zeta_{\lambda_2}} \ , \\
\sigma_{q_0} :=& \frac{\left(\sqrt[3]{\lambda_1\lambda_2\lambda_3}\right)^2}{3\lambda_1\lambda_2\lambda_3} \frac{(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)dz}{w^2 d\zeta_0} \ , \\
\sigma_{q_1} :=& \frac{\left(\sqrt[3]{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}\right)^2}{3(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)} \frac{z(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)dz}{w^2 d\zeta_1} \ ,
\end{aligned}$$

where the argument a of each root is $0 \le a < 2\pi i/3$ and $\zeta_{\lambda_1}, \zeta_{\lambda_2}, \zeta_0, \zeta_1$ are local coordinates centered in p_1, p_2, q_0, q_1, respectively, such that

$$\begin{aligned}
& z(p) = \lambda_1 + \zeta_{\lambda_1}^3(p) \ , \\
& w(p) = \sqrt[3]{\lambda_1(\lambda_1-1)(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2}\zeta_{\lambda_1}^2 + O(\zeta_{\lambda_1}^3) \ , && \text{for } p \sim p_1 \ , \\
& z(p) = \lambda_2 + \zeta_{\lambda_2}^3(p) \ , \\
& w(p) = \sqrt[3]{\lambda_2(\lambda_2-1)(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_1)^2}\zeta_{\lambda_2}^2 + O(\zeta_{\lambda_2}^3) \ , && \text{for } p \sim p_2 \ , \\
& z(p) = \zeta_0^3(p) \ , \\
& w(p) = \sqrt[3]{\lambda_1\lambda_2\lambda_3}\zeta_0 + O(\zeta_0^2) \ , && \text{for } p \sim q_0 \ , \\
& z(p) = 1 + \zeta_1^3(p) \ , \\
& w(p) = \sqrt[3]{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}\zeta_1 + O(\zeta_1^2) \ , && \text{for } p \sim q_1 \ ,
\end{aligned}$$

with the same convention as before for the third roots.

By using Eq.(4.9),

$$\begin{aligned}
\sigma_{p_1} &= \frac{\sum_i^4 \theta_i(p_2+q_0+q_1)\omega_i}{\sum_i^4 \theta_i(p_2+q_0+q_1)\omega_i(p_1)} \ , \quad \sigma_{p_2} = \frac{\sum_i^4 \theta_i(p_1+q_0+q_1)\omega_i}{\sum_i^4 \theta_i(p_1+q_0+q_1)\omega_i(p_2)} \ , \\
\sigma_{q_0} &= \frac{\sum_i^4 \theta_i(p_1+p_2+q_1)\omega_i}{\sum_i^4 \theta_i(p_1+p_2+q_1)\omega_i(q_0)} \ , \quad \sigma_{p_2} = \frac{\sum_i^4 \theta_i(p_1+p_2+q_0)\omega_i}{\sum_i^4 \theta_i(p_1+p_2+q_0)\omega_i(q_1)} \ .
\end{aligned}$$

Note that $\sigma_{p_1}, \sigma_{p_2} \in U$, whereas $\sigma_{q_0}, \sigma_{q_1} \in V$, so that, by using the decomposition of the canonical basis under $H^0(K_C) \to U \oplus V$ given by Eq.(8.1), we obtain the following identities

$$\begin{aligned}
0 =& (\theta_1(p+q_0+q_1) - \zeta\theta_3(p+q_0+q_1))v_1 \\
& + (\theta_2(p+q_0+q_1) - \zeta\theta_4(p+q_0+q_1))v_2 \ , \\
0 =& (\theta_1(q+p_1+p_2) - \zeta^2\theta_3(q+p_1+p_2))u_1 \\
& + (\theta_2(q+p_1+p_2) - \zeta^2\theta_4(q+p_1+p_2))u_2 \ ,
\end{aligned}$$

for arbitrary $p, q \in C$. Since u_1, u_2, v_1, v_2 are linearly independent, we conclude that

$$\begin{aligned}
\frac{\theta_3(p+q_0+q_1)}{\theta_1(p+q_0+q_1)} &= \zeta = \frac{\theta_4(p+q_0+q_1)}{\theta_2(p+q_0+q_1)} \ , \\
\frac{\theta_3(q+p_1+p_2)}{\theta_1(q+p_1+p_2)} &= \zeta^2 = \frac{\theta_4(q+p_1+p_2)}{\theta_2(q+p_1+p_21)} \ ,
\end{aligned}$$

for all $p, q \in C$. Hence,

$$\sigma_{p_1} = \frac{\sum_i^2 \theta_i(p_2+q_0+q_1)u_i}{\sum_i^2 \theta_i(p_2+q_0+q_1)u_i(p_1)} \ , \quad \sigma_{p_2} = \frac{\sum_i^2 \theta_i(p_1+q_0+q_1)u_i}{\sum_i^2 \theta_i(p_1+q_0+q_1)u_i(p_2)} \ ,$$
$$\sigma_{q_0} = \frac{\sum_i^2 \theta_i(p_1+p_2+q_1)v_i}{\sum_i^2 \theta_i(p_1+p_2+q_1)v_i(q_0)} \ , \quad \sigma_{p_2} = \frac{\sum_i^2 \theta_i(p_1+p_2+q_0)v_i}{\sum_i^2 \theta_i(p_1+p_2+q_0)v_i(q_1)} \ .$$

Note that $K(p_1,p_2) = 0 = K(q_0,q_1)$, so that

$$k_4 = \frac{\left(\det \begin{pmatrix} k(p_1,q_0) & k(p_1,q_1) \\ k(p_2,q_0) & k(p_2,q_1) \end{pmatrix}\right)^2}{\left(\det \omega(p_1,p_2,q_0,q_1)\right)^2} \ .$$

An alternative formula for $k(p_1,q_0)$

$$\frac{c_{g,2} S(p_1+p_2+q_0+q_1)^4 E(p_1,q_0)^2 E(p_3,q_\infty) E(p_3,p_1)^2 E(q_0,q_\infty)^2 E(q_0,p_1)^4}{E(p_2,q_1)^3 \sigma(p_2)^4 \sigma(q_1)^4 \big(E(p_2,q_\infty)E(p_2,p_1)^2 E(q_1,p_3) E(q_1,q_0)^2\big)^3}$$

8.2 Computation of K^{q_∞}.

Set $\mu_1 := 0$ and $\mu_2 := 1$. Then, note that

$$\int_{\alpha_i} \eta(x) \int_{q_\infty}^x \rho = \int_\infty^{\lambda_i} \eta(x) \int_\infty^x \rho + \int_{\lambda_i}^{\mu_i} \phi^* \eta(x) \int_{\lambda_i}^x \phi^* \rho + \int_{\mu_i}^\infty \eta(x) \int_{\mu_i}^x \rho$$
$$+ \Big(\int_{\lambda_i}^{\mu_i} \phi^* \eta\Big)\Big(\int_\infty^{\lambda_i} \rho\Big) + \Big(\int_{\mu_i}^\infty \eta\Big)\Big(\int_\infty^{\lambda_i} \rho + \int_{\lambda_i}^{\mu_i} \phi^* \rho\Big) \ ,$$
$$\int_{\alpha_{i+2}} \eta(x) \int_{q_\infty}^x \rho = \int_{\alpha_i} \phi^* \eta(x) \int_{q_\infty}^x \phi^* \rho \ ,$$

