Università degli Studi di Padova
Scuola di Dottorato in Scienze matematiche
Indirizzo: Matematica Computazionale
Ciclo: XXVII
Sciences mathématiques de Paris Centre - ED 386
Thése en cotutelle pour l'obtention du
Diplôme de Docteur de l'Université Paris Diderot

A Study Of Arbitrage Opportunities In Financial Markets Without Martingale Measures

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Acknowledgments

First and foremost, I would like to express my sincerest gratitude to Professor Wolfgang Runggaldier and Professor Peter Tankov for serving as my advisors in my doctoral studies. I thank them for always sharing their excellence ideas, helpful suggestions, constructive feedbacks, for introducing me to such interesting problems. I have had many enlightening discussions with them over the course of my graduate studies, which sharpened my understanding of the subject matter of this thesis.

I am very thankful to Professor Monique Jeanblanc and Dr. Johannes Ruf for spending time for reading my manuscript and giving useful comments. I am also deeply indebted to the members of my dissertation committee. Without the financial supports of the scholarship from Padova for three years and of Natixis for the last year, I would never have completed my doctoral studies. I am very thankful to their supports.

I am very grateful to all the staff and postgraduates of Department of Mathematics, Padova University and LPMA, University Paris Diderot, who have been unfailingly helpful and friendly.

Finally, I thank my family and my wife for their support. They have always given me unconditional support in all of my endeavors and their faith in me has always been a driving and influential force in my life. Nothing in this thesis would have been written without them.

Contents

Introduction in English

This Ph.D. thesis consists of four dependent chapters and is devoted to a systematic study of arbitrage opportunities, with particular attention to general and incomplete market models with càdlàg semimartingales.

In Chapter 1, we state our motivation, and then briefly review the theory of no-arbitrage, and the previous studies of arbitrage opportunities in the literature. We introduce a general framework, which will be used throughout this dissertation. We discuss no-arbitrage conditions, utility optimization problems and recall results from the literature. Finally, we state three research questions and summarize new results.

Chapter 2 solves the problem of finding arbitrages when investors are heterogeneous in the sense that their beliefs correspond to non-equivalent probabilities. Optimal arbitrage profit and the corresponding strategy are carefully investigated by techniques of non-equivalent measure changes. We also discuss the financial implications of this study and give some meaningful examples. In contrast to typical Brownian models in which arbitrages (if exist) are fragile, some of our arbitrage examples are shown to be robust if market's frictions such as transaction costs and model misspecification are taken into account.

In Chapter 3, we study the problem of optimal investment with the possibility of intermediate consumption and stochastic field utility. We show that the no unbounded profit with bounded risk condition suffices to establish the key duality relations of utility maximization.

In Chapter 4, we investigate insider trading activities. Suppose that there exists an insider, who has access to some private information at the beginning of trading.

Financial mathematics uses the terminology ”initial enlargement of filtration” to explain this circumstance. We first consider the problem of logarithmic utility optimization for the insider. We are able to characterize the insider’s expected utility by duality method and hence give a new sufficient condition for the condition no unbounded profit with bounded risk. Thanks to the tools of non-equivalent measure changes in Chapter 2, we compute the superhedging price of any claim in the view of the insider and examine the question of optimal arbitrage profit.

Introduzione in Italiano

La presente tesi di dottorato è costituita da quattro capitoli tra loro dipendenti ed è dedicata allo studio sistematico di opportunità d'arbitraggio, con particolare attenzione a modelli di mercato generali e incompleti, in presenza di semimartingale cadlag.

Nel Capitolo 1 sono riportate le motivazioni al presente lavoro di ricerca, insieme ad un breve riepilogo della teoria del non arbitraggio e della letteratura riguardante le opportunità di arbitraggio. Introduciamo nozioni e concetti generali, che saranno utilizzati ovunque nella tesi. Discutiamo condizioni di non arbitraggio e problemi di ottimizzazione dell'utilità, richiamando risultati noti in letteratura. Infine, enunciamo tre problemi aperti e riassumiamo i risultati nuovi ottenuti nella tesi.

Nel Capitolo 2 forniamo una soluzione al problema di determinare arbitraggi quando gli investitori sono eterogenei, nel senso che le loro aspettative sono descritte tramite misure di probabilità non equivalenti. Il profitto derivante da un arbitraggio ottimale e la corrispondente strategia sono studiati attentamente per mezzo di tecniche legate al cambio di misura non equivalente. Discutiamo inoltre le implicazioni finanziarie di tale studio e forniamo alcuni esempi significativi. Contrariamente a quanto accade nei modelli browniani, in cui gli arbitraggi (se esistono) sono "fragili", alcuni degli arbitraggi forniti nei nostri esempi sono robusti quando le frizioni del mercato, come i costi di transazione oppure l'errata specificazione del modello ("model misspecification"), sono presi in considerazione.

Nel Capitolo 3 studiamo il problema di investimento ottimale con possibilità di consumo intertemporale. Mostriamo che la condizione "no unbounded profit with bounded risk” è sufficiente a stabilire le relazioni di dualità fondamentali per la massimizzazione dell’utilità.

Nel Capitolo 4 analizziamo le attività di ”insider trading”. Supponiamo che esista un insider, il quale ha accesso ad alcune informazioni private nel momento in cui inizia l’attività di trading. In finanza matematica si utilizza la terminologia ”allargamento iniziale della filtrazione” per denotare questa circostanza. Consideriamo innanzitutto il problema di ottimizzazione con utilità logaritmica per un insider. Siamo in grado di caratterizzare l’utilità attesa dell’insider attraverso il metodo di dualità e, quindi, di fornire una nuova condizione sufficiente per ”no unbounded profit with bounded risk”. Grazie alle tecniche di cambio di misura non equivalente presentate nel Capitolo 2, calcoliamo il prezzo ”superhedging” per l’insider di qualsiasi prodotto derivato ed esaminiamo il profitto derivante da un arbitraggio ottimale [1].

[1]Thanks to Andrea Cosso for his help on this translation

Introduction en Français

Cette thèse de doctorat constituée de quatre chapitres dépendants est dédiée à l'étude systématique des opportunités d'arbitrage dans les marchés financiers incomplets modélisés par les semi-martingales générales.

Dans le Chapitre 1, nous énonçons nos motivations, puis faisons un bref rappel de la théorie d'absence d'arbitrage et des précédentes études sur les opportunités d'arbitrage dans la littérature. Nous présentons ensuite le cadre théorique général de cette dissertation et les différentes conditions de non-arbitrage proposées dans la littérature, ainsi que leur lien avec les problèmes d'optimisation d'utilité. Enfin, nous formulons trois problèmes de recherche résolus dans cette thèse et donnons un résume des résultats obtenus.

Chapitre 2 est consacré aux opportunités d'arbitrages apparaissant en présence des investisseurs hétérogènes, dans le sens que leurs croyances correspondent à des probabilités non-équivalentes. Le profit d'arbitrage optimal et la stratégie correspondante sont étudiés par des techniques de changement de mesure non-quivalent. Nous discutons également les implications financières de cette étude et donnons quelques exemples pertinents. Contrairement aux modèles browniens classiques, pour lesquels les arbitrages (s'ils existent) sont fragiles, nous montrons que certains de nos exemples sont robustes en présence des frictions du marché, comme les coûts de transaction et les erreurs de spécification de modéle.

Dans le Chapitre 3, nous étudions le problème d'investissement optimal avec la possibilité de consommation intermédiaire. Nous montrons que la condition "no unbounded profit with bounded risk" est suffisante pour établir les relations de dualité classiques de maximisation d'utilité.

Dans le Chapitre 4, nous étudions les stratégies d'arbirtage des agents possedant une information privée. Nous supposons qu'un agent initié a accès à une information privée dès le début de trading. En termes mathématiques cette situation est décrite par un grossissement initial de filtration (par opposition au grossissement progressif, lorsque l'information privée devient disponible au fur et à mesure de trading). Nous considérons d'abord le problème d'optimisation d'utilité logarithmique pour l'agent initié. La caractérisation de l'utilité espérée de l'initié par la méthode de dualité nous permet de donner une nouvelle condition suffisante pour la condition "no unbounded profit with bounded risk". Grâce aux outils de changement de mesure non-équivalente du Chapitre 2, nous calculons le prix de sur-couverture d'un actif contingent du point de vue de l'initié et examinons la question de profit optimal d'arbitrage [2].

[2]Thanks to David Krief for his help on this translation

Chapter 1

Review of literature and new results

1.1 Markets with heterogeneous investors

Financial mathematics literature typically assumes that all investors are homogeneous in the sense that they share the same set of information, the same level of risk aversion, the same beliefs of market's structure, etc. This assumption has been the basis for fruitful development in finance. However, one can easily argue that this homogeneous assumption is too restrictive if one wants to model realistic markets. For example, it is common that people take different views on everything, from very significant issues to very simple ones. Hence, a more satisfactory model should take into account discrepancies among investors because it is a fact of life.

Plenty of important phenomena (such as bubbles, arbitrages, speculation) can be better perceived when all investors are not identical. Such formulations have been introduced in literature a long time ago. For example, Harrison and Kreps [1978] use heterogeneous expectation to explain the behavior of speculative investors, Scheinkman and Xiong [2003] propose a model in which each agent is overconfident on the informativeness of her own signal to get an aspect of bubbles. An incomplete list of studies on the topic of heterogeneous investors includes: Jarrow [1980], Constantinides and Duffie [1996], Scheinkman and Xiong [2003], Basak [2000, 2005], Jouini and Napp [2007], Nishide and Rogers [2011], Cvitanić et al. [2012], Larsson [2013], etc.

Models with heterogeneous investors can be classified into two main categories: heterogeneous beliefs and asymmetric information. In the first category, investors form different (or opposite) views about the future performance of the world and bet against each other. In the second category, investors differ in their information. Some investors might know more than others, and even if all investors hear the same news from public announcements, they still might interpret it differently.

Heterogeneity is a natural source of mispricings, which are documented in some empirical studies. For example, Yadav and Pope [1994] provide evidence on stock index futures and conclude that potential arbitrage opportunities are exploitable and economically significant. Financial literature studies optimal behavior in the presence of mispricings, but says little on arbitrage opportunities. Our objective is to continue along this path by developing a structural framework to address fascinating questions of arbitrages.

1.2 Overview of arbitrage pricing theory

The concept of arbitrage plays a very crucial role in the theory of modern finance. Informally speaking, an arbitrage opportunity is the possibility of making money out of nothing without taking any risk. Clearly, such strategies should be excluded in order to ensure market viability. The link between theories of no arbitrage and asset pricing has a long history and was established through seminal works (with no claim of being complete) of Black and Scholes [1973], Harrison and Kreps [1979], Kreps [1981], Harrison and Pliska [1981], Dalang et al. [1990]... The first rigorous formulation with general semimartingale models is given in Delbaen and Schachermayer [1994, 1998]. The authors prove the equivalence between the No Free Lunch with Vanishing Risk (NFLVR) condition and the existence of an equivalent sigma-martingale measure, i.e. a new probability measure under which the discounted asset price process is a sigma-martingale. We refer to the book of Delbaen and Schachermayer [2006] for the notion of NFLVR and all results in this theory.

The NFLVR condition provides a sound theoretical framework to solve problems of pricing, hedging or portfolio optimization. However, for some applications, requiring total absence of free lunches turns out to be too restrictive and it seems reasonable to assume that limited arbitrage opportunities exist in financial markets. This is one of the reasons why market models with arbitrage opportunities have appeared in the literature, starting with the three-dimensional Bessel process model of Delbaen and Schachermayer [1995a]. Without relying on the concept of equivalent martingale measure, Platen [2006], see also Platen and Heath [2006], developed the Benchmark Approach, a new asset pricing theory in which the physical measure becomes the main ingredient. In the context of Stochastic Portfolio Theory [Karatzas and Fernholz, 2009], the NFLVR condition is not imposed and arbitrage opportunities arise in relative sense. These works suggest that NFLVR condition can be replaced by another weaker notion while preserving the solvability of the economics problems mentioned above.

Numerous studies are devoted to proposing some new notions of arbitrage. We do not discuss all of these concepts here and refer to Fontana [2013] for an overview. If one is interested in utility maximization, Karatzas and Kardaras [2007] and Choulli et al. [2012] prove that the minimal no free lunch type condition making this problem well posed is the No Unbounded Profit with Bounded Risk (NUPBR) condition. This condition has also been referred to as BK in Kabanov [1997] and it is also equivalent to the No Asymptotic Arbitrage of the 1st kind (NAA1) condition of Kabanov and Kramkov [1994] taken with respect to a fixed probability measure. It is known that the NFLVR is equivalent to NUPBR plus the classical no arbitrage assumption (see Corollary 3.4 and Corollary 3.8 of Delbaen and Schachermayer [1994] or Proposition 4.2 of Karatzas and Kardaras [2007]). This means that markets satisfying only NUPBR may admit arbitrage opportunities. It is naturally thought that the existence of arbitrage is inconsistent with market's viability. However, it is not the case because these riskless profits are not scalable and arbitrageurs are financially constrained. In other words, arbitrageurs face a certain amount of risk before making money so that they cannot invest into arbitrage positions as much as they want.

The starting point of this dissertation is the gap between NFLVR and NUPBR. Assume a market only satisfies NUPBR, what conclusions could we make about arbitrage opportunities? This thesis contributes to a better understanding of this gap in some interesting situations.

1.3 Studies of arbitrages in literature

Delbaen and Schachermayer [1995a] discuss arbitrages and strict local martingales in the three dimensional Bessel model. They show that with respect to simple integrands, the three dimensional Bessel process satisfies the no-arbitrage property. However, the process in its natural filtration permits arbitrage with respect to general admissible integrands. It is very interesting to note that its inverse process is a (strict) local martingale and hence is arbitrage free. The situation where the inverse of an arbitrage-free asset admits arbitrage opportunities could happen in foreign exchange markets. Let us recall the comments from their paper. Assume that the price of one Euro in dollars is modeled by the inverse of three dimensional Bessel process, which yields that there are no arbitrage opportunities for European traders. However, there are such possibilities for American traders. The reason is that the admissibility of investment strategies depends on which currency is used as numéraire. It means that agents in one country can use some strategies that agents in the other country cannot. Furthermore, in order to exclude this circumstance, Delbaen and Schachermayer [1995c] give some criteria for the stability of no arbitrage conditions under a change of numéraire.

The three dimensional Bessel model is further studied in Karatzas and Kardaras [2007]. Although the market permits arbitrage, it still viable in the sense that one can find solutions for utility optimization problems. This property is then connected with the weaker no arbitrage condition NUPBR. Unlike in Delbaen and Schachermayer [1995a], where the presence of arbitrage relies on the predictable representation property, the authors are able to construct an arbitrage strategy (see Example 4.6 therein), which corresponds to a solution of a pricing PDE. Finally, optimal arbitrage is given in Ruf [2013] also by the PDE method. This is due to the fact that the pricing equation in this situation has multiple solutions.

In the context of Stochastic Portfolio Theory, see Fernholz et al. [2009], two portfolios define a relative arbitrage when one portfolio outperforms the other. In Fernholz et al. [2005], relative arbitrage is linked with weak diversity, a market property which means that no single stock is allowed to dominate the entire market in terms of relative capitalization, or with volatility-stabilized markets as in Fernholz and Karatzas [2005].

To benefit from potential arbitrage, one needs to characterize explicitly the arbitrage strategy, and also to devise a method to compare different strategies, so as to exploit the arbitrage opportunity in the most efficient way. An important step in this direction was made in Fernholz and Karatzas [2010]. In this paper, the authors introduce the notion of optimal relative arbitrage with respect to the market portfolio and characterize the optimal relative arbitrage in continuous Markovian market models in terms of the smallest positive solution to a parabolic partial differential inequality. The idea is then extended in Fernholz and Karatzas [2011] by considering market models with uncertainty regarding the relative risk and covariance structure of its assets, or in Bayraktar et al. [2012] when an investor wants to beat the market portfolio with a certain probability. Optimal relative arbitrage turns out to be related to the minimal cost needed to superhedge the market portfolio in an almost sure way. In continuous diffusion settings, the problem of hedging in markets with arbitrage opportunities is studied in detail in Ruf [2013]. That paper shows in particular that delta hedging is still the optimal hedging strategy in continuous Markovian markets which admit no equivalent local martingale measure but only a square-integrable market price of risk.

Arbitrages appear naturally in models of insider trading, in particular in enlargement of filtration theory. The seminal work of Pikovsky and Karatzas [1996] is devoted to analyzing the additional logarithmic utility for an insider when he gains some private information from the beginning of trading. Imkeller [2003]; Imkeller et al. [2001] use Malliavin calculus to derive the preservation of the semimartingale property and construct explicit arbitrage strategies. In Imkeller [2002], Zwierz [2007], an insider possesses some additional knowledge which allows him to stop at a random time τ which is not accessible to regular agents. The authors show that for a class of random times, the market price of risk for the insider is not square integrable on a set of positive measure and thus prove the existence of arbitrage opportunities. In Fontana et al. [2014], explicit constructions of arbitrages are given if the market for regular agent is complete. Aksamit et al. [2013] study various kinds of honest times and non honest times. Other studies focused on arbitrage include: Elliott and Jeanblanc [1999], Grorud and Pontier [1998], Grorud and Pontier [2001], Kohatsu-Higa [2007]; Kohatsu-Higa and Yamazato [2008, 2011], etc.

1.4 General settings of the thesis

The notation in this section will be used throughout the dissertation. For the theory of stochastic process and stochastic integration, we refer to Jacod and Shiryaev [2002] and Protter [2003].

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a given filtered probability space, where the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ is assumed to satisfy the condition of right-continuity. For any adapted RCLL process S, we denote by S_- its predictable left-continuous version and by $\Delta S := S - S_-$ its jump process. For a d-dimensional semimartingale S and a predictable process H, we denote by $H \cdot S$ the vector stochastic integral of H with respect to S. We fix a finite planning horizon $T < \infty$ (a stopping time) and assume that after T all price processes are constant and equal to their values at T.

On the stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, we consider a financial market with an $\mathbb{R}^d$-valued nonnegative semimartingale process $S = (S^1, ..., S^d)$ whose components model the prices of d risky assets. The riskless asset is denoted by S^0 and we assume that $S^0 \equiv 1$, that is, all price processes are already discounted. We suppose that the financial market is frictionless, meaning that there are no trading restrictions, transaction costs, or other market imperfections.

Let $L(S)$ be the set of all $\mathbb{R}^d$-valued S-integrable predictable processes. It is the most reasonable class of strategies that investors can choose, but another constraint, which is described below, is needed in order to rule out doubling strategies.

Definition 1.4.1. *Let $x \in \mathbb{R}_+$. An element $H \in L(S)$ is said to be an x-admissible strategy if $H_0 = 0$ and $(H \cdot S)_t \geq -x$ for all $t \in [0,T]$ $\mathbb{P}$-a.s. An element $H \in L(S)$ is said to be an admissible strategy if it is an x-admissible strategy for some $x \in \mathbb{R}_+$.*

Remark 1.4.2.

- *We would like to emphasize that $H \cdot S$ has to be understood as the vector stochastic integral of H with respect to S, see the discussion of this concept in Shiryaev and Cherny [2002]. The notion of vector stochastic integral is a generalization of the notion of componentwise stochastic integral $\sum_{i=1}^{d} H^i \cdot S^i$ in order to obtain a closed space of stochastic integrals. It should not to be confused with the notion of vector-valued stochastic integral $(H^1 \cdot S^1, ..., H^d \cdot S^d)$.*

- *The vector-valued integral is introduced in Shiryaev and Cherny [2002] without the assumption of completeness of the filtration. Moreover, Appendix A in Perkowski and Ruf [2013] argues that the completeness assumption usually does not matter.*

For $x \in \mathbb{R}_+$, we denote by $\mathscr{A}_x$ the set of all x-admissible strategies and by $\mathscr{A}$ the set of all admissible strategies. As usual, H_t is assumed to represent the number of risky asset held at time t. For $(x,H) \in \mathbb{R}_+ \times \mathscr{A}$, we define the portfolio value process $V_t^{x,H} := x + (H \cdot S)_t$. This is equivalent to requiring that portfolios are only generated by self-financing admissible strategies.

Given the semimartingale S, we denote by $\mathscr{K}_x$ the set of all outcomes that one can realize by x-admissible strategies starting with zero initial cost:

$$\mathscr{K}_x := \{(H \cdot S)_T | H \text{ is } x\text{-admissible}\} \tag{1.1}$$

and by $\mathscr{X}_x$ the set of outcomes of x-admissible strategies with initial cost x:

$$\mathscr{X}_x := \{x + (H \cdot S)_T | H \text{ is } x\text{-admissible}\}.$$

Remark that all elements in $\mathscr{X}_x$ are nonnegative. The unions of $\mathscr{K}_x$ and all $\mathscr{X}_x$ over all $x \in \mathbb{R}_+$ are denoted by $\mathscr{K}$ and $\mathscr{X}$, respectively. All bounded claims which can be superreplicated by admissible strategies with zero initial cost are contained in

$$\mathscr{C} := \left(\mathscr{K} - L^0_+\right) \cap L^\infty.$$

1.4.1 No arbitrage conditions

Now, we recall some no-free-lunch conditions, which are studied in the works of Delbaen and Schachermayer [1994], Karatzas and Kardaras [2007] and Kardaras [2012].

Definition 1.4.3 (NA)**.** *We say that the market satisfies the No Arbitrage (NA) condition with respect to general admissible integrands if*

$$\mathscr{C} \cap L^\infty_+ = \{0\}.$$

Definition 1.4.4 (NFLVR)**.** *We say that the market satisfies the No Free Lunch with Vanishing Risk (NFLVR) property, with respect to general admissible integrands, if*

$$\overline{\mathscr{C}} \cap L^\infty_+ = \{0\},$$

where the bar denotes the closure in the supnorm topology of L^∞.

We recall the concept of sigma-martingales.

Definition 1.4.5 (Sigma-martingale)**.** *A $\mathbb{R}^d$-valued semimartingale $X = (X_t)_{t\in\mathbb{R}_+}$ is called a sigma-martingale if there exists an $\mathbb{R}^d$ -valued martingale M and an M -integrable predictable $\mathbb{R}_+$-valued process φ such that $X = \varphi \cdot M$.*

Theorem 1.4.6 (Fundamental Theorem of Asset Pricing (FTAP), Delbaen and Schachermayer [1994, 1998])**.** *The asset S satisfies the NFLVR condition if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that S is a sigma-martingale with respect to $\mathbb{Q}$.*

By FTAP, the NFLVR condition is equivalent to the existence of an equivalent sigma-martingale measure, see Theorem 1.1 of Delbaen and Schachermayer

[1998]. However, nonnegative sigma-martingales are local martingales, see for example Exercise 41, page 241 of Protter [2003]. Thus, the limitation to nonnegative processes allows us to work with local martingales instead of sigma-martingales.

If one is interested in utility maximization, it has been shown [Choulli et al., 2012; Karatzas and Kardaras, 2007] that the minimal no free lunch type condition making this problem well posed is the NUPBR condition.

Definition 1.4.7 (NUPBR)**.** *There is No Unbounded Profit With Bounded Risk (NUPBR) if the set $\mathscr{K}_1$ is bounded in L^0, that is, if*

$$\lim_{c\to\infty}\sup_{H\in\mathscr{A}_1}\mathbb{P}\left[V_T^{0,H}>c\right]=0$$

holds.

This condition has also been referred to as BK in Kabanov [1997] and it is also equivalent to the No Asymptotic Arbitrage of the 1st kind (NAA1) condition of Kabanov and Kramkov [1994] taken with respect to a fixed probability measure or to the condition No Arbitrage of The First Kind (NA1) of Kardaras [2012].

Definition 1.4.8 (NA1)**.** *An $\mathscr{F}_T$-measurable random variable ξ is called an arbitrage of the first kind if $\mathbb{P}[\xi\geq 0]=1, \mathbb{P}[\xi>0]>0$, and for all $x>0$, there exists $H\in\mathscr{A}_x$ such that $V_T^{x,H}\geq\xi$. If there exists no arbitrage of the first kind in the market, we say that condition NA1 holds.*

It is known that the NFLVR is equivalent to NUPBR plus the classical no arbitrage assumption (see Corollary 3.4 and Corollary 3.8 of Delbaen and Schachermayer [1994] or Proposition 4.2 of Karatzas and Kardaras [2007]). This means that markets satisfying only NUPBR may admit arbitrage opportunities.

The economic interpretation of no arbitrage type conditions above can be described as follows. Classical arbitrage means that one can make something out of nothing without risk. If there is a FLVR, starting with zero capital, one can find a sequence of wealth processes such that the terminal wealths converge to a nonnegative random variable which is not identical to zero and the risk of the trading strategies becomes arbitrarily small. If an UPBR exists, one can find a sequence of wealth processes with bounded (or indeed arbitrarily small) risk whose terminal wealths are unbounded with a fixed probability.

Definition 1.4.9. *An equivalent local martingale deflator (ELMD) is a nonnegative process Z with* $Z_0 = 1$ *and* $Z_T > 0$ *such that* $ZV^{x,H}$ *is a local martingale for all* $H \in \mathscr{A}_x, x \in \mathbb{R}_+$.

In particular, an ELMD is a nonnegative local martingale. Fatou's Lemma implies that it is also a supermartingale and its expectation is less or equal to one. Hence, if there exists an ELMD with constant expectation one, the NFLVR condition holds. It is worth to remark that the situation when the ELMD is a strict local martingale is very different from a market with a bubble. Indeed, an asset price is said to be a bubble if it is a strict local martingale under the risk-neutral measure, see Heston et al. [2007], Cox and Hobson [2005], Jarrow et al. [2007], Jarrow et al. [2010], which means that the NFLVR condition is valid.

The following result has recently been proven in Kardaras [2012] in the one dimensional case. An alternative proof in the multidimensional case has been given in Takaoka and Schweizer [2014] by a suitable change of numéraire argument in order to apply the classical results of Delbaen and Schachermayer [1994], and in Song [2013] by only using the properties of the local characteristics of the asset process.

Theorem 1.4.10. *The NUPBR condition is equivalent to the existence of at least one ELMD.*

As discussed, the condition NUPBR is the minimal requirement for market's viability. In this dissertation, it is always observed that our market models satisfy this condition. In the next subsection, we will recall some optimization problems which are important in theoretical as well as practical purposes.

1.4.2 Utility maximization

One can say that one of the most important problems in mathematical finance is the utility maximization: an investor who wants to invest and consume in a way that maximizes his expected utility.

In the seminal work, Merton [1969] explicitly solved an optimal investment problem via dynamic programing arguments. The notion of equivalent martingale measures introduced martingale duality method for solving such optimization problems. Karatzas et al. [1987] developed this method under the assumption of complete markets. The difficult case with incomplete markets was studied in Karatzas et al. [1991] for Brownian settings, and in Kramkov and Schachermayer [1999, 2003] for general semimartingale settings. Karatzas and Žitković [2003] and Žitković et al. [2005] obtain sufficient condition with the possibility of intermediate consumption and stochastic utility for incomplete markets. An incomplete list of studies in this theory includes: Cvitanic, Schachermayer and Wang (2001), Hugonnier and Kramkov (2004), Mostovyi [2015], etc.

Maximization of expected utility from terminal wealth

The agent's preferences are described by a utility function: that is a concave and strictly increasing function $U : (0,\infty) \mapsto \mathbb{R}$. We define $U(0) = U(0+)$ by continuity. Starting with initial capital $x > 0$, the investor wants to solve the following problem

$$u(x) := \sup_{H \in \mathscr{A}_x} \mathbb{E}[U(V_T^{x,H})]. \tag{1.2}$$

We use the usual convention $\mathbb{E}[U(V_T^{x,H})] = -\infty$ whenever $\mathbb{E}[(U(V_T^{x,H}))^-] = \infty$. The optimization problem (1.2) makes sense only if $u(x) < \infty$. Also note that $u(x) > -\infty$ for every $x > 0$ because $u(x) \geq U(x)$. Among all possible utility functions, an interesting one is probably the logarithmic utility function $U(x) = \log x$.

Definition 1.4.11. *An element $V^{x,H^{log}} \in \mathscr{K}_x$ is called log-optimal portfolio if*

$$\mathbb{E}[\log V_T^{x,H}] \leq \mathbb{E}[\log V_T^{x,H^{log}}]$$

for all $V^{x,H} \in \mathscr{K}_x$ such that $\mathbb{E}[(\log V^{1,H})_T^-] < \infty$.

The log-optimal portfolio maximizes the instantaneous growth rate (defined as the drift of a portfolio at log scale) among all portfolios. In long term, it will have higher growth rate than any other strategies. For more historical facts and details about log-optimal portfolio, we refer to Christensen [2005].

Finding the log-optimal strategy is not always an easy task. When asset prices are continuous, an explicit solution is possible. For example, for continuous diffusion cases, the problem is much easier and was solved by Merton [1971]. The optimal portfolio fraction turns out to relate to the market price of risk. This however cannot be extended to general cases with discontinuous assets. In Goll and Kallsen [2003], the optimal solution is provided explicitly in terms of the semimartingale characteristics of the price process.

Definition 1.4.11 does not include the case with infinite expected growth rate. The definition is then modified as follows in order to cover interesting cases in which log-investor can trade to infinity but the modified definition is still well-defined.

Definition 1.4.12. *A portfolio H^{go} is called growth-optimal portfolio (GOP) or relatively log-optimal if*

$$\mathbb{E}\left[\log \frac{V_T^{1,H}}{V_T^{1,H^{go}}}\right] \leq 0, \qquad \forall H \in \mathscr{A}_1. \tag{1.3}$$

If the log-optimal portfolio exists and finite, then GOP exists. Nevertheless, the converse implication is not true and we will see it soon. The GOP enjoys impressive properties of log-optimal portfolios as well as the so-called numéraire property. The GOP is understood as the best investment decision that an investor can make, so that other portfolios cannot dominate its performance. In terms of mathematics, any portfolio is a supermartingale when discounted by GOP.

Now, we summarize some connections between the existence of solutions of these optimization problems and no-arbitrage conditions. If the condition NFLVR holds, then GOP and the log-optimal portfolio coincide, see for example in Proposition 4.3 of Becherer [2001]. However, if only the condition NUPBR holds, the gap between GOP and log-optimal portfolio appears: GOP coincides with the relatively log-optimal portfolio, as in Proposition 3.19 of Karatzas and Kardaras [2007], and the log-optimal portfolio does not necessarily exist. This point is illustrated in Example 4.3 of Christensen and Larsen [2007] with the three dimensional Bessel process. In their example, the condition NUPBR holds (GOP exists) but the log-investor can trade to infinite utility. Also in Example 20 of Karatzas and Kardaras [2007], the classical log-utility optimization problem is not well-posed (infinite utility) but NUPBR holds. To conclude, NUPBR condition does not imply that the log-utility is finite.

Conversely, if the log-utility problem is finite, what conclusion should we make about the markets? It is proved in Proposition 4.19 of Karatzas and Kardaras [2007] that if the condition NUPBR fails, then $u(x) = \infty$ for all $x > 0$. The converse implication means if there exists $x > 0$ such that $u(x)$ is finite then the condition NUPBR holds. We will often use this result in the dissertation.

1.5 Contributions of the thesis

In this thesis, we propose to analyze financial markets with two possible sources of heterogeneity among agents: they may differ in their beliefs, or in their level of information. In the following, we state three research questions, which will be discussed in detail.

Question 1. Assume that the insider has been informed that a certain event cannot happen. How could she extract profit from this private information in an efficient way?

Question 2. Does the condition NUPBR suffice to establish the key duality relations of the utility maximization problem?

Question 3. Assume that the insider knows the terminal value of the underlying asset, for example S_T at time 0. Is there an optimal way to use this kind of information? Can the insider make an arbitrage profit? What about the optimal strategy?

In conjunction with literature, especially with Section 1.3, answering these proposed questions will considerably improve our knowledge about arbitrages in many directions. First, most previous studies about arbitrage focus on restrictive settings. For example, one usually assumes that the market is complete or asset prices are continuous (Brownian settings), which makes things easier, see for example Imkeller et al. [2001], Imkeller [2002], Fontana et al. [2014], Aksamit et al. [2013] and others. Second, arbitrage opportunities often appear implicitly, in the sense that one establishes the existence of such arbitrage profits without giving a constructive way to exploit them. In rare special cases, one can use simple buy and hold strategies to make profits, or may use market's completeness or other fine properties of the market to guess arbitrage strategies. Third, arbitrages found in these ways are far from being optimal. In this thesis, we propose a systematic way for exploiting optimal arbitrage profits and the corresponding strategies in fully general semimartingale settings with particular attention to incomplete markets. Finally, the positive answer of **Question 2** will meaningfully improve the existing results on optimization. More precisely, we show that the key conclusions of the utility maximization theory hold under NUPBR in full generality.

1.5.1 Arbitrages arising when agents have non-equivalent beliefs

Chapter 2 of the thesis, based on Chau and Tankov [2015], aims at giving the answer to Question 1. We consider an economy in which agents have different beliefs about the world. For simplicity, assume that there are two agents acting in the economy: an ordinary agent and an insider. The ordinary agent is assumed to be risk neutral and choose investment on the probability basis $(\Omega, \mathbb{F}, \mathbb{Q}, S)$. The insider, who has different belief about the market, makes her decisions on $(\Omega, \mathbb{F}, \mathbb{P}, S)$, where $\mathbb{P} \ll \mathbb{Q}$. As in Larsson [2013], we employ the idea that heterogeneity in beliefs is pushed to an extreme degree: agents do not agree about zero probability events, i.e. certain events are possible in the view of one agent but not the other. In mathematical terms, we say that the measure $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ but not equivalent to it. Let M be the density of $\mathbb{P}$ with respect to $\mathbb{Q}$

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathscr{F}_t} = M_t, \qquad t \in [0,T].$$

The following theorem is the main result of this chapter.

Theorem 1.5.1. *If the density M does not jump to zero, then the insider satisfies NUPBR condition. Furthermore, the superhedging price of a nonnegative claim f for the insider equals to the superhedging price of the claim $f 1_{M_T>0}$ for the ordinary agent.*

Because $\mathbb{P} \ll \mathbb{Q}$, it may happen that some events have positive probability under $\mathbb{Q}$ but zero probability under $\mathbb{P}$. As a consequence, the insider does not need to replicate the claim f on the events which are assigned measure zero (i.e the event $\{M_T = 0\}$). Obviously, the price of f for the insider is smaller than the price of f for the ordinary agent. The theorem allows us to compute exactly the price for the insider as the price of $f 1_{M_T>0}$ in the view of the ordinary agent. This transformation is interesting because martingale pricing theory is applicable for the ordinary agent. Furthermore, optimal hedging strategy for the insider is the hedging strategy for the ordinary agent with the corresponding claim.

In particular, we provide some meaningful examples when the martingale M is associated with a predictable stopping time, for example a default time, or a hitting time of the asset's volatility. We also comment about fragility/robustness of arbitrages with respect to small transaction costs or model misspecification. We show that arbitrages in some of our examples are robust, in contrast to models satisfying conditions of Guasoni and Rásonyi [2015].

1.5.2 Optimal investment with intermediate consumption under no unbounded profit with bounded risk

Chapter 3, based on joint work with Andrea Cosso, Claudio Fontana and Oleksii Mostovyi, gives the positive answer to Question 2.

There are some papers which point out that the problem of utility maximization from terminal wealth could be studied without relying on the existence of equivalent local martingale measures. Indeed, Karatzas et al. [1991] studied this problem in an incomplete Itô process setting under a finite time horizon and established a duality theory which does not require the full strength of the NFLVR condition. In a continuous semimartingale setting, the results of Kramkov and

Schachermayer [1999] have been extended in Larsen [2009] by weakening the NFLVR requirement. Finally, in a general semimartingale setting, Larsen and Žitković [2013] have established a general duality theory for the problem of maximizing expected utility from terminal wealth (for a deterministic utility function) in the presence of trading constraints without the full strength of the NFLVR condition.

As discussed, the condition NUPBR cannot be relaxed in order to solve the problem of utility maximization, see Proposition 4.19 of Karatzas and Kardaras [2007]. In this chapter, we show that the key duality relations of the utility maximization theory hold under the minimal assumptions of NUPBR and of the finiteness of both primal and dual value functions. We adopt the setting of Mostovyi [2015] which allows a stochastic field utility and intermediate consumption occurring according to some stochastic clock in order to include certain classical problems. The result does not rely on the asymptotic elasticity of the utility.

We fix a *stochastic clock* $\kappa = (\kappa_t)_{t\geq 0}$ which is a nondecreasing, càdlàg adapted process such that

$$\kappa_0 = 0, \qquad \mathbb{P}(\kappa_\infty > 0) > 0 \qquad \text{and} \qquad \kappa_\infty \leq A, \tag{1.4}$$

for some finite constant A. The stochastic clock κ represents the notion of time according to which consumption occurs.

We consider a *stochastic utility field* $U = U(t,\omega,x) : [0,\infty)\times\Omega\times[0,\infty) \to \mathbb{R}\cup\{-\infty\}$ satisfying the following assumption (see Assumption 2.1 of Mostovyi [2015])

Assumption 1.5.2. *For every $(t,\omega)\in[0,\infty)\times\Omega$, the function $x\mapsto U(t,\omega,x)$ is strictly concave, strictly increasing, continuously differentiable on $(0,\infty)$ and satisfies the Inada conditions*

$$\lim_{x\downarrow 0} U'(t,\omega,x) = +\infty \qquad \textit{and} \qquad \lim_{x\to+\infty} U'(t,\omega,x) = 0,$$

with U' denoting the partial derivative of U with respect to its third argument. By continuity, at $x=0$ we have that $U(t,\omega,0) = \lim_{x\downarrow 0} U(t,\omega,x)$ (note this value may be $+\infty$). Finally, for every $x\geq 0$, the stochastic process $U(\cdot,\cdot,x)$ is optional.

For a given initial capital $x > 0$, the associated *value function* is denoted by

$$u(x) := \sup_{c \in \mathscr{A}(x)} \mathbb{E}\left[\int_0^\infty U(t,\omega,c_t)\,\mathrm{d}\kappa_t\right], \tag{1.5}$$

where $c = (c_t)_{t\geq 0}$ is a nonnegative optional process representing the consumption and $\mathscr{A}(x)$ is the set of all admissible consumption rates. The *stochastic field V conjugate to U* is defined as

$$V(t,\omega,y) := \sup_{x>0}\big(U(t,\omega,x) - xy\big), \qquad (t,\omega,y) \in [0,\infty)\times\Omega\times[0,\infty).$$

We define the set of *equivalent local martingale deflators (ELMD)* as follows:

$$\mathscr{Z} := \big\{Z > 0 : Z \text{ is a càdlàg local martingale such that } Z_0 = 1 \text{ and } ZX \text{ is a local martingale for every } X \in \mathscr{X}\big\}.$$

We also denote

$$\mathscr{Y}(y) := \mathrm{cl}\big\{Y : Y \text{ is càdlàg adapted and } 0 \leq Y \leq yZ \ (\mathrm{d}\kappa\otimes\mathbb{P})\text{-a.e. for some } Z \in \mathscr{Z}\big\},$$

where the closure is taken in the topology of convergence in measure $(\mathrm{d}\kappa\otimes\mathbb{P})$ on the space of real-valued optional processes. For $y > 0$, the value function of the dual optimization problem is defined as

$$v(y) := \inf_{Y \in \mathscr{Y}(y)} \mathbb{E}\left[\int_0^\infty V(t,\omega,Y_t)\,\mathrm{d}\kappa_t\right]. \tag{1.6}$$

We are now in a position to state the following theorem, which establishes a full duality theory for a general optimal investment/consumption problem under the condition NUPBR.

