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ON REGULAR AND SINGULAR POINTS OF THE MINIMUM TIME FUNCTION

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To my parents

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Abstract

In this thesis, we study the regularity of the minimum time function $\mathcal{T}$ for both linear and nonlinear control systems in Euclidean space.

We first consider nonlinear problems satisfying Petrov condition. In this case, $\mathcal{T}$ is locally Lipschitz and then is differentiable almost everywhere. In general, $\mathcal{T}$ fails to be differentiable at points where there are multiple time optimal trajectories and its differentiability at a point does not guarantee continuous differentiability around this point. We show that, under some regularity assumptions, the non-emptiness of proximal subdifferential of the minimum time function at a point x implies its continuous differentiability on a neighborhood of x. The technique consists of deriving sensitivity relations for the proximal subdifferential of the minimum time function and excluding the presence of conjugate points when the proximal subdifferential is nonempty.

We then study the regularity the minimum time function $\mathcal{T}$ to reach the origin under controllability conditions which do not imply the Lipschitz continuity of $\mathcal{T}$. Basing on the analysis of zeros of the switching function, we find out singular sets (e.g., non - Lipschitz set, non - differentiable set) and establish rectifiability properties for them. The results imply further regularity properties of $\mathcal{T}$ such as the SBV regularity, the differentiability and the analyticity. The results are mainly for linear control problems.

Sommario

La presente tesi è dedicata allo studio della regolarità della funzione tempo minimo $\mathcal{T}$ per sistemi di controllo sia lineari che non lineari in dimensione finita.

Si considerano dapprima problemi non lineari in cui la condizione di controllabilità detta di Petrov è soddisfatta. Come è ben noto, in questo caso $\mathcal{T}$ è localmente Lipschitziana e quindi è differenziabile quasi ovunque. In generale, $\mathcal{T}$ non è differenziabile nei punti dai quali escono diverse traiettorie ottimali e inoltre il fatto che $\mathcal{T}$ è differenziabile in un punto non garantisce che lo sia in un intorno (l'insieme dei punti di differenziabilità non è aperto). Imponendo alcune condizioni di regolarità sulla dinamica, si dimostra che se il sottodifferenziale prossimale di $\mathcal{T}$ è non vuoto in un punto x, allora $\mathcal{T}$ è differenziabile in tutto un intorno di x. La tecnica usata consiste nel derivare relazioni di sensitività per il sottodifferenziale prossimale di $\mathcal{T}$ e nell'escludere la presenza di punti coniugati dove tale sottodifferenziale è non vuoto.

In secondo luogo si studia la regolarità di $\mathcal{T}$ sotto condizioni di controllabilità più generali, tali da non imporre la Lipschitzianità. In questo caso il bersaglio è l'origine e la dinamica è – principalmente – lineare a coefficienti costanti. Si identificano alcuni insiemi singolari (cioè dove $\mathcal{T}$ non è differenziabile), ad esempio l'insieme dove $\mathcal{T}$ non è Lipschitz e l'insieme dei punti dove l'insieme raggiungibile presenta più di un versore normale, e si dimostrano risultati di rettificabilità, in questo modo mostrando che sono "molto piccoli". Come conseguenza si ricavano ulteriori risultati di regolarità per $\mathcal{T}$, fra i quali la regolarità SBV e la differenziabilità e l'analiticità in aperti il cui complementare ha dimensione inferiore a quella dello spazio degli stati. La tecnica usata è basata principalmente su un'analisi accurata degli zeri della cosiddetta funzione di *switching*.

Contents

Chapter 1

Introduction

We consider the minimum time problem for a control system

$$\begin{cases} y'(t) &= f(y(t), u(t)), \quad \text{for a.e. } t > 0, \\ y(0) &= x \in \mathbb{R}^N, \end{cases} \tag{1.0.1}$$

where $u(\cdot)$ is a measurable control taking values in a compact subset U of the Euclidean space $\mathbb{R}^M$. The target $\mathcal{K}$ is assumed to be a closed subset of the state space $\mathbb{R}^N$.

Under suitable assumptions on f, for each measurable control $u(\cdot)$, the system (1.0.1) has a unique solution $y(\cdot; x, u)$ called the trajectory corresponding to the control $u(\cdot)$. We set

$$\theta(x, u) := \inf\{t \geq 0 | \; y(t; x, u) \in \mathcal{K}\},$$

with the convention $\inf \emptyset = +\infty$. The minimum time function $\mathcal{T} : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{T}(x) := \inf\{\theta(x, u) | \; u : [0, \infty) \to U \text{ is measurable}\}.$$

The reachable set $\mathcal{R}$ consists of all points $x \in \mathbb{R}^N$ such that $\mathcal{T}(x) < \infty$.

The regularity of the minimum time function is a classical and wide-studied topic. It is related to the controllability properties of the system (1.0.1) as well as to the regularity of the target and of the dynamics, together with suitable relations between them. Cannarsa and Sinestrari prove in [16] that if the dynamics is smooth enough and the target satisfies an internal sphere condition, then $\mathcal{T}$ is semiconcave provided *Petrov condition* holds. Here Petrov condition tells that for all x in the boundry of $\mathcal{K}$ there exists a control u_x such that $f(x, u_x)$ makes a negative (bounded away from zero) scalar product with external normals to $\mathcal{K}$ at x.They also prove, in the case of convex targets and linear systems, that $\mathcal{T}$ is semiconvex if Petrov condition holds true.

It well-known that Petrov condition is equivalent to the local Lipschitz continuity of $\mathcal{T}$ (see, e.g., [12], Section 8.2). If Petrov condition holds true, i.e., $\mathcal{T}$ is locally Lipschitz, then $\mathcal{T}$ is twice differentiable a.e. on the reachable set, thanks to semiconcavity/semiconvexity. However, it may fail to be everywhere differentiable and its differentiability at a point does not guarantee continuous differentiability around this points. A natural question is trying to identify hypotheses on the dynamic data and the target to ensure continuous differentiability of $\mathcal{T}$ around a given point. This question is investigated in Chapter 3 of this thesis.

Moreover, simple examples show that the minimum time function is not Lipschitz even for a point-target (see, e.g., Section 4.3.4). This happens, in particular, when $f(x,u) = g(x)+h(x)u$ and $\mathcal{K}$ is an equilibrium point of the dynamics. If the Lie bracket $[g(x), h(x)]$ can be approximated by admissible trajectories of the controlled dynamics, by switching between suitable controls u and $-u$, then one can prove that it is possible to reach $\mathcal{K}$ in finite time from a neighborhood, and $\mathcal{T}$ is Hölder continuous with a suitable exponent depending on the maximal order of the Lie brackets. This is the case, for example, if $\mathcal{K} = \{0\}$ and the dynamics is linear and satisfies the classical Kalman rank condition. We call this one *higher order controllability condition*. A natural question is trying to find the set of non-Lipschitz points $\mathcal{S}$ of $\mathcal{T}$ and to study the structure of $\mathcal{S}$. This is an important problem because once we know the set $\mathcal{S}$ and its structure we can establish further regularity properties of $\mathcal{T}$ such as the SBV regularity, the differentiability and the analyticity. One other hand, knowing the non-Lipschitz set without knowing the minimum time function may be useful in efficiently designing feedback control as well as numerical methods to compute the minimum time function.

This thesis consists of two main parts: Chapter 3 and Chapter 4. In Chapter 3, we study the local regularity of the minimum time function $\mathcal{T}$ for nonlinear systems under conditions which imply the Lipchitz continuity of $\mathcal{T}$. Chapter 4 is concerned with the singularity and regularity properties of the minimum time function to reach the origin under conditions which do not imply the Lipchitz continuity. In Chapter 2, we fix the notation and recall definitions and preliminaries we need in the sequel. Chapter 5 is Appendix where we give and prove some rectifiability results.

In what follows, results appearing in Chapter 3 and 4 are discussed informally.

Local regularity of the minimum time function

This chapter is devoted to local regularity properties of the minimum time function. In [11], Cannarsa and Sinestrari prove that if the data is smooth enough and if $\mathcal{T}$ is differentiable at a point x in its domain which is not a *conjugate point* (see Definition 5.1 in [11]), then $\mathcal{T}$ is smooth in a neighborhood of x. Therefore $\mathcal{T}$ is smooth in a neighborhood of a point x once we know that $\mathcal{T}$ is differentiable at x and that x is not a conjugate point. Recently, in [7], [8], it was shown that value functions of Bolza problems arising in calculus of variations and in optimal control are smooth in neighborhoods of points having nonempty proximal subdifferential with respect to x. For this aim it was proved that such points cannot be conjugate. However, an essential assumption imposed in [7], [8] was the smoothness and the strong convexity of the Hamiltonian of the Bolza problem with respect to its last variable p. It is the very nature of time optimal control problems not to have a strongly convex Hamiltonian : if $H(x, \cdot)$ is smooth outside of zero, then $p \neq 0$ is always an element of the kernel of $H_{pp}(x,p)$, where the Hamiltonian associated to the time optimal problem is defined by

$$H(x,p) := \sup_{u \in U} \left\{ \langle -f(x,u), p \rangle \right\}.$$

For this reason, the arguments used for the Bolza problem do not apply to the minimum time problem.

In his thesis [37], F. Marino has bypassed this difficulty by introducing a new Hamiltonian

$$\mathcal{H}(x,p) = \frac{1}{2}(H(x,p)^2 - 1), \ \ \forall x, p \in \mathbb{R}^n,$$

and assuming that $\mathcal{H}_{pp}(x,p) > 0$, for all $x, p \in \mathbb{R}^n$. In this setting, by using the same argument as for the Bolza problem but for the backward Hamiltonian system, he proved that if the proximal subdifferential of the minimum time function $\mathcal{T}$ at a point x is nonempty, then x is not a conjugate point. It follows then that $\mathcal{T}$ is continuously differentiable in a neighborhood of x. However, the assumptions imposed on $\mathcal{H}$, being quite restrictive, narrow their applicability. It is natural then to pursue this investigation.

We deal directly with the Hamiltonian H under assumptions implying that $\mathcal{T}$ is semiconcave. In particular, the nonemptiness of the subdifferential of $\mathcal{T}$ at a point x implies that $\mathcal{T}$ is differentiable at x. We provide sufficient conditions for the local C^1 regularity of $\mathcal{T}$ at points with nonempty proximal subdifferential. Since such points are dense in the domain of $\mathcal{T}$, this implies that $\mathcal{T}$ is locally C^1 on a dense subset of its domain.

If the kernel of $H_{pp}(x,p)$ has the dimension equal to 1 for all $(x,p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, then, using the fact that the Hamiltonian is constant along the extremals of the time optimal problem, we show that a point at which $\mathcal{T}$ has a nonempty proximal subdifferential cannot be conjugate. That is if $\ker H_{pp}(x,p)$ is spanned by p for all $(x,p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, then our result is similar to the one known for the Bolza problem. We would like to stress here that our definition of conjugate point differs from the one of [11], because it involves tangents to the boundary of the target only instead of the whole space $\mathbb{R}^n$, see Definition 3.4.1 below.

When the dimension of the kernel of $H_{pp}(x,p)$ is greater than 1, then we propose an alternative assumption on the Hamiltonian. Namely, let $p_x \in \mathbb{R}^n$ be such that $H(x,p_x) = 1$. Consider the solution (y,p) to the Hamiltonian system

$$\begin{cases} y'(t) &= -H_p(y(t),p(t)), \quad y(0) = x \\ p'(t) &= H_x(y(t),p(t)), \quad p(0) = p_x \end{cases}$$

and define the linear operator $A(x,p_x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ by

$$A(x,p_x) := \frac{d}{dt} H_{pp}(y(\cdot),p(\cdot))_{|t=0}.$$

We show then that if $A(x,p_x)$ is positive definite on the space $\ker H_{pp}(x,p_x) \cap \{H_p(x,p_x)\}^\perp$ for all x, p_x as above, then the minimum time function is continuously differentiable on a neighborhood of any point having nonempty proximal subdifferential. In fact we shall provide a more precise statement in Section 3.