$i = 1, 2$, for arbitrary $\eta, \rho \in H^0(K_C)$. Consider the $4g$-edged polygon obtained by the canonical dissection of C along the chosen basis of $\pi^1(C, q_\infty)$. Let U_i, V_i, $i = 1, 2$ be holomorphic functions on such a polygon such that $u_i = dU_i$ and $v_i = dV - i$. By

$$U_i(\alpha_i(q_\infty)) - U_i(q_\infty) \equiv \int_{\alpha_i} u_i = \frac{1}{1-\zeta} \ ,$$
$$V_i(\alpha_i(q_\infty)) - V_i(q_\infty) \equiv \int_{\alpha_i} v_i = \frac{1}{1-\zeta^2} \ ,$$

we obtain

$$\int_{\alpha_i} u_i(x) \int_{q_\infty}^x u_i = \frac{1}{2} \int_{q_\infty}^{\alpha_i(q_\infty)} dU_i^2 - U_i(q_\infty) \int_{\alpha_i} u_i = \frac{1}{2(1-\zeta)^2} = -\frac{\zeta^2}{6} \ ,$$

and analogously

$$\int_{\alpha_i} v_i(x) \int_{q_\infty}^x v_i = -\frac{\zeta}{6} \ .$$

Let us compute

$$
\begin{aligned}
\int_{\alpha_i} u_k(x) \int_{q_\infty}^x v_j = & \int_\infty^{\lambda_i} u_k(x) \int_\infty^x v_j + \int_{\lambda_i}^{\mu_i} u_k(x) \int_{\lambda_i}^x v_j + \int_{\mu_i}^\infty u_k(x) \int_{\mu_i}^x v_j \\
& + \zeta^2 \Big(\int_{\lambda_i}^{\mu_i} u_k\Big)\Big(\int_\infty^{\lambda_i} v_j\Big) + \Big(\int_{\mu_i}^\infty u_k\Big)\Big(\int_\infty^{\lambda_i} v_j + \zeta \int_{\lambda_i}^{\mu_i} v_j\Big) \\
= & - V_j(\infty) \int_\infty^{\lambda_i} u_k - V_j(\lambda_i) \int_{\lambda_i}^{\mu_i} u_k \\
& - V_j(\mu_i) \int_{\mu_i}^\infty u_k + \zeta^2 (V_j(\lambda_i) - V_j(\infty)) \int_{\lambda_i}^{\mu_i} u_k \\
& + (V_j(\lambda_i) - V_j(\infty)) \int_{\mu_i}^\infty u_k + \zeta (V_j(\mu_i) - V_j(\lambda_i)) \int_{\mu_i}^\infty u_k \\
= & (\zeta^2 - 1)(V_j(\infty) - V_j(\lambda_i)) \int_{\mu_i}^{\lambda_i} u_k + (\zeta - 1)(V_j(\mu_i) - V_j(\lambda_i)) \int_{\mu_i}^\infty u_k \\
= & (\zeta^2 - 1) \int_{\mu_i}^{\lambda_i} u_k \int_{\lambda_i}^\infty v_j + (1 - \zeta) \int_{\mu_i}^\infty u_k \int_{\mu_i}^{\lambda_i} v_j \ ,
\end{aligned}
$$

for all $i, j, k = 1, 2$. By an analogous calculation, or by noting that

$$
\begin{aligned}
\int_{\alpha_i} & u_k(x) \int_{q_\infty}^x v_j + \int_{\alpha_i} v_j(x) \int_{q_\infty}^x u_k \\
& = \int_{q_\infty}^{\alpha_i(q_\infty)} d(U_k V_j) - V_j(q_\infty) \int_{\alpha_i} u_k - U_k(q_\infty) \int_{\alpha_i} v_j \\
& = (U_k(q_\infty) + \int_{\alpha_i} u_k)(V_j(q_\infty) + \int_{\alpha_i} v_j) - U_k(q_\infty) V_j(q_\infty) \\
& \quad - V_j(q_\infty) \int_{\alpha_i} u_k - U_k(q_\infty) \int_{\alpha_i} v_j \\
& = \int_{\alpha_i} u_k \int_{\alpha_i} v_j = (1 - \zeta)(1 - \zeta^2) \int_{\mu_i}^{\lambda_i} u_k \int_{\mu_i}^{\lambda_i} v_j \\
& = [(1 - \zeta) + (1 - \zeta^2)] \int_{\mu_i}^{\lambda_i} u_k \int_{\mu_i}^{\lambda_i} v_j \ ,
\end{aligned}
$$

one obtains

$$
\int_{\alpha_i} v_j(x) \int_{q_\infty}^x u_k = (\zeta - 1) \int_{\mu_i}^{\lambda_i} v_j \int_{\lambda_i}^\infty u_k + (1 - \zeta^2) \int_{\mu_i}^\infty v_j \int_{\mu_i}^{\lambda_i} u_k \ ,
$$

for all $i,j,k = 1,2,3$. Hence, $\sum_{i\neq 1}\int_{\alpha_i}\omega_i(x)\int_{q_\infty}^x \omega_1$ is given by

$$\begin{aligned}
&\int_{\alpha_2} u_2(x)\Big(\int_{q_\infty}^x u_1 + \int_{q_\infty}^x v_1\Big) + \int_{\alpha_2} v_2(x)\Big(\int_{q_\infty}^x u_1 + \int_{q_\infty}^x v_1\Big)\\
&-\zeta^2\int_{\alpha_1}\phi^* u_1(x)\Big(\int_{q_\infty}^x \phi^* u_1 + \int_{q_\infty}^x \phi^* v_1\Big) - \zeta\int_{\alpha_1}\phi^* v_1(x)\Big(\int_{q_\infty}^x \phi^* u_1 + \int_{q_\infty}^x \phi^* v_1\Big)\\
&-\zeta^2\int_{\alpha_2}\phi^* u_2(x)\Big(\int_{q_\infty}^x \phi^* u_1 + \int_{q_\infty}^x \phi^* v_1\Big) - \zeta\int_{\alpha_2}\phi^* v_2(x)\Big(\int_{q_\infty}^x \phi^* u_1 + \int_{q_\infty}^x \phi^* v_1\Big)\\
=&-\int_{\alpha_1} u_1(x)\int_{q_\infty}^x u_1 - \int_{\alpha_1} v_1(x)\int_{q_\infty}^x v_1 - \zeta^2\int_{\alpha_1} u_1(x)\int_{q_\infty}^x v_1 - \zeta\int_{\alpha_1} v_1(x)\int_{q_\infty}^x u_1\\
&+(1-\zeta^2)\int_{\alpha_2} u_2(x)\int_{q_\infty}^x v_1 + (1-\zeta)\int_{\alpha_2} v_2(x)\int_{q_\infty}^x u_1\ ,
\end{aligned}$$