Theorem 1.5.3. *Assume that conditions 1.4 and NUPBR hold true and let U be a utility stochastic field satisfying Assumption 3.2.3. Suppose that*

$$v(y) < \infty \quad \textit{for all } y > 0 \quad \textit{and} \quad u(x) > -\infty \quad \textit{for all } x > 0. \tag{1.7}$$

Then the value function u and the dual value function v defined in (3.2) *and* (3.3)*, respectively, satisfy the following properties:*

(i) $u(x) < \infty$, *for all* $x > 0$, *and* $v(y) > -\infty$, *for all* $y > 0$. *Moreover, the functions* u *and* v *are conjugate, i.e.,*

$$v(y) = \sup_{x>0}\big(u(x) - xy\big), \quad y > 0,$$
$$u(x) = \inf_{y>0}\big(v(y) - yx\big), \quad x > 0;$$

(ii) the functions u *and* $-v$ *are continuously differentiable on* $(0,\infty)$, *strictly concave, strictly increasing and satisfy the Inada conditions*

$$u'(0) := \lim_{x\downarrow 0} u'(x) = +\infty, \qquad -v'(0) := \lim_{y\downarrow 0} -v'(y) = +\infty,$$
$$u'(\infty) := \lim_{x\to+\infty} u'(x) = 0, \qquad -v'(\infty) := \lim_{y\to+\infty} -v'(y) = 0.$$

Moreover, for every $x > 0$ *and* $y > 0$, *the solutions* $\hat{c}(x)$ *to* (3.2) *and* $\hat{Y}(y)$ *to* (3.3) *exist and are unique and, if* $y = u'(x)$, *we have the dual relations*

$$\hat{Y}_t(y)(\omega) = U'\big(t,\omega,\hat{c}_t(x)(\omega)\big), \qquad \mathrm{d}\kappa\otimes\mathbb{P}\text{-}a.e.,$$

and

$$\mathbb{E}\left[\int_0^\infty \hat{c}_t(x)\hat{Y}_t(y)\,\mathrm{d}\kappa_t\right] = xy.$$

Finally, the dual value function v *can be equivalently represented as*

$$v(y) = \inf_{Z\in\mathscr{Z}} \mathbb{E}\left[\int_0^\infty V(t,\omega,yZ_t)\,\mathrm{d}\kappa_t\right], \quad y > 0. \tag{1.8}$$

1.5.3 Optimal arbitrage for initial filtration enlargement

Chapter 4, based on joint work with Prof. Peter Tankov and Prof. Wolfgang Runggaldier, investigates Question 3. Again, we suppose there are an ordinary agent and an insider as in Chapter 2. Instead of having different probability measures, the investors here are assumed to possess different levels of information. The ordinary agent chooses investments on the financial market $(\Omega,\mathbb{F},\mathbb{P},S)$ while the insider decides hers on $(\Omega,\mathbb{G},\mathbb{P},S)$ where $\mathbb{F}\subset\mathbb{G}$. We assume that at time zero the insider knows the realization of a random variable G which is observed by the ordinary agent only at the end of trading. This idea is formulated in mathematical terms by $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(G)$ for all $t \in [0,T]$. To begin, we recall some technical assumptions.

Assumption 1.5.4. *(Jacod's Condition) For all $t \in [0,T)$, the regular conditional distribution of G given $\mathscr{F}_t$ is absolutely continuous with respect to the law of G, i.e. we have*

$$\mathbb{P}[G \in dx|\mathscr{F}_t](\omega) \ll \mathbb{P}[G \in dx], \qquad \text{for all } \omega \in \Omega. \tag{1.9}$$

Let $(p_t^x)_{t\in[0,T)}$ be the densities extracted from the relation (1.9). Assumption 1.5.4 ensures a $\mathbb{F}$-local martingale remains a $\mathbb{G}$-semimartingales. The process p^x plays an important role in the semimartingale decomposition, see in Proposition 4.2.3, and then in theory of initial enlargement of filtration. It is worth to notice that our setting is different from previous studies in two directions.

First, we do not require the equivalence in (1.9), i.e. we weaken the following assumption

$$\mathbb{P}[G \in dx|\mathscr{F}_t](\omega) \sim \mathbb{P}[G \in dx], \qquad \text{for all } \omega \in \Omega. \tag{1.10}$$

which is used in most of previous discussion. An important message from the stronger formulation (1.10) is that there exists a probability measure $\mathbb{P}^G$ equivalent to $\mathbb{P}$ such that under $\mathbb{P}^G$ the sigma algebra $\mathscr{F}_t$ and $\sigma(G)$ are independent. Under (1.10), Amendinger [2000] shows martingale representation theorems for the filtration $\mathbb{G}$ and deduce that in complete markets, there can be no free lunch with vanishing risk and that the insider has no possibilities of exercising arbitrage. Or in Amendinger et al. [2003a], utility indifference prices of the additional information is computed for common utility functions.

Second, we do not even require the relation (1.9) holds at time T. Thus, the process p^x is not well-defined at time T and things get much difficult. However, we cannot avoid this issue in order to cover interesting cases, for example when the insider knows the value of S_T.

The following condition is crucial in our discussion, and its meaning is explained before Section 4.3.

Assumption 1.5.5. *For every x, the process p^x does not jump to zero.*

We make use of techniques from nonequivalent measure changes and our contributions in this chapter are:

- A representation of the expected log-utility of the insider by duality. It helps us to explain an observed phenomenon that the insider's utility is usually infinite. Furthermore, it leads to a new sufficient condition to check NUPBR for the insider.

- A tractable formula of superhedging prices for the insider and an explicit approach for associated strategies even in quite general incomplete market models with càdlàg semi-martingales. Hence, optimal arbitrage is obtained in a systematic way.

More precisely, if G is a discrete random variable, the results read as the following.

Theorem 1.5.6. *Under Assumption (1.5.5), the $(\mathbb{G},\mathbb{P})$-market satisfies NUPBR. The expected log-utility of the insider is*

$$\sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] = -\sum_{i=1}^{n} \mathbb{P}[G=g_i]\log\mathbb{P}[G=g_i], + \sum_{i=1}^{n} \inf_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{G=g_i}\log\frac{1}{Z_T}\right].$$

For any claim $f\geq 0$, the superhedging price of f for the insider is given by

$$x_*^{\mathbb{G},\mathbb{P}}(f) = \sum_{i=1}^{n} x_*^{\mathbb{F},\mathbb{P}}(f1_{G=g_i})1_{G=g_i},$$

and the associated hedging strategy on the event $\{G=g_i\}$ is $H^{\mathbb{F},i}1_{G=g_i}$, where $H^{\mathbb{F},i}$ is the suphedging strategy for $f1_{G=g_i}$ in the $(\mathbb{F},\mathbb{P})$-market, i.e.

$$x_*^{\mathbb{F},\mathbb{P}}(f1_{G=g_i})1_{G=g_i} + (H^{\mathbb{F},i}1_{G=g_i}\cdot S)_T \geq f1_{G=g_i}, \mathbb{P}-a.s.$$

The result leads to some interesting consequences. First, it points out the key factor which draws the insider's profit to infinity. That is the additional term, involving the entropy of G, which appears in the duality result. In order to get finite utility, the second term of the RHS (which depends on the market's structure) should compensate the first term of the RHS (which depends only on G, not the market) in the duality relation. The insider's utility is the sum of the two terms. This hints at a way to understand the value of initial information: in some markets, the information is more advantageous rather than in others. Second, superhedging prices can be computed by tools of the ordinary agents. In this discrete case, initial enlargement can be viewed as a combination of non-equivalent change of measures.

The case when G is not a purely atomic is more difficult. We first show that if the set of all equivalent local martingale measures for regular agents is uniformly integrable, then the condition NUPBR always fails under $\mathbb{G}$, i.e. the insider has unbounded profits. Next, we turn to the case where the condition NUPBR holds for the insider. We then approximate $\mathbb{G}$ by a sequence of increasing filtration $(\mathbb{G}^n)_{n\in\mathbb{N}}$ so that each $\mathbb{G}^n$ is obtained by enlarging $\mathbb{F}$ with a discrete random variable. Now, we apply Theorem 1.5.6 to the market with the filtration $\mathbb{G}^n$ and take the limit as n tends to infinity. The main result is reported as follows.

Theorem 1.5.7. *Under Assumption 1.5.5, and suppose that NUPBR holds for $\mathbb{G}$, the expected log-utility in the market with the filtration $\mathbb{G}^n$ tends to the expected log-utility in the market with filtration $\mathbb{G}$*

$$\lim_{n\to\infty} \sup_{H\in\mathscr{A}_1^{\mathbb{G}^n}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] = \sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}].$$

For any claim $f \geq 0$, the superhedging price of f in the market with the filtration $\mathbb{G}^n$ tends to the superhedging price of f under $\mathbb{G}$,

$$\lim_{n\to\infty} x_*^{\mathbb{G}^n,\mathbb{P}}(f) = x_*^{\mathbb{G},\mathbb{P}}(f) := x_*.$$

Chapter 2

Arbitrages arising when agents have non-equivalent beliefs

Abstract: We construct and study market models admitting optimal arbitrage. We say that a model admits optimal arbitrage if it is possible, in a zero-interest rate setting, starting with an initial wealth of 1 and using only positive portfolios, to superreplicate a constant $c > 1$. The optimal arbitrage strategy is the strategy for which this constant has the highest possible value. Our definition of optimal arbitrage is similar to that in Fernholz and Karatzas [2010], where optimal relative arbitrage with respect to the market portfolio is studied. In this work we present a systematic method to construct market models where the optimal arbitrage strategy exists and is known explicitly. We then develop several new examples of market models with arbitrage, which are based on economic agents views concerning the impossibility of certain events rather than ad hoc constructions. We also explore the robustness of arbitrage strategies with respect to small perturbations of the price process and provide new examples of arbitrage models which are robust in this sense.

Key words: optimal arbitrage, no unbounded profits with bounded risk, strict local martingales, incomplete markets, robustness of arbitrage

2.1 Introduction

The goal of this study is to propose a new methodology for building models admitting optimal arbitrage, with an explicit characterization of the optimal arbitrage strategy. To do so, we start with a probability measure $\mathbb{Q}$ under which the NFLVR condition holds. We then construct a new probability measure $\mathbb{P}$, not equivalent to $\mathbb{Q}$, under which NFLVR no longer holds but NUBPR is still satisfied. This procedure is not new and goes back to the construction of the Bessel process by Delbaen and Schachermayer [1995a]. However, we extend it in two directions.

First, from the theoretical point of view, we provide a characterization of the superhedging price of a claim under $\mathbb{P}$ in terms of the superhedging price of a related claim under $\mathbb{Q}$. This allows us to characterize the optimal arbitrage profit under $\mathbb{P}$ in terms of the superhedging price under $\mathbb{Q}$, which is much easier to compute using the equivalent local martingale measures.

Second, from the economic point of view, we provide an economic intuition for the new arbitrage model as a model implementing the view of the economic agent concerning the impossibility of certain market events. In other words, if an economic agent considers that a certain event (such as a sovereign default) is impossible, but it is actually priced in the market, our method can be used to construct a new model incorporating this arbitrage opportunity, and to compute the associated optimal arbitrage strategy. Note that the presence of such heterogeneous beliefs is compatible with an economic equilibrium, as shown in a recent paper [Larsson, 2013]. It may of course happen that the economic agent *incorrectly believes* that a certain event is impossible while in reality it has a non-zero probability. This agent would then see an 'illusory' arbitrage opportunity in the market. In this case, the strategies given in this paper will lead to a loss if the event deemed impossible by the agent is realized.

We then combine these two ideas to develop several new classes of examples of models with optimal arbitrage, allowing for a clear economic interpretation, with a special focus on incomplete markets. We also discuss the issue of robustness of these arbitrages to small transaction costs / small observation errors, related to the notion of fragility of arbitrage introduced in Guasoni and Rásonyi

[2015], and show that some of our examples are robust in this sense.

The chapter is organized as follows. In Section 2.2, we state the main assumptions and discuss about robustness/fragility of arbitrages. In Section 2.3, optimal arbitrage profit is introduced and related to a superhedging problem. In Section 2.4, we use an absolutely continuous measure change to build markets with optimal arbitrage. Finally, several new examples built using this construction are gathered in Section 2.5.

2.2 General setting

We recall the general setting in Chapter 1. The financial market consists of a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ and a d-dimensional semimartingale S. The following assumption is forced to be true throughout this chapter.

Assumption 2.2.1. *The market satisfies the condition NUPBR under the physical measure* $\mathbb{P}$*.*

Fragile and robust arbitrages. Assume that one has come up with a model for the financial market which does not admit an equivalent local martingale measure and therefore admits unscalable arbitrage opportunities. The next step is to exploit these unscalable arbitrage opportunities. Here, two situations may arise:

- The arbitrage is robust with respect to small perturbations of the price process. This means that even if small market frictions are present, or the prices are recorded with small observation errors, the arbitrage strategy will still yield a profit with zero initial investment and no risk.

- The arbitrage is not robust with respect to small perturbations of the price process. This means that in the presence of transaction costs or observation errors, however small, the risk-free profit may disappear. The arbitrage strategy may still be attractive from the practical point of view, as it may still generate a high profit with low risk, but it is no longer an arbitrage in the mathematical sense of this word.

It is clearly important to distinguish between the above two situations, although both types of trading strategies may be of interest to practitioners. This type of robustness was studied in Guasoni and Rásonyi [2015] and can be characterized through the following two definitions.

Definition 2.2.2. *For $\varepsilon > 0$, two strictly positive processes $S, \tilde{S}$ are ε-close if*

$$\frac{1}{1+\varepsilon} \leq \frac{\tilde{S}_t}{S_t} \leq 1+\varepsilon \qquad a.s. \text{ for all } t \in [0,T].$$

Definition 2.2.3 (Fragility/Robustness)**.** *We say that the $(\mathbb{P}, S)$-market with arbitrage opportunities is fragile if for every $\varepsilon > 0$ there exists a process $\tilde{S}$, which is ε-close to S, such that the $(\mathbb{P}, \tilde{S})$-market satisfies NFLVR. If the $(\mathbb{P}, S)$-market is not fragile we say that it is robust.*

[Guasoni and Rásonyi, 2015, Theorems 1 and 2] show that in a diffusion market model, if the coefficients of the log-price process are locally bounded, then the market with arbitrages is fragile. For instance, when we introduce small frictions in the Bessel process example of Delbaen and Schachermayer [1995a], the arbitrage disappears.

Bender [2012] defines a *simple obvious arbitrage* as a buy and hold strategy, which guarantees the investor a profit of at least $\varepsilon > 0$ if the investor trades at all. It can be shown that if a market admits simple obvious arbitrage strategies, which are in $\mathscr{K}$ (see (1.1)), then the market is always robust. Indeed, assume that there exists a simple obvious arbitrage, i.e. there is a stopping time σ, an $\varepsilon > 0$ and an $\mathscr{F}_\sigma$-measurable random variable H such that $\mathbb{P}[\sigma < T] > 0$ and

$$\mathbb{P}\left[\{\sigma < T\} \cap \{\sup_{t\in[\sigma,T]} H(S_t - S_\sigma) < \varepsilon\}\right] = 0.$$

Without loss of generality, one may assume that $|HS_\sigma| < N$ for some $N < \infty$ (we do not trade when this condition is not fulfilled). Let $\tilde{S}$ be a process which is

δ-close to S for some $\delta > 0$. Then, on the event $\sigma < T$,

$$\begin{aligned}
&\sup_{t\in[\sigma,T]} H(\tilde{S}_t - \tilde{S}_\sigma)\\
&\geq \mathbf{1}_{H>0} \sup_{t\in[\sigma,T]} \Big(\frac{HS_t}{1+\delta} - HS_\sigma(1+\delta)\Big) + \mathbf{1}_{H\leq 0} \sup_{t\in[\sigma,T]} \Big(HS_t(1+\delta) - \frac{HS_\sigma}{1+\delta}\Big)\\
&\geq \mathbf{1}_{H>0}\Big(\frac{HS_\sigma+\varepsilon}{1+\delta} - HS_\sigma(1+\delta)\Big) + \mathbf{1}_{H\leq 0}\Big((HS_\sigma+\varepsilon)(1+\delta) - \frac{HS_\sigma}{1+\delta}\Big)\\
&\geq \frac{\varepsilon}{1+\delta} - 2N\delta \geq \frac{\varepsilon}{2}
\end{aligned}$$

for δ small enough.

2.3 Optimal arbitrage

It is well known that NFLVR holds if and only if both NUPBR and NA hold, see Corollary 3.4 and 3.8 of Delbaen and Schachermayer [1994] or Proposition 4.2 of Karatzas and Kardaras [2007]. Moreover, Lemma 3.1 of Delbaen and Schachermayer [1995b] shows that if NA fails then the market admits an arbitrage created by a strategy either in $\mathscr{A}_0^{\mathbb{P}}$ or in $\mathscr{A}_x^{\mathbb{P}}$ with $x > 0$. An arbitrage created by a strategy H in $\mathscr{A}_0^{\mathbb{P}}$ can be freely scaled to obtain the sequence of strategies $(nH) \subset \mathscr{A}_0$ which produces arbitrarily large levels of profit. Therefore, from the economic point of view it makes sense to exclude such scalable arbitrages by imposing our Assumption 2.2.1. Thus, in our market, it is only possible to exploit unscalable arbitrages, which are generated by strategies in the set $\mathscr{A}_x^{\mathbb{P}}$ with $x > 0$. For a given line of credit, the gains from such an unscalable arbitrage are limited and the question of optimal arbitrage profit arises naturally.

Definition 2.3.1. *For a fixed time horizon T, we define*

$$U(T) := \sup\Big\{c > 0 : \exists H \in \mathscr{A}_1^{\mathbb{P}}, V_T^{1,H} \geq c, \mathbb{P}-a.s\Big\} \geq 1.$$

If $U(T) > 1$, we call $U(T)$ optimal arbitrage profit.

The quantity $U(T)$ is the maximum deterministic amount that one can realize at time T starting from unit initial capital. Obviously, this value is bounded from below by 1. This definition goes back to the paper of Fernholz and Karatzas [2010]. In diffusion setting, these authors characterize the following value

$$\sup\left\{c>0: \exists H \in \mathscr{A}_1^{\mathbb{P}}, V_T^{1,H} \geq c\sum_{i=1}^{d} S_T^i, \mathbb{P}-a.s.\right\},$$

which is the highest return that one can achieve relative to the market capitalization.

2.3.1 Optimal arbitrage and the superhedging price

Definition 2.3.2. *Given a claim $f \geq 0$, we define*

$$SP_+^{\mathbb{P}}(f) := \inf\left\{x \geq 0 : \exists H \in \mathscr{A}_x^{\mathbb{P}}, V_T^{x,H} \geq f, \mathbb{P}-a.s\right\},$$

that is the minimal amount starting from which one can superhedge f by a non-negative wealth process.

Let us compare this definition with the usual definition of the superhedging price found in the literature. The superhedging price of a given claim f is commonly defined using wealth processes which may be negative but are uniformly bounded from below:

$$SP^{\mathbb{P}}(f) := \inf\left\{x \geq 0 : \exists H \in \mathscr{A}^{\mathbb{P}}, V_T^{x,H} \geq f, \mathbb{P}-a.s\right\}.$$

Clearly,

$$SP^{\mathbb{P}}(f) \leq SP_+^{\mathbb{P}}(f). \tag{2.1}$$

In markets that satisfy NA, $SP_+^{\mathbb{P}}(f) = SP^{\mathbb{P}}(f)$. Indeed, if NA holds, for every admissible integrand H we have $\|(H\cdot S)_t^-\|_\infty \leq \|(H\cdot S)_T^-\|_\infty$, see Proposition 3.5 in Delbaen and Schachermayer [1994]. If $x+(H\cdot S)_T \geq f$ then $(H\cdot S)_T \geq f-x \geq -x$ so that $\|(H\cdot S)_T^-\|_\infty \leq x$. This implies that $\|(H\cdot S)_t^-\|_\infty \leq x$ or $(H\cdot S)_t \geq -x$, for all $t \in [0,T]$.

On the other hand, in our market model with arbitrage, the inequality in (2.1) may be strict. The difference between the two superhedging prices is discussed in Khasanov [2013].

The following lemma is simple but useful to our problem.

Lemma 2.3.3. $U(T) = 1/SP_+^{\mathbb{P}}(1)$.

Proof. Take any $c > 0$ such that there exists a strategy H which satisfies

- $V_T^{1,H} = 1 + (H \cdot S)_T \geq c, \mathbb{P} - a.s.$
- $(H \cdot S)_t \geq -1$ for all $0 \leq t \leq T$.

Scaling this strategy by a factor $1/c$, we get an admissible strategy allowing to superhedge 1 at cost $1/c$, which shows that

$$U(T) \leq \frac{1}{SP_+^{\mathbb{P}}(1)}.$$

The converse inequality can be proved by the same argument. □

The above lemma has two consequences. First, the knowledge of $SP_+^{\mathbb{P}}(1)$ is enough to find optimal arbitrage profit. Second, one should find the strategy to superhedge 1 in order to realize optimal arbitrage.

Obviously, $SP_+^{\mathbb{P}}(1) \leq 1$. If $SP_+^{\mathbb{P}}(1) < 1$, there is optimal arbitrage. If $SP_+^{\mathbb{P}}(1) = 1$, optimal arbitrage does not exist, but arbitrages may still exist. In Example 9 of Ruf [2011], the cheapest price to hold 1 is 1, but we can achieve a terminal wealth larger than 1 with positive probability.

Remark 2.3.4. *Under the NUPBR assumption, it is necessary that* $SP_+^{\mathbb{P}}(1) > 0$. *Indeed, let x be a nonnegative number and assume that we can find a strategy* $H \in \mathscr{A}_x^{\mathbb{P}}$ *such that*

$$x + (H \cdot S)_T \geq 1, \qquad \mathbb{P} - a.s..$$

Multiplying both sides of the above inequality with an ELMD Z and using its supermartingale property, we have that

$$x \geq \mathbb{E}\left[Z_T(x + (H \cdot S)_T)\right] \geq \mathbb{E}[Z_T] > 0.$$

The last inequality is due to the fact that $Z_T > 0, \mathbb{P} - a.s.$. *Therefore,* $SP_+^{\mathbb{P}}(1) \geq \mathbb{E}[Z_T] > 0$.

Furthermore, Khasanov [2013] shows that $SP_+^{\mathbb{P}}(1) = \sup_Z \mathbb{E}[Z_T]$, *where the* sup *is taken over all ELMD. However, if we do not restrict ourselves to nonnegative wealth processes, it may be possible to superhedge* 1 *at zero price, that is,* $SP^{\mathbb{P}}(1)$ *may be equal to zero.*

2.4 Constructing market models with optimal arbitrage

In this section we present a construction of market models with optimal arbitrage. It works by starting with a probability measure $\mathbb{Q}$ under which the price process satisfies NFLVR and making a non-equivalent measure change to construct a new measure $\mathbb{P}$ allowing for arbitrage. Arbitrage opportunities constructed with an absolutely continuous measure change have been studied in earlier works. The first example of this kind of technique is the Bessel model, which is given in Delbaen and Schachermayer [1995a]. This technique is generalized in Osterrieder and Rheinländer [2006] and Ruf and Runggaldier [2013]. The same idea is used in Kardaras et al. [2015], Pal and Protter [2010] for the construction of strict local martingales. However, we push this idea further by characterizing the superhedging price under $\mathbb{P}$ in terms of the superhedging price under $\mathbb{Q}$, which enables us to describe optimal arbitrages as well as the corresponding optimal strategy (under $\mathbb{P}$).

2.4.1 A construction based on a nonnegative martingale

Let $\mathbb{Q}$ be a probability measure on the filtered measure space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}\right)$ described in the beginning of Section 2.2, and assume that under $\mathbb{Q}$, the following are true:

- The risky asset process S satisfies NFLVR.
- There exists a nonnegative RCLL martingale M with $M_0 = 1$,

$$\mathbb{Q}[\tau \leq T] > 0 \quad \text{and} \quad \mathbb{Q}[\{\tau \leq T\} \cap \{M_{\tau-} > 0\}] = 0, \tag{2.2}$$

  where $\tau := \inf\{t \geq 0 : M_t = 0\}$ with the convention that $\inf \emptyset = +\infty$.

Since M is right-continuous, condition (2.2) means that M may only hit zero continuously on $[0,T]$. Using M as a Radon-Nikodym derivative, we define a new probability measure via

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathscr{F}_t} := M_t.$$

Then $\mathbb{P}$ is only absolutely continuous (but not equivalent) with respect to $\mathbb{Q}$. In fact, M can reach zero under $\mathbb{Q}$ but it is always positive under $\mathbb{P}$ (up to negligible set), because $\mathbb{P}[\tau \leq T] = \mathbb{E}^{\mathbb{Q}}[M_T 1_{\tau \leq T}] = 0$.

Theorem 2.4.1. *Assume that the $(\mathbb{Q}, S)$-market satisfies the condition NFLVR and the condition (2.2) holds true. Then the $(\mathbb{P}, S)$-market satisfies the condition NUPBR, and for any $\mathscr{F}_T$-measurable claim $f \geq 0$, we have*

$$SP_+^{\mathbb{P}}(f) = SP_+^{\mathbb{Q}}(f 1_{M_T > 0}).$$

Corollary 2.4.2. *Denote by $ELMM(\mathbb{Q}, S)$ the set of all equivalent local martingale measures for the $(\mathbb{Q}, S)$-market. Under the assumptions of the theorem let*

$$\sup_{\overline{\mathbb{Q}} \in ELMM(\mathbb{Q},S)} \mathbb{E}^{\overline{\mathbb{Q}}}[1_{M_T > 0}] < 1. \tag{2.3}$$

Then the $(\mathbb{P}, S)$-market admits optimal arbitrage and the optimal arbitrage strategy is a multiple of the superhedging strategy of the claim $1_{M_T > 0}$ in the $(\mathbb{Q}, S)$-market, as shown in the proof of Lemma 2.3.3. The existence of the hedging strategy for the claim $1_{M_T > 0}$ starting from the capital in (2.3) is given in Corollary 10 of Delbaen and Schachermayer [1995c].

Proof. By the standard super-replication theorem under absence of arbitrage (Theorem 5.12 of Delbaen and Schachermayer [1998]) ,

$$SP_+^{\mathbb{Q}}(1_{M_T > 0}) = SP^{\mathbb{Q}}(1_{M_T > 0}) = \sup_{\overline{\mathbb{Q}} \in ELMM(\mathbb{Q},S)} \mathbb{E}^{\overline{\mathbb{Q}}}[1_{M_T} > 0].$$

The first equality is due to the fact that NA holds with respect to $\mathbb{Q}$. The optimal arbitrage strategy is given as in Lemma 2.3.3. □

Remark 2.4.3. *The condition (2.3) is exactly the condition introduced in Proposition 2.8 of Osterrieder and Rheinländer [2006] in order to obtain an arbitrage opportunity under the new measure. However, our approach not only allows us to show the existence of arbitrage opportunities, but also of an optimal arbitrage.*

Proof of Theorem 2.4.1. Let $\overline{\mathbb{Q}}$ be a local martingale measure equivalent to $\mathbb{Q}$, and denote by $\overline{Z}$ its density with respect to $\mathbb{Q}$.
Step 1: we prove that the $(\mathbb{P},S)$-market satisfies NUPBR by showing that $\overline{Z}/M$ is an ELMD. The strategy of this step of the proof is similar to the proof of Theorem 5.3 in Fontana [2014]; for an analogous result, see also Proposition 2.3 of Carr et al. [2014].

We define

$$\tau_n := \inf\{t \geq 0 : M_t < \frac{1}{n}\}$$

with the convention $\inf\emptyset = +\infty$. Since, by condition (2.2), M does not jump to zero, we have that $M_{t\wedge\tau_n} > 0\ \forall t \geq 0$ $\mathbb{Q}$-a.s.

We remark that $\mathbb{Q} \ll \mathbb{P}$ on $\mathscr{F}_{t\wedge\tau_n}$. Indeed, take any $A \in \mathscr{F}_{t\wedge\tau_n}$ such that $\mathbb{P}(A) = 0$, we compute

$$\mathbb{Q}[A] = \mathbb{E}^{\mathbb{Q}}\left[1_A \frac{M_{t\wedge\tau_n}}{M_{t\wedge\tau_n}}\right] = \mathbb{E}^{\mathbb{P}}\left[1_A \frac{1}{M_{t\wedge\tau_n}}\right] = 0.$$

This means $\mathbb{P}$ is equivalent to $\mathbb{Q}$ on $\mathscr{F}_{t\wedge\tau_n}$.

By Corollary 3.10, page 168 of Jacod and Shiryaev [2002], to prove that a process N is a $\mathbb{P}$-local martingale with localizing sequence (τ_n), we need to prove that $(NM)^{\tau_n}$ is a $\mathbb{Q}$-local martingale for every $n \geq 1$.

Let V be a $\mathbb{P}$-admissible wealth process. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $\mathscr{F}_{t\wedge\tau_n}$, we obtain that V^{τ_n} is a $\mathbb{Q}$-admissible wealth process. Therefore, for each n, we have that $\overline{Z}V^{\tau_n}$ and so also $(\overline{Z}V)^{\tau_n}$ is a $\mathbb{Q}$-local martingale. This shows that $\frac{\overline{Z}V}{M}$ is a local martingale under $\mathbb{P}$.
Step 2: we prove the equality $SP_+^{\mathbb{P}}(f) = SP_+^{\mathbb{Q}}(f1_{M_T>0})$.
$(\leq)$ Take any $x > 0$ such that there exists a strategy $H \in \mathscr{A}_x^{\mathbb{Q}}$ which satisfies $V_T = x + (H\cdot S)_T \geq f1_{M_T>0}, \mathbb{Q}-a.s.$ Since $\mathbb{P} \ll \mathbb{Q}$, Theorem 25, page 170 of Protter [2003] shows that $H \in L(S)$ under $\mathbb{P}$ as well and $H_{\mathbb{Q}}\cdot S = H_{\mathbb{P}}\cdot S, \mathbb{P}-a.s.$. We also see that $x + (H\cdot S)_t \geq 0, \mathbb{P}-a.s$ and $x + (H\cdot S)_T \geq f1_{M_T>0} = f, \mathbb{P}-a.s.$ This means that

$$SP_+^{\mathbb{P}}(f) \leq SP_+^{\mathbb{Q}}(f1_{M_T>0}). \tag{2.4}$$

$(\geq)$ For the converse inequality, take any $x > 0$ such that there exists a strategy $H \in \mathscr{A}_x^{\mathbb{P}}$ and $V_T^{\mathbb{P}} = x + (H\cdot S)_T \geq f, \mathbb{P}-a.s.$ We will show that $x \geq SP_+^{\mathbb{Q}}(f1_{M_T>0})$.

Define $H^n := H1_{t\leq\tau_n}$, then H^n is S-integrable and x-admissible under $\mathbb{Q}$. From Step 1, we see that $\tau_n \wedge T \nearrow T$, $\mathbb{P}$-a.s. and therefore $V_T^n := x + (H^n \cdot S)_T \to V_T^{\mathbb{P}}$, $\mathbb{P}$-a.s. or $V_T^n 1_{M_T>0} \to V_T^{\mathbb{P}} 1_{M_T>0} \geq f 1_{M_T>0}$, $\mathbb{Q}$-a.s. The following convergence holds

$$V_T^n - V_T^n 1_{M_T=0} = V_T^n 1_{M_T>0} \to V_T^{\mathbb{P}} 1_{M_T>0} \geq f 1_{M_T>0}, \mathbb{Q}-a.s.$$

The sequence $V_T^n - x - V_T^n 1_{M_T=0} = (H^n \cdot S)_T - V_T^n 1_{M_T>0}$ is in the set $\mathscr{K} - L_+^0$ (under $\mathbb{Q}$) and uniformly bounded from below by $-x$. Because the $(\mathbb{Q}, S)$-market satisfies NFLVR condition, the set $\mathscr{K} - L_+^0$ is Fatou-closed (see Theorem 3.1 of Kabanov [1997]) and we obtain $V_T^{\mathbb{P}} 1_{M_T>0} - x \in \mathscr{K} - L_+^0$ or $x \geq SP_+^{\mathbb{Q}}(f 1_{M_T>0})$. In other words,

$$SP_+^{\mathbb{P}}(f) \geq SP_+^{\mathbb{Q}}(f 1_{M_T>0}). \tag{2.5}$$

From (2.4) and (2.5), the proof is complete. □

Remark 2.4.4. *If the martingale used to construct the probability measure $\mathbb{P}$ does not satisfy condition* (2.2)*, that is, may reach zero by a jump, the NUPBR property may or may not hold under the measure $\mathbb{P}$. More precisely, it is shown in Proposition 5.4 of Fontana [2014] that when the martingale M with the predictable representation property under $\mathbb{Q}$ is the price process itself, that is, $M_t = S_t$ for all t, the failure of the condition (2.2) implies that the condition NUPBR is not satisfied under $\mathbb{P}$. On the other hand, one can construct examples when M jumps to zero yet the NUPBR property holds in the $\mathbb{P}$-market. A trivial example of this situation is when M and S are $\mathbb{Q}$-independent. Another example (where $M = S$) might be constructed in the spirit of Example 4.7 of Fisher et al. [2015] by combining a Brownian motion with an independent jump. We leave a detailed study of this question for further research.*

Remark 2.4.5 (A connection with Föllmer's exit measure for local martingales)**.** *A number of authors [Carr et al., 2014; Delbaen and Schachermayer, 1995a; Fernholz and Karatzas, 2010; Pal and Protter, 2010] analyze financial models based on strict local martingales using Föllmer's exit measure for supermartingales [Föllmer, 1972; Meyer, 1972]. In particular, given a nonnegative local martingale X under the measure $\mathbb{P}$, there exists, under certain assumptions, a* *unique measure $\mathbb{Q}$, such that $1/X$ is the density of $\mathbb{P}$ with respect to $\mathbb{Q}$, and X explodes under $\mathbb{Q}$ in finite time if and only if it is $\mathbb{P}$-strict local martingale.*

In the framework of Theorem 2.4.1, let $\overline{Z}$ be the density of an equivalent local martingale measure. Then, as shown in the proof of this theorem, $\frac{\overline{Z}}{M}$ is a local martingale under $\mathbb{P}$ and therefore, the equivalent martingale measure $\overline{\mathbb{Q}}$ is the Föllmer's exit measure for $(\frac{\overline{Z}}{M}, \mathbb{P})$. In particular, $\overline{Z}/M$ is a true martingale under $\mathbb{P}$ if and only if $\overline{\mathbb{Q}}[M_T/\overline{Z}_T = 0] = 0$, in other words, M does not hit zero under the original measure $\mathbb{Q}$. Under our assumption $\mathbb{Q}[\tau \leq T] > 0$, therefore, $\overline{Z}/M$ is a strict local martingale deflator for every ELMM $\overline{\mathbb{Q}}$.

If one assumes that $ELMM(\mathbb{Q}, S)$ is a singleton (contains only one measure $\overline{\mathbb{Q}}$), then optimal arbitrage exists if and only if $\overline{\mathbb{Q}}[M_T > 0] < 1$, or, equivalently, $\overline{\mathbb{Q}}[M_T = 0] > 0$. Therefore, in this context, optimal arbitrage exists whenever $\overline{Z}/M$ is not a true martingale under $\mathbb{P}$; in other words, either NFLVR holds under $\mathbb{P}$ or there is an optimal arbitrage.

It is also possible to turn the construction around, that is, start with the probability $\mathbb{P}$ under which the financial market satisfies NUPBR and admits a strict local martingale deflator M, and construct a "generalized martingale measure" as the Föllmer's exit measure for $(1/M, \mathbb{P})$. This approach, which is taken for example in Fernholz and Karatzas [2010], yields the optimal arbitrage profit directly in terms of Föllmer's exit measure. However, this representation depends crucially on the martingale representation property (Assumption A in Fernholz and Karatzas [2010]); in our incomplete market setting one would have to consider Föllmer's exit measure with respect to every possible local martingale deflator, which makes this approach less appealing in general. It's remarked that in continuous Markovian models, this approach is still useful because only one deflator has to be checked, even if the model is incomplete, see Proposition 3.1 of Ruf [2013].

2.4.2 A construction based on a predictable stopping time

As before, we consider a measure $\mathbb{Q}$ on the space $(\Omega, \mathscr{F}, (\mathscr{F})_{t\geq 0})$. Let σ be a stopping time such that $\mathbb{Q}(\sigma > T) > 0$. We define a new probability measure, absolutely continuous with respect to $\mathbb{Q}$, by

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathscr{F}_t} = \frac{\mathbb{Q}[\sigma > T|\mathscr{F}_t]}{\mathbb{Q}[\sigma > T]} = M_t. \tag{2.6}$$

Under the new measure, $\mathbb{P}(\sigma > T) = \mathbb{E}^{\mathbb{Q}}(M_T 1_{\sigma>T}) = 1$.

This construction has the following economic interpretation. Consider an event (E), such as the default of a company or a sovereign state, whose occurrence is characterized by a stopping time σ. Given a planning horizon T, we are interested in the occurrence of this event (E) before the planning horizon. Suppose that the market agents have common anticipations of the probability of future scenarios, which correspond to the arbitrage-free probability measure $\mathbb{Q}$, and that under this probability, the event (E) has nonzero probabilities of occurring both before and after the planning horizon. Consider now an informed economic agent who believes that the event (E) will not happen before the planning horizon T. For instance, the agent may believe that the company or the state in question will be bailed out in case of potential default. Our informed agent may then want to construct an alternative model $\mathbb{P}$, in which the arbitrage opportunity due to mispricing may be exploited and the arbitrage strategy may be constructed in a rigorous manner.[1] The following corollary provides a method for constructing such a model.

Corollary 2.4.6. *Assume that the following conditions hold*

- *The risky asset process S satisfies NFLVR under $\mathbb{Q}$.*
- *The filtration $\mathbb{F}$ is quasi-left continuous.*
- *σ is a predictable stopping time such that for any stopping time θ,*

$$\mathbb{E}^{\mathbb{Q}}[1_{\sigma>T}|\mathscr{F}_\theta] > 0, \mathbb{Q}-a.s. \qquad on\ \{\sigma > \theta\}.$$

[1] The "informed agent" interpretation of our arbitrage construction hints at possible connections with the research on arbitrage opportunities arising from enlargement of the underlying filtration with additional information, see e.g. Fontana et al. [2014]; Imkeller et al. [2001] and section 7 of Larsson et al. [2014] as well as the discussion at the end of Section 2.5.1 of the present chapter.