To prove our results we need the following property:

(P) The minimum time function is differentiable at a point x if and only if there exists a unique optimal trajectory starting at x,

which does not hold true in general. In [11], the authors provide some conditions to ensure that the property (P) holds true: namely, they suppose that the target $\mathcal{K}$ satisfies an interior sphere condition (i.e., there exists $r > 0$ such that $\forall x \in \mathcal{K}, \exists x_0$ with $x \in \bar{B}_r(x_0) \subset \mathcal{K}$) and that for any $x \in \mathbb{R}^n$ the set $f(x,U)$ is strictly convex and has a $C^{1,1}$ boundary. In Section 2, we show that the property (P) still holds true if instead of requiring the smoothness of $f(x,U)$, we assume that the signed distance to the target $\mathcal{K}$ is of class $C^{1,1}$ and that the Hamiltonian is sufficiently smooth. In particular, like in [11], our target cannot be a singleton.

Non-Lipschitz singularities, the SBV regularity and the differentiability of the minimum time function

In this chapter we concentrate mainly on the lack of Lipschitz continuity of the minimum time function $\mathcal{T}$ to reach the origin for an affine control system

$$\begin{cases} \dot{x} &= F(x) + G(x)u, \qquad |u| \leq 1, \ x \in \mathbb{R}^N, \\ x(0) &= \xi. \end{cases} \tag{1.0.2}$$

which is essentially due to the lack of first order controllability. More precisely, even if at some x the right hand side of (1.0.2) does not point towards the origin, i.e., the scalar product between any vector in $F(x)+G(x)\mathcal{U}$ and x is merely nonnegative, it is still possible that a trajectory through x reach the origin, provided the Lie bracket $[F(x), G(x)]$ has non-vanishing scalar product with the missing direction x (this is usually called a *higher order controllability condition*). The price to pay is a slower approaching to the origin: one needs to switch between G and $-G$, like a sailor which has to beat to windward. The simple example $\ddot{x} = u \in [-1, 1]$ exhibits this behavior: at every point of the x_1-axis (we set $\dot{x}_1 = x_2$, $\dot{x}_2 = u$) the right hand side of (1.0.2) is vertical and $\mathcal{T}$ is not locally Lipschitz in the whole of $\mathbb{R}^n$ (but at the points of the x_1-axis $\mathcal{T}$ is indeed Lipschitz). By introducing the minimized Hamiltonian

$$h(x,\zeta) := \langle F(x), \zeta\rangle + \min_{u\in\mathcal{U}} \langle G(x)u, \zeta\rangle,$$

the condition of non-pointing towards the origin can be better identified as

$$h(x,\zeta) \geq 0,$$

where ζ is a normal to the sublevel of $\mathcal{T}$ corresponding to $\mathcal{T}(x)$. Since the minimized Hamiltonian is constant and nonpositive along every optimal trajectory, it is natural to expect that non-Lipschitz points of $\mathcal{T}$ lie exactly where such Hamiltonian vanishes. In fact, in Section 4.2.2 we prove this characterization.

Let $\mathcal{S}$ be the set of non-Lipschitz points of $\mathcal{T}$. In Sections 4.3 and 4.4 we characterize $\mathcal{S}$ using points which belong to an optimal pair (i.e., an optimal trajectory together with a corresponding adjoint arc) of (1.0.2) with vanishing Hamiltonian, and, for the linear case, we give an explicit representation of $\mathcal{S}$. As a consequence, we show that at each $\bar{x} \in \mathcal{S}$ the sublevel $\mathcal{R}(\mathcal{T}(\bar{x}))$ is tangent to $\mathcal{S}$, in the sense that there exists a normal vector to $\mathcal{R}(\mathcal{T}(\bar{x}))$ at $\bar{x}$ which is also orthogonal to the optimal trajectory reaching $\bar{x}$ from the origin. The result is valid for both normal linear systems with constant coefficients in any space dimension (see Theorem 4.3.8) and for smooth nonlinear two dimensional systems such that the origin is an equilibrium point, the linearization at 0 is normal and furthermore $DG(0) = 0$ (see Theorem 4.4.2). The condition $DG(0) = 0$ ensures that the nonlinearity is sufficiently mild to preserve a linear like behavior in a neighborhood of the origin whose size can be estimated. In both cases it is known that the epigraph of $\mathcal{T}$ has locally positive reach (see [21, 23]). Reasons for the restriction to two space dimensions in the nonlinear case are discussed in the paper [23], to which the present work owes some results.

Our first result is the $\mathcal{H}^{N-1}$-rectifiability of $\mathcal{S}$ for the linear single input case, see Theorem 4.3.9, and, respectively, the $\mathcal{H}^1$-rectifiability for the nonlinear two dimensional case, see Theorem 4.4.2. For the linear case, the switching function

$$g_\zeta(t) = \langle \zeta, e^{At} b \rangle,$$

where b is a column of the matrix B, plays an important role. Actually we partition $\mathcal{S}$ according to the multiplicity of zeros of g_ζ and embed each part into a locally Lipschitz graph. The nonlinear case is handled by showing that $\mathcal{S}$ consists of optimal trajectories with vanishing Hamiltonian. Since, due to the space dimension restriction, such trajectories are at most two for the single input case, the $\mathcal{H}^1$-rectifiability is clear. We observe that the investigation of the regularity of the *Minimum time front*, i.e., the boundary of $\mathcal{R}(t)$, performed in [4, Chapter 3] cannot provide information on non-Lipschitz points of $\mathcal{T}$, since an analysis of sublevels is not enough to describe the epigraph of a function.

These rectifiability results shade some light on the propagation of singularities for minimum time functions. In fact, the positive reach property of epi($\mathcal{T}$) implies that $\mathcal{T}$ is locally semiconvex outside the closed set $\mathcal{S}$ (see [19, Theorem 5.1]). The structure of singularities of semiconvex functions is well understood (see [**?**, Chapter 4]): in particular, the subgradient of a locally semiconvex function is a singleton and has closed graph outside a $\mathcal{H}^{N-1}$-rectifiable set. Therefore our rectifiability results for $\mathcal{S}$ imply that property for $\mathcal{T}$. We observe that, for general functions whose epigraph satisfies a uniform external sphere condition, the Hausdorff dimension of the set of non-Lipschitz points was proved to be less or equal to $n - 1/2$, with an example showing the sharpness of the estimate (see [36, Theorem 1.3 and Proposition 7.3]). The present paper therefore improves that result, for the particular case of a minimum time function. We prove also a converse (propagation) result: for $\mathcal{H}^{N-1}$-a.e. $x \in \mathcal{S}$ such that $\mathcal{T}(x)$ is small enough for any neighborhood V of x, we have $\mathcal{H}^{N-1}(\mathcal{S} \cap V) > 0$. In particular, for single input normal linear systems we show that, for all $t > 0$ small enough, the set $\mathcal{S} \cap \mathrm{bdry}\mathcal{R}(t)$, up to a $\mathcal{H}^{N-2}$-negligible subset, is a $\mathcal{C}^1$-surface of dimension $N-2$. Apparently, this is the first result in the literature concerning propagation of non-Lispchitz singularities.

The positive reach property of $\mathrm{epi}(\mathcal{T})$ implies also that $\mathcal{T}$ has locally bounded variation (see [19, Proposition 7.1]). Therefore, as a consequence of the above analysis we obtain that $\mathcal{T}$ belongs to the smaller class of locally SBV functions, namely the Cantor part $D_c\mathcal{T}$ of its distributional derivative, which is a Radon measure by definition of BV, vanishes. In fact, on one hand $D_c\mathcal{T}$ must be concentrated on the set $\mathcal{S}$ of non-Lipschitz points of $\mathcal{T}$, on the other the rectifiability properties that we proved for $\mathcal{S}$ yield exactly the SBV regularity of $\mathcal{T}$. To our best knowledge this property of $\mathcal{T}$ is observed here for the first time.

Since the minimum time function $\mathcal{T}$ is locally semiconvex outside the non-Lipschitz set $\mathcal{S}$, it is twice differentiable a.e. on $\mathcal{R} \setminus \mathcal{S}$. A natural question is trying to find the set of points where $\mathcal{T}$ is differentiable and study the structure of its complement in the reachable set. In the Section 4.6, by developing some techniques on the analysis of zeros of the switching function, we can take out some countably $\mathcal{H}^{N-1}$-rectifiable sets such that the union of them with $\mathcal{S}$ is a closed set and we prove that $\mathcal{T}$ is of class C^1 on the complement of that set in $\mathcal{R}$, say Ω. By further analysis, we can take out from Ω another closed, countably $\mathcal{H}^{N-1}$-rectifiable set and we finally get a set where $\mathcal{T}$ is analytic on. More precisely, we prove the following theorem

Theorem 1.0.1. *The minimum time function $\mathcal{T}$ is analytic on an open set whose complement in the reachable set is countably $\mathcal{H}^{N-1}$ - rectifiable.*

The analyticity of the minimum time function for normal linear systems on an open set whose complement has lower dimension is known in literature (see, e.g., [6, 42, 43, 44]).

However, our approach is different and we know the complement set as well as its structure.

Chapter 2

Preliminaries

2.1 Notions

In this section, we fix notions which are used in the thesis.

We denote by $|\cdot|$ the Euclidean norm, by $\langle \cdot, \cdot \rangle$ the inner product, by $\mathbb{S}^{N-1}$ the unit sphere in $\mathbb{R}^N$, by $B(x_0, \varepsilon)$ the open ball of center x_0 with radius ε, by $\mathbb{R}^{N\times M}$ the set of all matrices of N rows and M columns, by A^T the transpose of a given matrix A and by $\|A\|$ its norm, as the linear operator.

Let $f : \mathbb{R}^N \to \mathbb{R}$. If f is differentiable, we denote by ∇f its derivative.

For a function $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ associating to each $x \in \mathbb{R}^N, y \in \mathbb{R}^M$ a real, denote by f_x, f_y its partial derivatives (when they do exist). If f is twice differentiable, the second order partial derivatives are denoted by f_{xx}, f_{xy}, f_{yy}. Similarly, for a function $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^k$ associating to each $x \in \mathbb{R}^N, y \in \mathbb{R}^M$ an element in $\mathbb{R}^k$, we denote by Df its Jacobian matrix and by $D_x f, D_y f$ its partial Jacobians.

For a nonempty subset K of $\mathbb{R}^N$, we denote by $\bar{K}$ the closure of K, by $\mathrm{Int}K$ its interior, by $\mathrm{bdry}K$ its boundary, by K^c its complement and by $\mathrm{conv}\, K$ its convex hull. The distance function from K, $d_K : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$d_K(x) := \inf_{y\in K} |y - x|, \ \ \forall x \in \mathbb{R}^N,$$

the signed distance function $b_K : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$b_K(x) := d_K(x) - d_{K^c}(x), \quad \forall x \in \mathbb{R}^N$$

whenever $K \neq \mathbb{R}^N$ and the metric projection into K, $\pi_K : \mathbb{R}^N \to K$ is defined by

$$\pi_K(x) := \{y \in K : |y - x| = d_K(x)\}.$$

2.2 Nonsmooth analysis and sets with positive reach

In this section we recall some basic concepts of nonsmooth analysis. Standard references are in [18, 40].

Let $K \subset \mathbb{R}^N$ be closed with boundary bdry$K$. Given $x \in K$ and $v \in \mathbb{R}^N$, we say that v is a *proximal normal* to K at x, and denote this fact by $v \in N_K(x)$, provided there exists $\sigma = \sigma(v, x) \geq 0$ such that

$$\langle v, y - x \rangle \leq \sigma |y - x|^2, \quad \text{for all } y \in K.$$

Equivalently, $v \in N_K(x)$ if and only if there is some $\lambda > 0$ such that $\pi_K(x + \lambda v) = \{x\}$. If K is convex, then $N_K(x)$ coincides with the normal cone of Convex Analysis.