and we finally obtain

$$\sum_{i\neq 1}\int_{\alpha_i}\omega_i(x)\int_{q_\infty}^x \omega_1 = \frac{\zeta^2}{6} + \frac{\zeta}{6} + \frac{1-\zeta}{9} + \frac{\zeta^2-1}{9} + \frac{1-\zeta^2}{3}c + \frac{\zeta-1}{3}c = \frac{1}{6} - c\ ,$$

where $c = \frac{\tau_{12}}{2} = \frac{\tau_{34}}{2} = \tau_{14} = \tau_{23}$. Similar computations yield

$$\begin{aligned}
\sum_{i\neq 2}\int_{\alpha_i}\omega_i(x)\int_{q_\infty}^x \omega_2 &= \frac{1}{6} - c\ ,\\
\sum_{i\neq 3}\int_{\alpha_i}\omega_i(x)\int_{q_\infty}^x \omega_3 &= -\frac{1}{6} - c\ ,\\
\sum_{i\neq 4}\int_{\alpha_i}\omega_i(x)\int_{q_\infty}^x \omega_4 &= -\frac{1}{6} - c\ ,
\end{aligned}$$

so that

$$K^{q_\infty} = \begin{pmatrix} 1/2+a-1/6+c\\ 1/2+b-1/6+c\\ 1/2+a+1/6+c\\ 1/2+b+1/6+c\end{pmatrix} = \frac{1}{3}\left[\begin{pmatrix}1\\1\\2\\2\end{pmatrix} + \tau\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right]\ .$$

8.3 The prime form

Set

$$f_i := \Big[\frac{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}{w}\Big]^{2-i}(dz)^{1/2}\ ,\qquad i = 1,2,3\ .$$

Note that f_1, f_2, f_3 are meromorphic sections of the same line bundle L (since f_i/f_j is a meromorphic function, for all $i,j\in I_3$). Furthermore, since

$$\begin{aligned}
(f_1) &= 2p_1 + 2p_2 + 2p_3 - 3q_\infty\ ,\\
(f_2) &= q_0 + q_1 + p_1 + p_2 + p_3 - 2q_\infty\ ,\\
(f_3) &= 2q_0 + 2q_1 - q_\infty\ ,
\end{aligned}$$

and

$$I(2q_0 + 2q_1 - q_\infty) \in \mathbb{Z}^4 + \tau\mathbb{Z}^4\ ,$$

it follows that the sections of L are the single-valued 1/2-differentials corresponding to the spin structure $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

Now, consider the meromorphic function $z(p)-z(q)$, $(p,q)\in C\times C$. Its divisor with respect to p is $q+\phi(q)+\phi^2(q)-3q_\infty$. Let us define

$$F(p,q):=\frac{\sum_{i\in I_3} f_i(p)f_{4-i}(q)}{3(z(p)-z(q))}\ ,$$

It is a meromorphic section of $\pi_1^*L\otimes\pi_2^*L$ on $C\times C$, where π_i, $i=1,2$, is the projection of $C\times C$ on the i-th component. Let us show that $F(p,q)$ has only a single pole at $p=q$. In facts, the only possible poles are q, $\phi(q)$, $\phi^2(q)$. On the other hand, by using $w(\phi^r(q))=\zeta^r w(q)$, $r=0,1,2$, one obtains

$$\sum_{i\in I_3} f_i(p)f_{4-i}(q)\overset{p\to\phi^r(q)}{\sim}\sqrt{dz(p)}\sqrt{dz(q)}(\zeta^{-r}+1+\zeta^r)\ ,$$

which vanishes if $r=1,2$.

For each non-singular even spin structure δ, define the Szegö kernel

$$S_\delta(p,q):=\frac{\theta[\delta](p-q)}{\theta[\delta](0)E(q,p)}\ .$$

This is a meromorphic section of $\pi_1^*L_\delta\otimes\pi_2^*L_\delta$ on $C\times C$ with a unique pole in $p=q$, where the sections of L_δ are the 1/2-differentials with spin structure δ. Set $S(p,q):=S_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}(p,q)$.

Proposition 8.1.

$$F(p,q)=S(p,q)\ .$$

Proof. Note that $F(p,q)$ and $S(p,q)$ are meromorphic sections of the same line bundle $\pi_1^*L\otimes\pi_2^*L$ on $C\times C$. Fix a point $q\in C$ and a local coordinate ζ centered in q. In the limit $p\to q$, $\theta(p-q)\sim\theta(0)+\mathcal{O}(\zeta(p)^2)$ so that, by considering the expansion of $E(q,p)$, we have

$$S(p,q)\overset{p\to q}{\sim}\frac{\sqrt{d\zeta(p)}\sqrt{d\zeta(q)}}{\zeta(p)}(1+\mathcal{O}(\zeta^2(p)))\ .$$

Let us consider the expansion of $F(p,q)$ in the same limit. If q is distinct from the branching points, then z is a good coordinate around q and we have

$$F(p,q)\overset{p\to q}{\sim}\frac{\sqrt{dz(p)}\sqrt{dz(q)}}{z(p)-z(q)}\big(1+\mathcal{O}(z(p)-z(q))^2\big)\ .$$

On the contrary, if q coincide with a branching point, for example p_1, let us consider a local coordinate ζ on a neighborhood U of p_1 such that $z(p)=\lambda_1+\zeta^3(p)$ for $p\in U$. Then $dz(p)=3\zeta^2(p)d\zeta(p)$ and

$$F(p,p_1)=\frac{f_1(p)f_3(p_1)}{3(z(p)-\lambda_1)}\overset{p\to p_1}{\sim}\frac{\sqrt{d\zeta(p_1)}\sqrt{d\zeta(p)}}{\zeta(p)}(1+\mathcal{O}(\zeta^2(p)))\ .$$

By comparing the expansions of $S(p,q)$ and $F(p,q)$ around their unique pole, we conclude that

$$S(p,q)-F(p,q)\ ,$$

is a holomorphic section of $\pi_1^* L \otimes \pi_2^* L$. On the other hand, since $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ is even and non-singular (the only singular points are K^∞ and $-K^\infty$ and they are not half-periods) it follows that $h^0(L) = 0$ and then also $h^0(\pi_1^* L \otimes \pi_2^* L) = 0$. Hence, $S(p,q) - F(p,q)$ is the constant 0 and the proposition follows. $\square$

APPENDIX

A. VARIETIES

A.1 Analytic and algebraic varieties

Definition A.1. An *analytic variety* in an open set $U \subseteq \mathbb{C}^n$ is a subset $V \subseteq U$ such that for each $p \in U$ there exists an open neighborhood $U' \subseteq U$ such that $U' \cap V$ is the set of zeros of a finite collection $\{f_1, \ldots, f_k\}$ of holomorphic functions on U'.