Then the $(\mathbb{P}, S)$*-market satisfies NUPBR. Given a* $\mathscr{F}_T$*-measurable claim* $f \geq 0$*, we have*

$$SP_+^{\mathbb{P}}(f) = SP_+^{\mathbb{Q}}(f 1_{\sigma > T}).$$

In addition, if

$$SP_+^{\mathbb{Q}}(1_{\sigma > T}) = \sup_{\overline{\mathbb{Q}} \in ELMM(\mathbb{Q}, S)} \mathbb{E}^{\overline{\mathbb{Q}}}[1_{\sigma > T}] < 1,$$

then the $(\mathbb{P}, S)$*-market admits optimal arbitrage.*

Proof. This result will follow from Theorem 2.4.1 after checking the condition (2.2) on M. Let $\tau = \inf\{t > 0 : M_t = 0\}$. By construction, $M_\sigma = 0$ on $\{\sigma \leq T\}$ and $M_t > 0$ for $t < \sigma$. This means that

$$\tau = \begin{cases} \sigma, & \sigma \leq T \\ +\infty, & \text{otherwise.} \end{cases}$$

Since the filtration $\mathbb{F}$ is quasi left continuous and σ is a predictable stopping time, M does not jump at σ (see Protter [2003], page 190). This means that

$$\mathbb{Q}[\{\tau \leq T\} \cap \{M_{\tau-} > 0\}] = 0$$

and condition (2.2) is satisfied. □

2.5 Examples

2.5.1 A complete market example

Let $W^{\mathbb{Q}}$ be a Brownian motion and let $\mathbb{F}$ be its natural filtration. We assume that the price of a risky asset evolves as follows

$$S_t = 1 + W_t^{\mathbb{Q}}$$

and define a predictable stopping time by $\sigma = \inf\{t > 0 : S_t \leq 0\}$. Using the law of infimum of Brownian motion, we get

$$\mathbb{Q}[\sigma > T] = \mathbb{Q}[\inf_{0 \leq t \leq T} W_t^{\mathbb{Q}} > -1] = 1 - 2\mathscr{N}\left(-\frac{1}{\sqrt{T}}\right) > 0,$$

where $\mathscr{N}$ denotes the standard normal distribution function.

Next, by the Markov property we compute

$$\mathbb{E}^{\mathbb{Q}}[1_{\sigma>T}|\mathscr{F}_t] = \mathbb{Q}[\inf_{0\leq t\leq T} W_t^{\mathbb{Q}} > -1|\mathscr{F}_t] = \begin{cases} 0 & \text{on } \sigma \leq t \\ 1-2\mathscr{N}\left(-\frac{S_t}{\sqrt{T-t}}\right) > 0 & \text{on } \sigma > t. \end{cases} \tag{2.7}$$

Hence, on $\{\sigma > t\}$, we obtain $\mathbb{E}^{\mathbb{Q}}[1_{\sigma>T}|\mathscr{F}_t] > 0$. This means that the construction of Section 2.4.2 applies and we may define a new measure $\mathbb{P}$ via (2.6). Since the $(\mathbb{Q},S)$-market is complete and $ELMM(\mathbb{Q},S) = \{\mathbb{Q}\}$, the superhedging price of the claim $1_{\sigma>T}$ is

$$\mathbb{Q}[\sigma > T] = 1-2\mathscr{N}\left(-\frac{1}{\sqrt{T}}\right) < 1,$$

which means that the $\mathbb{P}$-market admits optimal arbitrage.

Applying the Itô formula to (2.7), we get the martingale representation:

$$\mathbb{E}^{\mathbb{Q}}[1_{\sigma>T}|\mathscr{F}_t] = \mathbb{Q}[\sigma > T] + \sqrt{\frac{2}{\pi}} \int_0^{\sigma\wedge t} \frac{1}{\sqrt{T-s}} e^{-\frac{S_s^2}{2(T-s)}} dW_s^{\mathbb{Q}}. \tag{2.8}$$

Therefore,

$$H_t = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{T-t}} e^{-\frac{S_t^2}{2(T-t)}} \mathbf{1}_{t\leq\sigma}$$

is the optimal arbitrage strategy, that is, the hedging strategy for $1_{\sigma>T}$ in the $(\mathbb{Q},S)$-market as well as the hedging strategy for 1 in the $(\mathbb{P},S)$-market.

Let us now compute the dynamics of S under $\mathbb{P}$. By Girsanov's Theorem (see, e.g., Theorem 41 on page 136 of Protter [2003]),

$$W_t^{\mathbb{P}} = W_t^{\mathbb{Q}} - \frac{2}{\mathbb{Q}[\sigma > T]\sqrt{2\pi}} \int_0^{\sigma\wedge t} \frac{1}{M_s} e^{-\frac{S_s^2}{2(T-s)}} \frac{1}{\sqrt{T-s}} ds$$

is a $\mathbb{P}$-Brownian motion. The dynamics of S under $\mathbb{P}$ are therefore given by

$$S_t = 1 + W_t^{\mathbb{P}} + \frac{2}{\mathbb{Q}[\sigma > T]\sqrt{2\pi}} \int_0^{\sigma\wedge t} \frac{e^{-\frac{S_s^2}{2(T-s)}}}{M_s\sqrt{T-s}} ds \tag{2.9}$$

$$= 1 + W_t^{\mathbb{P}} + \sqrt{\frac{2}{\pi}} \int_0^{\sigma\wedge t} \frac{1}{1-2\mathcal{N}\left(-\frac{S_s}{\sqrt{T-s}}\right)} \frac{e^{-\frac{S_s^2}{2(T-s)}}}{\sqrt{T-s}} ds \tag{2.10}$$

Fragility and robustness Now, let us discuss the robustness of the $(\mathbb{P}, S)$-market in this example in the sense of Guasoni and Rásonyi [2015]. The optimal arbitrage constructed using the predictable stopping time $\sigma = \inf\{t > 0 : S_t \leq 0\}$ is not robust. Indeed, from (2.9) we can write the dynamics of $X_t := \log S_t$ as follows

$$dX_t = e^{-X_t} dW_t^{\mathbb{P}} + \left[\frac{2}{\mathbb{Q}[\sigma > T]\sqrt{2\pi}} e^{-X_t} \frac{1}{M_t} e^{-\frac{S_t^2}{2(T-t)}} \frac{1}{\sqrt{T-t}} - \frac{1}{2} e^{-2X_t}\right] dt \tag{2.11}$$

Since in (2.11), the drift is locally bounded and the volatility is continuous and nonsingular, by Theorem 2 of Guasoni and Rásonyi [2015], we conclude that the $(\mathbb{P}, S)$-market is fragile.

However, we can slightly modify the stopping time σ to construct an arbitrage which is not destroyed by small perturbations of the price process as above. More precisely, we choose the predictable stopping time σ as the first time when S_t hits a line with positive slope, that is

$$\sigma = \inf\{t \geq 0 : S_t \leq \alpha t\}$$

with $\alpha > 0$. By Proposition 3.2.1.1 in Jeanblanc et al. [2009],

$$\mathbb{Q}[\sigma > T] = \mathbb{Q}\left[\inf_{0\leq t\leq T}(W_t^{\mathbb{Q}} - \alpha t) > -1\right]$$
$$= \mathcal{N}\left(\frac{1-\alpha T}{\sqrt{T}}\right) - e^{2\alpha}\mathcal{N}\left(\frac{-1-\alpha T}{\sqrt{T}}\right) \in (0,1),$$

and we can define a measure $\mathbb{P}$ admitting optimal arbitrage via (2.6). We are going to compute the dynamics of $\log S$ in this case and compare to the results of

Guasoni and Rásonyi [2015]. By the Markov property and the law of infimum of Brownian motion with drift, we compute the conditional probability

$$\mathbb{E}^{\mathbb{Q}}[1_{\sigma>T}|\mathscr{F}_t] = \begin{cases} 0 & \text{if } \sigma \le t \\ \mathscr{N}\left(\frac{S_t-\alpha T}{\sqrt{T-t}}\right) - e^{2\alpha(S_t-\alpha t)}\mathscr{N}\left(\frac{-S_t+2\alpha t-\alpha T}{\sqrt{T-t}}\right) & \text{if } \sigma > t. \end{cases}$$

Denoting

$$Y_t^1 = \frac{S_t-\alpha T}{\sqrt{T-t}}, \qquad Y_t^2 = \frac{-S_t+2\alpha t-\alpha T}{\sqrt{T-t}}$$

and applying the Itô formula, we obtain the dynamics of the conditional law on $\sigma < t$:

$$d\mathbb{E}^{\mathbb{Q}}[1_{\sigma>T}|\mathscr{F}_t] = \left[\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^1)^2}{2}}}{\sqrt{T-t}} + \frac{e^{2\alpha(S_t-\alpha t)}}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^2)^2}{2}}}{\sqrt{T-t}} - \mathscr{N}(Y_t^2)2\alpha e^{2\alpha(S_t-\alpha t)}\right] dW_t^{\mathbb{Q}},$$

and the dynamics of M_t:

$$dM_t = \frac{1}{\mathbb{Q}[\sigma>T]}\left[\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^1)^2}{2}}}{\sqrt{T-t}} + \frac{e^{2\alpha(S_t-\alpha t)}}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^2)^2}{2}}}{\sqrt{T-t}} - \mathscr{N}(Y_t^2)2\alpha e^{2\alpha(S_t-\alpha t)}\right] dW_t^{\mathbb{Q}}.$$

By Girsanov's Theorem,

$$dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} - \frac{1}{M_t\mathbb{Q}[\sigma>T]}\left[\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^1)^2}{2}}}{\sqrt{T-t}} + \frac{e^{2\alpha(S_t-\alpha t)}}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^2)^2}{2}}}{\sqrt{T-t}} - \mathscr{N}(Y_t^2)2\alpha e^{2\alpha(S_t-\alpha t)}\right] dt$$

is a $\mathbb{P}$-Brownian motion. Finally, the dynamic of S under $\mathbb{P}$ is

$$dS_t = dW_t^{\mathbb{P}} + \frac{1}{M_t\mathbb{Q}[\sigma>T]}\left[\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^1)^2}{2}}}{\sqrt{T-t}} + \frac{e^{2\alpha(S_t-\alpha t)}}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^2)^2}{2}}}{\sqrt{T-t}} - \mathscr{N}(Y_t^2)2\alpha e^{2\alpha(S_t-\alpha t)}\right] dt.$$

Applying Itô's formula once again, we see that $X_t = \log S_t$ satisfies

$$\begin{aligned} dX_t &= e^{-X_t}dS_t - \frac{1}{2}e^{-2X_t}dt \\ &= \frac{e^{-X_t}}{M_t\mathbb{Q}[\sigma>T]}\left[\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^1)^2}{2}}}{\sqrt{T-t}} + \frac{e^{2\alpha(S_t-\alpha t)}}{\sqrt{2\pi}}\frac{e^{-\frac{(Y_t^2)^2}{2}}}{\sqrt{T-t}} - \mathscr{N}(Y_t^2)2\alpha e^{2\alpha(S_t-\alpha t)}\right] dt \\ &\quad - \frac{1}{2}e^{-2X_t}dt + e^{-X_t}dW_t^{\mathbb{P}}. \end{aligned} \tag{2.12}$$

The drift in (2.12) can be written as a function of (t, X_t), and it is not locally bounded, for example, $1/M$ is unbounded in a neighborhood of $(t, \log(\alpha t))$. So the result of Guasoni and Rásonyi [2015] breaks down. Hence, Theorem 2 of Guasoni and Rásonyi [2015] cannot be applied.

It is easy to see that $S_T > \alpha T$, $\mathbb{P}$-a.s. This allows to construct a simple buy-and-hold arbitrage strategy.

- If $\alpha T > 1$, buy one unit of S in the beginning and hold it until T. This strategy yields a profit of $\alpha T - 1$ with probability 1.

- If $\alpha T \leq 1$, introduce the stopping time $\sigma_1 := \inf\{t > 0 : S_t = \frac{\alpha T}{2}\}$. If $\sigma_1 \leq \frac{T}{2}$, buy one unit of S at σ_1 and hold it until T. Otherwise, do nothing. It is easy to see that $\mathbb{P}[\sigma_1 \leq T/2] = \mathbb{Q}[\sigma_1 \leq T/2, \sigma > T] > 0$, which means that this strategy yields a profit $S_T - S_{\sigma_1} \geq \frac{\alpha T}{2}$ with positive probability.

This strategy is always admissible and a simple obvious arbitrage in the sense of Bender [2012], which means that the market is robust and not fragile (see discussion at the end of section 2.2). Note however that the strategy which realizes the optimal arbitrage is different from the robust buy-and-hold strategy. This means that the profit from the strategy which realizes the optimal arbitrage may decrease or even disappear after a small perturbation of the price process.

A connection with initial enlargement of filtrations. Let us consider again the case $\sigma = \inf\{t > 0 : S_t \leq 0\}$. The ability to know that the asset is always nonnegative under $\mathbb{Q}$ is similar to the knowledge of the $\mathscr{F}_T$-measurable random variable $L := 1_{\inf_{0 \leq t \leq T} S_t > 0} = 1_{\sigma > T}$ at the beginning of trading. Motivated by this observation, we investigate the enlarged market $(\Omega, \mathscr{G}, \mathbb{G}, \mathbb{Q})$ where $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(L)$.

Because L is a discrete random variable, Jacod's condition is fulfilled and every $(\mathbb{F}, \mathbb{Q})$-semimartingale is also a $(\mathbb{G}, \mathbb{Q})$-semimartingale, see for example Theorem 10, page 363 of Protter [2003]. The conditional probability of the event

$\{L=1\}$ given $\mathscr{F}_t$ is exactly given by (2.7). We define

$$p_t^1 := \frac{\mathbb{Q}[\sigma > T|\mathscr{F}_t]}{\mathbb{Q}[\sigma > T]} = \frac{1_{\sigma>t}}{\mathbb{Q}[\sigma > T]}\left(1-2\mathscr{N}\left(-\frac{S_t}{\sqrt{T-t}}\right)\right),$$
$$p_t^0 := \frac{\mathbb{Q}[\sigma \le T|\mathscr{F}_t]}{\mathbb{Q}[\sigma \le T]} = \frac{1_{\sigma>t}}{\mathbb{Q}[\sigma \le T]}2\mathscr{N}\left(-\frac{S_t}{\sqrt{T-t}}\right)+\frac{1_{\sigma\le t}}{\mathbb{Q}[\sigma \le T]}.$$

Using (2.8), we get

$$p_t^1 = 1+\frac{1}{\mathbb{Q}[\sigma > T]}\sqrt{\frac{2}{\pi}}\int_0^{\sigma\wedge t}\frac{1}{\sqrt{T-s}}e^{-\frac{S_s^2}{2(T-s)}}dW_s^{\mathbb{Q}}, \tag{2.13}$$

$$p_t^0 = 1-\frac{1}{\mathbb{Q}[\sigma \le T]}\sqrt{\frac{2}{\pi}}\int_0^{\sigma\wedge t}\frac{1}{\sqrt{T-s}}e^{-\frac{S_s^2}{2(T-s)}}dW_s^{\mathbb{Q}}, \tag{2.14}$$

and then $p_t^L = p_t^1 1_{\sigma>T} + p_t^0 1_{\sigma\le T}$. Because $W^{\mathbb{F},\mathbb{Q}}$ is a $(\mathbb{F},\mathbb{Q})$-martingale, we have that

$$\begin{aligned}
W_t^{\mathbb{G},\mathbb{Q}} &= W_t^{\mathbb{F},\mathbb{Q}} - \int_0^t \frac{d\left\langle p^L, W^{\mathbb{F},\mathbb{Q}}\right\rangle_s}{p_s^L} \\
&= W_t^{\mathbb{F},\mathbb{Q}} - 1_{\sigma>T}\int_0^t \frac{1}{p_s^1}\frac{1}{\mathbb{Q}[\sigma > T]}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T-s}}e^{-\frac{S_s^2}{2(T-s)}}ds \\
&\qquad + 1_{\sigma\le T}\int_0^{t\wedge\sigma} \frac{1}{p_t^0}\frac{1}{\mathbb{Q}[\sigma \le T]}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T-s}}e^{-\frac{S_s^2}{2(T-s)}}ds
\end{aligned}$$

is a $(\mathbb{G},\mathbb{Q})$-martingale. To simplify notations, we denote

$$\begin{aligned}
\mu_t^{\mathbb{G}} := {}& 1_{\sigma>T}\frac{1}{p_s^1}\frac{1}{\mathbb{Q}[\sigma > T]}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T-s}}e^{-\frac{S_s^2}{2(T-s)}} \\
& - 1_{\sigma\le T}1_{t\le\sigma}\frac{1}{p_t^0}\frac{1}{\mathbb{Q}[\sigma \le T]}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{T-s}}e^{-\frac{S_s^2}{2(T-s)}}.
\end{aligned}$$

The process $\mu^{\mathbb{G}}$ is called *the information drift* corresponding to the insider information $\mathbb{G}$, see for example Imkeller [2003]. Thus, the evolution of S under the filtration $\mathbb{G}$ follows easily

$$S_t^{\mathbb{G}} = 1 + W_t^{\mathbb{G},\mathbb{Q}} - \int_0^t \mu_s^{\mathbb{G}} ds. \tag{2.15}$$

Let us make some comments. In this example, the initial enlargement of filtrations argument can be interpreted as a combination of two absolutely continuous measure changes.

- The martingale M used for the absolutely continuous change of measure in (2.6) is exactly the density process p^L in Jacod's condition when restricted on $1_{\sigma>T}$ in (2.13).
- The evolutions for the asset price S in (2.9) and in (2.15) are the same, although they are defined on different probability spaces: $(\mathbb{F},\mathbb{P})$ and $(\mathbb{G},\mathbb{Q})$, respectively.
- The superhedging price of 1 can be found on each event $\{\sigma > T\}$ and $\{\sigma \leq T\}$ separately . From (2.13) and (2.14) we have that

$$1 = 1_{\sigma>T}\left(\mathbb{Q}[\sigma > T] + \sqrt{\frac{2}{\pi}} \int_0^{\sigma\wedge t} \frac{1}{\sqrt{T-s}} e^{-\frac{S_s^2}{2(T-s)}} dS_s\right)$$
$$+ 1_{\sigma\leq T}\left(\mathbb{Q}[\sigma \leq T] - \sqrt{\frac{2}{\pi}} \int_0^{\sigma\wedge t} \frac{1}{\sqrt{T-s}} e^{-\frac{S_s^2}{2(T-s)}} dS_s\right).$$

2.5.2 A robust arbitrage based on the Poisson process

Another way to ensure robustness of arbitrage with respect to small perturbations is to introduce jumps into the price process dynamics. Let N be a standard Poisson process under $\mathbb{Q}$ and assume that $\mathbb{F} = \mathbb{F}^N$, which is a quasi left-continuous filtration. Then $S_t = 1 + N_t - t$ is a $\mathbb{Q}$-martingale. We define a predictable stopping time $\tau = \inf\{t > 0 : S_t \leq 0\}$ and a new probability measure $\mathbb{P} \ll \mathbb{Q}$ via $d\mathbb{P}|_{\mathscr{F}_t} = S_{t\wedge\tau} d\mathbb{Q}|_{\mathscr{F}_t}$. The $(\mathbb{P},S)$-market admits optimal arbitrage provided $T > 1$, because in this case $SP^{\mathbb{Q}}(1_{S_T>0}) = \mathbb{Q}[S_T > 0] < 1$.

Here, unlike the first example of this section or the Bessel process example discussed in Guasoni and Rásonyi [2015], we can prove that the arbitrage is robust. Indeed, we fix a real number $\varepsilon > 0$ and construct a simple buy-and-hold arbitrage strategy as follows:

- if S jumps on $[0,\varepsilon]$ then we do nothing.
- if S does not jump on $[0,\varepsilon]$, we buy one unit of S at ε and hold it until the first jump time of S.

Assuming that $1 < T \notin \mathbb{N}$, the process N must jump before T, because $S_t > 0, \mathbb{P}$-a.s. This means that this strategy generates a profit greater than ε with positive probability. Therefore, this is a simple obvious arbitrage in the sense of Bender [2012], and so the $(\mathbb{P},S)$-market is robust (see discussion at the end of section 2.2).

Let us compute the investment strategy realizing the optimal arbitrage in this example. First, we have

$$\mathbb{Q}[S_T > 0|\mathscr{F}_t] = \mathbb{Q}[N_T - N_t > T-1-N_t|N_t] = \sum_{k\geq 0} \frac{(T-t)^k e^{t-T}}{k!} \mathbf{1}_{k>T-1-N_t}$$

Applying the Itô formula to the right hand side, we get:

$$\begin{aligned}\mathbb{Q}[S_T > 0|\mathscr{F}_t] = \mathbb{Q}[S_T > 0] &- \int_0^t \sum_{k\geq 0} \frac{(T-s)^k e^{s-T}}{k!} \mathbf{1}_{k+1>T-1-N_s\geq k} ds \\ &+ \sum_{0\leq s\leq t:\Delta N_s\neq 0} \sum_{k\geq 0} \frac{(T-t)^k e^{t-T}}{k!} \mathbf{1}_{k+1>T-1-N_{s-}\geq k} \\ &= \mathbb{Q}[S_T > 0] + \int_0^t \frac{(T-s)^{[T-1-N_{s-}]} e^{s-T}}{[T-1-N_{s-}]!} \mathbf{1}_{T-1-N_{s-}\geq 0} (dN_s - ds).\end{aligned}$$

Therefore, the investment strategy realizing the optimal arbitrage profit in this case consists in investing the amount

$$H_t = \frac{(T-t)^{[T-1-N_{t-}]} e^{t-T}}{[T-1-N_{t-}]!} \mathbf{1}_{T-1-N_{t-}\geq 0}$$

into the risky asset at every date $t \in [0,T]$. This strategy is not buy-and-hold, but it is of finite variation, which means that a small perturbation of the price process will correspond to a small modification of the arbitrage gain, showing that not only the mere presence of the arbitrage is robust to frictions, but also the strategy realizing the optimal arbitrage is robust.

The optimal wealth. In this example, the optimal wealth process can be computed explicitly. By Markov property, we have

$$E^{Q}\left[1_{\tau>T} | F_t\right] = Q\left[S_t + N_s - \lambda s > 0, 0 \leq s \leq T-t\right] := h\left(t, S_t\right).$$

Next, we denote $\tau_x = \inf\{t > 0 : N_t \leq -x + \lambda t\}$ and the quantity $h(t,x)$ becomes $Q[\tau_x > T-t]$. We observe that τ_x is a discrete random variable with values in $\left\{\frac{x}{\lambda}, \frac{x+1}{\lambda}, \ldots\right\}$ and its distribution is given below,

$$\begin{aligned}
Q\left[\tau_x = \frac{x}{\lambda}\right] &= Q\left[N_{\frac{x}{\lambda}} = 0\right] = e^{-\lambda \frac{x}{\lambda}} = e^{-x}, \\
Q\left[\tau_x = \frac{x+1}{\lambda}\right] &= Q\left[N_{\frac{x+1}{\lambda}} = 1, N_{\frac{x}{\lambda}} = 1\right] \\
&= Q\left[N_{\frac{x+1}{\lambda}} - N_{\frac{x}{\lambda}} = 0 \middle| N_{\frac{x}{\lambda}} = 1\right] Q\left[N_{\frac{x}{\lambda}} = 1\right] \\
&= Q\left[N_{\frac{1}{\lambda}} = 0\right] Q\left[N_{\frac{x}{\lambda}} = 1\right] = e^{-\lambda \frac{1}{\lambda}} e^{-\lambda \frac{x}{\lambda}} \lambda \frac{x}{\lambda} = x e^{-x-1}, \ldots
\end{aligned}$$

Remark 2.5.1. *Arbitrage opportunities can be exploited by a differential-difference equation. We consider the following function*

$$h(t,x) = E^{Q}\left[1_{\{x+N_{T-t}-\lambda(T-t)\geq 0\}}\right] = \sum_{k=0}^{\infty} e^{-\lambda(T-t)} \frac{\left(\lambda\left(T-t\right)\right)^k}{k!} 1_{\{x+k-\lambda(T-t)\geq 0\}}.$$

The function h is not continuous on the lines $x+k-\lambda\left(T-t\right)=0$. Outside these lines, h is differentiable. The equation for h is

$$\begin{aligned}
\frac{\partial h}{\partial t} - \frac{\partial h}{\partial x}\lambda + \left\{h\left(t,x+1\right) - h\left(t,x\right)\right\}\lambda &= 0 \\
h(T,x) &= 1_{x\geq 0}.
\end{aligned}$$

We assume $1 < \lambda T \notin \mathbb{N}$ to ensure that the trajectories (t, S_t) never lie on the discontinuous lines of h, then we can apply Itô's formula to $h(t, S_t)$

$$\begin{aligned} dh(t,S_t) &= \frac{\partial h}{\partial t}dt - \frac{\partial h}{\partial x_1}\lambda dt + \{h(t,S_t) - h(t,S_{t-})\} dN_t \\ &= \{h(t,S_{t-}+1) - h(t,S_{t-})\}(dN_t - \lambda dt). \end{aligned}$$

If we start with the initial money $h(0,1) = Q[1 + N_T - \lambda T \geq 0] < 1$ and hold the amount of $h(t,S_{t-}+1) - h(t,S_{t-})$ on the risky asset, we will obtain the outcome $h(T,S_T) = 1$ at terminal time T. However this arbitrage is not optimal.

2.5.3 Extension to incomplete markets

Assume that S is a nonnegative $\mathbb{Q}$ local martingale with only positive jumps starting at 1. Suppose that the conditions in Corollary 2.4.6 are fulfilled.

Let a be a positive number such that $aT/2 > 1$ and define a predictable stopping time by $\sigma = \inf\{t > 0 : S_t \leq at\}$. Suppose that market agents commonly describe S in a way such that it can go below the line αt, for example $0 < Q[\tau < T] < 1$. Nevertheless, a trader believes this phenomena does not occur. He will modify the common perspective by introducing his measure The measure $\mathbb{P}$ is defined by

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathscr{F}_t} = \frac{\mathbb{Q}[\sigma > T | \mathscr{F}_t]}{\mathbb{Q}[\sigma > T]}.$$

In his point of view, S is always above the line αt because $\mathbb{P}[\tau = T] = 1$. From the economic point of view, this arbitrage represents a bet that the asset price will remain above the line αt. Then for any equivalent local martingale measure $\overline{\mathbb{Q}}$ such that S is a nonnegative $\overline{\mathbb{Q}}$ local martingale, we have

$$\begin{aligned} \overline{\mathbb{Q}}[\sigma \leq T] &\geq \overline{\mathbb{Q}}[S_{T/2} \leq aT/2] = 1 - \overline{\mathbb{Q}}[S_{T/2} > aT/2] \\ &\geq 1 - \frac{\mathbb{E}^{\overline{\mathbb{Q}}}[S_{T/2}]}{aT/2} \geq 1 - \frac{1}{aT/2}. \end{aligned}$$

The superhedging price is $SP^{\mathbb{Q}}(1_{\sigma > T}) = \sup_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}[\sigma > T] \leq \frac{1}{aT/2} < 1$. Therefore, the $(\mathbb{P}, S)$-market admits optimal arbitrage. Since $S_T > aT > 2, \mathbb{P} - a.s.$, the presence of arbitrage is robust with respect to small perturbations of S.

2.5.4 A variation: building an arbitrage from a bubble

Let S be a nonnegative $\mathbb{Q}$ local martingale with no positive jumps, satisfying $S_0 = 1$ and $S_t \le \varepsilon < 1$, $\mathbb{Q}$-a.s for $t \ge T$. Because the process S is a strict local martingale under $\mathbb{Q}$ (its expectation at time T is less than 1), the market admits a bubble.

We define $\sigma = \inf\{t \ge 0 : S_t > K\}$ for $K > 1$. In this example, a trader believes that the price of S may not exceed an upper bound K. As in previous examples, this trader may construct an arbitrage model $\mathbb{P}$ as in (2.6), provided that the conditions in Corollary 2.4.6 are fulfilled. Under any ELMM $\overline{\mathbb{Q}}$, $S_{\sigma\wedge t}$ is a bounded $\overline{\mathbb{Q}}$ local martingale and hence a $\overline{\mathbb{Q}}$ martingale. So we get

$$1 = \mathbb{E}^{\overline{\mathbb{Q}}}[S_{\sigma\wedge T}] = K\overline{\mathbb{Q}}[\sigma \le T] + \mathbb{E}^{\overline{\mathbb{Q}}}[S_T 1_{\sigma>T}],$$

and therefore

$$\overline{\mathbb{Q}}[\sigma \le T] = \frac{1 - \mathbb{E}^{\overline{\mathbb{Q}}}[S_T 1_{\sigma>T}]}{K} > \frac{1-\varepsilon}{K}.$$

The superhedging price of $1_{\sigma>T}$ is

$$\sup_{\overline{\mathbb{Q}}\in ELMM(\mathbb{Q},S)} \mathbb{E}^{\overline{\mathbb{Q}}}[1_{\sigma>T}] < 1 - \frac{1-\varepsilon}{K} < 1.$$

2.5.5 A joint bet on an asset and its volatility

Let S and ξ be continuous Itô processes with dynamics,

$$dS_t = \xi_t dW_t^{\mathbb{Q}}, \quad S_0 = 1,$$
$$d\xi_t = a(t,\xi_t)\xi_t dt + b(t,\xi_t)\xi_t d\overline{W}_t^{\mathbb{Q}},$$

where $(W^{\mathbb{Q}}, \overline{W}^{\mathbb{Q}})$ is a standard two-dimensional Brownian motion and the coefficients a and b are such that the equation for ξ admits a strong solution and in addition

$$K^{-1} < b(t,x) < K \quad \text{and} \quad a(t,x) < K$$

for all t, all $x > 0$ and some constant $K > 1$. We assume that the filtration $\mathbb{F} = (\mathscr{F}_t)_{0\le t\le T}$ is given by $\mathscr{F}_t = \mathscr{F}_t^{W^{\mathbb{Q}}} \vee \mathscr{F}_t^{\overline{W}^{\mathbb{Q}}}$.

Let $\underline{\xi} < \xi_0$ and define stopping times as follows:

$$\sigma_1 := \inf\{t > 0 : S_t \leq 0\}, \qquad \sigma_2 := \inf\{t > 0 : \xi_t \leq \underline{\xi}\}, \qquad \sigma = \sigma_1 \wedge \sigma_2.$$

This choice of the stopping time represents a bet that S will not hit 0 and its volatility will stay above $\underline{\xi}$ up to time T.

Let us check the conditions in Corollary 2.4.6. Given that S and ξ are continuous, the only nontrivial part is to show that for every stopping time $\theta \in [0,T]$,

$$\mathbb{Q}[\sigma > T|\mathscr{F}_\theta] = \mathbb{Q}[\sigma_1 > T, \sigma_2 > T|\mathscr{F}_\theta] > 0 \tag{2.16}$$

on $\{\sigma > \theta\}$.

By Theorem 3.2 in Pakkanen [2010], $\log\xi$ has the $\mathbb{F}^{\overline{W}^{\mathbb{Q}}}$-conditional full support property, which implies (by Lemma 2.1 in the above reference), that the probability that $\log\xi$ stays in a small ball around any continuous function between time t and time T, conditional on $\mathscr{F}_t^{\overline{W}^{\mathbb{Q}}}$ is strictly positive. This in turn means that

$$\mathbb{Q}[\sigma_2 > T|\mathscr{F}_\theta^{\overline{W}^{\mathbb{Q}}}] > 0$$

on $\{\sigma_2 > \theta\}$. Since $W^{\mathbb{Q}}$ is independent from ξ, by Theorem 3.1 in the above reference, S has the conditional full support property also with respect to the filtration $\mathscr{F}_t \vee \sigma(\xi_s, 0 \leq s \leq T)$, i.e.

$$\mathbb{Q}[\sigma_1 > T|\mathscr{F}_\theta \vee \sigma(\xi_s, 0 \leq s \leq T)] > 0$$

on $\{\sigma > \theta\}$. Together with the full support property of $\log\xi$, this implies (2.16), i.e. on $\{\sigma > \theta\}$

$$\mathbb{Q}[\sigma_1 > T, \sigma_2 > T|\mathscr{F}_\theta] = \mathbb{E}[\mathbb{E}[1_{\sigma_1 > T}|\mathscr{F}_t \vee \sigma(\xi_s, 0 \leq s \leq T)]1_{\sigma_2 > T}|\mathscr{F}_\theta] > 0.$$

Then, under any equivalent local martingale measure $\overline{\mathbb{Q}}$, we have

$$\begin{aligned}\overline{\mathbb{Q}}[\sigma \leq T] &= \overline{\mathbb{Q}}[\sigma_1 \leq T, \sigma_2 > T] + \overline{\mathbb{Q}}[\sigma_2 \leq T] \\ &\geq \overline{\mathbb{Q}}[\inf_{0 \leq u \leq \underline{\xi}^2 T} B_u \leq -1, \sigma_2 > T] + \overline{\mathbb{Q}}[\sigma_2 \leq T] \geq \overline{\mathbb{Q}}[\inf_{0 \leq u \leq \underline{\xi}^2 T} B_u \leq -1] > 0,\end{aligned}$$

where B is the Brownian motion such that

$$\int_0^t \xi_u dW_u^{\mathbb{Q}} = B_{\int_0^t \xi_u^2 du}$$

a.s. for all $t \geq 0$. Thus, $\overline{\mathbb{Q}}[\sigma \leq T]$ is bounded from below by the quantity $\overline{\mathbb{Q}}[\inf_{0 \leq u \leq \underline{\xi}^2 T} B_u \leq -1]$. This quantity is positive and does not depend on $\overline{\mathbb{Q}}$, since B is a Brownian motion under $\overline{\mathbb{Q}}$. Therefore $\sup_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}[\sigma > T]$ is bounded away from one and the superhedging price satisfies $SP^{\mathbb{Q}}(1_{\sigma > T}) = \sup_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}[\sigma > T] < 1$.

2.5.6 A variation: betting on the square bracket

The construction in Example 2.5.5 can be modified as follows. Let S be a continuous $\mathbb{Q}$-local martingale with $S_0 = 1$. We define

$$\sigma_1 := \inf\{t > 0 : S_t \leq 0\}, \quad \sigma_2 := \inf\{t > 0 : [S]_t \leq -a + bt\}, \quad \sigma = \sigma_1 \wedge \sigma_2,$$

where a, b are positive constants. Suppose that the conditions in Corollary 2.4.6 are fulfilled (they can be checked under suitable assumptions using the conditional full support property similarly to how this was done in the previous example). Then under any ELMM $\overline{\mathbb{Q}}$, we compute

$$\begin{aligned}
\overline{\mathbb{Q}}[\sigma \leq T] &= \overline{\mathbb{Q}}[\sigma_1 \leq T, \sigma_2 > T] + \overline{\mathbb{Q}}[\sigma_2 \leq T] \\
&= \overline{\mathbb{Q}}[\inf_{0 \leq u \leq [S]_T} B_u \leq -1, \sigma_2 > T] + \overline{\mathbb{Q}}[\sigma_2 \leq T] \geq \overline{\mathbb{Q}}[\inf_{0 \leq u \leq bT - a} B_u \leq -1] > 0,
\end{aligned}$$

where B is the Brownian motion such that $S_t = B_{[S]_t}$ a.s. for all $t \geq 0$. Thus the superhedging price satisfies $SP^{\mathbb{Q}}(1_{\sigma > T}) = \sup_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}[\sigma > T] < 1$.

Chapter 3

Optimal investment with intermediate consumption under no unbounded profit with bounded risk

Abstract: We consider the problem of optimal investment with intermediate consumption in a general semimartingale model of an incomplete market, with preferences being represented by a utility stochastic field. We show that the key conclusions of the utility maximization theory hold under the assumptions of no unbounded profit with bounded risk (NUPBR) and of the finiteness of both primal and dual value functions.

Key words: utility maximization, unbounded profit with bounded risk, arbitrage of the first kind, local martingale deflator, duality theory, semimartingale, incomplete markets.

This chapter is based on joint work with Andrea Cosso, Claudio Fontana and Oleksii Mostovyi.

3.1 Introduction

Since the pioneering work of Harrison and Kreps [1979], equivalent local/sigma martingale measures play a prominent role in the problems of pricing and portfolio optimization. Their existence is equivalent to the absence of arbitrage in the sense of *no free lunch with vanishing risk* (NFLVR) and this represents the crucial no-arbitrage-type assumption in the classical duality approach to optimal investment problems (see e.g. Karatzas and Žitković [2003]; Kramkov and Schachermayer [1999, 2003]; Žitković et al. [2005]). In a general semimartingale setting, necessary and sufficient conditions for the validity of the key assertions of the utility maximization theory (with the possibility of intermediate consumption) have been recently established in Mostovyi [2015]. More specifically, such assertions have been proven in Mostovyi [2015] under the assumptions that the primal and dual value functions are finite and that there exists an *equivalent martingale deflator*. In particular, in a finite time horizon, the latter assumption is equivalent to the validity of NFLVR.

In this chapter, we consider a general semimartingale setting in an infinite time horizon where preferences are modeled via a utility stochastic field, allowing for intermediate consumption. Building on the abstract theorems of Mostovyi [2015], our main result shows that the standard assertions of the utility maximization theory hold true as long as there is *no unbounded profit with bounded risk* (NUPBR) and both primal and dual value functions are finite. In general, NUPBR is weaker than NFLVR and can be shown to be equivalent to the existence of an *equivalent local martingale deflator*. Our results give an affirmative answer to a widespread conjecture in the mathematical finance community stating that the key conclusions of the utility maximization theory hold under NUPBR.

The proofs rely on certain characterizations of the dual feasible set. Thus, in Lemma 3.3.2 we give a polarity description, show its closedness under countable convex combinations in Lemma 3.3.3, and demonstrate in Proposition 3.2.1 that nonemptyness of the set that generates the dual domain is equivalent to NUPBR. Upon that, we prove the bipolar relations between primal and dual feasible sets and apply the abstract theorems from Mostovyi [2015]. As an implication of the bipolar relations, we also show how Theorem 2.2 in Kramkov and Schachermayer [1999] can be extended to hold under NUBPR (instead of NFLVR), see Remark 3.2.5 below for details.