The set of *Frétchet normal* to K at x is denoted by $N_K^F(x)$, and consists of those $v \in \mathbb{R}^N$ for which

$$\limsup_{K \ni y \to x} \langle v, \frac{y - x}{|y - x|} \rangle \leq 0.$$

The set of *limiting normals* to K at x is denoted by $N_K^L(x)$, and consists of those $v \in \mathbb{R}^N$ for which there exist sequences $\{x_i\}$, $\{v_i\}$ with $x_i \to x$, $v_i \to v$, and $v_i \in N_K(x_i)$.

The *Clarke normal* cone $N_K^C(x)$ equals $\overline{\text{co}} N_K^L(x)$, where "$\overline{\text{co}}$" means the closed convex hull.

Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. The epigraph of f is the set $\text{epi}(f) = \{(x, y) \in \Omega \times \mathbb{R} : y \geq f(x)\}$, while the hypograph of f is the set $\text{hyp}(f) = \{(x, y) \in \Omega \times \mathbb{R} : y \leq f(x)\}$. By using epigraph and hypograph of f, we can define some concepts of generalized differential for f at $x \in dom(f) = \{x : f(x) < +\infty\}$.

(i) The *proximal subdifferential* $\partial^P f(x)$ of f at a point $x \in dom(f)$ is the set of vectors $v \in \mathbb{R}^N$ such that

$$(v, -1) \in N_{\text{epi}(f)}(x, f(x)).$$

Equivalently,

$$\partial^P f(x) = \Big\{ v \in \mathbb{R}^N : \text{ there exist } c, \rho > 0 \text{ such that} \\ f(y) - f(x) - \langle v, y - x \rangle \geq -c|y - x|^2, \ \forall y \in B(x, \rho) \Big\}.$$

(ii) The *subdifferential* $D^-f(x)$ of f at a point $x \in dom(f)$ is the set

$$D^-f(x) = \left\{ p \in \mathbb{R}^N : \liminf_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}.$$

(iii) The *superdifferential* $D^-f(x)$ of f at a point $x \in dom(f)$ is the set

$$D^+f(x) = \left\{ p \in \mathbb{R}^N : \limsup_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}.$$

Note that $\partial^P f(x) \subseteq D^-f(x)$ and it is well known that, for a lower semicontinuous function f, $\partial^P f(x)$ is nonempty on a dense set of points $x \in \Omega$. For any function f, $D^-f(x)$ and $D^+f(x)$ are closed convex sets, possibly empty. It is well known that $D^-f(x)$ and $D^+f(x)$ are both nonempty if and only if f is differentiable at x. In this case

$$D^-f(x) = D^+f(x) = \{\nabla f(x)\}.$$

Let Ω be an open subset of $\mathbb{R}^N$ an $f : \Omega \to \mathbb{R}$ be a locally Lipschitz function. Then f is differentiable almost everywhere in Ω. The reachable gradient of f at a point $x \in \Omega$ is defined by

$$D^\star f(x) := \operatorname*{Lim\,sup}_{y \to x} \{\nabla f(y)\},$$

where Lim sup is the Painlevé - Kuratowski upper limit. Equivalently, a vector $p \in \mathbb{R}^N$ belongs to $D^\star f(x)$ if a sequence $\{x_k\} \subset A \setminus \{x\}$ exists such that f is differentiable at x_k for each $k \in \mathbb{N}$ and

$$\lim_{k \to \infty} x_k = x \qquad \lim_{k \to \infty} Df(x_k) = p.$$

We notice that for any locally Lipschitz function f, $D^\star f(x)$ is compact at each point $x \in \Omega$: it is closed by definition and it is bounded since f is Lipschitz. From Rademacher's Theorem it follows that $D^\star f(x) \neq \emptyset$ for every $x \in \Omega$.

The *horizon subdifferential* $\partial^\infty f(x)$ of f at a point $x \in \mathrm{dom}(f)$ is the set of vectors $v \in \mathbb{R}^N$ such that

$$(v, 0) \in N_{\mathrm{epi}(f)}(x, f(x)).$$

This concept is connected with the lack of Lipschitz continuity of f around x (see, e.g., [41, Chapter 9]) and will be used mainly in Section 4.2.

Sets with positive reach will play an important role in the sequel. The definition was first given by Federer in [28] and later studied by several authors (see the survey paper [26]).

Definition 2.2.1. *Let $K \subset \mathbb{R}^N$ be locally closed. We say that K has locally positive reach provided there exists a continuous function $\varphi : K \to [0, +\infty)$ such that the inequality*

$$\langle v, y - x \rangle \leq \varphi(x)|v|\,|y - x|^2 \tag{2.2.1}$$

holds for all $x, y \in K$ and $v \in N_K(x)$.

In particular, every convex set has positive reach: it suffices to take $\varphi \equiv 0$ in (2.2.1). Continuous functions whose epigraph has locally positive reach will be crucial in our analysis. Such functions enjoy several regularity properties, mainly studied in [19]. We list two of them which will be used in the sequel.

Theorem 2.2.2. *Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be continuous and such that epi(f) has locally positive reach. Then*

(i) f is a.e. differentiable in Ω,

(ii) f has locally bounded variation in Ω.

In optimal control theory, semiconcave functions play an important role. In this section, we just introduce some basic concepts and properties concerning with semiconcave function needed our results. For further properties and characterizations of semiconcave functions, we refer to [12].

Definition 2.2.3. *Let $\Omega \subset \mathbb{R}^N$. A function $f : \Omega \to \mathbb{R}$ is called (linearly) semiconcave iff there exists a constant $c \in \mathbb{R}$ such that*

$$\lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y) \leq c\lambda(1 - \lambda)|x - y|^2, \tag{2.2.2}$$

for all $x, y \in \Omega$ such that the line segment $[x, y] \subseteq \Omega$. We say that f is locally semiconcave in Ω iff f is semiconcave on every compact subset $K \subset \Omega$.

Proposition 2.2.4 ([12])**.** *Let Ω be an open subset of $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$ be locally semiconcave. Then f is locally Lipschitz and*

$$D^+ f(x) = \operatorname{conv} D^\star f(x)$$

for all $x \in \Omega$.

It follows that if f is a locally semiconcave function, then $D^+ f(x)$ is nonempty at each point x. Thus if $\partial^P f(x) \neq \emptyset$ (or if $D^- f(x) \neq \emptyset$), then f is differentiable at x.

2.3 Geometric measure theory

In this section we introduce some definitions and prove some results which will be used for our results. For basic concepts of geometric measure theory, we refer to [29].

Definition 2.3.1. *Let $k \in [0, \infty)$ and $E \subset \mathbb{R}^N$. The k - dimensional Hausdorff measure of E is given by*

$$\mathcal{H}^k(E) := \lim_{\delta \to 0} \mathcal{H}^k_\delta(E),$$

where, for $0 < \delta \leq \infty$, $\mathcal{H}^k_\delta(E)$ is defined by

$$\mathcal{H}^k_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} \alpha(k) \left(\frac{diam E_i}{2} \right)^k : diam E_i \leq \delta, E \subset \bigcup_{i=1}^{\infty} E_i \right\},$$

with

$$\alpha(k) = \frac{\pi^{k/2}}{\Gamma(1 + k/2)}, \text{ and } \Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx,\ 0 < s < \infty.$$

Definition 2.3.2. *The Hausdorff dimension of $E \subset \mathbb{R}^N$ is given by*

$$\mathcal{H} - \dim(E) := \inf\{k \geq 0 : \mathcal{H}^k(E) = 0\}.$$

It well known that $\mathcal{H}^k$ is a Borel measure on $\mathbb{R}^N$, $\mathcal{H}^0$ is the counting measure.

Definition 2.3.3. *A set $E \subset \mathbb{R}^N$ is said to be countably $\mathcal{H}^k$ - rectifiable if there exist countably many sets $A_i \subset \mathbb{R}^k$ and countably many Lipschitz functions $f_i : A_i \to \mathbb{R}^N$ such that*

$$\mathcal{H}^k \left(E \setminus \bigcup_{i=1}^{\infty} f_i(A_i) \right) = 0.$$

Since each Lipschitz function $f_i : A_i \to \mathbb{R}^N$ can be extended upto $\mathbb{R}^k$, we have a equivalent definition: A set $E \subset \mathbb{R}^N$ is said to be countably $\mathcal{H}^k$ - rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^N$ such that

$$\mathcal{H}^k \left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k) \right) = 0.$$

For any contably $\mathcal{H}^k$-rectifiable set A, it is well known that if $f : A \to \mathbb{R}^N$ is Lipschitz continuous then $f(A)$ is countably $\mathcal{H}^k$-rectifiable. In what follows, we prove some rectifiability results which will be used in the sequel.

Theorem 2.3.4. *Let $f : \mathbb{R}^N \to \mathbb{R}^k$, $k < N$, be a smooth function such that $Df(x)$ has rank k for all $x \in \mathbb{R}^N$. Then $f^{-1}\{y\}$ is countably $\mathcal{H}^{N-k}$ - rectifiable for all y.*

Proof. By Lemma 3.2.9 ([29], p. 247), $\mathbb{R}^N$ has a countable partition $(B_i)_{i\in\mathbb{N}}$ of Borel sets and there exist countable projections $(P_i)_{i\in\mathbb{N}}$, $P_i \in O^\star(N, N-k)$ and Lipschitz mappings $u_i : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $v_i : \mathbb{R}^k \times \mathbb{R}^{N-k} \to \mathbb{R}^N$ with $u_i(x) = (f(x), p_i(x))$ and $v_i[u_i(x)] = x$ for all $x \in B_i$, for all i. Also by Lemma 3.2.10 ([29] p.248), we have

$$B_i \cap f^{-1}\{y\} = v_i[\{y\} \times P_i[B_i \cap f^{-1}\{y\}], \ \forall y \in \mathbb{R}^k.$$

We have $\mathcal{H}^{N-k}(f^{-1}\{y\} \setminus B) = 0$ for all $y \in \mathbb{R}^k$. It follows that $f^{-1}\{y\}$ is countably $\mathcal{H}^{N-k}$ - rectificble. □

Theorem 2.3.5. *Let $1 < M \leq N$ and let $\Omega \subset \mathbb{R}^M$ be open. Assume that $G : \Omega \to \mathbb{R}^N$ be an analytic and injective function. Define*

$$D = \{x \in \Omega : \text{rank} DG(x) \leq M - 1\}.$$

Then D is closed and countably $\mathcal{H}^{M-1}$-rectifiable.

Proof. The closedness of D is obvious. We first show that there exists at least one square submatrix $B(x)$ of order $M-1$ of $DG(x)$ such that $\det B(\cdot) \not\equiv 0$ in Ω. Assume, by contradiction, that all square submatrices of order $M-1$ of $DG(x)$ are identically 0 in Ω. Then $\dim\ker DG(x) \geq 1, \forall x \in \Omega$. Denote by κ the minimal dimension of $\dim\ker DG(x), x \in \Omega$. Set

$$V = \{x \in \Omega : \dim\ker DG(x) = \kappa\}.$$

Then V is nonempty and furthermore we have the following Claim:

Claim: V is open.

Proof of Claim. Let $x_0 \in V$. Assume that there exists a sequence $\{x_n\}$ with $x_n \notin V$ for all n such that $x_n \to x_0$ as $n \to \infty$. For each n, since $x_n \notin V$, $\dim\ker DG(x_n) \geq \kappa + 1$. We take $\kappa + 1$ orthonormal vectors in $\ker DG(x_n)$, say $v_n^1, \cdots, v_n^{\kappa+1}$. By passing up to subsequences, we may assume that $v_n^i \to v^i, i = 1, \cdots, \kappa + 1$. It is easy to see that $v^1, \cdots, v^{\kappa+1}$ belong to $\ker DG(x_0)$ and they are orthonormal. This contradicts to $\dim\ker DG(x_0) = \kappa$. Claim is proved.