A analytic variety $V \subset U \subseteq \mathbb{C}^n$ is

- *irreducible* if it cannot be written as the union $V = V_1 \cup V_2$ of analytic varieties $V_1, V_2 \subseteq U$, with $V_1, V_2 \neq V$.
- *smooth* at $p \in V$ if there exists a neighborhood $U' \subseteq U$ such that $U' \cap V$ is the set of zeroes of k holomorphic functions $f_1, \ldots, f_k$ on U', such that the matrix $\partial f_i / \partial z_j$, where $z_1, \ldots, z_n$ are coordinates in $\mathbb{C}^n$, has rank k.

It can be proved that any analytic variety is the finite union of irreducible components.

Definition A.2. A *complex manifold* is a differentiable manifold admitting an open covering $\{U_\alpha\}$ and a collection of coordinate maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\phi_\alpha \circ \phi_\beta^{-1}$ is holomorphic on $\phi_\beta(U_\alpha \cap U_\beta)$ for all α, β.

A complex manifold of dimension 1 is a Riemann surface.

A holomorphic function on a complex manifold M is a function such that $f_{|U_\alpha} \circ \phi_\alpha^{-1}$ is a holomorphic function on $\phi_\alpha(U_\alpha) \subset \mathbb{C}^n$.

As a generalization of definition A.1, an analytic subvariety of a complex manifold M is locally defined as the set of zeroes of a collection of holomorphic functions.

An example of complex manifold is given by the complex projective space $\mathbb{P}^n \equiv \mathbb{P}\mathbb{C}^{n+1}$, defined as the space of 1-dimensional subspaces of $\mathbb{C}^{n+1}$. More generally, we denote by $\mathbb{P}V$ the space of 1-dimensional subspaces of a vector space V. Any complex homogeneous polynomial in $n+1$ variables is well defined as a polynomial in $\mathbb{P}^n$. Such homogeneous form a graded ring $\mathbb{C}[X_0, \ldots, X_n]$, the grading being given by the degree of the polynomial.

Definition A.3. An *algebraic variety* is the locus of zeroes of a collection of homogeneous polynomials in $\mathbb{P}^n$.

An algebraic variety is obviously an analytic subvariety of $\mathbb{P}^n$. The converse is also true, by the following theorem.

Theorem A.1 (Chow's Theorem)**.** *Any analytic subvariety of* $\mathbb{P}^n$ *is algebraic.*

To each subvariety V of $\mathbb{P}^n$, one cen attach the ideal $I(V)$ of homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ whose zero locus contains V. Note that $I(V)$ inherits the grading from $\mathbb{C}[X_0, \ldots, X_n]$.

A.2 Sheaves

Definition A.4. Let X be a topological space and C be a category. Then, a C-valued *pre-sheaf* $\mathcal{F}$ on X is a controvariant functor from the category of open sets on X with inclusion morphisms to the category C. In other words, a presheaf is given by:

- to each open set U of X is associated an object $\mathcal{F}(U)$ of C.
- for each pair U, V of open sets of X, with $V \subseteq U$, a morphism (*restriction morphisms*) $r_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ is defined, such that
  - $r_{U,U} = \mathrm{id}_U$ for all open sets U;
  - for all the open sets $W \subseteq V \subseteq U$, $r_{W,U} = r_{W,V} \circ r_{V,U}$.

In general, one considers categories C of rings, groups or fields. For each open subset U of X, the object $\mathcal{F}(U)$ is called the *sections of* $\mathcal{F}$ *over* U. If C is a concrete category, i.e., roughly speaking, its objects are sets with some additional structure and the morphisms are functions compatible with such a structure, then each element of the set $\mathcal{F}(U)$ is called a *section* of $\mathcal{F}$. Sections of $\mathcal{F}$ on U are also denoted by $\Gamma(U, \mathcal{F})$. In the following, we will only consider concrete categories C.

Definition A.5. For each topological space X and a concrete category C, a C-valued pre-sheaf $\mathcal{F}$ over X is a *sheaf* if it satisfies the following conditions:

- *Normalization*: $\mathcal{F}(\varnothing)$ is the terminal object of C.
- *Gluing*: Let $\{U_i\}$ be an arbitrary family of open subsets of X and fix a section s_i of $\mathcal{F}$ on each U_i, in such a way that, for all the intersections $U_i \cap U_j$, $r_{U_i \cap U_j, U_i}(s_i) = r_{U_i \cap U_j, U_j}(s_j)$. Then, there exists a unique section $s \in \mathcal{F}(U)$, with $U := \bigcup_i U_i$, such that $r_{U_i,U} s = s_i$.

A.3 Curves and divisors

By a curve C, we mean a projective algebraic variety of dimension 1. We will only consider smooth irreducible curves, which are in one-to-one correspondence with Riemann surfaces. We denote by g its geometric genus, which corresponds to half its first Betti number

$$g = \frac{1}{2} \operatorname{rank} H^1(C, \mathbb{Z}) \ .$$

In the following we will identify invertible sheaves on C with line bundles and freely-generated sheaves with vector bundles. For each sheaf of $\mathbb{C}$-vector space on the topological space V, we set

$$h^i(V, \mathcal{F}) := \dim_{\mathbb{C}} H^i(V, \mathcal{F}) \ .$$

For sheaves $\mathcal{F}_C$ on a smooth curve C, we will often use the shorthand notation

$$H^i(\mathcal{F}_C) := H^i(C, \mathcal{F}_C) \ .$$

A divisor on C is a formal sum

$$d := \sum_{p\in C} n(p)p \ ,$$

where $n(p) \in \mathbb{Z}$ are non-zero for a finite number of points in $\mathbb{C}$. On the set of divisors on C is naturally the structure of abelian group $\operatorname{Div} C$ with respect to the sum, with a grading given by the homomorphism

$$\begin{aligned} \deg\colon \operatorname{Div} C &\to \mathbb{Z} \\ \sum_{p\in C} n(p)p &\mapsto \sum_{p\in C} n(p) \ . \end{aligned}$$

A divisor $d = \sum_{p\in C} n(p)p$ is effective or positive if $n(p) \geq 0$ for all $p \in C$; in this case, we write $d \geq 0$. The divisor d is greater than d', and we write $d \geq d'$ if and only if $d - d' \geq 0$. The set of effective divisors inherits the structure of abelian semigroup. The set of effective divisors of a given degree $n \geq 0$ is naturally identified with the space $C_n := \operatorname{Sym}^n C$, which is the symmetrization of the cartesian product $C^n \equiv C \times \ldots \times C$; such a space is endowed with the topology and complex structure induced by the Riemann surface C.