Neither NFLVR, nor NUPBR by itself guarantee the existence of solutions to utility maximization problems, see [Kramkov and Schachermayer, 1999, Example 5.2] and [Christensen and Larsen, 2007, Example 4.3] for counterexamples. This is why finiteness of the value functions is needed in the formulation of our main result. However, it is shown in Choulli et al. [2015] that NUPBR holds if and only if, for every sufficiently nice deterministic utility function, the problem of maximizing expected utility from terminal wealth admits a solution under an equivalent probability measure, which can be chosen to be arbitrarily close to the original measure (see [Choulli et al., 2015, Theorem 2.8] for details). Besides, no unbounded profit with bounded risk represents the minimal no-arbitrage-type assumption that allows for the standard conclusions of the theory to hold for the utility maximization problem from terminal wealth. Indeed, by [Karatzas and Kardaras, 2007, Proposition 4.19], the failure of NUPBR implies that there exists a time horizon such that the corresponding utility maximization problem either does not have a solution, or has infinitely many. Our work complements these papers by providing the convex duality results under NUPBR, also allowing for stochastic preferences as well as intermediate consumption.

The problem of utility maximization without relying on the existence of martingale measures has already been addressed in the literature. In the very first paper Merton [1969] on expected utility maximization in continuous time settings, an optimal investment problem is explicitly solved even though an equivalent martingale measure does not exist in general in the infinite time horizon case. In an incomplete Itô process setting under a finite time horizon, Karatzas et al. [1991] have considered the problem of maximization of expected utility from terminal wealth and established the existence results for an optimal portfolio via convex duality theory without the full strength of NFLVR (see also [Fernholz et al., 2009, Section 10.3] and [Fontana and Runggaldier, 2013, Section 4.6.3]). In particular, in view of [Kardaras, 2010, Theorem 4], Assumption 2.3 in Karatzas et al.

[1991] is equivalent to the nonemptyness of the set of equivalent local martingale deflators. Passing from an Itô process to a continuous semimartingale setting, the results of Kramkov and Schachermayer [1999] have been extended by weakening the NFLVR requirement in Larsen [2009] (note that [Larsen, 2009, Assumption 2.1] is equivalent to NUPBR). In a general semimartingale setting, Larsen and Žitković [2013] have established convex duality results for the problem of maximizing expected utility from terminal wealth (for a deterministic utility function) in the presence of trading constraints without relying on the existence of martingale measures. In particular, in the absence of trading constraints, the no-arbitrage-type requirement adopted in Larsen and Žitković [2013] turns out to be equivalent to NUPBR. Indeed, [Larsen and Žitković, 2013, Assumption 2.3] requires the $\mathbf{L}^0_+$-solid hull[1] of the set of all terminal wealths generated by admissible strategies with initial wealth x, denoted by $\mathscr{C}(x)$, to be convexly compact[2] for all $x \in \mathbb{R}$ and nonempty for some $x \in \mathbb{R}$. In the absence of trading constraints, [Kardaras, 2010, Theorem 2] shows that the boundedness in $\mathbf{L}^0$ of $\mathscr{C}(x)$ already implies its closedness in $\mathbf{L}^0$, thus in such a framework the convex compactness of $\mathscr{C}(x)$ holds if and only if the NUPBR condition does.

The chapter is structured as follows. Section 3.2 begins with a description of the general setting (Subsection 3.2.1), introduces and characterizes the NUPBR condition (Subsection 3.2.2) and then proceeds with the statement of the main results (Subsection 3.2.3). Section 3.3 contains the proofs of our results.

[1] As usual, $\mathbf{L}^0$ denotes the space of equivalence classes of real-valued random variables on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, equipped with the topology of convergence in probability; $\mathbf{L}^0_+$ is the positive orthant of $\mathbf{L}^0$.

[2] See [Žitković, 2010, Definition 2.1] for the definition of convex compactness, a convenient characterization of which is given by [Žitković, 2010, Theorem 3.1]: "a closed and convex subset of $\mathbf{L}^0_+$ is convexly compact if and only if it is bounded in probability".

3.2 Setting and main results

3.2.1 Setting

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ be a complete stochastic basis, with $\mathscr{F}_0$ being the completion of the trivial σ-algebra, and $S = (S_t)_{t\geq 0}$ an $\mathbb{R}^d$-valued semimartingale [3], representing the discounted prices of d risky assets. We fix a *stochastic clock* $\kappa = (\kappa_t)_{t\geq 0}$ which is a nondecreasing, càdlàg adapted process such that

$$\kappa_0 = 0, \qquad \mathbb{P}(\kappa_\infty > 0) > 0 \qquad \text{and} \qquad \kappa_\infty \leq A, \tag{3.1}$$

for some finite constant A. The stochastic clock κ represents the notion of time according to which consumption occurs. By suitably specifying the clock process κ, several different formulations of investment problems, with or without intermediate consumption, can be recovered from the present setting (compare e.g. with [Žitković et al., 2005, Section 2.8] and [Mostovyi, 2015, Examples 2.5-2.9]).

A *portfolio* is defined by a triplet $\Pi = (x, H, c)$, where $x \in \mathbb{R}$ represents an initial capital, $H = (H_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued predictable S-integrable process representing the holdings in the d risky assets and $c = (c_t)_{t\geq 0}$ is a nonnegative optional process representing the consumption rate. The discounted value process $V = (V_t)_{t\geq 0}$ of a portfolio $\Pi = (x, H, c)$ is defined as

$$V_t := x + \int_0^t H_u \, \mathrm{d}S_u - \int_0^t c_u \, \mathrm{d}\kappa_u, \qquad t \geq 0.$$

We let $\mathscr{X}$ be the collection of all nonnegative value processes associated to portfolios of the form $\Pi = (1, H, 0)$, i.e.,

$$\mathscr{X} := \left\{ X \geq 0 : X_t = 1 + \int_0^t H_u \, \mathrm{d}S_u, t \geq 0 \right\}.$$

For a given initial capital $x > 0$, a consumption process c is said to be *x-admissible* (written as $c \in \mathscr{A}_x$) if there exists an $\mathbb{R}^d$-valued predictable S-integrable process

[3] It's remarked that the assumptions of this present chapter is slightly different from the general setting introduced in Section 1.4.

H such that the value process V associated to the portfolio $\Pi = (x,H,c)$ is non-negative. For brevity, we let $\mathscr{A} := \mathscr{A}_1$. We remark that the set $\mathscr{A}$ in this present chapter involves a property for consumption processes, not for strategies as in the general setting in Section 1.4.

3.2.2 No unbounded profit with bounded risk

In this paper, we shall assume the validity of the following no-arbitrage-type condition:

$$\text{the set } \mathscr{X}(T) := \left\{X_T : X \in \mathscr{X}\right\} \text{ is bounded in probability, for all } T \geq 0. \tag{NUPBR}$$

For each $T \geq 0$, the boundedness in probability of the set $\mathscr{X}(T)$ has been named *no unbounded profit with bounded risk* in Karatzas and Kardaras [2007] and, as shown in Proposition 1 of Kardaras [2010], is equivalent to the absence of *arbitrages of the first kind* on $[0,T]$. Hence, condition NUPBR is equivalent to the absence of arbitrages of the first kind in the sense of Definition 1 of Kardaras [2014].

We define the set of *equivalent local martingale deflators (ELMD)* as follows:

$$\mathscr{Z} := \big\{Z > 0 : Z \text{ is a càdlàg local martingale such that } Z_0 = 1 \text{ and}$$
$$ZX \text{ is a local martingale for every } X \in \mathscr{X}\big\}.$$

The following result is already known in the one-dimensional case in finite horizon (see e.g. [Kardaras, 2012, Theorem 2.1]). Since we could not find a specific reference for our formulation in the multi-dimensional case in infinite horizon, we provide a detailed proof in Section 3.3. The condition (NUPBR) in infinite horizon is also discussed in Definition 2.1 and Remark 2.2 of Aksamit et al. [2014].

Proposition 3.2.1. *NUPBR holds if and only if $\mathscr{Z} \neq \emptyset$.*

Remark 3.2.2. *In Mostovyi [2015], it is assumed that the set*

$$\{Z \in \mathscr{Z} : Z \text{ is a martingale}\} \neq \emptyset.$$

In view of Proposition 3.2.1, the latter condition is stronger than NUPBR. A classical example where NUPBR holds but $\{Z \in \mathscr{Z} : Z \text{ is a martingale}\} = \emptyset$ is provided by the three-dimensional Bessel process (see e.g. Delbaen and Schachermayer [1995a] and Example 4.6 of Karatzas and Kardaras [2007]).

3.2.3 Optimal investment with intermediate consumption

We now proceed to show that the key conclusions of the utility maximization theory can be established under condition (NUPBR). We assume that preferences are represented by a *utility stochastic field* $U = U(t,\omega,x) : [0,\infty) \times \Omega \times [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ satisfying the following assumption (see Assumption 2.1 of Mostovyi [2015]).

Assumption 3.2.3. *For every $(t,\omega) \in [0,\infty) \times \Omega$, the function $x \mapsto U(t,\omega,x)$ is strictly concave, strictly increasing, continuously differentiable on $(0,\infty)$ and satisfies the Inada conditions*

$$\lim_{x\downarrow 0} U'(t,\omega,x) = +\infty \qquad \textit{and} \qquad \lim_{x\to+\infty} U'(t,\omega,x) = 0,$$

with U' denoting the partial derivative of U with respect to its third argument. By continuity, at $x=0$ we have that $U(t,\omega,0) = \lim_{x\downarrow 0} U(t,\omega,x)$ (note this value may be $+\infty$). Finally, for every $x \geq 0$, the stochastic process $U(\cdot,\cdot,x)$ is optional.

To a utility stochastic field U satisfying Assumption 3.2.3, we associate the *primal value function*, defined as

$$u(x) := \sup_{c\in\mathscr{A}_x} \mathbb{E}\left[\int_0^\infty U(t,\omega,c_t)\,\mathrm{d}\kappa_t\right], \qquad x > 0, \tag{3.2}$$

with the convention $\mathbb{E}[\int_0^\infty U(t,\omega,c_t)\,\mathrm{d}\kappa_t] := -\infty$ if $\mathbb{E}[\int_0^\infty U^-(t,\omega,c_t)\,\mathrm{d}\kappa_t] = +\infty$.

In order to construct the dual value function, we define as follows the *stochastic field V conjugate to U*:

$$V(t,\omega,y) := \sup_{x>0}\big(U(t,\omega,x) - xy\big), \qquad (t,\omega,y) \in [0,\infty) \times \Omega \times [0,\infty).$$

We also introduce the following set of dual processes:

$$\mathscr{Y}(y) := \mathrm{cl}\big\{Y : Y \text{ is càdlàg adapted and}$$
$$0 \le Y \le yZ \ (\mathrm{d}\kappa \otimes \mathbb{P})\text{-a.e. for some } Z \in \mathscr{Z}\big\},$$

where the closure is taken in the topology of convergence in measure $(\mathrm{d}\kappa \otimes \mathbb{P})$ on the space of real-valued optional processes. We write $\mathscr{Y} := \mathscr{Y}(1)$ for brevity. The value function of the dual optimization problem (*dual value function*) is then defined as

$$v(y) := \inf_{Y \in \mathscr{Y}(y)} \mathbb{E}\left[\int_0^\infty V(t,\omega,Y_t)\,\mathrm{d}\kappa_t\right], \qquad y > 0, \tag{3.3}$$

with the convention $\mathbb{E}[\int_0^\infty V(t,\omega,Y_t)\,\mathrm{d}\kappa_t] := +\infty$ if $\mathbb{E}[\int_0^\infty V^+(t,\omega,Y_t)\,\mathrm{d}\kappa_t] = +\infty$.

We are now in a position to state the following theorem, which is the main result of this chapter.

Theorem 3.2.4. *Assume that conditions* (3.1) *and (NUPBR) hold true and let U be a utility stochastic field satisfying Assumption 3.2.3. Suppose that*

$$v(y) < \infty \quad \text{for all } y > 0 \quad \text{and} \quad u(x) > -\infty \quad \text{for all } x > 0. \tag{3.4}$$

Then the value function u and the dual value function v defined in (3.2) *and* (3.3)*, respectively, satisfy the following properties:*

(i) $u(x) < \infty$*, for all* $x > 0$*, and* $v(y) > -\infty$*, for all* $y > 0$*. Moreover, the functions u and v are conjugate, i.e.,*

$$v(y) = \sup_{x>0}\big(u(x) - xy\big), \quad y > 0,$$
$$u(x) = \inf_{y>0}\big(v(y) - yx\big), \quad x > 0;$$

(ii) the functions u and $-v$ *are continuously differentiable on* $(0,\infty)$*, strictly concave, strictly increasing and satisfy the Inada conditions*

$$u'(0) := \lim_{x\downarrow 0} u'(x) = +\infty, \qquad -v'(0) := \lim_{y\downarrow 0} -v'(y) = +\infty,$$
$$u'(\infty) := \lim_{x\to+\infty} u'(x) = 0, \qquad -v'(\infty) := \lim_{y\to+\infty} -v'(y) = 0.$$

Moreover, for every $x > 0$ and $y > 0$, the solutions $\hat{c}(x)$ to (3.2) *and $\hat{Y}(y)$ to* (3.3) *exist and are unique and, if $y = u'(x)$, we have the dual relations*

$$\hat{Y}_t(y)(\omega) = U'\big(t, \omega, \hat{c}_t(x)(\omega)\big), \qquad \mathrm{d}\kappa \otimes \mathbb{P}\text{-a.e.},$$

and

$$\mathbb{E}\left[\int_0^\infty \hat{c}_t(x)\hat{Y}_t(y)\,\mathrm{d}\kappa_t\right] = xy.$$

Finally, the dual value function v can be equivalently represented as

$$v(y) = \inf_{Z \in \mathscr{Z}} \mathbb{E}\left[\int_0^\infty V(t, \omega, yZ_t)\,\mathrm{d}\kappa_t\right], \quad y > 0. \tag{3.5}$$

Remark 3.2.5. *A close look at the proof of Theorem 3.2.4 reveals that for κ corresponding to utility maximization from terminal wealth, the sets $\mathscr{A}$ and $\mathscr{Y}$ satisfy the assumptions of Proposition 3.1 in Kramkov and Schachermayer [1999]. This implies that for a deterministic utility U satisfying the Inada conditions and such that $AE(U) < 1$ (in the terminology of Kramkov and Schachermayer [1999]), under an additional assumption of finiteness of $u(x)$ for some $x > 0$, the assertions of Theorem 2.2 in Kramkov and Schachermayer [1999] hold under (NUPBR) (and possibly without NFLVR). This is a consequence of "abstract" Theorem 3.1 in Kramkov and Schachermayer [1999].*

3.3 Proofs

Proof of Proposition 3.2.1. Suppose that NUPBR holds. Then, for every $n \in \mathbb{N}$, the set $\mathscr{X}_n$ is bounded in L^0 and, by [Takaoka and Schweizer, 2014, Theorem 2.6], there exists a strictly positive càdlàg local martingale Z^n such that $Z^n_0 = 1$ (since $\mathscr{F}_0$ is trivial) and the $\mathbb{R}^d$-valued process Z^nS is a sigma-martingale on $[0,n]$. As a consequence of [Ansel and Stricker, 1994, Corollary 3.5] (see also [Choulli et al., 2015, Remark 2.4]), it holds that Z^nX is a local martingale on $[0,n]$, for every $X \in \mathscr{X}$ and $n \in \mathbb{N}$. For all $t \geq 0$, let then $n(t) := \min\{n \in \mathbb{N} : n > t\}$ and define the càdlàg process $Z = (Z_t)_{t\geq 0}$ via

$$Z_t := \prod_{k=1}^{n(t)} \frac{Z^k_{k\wedge t}}{Z^k_{(k-1)\wedge t}}, \qquad t \geq 0.$$

We now claim that $Z \in \mathscr{Z}$. Since $X \equiv 1 \in \mathscr{X}$ and in view of [Jacod and Shiryaev, 2002, Lemma I.1.35], it suffices to show that, for every $X \in \mathscr{X}$, the process ZX is a local martingale on $[0,m]$, for each $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$. Consider an arbitrary $X \in \mathscr{X}$ and let $\{\tau_k^n\}_{k\in\mathbb{N}}$ be a localizing sequence for the local martingale $Z^n X$ on $[0,n]$, for each $n \in \{1,\ldots,m\}$. Let $\tau^j_{k\,\{\tau^j_k<j\}} := \tau^j_k \mathbb{I}_{\{\tau^j_k<j\}} + \infty \mathbb{I}_{\{\tau^j_k=j\}}$ (which is a stopping time by [Jacod and Shiryaev, 2002, §I.1.15]), for $j = 1,\ldots,m$ and $k \in \mathbb{N}$, and define the stopping times

$$T_k^m := \min\left\{\tau^1_{k\,\{\tau^1_k<1\}},\ldots,\tau^m_{k\,\{\tau^m_k<m\}},m\right\}, \qquad k \in \mathbb{N}.$$

Similarly as in [Fontana et al., 2015, proof of Theorem 4.10], it can be readily verified that the stopped proces $(ZX)^{T_k^m}$ is a martingale on $[0,m]$, for all $k \in \mathbb{N}$. Since $\lim_{k\to+\infty} \mathbb{P}(T_k^m = m) = 1$, this shows that ZX is a local martingale on $[0,m]$. By the arbitrariness of m, this proves the claim.

To prove the converse implication, note that, for any $X \in \mathscr{X}$ and $Z \in \mathscr{Z}$, the process ZX is a supermartingale and, hence, for every $T \geq 0$, it holds that $\mathbb{E}[Z_T X_T] \leq 1$. This shows that the set $Z_T \mathscr{X}_T$ is bounded in L^1 and, hence, in L^0. Since the multiplication by the finite random variable Z_T does not affect the boundedness in L^0, this implies that $\mathscr{X}_T$ is bounded in L^0, for all $T \geq 0$. □

Let us now turn to the proof of Theorem 3.2.4. Together with the abstract results established in [Mostovyi, 2015, Section 3], the key step is represented by the following lemma, which generalizes [Mostovyi, 2015, Lemmata 4.2 and 4.3] by relaxing the no-arbitrage-type requirement into condition NUPBR.

As a preliminary, for a semimartingale $\widetilde{S}$, let us denote by $\mathscr{H}(\widetilde{S})$ the set of all *admissible* integrands, in the sense of [Delbaen and Schachermayer, 1994, Definition 2.7], and define the following sets of equivalent probability measures:

$$\mathscr{M}_\sigma(\widetilde{S}) := \left\{\mathbb{Q} \sim \mathbb{P} : \widetilde{S} \text{ is a } \mathbb{Q}\text{-sigma-martingale}\right\},$$

$$\mathscr{M}_{loc}(\widetilde{S}) := \left\{\mathbb{Q} \sim \mathbb{P} : \int H \,\mathrm{d}\widetilde{S} \text{ is a } \mathbb{Q}\text{-local martingale for every } H \in \mathscr{H}(\widetilde{S})\right\},$$

$$\mathscr{M}_s(\widetilde{S}) := \left\{\mathbb{Q} \sim \mathbb{P} : \int H \,\mathrm{d}\widetilde{S} \text{ is a } \mathbb{Q}\text{-supermartingale for every } H \in \mathscr{H}(\widetilde{S})\right\}.$$

Lemma 3.3.1. *Let $\widetilde{S}$ be a semimartingale. Then it holds that*

$$\mathscr{M}_\sigma(\widetilde{S}) \subseteq \mathscr{M}_{loc}(\widetilde{S}) \subseteq \mathscr{M}_s(\widetilde{S}).$$

Moreover, if $\mathscr{M}_s(\widetilde{S}) \neq \emptyset$, then $\mathscr{M}_\sigma(\widetilde{S}) \neq \emptyset$ and the set $\mathscr{M}_\sigma(\widetilde{S})$ is dense in $\mathscr{M}_s(\widetilde{S})$ with respect to the norm of $L^1(\Omega, \mathscr{F}, \mathbb{P})$.

Proof. Since the class of sigma-martingales is stable with respect to stochastic integration (see [Jacod and Shiryaev, 2002, Proposition III.6.42]), the first inclusion follows from [Ansel and Stricker, 1994, Corollary 3.5]. The second inclusion follows from the fact that any local martingale bounded from below is a supermartingale. Finally, the last assertion follows from [Delbaen and Schachermayer, 1998, Proposition 4.7], by noting that, if $\mathscr{M}_s(\widetilde{S}) \neq \emptyset$, then $\widetilde{S}$ satisfies NFLVR, so that $\mathscr{M}_\sigma(\widetilde{S}) \neq \emptyset$ by [Delbaen and Schachermayer, 1998, Theorem 1.1]. □

The following lemma provides a polarity characterization of attainable consumption streams.

Lemma 3.3.2. *Let c be a nonnegative optional process and κ a stochastic clock. Under assumptions* (3.1) *and NUPBR, the following conditions are equivalent:*

(i) $c \in \mathscr{A}$*;*

(ii) $\sup_{Z \in \mathscr{Z}} \mathbb{E}[\int_0^\infty c_t Z_t \, \mathrm{d}\kappa_t] \leq 1$.

Proof. If $c \in \mathscr{A}$, there exists an $\mathbb{R}^d$-valued predictable S-integrable process H such that

$$1 + \int_0^t H_u \, \mathrm{d}S_u \geq \int_0^t c_u \, \mathrm{d}\kappa_u \geq 0, \qquad t \geq 0.$$

We define $C_t := \int_0^t c_u \, \mathrm{d}\kappa_u$, for $t \geq 0$, and observe that C is an increasing process. For an arbitrary $Z \in \mathscr{Z}$, the process $(\int_0^t C_{u-} \, \mathrm{d}Z_u)_{t \geq 0}$ is a local martingale and we let $\{\tau_n\}_{n \in \mathbb{N}}$ be a localizing sequence such that $(\int C_- \, \mathrm{d}Z)^{\tau_n}$ is a uniformly integrable martingale, for all $n \in \mathbb{N}$. Using the supermartingale property of $Z(1 + \int H \, \mathrm{d}S)$, we obtain for every $n \in \mathbb{N}$

$$1 \geq \mathbb{E}\left[Z_{\tau_n}\left(1 + \int_0^{\tau_n} H_u \, \mathrm{d}S_u\right)\right] \geq \mathbb{E}\left[Z_{\tau_n} C_{\tau_n}\right] = \mathbb{E}\left[\int_0^{\tau_n} Z_u \, \mathrm{d}C_u + \int_0^{\tau_n} C_{u-} \, \mathrm{d}Z_u\right],$$

where the last equality follows by integration by parts (see [Jacod and Shiryaev, 2002, Proposition I.4.49]). Since $\{\tau_n\}_{n\in\mathbb{N}}$ is a localizing sequence for $\int C_- \, \mathrm{d}Z$, it holds that $\mathbb{E}[\int_0^{\tau_n} C_{u-} \, \mathrm{d}Z_u] = 0$, for all $n \in \mathbb{N}$. Hence:

$$1 \geq \mathbb{E}\left[\int_0^{\tau_n} Z_u \, \mathrm{d}C_u\right], \qquad \text{for all } n \in \mathbb{N}.$$

By the monotone convergence theorem, we get that $1 \geq \lim_{n\to\infty} \mathbb{E}[\int_0^{\tau_n} Z_u \, \mathrm{d}C_u] = \mathbb{E}[\int_0^{\infty} Z_u \, \mathrm{d}C_u]$. Since $Z \in \mathscr{Z}$ is arbitrary, this proves the implication (i)$\Rightarrow$(ii).

Suppose now that $\sup_{Z\in\mathscr{Z}} \mathbb{E}[\int_0^\infty c_t Z_t \, \mathrm{d}\kappa_t] \leq 1$. Take an arbitrary $Z \in \mathscr{Z}$ and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of stopping times increasing $\mathbb{P}$-a.s. to infinity such that Z^{ρ_n} is a uniformly integrable martingale, for all $n \in \mathbb{N}$. By Lemma 3.3.1, it holds that $\mathscr{M}_\sigma(S^{\rho_n}) \neq \emptyset$, for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, let $\mathbb{Q} \in \mathscr{M}_\sigma(S^{\rho_n})$ and denote by $M = (M_t)_{t\geq 0}$ its density process (i.e., $\mathrm{d}\mathbb{Q}|_{\mathscr{F}_t} = M_t \, \mathrm{d}\mathbb{P}|_{\mathscr{F}_t}$, for all $t \geq 0$). Letting $Z' := M^{\rho_n} Z (Z^{\rho_n})^{-1}$, the same arguments of [Stricker and Yan, 1998, Lemma 2.3] imply that $Z' \in \mathscr{Z}$. For an arbitrary stopping time $\tau \in \mathscr{T}$ (with $\mathscr{T}$ denoting the set of all finite stopping times), it then holds that

$$\mathbb{E}^{\mathbb{Q}}[C_{\tau\wedge\rho_n}] = \mathbb{E}[M_{\tau\wedge\rho_n} C_{\tau\wedge\rho_n}] = \mathbb{E}[Z'_{\tau\wedge\rho_n} C_{\tau\wedge\rho_n}] \leq 1,$$

where the last inequality follows from the assumption that $\sup_{Z\in\mathscr{Z}} \mathbb{E}[\int_0^\infty c_t Z_t \, \mathrm{d}\kappa_t] \leq 1$ by the same arguments used in the first part of the proof together with an application of Fatou's lemma. By the arbitrariness of $\mathbb{Q} \in \mathscr{M}_\sigma(S^{\rho_n})$ and $\tau \in \mathscr{T}$ together with the denseness of $\mathscr{M}_\sigma(S^{\rho_n})$ in $\mathscr{M}_s(S^{\rho_n})$ (see Lemma 3.3.1), it then follows that

$$\sup_{\mathbb{Q}\in\mathscr{M}_s(S^{\rho_n})} \sup_{\tau\in\mathscr{T}} \mathbb{E}^{\mathbb{Q}}[C_{\tau\wedge\rho_n}] = \sup_{\mathbb{Q}\in\mathscr{M}_\sigma(S^{\rho_n})} \sup_{\tau\in\mathscr{T}} \mathbb{E}^{\mathbb{Q}}[C_{\tau\wedge\rho_n}] \leq 1.$$

[Föllmer and Kramkov, 1997, Proposition 4.2] (together with Examples 2.2 and 4.1 therein) then gives the existence of an adapted càdlàg process V^n such that $V_t^n \geq C_{t\wedge\rho_n}$, for all $t \geq 0$, and admitting a decomposition of the form

$$V_t^n = V_0^n + \int_0^t H_u^n \, \mathrm{d}S_u^{\rho_n} - A_t^n, \qquad t \geq 0,$$

where H^n is an $\mathbb{R}^d$-valued predictable S^{ρ_n}-integrable process, A^n is an adapted increasing process with $A_0^n = 0$ and $V_0^n = \sup_{\mathbb{Q}\in\mathscr{M}_s(S^{\rho_n}),\tau\in\mathscr{T}} \mathbb{E}^{\mathbb{Q}}[C_{\tau\wedge\rho_n}] \leq 1$. Therefore, for every $n \in \mathbb{N}$, we obtain

$$1+\int_0^t H_u^n \,\mathrm{d}S_u \geq V_0^n + \int_0^t H_u^n \,\mathrm{d}S_u = V_t^n + A_t^n \geq V_t^n \geq C_t, \qquad 0 \leq t \leq \rho_n.$$

Let $\bar{H}^n := H^n \mathbb{I}_{[\![0,\rho_n]\!]}$, for all $n \in \mathbb{N}$. By [Föllmer and Kramkov, 1997, Lemma 5.2], we can construct a sequence of processes $\{Y^n\}_{n\in\mathbb{N}}$, with $Y^n \in \mathrm{conv}(1+\bar{H}^n \cdot S, 1+\bar{H}^{n+1}\cdot S, \ldots)$, $n \in \mathbb{N}$, and a càdlàg process Y such that $\{ZY^n\}_{n\in\mathbb{N}}$ is Fatou-convergent [4] to a supermartingale ZY, for every strictly positive càdlàg local martingale Z such that ZX is a supermartingale for every $X \in \mathscr{X}$. Note that $Y_t \geq C_t$, for all $t \geq 0$, and $Y_0 \leq 1$. Similarly as above, applying then [Föllmer and Kramkov, 1997, Theorem 4.1] to the stopped process Y^{ρ_n}, for $n \in \mathbb{N}$, we obtain the decomposition

$$Y_t^{\rho_n} = Y_0 + \int_0^t G_u^n \,\mathrm{d}S_u^{\rho_n} - B_t^n, \qquad t \geq 0,$$

where G^n is an $\mathbb{R}^d$-valued predictable S^{ρ_n}-integrable process and B^n is an adapted increasing process with $B^n = 0$, for $n \in \mathbb{N}$. Without loss of generality, we can assume that $G^n \mathbb{I}_{]\!]\rho_n,+\infty[\![} = 0$, for all $n \in \mathbb{N}$. Letting $G := G^1 + \sum_{n=1}^{\infty}(G^{n+1} - G^n)\mathbb{I}_{]\!]\rho_n,+\infty[\![} = G^1 + \sum_{n=1}^{\infty} G^{n+1}\mathbb{I}_{]\!]\rho_n,\rho_{n+1}]\!]}$, it follows that $1+\int_0^t G_u \,\mathrm{d}S_u \geq C_t$, for all $t \geq 0$, thus establishing the implication (ii)⇒(i). □

We are now in a position to complete the proof of Theorem 3.2.4, which generalizes the results of [Mostovyi, 2015, Theorems 2.3 and 2.4] to the case where only (NUPBR) is assumed to hold. As a preliminary, we need the following result on the set $\mathscr{Z}$.

Lemma 3.3.3. *Under* (NUPBR)*, the set $\mathscr{Z}$ is closed under countable convex combinations. If in addition* (3.1) *holds, then for every $c \in \mathscr{A}$, we have*

$$\sup_{Z\in\mathscr{Z}} \mathbb{E}\left[\int_0^{\infty} c_t Z_t \,\mathrm{d}\kappa_t\right] = \sup_{Y\in\mathscr{Y}} \mathbb{E}\left[\int_0^{\infty} c_t Y_t \,\mathrm{d}\kappa_t\right] \leq 1. \tag{3.6}$$

[4] See [Föllmer and Kramkov, 1997, Definition 5.2] for the definition of Fatou convergence of stochastic processes.

Proof. Let $\{Z^n\}_{n\in\mathbb{N}}$ be a sequence of processes belonging to $\mathscr{Z}$ and $\{\lambda^n\}_{n\in\mathbb{N}}$ a sequence of positive numbers such that $\sum_{n=1}^{\infty}\lambda^n = 1$. Letting $Z := \sum_{n=1}^{\infty}\lambda^n Z^n$, we need to show that $Z \in \mathscr{Z}$. For each $N \in \mathbb{N}$, define $\widetilde{Z}^N := \sum_{n=1}^{N}\lambda^n Z^n$. For every $X \in \mathscr{X}$, $\{\widetilde{Z}^N X\}_{N\in\mathbb{N}}$ is an increasing sequence of nonnegative local martingales (i.e. $\widetilde{Z}^{N+1}_t X_t \geq \widetilde{Z}^N_t X_t$, for all $N \in \mathbb{N}$ and $t \geq 0$), such that $\widetilde{Z}^N_t X_t$ converges a.s. to $Z_t X_t$ as $N \to +\infty$, for every $t \geq 0$, and $Z_0 X_0 = 1$. The local martingale property of ZX then follows from [Klein et al., 2014, Proposition 5.1] (note that its proof carries over without modifications to the infinite horizon case), whereas [Dellacherie and Meyer, 1982, Theorem VI.18] implies that ZX is a càdlàg process. Since $X \in \mathscr{X}$ is arbitrary and $X \equiv 1 \in \mathscr{X}$, this proves the claim.

Finally, relation (3.6) follows by the same arguments used in [Mostovyi, 2015, Lemma 4.3]. □

We denote by $\mathbf{L}^0(\mathrm{d}\kappa \times \mathbb{P})$ be the linear space of equivalence classes of real-valued optional processes on the stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,\infty)}, \mathbb{P})$, equipped with the topology of convergence in measure $(\mathrm{d}\kappa \times \mathbb{P})$. Let $\mathbf{L}^0_+(\mathrm{d}\kappa \times \mathbb{P})$ be the positive orthant of $\mathbf{L}^0(\mathrm{d}\kappa \times \mathbb{P})$.

Proof of Theorem 3.2.4. It is clear that the sets $\mathscr{A}$ and $\mathscr{Y}$ are convex solid subsets of $\mathbf{L}^0_+(\mathrm{d}\kappa \times \mathbb{P})$. By definition, $\mathscr{Y}$ is closed in the topology of convergence in measure $(\mathrm{d}\kappa \times \mathbb{P})$. As in [Mostovyi, 2015, part (i) of Proposition 4.4], a simple application of Fatou's lemma together with Lemma 3.3.2 allows to show that $\mathscr{A}$ is also closed in the same topology. Moreover, by the same arguments used in [Mostovyi, 2015, part (ii) of Proposition 4.4], Lemma 3.3.2 together with the bipolar theorem of Brannath and Schachermayer [1999] implies that $\mathscr{A}$ and $\mathscr{Y}$ satisfy the bipolar relations

$$c \in \mathscr{A} \quad \Longleftrightarrow \quad \mathbb{E}\left[\int_0^\infty c_t Y_t \,\mathrm{d}\kappa_t\right] \leq 1 \quad \text{for all } Y \in \mathscr{Y}, \tag{3.7}$$

$$Y \in \mathscr{Y} \quad \Longleftrightarrow \quad \mathbb{E}\left[\int_0^\infty c_t Y_t \,\mathrm{d}\kappa_t\right] \leq 1 \quad \text{for all } c \in \mathscr{A}. \tag{3.8}$$

Since $X \equiv 1 \in \mathscr{X}$ and $\mathscr{Z} \neq \emptyset$, both $\mathscr{A}$ and $\mathscr{Y}$ contain at least one strictly positive element. In view of Lemma 3.3.3, Theorem 3.2.4 then follows directly from the abstract results of [Mostovyi, 2015, Theorems 3.2 and 3.3]. □

Acknowledgement

Oleksii Mostovyi's work is supported by the National Science Foundation under award number DMS-1600307. Huy N. Chau's work is supported by Natixis Foundation for Quantitative Research.

Chapter 4

Optimal arbitrage for initial filtration enlargement

Abstract: This chapter studies optimal trading for an insider who has some additional information at the beginning of the trading. Using the technique of non-equivalent measure change, we are able to provide a new criterion ensuring that the market for the insider satisfies the condition No Unbounded Profit with Bounded Risk (NUPBR), which excludes 'scalable' arbitrages. Furthermore, if the condition NUPBR holds, we present a systematic way to exploit optimal arbitrage opportunities.

Key words: No Unbounded Profits with Bounded Risk, optimal arbitrage, initial enlargement of filtration, incomplete markets, hedging.

This chapter is based on joint work with Peter Tankov and Wolfgang Runggaldier.

Acknowledgement

Huy N. Chau's work is supported by Natixis Foundation for Quantitative Research.

4.1 Introduction

In financial mathematics, theory of filtration enlargement is often used to model insider trading activities. This theory was developed in the seminal works of Itô, Barlow, Jacod, Jeulin and Yor and we refer to the book of Mansuy and Yor [2006] for an overview. By definition, an insider is a person who can access the private information that others cannot. Consequently, the insider quickly incorporates such information to her trading strategies. One of the first papers on this topic is Pikovsky and Karatzas [1996], where the authors study the logarithmic utility optimization problem for the insider, who is able to anticipate the future, i.e. the terminal values of asset's prices in Brownian settings. Whenever the logarithmic utility maximization problem for the insider is finite, the authors are able to compute it explicitly and give the optimal portfolio in closed form. Grorud and Pontier [1998] also compute logarithmic utility of the insider with consumption and propose a statistical test whether or not a trader is an insider. Amendinger et al. [1998] relate the insider's additional expected logarithmic utility with a relative entropy. This relation is then extended in Ankirchner et al. [2006]. For further references, we refer to Elliott and Jeanblanc [1999], Amendinger et al. [2003b], Hillairet [2005], Ankirchner et al. [2006], Ankirchner and Zwierz [2011], Danilova et al. [2010]... and others.

Apart from portfolio optimization, the question of arbitrage has also received considerable attention. Amendinger [2000] studies martingale representation properties in the larger filtration. Imkeller et al. [2001] and Imkeller [2003] make use of Malliavin's calculus in order to find semimartingale decompositions for some classes of additional information which do not satisfy Jacod's condition. In addition, they prove that arbitrage opportunities exist in dramatic manners, in the sense that the so-called information drift has possibility of explosion. However, we will see later that the exploding information drift violates NUPBR (no unbounded profit with bounded risk) condition. Recently, Acciaio et al. [2014] investigated the stability of NUPBR for the insider with infinite horizon settings. They give a sufficient condition for the stability of NUPBR under initial filtration enlargement. Basically, it is shown that if the risky asset and the density process in Jacod's condition do not jump at the same time, then the enlarged market satisfies NUPBR. However, Jacod's condition is required at all times in their results, which cannot be always satisfied in finite horizon settings.

In this chapter, we focus on incomplete markets and examine the logarithmic utility maximization problems by using duality approach, rather than using the information drift or relative entropy as in all previous studies. By the technique of non-equivalent measure change as in Chapter 2, we are able to give a new dual representation for the insider's expected log-utility in quite general market models. This duality result differs from classical results (see for example, Kramkov and Schachermayer [1999]) as it contains an extra term involving the entropy of the information. It is usually observed that the extra factor is the main reason such that the insider's expected utility blows up at the time when the private information becomes public. This new finding allows us to introduce a new criterion for checking the condition NUPBR in the enlarged market with finite horizon settings. To do this, we first show that NUPBR always fails if the set of all martingale densities for regular agents is uniformly integrable and an explicit construction of unbounded profits is also given. Hence, the non-uniform integrability property is a necessary condition for the finiteness of the insider's expected log-utility. It helps to compensate explosive profits coming from the additional information, which in turn ensures NUPBR.

In the case NUPBR holds but not NFLVR, we present a new systematic approach to find optimal arbitrage for the insider. We first study the case with discrete information, i.e. when the private information is given by a discrete random variable. The case with continuous random variable is treated by an approximating procedure. Some illustrative examples are given.

The chapter is organized as follows. In Section 4.2, we provide some preliminaries on initial enlargement of filtrations. Section 4.3 introduces the notion of superhedging and optimal arbitrage. Section 4.4 computes log-utility and optimal arbitrage when the information is given by a discrete random variable. Finally, the same questions for the case with continuous random variables are discussed in Section 4.5.

4.2 Preliminaries on initial enlargement of filtrations

Assume that the filtration $\mathbb{F}$ models the public information based on which ordinary agents make their decisions. We also suppose that the ordinary agents are risk neutral, i.e. the $(\mathbb{F},\mathbb{P})$-market satisfies the condition NFLVR. The filtration $\mathbb{F}$ is not necessarily the natural filtration of the stock prices. Suppose that there is an insider who possesses from the beginning an additional information about the value of some $\mathscr{F}_T$-measurable $\mathbb{R}$-valued random variable G. In mathematical terms, we model her knowledge by the enlarged filtration $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$ where

$$\mathscr{G}_t := \bigcap_{\varepsilon>0} (\mathscr{F}_{t+\varepsilon} \vee \sigma(G)).$$

Note that we assume both that G is $\mathscr{F}_T$-measurable and that trading is possible up to the terminal time T. This situation is known to be "difficult" (see e.g., the introduction of Grorud and Pontier [1998]) and often leads to arbitrages of the first kind Amendinger et al. [1998]; our aim is to explore this setting in detail and identify the cases where the NUPBR holds and optimal arbitrage strategies may be found.