Since $\dim \ker DG(x) = \kappa$, for all $x \in V$, we can find an analytic selection $g(\cdot)$, never vanishing in V, of $\ker DG(\cdot)$. Take $y_0 \in V$. For $s_0 > 0$, small, the equation

$$\dot{y}(s) = g(y(s)), \quad y(0) = y_0, \quad s \in [0, s_0]$$

has unique solution $y(\cdot)$ and $y(s) \in V$ for all $s \in [0, s_0]$.

Now, for $s \in [0, s_0]$, we have

$$\frac{d}{ds} G(y(s)) = DG(y(s))\dot{y}(s) = DG(y(s))g(y(s)) = 0.$$

It means that $G(\cdot)$ is constant along the curve $y(s)$. This contradicts to the injection of G.

Now let $B(x)$ is a square submatrix of order $M-1$ of $DG(x)$ such that $\det B(x) \not\equiv 0$ in Ω. Consider the function $f : \Omega \to \mathbb{R}$ defined by $f(x) = \det B(x), x \in \Omega$. Then f is analytic in Ω and we have $D \subset D_0 := \{x \in \Omega : f(x) = 0\}$. For each $h \in \mathbb{N}^\star$, set

$$D_h = \left\{ x = (x_1, \cdots, x_M) \in D_0 : \frac{\partial^i f}{\partial x_1^i}(x) = 0, i = 1, \cdots, h-1 \text{ and } \frac{\partial^h f}{\partial x_1^h}(x) \neq 0 \right\}.$$

Then $D_0 = \bigcup_{h=1}^{\infty} D_h$.

By implicit function theorem, we observe that D_h is countably $\mathcal{H}^{M-1}$ - rectifiable for all $h = 1, 2, \cdots$. It follows that D is countably $\mathcal{H}^{M-1}$ - rectifiable. $\square$

2.4 Control theory

2.4.1 Control systems

In this thesis we only consider autonomous control systems

Definition 2.4.1. *A control system is a pair (f, U) where $U \subset \mathbb{R}^M$ is a closed and $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ is a continuous function. The set U is called the control set, while f is called the dynamics of the system. The state equation associated with the system is*

$$\begin{cases} y'(t) &= f(y(t), u(t)) \\ u(t) &\in \mathcal{U}_{ad} \\ y(0) &= x, \end{cases} \tag{2.4.1}$$

where $\mathcal{U}_{ad}$ the set of admissible controls i.e., the measure functions $u : \mathbb{R} \to \mathbb{R}^M$, such that $u(t) \in U$ a.e.

Usually, we give following assumptions on the control system

(H1) The control set U is nonempty and compact.

(H2) The dynamics f satisfies

$$|f(x,u) - f(y,u)| \leq L_1|x-y|, \ \ \forall x, y \in \mathbb{R}^n, \, \forall u \in U,$$

for some positive constant L_1.

(iii) $D_x f$ exists and is continuous; in addition, there exists $L_2 > 0$ such that

$$||D_x f(x,u) - D_x f(y,u)|| \leq L_2|x-y|, \ \ \forall x, y \in \mathbb{R}^N, \forall u \in U.$$

Under assumption (H2), the system (2.4.1) has a unique Carathéodory solution for each $u(\cdot) \in \mathcal{U}_{ad}$ and we denote the solution by $y(\cdot; x, u)$. We call $y(\cdot; x, u)$ the trajectory starting at x corresponding to the control $u(\cdot)$.

The attainable set $\mathcal{A}^T(x)$ from x in time T is defined by

$$\mathcal{A}^T(x) := \{y(t; x, u) : t \leq T, u(\cdot) \in \mathcal{U}_{ad}\}.$$

Observe that assumptions (H1) and (H2), together with the continuity of f, imply

$$|f(x,u)| \leq C + L_1|x|, \ \forall x \in \mathbb{R}^N, u \in U, \tag{2.4.2}$$

where $C = \max_{u \in U} |f(0,u)|$. Therefore $\mathcal{A}^T(x)$ is bounded for all $x \in \mathbb{R}^N$ and $T < \infty$. In proving our results, we can replace the global Lipschitzianity (H2) by 2.4.2) and a locally Lipschitz condition on f.

2.4.2 Minimum time function and controllability

Together with the control system (2.4.1), we consider a nonempty closed subset $\mathcal{K}$ of $\mathbb{R}^N$ which is called the target.

For a given point $x \in \mathbb{R}^N \setminus \mathcal{K}$ and $u(\cdot) \in \mathcal{U}_{ad}$, we define

$$\theta(x,u) := \min\{t \geq 0 : y(t; x, u) \in \mathcal{K}\}.$$

Observe that $\theta(x,u) \in [0,\infty]$ and $\theta(x,u)$ is the time taken for the trajectory $y(t;x,u)$ to reach the target for the first time provided $\theta(x,u) < \infty$. The minimum time function $\mathcal{T}(x)$ to reach $\mathcal{K}$ from x is defined by

$$\mathcal{T}(x) := \inf\{\theta(x,u) : u(\cdot) \in \mathcal{U}_{ad}\}. \tag{2.4.3}$$

If $x \in \mathcal{K}$, then we set $\mathcal{T}(x) = 0$.

In general, the infimum in (2.4.3) is not attained. The following theorem gives a condition for which the infimum is attained.

Theorem 2.4.2. *Assume that (H1) and (H2) hold and that $f(z,U)$ is convex for every $z \in \mathbb{R}^N$. Then one has*

$$\mathcal{T}(x) := \min\{\theta(x,u) : u(\cdot) \in \mathcal{U}_{ad}\}. \tag{2.4.4}$$

Proof. See, e.g, Theorem 8.1.2 in [12]. □

A minimizing control in (2.4.4), say $u^*(\cdot)$, is called an optimal control for x. The trajectory $y(\cdot;x,u^*)$ corresponding to $u^*(\cdot)$ is called an optimal trajectory.

The following result is called *Dynamic Programming Principle*. It is an important tool for the study of the properties of the minimum time function.

Theorem 2.4.3. *Assume that the control system satisfies (H1) and (H2). For $x \in \mathbb{R}^N \setminus \mathcal{K}$ and for $0 \le t \le \mathcal{T}(x)$, one has*

$$\mathcal{T}(x) = t + \inf\{\mathcal{T}(y) : y \in \mathcal{A}^t(x)\}. \tag{2.4.5}$$

Equivalently, for all $u(\cdot) \in \mathcal{U}_{ad}$, if we set $x(\cdot) = y(\cdot;x,u)$ then the function $t \mapsto t + \mathcal{T}(x(t))$ is increasing in $[0,\mathcal{T}(x)]$.

Moreover, if $x(\cdot)$ is an optimal trajectory then $t \mapsto t + \mathcal{T}(x(t))$ is constant in $[0,\mathcal{T}(x)]$, i.e.,

$$\mathcal{T}(x(t)) = t - s + \mathcal{T}(x(s)), \ \text{for} \ 0 \le s \le t \le \mathcal{T}(x).$$

Proof. See e.g., [12]. □

We now introduce the following important notations

$$\begin{aligned} \mathcal{R}(t) &:= \{x \in \mathbb{R}^N : \mathcal{T}(x) \le t\}, \ t > 0, \\ \mathcal{R} &:= \bigcup_{t>0} \mathcal{R}(t) = \{x \in \mathbb{R}^N : \mathcal{T}(x) < \infty\}. \end{aligned}$$

We call $\mathcal{R}$ the *reachable set* and $\mathcal{R}(t)$, for $t > 0$, the reachable set at time t.

We now give some concepts on controllability and its relations with the continuity of the minimum time function.

Definition 2.4.4. *The control system* (f, U) *is small time controllable on* $\mathcal{K}$ *(shortly, STCS) if* $\mathcal{K} \subseteq \mathrm{Int}\mathcal{R}(t)$ *for all* $t > 0$. *If* $\mathcal{K} = \{0\}$ *this property is called small time local controllability (shortly, STLC).*

Proposition 2.4.5. *Assume that the control system* (f, U) *satisfies (H1)-(H2) and the target* $\mathcal{K}$ *is compact. Then the following are equivalent*

(i) The system (f, U) *is STCS.*

(ii) $\mathcal{T}$ *is continuous at* x *for all* $x \in \mathrm{bdry}\mathcal{K}$.

Proof. See, e.g., [3]. □

Notice that under assumptions in Propostion 2.4.5, $\mathcal{T}(x) > 0$ for all $x \notin \mathcal{K}$.

Proposition 2.4.6. *Under assumptions in Proposition 2.4.5, if the control system* (f, U) *is STCS then:*

(i) The reachable set $\mathcal{R}$ *is open.*

(ii) $\mathcal{T}$ *is continuous on* $\mathcal{R}$.

(iii) $\lim_{x \to x_0 \in \mathrm{bdry}\mathcal{K}} \mathcal{T}(x) = +\infty$.

Proof. See, e.g., [3]. □

Finally, we recall Petrov condition which implies the Lipschitz continuity of the minimum time function.

Definition 2.4.7. *We say that the control system* (f, U) *and the target* $\mathcal{K}$ *satisfy the Petrov condition if, for any* $R > 0$, *there exists* $\mu > 0$ *such that*

$$\min_{u \in U} \langle f(x, u), \zeta \rangle < -\mu |\zeta|, \quad \forall\, x \in \mathrm{bdry}\mathcal{K} \cap B(0, R), \zeta \in N_{\mathcal{K}}(x). \tag{2.4.6}$$

Theorem 2.4.8. *Assume that (H1)-(H2) hold and that* $\mathcal{K}$ *is compact. If the system* (f, U) *and* $\mathcal{K}$ *staisfy Petrov condition then the minimum time function* $\mathcal{T}$ *is locally Lipschitz on* $\mathcal{R}$.

Proof. See, e.g., Chapter 8 in [12]. □

Chapter 3

Local regularity of the minimum time function

This chapter is devoted to the study of local regularity of the minimum time function $\mathcal{T}$ for nonlinear control systems under conditions which imply Lipschitz continuity of $\mathcal{T}$.

3.1 Nonlinear control systems

We consider throughout the chapter a nonlinear control system of the following form

$$\begin{cases} y'(t) &= f(y(t), u(t)) \\ u(t) &\in U \\ y(0) &= x, \end{cases} \tag{3.1.1}$$

where the function $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ and the control set U, a compact nonempty subset of $\mathbb{R}^M$, are given. We denote by $\mathcal{U}_{ad}$ the set of admissible controls, i.e., the measurable functions $u : \mathbb{R}_+ \to \mathbb{R}^M$, such that $u(t) \in U$ a.e. Below, suitable conditions will be made on f to ensure the global existence of a unique solution to (3.1.1). For each admissible control $u(\cdot) \in \mathcal{U}_{ad}$, the corresponding solution of (3.1.1) is denoted by $y(\cdot; x, u)$.

We now assume that a closed nonempty set $\mathcal{K} \subset \mathbb{R}^N$ different from $\mathbb{R}^N$ is given which is called the target. For each $x \in \mathbb{R}^N \setminus \mathcal{K}$ and $u(\cdot) \in \mathcal{U}_{ad}$ such that $y(t; x, u) \in \mathcal{K}$ for some $t > 0$, we set

$$\theta(x, u) := \min\{t \geq 0 : y(t; x, u) \in \mathcal{K}\}.$$

If for every $t > 0$, $y(t, x, u) \notin \mathcal{K}$, then we set $\theta(x, u) = +\infty$. Thus $\theta(x, u) \in [0, +\infty]$, and $\theta(x, u)$ is the time at which the trajectory $y(\cdot; x, u)$ reaches the target $\mathcal{K}$ for the first time, provided $\theta(x, u) < +\infty$. The reachable set $\mathcal{R}$ the set of all x such that $\theta(x, u) < +\infty$ for some admissible control $u(\cdot)$. The minimum time function $\mathcal{T} : \mathcal{R} \to [0, \infty)$ is defined by

$$\mathcal{T}(x) := \inf\{\theta(x, u) : u(\cdot) \in \mathcal{U}_{ad}\}. \tag{3.1.2}$$

The Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ associated with the system (3.1.1) is defined by

$$H(x, p) := \max_{u \in U} \{\langle -f(x, u), p\rangle\}, \qquad (x, p) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{3.1.3}$$

In the results below we will often require some of the following assumptions

(A1) U is compact and the set $f(x, U)$ is convex for any $x \in \mathbb{R}^N$.