Any holomorphic function f defined on an open neighborhood U of $p \in C$ can always be written as $f(z) = (z - z(p))^n g(z)$, where z is a local coordinate centered in p and g is a holomorphic function with $g(z(p)) \neq 0$. The integer n is defined to be the multiplicity of f at p. Such a definition extends to the case of holomorphic sections of line bundles, since the multiplicity does not depend on the local trivialization. Then, to each section s is associated a divisor

$$(s) := \sum_{p} m(p)p \ ,$$

where $m(p)$ is the multiplicity of s at p.

A meromorphic function f on C is defined locally as the ratio of two holomorphic functions. More precisely, given an open covering $\{U_\alpha\}$ of C, f is given by a collection of holomorphic functions $\{h_\alpha, h'_\alpha\}$ such that h_α, h'_α are relatively prime and $h_\alpha/h'_\alpha = h_\beta/h'_\beta$ on $U_\alpha \cap U_\beta$, for all α, β. Roughly speaking, the restriction of f to U_α should be identified with the ratio h_α/h'_α. The multiplicity of f at $p \in U_\alpha$ is well defined as the difference of the multiplicities of h_α and h'_α. The set $\mathfrak{M}$ of meromorphic functions on C is a field and the map $f \mapsto (f)$ which maps $f \in \mathfrak{M}$ to its divisor is a homomorphism between $\mathfrak{M}$, seen as a multiplicative group, and $\operatorname{Div} C$. More generally, a meromorphic section of a line bundle $\mathcal{L}$ on C is given by a collection of holomorphic functions $\{h_\alpha, h'_\alpha\}$ such that

$$h_\alpha/h'_\alpha = g_{\alpha\beta} h_\beta/h'_\beta \ ,$$

on $U_\alpha \cap U_\beta$, where $\{g_{\alpha\beta}\}$ are the transition functions of the line bundle $\mathcal{L}$ with respect to the covering $\{U_\alpha\}$. For each arbitrary $\mathcal{L}$, the space $\mathfrak{M}(\mathcal{L})$ of meromorphic sections of $\mathcal{L}$ is non-empty (as a consequence of the Riemann-Roch Theorem below) and is a one-dimensional vector space over the field $\mathfrak{M}$.

An element $d \in \operatorname{Div} C$ is a *principal* divisor if it is the divisor of a meromorphic function on C. For a compact Riemann surface C, the principal divisors have degree 0. Two elements $d, d' \in \operatorname{Div} C$ are linearly equivalent if their difference is a principal divisor; the class of divisors in $\operatorname{Div} C$ linearly equivalent to d is called the divisor class of d and is denoted by $[d]$.

To each divisor d, one can attach the sheaf $\mathcal{O}(d)$ whose sections on the open set $U \subset C$ are given by

$$\Gamma(U, \mathcal{O}(d)) := \{f \in \mathfrak{M}_U \mid (f) + d \geq 0\} ,$$

where $\mathfrak{M}_U$ denotes the field of meromorphic functions on U. One can always choose a covering $\{U_\alpha\}$ and a collection of meromorphic functions $\{h_\alpha\}$ such that

$$d_{|U_\alpha} = (h_\alpha) ,$$

where the restriction of a divisor $d \equiv \sum_{p\in C} n(p)p$ to an open subset $U \subseteq C$ is a divisor on U given by $d_U := \sum_{p\in U} n(p)p$. Furthermore, we can require that $d_{U_\alpha \cap U_\beta} = 0$ for all α, β. Therefore, the maps $g_{\alpha\beta}$ defined on $U_\alpha \cap U_\beta$ by

$$g_{\alpha\beta} = \frac{h_\alpha}{h_\beta} ,$$

are the transition functions of a line bundle $\mathcal{L}(d)$.

Conversely, given a line bundle $\mathcal{L}$ the choice of a meromorphic section s of $\mathcal{L}$ determines an isomorphism $H^0(C, \mathcal{L}) \xrightarrow{\cong} \mathcal{O}((s))$ by $t \mapsto t/s$, $t \in H^0(C, \mathcal{L})$. Therefore, any line bundle $\mathcal{L}$ can be written as $\mathcal{L}(d)$ for some d. Furthermore, $\mathcal{L}(d)$ and $\mathcal{L}(d')$ are isomorphic if and only if d is linearly equivalent to d'.

The degree $\deg(d)$ corresponds to the first Chern class of $\mathcal{L}(d)$, which we also denote by $\deg \mathcal{L}(d)$

$$c_1(\mathcal{L}(d)) \equiv \deg \mathcal{L}(d) = \deg(d) .$$

Let K_C denote the canonical line bundle on C, whose sections are the holomorphic 1-differentials. Therefore, $H^0(K_C) \equiv H^0(C, K_C)$ denotes the space of holomorphic abelian differentials on C. The following fundamental theorem holds.

Theorem A.2 (Riemann-Roch)**.** *For any line bundle $\mathcal{L}$ on a smooth curve C of genus g*

$$h^0(\mathcal{L}) - h^0(K_C \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1 .$$

Since the only holomorphic functions on a closed Riemann surface C are the constants, it follows that $h^0(\mathcal{O}) = 1$, where $\mathcal{O} \equiv \mathcal{O}(0)$. Then by the Riemann-Roch Theorem, we have

$$h^0(K_C) = g .$$

By considering $\mathcal{L} \equiv K_C$, this implies

$$\deg K_C = 2g - 2 .$$

Let us define

$$l(d) := h^0(\mathcal{O}(d)) , \quad i(d) := h^0(K_C \otimes \mathcal{O}(-d)) ,$$

where $i(d)$ is called the *index of specialty* of d. Then, the Riemann-Roch theorem can be restated as

$$l(d) - i(d) = \deg(d) - g + 1 .$$

In the case of an effective divisor $d = \sum n(p)p$, $H^0(K_C \otimes \mathcal{O}(-d))$ is the space of holomorphic 1-differentials vanishing at each point $p \in C$ with multiplicity at least $n(p)$ and $H^0(\mathcal{O}(d))$ is the space of meromorphic functions with poles of order at most $n(p)$ at each $p \in C$.

An effective divisor d is called *special* if $i(d) > 0$. The following relations hold for any effective divisor d

$$\begin{aligned} i(d) =&\geq g - \deg d && \text{for } \deg d < g \\ i(d) \geq& 0 && \text{for } g \leq \deg d \leq 2g-2\ , \\ i(d) =& 0 && \text{for } g > 2g-2\ . \end{aligned}$$

Any effective divisor d, with $\deg d \leq 2g-2$, for which the disequalities above hold in strict sense is called an *exceptional* special divisor. The subset of exceptional special divisors of degree d is a subvariety of non-zero codimension in the space C_d of effective divisors. In particular, the subvariety of divisors of degree d and index of specialty $i(d) = d - g + r$, $r \geq 1$ is denoted by C_d^r. By the Riemann-Roch theorem, the condition $i(d) = \deg d - g + r$ corresponds to the existence of r independent meromorphic functions with divisor greater than $-d$. It can be proved that a meromorphic function with only one pole never exists on a Riemann surface C of genus $g > 0$. Therefore, an effective divisor of degree 1 is never exceptional.