Because the semimartingale property is widely accepted in financial modeling, we first recall a condition of Jacod [1985] which ensures that a $\mathbb{F}$-local martingale remains a $\mathbb{G}$-semimartingale. For all $t \in [0,T)$, let $\nu_t : \Omega \times \mathbb{R} \to [0,1]$ be a regular version of the $\mathscr{F}_t$ -conditional law of G and ν be the law of G. The following assumption plays an important role in most studies in this theory.

Assumption 4.2.1. *(Jacod's Condition) For all $t \in [0,T)$, the regular conditional distribution of G given $\mathscr{F}_t$ is absolutely continuous with respect to the law of G, i.e. we have*

$$\nu_t \ll \nu, \qquad \mathbb{P}-a.s. \tag{4.1}$$

In particular, this assumption implies that $G \notin \mathscr{F}_t$ for all $t < T$. We emphasize that this is a "weak" version of Jacod's condition in the following sense.

- The absolute continuity of the measure ν_t with respect to the law of G is imposed only before the terminal time T. Of course, in our setting when

$G \in \mathscr{F}_T$ the absolute continuity cannot hold at the terminal date. When $G \notin \mathscr{F}_T$ and $\nu_T \ll \nu$, the NUPBR property can often be shown by constructing a local martingale deflator from the density process of ν_t with respect to ν; in our setting this method is not avaiable.

- Only absolute continuity is imposed, rather than equivalence. If $\nu_t \sim \nu$ for all $t \in [0,T]$, then the density of ν_t with respect to ν is strictly positive and one can show that NFLVR property holds by constructing an equivalent martingale measure from the density process. See Grorud and Pontier [1998], Theorem 3.2 of Amendinger [2000] or Föllmer and Imkeller [1993] for details and related results. The situation when $\nu_t \sim \nu$ for $t \in [0,T)$ and $\nu_T \ll \nu$ is not fundamentally different from the situation when only absolute continuity is imposed on $[0,T]$.

The density of ν_t with respect to ν in Assumption 4.2.1 plays an important role in enlargement of filtration theory. Let $\mathscr{O}(\mathbb{F})$ be the $\mathbb{F}$-optional sigma field on $\Omega \times \mathbb{R}_+$. It is shown in Lemme 1.8 and Corollaire 1.11 of Jacod [1985] that we can even choose a nice version of the densities extracted from (4.1).

Lemma 4.2.2. *Under Assumption 4.2.1, there exists a nonnegative $\mathscr{B} \otimes \mathscr{O}(\mathbb{F})$-measurable function $\mathbb{R} \times \Omega \times \mathbb{R}_+ \ni (x,\omega,t) \mapsto p_t^x(\omega) \in [0,\infty)$, càdlàg in t such that*

1. *for every $t \in [0,T)$, we have $\nu_t(dx) = p_t^x(\omega)\nu(dx)$.*
2. *for each $x \in \mathbb{R}$, the process $(p_t^x(\omega))_{t\in[0,T)}$ is a $(\mathbb{F},\mathbb{P})$-martingale.*
3. *The processes p^x, p_-^x are strictly positive on $[0,\tau^x)$ and $p^x = 0$ on $[\tau^x,T)$, where*
$$\tau^x := \inf\{t \geq 0 : p_{t-}^x = 0 \text{ or } p_t^x = 0\} \wedge T.$$
Furthermore, if we define $\tau^G(\omega) := \tau^{G(\omega)}(\omega)$ then $\mathbb{P}[\tau^G = T] = 1$.

The conditional density process p^G is the key to find the semimartingale decomposition of a $\mathbb{F}$-local martingale in the enlarged filtration $\mathbb{G}$.

Proposition 4.2.3. *Under Assumption 4.2.1, every $\mathbb{F}$-local martingale X is a $\mathbb{G}$-semimartingale on $[0,T)$ with decomposition*

$$X_t = X_t^{\mathbb{G}} + \int_0^t \frac{d\left\langle X, p_\cdot^G \right\rangle_s^{\mathbb{F}}}{p_{s-}^G} \tag{4.2}$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale.

Passing a $\mathbb{F}$-local martingale to the filtration $\mathbb{G}$ introduces a drift, i.e. a finite variation term, in its semimartingale decomposition. This extra term measures the difference of the information in the two filtrations.

Definition 4.2.4 (Information drift, Ankirchner et al. [2006])**.** *Assume that S has $\mathbb{F}$-semimartingale decomposition $S = M + \alpha \cdot \langle M, M \rangle$. The $\mathbb{G}$-predictable process μ satisfying*

$$M - \mu \cdot \langle M, M \rangle \text{ is a } \mathbb{G}\text{-local martingale}$$

is called information drift of M in the filtration $\mathbb{G}$ with respect to the filtration $\mathbb{F}$.

In order to better understand the insider's activities, we also need to study her strategies. The structure of $\mathbb{G}$-optional and $\mathbb{G}$-predictable processes is given in Proposition 2.3.1 of Jeanblanc [2010], reproduced below (this result was established in Jeulin [1980b] for discrete random variables and in Jeulin [1980a] for progressive enlargement with a stopping time).

Proposition 4.2.5. *Every $\mathbb{G}$-optional process Y is of the form $Y_t(\omega) = y_t(\omega, G(\omega))$ for some $\mathbb{F} \otimes \mathscr{B}(\mathbb{R})$-optional process $y_t(\omega, u)$.*

Every $\mathbb{G}$-predictable process Y is of the form $Y_t(\omega) = y_t(\omega, G(\omega))$ where $(t, \omega, u) \mapsto y_t(\omega, u)$ is a $\mathscr{P}(\mathbb{F}) \otimes \mathscr{B}(\mathbb{R})$-measurable function.

Finally, we make the following technical assumption.

Assumption 4.2.6. *For every x, the process p^x does not jump to zero, i.e.*

$$\mathbb{P}[\tau^x < T, p_{\tau^x -}^x > 0] = 0.$$

This assumption is introduced in Kardaras et al. [2015] for a general construction of strict local martingales, in Ruf and Runggaldier [2013] for a construction of markets with arbitrages and in Chapter 2 for the study of optimal arbitrage. Acciaio et al. [2014] also use this assumption to prove the preservation of NUPBR under the enlarged market. Basically, it requires that the conditional density p^x goes to zero continuously. The meaning of this assumption can be interpreted as follows. Let us consider an event $\{G \in dx\}$ for some $x \in \mathbb{R}$. This information is given for the insider at time zero, but not for regular agents, who can observe it at time T. However, regular agents can estimate the possibility of its occurrence, given their information. Hence, their perception is formulated by the density process $p_t^x = \mathbb{P}[G \in dx|\mathscr{F}_t]/\mathbb{P}[G \in dx]$. If the process p^x jumps to zero at time τ^x, it means that strictly before time τ^x, regular agents think that the event $\{G \in dx\}$ is possible but at time τ^x, they suddenly realize that they are totally wrong. In the view of the insider, the event $\{G \in dx\}$ is overestimated by regular agents and hence, the insider could make a profit by trading against them. Conversely, if the density p^x goes to zero continuously, regular agents realize that the probability of the event $\{G \in dx\}$ gets smaller and smaller. The insider may gain some profit by trading against regular agents, however, the insider's profit is not big (which implies no unbounded profit) because regular agents have time to correct their beliefs.

4.3 Optimal arbitrage

In Chapter 2, it is shown that optimal arbitrage is the inverse of the superhedging price of the claim 1. However, in the present chapter, we will emphasize that there is a little difference in these concepts. First, we slightly adapt the definition of superhedging price.

Definition 4.3.1. *Let $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$ and let $f \geq 0$ be a given claim. A $\mathscr{H}_0$-measurable random variable $x_*(f)$ is called the superhedging price of f with respect to $\mathbb{H}$ if* *there exists a $\mathbb{H}$-predictable strategy H such that*

$$x_*(f) + (H \cdot S)_t \geq 0, \qquad \mathbb{P}-a.s, \forall t \in [0,T], \tag{4.3}$$

$$x_*(f) + (H \cdot S)_T \geq f, \qquad \mathbb{P}-a.s. \tag{4.4}$$

and for each $x \in \mathscr{H}_0$ which satisfies

$$x + (H \cdot S)_t \geq 0, \qquad \mathbb{P}-a.s, \forall t \in [0,T],$$
$$x + (H \cdot S)_T \geq f, \qquad \mathbb{P}-a.s,$$

one has $x_*(f) \leq x, \mathbb{P}-a.s.$

In other words, the superhedging price of f is the minimal amount starting from which one can superhedge f by a nonnegative wealth process. However, the price can be a true random variable if $\mathscr{H}_0$ is nontrivial. The definition of optimal arbitrage is given as follows, in conjunction with Lemma 2.3.3 of Chapter 2.

Definition 4.3.2 (Optimal arbitrage)**.** *We say that there is optimal arbitrage in the market if $x_*(1) \leq 1$ and $\mathbb{P}[x_*(1) < 1] > 0$. If $x_*(1) < 1, \mathbb{P}-a.s.$, the optimal arbitrage is strong.*

The condition NUPBR implies that $x_*(1)$ is strictly positive, see Remark 2.3.4. The converse implication is not true. Indeed, we can find a market which satisfies NA but not NUPBR, see Section 4 of Levental and Skorohod [1995]. Because the condition NA implies that $x_*(1) > 0$, the fact $x_*(1) > 0$ does not imply that NUPBR holds.

4.4 Initial enlargement with a discrete random variable

Let us assume that G is a discrete random variable taking a finite number of values $\{g_1, ..., g_n\}$ with nonzero probability. This is a "classical" case of initial filtration enlargement, studied, e.g., in Aksamit [2014]; Jeulin [1980a,b]; Meyer [1978].

In particular, it is known that every $\mathbb{F}$-local martingale is a $\mathbb{G}$-semimartingale on $[0,T]$ with decomposition

$$X_t = X_t^{\mathbb{G}} + \sum_{i=1}^{n} 1_{G=g_i} \int_0^t \frac{d\langle X, p^{g_i}\rangle_s^{\mathbb{F}}}{p_{s-}^{g_i}},$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale. Therefore, it is not necessary to impose Jacod's condition in this section. However, the additional assumption 4.2.6 will be imposed unless stated otherwise.

When the insider is informed that G will take the value g_i where $i \in \{1,...,n\}$, she recognizes that all scenarios for the market's evolution are contained in the event $\{G = g_i\}$. Consequently, she has no reason to keep using the original belief (probability measure $\mathbb{P}$) in order to determine her investment strategy. The insider would like to update her belief by doing a measure change, i.e by dismissing all scenarios contained in $\{G = g_j\}$ for all $j \neq i$. To make this more concrete, we consider the following measure transformation

$$\left.\frac{d\mathbb{Q}^i}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{\mathbb{P}[G = g_i|\mathscr{F}_t]}{\mathbb{P}[G = g_i]} := p_t^{g_i}, \qquad i = 1,...n. \tag{4.5}$$

The measure $\mathbb{Q}^i$ gives total mass to the event $\{G = g_i\}$ and is absolutely continuous with respect to $\mathbb{P}$ but not equivalent to it. We shall use the techniques developed in Chapter 2 in this initial enlargement setting. In Section 4.4.1, we compute the expected log-utility of the insider and then study under which circumstances, the market for the insider satisfies the condition NUPBR. Next, we find superhedging prices for the insider in Section 4.4.2. In Section 4.4.3 we give an example with a complete market where all computations are carried out. Section 4.4.4 gives an incomplete market example.

4.4.1 NUPBR and log-utility maximization

In this section, we will compute the insider's expected log-utility. First, we relate the expected utility of the insider to the expected utility of regular agents when restricted to an event $\{G = g_i\}$.

Lemma 4.4.1. *The expected logarithmic utility for the insider can be represented as follows*

$$\sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] = \sum_{i=1}^{n} \sup_{H\in\mathscr{A}_1^{\mathbb{F}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H} 1_{\{G=g_i\}}]. \tag{4.6}$$

Proof. The proof is easy and only requires representation of $\mathbb{G}$-predictable processes as in Proposition 4.2.5. We do not use Assumption 4.2.6 here.
($\leq$) Let $H^{\mathbb{G}} \in \mathscr{A}^{\mathbb{G}}$ be a $\mathbb{G}$-predictable strategy. By Proposition 4.2.5, the process $H^{\mathbb{G}}$ is of the form $H_t^{\mathbb{G}}(\omega) = h_t(\omega, G(\omega))$ where $h_t(\omega,x)$ is a $\mathscr{P}(\mathbb{F}) \times \mathscr{B}(\mathbb{R})$-measurable function. Then, $H^{\mathbb{F},i} = h(\omega, g_i)$ is $\mathbb{F}$-predictable and $H^{\mathbb{G}} 1_{\{G=g_i\}} = H^{\mathbb{F},i} 1_{\{G=g_i\}}$ a.s. Hence, we have that $H^{\mathbb{G}} = \sum_{i=1}^{n} H^{\mathbb{F},i} 1_{\{G=g_i\}}$ and therefore

$$\int_0^T H_t^{\mathbb{G}} dS_t = \sum_{i=1}^{n} 1_{\{G=g_i\}} \int_0^T H_t^{\mathbb{F},i} dS_t,$$

where the equality follows from the fact that S is a $\mathbb{G}$-semimartingale.

We then compute

$$\mathbb{E}^{\mathbb{P}}[\log V_T^{1,H^{\mathbb{G}}}] = \sum_{i=1}^{n} \mathbb{E}^{\mathbb{P}} \left[1_{\{G=g_i\}} \log V_T^{1,H^{\mathbb{F},i}} \right].$$

Taking the supremum over the set of all $\mathbb{G}$-admissible strategies we obtain the inequality ($\leq$) in (4.6).
($\geq$) For all $H^{\mathbb{F},i}$, the following inequality holds true

$$\sum_{i=1}^{n} \mathbb{E}^{\mathbb{P}} \left[1_{\{G=g_i\}} \log V_T^{1,H^{\mathbb{F},i}} \right] = \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H^{\mathbb{G}}}] \leq \sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}].$$

So, the proof is complete. □

The usefulness of this lemma is that it transforms the insider's maximization problem to many problems of regular agents under measure changes as in (4.5), up to a constant. As in the framework of NFLVR, it would be interesting to relate the problem of maximizing log-utility under $\mathbb{Q}^i$ to the problem of minimizing an appropriate function of deflators by duality. However, the equivalence between $\mathbb{P}$

and $\mathbb{Q}^i$ is not preserved and it is no longer true that the condition NFLVR holds under $\mathbb{Q}^i$. The classical results, see for example Kramkov and Schachermayer [1999, 2003], cannot be applied here. This is not really bad news because as soon as Assumption 4.2.6 is valid, it can be proved that the $(\mathbb{F},\mathbb{Q}^i)$-market satisfies the condition NUPBR and the duality approach is still applicable, see Chapter 3. Nevertheless, coming back to the original measure $\mathbb{P}$ introduces an extra term, as seen in the next lemma.

Lemma 4.4.2. *Under Assumption 4.2.6,*

$$\sup_{H\in\mathscr{A}_1^{\mathbb{F}}} \mathbb{E}^{\mathbb{P}}[1_{\{G=g_i\}}\log V_T^{1,H}] = -\mathbb{P}[G=g_i]\log\mathbb{P}[G=g_i] \\ + \inf_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{G=g_i\}}\log\frac{1}{Z_T}\right], \quad i\in\{1,\dots n\}.$$

Proof. For each $i\in\{1,\dots,n\}$, the $(\mathbb{F},\mathbb{Q}^i)$-market is obtained from the $(\mathbb{F},\mathbb{P})$-market by an absolutely continuous measure change, see Equation (4.5). Furthermore, the density process p^{g_i} does not jump to zero, by Assumption 4.2.6. Theorem 2.4.1 in Chapter 2 shows that the $(\mathbb{F},\mathbb{Q}^i)$-market satisfies the condition NUPBR and for any local martingale density $Z\in ELMM(\mathbb{F},\mathbb{P})$, the process Z/p^{g_i} is a local martingale deflator for the $(\mathbb{F},\mathbb{Q}^i)$-market (note that on $\{G=g_i\}$, $p_t^{g_i}>0$ for all t). Let us introduce the following subsets of L^0_+

$$\mathscr{C}(x) := \{v\in L^0_+ : 0\le v\le xV_T^{1,H^{\mathbb{F}}},\ \text{for some } H^{\mathbb{F}}\in\mathscr{A}_1\},$$
$$\mathscr{D}(y) := \left\{z\in L^0_+ : 0\le z\le yZ_T,\ \text{for some } Z\in ELMM(\mathbb{F},\mathbb{P})\right\},$$
$$\mathscr{D}^i(y) := \left\{z^i=\frac{z}{p_T^{g_i}}, z\in\mathscr{D}(y)\right\}.$$

Because the $(\mathbb{F},\mathbb{P})$-market satisfies NFLVR, Proposition 3.1 of Kramkov and Schachermayer [1999] implies that $\mathscr{C}$ and $\mathscr{D}$ are convex with the following properties

$$v\in\mathscr{C}(1) \iff \mathbb{E}^{\mathbb{P}}[vz]\le 1,\ \text{for all } z\in\mathscr{D}(1),$$
$$z\in\mathscr{D}(1) \iff \mathbb{E}^{\mathbb{P}}[vz]\le 1,\ \text{for all } v\in\mathscr{C}(1).$$

These imply that

$$v \in \mathscr{C}(1) \iff \mathbb{E}^{\mathbb{Q}^i}[vz^i] \le 1, \text{ for all } z^i \in \mathscr{D}^i(1),$$
$$z^i \in \mathscr{D}^i(1) \iff \mathbb{E}^{\mathbb{Q}^i}[vz^i] \le 1, \text{ for all } v \in \mathscr{C}(1)$$

and thus the condition (3.1) of Mostovyi [2015] holds under the measure $\mathbb{Q}^i$. In addition, $\mathscr{C}$ and $\mathscr{D}^i$ contain at least one strictly positive element. For all $y > 0$, the finiteness of the dual optimization

$$\inf_{z^i \in \mathscr{D}^i(y)} \mathbb{E}^{\mathbb{Q}^i}\left[\log \frac{1}{z^i}\right]$$

is deduced from $\inf_{Z \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}[\log(1/Z_T)] < \infty$. Furthermore, for all $x > 0$, we have that $\sup_{v \in \mathscr{C}(x)} \mathbb{E}^{\mathbb{Q}^i}[\log v] > -\infty$. An application of Theorem 3.2 of Mostovyi [2015] shows that

$$\sup_{H \in \mathscr{A}_1^{\mathbb{F}}} \mathbb{E}^{\mathbb{Q}^i}[\log V_T^{1,H}] = \inf_{Z \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{Q}^i}\left[\log \frac{p_T^{g_i}}{Z_T}\right]$$

and both sides of the equality are attainable. Hence,

$$\begin{aligned}
\sup_{H \in \mathscr{A}_1^{\mathbb{F}}} \mathbb{E}^{\mathbb{P}}[1_{\{G=g_i\}} \log V_T^{1,H}] &= \inf_{Z \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{G=g_i\}} \log \frac{p_T^{g_i}}{Z_T}\right] \\
&= \inf_{Z \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{G=g_i\}} \log p_T^{g_i} + 1_{\{G=g_i\}} \log \frac{1}{Z_T}\right] \\
&= -\mathbb{P}[G=g_i] \log \mathbb{P}[G=g_i] + \inf_{Z \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{G=g_i\}} \log \frac{1}{Z_T}\right].
\end{aligned}$$

The proof is complete. □

Remark 4.4.3. *Because the $(\mathbb{F},\mathbb{Q}^i)$-market satisfies the condition NUPBR, one can use the results in Chapter 3 for the log-utility optimization problem under $(\mathbb{F},\mathbb{Q}^i)$, i.e. the set of local martingale deflators of the $(\mathbb{F},\mathbb{Q}^i)$-market will be concerned. However, we would like to represent the insider's value function in terms of ordinary agents' martingale densities. So we use the abstract results of Mostovyi [2015].*

Lemma 4.4.1 and Lemma 4.4.2 lead to a new characterization of the expected log-utility of the insider in terms of the additional information G and the set of all local martingale densities of the $(\mathbb{F},\mathbb{P})$-market.

Theorem 4.4.4. *Under Assumption (4.2.6),*

$$\sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] = -\sum_{i=1}^{n} \mathbb{P}[G=g_i]\log\mathbb{P}[G=g_i], \\ +\sum_{i=1}^{n} \inf_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{G=g_i\}}\log\frac{1}{Z_T}\right]. \tag{4.7}$$

It is remarked that whenever each optimization problem under $\mathbb{Q}^i$ is attainable then the optimization for the insider is also attainable. This theorem is useful for several reasons. First, it shows how to compute the insider's profit. Let us consider the two components in the RHS of (4.7). The first component, which is the entropy of G, always contributes to the profit of the insider and does not depend on the structure of the $(\mathbb{F},\mathbb{P})$-market. The second component can be interpreted as the value of G with respect to the $(\mathbb{F},\mathbb{P})$-market. The expected log-utility for the insider is finite if each component is finite, or if the second component compensates the first component. As a result, this theorem provides a tool to check NUPBR under $\mathbb{G}$.

Let us compare our results with the results in Amendinger et al. [1998]. The additional expected log-utility of the insider is denoted by

$$\Delta(\mathbb{F},\mathbb{G}) := \sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] - \sup_{H\in\mathscr{A}_1^{\mathbb{F}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}].$$

In their approach, the quantity $\Delta(\mathbb{F},\mathbb{G})$ is represented by the information drift, see Definition 3.6 in their paper, and in our approach, it can be expressed as

$$-\sum_{i=1}^{n} \mathbb{P}[G=g_i]\log\mathbb{P}[G=g_i] + \sum_{i=1}^{n} \inf_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{G=g_i\}}\log\frac{1}{Z_T}\right] \\ - \inf_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[\log\frac{1}{Z_T}\right]. \tag{4.8}$$

If the market is complete, the two approaches end up with the same result: the quantity in (4.8) reduces to the entropy of G, which is exactly what is stated in Theorem 4.1 of Amendinger et al. [1998].

4.4.2 Superhedging and optimal arbitrage

In this section, we turn to the problem of superhedging for the insider. Under $\mathbb{G}$, we note that the initial capital is a $\mathscr{G}_0$-measurable random variable, which means that we do not start with the same capital for all scenarios since $\mathscr{G}_0$ is not a trivial sigma algebra. However, it is not surprising that the capital is constant on each event $\{G = g_i\}$.

Theorem 4.4.5. *Under Assumption 4.2.6, the* $(\mathbb{G}, \mathbb{P})$*-market satisfies NUPBR and the superhedging price of a claim* f *in this market is given by*

$$x_*^{\mathbb{G},\mathbb{P}}(f) = \sum_{i=1}^{n} x_*^{\mathbb{F},\mathbb{P}}(f 1_{\{G=g_i\}}) 1_{\{G=g_i\}},$$

and the associated hedging strategy is $H^{\mathbb{F},i} 1_{\{G=g_i\}}$*, where* $H^{\mathbb{F},i}$ *is the suphedging strategy for* $f 1_{\{G=g_i\}}$ *in the* $(\mathbb{F}, \mathbb{P})$*-market, i.e.*

$$x_*^{\mathbb{F},\mathbb{P}}(f 1_{\{G=g_i\}}) 1_{\{G=g_i\}} + (H^{\mathbb{F},i} 1_{\{G=g_i\}} \cdot S)_T \geq f 1_{\{G=g_i\}}, \mathbb{P} - a.s.$$

Proof. The first statement is proved by contradiction, noticing that NUPBR is equivalent to NA1. Assume that there is an arbitrage of the first kind in the $(\mathbb{G}, \mathbb{P})$-market, i.e. we can find a $\mathscr{F}_T$-random variable ξ (because $\mathscr{F}_T = \mathscr{G}_T$) such that $\mathbb{P}[\xi \geq 0] = 1, \mathbb{P}[\xi > 0] > 0$ and for all $\varepsilon > 0$, there exists a $\mathbb{G}$-predictable strategy $H^{\mathbb{G},\varepsilon}$ which satisfies

$$\varepsilon + (H^{\mathbb{G},\varepsilon} \cdot S)_T \geq \xi, \mathbb{P} - a.s. \tag{4.9}$$

Choose an index i such that $\mathbb{P}[\{\xi > 0\} \cap \{G = g_i\}] > 0$. The inequality (4.9) still holds true under $\mathbb{Q}^i$

$$\varepsilon + (H^{\mathbb{G},\varepsilon} 1_{\{G=g_i\}} \cdot S)_T \geq \xi, \mathbb{Q}^i - a.s. \tag{4.10}$$

Let us look at the hedging strategy $H^{\mathbb{G},\varepsilon}1_{\{G=g_i\}}$ under $\mathbb{Q}^i$. Using the argument as in the proof of Lemma 4.4.1, we have that $H^{\mathbb{G},\varepsilon}1_{\{G=g_i\}} = \tilde{H}^{\mathbb{F},i,\varepsilon}1_{\{G=g_i\}}$, where $\tilde{H}^{\mathbb{F},i,\varepsilon}$ is a $\mathbb{F}$-predictable strategy. Thus, (4.10) implies that ξ is an arbitrage of the first kind in the $(\mathbb{F},\mathbb{Q}^i)$-market, which is equivalent to the failure of NUPBR. However, by Theorem 2.4.1 of Chapter 2, the condition NUPBR holds for the $(\mathbb{F},\mathbb{Q}^i)$-market, see the argument in the proof of Lemma 4.4.2. This contradiction means that the first statement is proved.

For the second statement, we compute the superhedging prices of f under the new measures $\mathbb{Q}^i, i = 1..,n$, again by using Theorem 2.4.1 of Chapter 2

$$x_*^{\mathbb{F},\mathbb{Q}^i}(f) = x_*^{\mathbb{F},\mathbb{P}}(f1_{\{G=g_i\}}).$$

For each i, we denote by $H^{\mathbb{F},i}$ the $\mathbb{F}$- strategy which superhedges f under the $(\mathbb{F},\mathbb{Q}^i)$-market, that is

$$x_*^{\mathbb{F},\mathbb{Q}^i}(f) + (H^{\mathbb{F},i}\cdot S)_T \geq f, \mathbb{Q}^i - a.s.$$

This inequality holds also under $\mathbb{P}$ when restricted on $\{G = g_i\}$

$$x_*^{\mathbb{F},\mathbb{P}}(f1_{\{G=g_i\}})1_{\{G=g_i\}} + (H^{\mathbb{F},i}1_{\{G=g_i\}}\cdot S)_T \geq f1_{\{G=g_i\}}, \mathbb{P} - a.s.$$

Summing up these inequalities we obtain

$$\left(\sum_i x_*^{\mathbb{F},\mathbb{P}}(f1_{\{G=g_i\}})1_{\{G=g_i\}}\right) + \left(\sum_i H^{\mathbb{F},i}1_{\{G=g_i\}}\right)\cdot S_T \geq f, \mathbb{P} - a.s.$$

The hedging strategy $\left(\sum_i H^{\mathbb{F},i}1_{\{G=g_i\}}\right)$ is $\mathbb{G}$-predictable.

Finally, we prove that the initial capital $\sum_i x_*^{\mathbb{F},\mathbb{P}}(f1_{\{G=g_i\}})1_{\{G=g_i\}}$ is exactly the superhedging price of f in the $(\mathbb{G},\mathbb{P})$-market. Assume that y is a $\mathscr{G}_0$-measurable random variable such that

$$y + (H^{\mathbb{G}}\cdot S)_T \geq f, \mathbb{P} - a.s.$$

where $H^{\mathbb{G}}$ is a $\mathbb{G}$-predictable strategy. Hence,

$$y1_{\{G=g_i\}} + (H^{\mathbb{G}}1_{\{G=g_i\}}\cdot S)_T \geq f1_{\{G=g_i\}}, \mathbb{P} - a.s.$$

and because $\mathbb{Q}^i \ll \mathbb{P}$ with $\mathbb{Q}^i[G = g_i] = 1$, we obtain

$$y + (H^{\mathbb{G}} 1_{\{G=g_i\}} \cdot S)_T \geq f, \mathbb{Q}^i - a.s.$$

By using the same argument as in the proof of the first statement, we can replace $H^{\mathbb{G}}$ by a $\mathbb{F}$-predictable strategy $\tilde{H}^{\mathbb{F},i}$ and then

$$y + (\tilde{H}^{\mathbb{F},i} \cdot S)_T \geq f, \mathbb{Q}^i - a.s.$$

By definition, the superhedging price of f under $\mathbb{Q}^i$ is smaller than y. We conclude that $\sum_i x_*^{\mathbb{F},\mathbb{P}}(f 1_{\{G=g_i\}}) 1_{\{G=g_i\}} \leq y, \mathbb{P} - a.s.$. □

It could also be useful to stress that, while Theorem 4.4.5 guarantees that under Assumption 4.2.6, the $(\mathbb{G},\mathbb{P})$-market satisfies NUPBR, it does not exclude that this market also satisfies NFLVR. However, if the insider could exploit arbitrage opportunities, then her riskless profits are deduced immediately from Theorem 4.4.5.

Corollary 4.4.6. *If there exists an index i such that $x_*^{\mathbb{F},\mathbb{P}}(1_{\{G=g_i\}}) < 1$, then the insider has an optimal arbitrage on the event $\{G = g_i\}$. If $x_*^{\mathbb{G},\mathbb{P}}(1) < 1, \mathbb{P} - a.s.$ then the insider has strong optimal arbitrage.*

4.4.3 A complete market example

Let us assume that the risky asset is a geometric Brownian motion

$$dS_t = S_t dW_t, \qquad t \in [0,1], \qquad S_0 = 1$$

and the public information $\mathbb{F}$ is the natural filtration generated by the Brownian motion W. The insider knows at time $t = 0$ whether W_1 will be above or below a real number c. In mathematical terminology, the additional information of the insider is given by the discrete random variable $G = 1_{[c,\infty)}(W_1)$. This example is considered in Pikovsky and Karatzas [1996] and we further study it by applying our results.

We first check Jacod's condition. Because G takes only two possible values, it is easy to see that the conditional laws of G given $\mathscr{F}_t$ are given by

$$p_t^1 = \frac{\mathbb{P}[G=1|\mathscr{F}_t]}{\mathbb{P}[G=1]}, \qquad p_t^0 = \frac{\mathbb{P}[G=0|\mathscr{F}_t]}{\mathbb{P}[G=0]}.$$

Let us denote by $\Phi(u) = \int\limits_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$ the standard normal distribution function. The densities p^1, p^0 can be computed explicitly for all $t \in [0,1)$

$$p_t^1 = \frac{1}{1-\Phi(c)} \frac{1}{\sqrt{2\pi(1-t)}} \int\limits_{c}^{\infty} \exp\left(-\frac{(u-W_t)^2}{2(1-t)}\right) du = \frac{1}{1-\Phi(c)} \Phi\left(\frac{W_t - c}{\sqrt{1-t}}\right), \tag{4.11}$$

$$p_t^0 = \frac{1}{\Phi(c)} \frac{1}{\sqrt{2\pi(1-t)}} \int\limits_{-\infty}^{c} \exp\left(-\frac{(u-W_t)^2}{2(1-t)}\right) du = \frac{1}{\Phi(c)} \Phi\left(\frac{c - W_t}{\sqrt{1-t}}\right) \tag{4.12}$$

At time 1, we have

$$p_1^1 = \frac{1_{\{W_1 \geq c\}}}{\mathbb{P}[W_1 \geq c]}, \qquad p_1^0 = \frac{1_{\{W_1 < c\}}}{\mathbb{P}[W_1 < c]}.$$

Here, the conditional law of G is absolutely continuous w.r.t the law of G for $t \in [0,1]$ and equivalent to the law of G only for $t \in [0,1)$. Applying Itô's formula to (4.11) and (4.12) gives the dynamics of p^1 and p^0

$$dp_t^1 = \frac{1}{1-\Phi(c)} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{(c-W_t)^2}{2(1-t)}} dW_t, \tag{4.13}$$

$$dp_t^0 = -\frac{1}{\Phi(c)} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{(c-W_t)^2}{2(1-t)}} dW_t. \tag{4.14}$$

Let us denote

$$\alpha_t = \begin{cases} \frac{1}{\Phi\left(\frac{W_t - c}{\sqrt{(1-t)}}\right)} \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(c-W_t)^2}{2(1-t)}\right), & W_1 \geq c, \\ -\frac{1}{\Phi\left(\frac{c - W_t}{\sqrt{(1-t)}}\right)} \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(c-W_t)^2}{2(1-t)}\right), & W_1 < c. \end{cases} \tag{4.15}$$

The process α is taken from the martingale representation of p^G, that is $dp_t^G = p_t^G \alpha_t dW_t$. Proposition 4.2.3 implies that α is the information drift of $\mathbb{G}$ with respect to $\mathbb{F}$,

$$\frac{d\left\langle p^G, W\right\rangle_t}{p_t^G} = \alpha_t dt.$$

Assumption 4.2.6 is satisfied because p^1, p^0 are continuous. Hence, by Theorem 4.4.5, the insider's market satisfies the NUPBR condition.

We now compute the expected log-utility of the insider. Because the $(\mathbb{F},\mathbb{P})$-market is complete with the unique martingale density $Z \equiv 1$, this quantity may be deduced from Theorem 4.4.4,

$$\begin{aligned}\sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_1^{1,H}] &= -\mathbb{P}[W_1 \geq c]\log\mathbb{P}[W_1 \geq c] - \mathbb{P}[W_1 < c]\log\mathbb{P}[W_1 < c] \\ &\quad + \mathbb{E}^{\mathbb{P}}\left[1_{\{W_1\geq c\}}\log\frac{1}{Z_1}\right] + \mathbb{E}^{\mathbb{P}}\left[1_{\{W_1< c\}}\log\frac{1}{Z_1}\right] \\ &= -\mathbb{P}[W_1 \geq c]\log\mathbb{P}[W_1 \geq c] - \mathbb{P}[W_1 < c]\log\mathbb{P}[W_1 < c].\end{aligned}$$

Next, by applying Theorem 4.4.5 and the standard superreplication theorem (see Theorem 5.12 of Delbaen and Schachermayer [1998]), we compute the superhedging price of 1 under $\mathbb{G}$ as follows

$$\begin{aligned}x_*^{\mathbb{G},\mathbb{P}}(1) &= 1_{W_1\geq c} x_*^{\mathbb{F},\mathbb{P}}(1_{\{W_1\geq c\}}) + 1_{\{W_1<c\}} x_*^{\mathbb{F},\mathbb{P}}(1_{W_1<c}) \\ &= 1_{W_1\geq c}\mathbb{P}[W_1 \geq c] + 1_{W_1<c}\mathbb{P}[1_{W_1<c}].\end{aligned}$$

Now, we turn to hedging strategies. The wealth processes that replicate the two claims $1_{\{W_1\geq c\}}$ and $1_{\{W_1<c\}}$ in the $(\mathbb{F},\mathbb{P})$-market are respectively given by

$$\begin{aligned}\mathbb{E}[1_{W_1\geq c}|\mathscr{F}_t] &= \mathbb{P}[W_1 \geq c]\mathbb{E}\left[p_1^1|\mathscr{F}_t\right] = \mathbb{P}[W_1 \geq c]p_t^1, \\ \mathbb{E}[1_{W_1<c}|\mathscr{F}_t] &= \mathbb{P}[W_1 < c]\mathbb{E}\left[p_1^0|\mathscr{F}_t\right] = \mathbb{P}[W_1 < c]p_t^0.\end{aligned}$$

From (4.13) and (4.14), it is deduced that

$$1_{\{W_1\geq c\}}p_t^1 = 1_{\{W_1\geq c\}}\left(1 + \int_0^t p_u^1\alpha_u dW_u\right), \qquad 1_{\{W_1<c\}}p_t^0 = 1_{\{W_1<c\}}\left(1 + \int_0^t p_u^0\alpha_u dW_u\right).$$

Thus, the insider needs to follow the wealth processes p^1 or p^0 to exploit optimal arbitrage,

$$1_{\{W_1 \geq c\}}\mathbb{P}[W_1 \geq c]p_t^1 + \mathbb{P}[W_1 < c]1_{\{W_1<c\}}p_t^0 = 1_{\{W_1\geq c\}}\left(\mathbb{P}[W_1 \geq c] + \mathbb{P}[W_1 \geq c]\int_0^t \frac{p_u^1\alpha_u}{S_u}dS_u\right) + 1_{\{W_1<c\}}\left(\mathbb{P}[W_1 < c] + \mathbb{P}[W_1 < c]\int_0^t \frac{p_u^0\alpha_u}{S_u}dS_u\right). \tag{4.16}$$

Remark 4.4.7. *We cannot exploit arbitrage before 1. The process $1/p^G$ is a $\mathbb{G}$-strict local martingale on $[0,1]$ and a $\mathbb{G}$-martingale on $[0,1)$. For a fixed $t \in [0,1)$, we can define an ELMM $\mathbb{Q}$ such that*

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{G}_t} = \frac{1}{p_t^G},$$

and so the NFLVR condition holds before time 1.

Remark 4.4.8 (PDE characterization)**.** *The approach with PDE is still relevant here. In this example, the dynamics of S under $\mathbb{G}$ are $dS_t = S_t(dW_t^{\mathbb{G}} + \alpha_t dt)$. Theorem 4.1 of Ruf [2013] suggests that the pricing equation for the claim* 1 *is*

$$\begin{aligned}\frac{\partial h}{\partial t} + \frac{1}{2}x^2\frac{\partial^2 h}{\partial x^2} &= 0\\ h(1,x) &= 1.\end{aligned} \tag{4.17}$$

From (4.11) and (4.16) we observe that on the set $\{W_1 \geq c\}$, the portfolio process corresponding to the optimal arbitrage is

$$\frac{p_t^1}{p_1^1} = \Phi\left(\frac{\ln S_t + \frac{1}{2}t - c}{\sqrt{1-t}}\right).$$

Now, we check whether the function $g(t,x) := \Phi\left(\frac{\ln x + \frac{1}{2}t - c}{\sqrt{1-t}}\right)$ satisfies (4.17). Straight-

forward calculations show

$$\frac{\partial g}{\partial t} = \Phi'\left(\frac{\ln x + \frac{1}{2}t - c}{\sqrt{1-t}}\right)\left[\frac{1}{2\sqrt{1-t}} + \frac{1}{2}\frac{\ln x + \frac{1}{2}t - c}{(1-t)^{3/2}}\right],$$

$$\frac{\partial g}{\partial x} = \Phi'\left(\frac{\ln x + \frac{1}{2}t - c}{\sqrt{1-t}}\right)\frac{1}{x\sqrt{1-t}},$$

$$\frac{\partial^2 g}{\partial x^2} = \Phi''\left(\frac{\ln x + \frac{1}{2}t - c}{\sqrt{1-t}}\right)\frac{1}{x^2(1-t)} - \Phi'\left(\frac{\ln x + \frac{1}{2}t - c}{\sqrt{1-t}}\right)\frac{1}{x^2\sqrt{1-t}},$$

$$= -\Phi'\left(\frac{\ln x + \frac{1}{2}t - c}{\sqrt{1-t}}\right)\frac{\ln x + \frac{1}{2}t - c}{\sqrt{1-t}}\frac{1}{x^2(1-t)} - \Phi'\left(\frac{\ln x + \frac{1}{2}t - c}{\sqrt{1-t}}\right)\frac{1}{x^2\sqrt{1-t}}.$$

It is then easy to see that the function g satisfies (4.17). The case $\{W_1 < c\}$ is computed similarly. We note that the drift does not fulfill the condtion (A1) of Ruf [2013] at time $T = 1$ and so we cannot apply Theorem 4.7 therein.