(A2) f is continuous, locally Lipschitz with respect to x, uniformly in u and there exists $k > 0$ such that $\sup_{u \in U} |f(x, u)| \leq k(1 + |x|)$, for all $x \in \mathbb{R}^N$.

(A3) $D_x f(x, u)$ exists for all x, u and is locally Lipschitz in x, uniformly in u.

(A4) $b_{\mathcal{K}}$ is of class $C^{1,1}_{loc}$ on a neighborhood of bdry$\mathcal{K}$ and for any $z \in$ bdry $\mathcal{K}$,

$$\min_{u \in U} \langle f(z, u), n_z\rangle < 0, \tag{3.1.4}$$

where n_z denotes the unit outward normal to $\mathcal{K}$ at z.

We would like to underline here that under assumption (A4), $n_z = \nabla b_{\mathcal{K}}(z)$. Recall that if $\mathcal{K}$ is of class C^k, for some $k \geq 2$, then $b_{\mathcal{K}}$ is of class C^k on a neighborhood of bdry$\mathcal{K}$.

Remark 3.1.1. *In this chapter we use some results from [12, Chapter 8], where f and $D_x f$ are supposed to be globally Lipschitz in x uniformly in $u \in U$. The sublinear growth condition in the assumption (A2) implies that for any $r > 0, T > 0$ the set of trajectories $x(\cdot)$ of our control system defined on $[0, T]$ and satisfying $x(0) \in B(0, r)$ is bounded. For this reason the corresponding proofs in [12, Chapter 8] can be localized and results are valid also under our assumptions.*

An admissible control $u(\cdot)$ at which the infimum (3.1.2) is attained and the corresponding trajectory $y(\cdot; x, u)$ are called optimal.

Consider a family of continuous functions $y_k : [0, T_k] \to \mathbb{R}^N$ with $T_k \to T$. In this chapter when we say that $y_k(\cdot)$ converge uniformly to a continuous function $y : [0, T] \to \mathbb{R}^N$, we mean that for all $t_0 \in [0, T)$ and for all $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$\max_{t \in [0, t_0]} |y_k(t) - y(t)| \leq \varepsilon, \ \ \forall k \geq k_\varepsilon.$$

The following theorem gives a sufficient condition for the existence of optimal controls.

Theorem 3.1.2 ([12])**.** *Under assumptions (A1) - (A3), the infimum in (3.1.2) is attained. Furthermore, if the minimum time function $\mathcal{T}$ is continuous in $\mathcal{R}$, then for any sequence $x_k \in \mathcal{R}$ converging to some $x \in \mathcal{R}$, the uniform convergence of the optimal trajectories $y_k(\cdot) := y(\cdot, x_k, u_k)$ to some $y(\cdot) : [0, \mathcal{T}(x)] \to \mathbb{R}^N$, implies that $y(\cdot)$ is optimal for x.*

Proof. See e.g. [12]. □

3.2 Properties of the minimum time function

In this section, we present some properties of the minimum time function as well as its generalized differential. The first result is about the semiconcavity of $\mathcal{T}$.

Theorem 3.2.1. *If (A1)-(A4) hold true, then $\mathcal{R}$ is open and $\mathcal{T}$ is locally semiconcave in $\mathcal{R} \setminus \mathcal{K}$.*

Proof. See e.g. [16], [12]. □

The next result can be seen as a propagation property of the proximal subdifferential of the minimum time function. It will be an important tool for proving optimality conditions as well as the local regularity of the minimum time function.

Theorem 3.2.2. *Assume (A1) - (A3). Let $x_0 \in \mathcal{R} \setminus \mathcal{K}$ and $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair for x_0. Assume that $\partial^P \mathcal{T}(x_0) \neq \emptyset$ and let $p : [0, \mathcal{T}(x_0)] \to \mathbb{R}^n$ be a solution of*

$$p'(t) = -D_x f(\bar{y}(t), \bar{u}(t))^T p(t) \tag{3.2.1}$$

satisfying $p(0) \in \partial^P \mathcal{T}(x_0)$. Then for some $c > 0$ and for all $t \in [0, \mathcal{T}(x_0))$, there exists $r > 0$ such that, for every $x \in B(\bar{y}(t), r)$,

$$\mathcal{T}(x) - \mathcal{T}(\bar{y}(t)) \geq \langle p(t), x - \bar{y}(t) \rangle - c|x - \bar{y}(t)|^2.$$

Consequently, $p(t) \in \partial^P \mathcal{T}(\bar{y}(t))$ *for all* $t \in [0, \mathcal{T}(x_0))$.

Proof. Let $h \in \mathbb{S}^{N-1}$, $\bar{t}, \tilde{t} \in [0, \mathcal{T}(x_0))$ be such that $\bar{t} < \tilde{t}$ and let $w(\cdot)$ be the solution of the linear system

$$\begin{cases} w'(t) &= D_x f(\bar{y}(t), \bar{u}(t)) w(t) \\ w(\bar{t}) &= h \end{cases}$$

in $[0, \mathcal{T}(x_0)]$. Then $|w(t)| \leq K_1$ for all $t \in [0, \mathcal{T}(x_0)]$ and for some $K_1 \geq 0$ independent from h.

Now, for $0 < \varepsilon < 1$, let $y_\varepsilon(\cdot)$ be the solution of the system

$$\begin{cases} y_\varepsilon'(t) &= f(y_\varepsilon(t), \bar{u}(t)) \\ y_\varepsilon(\bar{t}) &= \bar{y}(\bar{t}) + \varepsilon h \end{cases}$$

in $[0, \mathcal{T}(x_0)]$. Then for all small $\varepsilon > 0$, $y_\varepsilon([0, \tilde{t}]) \cap \mathcal{K} = \emptyset$.

Moreover, there exists some $C > 0$ independent from h and ε such that

$$|y_\varepsilon(t) - \bar{y}(t) - \varepsilon w(t)| \leq C\varepsilon^2, \quad \forall t \in [0, \mathcal{T}(x_0)]. \tag{3.2.2}$$

Indeed, we have

$$\begin{aligned} &|y_\varepsilon(t) - \bar{y}(t) - \varepsilon w(t)| \\ &\quad\leq \int_{\bar{t}}^{t} |f(y_\varepsilon(s), \bar{u}(s)) - f(\bar{y}(s), \bar{u}(s)) - \varepsilon D_x f(\bar{y}(s), \bar{u}(s)) w(s)| ds \\ &\quad\leq \int_{\bar{t}}^{t} |f(\bar{y}(s) + \varepsilon w(s), \bar{u}(s)) - f(\bar{y}(s), \bar{u}(s)) - \varepsilon D_x f(\bar{y}(s), \bar{u}(s)) w(s)| ds \\ &\qquad + \int_{\bar{t}}^{t} |f(y_\varepsilon(s), \bar{u}(s)) - f(\bar{y}(s) + \varepsilon w(s), \bar{u}(s))| ds. \end{aligned} \tag{3.2.3}$$

By (A3), for all $s \in [0, \mathcal{T}(x_0)]$, we have

$$\begin{aligned} &|f(\bar{y}(s) + \varepsilon w(s), \bar{u}(s)) - f(\bar{y}(s), \bar{u}(s)) - \varepsilon D_x f(\bar{y}(s), \bar{u}(s)) w(s)| \\ &\quad\leq \varepsilon |w(s)| \int_0^1 |D_x f(\bar{y}(s) + \varepsilon\tau w(s), \bar{u}(s)) - D_x f(\bar{y}(s), \bar{u}(s))| d\tau \\ &\quad\leq K_2 \varepsilon^2 \end{aligned}$$

for some $K_2 > 0$ independent from h, ε. Similarly,

$$|f(y_\varepsilon(s), \bar{u}(s)) - f(\bar{y}(s) + \varepsilon w(s), \bar{u}(s))| \leq K_3 |y_\varepsilon(s) - \bar{y}(s) - \varepsilon w(s)|,$$

for all $s \in [0, \mathcal{T}(x_0)]$ and for some $K_3 > 0$ independent from $h,\ \varepsilon$.

Therefore, from (3.2.3) we deduce that

$$|y_\varepsilon(t) - \bar{y}(t) - \varepsilon w(t)| \le K_4\varepsilon^2 + \int_{\bar{t}}^{t} K_3|y_\varepsilon(s) - \bar{y}(s) - \varepsilon w(s)|ds,$$

for all $t \in [0, \mathcal{T}(x_0)]$ and for some $K_3, K_4 > 0$.

Applying the Gronwall inequality we obtain (3.2.2).

By the dynamic programming principle, for all small $\varepsilon > 0$, we have

$$\mathcal{T}(y_\varepsilon(0)) \le \mathcal{T}(y_\varepsilon(t)) + t \quad \text{and} \quad \mathcal{T}(x_0) = \mathcal{T}(\bar{y}(t)) + t, \quad \forall t \in [0, \tilde{t}] \tag{3.2.4}$$

Since $p(0) \in \partial^P \mathcal{T}(x_0)$, there exists $C_0 \ge 0$ such that for all $\varepsilon > 0$ sufficiently small

$$\mathcal{T}(y_\varepsilon(0)) - \mathcal{T}(x_0) \ge \langle p(0), y_\varepsilon(0) - x_0\rangle - C_0\varepsilon^2.$$

By (3.2.2) and (3.2.4), we have

$$\mathcal{T}(y_\varepsilon(t)) - \mathcal{T}(\bar{y}(t)) \ge \varepsilon\langle p(0), w(0)\rangle - c\varepsilon^2,$$

for some $c > 0$ and for all $t \in [0, \tilde{t}]$.

Since

$$\frac{d}{dt}\langle p(t), w(t)\rangle = \langle -D_x f(\bar{y}(t), \bar{u}(t))^T p(t), w(t)\rangle + \langle p(t), D_x f(\bar{y}(t), \bar{u}(t))w(t)\rangle = 0,$$

$\langle p(\cdot), w(\cdot)\rangle$ is constant on $[0, \mathcal{T}(x_0)]$. We have, for all $t \in [0, \tilde{t}]$,

$$\mathcal{T}(y_\varepsilon(t)) - \mathcal{T}(\bar{y}(t)) \ge \varepsilon\langle p(t), w(t)\rangle - c\varepsilon^2.$$

In particular,

$$\mathcal{T}(y_\varepsilon(\bar{t})) - \mathcal{T}(\bar{y}(\bar{t})) \ge \varepsilon\langle p(\bar{t}), w(\bar{t})\rangle - c\varepsilon^2$$

and therefore

$$\mathcal{T}(\bar{y}(\bar{t}) + \varepsilon h) - \mathcal{T}(\bar{y}(\bar{t})) \ge \varepsilon\langle p(\bar{t}), h\rangle - c\varepsilon^2.$$

This implies the conclusion. □ □

By using similar arguments to the above theorem, one can prove the following theorem

Theorem 3.2.3. *Assume (A1) - (A3). Let* $x_0 \in \mathcal{R} \setminus \mathcal{K}$ *and* $(\bar{y}(\cdot), \bar{u}(\cdot))$ *be an optimal pair for* x_0*. Assume that* $D^-\mathcal{T}(x_0) \neq \emptyset$ *and let* $p : [0, \mathcal{T}(x_0)] \to \mathbb{R}^n$ *be a solution of*

$$p'(t) = -D_x f(\bar{y}(t), \bar{u}(t))^T p(t) \tag{3.2.5}$$

satisfying $p(0) \in D^-\mathcal{T}(x_0)$*. Then one has* $p(t) \in D^-\mathcal{T}(\bar{y}(t))$ *for all* $t \in [0, \mathcal{T}(x_0))$*.*

Proposition 3.2.4. *Assume (A1) - (A3). If* $\zeta \in \partial^P \mathcal{T}(x_0)$ *for some* $x_0 \in \mathcal{R} \setminus \mathcal{K}$*, then* $H(x_0, \zeta) = 1$*.*

Proof. This proposition can be deduced from known results (e.g. Theorem 5.1 in [46]). We give a proof in our setting for the reader's convenience.