Definition A.6. A Riemann surface C of genus g is called

- hyperelliptic if it admits a meromorphic function with two poles
- trigonal if it admits a meromorphic function with three poles (but not less).
- more generally, n-gonal, $n > 2$, if it admits a meromorphic function with n poles (but not less).

Any Riemann surface C of genus g admits a meromorphic function with g poles. It follows that any Riemann surface of genus 2 is hyperelliptic. Furthermore, any Riemann surface C of genus $g > 3$ always admit a meromorphic function with $g-1$ poles. More precisely, the space of exceptional special divisors C_{g-1}^1 has dimension $g-3$ if C is hyperelliptic and $g-4$ otherwise.

B. THETA FUNCTIONS ON RIEMANN SURFACES

Set $A_Z := \mathbb{C}^g/L_Z$, $L_Z := \mathbb{Z}^g + Z\mathbb{Z}^g$, where Z belongs to the Siegel upper half-space

$$\mathfrak{H}_g := \{Z \in M_g(\mathbb{C}) | \, {}^tZ = Z, \, \mathrm{Im}\, Z > 0\} \ ,$$

and consider the theta function with characteristics

$$\theta \left[{a \atop b} \right] (z, Z) := \sum_{k \in \mathbb{Z}^g} e^{\pi i^t (k+a) Z (k+a) + 2\pi i^t (k+a)(z+b)} \tag{B.1}$$

$$= e^{\pi i^t a Z a + 2\pi i^t a(z+b)} \theta \left[{0 \atop 0} \right] (z + b + Za, Z) \ , \tag{B.2}$$

where $z \in A_Z$, $a, b \in \mathbb{R}^g$. It has the quasi-periodicity properties

$$\theta \left[{a \atop b} \right] (z + n + Zm, Z) = e^{-\pi i^t m Z m - 2\pi i^t m z + 2\pi i ({}^t a n - {}^t b m)} \theta \left[{a \atop b} \right] (z, Z) \ ,$$

$m, n \in \mathbb{Z}^g$. Denote by $\Theta \subset A_Z$ the divisor of $\theta(z, Z) := \theta \left[{0 \atop 0} \right] (z, Z)$ and by $\Theta_s \subset \Theta$ the locus where θ and its gradient vanish. If $\delta', \delta'' \in \{0, 1/2\}^g$, then $\theta [\delta] (z, \tau) := \theta \left[{\delta' \atop \delta''} \right] (z, \tau)$ has definite parity in z

$$\theta [\delta] (-z, \tau) = e(\delta) \theta [\delta] (z, \tau) \ ,$$

where $e(\delta) := e^{4\pi i^t \delta' \delta''}$. There are 2^{2g} different characteristics for which $\theta [\delta] (z, \tau)$ has definite parity. Note that, in particular, $\Theta = -\Theta$.

Geometrically $\theta \left[{a \atop b} \right] (z, Z)$ is the unique holomorphic section of the bundle $\mathcal{L}_{\Theta_{ab}}$ on A_Z defined by the divisor $\Theta_{ab} = \Theta + b + Za$ of $\theta \left[{a \atop b} \right] (z, Z)$. A suitable norm, continuous throughout A_Z, is given by

$$||\theta||^2 (z, Z) = e^{-2\pi^t \, \mathrm{Im}\, z (\mathrm{Im}\, Z)^{-1} \, \mathrm{Im}\, \bar{z}} |\theta|^2 (z, Z) \ .$$

Computing $c_1(\mathcal{L}_\Theta)$ and using the Hirzebruch-Riemann-Roch Theorem, it can be proved that θ is the unique holomorphic section of $\mathcal{L}_\Theta$. It follows that $(A_Z, \mathcal{L}_\Theta)$ is a principally polarized abelian variety (ppav). We denote by $\mathcal{A}_g := \mathfrak{H}_g/\Gamma_g$ the moduli space of ppav's.

B.1 Riemann theta functions and the prime form

Let $\{\alpha, \beta\} \equiv \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ be a symplectic basis of $H_1(C, \mathbb{Z})$ and $\{\omega_i\}_{i \in I_g}$ the basis of $H^0(K_C)$ satisfying the standard normalization condition $\oint_{\alpha_i} \omega_j = \delta_{ij}$, for all $i, j \in I_g$. Let $\tau \in \mathfrak{H}_g$ be the Riemann period matrix of C, $\tau_{ij} := \oint_{\beta_i} \omega_j$. A different choice of the symplectic basis of $H_1(C, \mathbb{Z})$ corresponds to a $\Gamma_g := Sp(2g, \mathbb{Z})$ transformation

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ , \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \ ,$$

$$\tau \mapsto \tau' = (A\tau + B)(C\tau + D)^{-1} . \tag{B.3}$$

Let us define the Abel-Jacobi map $I(d) := (I_1(d), \dots, I_g(d))$, acting on 0-degree divisors d on C, by

$$I_i(d) := \sum_{j=1}^{n} \int_{q_j}^{p_j} \omega_i \in J_0(C) ,$$

where $d := \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i$ and $J_0(C) := \mathbb{C}^g / L_\tau$ is the Jacobian variety associated to C. By Abel's theorem, $I(d) = 0$ if and only if d is a principal divisor (i.e. the divisor of a meromorphic function); hence, I is well defined as a map acting on divisor classes. For each fixed $p_0 \in C$, the map $p \mapsto I(p - p_0)$ is an embedding of C into the Jacobian; furthermore, by the Jacobi Inversion Theorem, the map

$$\begin{aligned} C_g &\to J_0(C) \\ d &\mapsto I(d - gp_0) , \end{aligned}$$

is surjective.

For each Riemann surface C, one can consider the Riemann theta function with characteristics $\theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z, \tau)$ associated to the ppav $J_0(C)$. For each $p \in C$ and $e \in J_0(C)$, the Riemann theta function $f(x) := \theta(I(x-p)-e) \equiv \theta(I(x-p)-e, \tau)$ is the section of a line bundle on C and has a well-defined divisor, which is completely characterized by a theorem by Riemann.

Definition B.1. The vector of Riemann constants is

$$\mathcal{K}_i^p := \frac{1}{2} + \frac{1}{2}\tau_{ii} - \sum_{j \neq i}^{g} \oint_{\alpha_j} \omega_j \int_p^x \omega_i , \tag{B.4}$$

$i \in I_g$, for all $p \in C$. For any p we define the Riemann divisor class Δ by

$$I((g-1)p - \Delta) := \mathcal{K}^p , \tag{B.5}$$

which has the property $2\Delta = K_C$.

Theorem B.1 (Riemann Vanishing Theorem)**.** *For any $p \in C$ and $e \in J_0(C)$*

i. if $\theta(e) \neq 0$, then the divisor d of $\theta(I(x-p)-e)$, is effective of degree g, with index of specialty $i(d) = 0$ and $e = I(d - p - \Delta)$;

ii. if $\theta(e) = 0$, then for some $\zeta \in C_{g-1}$, $e = I(\zeta - \Delta)$.