Remark 4.4.9 (Multiplicity of solutions)**.** *The equation (4.17) has a trivial solution $h(t,x) = 1$. Intuitively, two distinct solutions which exactly replicate a payoff at different costs will generate an arbitrage. Hence, the multiplicity of solutions is easily observed if there exist relative arbitrages or classical arbitrages. In asset pricing theory with bubbles, the multiplicity of solutions is discussed in Cox and Hobson [2005], Heston et al. [2007], Ekström and Tysk [2009] and others.*

4.4.4 An incomplete market example

Suppose that N^1 and N^2 are two independent standard Poisson processes. We consider a financial market with the risky asset $S_t = e^{N_t^1 - N_t^2}$ whose dynamics is given by

$$dS_t = S_{t-}\left((e-1)dN_t^1 + (e^{-1}-1)dN_t^2\right), \qquad S_0 = 1, \qquad t \in [0,T].$$

The public information $\mathbb{F}$ is generated by the two Poisson processes N^1, N^2. The $(\mathbb{F},\mathbb{P})$-market satisfies the NFLVR condition, and any martingale density Z is of the form

$$dZ_t = Z_{t-}\left((\alpha_t^1 - 1)(dN_t^1 - dt) + (\alpha_t^2 - 1)(dN_t^2 - dt)\right), \tag{4.18}$$

where α^1, α^2 are positive and $\alpha^1_t = e^{-1}\alpha^2_t$ in order for ZS to be a local martingale. Let us define $N_t := N^1_t - N^2_t$ and assume that the insider possesses the knowledge of N_T, and hence S_T, at the beginning of trading. The insider's filtration is $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(N_T) = \mathscr{F}_t \vee \sigma(S_T)$. Because the random variable N_T takes values in $\mathbb{Z}$, Jacod's condition is satisfied. An easy computation shows that for all $t \in [0,T)$,

$$p^x_t = \frac{\mathbb{P}[N_T = x|\mathscr{F}_t]}{\mathbb{P}[N_T = x]} = \frac{\sum_{k\geq 0} e^{-(T-t)}\frac{(T-t)^k}{k!} e^{-(T-t)}\frac{(T-t)^{k+x-N_t}}{(k+x-N_t)!} 1_{k+x-N_t\geq 0}}{\sum_{k\geq 0} e^{-T}\frac{T^k}{k!} e^{-T}\frac{T^{x+k}}{(x+k)!}} > 0$$

and for $t = T$

$$p^x_T = \frac{1_{\{N_T = x\}}}{\sum_{k\geq 0} e^{-T}\frac{T^k}{k!} e^{-T}\frac{T^{x+k}}{(x+k)!}}.$$

The filtration $\mathbb{F}$ is quasi-left continuous, which means that the density p^x does not jump to zero at the predictable stopping time T and thus Assumption (4.2.6) is fulfilled.

NUPBR and the expected log-utility. Theorem 4.4.5 allows us to conclude that the $(\mathbb{G},\mathbb{P})$-market satisfies NUPBR and the expected log-utility of the insider is

$$\sup_{H\in\mathscr{A}^{\mathbb{G}}_1} \mathbb{E}^{\mathbb{P}}[\log V^{1,H}_T] = -\sum_{x\in\mathbb{Z}} \mathbb{P}[N_T = x]\log\mathbb{P}[N_T = x] - \sum_{x\in\mathbb{Z}} \sup_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{N_T=x\}}\log Z_T\right].$$

Because the first term of the RHS in the above equation is explicit, we need to compute the second term, i.e. $\sup_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{N_T=x\}}\log Z_T\right]$ for each $x \in \mathbb{Z}$. Using the general formula of Z in (4.18) and taking the conditional expectation with respect to $\mathscr{G}_0$, we have that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[1_{\{N_T=x\}}\log Z_T] &= \mathbb{E}^{\mathbb{P}}\left[1_{\{N_T=x\}}\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{2}\int_0^T \log\alpha^i_t dN^i_t - (\alpha^i_t - 1)dt \,\middle|\, \mathscr{G}_0\right]\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[1_{\{N_T=x\}}\sum_{i=1}^{2}\int_0^T \left(\lambda^{\mathbb{G},i}_t \log\alpha^i_t - (\alpha^i_t - 1)\right)dt\right], \quad (4.19)\end{aligned}$$

where $\lambda^{\mathbb{G},1}, \lambda^{\mathbb{G},2}$ are intensities of N^1, N^2 under $\mathbb{G}$, respectively. In order to maximize this term, we substitute $\alpha_t^2 = e\alpha_t^1$ and differentiate with respect to α^1 to find the equation for optimal candidate

$$\frac{1}{\alpha_t^1}(\lambda_t^{\mathbb{G},1} + \lambda_t^{\mathbb{G},2}) = e + 1.$$

This equation gives us a solution $\alpha_t^1 = \frac{\lambda_t^{\mathbb{G},1} + \lambda_t^{\mathbb{G},2}}{e+1}$. Plugging the solution into the expectation (4.19), we obtain

$$\begin{aligned} &\sup_{Z \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{\{N_T = x\}} \log Z_T\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[1_{\{N_T=x\}} \int_0^T \left(\log\left(\frac{\lambda_t^{\mathbb{G},1} + \lambda_t^{\mathbb{G},2}}{e+1}\right)(\lambda_t^{\mathbb{G},1} + \lambda_t^{\mathbb{G},2}) - \lambda_t^{\mathbb{G},1} + 2\right) dt\right]. \end{aligned} \tag{4.20}$$

Now we need to compute the intensities $\lambda^{\mathbb{G},1}$ and $\lambda^{\mathbb{G},2}$ explicitly. To do this, we introduce a further larger filtration $\mathscr{H}_t = \mathscr{F}_t \vee \sigma(N_T^1, N_T^2)$. Under $\mathbb{H}$, we obtain that

$$dN_t^1 - \frac{N_T^1 - N_t^1}{T-t}dt, \qquad dN_t^2 - \frac{N_T^2 - N_t^2}{T-t}dt \tag{4.21}$$

are martingales under $\mathbb{H}$, see Theorem 3, page 356 of Protter [2003]. Now, Lemma 5.0.14 implies that the processes

$$dN_t^1 - \mathbb{E}\left[\left.\frac{N_T^1 - N_t^1}{T-t}\right| \mathscr{G}_t\right] dt, \qquad dN_t^2 - \mathbb{E}\left[\left.\frac{N_T^2 - N_t^2}{T-t}\right| \mathscr{G}_t\right] dt \tag{4.22}$$

are martingales under $\mathbb{G}$. This implies that

$$\lambda_t^{\mathbb{G},1} = \mathbb{E}\left[\left.\frac{N_T^1 - N_t^1}{T-t}\right| \mathscr{G}_t\right] \qquad = \frac{1}{T-t}\mathbb{E}[N_{T-t}^1 | N_{T-t}^1 - N_{T-t}^2].$$

On the event $\{N_{T-t}^1 - N_{T-t}^2 = y\}$, the random variable $\lambda^{\mathbb{G},1}$ becomes

$$\lambda_t^{\mathbb{G},1} = \frac{1}{T-t}\mathbb{E}[N_{T-t}^1 | N_{T-t}^1 - N_{T-t}^2 = y] = \frac{1}{T-t}\frac{\mathbb{E}[N_{T-t}^1 1_{\{N_{T-t}^1 - N_{T-t}^2 = y\}}]}{\mathbb{P}[N_{T-t}^1 - N_{T-t}^2 = y]}.$$

The computation can be done explicitly. For example, if $y > 0$, we compute

$$\begin{aligned}\frac{\mathbb{E}[N^1_{T-t}1_{\{N^1_{T-t}-N^2_{T-t}=y\}}]}{\mathbb{P}[N^1_{T-t}-N^2_{T-t}=y]} &= \frac{\sum_{k\geq 0}(y+k)\mathbb{P}[N^2_{T-t}=k]\mathbb{P}[N^1_{T-t}=y+k]}{\sum_{k\geq 0}\mathbb{P}[N^2_{T-t}=k]\mathbb{P}[N^1_{T-t}=y+k]} \\ &= \frac{\sum_{k\geq 0}(y+k)\frac{(T-t)^{2k+y}}{k!(y+k)!}}{\sum_{k\geq 0}\frac{(T-t)^{2k+y}}{k!(k+y)!}} = \frac{(T-t)I_{y-1}(2(T-t))}{I_y(2(T-t))},\end{aligned}$$

where $I_\alpha(x)$ is the modified Bessel functions of the first kind [1]. If $y = 0$,

$$\begin{aligned}\frac{\mathbb{E}[N^1_{T-t}1_{\{N^1_{T-t}-N^2_{T-t}=0\}}]}{\mathbb{P}[N^1_{T-t}-N^2_{T-t}=0]} &= \frac{\sum_{k\geq 0}k\mathbb{P}[N^2_{T-t}=k]\mathbb{P}[N^1_{T-t}=k]}{\sum_{k\geq 0}\mathbb{P}[N^2_{T-t}=k]\mathbb{P}[N^1_{T-t}=k]} \\ &= \frac{(T-t)^2\sum_{k\geq 0}\frac{(T-t)^{2k}}{k!(k+1)!}}{\sum_{k\geq 0}\frac{(T-t)^{2k}}{(k!)^2}} = \frac{(T-t)I_0(2(T-t))}{I_0(2(T_t))}.\end{aligned}$$

The case $y < 0$ is treated similarly. Finally, we have obtained explicit formulas for the intensities $\lambda^{\mathbb{G},1}, \lambda^{\mathbb{G},2}$ and then the expectation in (4.20) can be computed by numerical integration.

We remark that this argument only gives us the expected log-utility of the insider. In the following, we study the optimal strategy by a different approach.

A direct computation of the expected log-utility. Let $\pi^{\mathbb{G}}$ be a $\mathbb{G}$-predictable strategy and denote by $V^{1,\pi^{\mathbb{G}}}$ the corresponding wealth process whose dynamics are

$$\frac{dV_t^{1,\pi^{\mathbb{G}}}}{V_{t-}^{1,\pi^{\mathbb{G}}}} = \pi_t^{\mathbb{G}}\frac{dS_t}{S_{t-}} = \pi_t^{\mathbb{G}}\left((e-1)dN_t^1 + (e^{-1}-1)dN_t^2\right).$$

The logarithm of $V^{1,\pi^{\mathbb{G}}}$ satisfies

$$\begin{aligned}d\log V_t^{1,\pi^{\mathbb{G}}} &= \log\left(1+(e-1)\pi_t^{\mathbb{G}}\right)dN_t^1 + \log\left(1+(e^{-1}-1)\pi_t^{\mathbb{G}}\right)dN_t^2 \\ &= \left(\log\left(1+(e-1)\pi_t^{\mathbb{G}}\right)\lambda_t^{\mathbb{G},1} + \log\left(1+(e^{-1}-1)\pi_t^{\mathbb{G}}\right)\lambda_t^{\mathbb{G},2}\right)dt + \text{martingale parts.}\end{aligned}$$

[1] The modified Bessel functions of the first kind is given by the series representation $I_\alpha(x) = \sum_{m\geq 0}\frac{1}{m!(m+\alpha)!}\left(\frac{x}{2}\right)^{2m+\alpha}$, for a real number α.

Taking the expectation of both sides, we obtain that

$$\mathbb{E}^{\mathbb{P}}\left[\log V_T^{1,\pi^{\mathbb{G}}}\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left(\log\left(1+(e-1)\pi_t^{\mathbb{G}}\right)\lambda_t^{\mathbb{G},1} + \log\left(1+(e^{-1}-1)\pi_t^{\mathbb{G}}\right)\lambda_t^{\mathbb{G},2}\right)dt\right].$$

Differentiating with respect to $\pi^{\mathbb{G}}$ in the above formula, we get the equation

$$\frac{1}{1+(e-1)\pi_t^{\mathbb{G}}}(e-1)\lambda_t^{\mathbb{G},1} + \frac{1}{1+(e^{-1}-1)\pi_t^{\mathbb{G}}}(e^{-1}-1)\lambda_t^{\mathbb{G},2} = 0$$

Solving this equation, we obtain

$$\pi_t^{\mathbb{G}} = \frac{\lambda_t^{\mathbb{G},1}(e-1)+\lambda_t^{\mathbb{G},2}(e^{-1}-1)}{(e-1)(1-e^{-1})(\lambda_t^{\mathbb{G},1}+\lambda_t^{\mathbb{G},2})}$$

and the solution also satisfies the constraint $-\frac{1}{e-1} < \pi_t^{\mathbb{G}} < \frac{1}{1-e^{-1}}$ for the positivity of the wealth process. The expected log-utility for the insider is now computed by using the formulas of $\lambda^{\mathbb{G},1}$ and $\lambda^{\mathbb{G},2}$ as above.

Optimal arbitrage. By Theorem 4.4.5, the superhedging price of 1 under $\mathbb{G}$ is

$$x_*^{\mathbb{G},\mathbb{P}}(1) = \sum_{x\in\mathbb{Z}} x_*^{\mathbb{F},\mathbb{P}}(1_{\{N_T=x\}})1_{\{N_T=x\}}.$$

Now, we need to compute the quantity $x_*^{\mathbb{F},\mathbb{P}}(1_{\{N_T=x\}}) = \sup_{\overline{\mathbb{P}}\in ELMM(\mathbb{F},\mathbb{P})}\overline{\mathbb{P}}[N_T=x]$ for every $x\in\mathbb{Z}$ in order to find optimal arbitrage.

Proposition 4.4.10. *If $x\le 0$, we have*

$$x_*^{\mathbb{F},\mathbb{P}}(1_{\{N_T=x\}}) = \sup_{\overline{\mathbb{P}}\in ELMM(\mathbb{F},\mathbb{P})}\overline{\mathbb{P}}[N_T=x] = 1$$

and there is no arbitrage in the $(\mathbb{G},\mathbb{P})$-market. If $x>0$, we have that

$$x_*^{\mathbb{F},\mathbb{P}}(1_{\{N_T=x\}}) = \sup_{\overline{\mathbb{P}}\in ELMM(\mathbb{F},\mathbb{P})}\overline{\mathbb{P}}[N_T=x] = \frac{1}{e^x}$$

and the optimal arbitrage strategy in the $(\mathbb{G},\mathbb{P})$-market is the strategy which buys one unit of the risky asset and holds it until maturity.

Proof. First, we consider the case $x \leq 0$. Let us define $\tau := \inf\{t : N_t = x\}$. We choose $\alpha_t^1 = m1_{t\leq\tau}$ and denote by $\overline{\mathbb{P}}^m$ the corresponding martingale measure. This choice of α^1 makes the Poisson processes N^1 and N^2 jump more frequently. However, when $N_t^1 - N_t^2 = x$, the two Poisson processes will not jump anymore. In other words, the measure $\overline{\mathbb{P}}^m$ concentrates on the event $\{N_T = x\}$. For any $m > 0$, we have the following inequality

$$\sup_{\overline{\mathbb{P}}\in ELMM(\mathbb{F},\mathbb{P})} \overline{\mathbb{P}}[N_T = x] \geq \mathbb{E}^{\overline{\mathbb{P}}^m}[1_{\{N_T=x\}}] = \overline{\mathbb{P}}^m[\tau \leq T].$$

Under $\mathbb{P}^m$, the intensities of N^1 and N^2 are α^1 and α^2. We use the fact that a Poisson process with stochastic intensity λ_t can be viewed as a time change of a standard Poisson process $N_{\int_0^t \lambda_s ds}$ and $\alpha^2 = e\alpha^1$, then

$$\overline{\mathbb{P}}^m[\tau \leq T] = \mathbb{P}\left[\inf_{0\leq t\leq mT}(N_t^1 - \tilde{N}_t^2) \leq x\right]$$

where $\tilde{N}^2$ is a Poisson process with parameter e. Letting m go to infinity and using the Dominated Convergence theorem, we obtain

$$\sup_{\overline{\mathbb{P}}\in ELMM(\mathbb{F},\mathbb{P})} \overline{\mathbb{P}}[N_T = x] \geq \mathbb{P}\left[\inf_{t\geq 0}(N_t^1 - \tilde{N}_t^2) \leq x\right] = 1,$$

because $N_t^1 - \tilde{N}_t^2 \to -\infty$ as $t \to \infty$. So, the first statement holds true. For the case $x > 0$, we notice that e^{-x} is an upper bound for the supremum. Indeed, for any ELMM $\overline{\mathbb{P}}$, it holds that

$$\overline{\mathbb{P}}[N_T = x] \leq \overline{\mathbb{P}}[S_T \geq e^x] \leq \frac{\mathbb{E}^{\overline{\mathbb{P}}}[S_T]}{e^x} \leq \frac{1}{e^x}.$$

We repeat the computations as in the first case

$$\sup_{\overline{\mathbb{P}}\in ELMM(\mathbb{F},\mathbb{P})} \overline{\mathbb{P}}[N_T = x] \geq \mathbb{E}^{\overline{\mathbb{P}}^m}[1_{\{N_T=x\}}] = \overline{\mathbb{P}}^m[\tau \leq T] = \mathbb{P}\left[\sup_{0\leq t\leq mT}(N_t^1 - \tilde{N}_t^2) \geq x\right].$$

It suffices to show that

$$\mathbb{P}\left[\sup_{t\geq 0}(N_t^1 - \tilde{N}_t^2) \geq x\right] = \frac{1}{e^x}.$$

Indeed, let us denote $f(x) = \mathbb{P}\left[\sup_{t\geq 0}(N_t^1 - \tilde{N}_t^2) \geq x\right]$. Let τ_1, τ_2 be the first jump times of N^1 and $\tilde{N}^2$, respectively. Because $\tau_1 \sim Exp(1)$ and $\tau_2 \sim Exp(e)$ are independent, the random variable $\frac{\tau_1}{e\tau_2}$ has the density $\frac{1}{(1+t)^2}$, thanks to Lemma 5.0.12, and thus,

$$\mathbb{P}[\tau_1 < \tau_2] = \mathbb{P}\left[\frac{\tau_1}{e\tau_2} < \frac{1}{e}\right] = \int_0^{1/e} \frac{1}{(1+t)^2} dt = \frac{1}{1+e}.$$

From its definition, we have $f(0) = 1$ and for $x \geq 1$ it then follows that

$$\begin{aligned} f(x) &= \mathbb{P}\left[\sup_{t\geq 0}(N_t^1 - \tilde{N}_t^2) \geq x | \tau_1 > \tau_2\right] \mathbb{P}[\tau_1 > \tau_2] \\ &+ \mathbb{P}\left[\sup_{t\geq 0}(N_t^1 - \tilde{N}_t^2) \geq x | \tau_1 \leq \tau_2\right] \mathbb{P}[\tau_1 \leq \tau_2] \\ &= \frac{f(x-1)}{1+e} + \frac{ef(x+1)}{1+e}. \end{aligned}$$

Therefore, we obtain $f(x+1) - f(x) = \frac{f(x)-f(x-1)}{e}$ and thus

$$f(x) = 1 - (1 - f(1))\frac{1 - e^{-x}}{1 - e^{-1}}.$$

Because $\lim_{x\to\infty} f(x) = 0$, we have that $f(1) = e^{-1}$ and then $f(x) = e^{-x}$.

Now, we show that the buy and hold strategy is optimal. Because the insider knows the value of S_T, the buy and hold strategy gives her the wealth process which superreplicates the claim 1,

$$\frac{1}{S_T} + \frac{1}{S_T}\int_0^T 1 dS_u = 1.$$

It is clear that the insider needs the initial capital e^{-x} on the event $\{N_T = x\}$. $\square$

4.5 Initial enlargement with a general random variable

In this section, we investigate the finiteness of insider's expected log-utility (then NUPBR) and the question of optimal arbitrage when the $\mathscr{F}_T$-measurable random variable G is not purely atomic by analogy with Section 4.4.

It is usually observed that in these settings, the value of logarithmic utility of the insider is infinite, for example in Theorem 4.4 of Pikovsky and Karatzas [1996] where the insider has exact information about at least one stock's terminal price, or in Amendinger et al. [1998] where the insider's additional expected logarithmic utility is related to the entropy of G. This difficulty appears at T, the time when the conditional law of G given $\mathscr{F}_T$ is a Dirac measure and hence Jacod's condition fails. Because NUPBR is the minimal condition for well-posed maximization problems, Acciaio et al. [2014] give a sufficient condition so that NUPBR holds under $\mathbb{G}$ in infinite time horizon settings. Their idea is that if the processes p^x and S do not jump to zero at the same time, then one can construct an ELMD under $\mathbb{G}$. However, in finite horizon settings, it may happen that the process p^x is not well-defined at T, making it impossible to define an ELMD because p^x appears in the denominator of such an ELMD.

In Section 4.5.1, we show that if G is not atomic and the set of local martingale densities $ELMM(\mathbb{F},\mathbb{P})$ is uniformly integrable then there always exists an arbitrage of the first kind, and thus NUPBR fails. This negative message implies that the non-uniform integrability of $ELMM(\mathbb{F},\mathbb{P})$ is a necessary condition for NUPBR under $\mathbb{G}$. In Section 4.5.3, we introduce an approximation procedure which allows us to use the techniques from the case with discrete information in Section 4.4. In Section 4.5.4, we investigate the log-utility optimization problem in detail and give a new criterion for the validity of NUPBR for the insider in incomplete markets. In Section 4.5.5, we study superhedging prices and optimal arbitrage profit. Furthermore, some new examples are given to illustrate our points.

4.5.1 Arbitrage of the first kind

Assuming that the set $ELMM(\mathbb{F},\mathbb{P})$ is uniformly integrable, we will give an easy and explicit construction of arbitrages of the first kind in the next proposition. It is worth noticing that we do not use Assumption 4.2.6 in the proof.

Proposition 4.5.1. *(Arbitrage of the first kind) Assume that*

- *The law of G is not purely atomic,*
- *The set $\{Z_T = \frac{d\overline{\mathbb{P}}}{d\mathbb{P}} : \overline{\mathbb{P}} \in ELMM(\mathbb{F},\mathbb{P})\}$ is uniformly integrable.*

Then there exists an arbitrage of the first kind for the insider. In particular, if the market $(\mathbb{F},\mathbb{P})$ is complete, then NUPBR always fails under $\mathbb{G}$.

Proof. Let us choose $B \subset \mathbb{R}$ such that B does not contain any atoms of G and $\mathbb{P}[G \in B] = c > 0$. For each n, we find a partition $(B_i^n)_{1\leq i\leq n}$ of B such that $\mathbb{P}[G \in B_i^n] = c/n$ and we show that $1_{\{G\in B\}}$ is an arbitrage of the first kind as follows. First, we can compute the superhedging price of $1_{\{G\in B_i^n\}}$ and its associated hedging strategy $H^{\mathbb{F},i}$ in the $(\mathbb{F},\mathbb{P})$-market, see Corollary 10 of Delbaen and Schachermayer [1995c],

$$\sup_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}[Z_T 1_{\{G\in B_i^n\}}] + (H^{\mathbb{F},i}\cdot S)_T \geq 1_{\{G\in B_i^n\}}.$$

Therefore,

$$\sum_{i=1}^{n} \sup_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}[Z_T 1_{\{G\in B_i^n\}}]1_{\{G\in B_i^n\}} + \left(\sum_{i=1}^{n} H^{\mathbb{F},i} 1_{\{G\in B_i^n\}}\right)\cdot S_T \geq 1_{\{G\in B\}}. \tag{4.23}$$

Becasue the set of all local martingale densities $\{Z_T : Z \in ELMM(\mathbb{F},\mathbb{P})\}$ is uniformly integrable, for any $\varepsilon > 0$ there exists $K > 0$ such that

$$\sup_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}[Z_T 1_{\{Z_T>K\}}] \leq \varepsilon.$$

Then the initial capital in in (4.23) can be estimated by

$$\begin{aligned}
&\sum_{i=1}^{n} \sup_{Z\in ELMM(\mathbb{F},\mathbb{P})} \left(\mathbb{E}^{\mathbb{P}}[Z_T 1_{\{Z>K\}} 1_{\{G\in B_i^n\}}] + \mathbb{E}^{\mathbb{P}}[Z_T 1_{Z_T\le K} 1_{\{G\in B_i^n\}}]\right) 1_{\{G\in B_i^n\}} \\
&\le \sum_{i=1}^{n} (\varepsilon + K\mathbb{P}[G\in B_i^n])\, 1_{\{G\in B_i^n\}} \\
&= \sum_{i=1}^{n} \left(\varepsilon + K\frac{c}{n}\right) 1_{\{G\in B_i^n\}}.
\end{aligned}$$

We can choose ε and n such that the initial capital in (4.23) is arbitrarily small and thus the random variable $1_{\{G\in B\}}$ is an arbitrage of the first kind, in the sense of Definition 1.4.8. □

Remark 4.5.2 (A comparison to Amendinger et al. [1998])**.** *We recover their Theorem 4.4 by using a different approach. More precisely, in that paper the authors show that the insider's additional expected logarithmic utility up to time T becomes infinite and then NUPBR fails. However, their results apply only to continuous processes and require an even stronger condition than market completeness, namely that the inverse of p^G may be represented as a stochastic integral, see condition (45) therein. In our result, we are able to construct unbounded profits in general market settings. In particular, the following example shows that the property of uniform integrability holds also for some incomplete market models.*

An example with UI martingale densities in an incomplete market. We consider a risky asset whose price evolves as

$$dS_t = S_{t-}\sigma(t)(\theta dN_t^1 + (1-\theta)dN_t^2 - dt)$$

where $\theta \in (0,1)$ and $\sigma(t)$ is a non-constant continuous function. The filtration $\mathbb{F}$ is generated by two independent standard Poisson processes N^1 and N^2. Any martingale density has this form

$$Z_T = \mathscr{E}\left(\int_0^T (\phi_t^1 - 1)(dN_t^1 - dt)\right) \mathscr{E}\left(\int_0^T (\phi_t^2 - 1)(dN_t^2 - dt)\right),$$

where ϕ^1 and ϕ^2 are positive predictable processes satisfying $\theta\phi_t^1 + (1-\theta)\phi_t^2 = 1, \mathbb{P}-a.s.$ Therefore,

$$0 \le \phi^1 \le \frac{1}{\theta}, \qquad 0 \le \phi^2 \le \frac{1}{1-\theta}.$$

These inequalities lead to an upper bound for all martingale densities

$$Z_T = e^{-\int_0^T (\phi_t^1+\phi_t^2-2)dt} \prod_{i=1}^{N_T^1} \phi_{t_i}^1 \prod_{j=1}^{N_T^2} \phi_{t_j}^2 \le C(\theta)\frac{1}{\theta^{N_T^1}} \frac{1}{(1-\theta)^{N_T^2}} \le C(\theta)\frac{1}{\min\{\theta, 1-\theta\}}$$

where $C(\theta)$ is a constant depending on θ. As a result, the set of martingale densities is uniformly integrable. Let $(T_i^1)_{i\ge1}$ and $(T_j^2)_{j\ge1}$ be the jump times of N^1 and N^2 respectively. Because σ is a continuous function, the random variables $\sigma(T_i^1), \sigma(T_j^2)$ are continuous. This means that the information

$$G = S_T = \exp\left(-\int_0^T \sigma(s)ds\right) \prod_{i=1}^{N_T^1} \left(1+\theta\sigma(T_i^1)\right) \prod_{j=1}^{N_T^2} \left(1+(1-\theta)\sigma(T_j^2)\right)$$

is a nonatomic random variable and hence the market of the insider does not satisfy the NUPBR condition.

In the following, we give an example in which the additional information is a continuous random variable and the insider has no unbounded profit with bounded risk. This example ensures that the questions of NUPBR and optimal arbitrage make sense. The answers of these questions are introduced in Section 4.5.4 and Section 4.5.5 by using an approximating procedure illustrated in Section 4.5.3.

4.5.2 An example with a Lévy process with two-sided jumps

The idea of this example comes from Kohatsu-Higa and Yamazato [2011]. Because of the presence of non-predictable jumps, stock prices incorporate higher risks than that in Brownian motion, the optimal strategy of the insider is not highly oscillating. Consequently, the utility of the insider may be finite.

We assume that the public information $\mathbb{F}$ is the natural filtration generated by a Brownian motion W and two independent Poisson processes N^1, N^2 with intensity

$\lambda = 1$. The risky asset is $S_t = \exp(M_t)$ where $M_t = W_t + N_t^1 - N_t^2$ is a $\mathbb{F}$-martingale. The dynamic of S under $\mathbb{F}$ is

$$dS_t = S_{t-}\left(dW_t + \frac{1}{2}dt + (e-1)dN_t^1 + (e^{-1}-1)dN_t^2\right), \qquad S_0 = 1.$$

Let $H^{\mathbb{F}}$ be a $\mathbb{F}$- self-financing strategy. We denote by $\pi^{\mathbb{F}}$ the fraction of wealth invested in the stock, that is $\pi_t^{\mathbb{F}} := \frac{H_t^{\mathbb{F}} S_t}{V^{H_t^{\mathbb{F}}}}$. The associated wealth can be expressed as

$$\frac{dV_t^{\pi^{\mathbb{F}}}}{V_{t-}^{\pi^{\mathbb{F}}}} = \pi_t^{\mathbb{F}} \frac{dS_t}{S_{t-}}, \qquad V_0^{\pi^{\mathbb{F}}} = v.$$

The strategy $\pi^{\mathbb{F}}$ is admissible if for all $t \in [0,T]$ we have that $V_t^{v,\pi^{\mathbb{F}}} \geq 0, \mathbb{P}-a.s.$ This requirement is equivalent to

$$-\frac{1}{e-1} < \pi_t^{\mathbb{F}} < \frac{1}{1-e^{-1}}.$$

Now, we study the market of an insider with the additional information given by the final value of S, that is $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(S_T) = \mathscr{F}_t \vee \sigma(M_T)$. First, we check Jacod's condition. We compute by using the Markov property

$$\begin{aligned}
\mathbb{P}[M_T \in dx|\mathscr{F}_t] &= \mathbb{P}[M_T - M_t \in dx - M_t] \\
&= \sum_{i\geq 0, j\geq 0} \mathbb{P}[N_{T-t}^1 = i]\mathbb{P}[N_{T-t}^2 = j]\mathbb{P}[W_{T-t} \in dx - M_t - i + j] \\
&= \sum_{i\geq 0, j\geq 0} e^{-2(T-t)} \frac{(T-t)^{i+j}}{i!j!} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-M_t-i+j)^2}{2(T-t)}} dx, \\
\mathbb{P}[M_T \in dx] &= \sum_{i\geq 0, j\geq 0} e^{-2T} \frac{T^{i+j}}{i!j!} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x-i+j)^2}{2T}} dx,
\end{aligned}$$

and hence,

$$p_t^x = \frac{\mathbb{P}[M_T \in dx|\mathscr{F}_t]}{\mathbb{P}[M_T \in dx]} = \frac{\sum_{i\geq 0, j\geq 0} e^{-2(T-t)} \frac{(T-t)^{i+j}}{i!j!} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-M_t-i+j)^2}{2(T-t)}}}{\sum_{i\geq 0, j\geq 0} e^{-2T} \frac{T^{i+j}}{i!j!} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x-i+j)^2}{2T}}}.$$

The Jacod condition is satisfied. Now, the public filtration is quasi left-continuous, the $\mathbb{F}$-martingale p^x is positive if $t < T$ and it cannot jump at the predictable stopping time T. Thus, Assumption 4.2.6 is fulfilled.

We will compute the dynamics of S under $\mathbb{G}$ and then study the logarithmic utility maximization problem for the insider, which will also allow us to verify the NUPBR condition. First, we introduce a larger filtration $\mathscr{H}_t := \mathscr{F}_t \vee \sigma(W_T, N_T^1, N_T^2)$. By Theorem 3, page 356 of Protter [2003], the processes

$$dW_t - \frac{W_T - W_t}{T-t}dt, \qquad dN_t^1 - \frac{N_T^1 - N_t^1}{T-t}dt, \qquad dN_t^2 - \frac{N_T^2 - N_t^2}{T-t}dt$$

are martingales under $\mathbb{H}$. Lemma 5.0.14 shows that the processes

$$dW_t^{\mathbb{G}} := dW_t - \lambda_t^W dt, \qquad dN_t^1 - \lambda_t^{\mathbb{G},1}dt, \qquad dN_t^2 - \lambda_t^{\mathbb{G},2}dt$$

are martingales under $\mathbb{G}$, where

$$\lambda_t^W = \mathbb{E}\left[\frac{W_T - W_t}{T-t}\middle|\mathscr{G}_t\right],\ \lambda_t^{\mathbb{G},1} = \mathbb{E}\left[\frac{N_T^1 - N_t^1}{T-t}\middle|\mathscr{G}_t\right],\ \lambda_t^{\mathbb{G},2} = \mathbb{E}\left[\frac{N_T^2 - N_t^2}{T-t}\middle|\mathscr{G}_t\right].$$

We rewrite the dynamic of S under $\mathbb{G}$ as the sum of local martingales and a finite variation part

$$\begin{aligned} dS_t =& S_{t-}\left(dW_t^{\mathbb{G}} + (e-1)(dN_t^1 - \lambda_t^{\mathbb{G},1}dt) + (e^{-1}-1)(dN_t^2 - \lambda_t^{\mathbb{G},2}dt)\right) \\ &+ S_{t-}\left(\frac{1}{2} + \lambda_t^W + (e-1)\lambda_t^{\mathbb{G},1} + (e^{-1}-1)\lambda_t^{\mathbb{G},2}\right)dt. \end{aligned}$$

For a self-financing strategy $\pi_t^{\mathbb{G}}$, which is defined as $\pi^{\mathbb{F}}$, the corresponding wealth process is

$$\frac{dV_t^{\pi^{\mathbb{G}}}}{V_{t-}^{\pi^{\mathbb{G}}}} = \pi_t^{\mathbb{G}}\frac{dS_t}{S_{t-}}, \qquad V_0^{\pi^{\mathbb{G}}} = v.$$

In the $(\mathbb{G},\mathbb{P})$-market, the jump sizes do not change, so a $\mathbb{G}$-admissible strategy is also bounded from above and from below,

$$-\frac{1}{e-1} < \pi_t^{\mathbb{G}} < \frac{1}{1-e^{-1}}.$$

By Itô's formula, we have

$$\begin{aligned} d\log V_t^{\pi^{\mathbb{G}}} &= \pi_t^{\mathbb{G}}\left(dW_t^{\mathbb{G}} + (e-1)(dN_t^1 - \lambda_t^{\mathbb{G},1}dt) + (e^{-1}-1)(dN_t^2 - \lambda_t^{\mathbb{G},2}dt)\right) \\ &\quad + \pi_t^{\mathbb{G}}\left(\frac{1}{2} + \lambda_t^W + (e-1)\lambda_t^{\mathbb{G},1} + (e^{-1}-1)\lambda_t^{\mathbb{G},2}\right)dt \\ &\quad - \frac{1}{2}(\pi_t^{\mathbb{G}})^2 dt + \left(\log(1+(e-1)\pi_t^{\mathbb{G}}) - (e-1)\pi_t^{\mathbb{G}}\right)dN_t^1 \\ &\quad + \left(\log(1+(e^{-1}-1)\pi_t^{\mathbb{G}}) - (e^{-1}-1)\pi_t^{\mathbb{G}}\right)dN_t^2. \end{aligned}$$

All admissible strategies are bounded and the expected logarithmic utility for an insider is finite, because for any $\pi^{\mathbb{G}}$ it holds that

$$\begin{aligned} \mathbb{E}[\log V_T^{\pi^{\mathbb{G}}}] &\le \mathbb{E}\left[\int_0^T |\pi_t^{\mathbb{G}}|\left(\frac{1}{2} + \lambda_t^W + (e-1)\lambda_t^{\mathbb{G},1} + (e^{-1}-1)\lambda_t^{\mathbb{G},2}\right)dt\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_0^T (\pi_t^{\mathbb{G}})^2 dt\right] + \mathbb{E}\left[\int_0^T \left(\log(1+(e-1)\pi_t^{\mathbb{G}}) - (e-1)\pi_t^{\mathbb{G}}\right)\lambda_t^{\mathbb{G},1}dt\right] \\ &\quad + \mathbb{E}\left[\int_0^T \left(\log(1+(e^{-1}-1)\pi_t^{\mathbb{G}}) - (e^{-1}-1)\pi_t^{\mathbb{G}}\right)\lambda_t^{\mathbb{G},2}dt\right] < +\infty. \end{aligned}$$

In conclusion, the $(\mathbb{G},\mathbb{P})$-market satisfies NUPBR. Furthermore, the insider has arbitrage opportunities because she knows the final value of S. For example, she could buy the asset S (with $S_0 = 1$) and hold it until maturity if $S_T > 1$ for a riskless profit.

4.5.3 An approximation procedure

We have seen that the absence of uniform integrability is a necessary condition for NUPBR under $\mathbb{G}$. In the sequel, we introduce an approximation procedure to compute the log-utility of the insider and then a sufficient condition for NUPBR.

Let $\{g_i^n, i = 1,...,n\}$ be a finite increasing partition of $\mathbb{R}^+$ and denote

$$\sigma(G^n) = \sigma\left(\{G \in g_i^n\}, i = 1,...,n\right).$$

Motivated by the results in Section 4.4, we now approximate $\sigma(G)$ by the increasing sequence of sigma algebras $\sigma(G^n)$

$$\sigma(G^n) \subset \sigma(G^{n+1}) \subset ...\sigma(G) = \sigma\left(\bigcup_{n\geq 1}\sigma(G^n)\right).$$

Now, we define an increasing sequence of filtrations $\mathbb{G}^n = (\mathscr{G}_t^n)_{t\in[0,T]}$, where $\mathscr{G}_t^n := \bigcap_{\varepsilon>0}\mathscr{F}_{t+\varepsilon} \vee \sigma(G^n)$. The information from G is revealed from coarser levels to finer ones and we show how to use this additional information effectively at each level. For each n, we then proceed as in Theorem 4.4.4 and Theorem 4.4.5 to compute the log-utility value and superhedging prices under $\mathbb{G}^n$, the information of level n. Then, convergence results are applied to find solutions under $\mathbb{G}$.