Since $\zeta \in \partial^P \mathcal{T}(x_0)$, there exist $\sigma, \eta > 0$ such that for all $y \in B(x_0, \eta)$,

$$\mathcal{T}(y) - \mathcal{T}(x_0) - \langle \zeta, y - x_0 \rangle \geq -\sigma |y - x_0|^2. \tag{3.2.6}$$

Let $v \in U$ be such that

$$H(x_0, \zeta) = \langle -f(x_0, v), \zeta \rangle = \max_{u \in U} \langle -f(x_0, u), \zeta \rangle,$$

and $x(\cdot)$ be the solution of the differential equation

$$x'(t) = -f(x(t), v), \quad x(0) = x_0, \ t \in [0, \infty).$$

Since $x(\cdot)$ is continuous and $x_0 \notin \mathcal{K}$, there exists $\epsilon > 0$ such that $x(t) \notin \mathcal{K}$ and $|x(t) - x_0| < \eta$ for all $t \in [0, \epsilon]$. We now fix $t \in [0, \epsilon]$ and define $y(s) = x(t - s)$ for $s \in [0, t]$. Then $y(\cdot)$ is the solution of

$$y'(s) = f(y(s), v), \quad y(0) = x(t), \quad s \in [0, t].$$

It follows from the principle of optimality that

$$\mathcal{T}(x(t)) = \mathcal{T}(y(0)) \leq \mathcal{T}(y(t)) + t = \mathcal{T}(x_0) + t.$$

By (3.2.6), taking $y := x(t)$ and by the Gronwall inequality, we have

$$\langle \zeta, x(t) - x_0 \rangle \leq \sigma |x(t) - x_0|^2 + t \leq M t^2 + t,$$

for some $M > 0$. Equivalently,

$$\langle \zeta, \int_0^t f(x(s), v) ds \rangle \geq -M t^2 - t.$$

Dividing both sides of the latter inequality by $t > 0$ and letting $t \to 0+$, we get $\langle \zeta, f(x_0, v) \rangle \geq -1$. It follows that $H(x_0, \zeta) \leq 1$.

Now let $w(\cdot)$ be an optimal control for x_0 and $y(\cdot)$ be the corresponding trajectory. Then by the dynamic programming principle, we have, for all $t \in [0, \mathcal{T}(x_0)]$,

$$\mathcal{T}(x_0) = \mathcal{T}(y(t)) + t.$$

In (3.2.6), taking $y := y(t)$ for $t \in [0, \mathcal{T}(x_0)]$, we have

$$-t - \langle \zeta, y(t) - x_0 \rangle \geq -\sigma |y(t) - x_0|^2.$$

By the Gronwall inequality and Lipschitzianity of f, for $t \in [0, \mathcal{T}(x_0)]$, we have

$$\begin{aligned} tH(x_0, \zeta) &\geq \int_0^t \langle \zeta, -f(x_0, w(s)) \rangle ds = -\langle \zeta, \int_0^t f(x_0, w(s)) ds \rangle \\ &\geq -\sigma |y(t) - x_0|^2 + t - \langle \zeta, \int_0^t (f(x_0, w(s)) - f(y(s), w(s))) ds \rangle \\ &\geq -Kt^2 + t - L|\zeta| t^2, \end{aligned}$$

for some $K, L > 0$.

Dividing both sides of the latter inequality by $t > 0$ and letting $t \to 0+$, we obtain $H(x_0, \zeta) \geq 1$. This ends the proof. □ □

Proposition 3.2.4 yields the following corollary.

Corollary 3.2.5. *Assume (A1) - (A3). Then* $0 \notin \partial^P \mathcal{T}(x)$ *for all* $x \in \mathcal{R} \setminus \mathcal{K}$.

Proposition 3.2.6. *If (A1) - (A4) hold true, then* $0 \notin D^\star \mathcal{T}(x)$ *for all* $x \in \mathcal{R} \setminus \mathcal{K}$.

Proof. Given $x \in \mathcal{R} \setminus \mathcal{K}$, let q be any vector in $D^\star \mathcal{T}(x)$. Then there exists a sequence $\{x_k\} \subset \mathcal{R} \setminus \mathcal{K}$ converging to x such that $\mathcal{T}$ is differentiable at each x_k and $\nabla \mathcal{T}(x_k) \to q$ as $k \to \infty$.

It is known (see e.g. [2]) that $\mathcal{T}$ is a viscosity solution of the Hamilton - Jacobi - Bellman equation

$$H(x, \nabla \mathcal{T}(x)) = 1, \quad x \in \mathcal{R} \setminus \mathcal{K}.$$

Therefore, for every k, we have

$$H(x_k, \nabla \mathcal{T}(x_k)) = 1.$$

By the continuity of H, it follows that $H(x,q) = 1$. Thus $q \neq 0$ which ends the proof. □ □

Theorem 3.2.7. *Assume (A1) - (A4). Then $\nabla\mathcal{T}$ is continuous on the set of points in $\mathcal{R} \setminus \mathcal{K}$ at which $\mathcal{T}$ is differentiable.*

Proof. By Theorem 3.2.1, $\mathcal{T}$ is locally semiconcave on $\mathcal{R} \setminus \mathcal{K}$. Thus $D^\star\mathcal{T}(x)$ is equal to $\{\nabla\mathcal{T}(x)\}$ whenever $\mathcal{T}$ is differentiable at $x \in \mathcal{R} \setminus \mathcal{K}$. □ □

3.3 Optimality conditions

One of important tools for our analysis is the maximum principle. We now recall it in the following form

Theorem 3.3.1. *Assume (A1) - (A4). Let $x \in \mathcal{R} \setminus \mathcal{K}$, $u(\cdot)$ be an optimal control for x and $y(\cdot) := y(\cdot; x, u)$ be the corresponding optimal trajectory. Set $z = y(\mathcal{T}(x))$ and let ζ be an outer unit normal to $\mathcal{K}$ at z. Then for any $\mu > 0$, the solution of the system*

$$\begin{cases} p'(t) &= -D_x f(y(t), u(t))^T p(t) \\ p(\mathcal{T}(x)) &= \mu\zeta \end{cases} \tag{3.3.1}$$

satisfies

$$-\langle f(y(t), u(t)), p(t)\rangle = H(y(t), p(t)), \tag{3.3.2}$$

for a.e. $t \in [0, \mathcal{T}(x)]$.

Proof. See e.g. [9], [12]. □

A nonzero absolutely continuous function $p(\cdot)$ satisfying (3.3.1) for some $\mu > 0$ is called a dual arc associated to the optimal trajectory $y(\cdot; x, u)$. Theorem 3.3.3 below gives a connection between the dual arcs and the superdifferential of the minimum time function.

The following lemma could be deduced from Lemma 4.2 in [10] where an exit time problem was studied. We give a self-contained proof for the reader's convenience.

Lemma 3.3.2. *Assume (A1), (A4). Given $z \in bdry\mathcal{K}$, let ζ be the outer unit normal to $\mathcal{K}$ at z. Then there exists a unique $\mu > 0$ such that $H(z, \mu\zeta) = 1$.*

Proof. We have $H(z,0) - 1 < 0$.

On other hand, by (A4), there exists $w \in U$ such that $-\langle f(z,w), \zeta \rangle > 0$. For any $\mu > 0$, we have

$$H(z, \mu\zeta) - 1 \geq \mu(-\langle f(z,w), \zeta\rangle) - 1 \to +\infty \quad \text{as } \mu \to +\infty.$$

Therefore, by continuity of $H(x, \cdot)$, there exists $\mu > 0$ such that $H(z, \mu\zeta) = 1$. Furthermore, for any $\mu_1 > 0$,

$$H(z, \mu_1\zeta) = \frac{\mu_1}{\mu} H(z, \mu\zeta) = \frac{\mu_1}{\mu}.$$

Thus, the equality $H(z, \mu_1\zeta) = 1$, implies $\mu_1 = \mu$. □ □

Let $z \in \text{bdry}\mathcal{K}$. Then, under assumption (A4), the outer normal to $\mathcal{K}$ at z is equal to $\nabla b_{\mathcal{K}}(z)$. Let $\mu(z)$ be the positive number such that $H(z, \mu(z)\nabla b_{\mathcal{K}}(z)) = 1$. Then $\mu(\cdot)$ is a function from $\text{bdry}\mathcal{K}$ to $\mathbb{R}_+$ satisfying

$$\mu(z) = \frac{1}{H(z, \nabla b_{\mathcal{K}}(z))}. \tag{3.3.3}$$

Since $\nabla b_{\mathcal{K}}(\cdot)$ is locally Lipschitz in a neighborhood of $\text{bdry}\mathcal{K}$, under assumptions (A1) and (A4), $\mu(\cdot)$ is locally Lipschitz in a neighborhood of $\text{bdry}\mathcal{K}$.

Theorem 3.3.3 ([9, 12])**.** *In Theorem 3.3.1, if μ is so that $H(z, \mu\zeta) = 1$, then the dual arc $p(\cdot)$ satisfies*

$$p(t) \in D^+\mathcal{T}(y(t)), \quad \forall t \in [0, \mathcal{T}(x)).$$

Remark 3.3.4. *Observe that by Lemma 3.3.2 and Theorem 3.3.3, for each optimal trajectory $y(\cdot)$ starting at a point $x \in \mathcal{R} \setminus \mathcal{K}$, there exists a unique dual arc $p(\cdot)$ associated to $y(\cdot)$ satisfying the properties: $p(t) \in D^+\mathcal{T}(y(t))$ for all $t \in [0, \mathcal{T}(x))$ and $p(\mathcal{T}(x)) = \mu(y(\mathcal{T}(x)))\nabla b_{\mathcal{K}}(y(\mathcal{T}(x)))$ with $\mu(\cdot)$ as in (3.3.3).*

Corollary 3.3.5. *In Theorem 3.3.1, if $\partial^P\mathcal{T}(x) \neq \emptyset$, then $p(t) = \nabla\mathcal{T}(y(t))$ for all $t \in [0, \mathcal{T}(x))$.*

Similarly, we have

Corollary 3.3.6. *In Theorem 3.3.1, if $D^-\mathcal{T}(x) \neq \emptyset$, then $p(t) = \nabla\mathcal{T}(y(t))$ for all $t \in [0, \mathcal{T}(x))$.*

If the Hamiltonian is suitably smooth, we can derive further optimality conditions and establish additional relations between optimal trajectories as well as their associated dual arcs and the generalized gradients of the minimum time function. Observe that the Hamiltonian H is not differentiable whenever $p = 0$. We will consider the case where these are the only singularities of H. More precisely, we make the following assumption:

(H1) $H \in C^{1,1}_{loc}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$.

Lemma 3.3.7 ([12])**.** *If (H1) holds, then for any $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$, we have*

$$H_x(x,p) = -D_x f(x, u^\star(x,p))^T p, \quad H_p(x,p) = -f(x, u^\star(x,p)), \tag{3.3.4}$$

where $u^\star(x,p) \in U$ is any vector such that

$$-\langle f(x, u^\star(x,p)), p\rangle = H(x,p).$$

The lack of differentiability of H at $p = 0$ is not an obstacle since we are going to evaluate H along dual arcs which are nonzero. From Lemma 3.3.7 and Theorem 3.3.1, we have

Theorem 3.3.8. *Assume (A1) - (A4) and (H1). Let $y(\cdot)$ be an optimal trajectory for some $x \in \mathcal{R} \setminus \mathcal{K}$ and let $p(\cdot)$ be an associated dual arc. Then the pair $(y(\cdot), p(\cdot))$ solves the system*

$$\begin{cases} y'(t) &= -H_p(y(t), p(t)) \\ p'(t) &= H_x(y(t), p(t)) \end{cases} \tag{3.3.5}$$

in $[0, \mathcal{T}(x)]$. Moreover, $y(\cdot)$ and $p(\cdot)$ are of class C^1.