In view of the Riemann Vanishing Theorem, it is convenient to consider the following generalization the Abel-Jacobi map to divisors of general degree.

Definition B.2. For each divisor $d := \sum_{i=1}^{n} p_i - \sum_{i=1}^{m} q_i$ on C, with n, m non-negative integers, define the map $I(d) \equiv (I_1(d), \dots, I_g(d)) \in J_0(\mathbb{C})$ given by

$$I(d)_i := \sum_{j=1}^{n} \int_{p_0}^{p_j} \omega_i - \sum_{j=1}^{m} \int_{p_0}^{q_j} \omega_i + \frac{n-m}{g-1} \mathcal{K}^{p_0} .$$

Such a map does not depend on $p_0 \in C$ and reduces to the Abel-Jacobi map if $n = m$.

By such a definition and by the Riemann Vanishing Theorem, $\Theta \equiv I(C_{g-1})$. In the following, we will use the notation

$$\theta(d+e) := \theta(I(d)+e) \ ,$$

for all $e \in J_0(C)$ and divisors d of C.

For each half-integer theta characteristic $[\delta] \equiv \begin{bmatrix}\delta' \\ \delta''\end{bmatrix}$ and point $p \in C$, let d_p be the divisor of the Riemann theta function $\theta[\delta](x-p)$. It corresponds to the divisor of $\theta(x-p+\delta'+\tau\delta'')$, where $\delta'+\tau\delta''$ is a half-period. Hence, by the Riemann vanishing theorem and by $2\Delta = K_C$, it follows that $d_p - p$ corresponds to the divisor class of a spin bundle L_δ, with $L_\delta^2 \simeq K_C$, which only depends on δ (note that $\theta[\delta](x-p)/\theta[\delta](x-q)$ is a single-valued meromorphic function in x, so that $d_p - p$ and $d_q - q$ are equivalent divisors). In particular, Δ is the divisor class associated to $L_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}$. In other words, the theta characteristic on a Jacobian $J_0(C)$ are in one-to-one correspondence with the spin structures on C. There are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd spin structures.

By Riemann's Singularity Theorem it follows that the dimension of Θ_s for $g \geq 4$ is $g-3$ in the hyperelliptic case and $g-4$ if the curve is canonical. Furthermore, the following basic relation is easily proved.

Proposition B.2 (Riemann). *For all $e \in \Theta_s$,*

$$\sum_{i,j \in I_g} \theta_{ij}(e)\omega_i\omega_j(z) = 0 \ , \tag{B.6}$$

for all $z \in C$.

Let ν a non-singular odd characteristic. The holomorphic 1-differential

$$h_\nu^2(p) := \sum_1^g \omega_i(p)\partial_{z_i}\theta\,[\nu]\,(z)_{|z=0} \ , \tag{B.7}$$

$p \in C$, has $g-1$ double zeros. The prime form

$$E(z,w) := \frac{\theta\,[\nu]\,(w-z,\tau)}{h_\nu(z)h_\nu(w)} \ , \tag{B.8}$$

is a holomorphic section of a line bundle on $C \times C$, corresponding to a differential form of weight $(-1/2,-1/2)$ on $\tilde{C} \times \tilde{C}$, where $\tilde{C}$ is the universal cover of C. It has a first order zero along the diagonal of $C \times C$. In particular, if t is a local coordinate at $z \in C$ such that $h_\nu = dt$, then

$$E(z,w) = \frac{t(w)-t(z)}{\sqrt{dt(w)}\sqrt{dt(z)}}(1+\mathcal{O}((t(w)-t(z))^2)) \ .$$

Note that $I(z+{}^t\alpha n+{}^t\beta m) = I(z)+n+\tau m$, $m,n \in \mathbb{Z}^g$, and

$$E(z+{}^t\alpha n+{}^t\beta m, w) = \chi e^{-\pi i^t m\tau m - 2\pi i^t m I(z-w)}E(z,w) \ ,$$

where $\chi := e^{2\pi i({}^t\nu' n - {}^t\nu'' m)} \in \{-1,+1\}$, $m,n \in \mathbb{Z}^g$.

We will also consider the multi-valued $g/2$-differential $\sigma(z)$ on C with empty divisor, that is a holomorphic section of a trivial bundle on C, and satisfies the property

$$\sigma(z+{}^t\alpha n+{}^t\beta m) = \chi^{-g}e^{\pi i(g-1)^t m\tau m + 2\pi i^t m\mathcal{K}^z}\sigma(z) \ .$$

Such conditions fix $\sigma(z)$ only up to a factor independent of z; the precise definition, to which we will refer, can be given, following [24], on the universal covering of C (see also [22]). Furthermore,

$$\sigma(z,w) := \frac{\sigma(z)}{\sigma(w)} = \frac{\theta(\sum_1^g x_i - z)}{\theta(\sum_1^g x_i - w)} \prod_{i=1}^{g} \frac{E(x_i,w)}{E(x_i,z)} \ , \tag{B.9}$$

for all $z, w, x_1, \ldots, x_g \in C$, which follows by observing that the RHS is a nowhere vanishing section both in z and w with the same multi-valuedness of $\sigma(z)/\sigma(w)$.

Under the modular transformations $z \to z' = z(CZ+D)^{-1}$, $Z \to Z' = (AZ+B)(CZ+D)^{-1}$ the theta characteristics transform as

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \ ,$$

$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, for all $a, b, z \in \mathbb{C}^g$, and the theta functions transform as $\theta[^a_b](z,Z) \to \theta[^{a'}_{b'}](z',Z')$, with [37]

$$\theta[^{a'}_{b'}](z',Z') = \epsilon_G (\det(CZ+D))^{\frac{1}{2}} e^{2\pi i[\phi[^a_b](G) + \frac{1}{2}{}^t z (CZ+D)^{-1} C z]} \theta[^a_b](z,Z) \ , \tag{B.10}$$

where ϵ_G is an eighth root of 1 depending only on G,

$$\begin{pmatrix} a' \\ b' \end{pmatrix} := \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathrm{diag}\,(C^t D) \\ \mathrm{diag}\,(A^t B) \end{pmatrix} \ ,$$

and

$$2\phi[^a_b](G) := ({}^t a \ {}^t b) \begin{pmatrix} -{}^t BD & {}^t BC \\ {}^t BC & -{}^t AC \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \mathrm{diag}(A^t B) \cdot (Da - Cb) \ .$$

Let $\omega(z,w)$ be the unique symmetric differential on $C \times C$, with only a double pole along $z = w$, satisfying $\oint_{\alpha_j} \omega(z,w) = 0$ and $\oint_{\beta_j} \omega(z,w) = 2\pi i \omega_j$, $j \in I_g$. The latter conditions imply that under a modular transformation

$$\hat{\omega}(z,w) = \omega(z,w) - 2\pi i^t \omega(z)(C\tau + D)^{-1}\omega(w) \ .$$

Since $E(z,w)$ is the unique antisymmetric solution of $\partial_z \partial_w \log E = \omega(z,w)$ which is consistent with the expansion of $\omega(z,w)$ for $z \sim w$, it follows that

$$\hat{E}(z,w) = E(z,w) e^{\pi i (C\tau + D)^{-1} C \int_z^w \omega \cdot \int_z^w \omega} \ , \tag{B.11}$$

for all $z, w \in C$.