To begin, we prepare some preliminary results by showing that for any $\mathbb{G}$-predictable strategy $H^{\mathbb{G}}$, there exists a sequence of $(\mathbb{G}^n)$-predictable strategies $(H^{\mathbb{G}^n})_n$ which converge to $H^{\mathbb{G}}$ almost surely. The proof of this claim is broken into several lemmas, for the purpose of clear presentation.

Lemma 4.5.3. *Assume that $H^{\mathbb{G}}$ is a simple bounded $\mathbb{G}$-strategy. Then there is a sequence of $\mathbb{G}^n$-predictable strategies $(H^{\mathbb{G}^n})_n$ such that $H^{\mathbb{G}^n}$ converges uniformly to $H^{\mathbb{G}}$ a.s.*

Proof. We assume that $H_t^{\mathbb{G}}$ has the representation $\sum_{i=1}^k h_{T_i} 1_{]T_i,T_{i+1}]}(t)$ where h_{T_i} are bounded $\mathscr{G}_{T_i}$-measurable random variables, i.e. $|h_{T_i}| \leq K, i = 1,...,k$. We define

$$H_t^{\mathbb{G}^n} := \sum_{i=1}^k \mathbb{E}[h_{T_i}|\mathscr{G}_{T_i}^n] 1_{]T_i,T_{i+1}]}(t).$$

The process $H^{\mathbb{G}^n}$ is $\mathbb{G}^n$-predictable. Using Lévy's "Upward" Theorem (see Theorem 50.3 of Rogers and Williams [1979]), we obtain $\mathbb{E}[h_{T_i}|\mathscr{G}_{T_i}^n] \to h_{T_i}, \mathbb{P}-a.s.$ Because there are a finite number of indicator functions in the representation of $H^{\mathbb{G}}$, we can choose n such that

$$\sup_{t\in[0,T]} |H_t^{\mathbb{G}^n} - H_t^{\mathbb{G}}| \to 0, \mathbb{P}-a.s., \qquad n \to \infty.$$

Thus, the proof is complete. □

Lemma 4.5.4. *For any $\mathbb{G}$-predictable strategy $H^{\mathbb{G}}$, there exists a sequence of $\mathbb{G}^n$-predictable admissible strategies $H^{\mathbb{G}^n}$ such that for all $t \in [0,T]$*

$$H_t^{\mathbb{G}^n} \to H_t^{\mathbb{G}}, \qquad \mathbb{P}-a.s.$$

Proof. We can find a sequence of bounded simple predictable processes $(H^{\mathbb{G},n})_{n\in\mathbb{N}}$ such that

$$\forall t \in [0,T], \qquad H_t^{\mathbb{G},n} \to H_t^{\mathbb{G}}, \mathbb{P}-a.s. \tag{4.24}$$

For each n, Lemma 4.5.3 shows that there exists a sequence of $\mathbb{G}^m$-strategies $H^{n,\mathbb{G}^m}$ such that

$$f_{n,m} := \sup_{t\in[0,T]} |H_t^{n,\mathbb{G}^m} - H_t^{\mathbb{G},n}| \to 0, \mathbb{P}-a.s. \qquad m \to \infty.$$

Applying Lemma 5.0.11 to the sequences $f_{m,n}$ and $f_n = f = 0$, there exists a sequence $(m_n) \subset \mathbb{N}$ such that

$$f_{n,m_n} = \sup_{t\in[0,T]} |H_t^{n,\mathbb{G}^{m_n}} - H_t^{\mathbb{G},n}| \to 0, \mathbb{P}-a.s. \qquad n \to \infty. \tag{4.25}$$

Finally, (4.24 and (4.25) imply that $H_t^{n,\mathbb{G}^{m_n}} \to H_t^{\mathbb{G}}, \mathbb{P}-a.s.$ for all $t \in [0,T]$. □

4.5.4 NUPBR and Log-utility

By analogy to the discrete case, we prove NUPBR under Assumption 4.2.6 using the finiteness of log-utility. To begin with, we show the following result.

Lemma 4.5.5. *Under Assumption 4.2.6, for every $a < b$, the $\mathbb{F}$-martingale*

$$p_t^{(a,b)}(\omega) := \frac{\mathbb{P}[G \in (a,b)|\mathscr{F}_t](\omega)}{\mathbb{P}[G \in (a,b)]} = \frac{1}{\mathbb{P}[G \in (a,b)]} \int_a^b p_t^x(\omega)\mathbb{P}[G \in dx]$$

does not jump to zero.

Proof. The martingale $p^{(a,b)}$ is bounded by $\frac{1}{\mathbb{P}[G\in(a,b)]}$. Because $\lim_{s\nearrow t} p_s^x(\omega) = p_{t-}^x(\omega)$, and $\lim_{s\nearrow t} p_s^{(a,b)}(\omega) = p_{t-}^{(a,b)}(\omega)$ the reverse Fatou lemma implies that

$$\frac{1}{\mathbb{P}[G\in(a,b)]}\int_a^b p_{t-}^x(\omega)\mathbb{P}[G\in dx] \geq \frac{1}{\mathbb{P}[G\in(a,b)]}\limsup_{s\nearrow t}\int_a^b p_s^x(\omega)\mathbb{P}[G\in dx]$$
$$= p_{t-}^{(a,b)}(\omega).$$

We deduce that if $p_t^{(a,b)} = 0$ then we have that $p_t^x = 0$ for a.a. x except for a set having measure zero w.r.t the measure induced by G. Furthermore, if $p_{t-}^{(a,b)} > 0$ then $p_{t-}^x > 0$ on some set $J \subset (a,b)$ with positive measure (w.r.t the measure induced by G). Thus, if $p^{(a,b)}$ jumps to zero then p^x jumps to zero for all $x \in J$. On the other hand Assumption 4.2.6. requires p^x not to jump to zero for all x. Hence, the process $p^{(a,b)}$ does not jump to zero. □

Theorem 4.5.6. *We have that*

$$\lim_{n\to\infty}\sup_{H\in\mathscr{A}_1^{\mathbb{G}^n}}\mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] = \sup_{H\in\mathscr{A}_1^{\mathbb{G}}}\mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}].$$

Proof. Let $H^{\mathbb{G}}$ be an arbitrary $\mathbb{G}$-strategy and ε be a positive number and $(H^{\mathbb{G}^n})_n$ be a sequence of $\mathbb{G}^n$-predictable processes which tend to $H^{\mathbb{G}}$ almost surely. We denote

$$A(n,\varepsilon) := \{\omega : \sup_{t\in[0,T]} |(H^{\mathbb{G}^n}\cdot S)_t - (H^{\mathbb{G}}\cdot S)_t| \leq \varepsilon\}.$$

We now have the following estimation on $A(n,\varepsilon)$

$$\begin{aligned}(H^{\mathbb{G}^n}\cdot S)_t &= (H^{\mathbb{G}}\cdot S)_t + (H^{\mathbb{G}^n}\cdot S)_t - (H^{\mathbb{G}}\cdot S)_t \\ &\geq (H^{\mathbb{G}}\cdot S)_t - \varepsilon, \qquad \forall t\in[0,T].\end{aligned} \tag{4.26}$$

Denoting $\tau_n^\varepsilon := \inf\{t\in[0,T] : 1+2\varepsilon+(H^{\mathbb{G}^n}\cdot S)_t \leq 0\}$, we see that the strategy $H_{t\wedge\tau_n^\varepsilon}^{\mathbb{G}^n}$ with initial capital $1+2\varepsilon$ is admissible. The following inequality holds

$$1+2\varepsilon+(H^{\mathbb{G}^n}\cdot S)_T \geq 1_{A(n,\varepsilon)}V_T^{1,H^{\mathbb{G}}}$$

and so

$$\mathbb{E}^{\mathbb{P}}\left[\log\left(1+2\varepsilon+(H^{\mathbb{G}^n}\cdot S)_T\right)\right]\geq\mathbb{E}^{\mathbb{P}}\left[1_{A(n,\varepsilon)}\log V_T^{1,H^{\mathbb{G}}}\right]$$

or equivalently

$$\mathbb{E}^{\mathbb{P}}\left[\log\left(1+\frac{1}{1+2\varepsilon}(H^{\mathbb{G}^n}\cdot S)_T\right)\right]+\log(1+2\varepsilon)\geq\mathbb{E}^{\mathbb{P}}\left[1_{A(n,\varepsilon)}\log V_T^{1,H^{\mathbb{G}}}\right].$$

By construction, the stopped strategy $H^{\mathbb{G}^n}_{\cdot\wedge\tau_n^\varepsilon}$ is $(1+\varepsilon)$-admissible. From that, the strategy $\frac{1}{1+2\varepsilon}H^{\mathbb{G}^n}_{\cdot\wedge\tau_n^\varepsilon}$ is $(1+\varepsilon)/(1+2\varepsilon)$-admissible and hence 1-admissible. We deduce that

$$\sup_{H\in\mathscr{A}_1^{\mathbb{G}^n}}\mathbb{E}^{\mathbb{P}}\left[\log\left(1+(H\cdot S)_T\right)\right]+\log(1+2\varepsilon)\geq\mathbb{E}^{\mathbb{P}}\left[1_{A(n,\varepsilon)}\log V_T^{1,H^{\mathbb{G}}}\right]$$

By Dominated convergence theorem (see Theorem 32, page 174 of Protter [2003]), there exists a number $N(\varepsilon)$ such that for all $n>N(\varepsilon)$, it holds that $\mathbb{P}[A(n,\varepsilon)]>1-\varepsilon$. Letting n tend to infinity, we obtain

$$\lim_{n\to\infty}\sup_{H\in\mathscr{A}_1^{\mathbb{G}^n}}\mathbb{E}^{\mathbb{P}}\left[\log\left(1+(H\cdot S)_T\right)\right]+\log(1+2\varepsilon)\geq\mathbb{E}^{\mathbb{P}}\left[\log V_T^{1,H^{\mathbb{G}}}\right]$$

for any strategy $H^{\mathbb{G}}$ and thus

$$\lim_{n\to\infty}\sup_{H\in\mathscr{A}_1^{\mathbb{G}^n}}\mathbb{E}^{\mathbb{P}}\left[\log V_T^{1,H}\right]+\log(1+2\varepsilon)\geq\max_{H\in\mathscr{A}_1^{\mathbb{G}}}\mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}].$$

The conclusion follows easily. □

We will extend Theorem 4.4.4 in the next corollary.

Corollary 4.5.7. *Assumption 4.2.6 holds. Let G be a random variable with continuous density f and finite entropy. The insider's expected log-utility is*

$$\sup_{H\in\mathscr{A}_1^{\mathbb{G}}}\mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}]=-\int f(x)\log f(x)dx$$
$$+\lim_{n\to\infty}\sum_{i=1}^{n}\left(-\log|g_i^n|\mathbb{P}[G\in g_i^n]+\inf_{Z\in ELMM(\mathbb{F},\mathbb{P})}\mathbb{E}^{\mathbb{P}}\left[1_{\{G\in g_i^n\}}\log\frac{1}{Z_T}\right]\right).\tag{4.27}$$

Proof. Theorem 4.5.6 and Theorem 4.4.4 show that

$$\sup_{H\in\mathscr{A}_1^{\mathbb{G}}}\mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}]=\lim_{n\to\infty}\sum_{i=1}^{n}\left(-\mathbb{P}[G\in g_i^n]\log\mathbb{P}[G\in g_i^n]+\inf_{Z\in ELMM(\mathbb{F},\mathbb{P})}\mathbb{E}^{\mathbb{P}}\left[1_{\{G\in g_i^n\}}\log\frac{1}{Z_T}\right]\right) \tag{4.28}$$

Now we consider the first term in the RHS. Using mean value theorem, we have that $\mathbb{P}[G\in g_i^n]=f(x_i^n)|g_i^n|$ for some $x_i^n\in g_i^n$. Thus,

$$\begin{aligned}-\mathbb{P}[G\in g_i^n]\log\mathbb{P}[G\in g_i^n]&=-\mathbb{P}[G\in g_i^n]\log(f(x_i^n)|g_i^n|)\\&=-\mathbb{P}[G\in g_i^n]\log f(x_i^n)-\mathbb{P}[G\in g_i^n]\log|g_i^n|\\&=-f(x_i^n)\log f(x_i^n)|g_i^n|-\mathbb{P}[G\in g_i^n]\log|g_i^n|.\end{aligned}$$

Letting n tend to infinity, we get the result. □

As a consequence, the insider's log-utility problem is finite if G has finite entropy and for every event $\{G\in g_i^n\}$, there exists a martingale density Z_T such that the quantity $\mathbb{E}^{\mathbb{P}}[1_{\{G\in g_i^n\}}\log(1/Z_T)]$ can compensate the term $-\log|g_i^n|\mathbb{P}[G\in g_i^n]$. In complete markets, it is impossible to find such a martingale density for each event, implying that expected log-utility of the insider is infinite. In incomplete markets, the result provides us with a new criterion for NUPBR under $\mathbb{G}$.

Corollary 4.5.8. *Under Assumption 4.2.6, if there exists a constant $C<\infty$ such that for all a and all ε small enough,*

$$\begin{aligned}\sup_{Z\in ELMM}\mathbb{E}[1_{\{G\in(a,a+\varepsilon)\}}\log Z_T]\geq&-\mathbb{P}[G\in(a,a+\varepsilon)]\log\mathbb{P}[G\in(a,a+\varepsilon)]\\&-C\mathbb{P}[G\in(a,a+\varepsilon)]\end{aligned} \tag{4.29}$$

then the condition NUPBR holds under $\mathbb{G}$.

Proof. The equation (4.28) is still valid. Now, we apply (4.29) to (4.28) and obtain

$$\begin{aligned}\sup_{H\in\mathscr{A}_1^{\mathbb{G}}}\mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}]&=\lim_{n\to\infty}\sum_{i=1}^{n}\left(-\mathbb{P}[G\in g_i^n]\log\mathbb{P}[G\in g_i^n]+\inf_{Z\in ELMM(\mathbb{F},\mathbb{P})}\mathbb{E}^{\mathbb{P}}\left[1_{\{G\in g_i^n\}}\log\frac{1}{Z_T}\right]\right)\\&\leq\lim_{n\to\infty}\sum_{i=1}^{n}C\mathbb{P}[G\in g_i^n]=C.\end{aligned}$$

The expected log-utility of the insider is bounded and hence the condition NUPBR holds under $\mathbb{G}$. □

A counterexample

Assumption 4.2.6 cannot be dropped. We will illustrate this point by an example.

Proposition 4.5.9. *Let*

$$S_t = 1 + Y 1_{0.5 \le t}, \qquad t \in [0,1]$$

where $Y > -1$ is a continuous random variable with bounded density $0 < m \le f(y) \le M$ and $\mathbb{E}[Y] = 0$. We assume that the support of f is $(-1,L)$, where $L > 1$. Then the criterion (4.29) holds. In this case, the insider with the information S_T has infinite expected utility.

Proof. Letting $a \in (0, L+1)$ such that $1 \notin (a, a+\varepsilon)$, we choose

$$Z_1 = A 1_{Y \in (a-1, a-1+\varepsilon)} + B 1_{Y \in (b-1, b-1+\varepsilon)}(Y) + \delta 1_{Y \notin (a-1,a-1+\varepsilon) \cup (b-1,b-1+\varepsilon)}$$

where $0 < b < L+1$, $1 \notin (b, b+\varepsilon)$ and $(a-1)(b-1) < 0$. Now, we choose A, B, b, $0 < \delta < 1$ satisfying the following properties:

1) Z is a local martingale density, i.e. $Z_1 > 0$, $\mathbb{E}[Z_1] = 1$ and $\mathbb{E}[Z_1 S_1] = 1$,

2) the inequality (4.29) holds for the chosen Z_1.

Let us explain the meaning of this choice. Because the support of Y is $(-1,L)$, the support of Y under an equivalent local martingale measure is also $(-1,L)$. This means that Z_1 is defined appropriately. Furthermore, under the local martingale measure defined by Z_1, the random variable Y concentrates at two points $a-1$ and $b-1$. This implies the quantity $\mathbb{E}[1_{S_1 \in (a, a+\varepsilon)} \log Z_1] = \mathbb{E}[1_{Y \in (a-1, a-1+\varepsilon)} \log A]$ likely satisfies the condition (4.29) if A is big enough. Finally, the requirement $(a-1)(b-1) < 0$ is for the martingale property of S_1.

We define

$$q_1 := \mathbb{E}[1_{Y\in(a-1,a-1+\varepsilon)}], \qquad q_2 := \mathbb{E}[1_{Y\in(b-1,b-1+\varepsilon)}],$$
$$k_1 := \mathbb{E}[Y1_{Y\in(a-1,a-1+\varepsilon)}], \qquad k_2 := [Y1_{Y\in(b-1,b-1+\varepsilon)}].$$

The requirement $i)$ for Z_1 implies that

$$Aq_1 + Bq_2 + \delta(1-q_1-q_2) = 1, \qquad Ak_1 + Bk_2 + \delta(-k_1-k_2) = 0.$$

Solving these equations, we obtain

$$A = \frac{k_2(1-\delta)}{q_1k_2 - q_2k_1} + \delta, \qquad B = \frac{-k_1(1-\delta)}{q_1k_2 - q_2k_1} + \delta,$$

which are positive. Indeed, there are two possible cases:

- if $a > 1, b < 1$, then $q_1 > 0, q_2 > 0, k_1 > 0, k_2 < 0$. We compute $q_1k_2 - q_2k_1 < 0$ and hence A and B are positive.

- if $a < 1, b > 1$, it holds that $q_1 > 0, q_2 > 0, k_1 < 0, k_2 > 0$ and $q_1k_2 - q_2k_1 > 0$. Thus, A and B are positive.

We turn to the requirement $ii)$. Because $\mathbb{E}[1_{S_1\in(a,a+\varepsilon)}\log Z_1] = \mathbb{E}[1_{Y\in(a-1,a-1+\varepsilon)}\log A]$, we will show that there exists a constant C such that for all a and all $\varepsilon > 0$ small enough then

$$\log A \geq \log\frac{1}{\mathbb{P}[Y\in(a-1,a-1+\varepsilon)]} - C = \log\frac{1}{e^C\mathbb{P}[Y\in(a-1,a-1+\varepsilon)]}$$

or equivalently, $e^CA\mathbb{P}[Y\in(a-1,a-1+\varepsilon)] \geq 1$. Using the formula of A, we obtain the inequality

$$e^Cq_1\left(\frac{k_2(1-\delta)}{q_1k_2 - q_2k_1} + \delta\right) \geq 1.$$

It suffices to choose C such that

$$e^C(1-\delta)\frac{q_1k_2}{q_1k_2 - q_2k_1} \geq 1.$$

or equivalently

$$1-e^{C}(1-\delta)\leq\frac{q_2k_1}{q_1k_2}\leq 1. \tag{4.30}$$

For ε small enough, the mean value theorem implies that the condition (4.30) is equivalent to

$$1-e^{C}(1-\delta)\leq\frac{f(b-1+\varepsilon_1)(a-1+\varepsilon_2)f(a-1+\varepsilon_2)}{f(a-1+\varepsilon_3)(b-1+\varepsilon_4)f(b-1+\varepsilon_4)}\leq 1, \tag{4.31}$$

where $0\leq\varepsilon_i\leq\varepsilon, i\in\{1,2,3,4\}$. The second inequality in (4.31) is satisfied because the quantity in the middle of (4.31) is negative when $(a-1)(b-1)<0$. Now, we consider the first inequality in (4.31). There are two possible cases.

- If $0<a<1, 1<b<L+1$: in this case, we estimate
$$-2\frac{M^2}{m^2}\leq(a-1+\varepsilon_2)\frac{f(b-1+\varepsilon_1)f(a-1+\varepsilon_2)}{f(a-1+\varepsilon_3)f(b-1+\varepsilon_4)}.$$
Thus, it suffices to choose C and b such that
$$(1-e^{C}(1-\delta))(b-1+\varepsilon_4)\leq-2\frac{M^2}{m^2}. \tag{4.32}$$
If we choose C big enough then (4.32) is satisfied.

- If $1<a<L+1, 0<b<1$: we estimate
$$L\frac{M^2}{m^2}\geq(a-1+\varepsilon_2)\frac{f(b-1+\varepsilon_1)f(a-1+\varepsilon_2)}{f(a-1+\varepsilon_3)f(b-1+\varepsilon_4)},$$
the condition (4.31) implies we will choose C such that
$$(1-e^{C}(1-\delta))(b-1+\varepsilon_4)\geq L\frac{M^2}{m^2}.$$
This step can be done by choosing C big enough.

We conclude that the inequality (4.29) is satisfied. However, the condition NUPBR fails because the insider gains unbounded profits by holding a large amount of S if $S_1>1$. $\square$

4.5.5 Superhedging and optimal arbitrage

Let $f \geq 0$ be a given claim. We recall that $x_*^{\mathbb{F},\mathbb{P}}(f 1_{\{G\in g_i^n\}})$ and $H^{\mathbb{F},n,i}$ are the superhedging price and the associated superhedging strategy for the claim $f 1_{G\in g_i^n}$ under $(\mathbb{F},\mathbb{P})$, i.e.

$$x_*^{\mathbb{F},\mathbb{P}}(f 1_{\{G\in g_i^n\}}) + (H^{\mathbb{F},n,i}\cdot S)_T \geq f 1_{\{G\in g_i^n\}}, \mathbb{P}-a.s.$$

It is important to note that $x_*^{\mathbb{F},\mathbb{P}}(f 1_{\{G\in g_i^n\}})$ and $H^{\mathbb{F},n,i}$ are computed under $(\mathbb{F},\mathbb{P})$, where NFLVR holds. An application of Theorem 4.4.5 shows that the superhedging price of f under $\mathbb{G}^n$ is

$$x_*^{\mathbb{G}^n,\mathbb{P}}(f) = \sum_i x_*^{\mathbb{F},\mathbb{P}}(f 1_{\{G\in g_i^n\}}) 1_{\{G\in g_i^n\}}$$

and the corresponding wealth process

$$x_*^{\mathbb{G}^n,\mathbb{P}}(f) + \left(\sum_i H^{\mathbb{F},n,i} 1_{\{G\in g_i^n\}}\right)\cdot S_T \geq f, \mathbb{P}-a.s.$$

In particular, if $f=1$ and if $x_*^{\mathbb{F},\mathbb{P}}(1_{\{G\in g_i^n\}}) < 1$ for some i, then the insider has an arbitrage opportunity.

We observe that $(x_*^{\mathbb{G}^n,\mathbb{P}}(f))_n$ is a nonincreasing sequence and bounded from below by $x_*^{\mathbb{G},\mathbb{P}}(f) := x^*$. So the sequence converges to a limit and $\lim_{n\to\infty} x_*^{\mathbb{G}^n,\mathbb{P}}(f) \geq x_*^{\mathbb{G},\mathbb{P}}(f)$. In the sequel, we will show that this limit is actually the $\mathbb{G}$-superhedging price.

Let $H^{\mathbb{G}}$ be a superhedging strategy for f starting from x_* in the $(\mathbb{G},\mathbb{P})$-market, that is

$$x_* + (H^{\mathbb{G}}\cdot S)_T \geq f, \mathbb{P}-a.s.$$

By Lemma 4.5.4, there exists a sequence of $\mathbb{G}^n$-predictable processes $(H^{\mathbb{G}^n})_n$ which tend to $H^{\mathbb{G}}$ almost surely. Fixing $\varepsilon > 0$, we denote

$$A(n,\varepsilon) := \{\omega : \sup_{t\in[0,T]} |(H^{\mathbb{G}^n}\cdot S)_t - (H^{\mathbb{G}}\cdot S)_t| \leq \varepsilon\}$$

and $C_n := A(n,\varepsilon)\cap\{\mathbb{E}[x_*|\mathscr{G}_0^n] \geq x_* - \varepsilon\}$. We now have the estimation (4.26) on $A(n,\varepsilon)$, that is

$$(H^{\mathbb{G}^n}\cdot S)_t \geq (H^{\mathbb{G}}\cdot S)_t - \varepsilon, \qquad \forall t\in[0,T].$$

Lemma 4.5.10. *The following inequality holds true*

$$2\varepsilon + \mathbb{E}[x_*|\mathscr{G}_0^n] \geq x_*^{\mathbb{G}^n,\mathbb{P}}(f1_{C_n}) \tag{4.33}$$

Proof. Denoting $\tau_n^\varepsilon := \inf\{t \in [0,T] : 2\varepsilon + \mathbb{E}[x_*|\mathscr{G}_0^n] + (H^{\mathbb{G}^n} \cdot S)_t \leq 0\}$, we see that the strategy $H^{\mathbb{G}^n}_{t\wedge\tau_n^\varepsilon}$ with initial capital $2\varepsilon + \mathbb{E}[x_*|\mathscr{G}_0^n]$ is admissible. Furthermore, we deduce from (4.26) that on C_n,

$$2\varepsilon + \mathbb{E}[x_*|\mathscr{G}_0^n] + (H^{\mathbb{G}^n} \cdot S)_t \geq x_* + (H^{\mathbb{G}} \cdot S)_t \geq 0, \qquad \forall t \in [0,T]. \tag{4.34}$$

Thus $\tau_n^\varepsilon = T$ on the set C_n (recall that by definition, $\tau_n^\varepsilon \leq T$). Now, from (4.34) we have that

$$2\varepsilon + \mathbb{E}[x_*|\mathscr{G}_0^n] + (H^{\mathbb{G}^n} \cdot S)_T \geq f1_{C_n}$$

By definition of superhedging price, we conclude that for every n, the inequality in (4.33) holds true. □

Lemma 4.5.11. *Let f be a nonnegative claim satisfying*

$$\mathbb{P}[f > 0] > 0, \qquad \sup_Z \mathbb{E}[Z_T f] < \infty.$$

We fix ω and for each n, let i be the index such that $\omega \in \{G \in g_i^n\}$. There exists a subsequence $(n_k)_{k\geq 1}$ such that

$$\lim_{k\to\infty} \sup_Z \mathbb{E}^{\mathbb{P}}[Z_T f 1_{\{G\in g_i^{n_k}\}} 1_{C_{n_k}}] = \lim_{k\to\infty} \sup_Z \mathbb{E}^{\mathbb{P}}[Z_T f 1_{\{G\in g_i^{n_k}\}}]. \tag{4.35}$$

Proof. Obviously, the LHS of (4.35) is smaller than the RHS of (4.35) for any subsequence. We now consider the reverse inequality. For any $\delta > 0$, for all n, there exists an equivalent local martingale density Z^n such that

$$\sup_Z \mathbb{E}^{\mathbb{P}}[Z_T f 1_{\{G\in g_i^n\}}] - \delta \leq \mathbb{E}[Z_T^n f 1_{\{G\in g_i^n\}}]. \tag{4.36}$$

In order to prove (4.35), we need to prove there exists a subsequence $(n_k)_{k\geq 1}$ such that

$$\lim_{k\to\infty} \mathbb{E}[Z_T^{n_k} f (1_{\{G\in g_i^{n_k}\}} - 1_{\{G\in g_i^{n_k}\}} 1_{C_{n_k}})] = 0. \tag{4.37}$$

Indeed, if (4.37) holds true, then

$$\mathbb{E}[Z_T^{n_k} f 1_{\{G\in g^{n_k}\}}] \leq \mathbb{E}[Z_T^{n_k} f 1_{\{G\in g^{n_k}\}} 1_{C_{n_k}}] + \delta, \qquad \text{if } k \text{ is big enough.} \tag{4.38}$$

From (4.36) and (4.38), we get

$$\begin{aligned}\sup_Z \mathbb{E}^{\mathbb{P}}[Z_T f 1_{\{G\in g_i^{n_k}\}}] - \delta &\leq \mathbb{E}[Z_T^{n_k} f 1_{\{G\in g_i^{n_k}\}}] \leq \mathbb{E}[Z_T^{n_k} f 1_{\{G\in g^{n_k}\}} 1_{C_{n_k}}] + \delta \\ &\leq \sup_Z \mathbb{E}[Z_T f 1_{\{G\in g^{n_k}\}} 1_{C_{n_k}}] + \delta, \text{ if } k \text{ is big enough.}\end{aligned}$$

Letting k to infinity and using the arbitrariness of δ, we obtain (4.35).

Now, we prove (4.37). As n tends to infinity, Dominated convergence theorem (see Theorem 32, page 174 of Protter [2003]) implies that

$$\sup_{t\in[0,T]} |(H^{\mathbb{G}^n}\cdot S)_t - (H^{\mathbb{G}}\cdot S)_t| \to 0, \text{in probability.}$$

This means that there exists a subsequence, which is still denoted by (n), such that $\sup_{t\in[0,T]} |(H^{\mathbb{G}^n}\cdot S)_t - (H^{\mathbb{G}}\cdot S)_t| \to 0, a.s.$ On the other hand, Lévy's "Upward" Theorem (see Theorem 50.3 of Rogers and Williams [1979]) shows that $\mathbb{E}[x_*|\mathscr{G}_0^n] \to x_*, \mathbb{P}-a.s.$ Therefore, for each ω, there exists a number $N(\omega)$ such that $\omega \in C_n$, for all $n > N(\omega)$. Thus,

$$X_n := Z_T^n f(1_{\{G\in g_i^n\}} - 1_{\{G\in g_i^n\}} 1_{C_n}) \to 0, \qquad \mathbb{P}-a.s.$$

Because $0 \leq \mathbb{E}[X_n] \leq \mathbb{E}[Z_T^n f] \leq \sup_Z [Zf]$, for all n, Bolzano−Weierstrass theorem shows that we can find convergent subsequences

$$\lim_{k\to\infty} \mathbb{E}[X_{n_k}] := e_1, \qquad \text{and} \qquad \lim_{k\to\infty} \mathbb{E}[Z_T^{n_k} f] := e_2. \tag{4.39}$$

We observe that $X_{n_k} \leq Z_T^{n_k} f$. Furthermore, by the property of f, we have that $\sup_k \mathbb{E}[Z_T^{n_k} f] < \infty$. Komlós theorem, see Theorem 5.0.19, implies that there exists a random variable $\widehat{Z}f$ such that $0 \leq \widehat{Z}f \in L^1$ and a subsequence, which is still denoted by (n_k) such that

$$\frac{1}{k}\left(Z_T^{n_1} + ... + Z_T^{n_k}\right) f \to \widehat{Z}f, \qquad \mathbb{P}-a.s. \tag{4.40}$$

From (4.39) and Lemma 5.0.15, we have that

$$\lim_{k\to\infty}\frac{1}{k}\mathbb{E}\left[Z_T^{n_1}f+...+Z_T^{n_k}f\right]=e_2.$$

From (4.40) and Lemma 5.0.17, we obtain that $\mathbb{E}[\widehat{Z}f]=e_2$. By Lemma 5.0.15, the sequence $\left(\frac{1}{k}\sum_{i=1}^k X_{n_i}\right)_{k\geq 1}$ converges to zero almost surely. Applying the Extended dominated convergence theorem (see Theorem 5.0.18) to the sequences $\left(\frac{1}{k}\sum_{i=1}^k X_{n_i}\right)_{k\geq 1}$ and $\left(\frac{1}{k}\sum_{i=1}^k Z_T^{n_i}f\right)_{k\geq 1}$, we obtain

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^k\mathbb{E}[X_{n_i}]=0. \tag{4.41}$$

From (4.39), (4.41) and Lemma 5.0.16, we deduce that $e_1=0$ and hence (4.37) holds true. □

Theorem 4.5.12. *Under Assumption 4.2.6, suppose that NUPBR holds for $\mathbb{G}$. Let $f\geq 0$ be a given claim such that $\mathbb{P}[f>0]>0$ and $\sup_{Z\in ELMM}\mathbb{E}[Z_Tf]<\infty$. Then the following convergence holds*

$$\lim_{n\to\infty}x_*^{\mathbb{G}^n,\mathbb{P}}(f)=x_*,\mathbb{P}-a.s. \tag{4.42}$$

Proof. By Theorem 4.4.5, the quantity in the RHS of (4.33) and the quantity $x_*^{\mathbb{G}^n,\mathbb{P}}(f)$ can be computed by the following formulas

$$x_*^{\mathbb{G}^n,\mathbb{P}}(f1_{C_n})(\omega)=\sum_i 1_{\{G\in g_i^n\}}(\omega)\sup_Z\mathbb{E}^{\mathbb{P}}[Z_Tf1_{\{G\in g_i^n\}}1_{C_n}], \tag{4.43}$$

$$x_*^{\mathbb{G}^n,\mathbb{P}}(f)(\omega)=\sum_i 1_{\{G\in g_i^n\}}(\omega)\sup_Z\mathbb{E}^{\mathbb{P}}[Z_Tf1_{\{G\in g_i^n\}}]. \tag{4.44}$$

Fixing ω, Lemma 4.5.10 and Lemma 4.5.11 imply that there exists a sequence (n_k) such that

$$2\varepsilon+\lim_{k\to\infty}\mathbb{E}[x_*|\mathscr{G}_0^{n_k}](\omega)\geq\lim_{k\to\infty}x_*^{\mathbb{G}^{n_k},\mathbb{P}}(f1_{C_{n_k}})(\omega)=\lim_{k\to\infty}x_*^{\mathbb{G}^{n_k},\mathbb{P}}(f)(\omega)\geq x_*(\omega).$$

Applying Lévy's "Upward" Theorem again, we obtain

$$2\varepsilon+x_*(\omega)\geq\lim_{k\to\infty}x_*^{\mathbb{G}^{n_k},\mathbb{P}}(f)(\omega)\geq x_*(\omega).$$

By the arbitrariness of ε, this proves the result. □

This limiting procedure allows us to compute the superhedging price of the claim $f = 1$ for the insider with tools from theory of martingale pricing. Theoretically, if it happens that $G(\omega) = x$, the insider needs to compute

$$\lim_{\varepsilon_n \to 0} \sup_{\overline{\mathbb{P}} \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\overline{\mathbb{P}}}[1_{\{G \in (x-\varepsilon_n, x+\varepsilon_n)\}}]$$

for his initial capital in order to superhedge 1, and thus obtain optimal arbitrage. However, the corresponding strategy is more difficult to compute explicitly. From the practical point of view, a nearly optimal arbitrage is still good. For example, the insider can use an initial capital $\sup_{\overline{\mathbb{P}} \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\overline{\mathbb{P}}}[1_{\{G \in (x-\varepsilon, x+\varepsilon)\}}]$ for some ε small enough and the corresponding strategy $H^{\mathbb{F}}$ which superhedges $1_{\{G \in (x-\varepsilon, x+\varepsilon)\}}$ in the $(\mathbb{F},\mathbb{P})$-market:

$$\sup_{\overline{\mathbb{P}} \in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\overline{\mathbb{P}}}[1_{\{G \in (x-\varepsilon, x+\varepsilon)\}}] + (H^{\mathbb{F}} \cdot S)_T \geq 1_{\{G \in (x-\varepsilon, x+\varepsilon)\}}.$$

It is remarked that the strategy is computed under $\mathbb{F}$, where NFLVR holds, and we have tools to find such strategies.

4.6 Successive initial enlargement of filtrations

In this section, we will investigate the case when the insider receives information successively at deterministic fixed times. The additional information is represented by a random time τ, which is assumed to be $\mathscr{F}_T$-measurable. This situation can be considered a discrete version of the progressive enlargement of filtration setting. We refer to the thesis Aksamit [2014] for a study of progressive enlargement of filtrations with random times. In her thesis, some arbitrage profits are provided explicitly.

Let Δ_n be a partition of $[0,T]$, that is $\Delta_n := \{0 = T_0 \leq T_1 \leq ... \leq T_n = T\}$. We assume that the insider gains private information about τ only at T_i, for $i = 1, ...n$. In other words, her filtration is $\mathbb{G} = (\mathscr{G}_t)_t$, where

$$\mathscr{G}_t := \mathscr{F}_t \vee \sigma(1_{\tau \leq T_0}, ..., 1_{\tau \leq T_i}), \qquad t \in [T_i, T_{i+1}), \qquad i = 0, ..., n-1.$$

We observe that at the end of each period, the insider receives more information than what she has at the beginning of that period. This situation is slightly different from initial enlargement of filtration setting. We also define a further larger filtration $\mathbb{H}^n = (\mathscr{H}_t^n)_{t\in[0,T]}$, where $\mathscr{H}_t^n := \mathscr{F}_t \vee \sigma(1_{\tau\leq T_0},...,1_{\tau\leq T_n}), \forall t\in[0,T]$. The filtration $\mathbb{H}$ is initially enlarged by the set of information $\{\tau \leq T_i, i=0,...,(n-1)\}$. The study under the filtration $\mathbb{H}$ was developed in previous sections.

4.6.1 NUPBR and Log-utility

The following result is similar to Lemma 4.4.1. However, we only have an upper bound for the expected log-utility of the insider.

Lemma 4.6.1. *The expected log-utility of the insider under $\mathbb{G}$ is bounded by*

$$\sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] \leq \sum_{i=0}^{n} \sup_{H\in\mathscr{A}_1^{\mathbb{F}}} \mathbb{E}^{\mathbb{P}}[1_{T_i<\tau\leq T_{i+1}} \log V_T^{1,H}].$$

Proof. First, we note that a $\mathbb{G}$-strategy is of the form

$$\begin{aligned} H_t^{\mathbb{G}} &= H_t^{\mathbb{F},1} 1_{t\leq T_1} + (H_t^{\mathbb{F},2,1} 1_{\tau\leq T_1} + H_t^{\mathbb{F},2,2} 1_{\tau>T_1}) 1_{T_1<t\leq T_2} + ... \\ &= (H_t^{\mathbb{F},1} 1_{t\leq T_1} + H_t^{\mathbb{F},2,1} 1_{T_1<t\leq T_2} + ...) 1_{\tau\leq T_1} \\ &\quad + (H_t^{\mathbb{F},1} 1_{t\leq T_1} + H_t^{\mathbb{F},2,2} 1_{T_1<t\leq T_2} + ...) 1_{T_1<\tau\leq T_2} + ... \\ &= \tilde{H}_t^{\mathbb{F},1} 1_{\tau\leq T_1} + \tilde{H}_t^{\mathbb{F},2} 1_{T_1<\tau\leq T_2} + ... + H_t^{\mathbb{F},n} 1_{T_n-1<\tau\leq T_n} + \tilde{H}_t^{\mathbb{F},n+1} 1_{T_n<\tau}, \end{aligned}$$

where $\tilde{H}^{\mathbb{F},i}, i=1...(n+1)$ are $\mathbb{F}$-predictable. It means that a $\mathbb{G}$-strategy is also a $\mathbb{H}$-strategy. The proof continues as in Lemma 4.4.1 for the part ($\leq$). □

Remark 4.6.2. *The inverse inequality in Lemma 4.6.1 does not holds because in general, a $\mathbb{H}$-predictable strategy is not a $\mathbb{G}$-predictable strategy. Hence, we do not have the inverse inequality.*

By analogy to Theorem 4.4.4, we obtain an upper bound for the expected log-utility of the insider. Hence, if the upper bound is finite then the condition NUPBR for the insider holds true.