Note that if $y_1(\cdot)$ and $y_2(\cdot)$ are two distinct optimal trajectories for a point $x \in \mathcal{R} \setminus \mathcal{K}$ then $y_1(\mathcal{T}(x)) \neq y_2(\mathcal{T}(x))$. Indeed, if $y_1(\mathcal{T}(x)) = y_2(\mathcal{T}(x))$, then there exist dual arcs $p_1(\cdot), p_2(\cdot)$ associated to $y_1(\cdot)$, $y_2(\cdot)$ respectively, such that $(y_1(\cdot), p_1(\cdot))$ and $(y_2(\cdot), p_2(\cdot))$ solve (3.3.5) with the same final conditions. This implies that $y_1(\cdot)$ and $y_2(\cdot)$ do coincide.

Theorem 3.3.9. *Assume (A1) - (A4) and (H1). Let $x \in \mathcal{R} \setminus \mathcal{K}$ be such that $\mathcal{T}$ is differentiable at x. Consider the pair $(y(\cdot), p(\cdot))$ which solves (3.3.5) with the initial conditions*

$$\begin{cases} y(0) &= x \\ p(0) &= \nabla \mathcal{T}(x). \end{cases}$$

Then it is well defined on in $[0, \mathcal{T}(x)]$, $y(\cdot)$ *is an optimal trajectory for* x *and* $p(\cdot)$ *is a dual arc associated to* $y(\cdot)$ *with* $p(t) = \nabla\mathcal{T}(y(t))$, *for all* $t \in [0, \mathcal{T}(x))$ *and* $p(\mathcal{T}(x)) = \mu(y(\mathcal{T}(x)))\nabla b_{\mathcal{K}}(y(\mathcal{T}(x)))$, *where* $\mu(\cdot)$ *is as in (3.3.3). Moreover,* $y(\cdot)$ *is the unique optimal trajectory starting at* x.

Proof. Let $y(\cdot)$ be an optimal trajectory for x. By Remark 3.3.4, there exists a unique dual arc $p(\cdot)$ satisfying $p(t) \in D^+\mathcal{T}(y(t))$ for all $t \in [0, \mathcal{T}(x)[$ and $p(\mathcal{T}(x)) = \mu(y(\mathcal{T}(x)))\nabla b_{\mathcal{K}}(y(\mathcal{T}(x)))$. Then $p(0) = \nabla\mathcal{T}(x)$. By Theorem 3.3.8, the pair $(y(\cdot), p(\cdot))$ coincides on $[0, \mathcal{T}(x)]$ with the unique solution of system (3.3.5) with the initial condition $y(0) = x$, $p(0) = \nabla\mathcal{T}(x)$. Moreover, by Corollary 3.3.6, $p(t) = \nabla\mathcal{T}(y(t))$ for all $t \in [0, \mathcal{T}(x))$. □ □

Theorem 3.3.10. *Assume (A1)-(A4) and (H1). Let* $x \in \mathcal{R}\backslash\mathcal{K}$ *and* $q \in D^\star\mathcal{T}(x)$. *Assume that a solution* $(y(\cdot), p(\cdot))$ *of (3.3.5) is defined on* $[0, \mathcal{T}(x)]$ *with the initial conditions*

$$\begin{cases} y(0) = x \\ p(0) = q. \end{cases} \tag{3.3.6}$$

Then $y(\cdot)$ *is an optimal trajectory for* x *and* $p(\cdot)$ *is a dual arc associated to* $y(\cdot)$. *Moreover,* $p(t) \in D^\star\mathcal{T}(y(t))$ *for all* $t \in [0, \mathcal{T}(x))$ *and* $p(\mathcal{T}(x)) = \mu(y(\mathcal{T}(x)))\nabla b_{\mathcal{K}}(y(\mathcal{T}(x)))$, *where* $\mu(\cdot)$ *is as in (3.3.3).*

Proof. Let $\{x_k\} \subset \mathcal{R} \setminus \mathcal{K}$ be such that $\mathcal{T}$ is differentiable at x_k and

$$x_k \to x, \quad \nabla\mathcal{T}(x_k) \to q, \quad \text{as } k \to \infty.$$

By Theorem 3.3.9 there exists a unique solution $(y_k(\cdot), p_k(\cdot))$ of the system (3.3.5) defined on $[0, \mathcal{T}(x_k)]$ with the initial conditions

$$\begin{cases} y(0) = x_k \\ p(0) = \nabla\mathcal{T}(x_k). \end{cases}$$

Moreover, $y_k(\cdot)$ is the optimal trajectory for x_k and $p_k(\cdot)$ is a dual arc associated to $y_k(\cdot)$ satisfying $p_k(\mathcal{T}(x_k)) = \mu(y_k(\mathcal{T}(x_k)))\nabla b_{\mathcal{K}}(y_k(\mathcal{T}(x_k)))$ and $p_k(t) = \nabla\mathcal{T}(y_k(t))$ for all $t \in [0, \mathcal{T}(x_k))$.

By passing to subsequences, we may assume that $(y_k(\cdot), p_k(\cdot))$ converges uniformly to $(y(\cdot), p(\cdot))$, where (y, p) is the solution of (3.3.5) on $[0, \mathcal{T}(x)]$ with the initial conditions (3.3.6).

By Theorem 3.1.2, $y(\cdot)$ is an optimal trajectory for x. Since $x_k \to x$, by the continuity of $\mu(\cdot)$ and $\nabla b_{\mathcal{K}}(\cdot)$, we have

$$\begin{aligned} p(\mathcal{T}(x)) &= \lim_{k\to\infty} p_k(\mathcal{T}(x_k)) = \lim_{k\to\infty} \mu(y_k(\mathcal{T}(x_k)))\nabla b_{\mathcal{K}}(y_k(\mathcal{T}(x_k))) \\ &= \mu(y(\mathcal{T}(x)))\nabla b_{\mathcal{K}}(y(\mathcal{T}(x))). \end{aligned}$$

It follows that $p(\cdot)$ is a dual arc associated to $y(\cdot)$ and

$$p(t) = \lim_{k\to\infty} p_k(t) = \lim_{k\to\infty} \nabla\mathcal{T}(y_k(t)) \in D^{\star}\mathcal{T}(y(t)),\ t \in [0, \mathcal{T}(x)).$$

□ □

Theorem 3.3.11. *Assume (A1) - (A4) and (H1). Let $x \in \mathcal{R}\setminus\mathcal{K}$. If there is only one optimal trajectory starting at x, then $\mathcal{T}$ is differentiable at x.*

Proof. Let $\bar{x}(\cdot)$ be the unique optimal trajectory starting at x. Set $z = \bar{x}(\mathcal{T}(x))$. Assume to the contrary that $\mathcal{T}$ is not differentiable at x. Then there exist $q_1, q_2 \in D^{\star}\mathcal{T}(x)$ such that $q_1 \neq q_2$. Let $\{x_k\}, \{y_k\} \subset \mathcal{R}\setminus\mathcal{K}$ be such that $\mathcal{T}$ is differentiable at x_k, y_k and

$$x_k \to x, \quad \nabla\mathcal{T}(x_k) \to q_1, \quad y_k \to x, \quad \nabla\mathcal{T}(y_k) \to q_2 \ \text{ as } \ k \to \infty.$$

Let $(y_k(\cdot), p_k(\cdot))$ and $(\bar{y}_k(\cdot), \bar{p}_k(\cdot))$ be the solutions of (3.3.5) on $[0, \mathcal{T}(x_k)]$ and $[0, \mathcal{T}(y_k)]$ with initial conditions

$$(y_k(0), p_k(0)) = (x_k, \nabla\mathcal{T}(x_k)), \quad (\bar{y}_k(0), \bar{p}_k(0)) = (y_k, \nabla\mathcal{T}(y_k))$$

respectively. By Theorem 3.3.9,

$$\begin{aligned} p_k(\mathcal{T}(x_k)) &= \mu(y_k(\mathcal{T}(x_k)))\nabla b_{\mathcal{K}}(y_k(\mathcal{T}(x_k))), \\ \bar{p}_k(\mathcal{T}(y_k)) &= \mu(\bar{y}_k(\mathcal{T}(y_k)))\nabla b_{\mathcal{K}}(\bar{y}_k(\mathcal{T}(y_k))). \end{aligned}$$

By passing to subsequences, we may assume that $(y_k(\cdot), p_k(\cdot))$ and $(\bar{y}_k(\cdot), \bar{p}_k(\cdot))$ converge uniformly to $(y(\cdot), p(\cdot))$ and $(\bar{y}(\cdot), \bar{p}(\cdot))$ respectively, where $(y(\cdot), p(\cdot))$ and $(\bar{y}(\cdot), \bar{p}(\cdot))$ are solutions of (3.3.5) on $[0, \mathcal{T}(x)]$ with the initial conditions $(y(0), p(0)) = (x, q_1)$ and $(\bar{y}(0), \bar{p}(0)) = (x, q_2)$ respectively.

By Theorem 3.1.2, $y(\cdot)$ and $\bar{y}(\cdot)$ are optimal trajectories starting at x. Thus, by assumptions, $y(\cdot)$ and $\bar{y}(\cdot)$ coincide with $\bar{x}(\cdot)$. Moreover, $p(\cdot)$ and $\bar{p}(\cdot)$ are dual arcs associated to $\bar{x}(\cdot)$ satisfying $p(\mathcal{T}(x)) = \bar{p}(\mathcal{T}(x)) = \mu(z)\nabla b_{\mathcal{K}}(z)$. Thus $(\bar{x}(\cdot), p(\cdot))$ and $(\bar{x}(\cdot), \bar{p}(\cdot))$ are solutions of (3.3.5) with the same final conditions

$$\begin{cases} \bar{x}(\mathcal{T}(x)) &= z \\ p(\mathcal{T}(x)) &= \bar{p}(\mathcal{T}(x)) = \mu(z)\nabla b_{\mathcal{K}}(z). \end{cases}$$

It follows that $p(t) = \bar{p}(t)$ for all $t \in [0, \mathcal{T}(x)]$. This means that $p(0) = \bar{p}(0)$, i.e., $q_1 = q_2$, which is a contradiction. The proof is complete. □ □

As a consequence of Theorems 3.3.9 and 3.3.11, we have

Corollary 3.3.12. *Assume (A1) - (A4) and (H1). The minimum time function is differentiable at a point $x \in \mathcal{R} \setminus \mathcal{K}$ if and only if there exists a unique optimal trajectory starting at x.*

3.4 Conjugate times and local regularity of the minimum time function

In this section we always assume (A1) - (A4) and the following additional assumptions

(A5) $\mathcal{K}$ is of class C^2.

(H2) $H \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$.

Below we denote by $T_{\mathrm{bdry}\,\mathcal{K}}(z)$ the tangent space to the $(N-1)-$dimensional C^2-manifold bdry $\mathcal{K}$ at $z \in$ bdry $\mathcal{K}$.

Consider the Hamiltonian system

$$\begin{cases} -x'(t) &= H_p(x(t), p(t)) \\ p'(t) &= H_x(x(t), p(t)), \end{cases} \tag{3.4.1}$$

on $[0, T]$ for some $T > 0$, with the final conditions

$$\begin{cases} x(T) &= z \\ p(T) &= \varphi(z), \end{cases} \tag{3.4.2}$$

where z is in a neighborhood of bdry $\mathcal{K}$ and $\varphi(z) = \mu(z)\nabla b_{\mathcal{K}}(z)$ with $\mu(\cdot)$ is as in (3.3.3). Note that, by (A5), $\varphi(\cdot)$ is of class C^1 in a neighborhood of bdry$\mathcal{K}$.