Lemma B.3 (Fay [24]). *If $\{\alpha, \beta\}$ and $\{\tilde{\alpha}, \tilde{\beta}\}$ are two markings of C related by* (B.3) *and $\mathcal{K}^q$ and $\mathcal{K}^{q\,'}$ denote the respective vectors of Riemann constants for $q \in C$, then there are $a_0, b_0 \in (\frac{1}{2}\mathbb{Z})^g$, depending on the markings, such that*

$$a_0 - \frac{1}{2}\mathrm{diag}\,(C^t D) \in \mathbb{Z}^g \ , \qquad b_0 - \frac{1}{2}\mathrm{diag}\,(A^t B) \in \mathbb{Z}^g \ ,$$

$$\mathcal{K}^{q\,'} = {}^t(C\tau + D)^{-1}\mathcal{K}^q + \tau' a_0 + b_0 \in \mathbb{C}^g \ ,$$

and

$$\begin{aligned}\theta(z' + \mathcal{K}^{q'}, \tau') =& \epsilon'(\det(C\tau + D))^{\frac{1}{2}} \\ & e^{\pi i[{}^t(z+\mathcal{K}^q)(C\tau+D)^{-1}C(z+\mathcal{K}^q) - {}^t a_0 \tau' a_0 - 2\,{}^t(C\tau+D)^{-1}(z+\mathcal{K}^q)]}\theta(z,\tau) ,\end{aligned}$$

for all $z = {}^t(C\tau + D)z' \in \mathbb{C}^g$*, with* ϵ' *an eighth root of 1 depending on the markings.*

Theta functions and, in particular, *Thetanullwerte*, i.e. theta constants $\theta[\delta](0)$, with δ even characteristics, can be used to construct modular forms, i.e. meromorphic functions on $\mathfrak{H}_g$ which are invariant under modular transformations. Some regularity conditions at infinity are also required for $g = 1$, which are not necessary for $g > 1$ due to the Koecher principle. More generally, one considers modular forms of weight $k < 0$, i.e. holomorphic functions f on $\mathfrak{H}_g$ which transform as

$$f(Z') = \det(CZ + D)^{-k} f(Z) , \tag{B.12}$$

under modular transformations or other discrete subgroups of $\mathrm{Sp}(2g, \mathbb{R})/\mathbb{Z}_2$, the group of automorphisms of $\mathfrak{H}_g$.

The relationship between the Thetanullwerte and the Jacobi Nullwerte, i.e. the space of theta derivatives $\theta_i[\nu](0)$, with $[\nu]$ odd spin structures, is analyzed in the following section.

B.2 Generalizations of Jacobi's derivative identity

In this section, we consider the higher genus generalizations of the Jacobi's derivative formula

$$\theta' \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0) = -\pi\theta \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](0)\theta' \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0)\theta' \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](0) ,$$

which holds for $g = 1$. For any $Z \in \mathfrak{H}_g$, $g \geq 1$, and any g-tuple $\nu_1, \ldots, \nu_g$ of odd spin structures, let us define

$$[\nu_1, \ldots, \nu_g] := \det_{i,j} \theta_i[\nu_j] .$$

The generalization of Jacobi's identity is the expression of $[\nu_1, \ldots, \nu_g]$ as a polynomial in the theta constants $\theta[\delta](Z)$, where δ are the even spin structures. More details can be found in [41, 33].

For $g = 2$, such a problem was considered by Thomae and Weber. The solution is given by the following theorem.

Theorem B.4 (Rosenhain's formula, [54, 32])**.** *For any* $Z \in \mathfrak{H}_2$ *and any pair of odd characteristics* ν_1, ν_2

$$[\nu_1, \nu_2](Z) = \pm\pi^2 \prod_{\substack{\nu\ odd \\ \nu\neq\nu_1,\nu_2}} \theta[\nu_1 + \nu_2 + \nu](Z) ,$$

where the sign does not depend on Z*.*

Similar identities have been found by Frobenius for $g = 3$ and $g = 4$ [26] and by Fay for $g = 5$ [23]; some cases up to genus 7 have also been studied by Riemann [53].

The results valid for all g were firstly found by Igusa; in [39] it was shown that the Jacobi Nullwerte is always a rational function of the Thetanullwerte. In [38], the following general theorem has been proved, which holds for all genera.

For each theta characteristic $\alpha \equiv \left[\begin{smallmatrix}\alpha'\\ \alpha''\end{smallmatrix}\right]$, set $e(\alpha) := \exp(4\pi i \alpha' \cdot \alpha'')$. Then, a sequence $\alpha_1, \ldots, \alpha_n$ of theta characteristic is defined to be azygetic if

$$e(\alpha_i)e(\alpha_j)e(\alpha_k)e(\alpha_i + \alpha_j + \alpha_k) = -1 \ ,$$

for all $1 \leq i < j < k \leq n$ and essentially independent if, for any choice of $1 \leq i_1 < \ldots < i_{2k} \leq n$, with $k \geq 1$, we have

$$\alpha_{i_1} + \ldots + \alpha_{i_n} \neq 0 \mod 2 \ .$$

A fundamental system is an azygetic sequence of $2g+2$ characteristics; a special fundamental system is a fundamental system such that the first g characteristics are odd and the other $g + 2$ are even.

Theorem B.5 (Igusa [38]). *Let $\nu_1, \ldots, \nu_g$ be g odd characteristics such that*

$$[\nu_1, \ldots, \nu_g](Z) := \det \theta_i[\nu_j](0, Z) \ ,$$

$Z \in \mathfrak{H}$, does not identically vanish on $\mathfrak{H}$ and is a polynomial in the theta constants. Then, $\nu_1, \ldots, \nu_g$ are azygetic and essentially independent. Furthermore,

$$[\nu_1, \ldots, \nu_g](Z) = \pi^g \sum \pm\theta[\delta_1] \cdots \theta[\delta_{g+2}] \ , \tag{B.13}$$

where the sum is over all the sets $\{\delta_1, \ldots, \delta_{g+2}\}$ of $g+2$ even theta characteristics such that $\nu_1, \ldots, \nu_g, \delta_1, \ldots, \delta_{g+2}$ is a special fundamental system. In particular, if Z is the Riemann period matrix of a hyperelliptic Riemann surface, then the sum on the right hand side of Eq.(B.13) *has exactly one non-vanishing term.*

In fact, Fay in [23] proved that the formula (B.13) does not hold for $g = 6$; together with Theorem B.5, this is enough to conclude that the determinant $[\nu_1, \ldots, \nu_6]$ is never a polynomial in the theta constants. A generalized formula, however, was proved by Igusa in [41].

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