Proposition 4.6.3. *Under Assumption (4.2.6), we have that*

$$\sup_{H\in\mathscr{A}_1^{\mathbb{G}}} \mathbb{E}^{\mathbb{P}}[\log V_T^{1,H}] \leq -\sum_{i=0}^{n} \mathbb{P}[T_i < \tau \leq T_{i+1}] \log \mathbb{P}[T_i < \tau \leq T_{i+1}] + \sum_{i=0}^{n} \inf_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{T_i<\tau\leq T_{i+1}} \log \frac{1}{Z_T}\right]. \tag{4.45}$$

Proof. Using similar arguments as in Lemma 4.4.2, we obtain for $i \in \{0,\dots n\}$ that

$$\sup_{H\in\mathscr{A}_1^{\mathbb{F}}} \mathbb{E}^{\mathbb{P}}[1_{T_i<\tau\leq T_{i+1}} \log V_T^{1,H}] = - \mathbb{P}[T_i < \tau \leq T_{i+1}] \log \mathbb{P}[T_i < \tau \leq T_{i+1}] + \inf_{Z\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\mathbb{P}}\left[1_{T_i<\tau\leq T_{i+1}} \log \frac{1}{Z_T}\right].$$

The proof is complete by using Lemma 4.6.1. □

4.6.2 Superhedging

Unlike the case with initial information, the additional information here is revealed progressively as time goes on. Thus, the insider has possibilities to update his strategy when new information is available. We now find a way to compute the superhedging price. Then, the concept of optimal arbitrage is defined in a similar way to Definition 4.3.2.

The usual superhedging price at time s of a $\mathscr{G}_t$-measurable claim f_t is defined as follows

$$x_{*,s,t}^{\mathbb{G}}(f_t) := \inf\left\{ x \text{ is } \mathscr{G}_s\text{-measurable } : \exists H^{\mathbb{G}} \in \mathscr{A}_x^{\mathbb{G}} : x + \int_s^t H_u^{\mathbb{G}} dS_u \geq f_t, \mathbb{P}-a.s. \right\}.$$

We first notice that the superhedging price can be computed by using backward recursion. More precisely, in order to superhedge f at time T_n, we replicate the capital needed for the claim f at time T_{n-1}, and so on. The following argument works well in general, not only for the filtration $\mathbb{G}$.

Lemma 4.6.4. *It holds that* $x_{*,T_0,T_n}^{\mathbb{G}}(f) = x_{*,T_0,T_1}^{\mathbb{G}}(\dots(x_{*,T_{n-1},T_n}^{\mathbb{G}}(f)))$, *where* f *is a* $\mathscr{G}_T$*-measurable claim.*

Proof. First, we find the initial capital and the corresponding strategy which are used at time T_{n-1} in order to hedge the claim f, i.e. $x^{\mathbb{G}}_{*,T_{n-1},T_n}(f)+\int_{T_{n-1}}^{T_n} H^{\mathbb{G}}_u du \geq f$. Next, we find the initial capital and the strategy to hedge the claim $x^{\mathbb{G}}_{*,T_{n-1},T_n}(f)$ at the time T_{n-2}, i.e. $x^{\mathbb{G}}_{*,T_{n-2},T_{n-1}}(x^{\mathbb{G}}_{*,T_{n-1},T_n}(f))+\int_{T_{n-1}}^{T_n} H^{\mathbb{G}}_u du \geq x^{\mathbb{G}}_{*,T_{n-1},T_n}(f)$. We repeat this argument and easily deduce that $x^{\mathbb{G}}_{*,T_0,T_1}(....(x^{\mathbb{G}}_{*,T_{n-1},T_n}(f))) \geq x^{\mathbb{G}}_{*,T_0,T_n}(f)$.

For the converse inequality, let $x^{\mathbb{G}}_{*,T_0,T_n}$ be the initial capital and $H^{\mathbb{G}}$ be the hedging strategy of f at time T_0, i.e. $x^{\mathbb{G}}_{*,T_0,T_n}(f)+\int_{T_0}^{T_{n-1}} H^{\mathbb{G}}_u dS_u+\int_{T_{n-1}}^{T_n} H^{\mathbb{G}}_u dS_u \geq f$. Therefore,

$$x^{\mathbb{G}}_{*,T_0,T_n}(f)+\int_{T_0}^{T_{n-1}} H^{\mathbb{G}}_u dS_u \geq x^{\mathbb{G}}_{*,T_{n-1},T_n}(f).$$

We rewrite the inequality as follows

$$x^{\mathbb{G}}_{*,T_0,T_n}(f)+\int_{T_0}^{T_{n-2}} H^{\mathbb{G}}_u dS_u+\int_{T_{n-2}}^{T_{n-1}} H^{\mathbb{G}}_u dS_u \geq x^{\mathbb{G}}_{*,T_{n-1},T_n}(f).$$

Repeating this argument leads to $x^{\mathbb{G}}_{*,T_0,T_n}(f) \geq x^{\mathbb{G}}_{*,T_0,T_1}(....(x^{\mathbb{G}}_{*,T_{n-1},T_n}(f)))$ and we get the result. □

Lemma 4.6.4 gives us a dynamic way to compute superhedging prices. The advantage of this approach is that it allows us to update our strategies at each period. We now apply it to our successive enlargement setting with the discretization Δ_n.

At time T_{n-1}, since τ is $\mathscr{F}_T$-measurable, we have no problem when computing the superhedging price at time T_{n-1} of the claim f. The insider is informed by the random variables $1_{\tau\leq T_1}, 1_{T_1<\tau\leq T_2}, ..., 1_{T_{n-1}<\tau\leq T_{n-2}}$ and $1_{T_{n-1}<\tau}$. We define new probability measures

$$\left.\frac{d\mathbb{Q}^i}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{\mathbb{P}[T_{i-1}<\tau\leq T_i|\mathscr{F}_t]}{\mathbb{P}[T_{i-1}<\tau\leq T_i]} := M^i_t, \qquad t\in[0,T], i=1,...,n-1. \tag{4.46}$$

$$\left.\frac{d\mathbb{Q}^n}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{\mathbb{P}[T_{n-1}<\tau|\mathscr{F}_t]}{\mathbb{P}[T_{n-1}<\tau]} := M^n_t \qquad t\in[0,T].$$

Theorem 4.4.5 gives us the price of f at time T_{n-1}

$$x^{\mathbb{G},\mathbb{P}}_{*,T_{n-1},T_n}(f) = \sum_{i=1}^{n-2} x^{\mathbb{F},\mathbb{P}}_{*,T_{n-1},T_n}(f 1_{T_{i-1}<\tau\leq T_i}) 1_{T_{i-1}<\tau\leq T_i} \tag{4.47}$$
$$+ x^{\mathbb{F},\mathbb{P}}_{*,T_{n-1},T_n}(f 1_{T_{n-2}<\tau\leq T_{n-1}}) 1_{T_{n-2}<\tau\leq T_{n-1}} + x^{\mathbb{F},\mathbb{P}}_{*,T_{n-1},T_n}(f 1_{T_{n-1}<\tau}) 1_{T_{n-1}<\tau}$$

where $x^{\mathbb{F},\mathbb{P}}_{*,T_{n-1},T_n}(f 1_{T_{i-1}<\tau\leq T_i}) = \sup_{\tilde{\mathbb{P}}\in ELMM(\mathbb{F},\mathbb{P})} \mathbb{E}^{\tilde{\mathbb{P}}}[f 1_{T_i-1<\tau\leq T_i} | \mathscr{F}_{T_{n-1}}]$.

At time T_{n-2}, i.e. on the period $[T_{n-2}, T_{n-1})$, we need to super-replicate the random variable $x^{\mathbb{G},\mathbb{P}}_{*,T_{n-1},T_n}(f)$. This situation differs from the case of initial enlargement in many aspects. First, the random variables $1_{\tau\leq T_1}, 1_{T_1<\tau\leq T_2}, \ldots$ and $1_{T_{n-2}<\tau}$ are not $\mathscr{F}_{T_{n-1}}$-measurable. Second, the $\mathscr{G}_{T_{n-1}}$-measurable random variable $1_{\tau\leq T_{n-1}}$ (at the end of the period) is not measurable with respect to $\mathscr{G}_{T_{n-2}}$ (the beginning of the period) and this implies our present setting is not the same as in the initial enlargement case. Furthermore, the results in Section 4.4.2 allow us to superhedge the claims which are measurable with respect to $\mathscr{F}_{T_{n-1}} \vee \sigma(1_{\tau\leq T_1}, \ldots, 1_{\tau\leq T_{n-2}}) \subsetneq \mathscr{G}_{T_{n-1}}$ and it is noticed that $x^{\mathbb{G},\mathbb{P}}_{*,T_{n-1},T_n}(f)$ is $\mathscr{G}_{T_{n-1}}$-measurable. To conclude, the results in Section 4.4.2 can not be applied directly here and it is necessary to find another way to get rid of this measurability issue.

Let f_{n-1} be the minimal random variable satisfying two conditions: it dominates $x^{\mathbb{G},\mathbb{P}}_{*,T_{n-1},T_n}(f)$ and has to be $\mathscr{F}_{T_{n-1}} \vee \sigma(1_{\tau\leq T_1}, \ldots, 1_{\tau\leq T_{n-2}})$-measurable.

Lemma 4.6.5. *We have that* $x^{\mathbb{G},\mathbb{P}}_{*,T_{n-2},T_{n-1}}\left(x^{\mathbb{G},\mathbb{P}}_{*,T_{n-1},T_n}(f)\right) = x^{\mathbb{G},\mathbb{P}}_{*,T_{n-2},T_{n-1}}(f_{n-1})$.

Proof. The inequality ($\leq$) is obvious because $x^{\mathbb{G},\mathbb{P}}_{*,T_{n-1},T_n}(f) \leq f_{n-1}$. For the converse inequality ($\geq$), let x be any $\mathscr{G}_{T_{n-2}}$-measurable random variable such that there exists a strategy $H^{\mathbb{G}}$ satisfying

$$x + \int_{T_{n-2}}^{T_{n-1}} H^{\mathbb{G}}_u dS_u \geq x^{\mathbb{G},\mathbb{P}}_{*,T_{n-1},T_n}(f).$$

We observe that the quantity in the LHS is $\mathscr{F}_{T_{n-1}} \vee \sigma(1_{\tau\leq T_1}, \ldots, 1_{\tau\leq T_{n-1}})$-measurable and it can not be smaller than f_{n-1},

$$x + \int_{T_{n-2}}^{T_{n-1}} H^{\mathbb{G}}_u dS_u \geq f_{n-1}.$$

It implies that $x \geq x^{\mathbb{G},\mathbb{P}}_{*,T_{n-2},T_{n-1}}(f_{n-1})$ and the proof is complete. $\square$

Lemma 4.6.5 tells us that in order to compute the price at time T_{n-2} of the claim $x^{\mathbb{G},\mathbb{P}}_{*,T_{n-1},T_n}(f)$, we need to compute the price at time T_{n-2} of the claim f_{n-1}. The first $(n-2)$ components in the RHS of (4.47) are $\mathscr{F}_{T_{n-1}} \vee \sigma(1_{\tau\leq T_1},...,1_{\tau\leq T_{n-2}})$-measurable but not the two last components. The claim f_{n-1} takes the following form

$$f_{n-1} = \sum_{i=1}^{n-2} x^{\mathbb{F},\mathbb{P}}_{*,T_{n-1},T_n}(f 1_{T_{i-1}<\tau\leq T_i}) 1_{T_{i-1}<\tau\leq T_i} + \tilde{f}^{n-1}_{T_{n-1}} 1_{T_{n-2}<\tau} \tag{4.48}$$

$$:= \sum_{i=1}^{n-2} \tilde{f}^{i}_{T_{n-1}} 1_{T_{i-1}<\tau\leq T_i} + \tilde{f}^{n-1}_{T_{n-1}} 1_{T_{n-2}<\tau} \tag{4.49}$$

where $\tilde{f}^{n-1}_{T_{n-1}} = \max\left(x^{\mathbb{F},\mathbb{P}}_{*,T_{n-1},T_n}(f 1_{T_{n-2}<\tau\leq T_{n-1}}), x^{\mathbb{F},\mathbb{P}}_{*,T_{n-1},T_n}(f 1_{T_{n-1}<\tau})\right)$.

Now, we compute the superhedging price of the claim f_{n-1} in (4.48) at time T_{n-2}. This step is done by the same argument as the one used at time T_{n-1} with small modifications, however, we will make a clear discussion because of problems with measurability. To do so, we keep using the probability measure $\mathbb{Q}^i, i = 1,...,(n-2)$ and define a new probability measure which works for the union of $\{T_{n-2} < \tau \leq T_{n-1}\}$ and $\{T_{n-1} < \tau\}$,

$$\left.\frac{d\tilde{\mathbb{Q}}^{n-1}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{\mathbb{P}[T_{n-2} < \tau|\mathscr{F}_t]}{\mathbb{P}[T_{n-2} < \tau]} := \tilde{M}^{n-1}_t, \qquad t \in [0,T].$$

Proposition 4.6.6. *The price of f_{n-1} at time T_{n-2} is*

$$x^{\mathbb{G},\mathbb{P}}_{*,T_{n-2},T_{n-1}}(f_{n-1}) = \sum_{i=1}^{n-2} x^{\mathbb{F},\mathbb{P}}_{*,T_{n-2},T_{n-1}}\left(\tilde{f}^{i}_{T_{n-1}} 1_{M^i_{T_{n-1}}>0}\right) 1_{T_{i-1}<\tau\leq T_i} + x^{\mathbb{F},\mathbb{P}}_{*,T_{n-2},T_{n-1}}\left(\tilde{f}^{n-1}_{T_{n-1}} 1_{\tilde{M}^{n-1}_{T_{n-1}}>0}\right) 1_{T_{n-2}<\tau}.$$

Proof. Using techniques of non-equivalent measure changes, we compute the price of f_{n-1} under the probability measures $\mathbb{Q}^i, i = 1,...,(n-2)$ and $\tilde{\mathbb{Q}}^{n-1}$. For $i = 1,...,(n-2)$ and under $\mathbb{Q}^i$, the claim f_{n-1} becomes $\tilde{f}^{i}_{T_{n-1}}$ and its price is

$x^{\mathbb{F},\mathbb{Q}^i}_{*,T_{n-2},T_{n-1}}(\tilde{f}^i_{T_{n-1}})$. The case with $\tilde{\mathbb{Q}}^{n-1}$ is computed similarly. Thus, repeating the argument in the proof of Theorem 4.4.5, the price of f_{n-1} under $\mathbb{G}$ is

$$x^{\mathbb{G},\mathbb{P}}_{*,T_{n-2},T_{n-1}}(f_{n-1}) = \sum_{i=1}^{n-2} x^{\mathbb{F},\mathbb{Q}^i}_{*,T_{n-2},T_{n-1}}(\tilde{f}^i_{T_{n-1}})1_{T_{i-1}<\tau\leq T_i} + x^{\mathbb{F},\tilde{\mathbb{Q}}^{n-1}}_{*,T_{n-2},T_{n-1}}\left(\tilde{f}^{n-1}_{T_{n-1}}\right)1_{T_{n-2}<\tau}.$$

By using Theorem 2.4.1 in Chapter 2, we obtain

$$x^{\mathbb{F},\mathbb{Q}^i}_{*,T_{n-2},T_{n-1}}\left(\tilde{f}^i_{T_{n-1}}\right) = x^{\mathbb{F},\mathbb{P}}_{*,T_{n-2},T_{n-1}}\left(\tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0}\right), \qquad i=1,...,(n-2) \quad (4.50)$$

$$x^{\mathbb{F},\tilde{\mathbb{Q}}^{n-1}}_{*,T_{n-2},T_{n-1}}\left(\tilde{f}^{n-1}_{T_{n-1}}\right) = x^{\mathbb{F},\mathbb{P}}_{*,T_{n-2},T_{n-1}}\left(\tilde{f}^{n-1}_{T_{n-1}}1_{\tilde{M}^{n-1}_{T_{n-1}}>0}\right),$$

and the proof is complete. □

Remark 4.6.7. *Let us discuss the optimal superhedging strategy. Let $H^{\mathbb{F},i}$ be the hedging strategy for the claim $\tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0}$ under $\mathbb{P}$, that is*

$$x^{\mathbb{F},\mathbb{P}}_{*,T_{n-2},T_{n-1}}(\tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0}) + \int_{T_{n-2}}^{T_{n-1}} H^{\mathbb{F},i}_u dS_u \geq \tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0}, \mathbb{P}-a.s.$$

By Proposition 4.6.6, this hedging strategy is used on the event $\{T_{i-1}<\tau\leq T_i\}$,

$$1_{T_{i-1}<\tau\leq T_i}\left(x^{\mathbb{F},\mathbb{P}}_{*,T_{n-2},T_{n-1}}(\tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0}) + \int_{T_{n-2}}^{T_{n-1}} H^{\mathbb{F},i}_u dS_u\right) \geq \tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0}1_{T_{i-1}<\tau\leq T_i}$$

In order to superhedge the claim f_{n-1}, we will prove that

$$\tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0}1_{T_{i-1}<\tau\leq T_i} = \tilde{f}^i_{T_{n-1}}1_{T_{i-1}<\tau\leq T_i}, \mathbb{P}-a.s.$$

By the minimum principle for nonnegative supermartingales, see Proposition II.3.4 of Revuz and Yor [1999], if the martingale M^i reaches zeros at time s, then $M^i_t = 0$ for all $t\geq s$ and hence, $1_{M^i_{T_{n-1}}>0}1_{M^i_{T_n}>0} = 1_{M^i_{T_n}>0} = 1_{T_{i-1}<\tau\leq T_i}$. Therefore, the insider will use the initial capital $x^{\mathbb{F},\mathbb{P}}_{*,T_{n-2},T_{n-1}}(\tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0})$ and the strategy $H^{\mathbb{F},i}$ in order to hedge f_{n-1} on the event $\{T_{i-1}<\tau\leq T_i\}$,

$$1_{T_{i-1}<\tau\leq T_i}\left(x^{\mathbb{F},\mathbb{P}}_{*,T_{n-2},T_{n-1}}(\tilde{f}^i_{T_{n-1}}1_{M^i_{T_{n-1}}>0}) + \int_{T_{n-2}}^{T_{n-1}} H^{\mathbb{F},i}_u dS_u\right) \geq \tilde{f}^i_{T_{n-1}}1_{T_{i-1}<\tau\leq T_i}.$$

The following theorem is the main result of this section.

Theorem 4.6.8. *The superhedging price of f at time zero is obtained by applying Lemma 4.6.5 and Proposition 4.6.6 recursively.*

4.6.3 An example with time of supremum on fixed time horizon

Assume that the risky asset is a geometric Brownian motion $dS_t = S_t dW_t, t \in [0,T]$ with $S_0 = 1$. We consider the random time $\tau = \sup\{t \le T : S_t = \sup_{[0,t]} S_u\}$. Let $\Delta_3 = \{T_0 = 0 < T_1 < T_2 < T_3 = T\}$ be a partition of the interval $[0,T]$. The insider will be informed about τ at T_1 and T_2. Her filtration is given by

$$\mathscr{G}_t = \begin{cases} \mathscr{F}_t, & \text{if } 0 \le t < T_1 \\ \mathscr{F}_t \vee \sigma(1_{\tau \le T_1}), & \text{if } T_1 \le t < T_2 \\ \mathscr{F}_t \vee \sigma(1_{\tau \le T_1}, 1_{\tau \le T_2}), & \text{if } T_2 < t \le T_3. \end{cases}$$

We denote

$$\left.\frac{d\mathbb{Q}^1}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{\mathbb{P}[\tau \le T_1|\mathscr{F}_t]}{\mathbb{P}[\tau \le T_1]} = M_t^1, \qquad \left.\frac{d\mathbb{Q}^2}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{\mathbb{P}[T_1 < \tau \le T_2|\mathscr{F}_t]}{\mathbb{P}[T_1 < \tau \le T_2]} = M_t^2,$$
$$\left.\frac{d\mathbb{Q}^3}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{\mathbb{P}[T_2 < \tau|\mathscr{F}_t]}{\mathbb{P}[T_2 < \tau]} = M_t^3, \qquad t \in [0,T].$$

We compute the price of the claim $f = 1$ at time T_2 using Theorem 4.4.5

$$\begin{aligned} x_{*,T_2,T}^{\mathbb{G},\mathbb{P}}(1) &= x_{*,T_2,T}^{\mathbb{F},\mathbb{P}}(1_{M_T^1>0})1_{\tau \le T_1} + x_{*,T_2,T}^{\mathbb{F},\mathbb{P}}(1_{M_T^2>0})1_{T_1<\tau \le T_2} + x_{*,T_2,T}^{\mathbb{F},\mathbb{P}}(1_{M_T^3>0})1_{T_2<\tau} \\ &= \mathbb{P}[\tau \le T_1|\mathscr{F}_{T_2}]1_{\tau \le T_1} + \mathbb{P}[T_1 < \tau \le T_2|\mathscr{F}_{T_2}]1_{T_1<\tau \le T_2} + \mathbb{P}[T_2 < \tau|\mathscr{F}_{T_2}]1_{T_2<\tau}. \end{aligned}$$

The price of $x_{*,T_2,T}^{\mathbb{G},\mathbb{P}}(1)$ is computed backward by using the approach in Section 4.6.2. Let us denote

$$f_2 = \mathbb{P}[\tau \le T_1|\mathscr{F}_{T_2}]1_{\tau \le T_1} + \max\left(\mathbb{P}[T_1 < \tau \le T_2|\mathscr{F}_{T_2}], \mathbb{P}[T_2 < \tau|\mathscr{F}_{T_2}]\right)1_{T_1<\tau}$$

and

$$\left.\frac{d\tilde{\mathbb{Q}}^2}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \frac{\mathbb{P}[T_1 < \tau|\mathscr{F}_t]}{\mathbb{P}[T_1 < \tau]} := \tilde{M}_t^2, \qquad t \in [0,T].$$

The price of the claim 1 at time T_1 can be computed by Proposition 4.6.6

$$x^{\mathbb{G},\mathbb{P}}_{*,T_1,T_2}(f_2) = \mathbb{E}^{\mathbb{P}}\left[\mathbb{P}[\tau \le T_1|\mathscr{F}_{T_2}]1_{M^1_{T_2}>0}|\mathscr{F}_{T_1}\right]1_{\tau\le T_1}$$
$$+\mathbb{E}^{\mathbb{P}}\left[\max\left(\mathbb{P}[T_1<\tau\le T_2|\mathscr{F}_{T_2}],\mathbb{P}[T_2<\tau|\mathscr{F}_{T_2}]\right)1_{\tilde{M}^2_{\tilde{T}_2}>0}|\mathscr{F}_{T_1}\right]1_{T_1<\tau}.$$

Finally, the price of the claim 1 at time 0 is

$$x^{\mathbb{G},\mathbb{P}}_{*,0,T}(f) = \mathbb{E}^{\mathbb{P}}\Big[\max\Big(\mathbb{E}^{\mathbb{P}}\left[\mathbb{P}[\tau \le T_1|\mathscr{F}_{T_2}]1_{M^1_{T_2}>0}|\mathscr{F}_{T_1}\right],$$
$$\mathbb{E}^{\mathbb{P}}\left[\max\left(\mathbb{P}[T_1<\tau\le T_2|\mathscr{F}_{T_2}],\mathbb{P}[T_2<\tau|\mathscr{F}_{T_2}]\right)1_{\tilde{M}^2_{\tilde{T}_2}>0}|\mathscr{F}_{T_1}\right]\Big)\Big].$$

The computation can be made explicitly. For example, we have

$$\mathbb{P}[\tau > T_1|\mathscr{F}_{T_2}] = 1_{\sup_{(T_1,T_2]}S_u>\sup_{[0,T_1]}S_u} + 1_{\sup_{(T_1,T_2]}S_u\le\sup_{[0,T_1]}S_u}\mathbb{P}\left[\sup_{[T_2,T]}S_u > \sup_{[0,T_1]}S_u\right]$$
$$= 1_{\sup_{(T_1,T_2]}S_u>\sup_{[0,T_1]}S_u} + 1_{\sup_{(T_1,T_2]}S_u\le\sup_{[0,T_1]}S_u}\mathbb{P}\left[\sup_{[0,T-T_2]}\tilde{S}_u > \frac{S^*_{T_1}}{S_{T_2}}\right],$$

$$\mathbb{P}[\tau \le T_1|\mathscr{F}_{T_2}] = 1-\mathbb{P}[\tau > T_1|\mathscr{F}_{T_2}] = 1_{\sup_{(T_1,T_2]}S_u\le\sup_{[0,T_1]}S_u}\left(1-\mathbb{P}\left[\sup_{[0,T-T_2]}\tilde{S}_u > \frac{S^*_{T_1}}{S_{T_2}}\right]\right),$$

$$\mathbb{P}[T_1<\tau \le T_2|\mathscr{F}_{T_2}] = \mathbb{P}[\tau > T_1|\mathscr{F}_{T_2}] - \mathbb{P}[\tau > T_2|\mathscr{F}_{T_2}] = 1_{\sup_{(T_1,T_2]}S_u>\sup_{[0,T_1]}S_u}$$
$$+1_{\sup_{(T_1,T_2]}S_u\le\sup_{[0,T_1]}S_u}\mathbb{P}\left[\sup_{[0,T-T_2]}\tilde{S}_u > \frac{S^*_{T_1}}{S_{T_2}}\right] - \mathbb{P}\left[\sup_{[0,T-T_2]}\tilde{S}_u > \frac{S^*_{T_2}}{S_{T_2}}\right],$$
$$= 1_{\sup_{(T_1,T_2]}S_u>\sup_{[0,T_1]}S_u}\left(1-\mathbb{P}\left[\sup_{[0,T-T_2]}\tilde{S}_u > \frac{S^*_{T_2}}{S_{T_2}}\right]\right),$$

$$\mathbb{P}[\tau > T_2|\mathscr{F}_{T_2}] = \mathbb{P}\left[\sup_{[0,T-T_2]}\tilde{S}_u > \frac{S^*_{T_2}}{S_{T_2}}\right]...$$

These conditional probabilities can be computed by using the law of drifted Brownian motion in Section 3.2.2 of Jeanblanc et al. [2009] [2]. Other quantities can be

[2] For $x \ge 1$, the law of drifted Brownian motion in Section 3.2.2 of Jeanblanc et al. [2009] gives us

$$\mathbb{P}\left[\sup_{[0,t]}S_u > x\right] = \mathbb{P}\left[\sup_{[0,t]}\left(W_u - \frac{1}{2}u\right) > \ln x\right] = \mathscr{N}\left(\frac{-\ln x - \frac{1}{2}t}{\sqrt{t}}\right) - x\mathscr{N}\left(\frac{-\ln x + \frac{1}{2}t}{\sqrt{t}}\right).$$

computed similarly.

Chapter 5

Appendix

Definition 5.0.9 (Almost uniform convergence)**.** *Let* $(X, \mathscr{B}, \mu)$ *be a measurable space. We say that a sequence of measurable functions* f_n *converges to* f almost uniformly *if for every* $\varepsilon > 0$*, there exists a measurable set* $E \in \mathscr{B}$ *of measure* $\mu(E) < \varepsilon$ *such that* f_n *converges uniformly to* f *on* $X \backslash E$*.*

If $\mu(X)$ is finite, then we have a nice result as below.

Theorem 5.0.10 (Egorov's theorem)**.** *Let* $(X, \mathscr{B}, \mu)$ *be a measurable space so that* $\mu(X) < \infty$*. Let* $f_n, f : X \to \mathbb{R}$ *be measurable functions. Then* f_n *converges to* f *pointwise almost everywhere if and only if* f_n *converges to* f *almost uniformly.*

Lemma 5.0.11. *Let* $(X, \mathscr{B}, \mu)$ *be a measurable space, let* $f_n : X \to \mathbb{R}$ *be a sequence of measurable functions converging pointwise almost everywhere as* $n \to \infty$ *to a measurable limit* $f : X \to \mathbb{R}$ *and for each n, let* $f_{n,m} : X \to \mathbb{R}$ *be a sequence of measurable functions converging pointwise almost everywhere as* $m \to \infty$ *to* f_n*. If* $\mu(X)$ *is finite, there exists a sequence* $(m_n)_{n \geq 1}$ *such that* f_{n,m_n} *converges pointwise almost everywhere to* f*.*

Proof. Fixing $\varepsilon > 0$, we will find a sequence (m_n) such that f_{n,m_n} converges to f almost uniformly. Then we completes the proof by using Theorem 5.0.10.

Because f_n converges to f pointwise almost everywhere, Theorem 5.0.10 implies that f_n converges to f almost uniformly. There exists a measurable set

$E \in \mathscr{B}, \mu(E) < \varepsilon$ such that f_n converges to f uniformly on $X \backslash E$, i.e.

$$\text{there exists } N_\varepsilon \text{ such that } |f_n(x) - f(x)| < \varepsilon, \qquad \forall n \geq N_\varepsilon, \forall x \in X \backslash E. \tag{5.1}$$

For each n, since $f_{n,m}$ converges to f_n pointwise almost everywhere, the same argument shows that there exists $E_n \in \mathscr{B}, \mu(E_n) < \varepsilon/2^n$ such that $f_{n,m}$ converges to f_n uniformly on $X \backslash E_n$, i.e.

$$\text{there exists } M_n \text{ such that } |f_{n,m}(x) - f_n(x)| < 1/n, \qquad \forall m \geq M_n, \forall x \in X \backslash E_n. \tag{5.2}$$

Let us choose the sequence $(M_n)_{n \geq \max\{N_\varepsilon, 1/\varepsilon\}}$ and a measurable set F such that

$$X \backslash F = (X \backslash E) \cap \bigcap_{n \geq \max\{N_\varepsilon, 1/\varepsilon\}} (X \backslash E_n).$$

From (5.1) and (5.2), we have the following estimation

$$\begin{aligned} |f_{n,M_n}(x) - f(x)| &\leq |f_{n,M_n}(x) - f_n(x)| + |f_n(x) - f(x)| \\ &\leq 2\varepsilon, \qquad \forall n \geq \max\{N_\varepsilon, 1/\varepsilon\}, \qquad \forall x \in X \backslash F. \end{aligned}$$

We now compute $F = E \cup \left(\bigcup_{n \geq \max\{N_\varepsilon, 1/\varepsilon\}} E_n \right)$ and

$$\mu(F) \leq \mu(E) + \sum_{n \geq \max\{N_\varepsilon, 1/\varepsilon\}} \mu(E_n) \leq \varepsilon + \sum_{n \geq \max\{N_\varepsilon, 1/\varepsilon\}} \varepsilon/2^n \leq 2\varepsilon.$$

This implies that the sequence f_{n,M_n} converges to f almost uniformly. □

Lemma 5.0.12. *Assume that X, Y are two independent exponential random variables with parameters α, β, respectively. Then the random variable $Z = \frac{\alpha X}{\beta Y}$ has density $1/(1+z)^2$.*

Proof. For $z > 0$. we compute the cumulative distribution of Z

$$\begin{aligned} \mathbb{P}[Z \leq z] = \mathbb{P}\left[Y \geq \frac{\alpha X}{\beta z} \right] &= \int_0^\infty \left(\int_{(\alpha x)/(\beta z)}^\infty \beta e^{-\beta y} dy \right) \alpha e^{-\alpha x} dx \\ &= \int_0^\infty e^{\frac{-\alpha x}{z}} \alpha e^{-\alpha x} dx = \frac{z}{1+z}. \end{aligned}$$

The density of Z is obtained by taking derivative of the cumulative distribution of Z with respect to z. □

Definition 5.0.13 (Optional projection - Definition 5.2.1 of Jeanblanc et al. [2009])**.** *Let X be a bounded (or positive) process, and $\mathbb{F}$ a given filtration. The optional projection of X is the unique optional process ^{o} which satisfies*

$$\mathbb{E}[X_\tau 1_{\tau<\infty}] = \mathbb{E}[{}^oX_\tau 1_{\tau<\infty}]$$

for any $\mathbb{F}$-stopping time τ.

The following result helps us to find the compensator of a process when passing to smaller filtrations.

Lemma 5.0.14. *Let $\mathbb{G},\mathbb{H}$ be filtrations such that $\mathscr{G}_t \subset \mathscr{H}_t$, for all $t \in [0,T]$. Suppose that the process $M_t := X_t - \int_0^t \lambda_u du$ is a $\mathbb{H}$-martingale, where $\lambda \geq 0$. Then the process $M_t^G := X_t - \int_0^t {}^o\lambda_u du$ is a $\mathbb{G}$-martingale, where ${}^o\lambda$ is the optional projection of λ onto $\mathbb{G}$.*

Proof. Since $\lambda_u \geq 0$, the optional projection ${}^o\lambda$ exists and for fixed u, it holds that ${}^o\lambda_u = \mathbb{E}[\lambda_u|\mathscr{G}_u]$ almost surely. If $0 \leq s < t$ and H is bounded and $\mathscr{G}_s$-measurable, then, by Fubini's Theorem

$$\begin{aligned}
\mathbb{E}[H(M_t^G - M_s^G)] &= \mathbb{E}[H(X_t - X_s)] - \int_s^t \mathbb{E}[H\mathbb{E}[\lambda_u|\mathscr{G}_u]]du \\
&= \mathbb{E}[H(X_t - X_s)] - \int_s^t \mathbb{E}[H\lambda_u]du \\
&= \mathbb{E}[H(M_t - M_s)] = 0.
\end{aligned}$$

Hence M^G is a $\mathbb{G}$-martingale. □

We provide some useful results which are used in the proof of Theorem 4.5.12.

Lemma 5.0.15. *If* $\lim_{n\to\infty} x_n = x$*, then*

$$\lim_{k\to\infty} \frac{1}{k}(x_1 + \dots + x_k) = x.$$

Proof. For every $\varepsilon > 0$, there exists N such that $|x_n - x| \le \varepsilon$, for all $n \ge N$. We have the following estimation for $k > N$

$$\begin{aligned}\left|\frac{1}{k}(x_1 + \dots + x_k) - x\right| &= \frac{1}{k}\left|\sum_{i=1}^{N}(x_i - x) + \sum_{i=N+1}^{k}(x_i - x)\right| \\ &\le \frac{1}{k}\left|\sum_{i=1}^{N}(x_i - x)\right| + \frac{(k-N)}{k}\varepsilon.\end{aligned}$$

Let us choose $K \ge N$ such that $\frac{1}{K}\left|\sum_{i=1}^{N}(x_i - x)\right| \le \varepsilon$ then

$$\left|\frac{1}{k}(x_1 + \dots + x_k) - x\right| \le 2\varepsilon$$

for all $k \ge K$. The proof is complete. □

Lemma 5.0.16. *If the sequence* $(x_n)_{n\ge 1}$ *has the following properties*

- *for every* n*,* $x_n \ge 0$*,*
- $\lim_{n\to\infty} x_n = x_1^*$*,*
- $\lim_{k\to\infty} \frac{1}{k}(x_1 + \dots + x_k) = x_2^*$*.*

Then we have $x_1^* = x_2^*$*.*

Proof. First, we assume that $x_1^* > x_2^*$. Because $x_n \to x_1^*$, there exists a number $N > 0$ such that $x_n \ge (x_1^* + x_2^*)/2$, for all $n \ge N$. We estimate as follows if $k > N$

$$\frac{1}{k}(x_1 + \dots + x_k) \ge \frac{1}{k}(x_{N+1} + \dots + x_k) \ge \frac{x_1^* + x_2^*}{2}\frac{(k-N)}{k}.$$

It means that

$$\lim_{k\to\infty} \frac{1}{k}(x_1 + \dots + x_k) \ge \frac{x_1^* + x_2^*}{2} > x_2^*,$$

which is a contradiction.

Second, we assume that $x_1^* < x_2^*$. Because $x_n \to x_1^*$, there exists a number N such that $x_n \le (x_1^*+x_2^*)/2$ for all $n \ge N$. The following estimation holds for $k > N$,

$$\begin{aligned}\frac{1}{k}(x_1+...+x_k) &= \frac{1}{k}(x_1+...+x_N)+\frac{1}{k}(x_{N+1}+...+x_k)\\ &\le \frac{1}{k}(x_1+...+x_N)+\frac{(k-N)}{k}\frac{(x_1^*+x_2^*)}{2}.\end{aligned}$$

It means that

$$\lim_{k\to\infty}\frac{1}{k}(x_1+...+x_k) \le \frac{x_1^*+x_2^*}{2} < x_2^*,$$

which is also a contradiction. Finally, it holds that $x_1^* = x_2^*$. □

Lemma 5.0.17. *Let (X_n) be a sequence of random variable such that $X_n \to X, a.s.$ and $\lim_{n\to\infty}\mathbb{E}[X_n] = x$. Then we have $\mathbb{E}[X] = x$.*

Proof. It is proved that convergence almost surely implies convergence in distribution. Let F_n and F be the cumulative distribution functions of the random variables X_n and X. For every number t at which F is continuous, we have that $\lim_{n\to\infty}F_n(t) = F(t)$. Dominated convergence theorem implies

$$\begin{aligned}x = \lim_{n\to\infty}\mathbb{E}[X_n] = \lim_{n\to\infty}\int_0^\infty(1-F_n(t))dt &= \int_0^\infty(1-\lim_{n\to\infty}F_n(t))dt\\ &= \int_0^\infty(1-F(t))dt = \mathbb{E}[X].\end{aligned}$$

□

Theorem 5.0.18 (The extended dominated convergence theorem)**.** *Let $(\Omega,\mathscr{F},\mu)$ be a measure space and let $f_n, g_n : \Omega \to \mathbb{R}$ be measurable functions such that $|f_n| \le g_n$, a.e. for all $n \ge 1$. Suppose that*

- *$g_n \to g, a.e$ and $f_n \to f, a.e.$*
- *$g_n, g \in L^1(\Omega)$ and $\int |g_n|d\mu \to \int |g|d\mu$ as $n \to \infty$.*

Then $f \in L^1(\Omega)$,

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu, \qquad \text{and} \qquad \lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

Proof. See Theorem 2.3.11 of Athreya and Lahiri [2006]. □

Theorem 5.0.19 (Komlós Theorem, Komlós [1967])**.** *If* (ξ_n) *is a sequence of random variables for which*

$$\liminf_{n\to\infty} \mathbb{E}[|\xi_n|] < +\infty,$$

then there exists a subsequence (n_k) *and an integrable random variable* η*, for which*

$$\frac{1}{k}(\xi_{n_1} + \xi_{n_2} + \ldots + \xi_{n_k}) \to \eta, \qquad a.s.$$

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