For a given z in a neighborhood of bdry $\mathcal{K}$, let $(x(\cdot;z), p(\cdot;z))$ be the solution of (3.4.1) - (3.4.2) defined on a time interval $[0,T]$ with $T > 0$. Consider the so-called variational system

$$\begin{cases} -X' &= H_{xp}(x(t),p(t))X + H_{pp}(x(t),p(t))P, \quad X(T) = I \\ P' &= H_{xx}(x(t),p(t))X + H_{px}(x(t),p(t))P, \quad P(T) = D\varphi(z). \end{cases} \tag{3.4.3}$$

Then the solution (X,P) of (3.4.3) is defined in $[0,T]$ and depends on z. Moreover

$$X(\cdot;z) = D_z x(\cdot;z) \quad \text{and} \quad P(\cdot;z) = D_z p(\cdot;z),$$

on $[0,T]$.

By well - known properties of linear systems, for any z, we have

$$\operatorname{rank}\begin{pmatrix} X(t,z) \\ P(t,z) \end{pmatrix} = \operatorname{rank}\begin{pmatrix} X(T,z) \\ P(T,z) \end{pmatrix} = N, \ \ \forall\, t \in [0,T].$$

Therefore, for any $t \in [0,T]$, any z in a neighborhood of bdry$\mathcal{K}$ and any $\theta \in \mathbb{R}^N \setminus \{0\}$,

$$X(t,z)\theta = 0 \Rightarrow P(t,z)\theta \neq 0.$$

We now introduce some definitions.

Definition 3.4.1. *For $z \in$ bdry$\mathcal{K}$, the time*

$$t_c(z) := \inf\{t \in [0,T] : X(s)\theta \neq 0, \forall\, 0 \neq \theta \in T_{bdry\mathcal{K}}(z), \forall s \in [t,T]\}$$

is said to be conjugate for z iff there exists $0 \neq \theta \in T_{bdry\mathcal{K}}(z)$ such that

$$X(t_c(z))\theta = 0.$$

In this case, the point $x(t_c(z))$ is called conjugate for z.

Remark 3.4.2. *In the classical definition of conjugate point it is required, for some $0 \neq \theta \in \mathbb{R}^N$, $X(t_c(z))\theta = 0$ (see e.g. [11, 17, 39]). Here, in the context of time optimal control problems, we have narrowed the set of such θ getting then a stronger result in Theorem 3.4.5 below than the one we would have with the classical definition.*

Proposition 3.4.3. *Assume (A5), (H2) and let $z_0 \in$ bdry$\mathcal{K}$. If there is no conjugate time in $[0, T]$ for z_0, then there exists $\rho > 0$ such that there is no conjugate time in $[0, T]$ for $z \in$ bdry$\mathcal{K} \cap B(z_0, \rho)$*

Proof. Assume, by contradiction, that there exists a sequence $\{z_i\} \subset \text{bdry}\,\mathcal{K}$ such that $z_i \to z$ and $t_i \in [0, T]$ is the conjugate time for z_i. By passing to a subsequence, we may assume that $(X_i(\cdot), P_i(\cdot))$ converges uniformly to $(X(\cdot), P(\cdot))$, where $(X_i(\cdot), P_i(\cdot))$, and $(X(\cdot), P(\cdot))$ are solution of (3.4.3) with $z = z_i$ and $z = z_0$, respectively.

Since t_i is conjugate for z_i, there exists $\theta_i \in T_{\text{bdry}\,\mathcal{K}}(z_i) \cap \mathbb{S}^{n-1}$ such that $X_i(t_i)\theta_i = 0$. By passing to subsequences, we may assume that $t_i \to t \in [0, T]$ and $\theta_i \to \theta \in \mathbb{S}^{N-1}$. Observe that $\theta \in T_{\text{bdry}\,\mathcal{K}}(z_0)$. Therefore, we have

$$X(t)\theta = \lim_{i \to \infty} X_i(t_i)\theta_i = 0,$$

which is a contradiction. □ □

Similarly, one can prove the following proposition

Proposition 3.4.4. *Assume (A5), (H2) and let $z_0 \in$ bdry$\mathcal{K}$. If there is no conjugate time in $[0, T]$ for z_0, then there exist $\rho, \varepsilon > 0$ such that there is no conjugate time in $[0, \tau]$ for $z \in$ bdry$\mathcal{K} \cap B(z_0, \rho)$ and $\tau \in [T, T + \varepsilon]$.*

Theorem 3.4.5. *Assume (A1) - (A5) and (H2). Let $\bar{x} \in \mathcal{R} \setminus \mathcal{K}$ be such that $\mathcal{T}$ is differentiable at $\bar{x}$ and $x(\cdot)$ be the optimal trajectory for $\bar{x}$. Set $\bar{z} = x(\mathcal{T}(\bar{x}))$. If there is no conjugate time in $[0, \mathcal{T}(\bar{x})]$ for $\bar{z}$ then $\mathcal{T}$ is of class C^1 in a neighborhood of $\bar{x}$.*

Proof. Recall that $\mathcal{T}$ is locally Lipschitz on the open set $\mathcal{R} \setminus \mathcal{K}$. We first prove that $\mathcal{T}$ is differentiable on a neighborhood of $\bar{x}$. Assume to the contrary that there exists a sequence $\{x_i\} \subset \mathcal{R} \setminus \mathcal{K}$ such that $x_i \to \bar{x}$ as $i \to \infty$ and $\mathcal{T}$ is not differentiable at x_i for all i.

Since $\mathcal{T}$ is not differentiable at x_i, there exist two distinct optimal trajectories starting at x_i, say $x_i^1(\cdot)$ and $x_i^2(\cdot)$, for each i. Set $z_i^1 = x_i^1(\mathcal{T}(x_i))$ and $z_i^2 = x_i^2(\mathcal{T}(x_i))$. Then $z_i^1 \neq z_i^2$. By extracting a subsequence, we may assume that $\{z_i^1\}$ and $\{z_i^2\}$ converge to some limits z^1 and z^2 respectively. Since $x_i \to \bar{x}$, we have that $z^1 = z^2 = \bar{z}$ and $\mathcal{T}(x_i) \to \mathcal{T}(\bar{x})$. Therefore, by Proposition 3.4.4, there is no conjugate time in $[0, \mathcal{T}(x_i)]$ for all $z \in B(\bar{z}, \rho) \cap \text{bdy}\,\mathcal{K}$, for some $\rho > 0$ and for i sufficiently large. For each $z \in$

$B(\bar{z},\rho)\cap \operatorname{bdy}\mathcal{K}$, let $(x(\cdot;z),p(\cdot;z))$ be the solution of the system

$$\begin{cases} -x'(t) &= H_p(x(t),p(t)), \quad x(\mathcal{T}(x_i)) = z \\ p'(t) &= H_x(x(t),p(t)), \quad p(\mathcal{T}(x_i)) = \varphi(z). \end{cases}$$

Then $x_i^1(t) = x(t;z_i^1)$ and $x_i^2(t) = x(t;z_i^2)$, for all $t \in [0,\mathcal{T}(x_i)]$. Since there is no conjugate time in $[0,\mathcal{T}(x_i)]$ for all $z \in B(\bar{z},\rho)\cap \operatorname{bdy}\mathcal{K}$, we deduce that for all $z \in B(\bar{z},\rho)\cap \operatorname{bdy}\mathcal{K}$, for all $t \in [0,\mathcal{T}(x_i)]$, we have $D_z x(t;z)\theta \neq 0$ for any $\theta \in T_{\operatorname{bdry}\mathcal{K}}(\bar{z})\cap \mathbb{S}^{n-1}$. This means that $x(t;\cdot)$ is injective on $B(\bar{z},\varepsilon)\cap \operatorname{bdry}\mathcal{K}$ for some $\varepsilon > 0$, for all $t \in [0,\mathcal{T}(x_i)]$ with i large enough contradicting to the equality $x(0;z_i^1) = x_i = x(0,z_i^2)$, for all i. Theorem 3.2.7 completes the proof. □ □

The following theorem is the main result of the chapter.

Theorem 3.4.6. *Assume (A1) - (A5), (H2) and that the kernel of $H_{pp}(x,p)$ has the dimension equal to 1 for every $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$. Let $\bar{x} \in \mathcal{R}\setminus\mathcal{K}$. If $\partial^P\mathcal{T}(\bar{x}) \neq \emptyset$, then $\mathcal{T}$ is of class C^1 in a neighborhood of $\bar{x}$.*

Proof. The first part of proof follows the lines of [7]. Set $T_0 := \mathcal{T}(\bar{x})$. Since $\partial^P\mathcal{T}(\bar{x}) \neq \emptyset$ and $\mathcal{T}$ is locally semiconcave in $\mathcal{R}\setminus\mathcal{K}$, $\mathcal{T}$ is differentiable at $\bar{x}$. Let $x_0(\cdot)$ be the optimal trajectory for $\bar{x}$. Then, thanks to Theorem 3.3.9, $\mathcal{T}$ is differentiable at $x_0(t)$ for all $t \in [0,T_0[$. Set $z_0 = x_0(T_0)$. Let $p_0(\cdot)$ be the dual arc associated to $x_0(\cdot)$ with $p_0(T_0) = \mu(z_0)\nabla b_{\mathcal{K}}(z_0)$. Then $(x_0(\cdot),p_0(\cdot))$ solves the system

$$\begin{cases} -x'(t) &= H_p(x(t),p(t)), \qquad x(T_0) = z_0 \\ p'(t) &= H_x(x(t),p(t)), \qquad p(T_0) = \varphi(z_0), \end{cases} \tag{3.4.4}$$

where $\varphi(\cdot)$ is defined by $\varphi(z) = \mu(z)\nabla b_{\mathcal{K}}(z)$, for all z close to z_0, with $\mu(\cdot)$ is as in (3.3.3).

Let (X_0,P_0) be the solution of the system

$$\begin{cases} -X' &= H_{xp}(x_0(t),p_0(t))X + H_{pp}(x_0(t),p_0(t))P, \quad X(T_0) = I \\ P' &= H_{xx}(x_0(t),p_0(t))X + H_{px}(x_0(t),p_0(t))P, \quad P(T_0) = D\varphi(z_0). \end{cases} \tag{3.4.5}$$

We will show that there is no conjugate time for z_0 in $[0,T_0]$. Assume to the contrary that there exists a conjugate time $t_c \in [0,T_0[$ for z_0. Then for some $\theta \in T_{\operatorname{bdry}\mathcal{K}}(z_0)\cap\mathbb{S}^{n-1}$, $X_0(t_c)\theta = 0$ and therefore $P_0(t_c)\theta \neq 0$.

For any $z \in \mathbb{R}^n$ near z_0, let $(x(\cdot;z), p(\cdot;z))$ be the solution of (3.4.4) with z replacing z_0. It is well defined on $[0, T_0]$ when z is sufficiently close to z_0. Then $x_0(t) = x(t; z_0),\ p_0(t) = p(t; z_0)$ for all $t \in [0, T_0]$. Moreover $x(\cdot, z)$ and $p(\cdot; z)$ converge uniformly to $x_0(\cdot)$ and $p_0(\cdot)$, respectively, in $[0, T_0]$ as $z \to z_0$. We claim that $H(x(t; z), p(t; z)) = 1$ for all $t \in [0, T_0]$ and for all z sufficiently close to z_0. Indeed, by fixing z and differentiating $H(x(\cdot; z), p(\cdot; z))$ we obtain

$$
\begin{aligned}
\frac{d}{dt} H(x(t;z), p(t;z)) &= \langle H_x(x(t;z), p(t;z)), x'(t;z)\rangle \\
&\quad + \langle H_p(x(t;z), p(t;z)), p'(t;z)\rangle \\
&= -\langle H_x(x(t;z), p(t;z)), H_p(x(t;z), p(t;z))\rangle \\
&\quad + \langle H_p(x(t;z), p(t;z)), H_x(x(t;z), p(t;z))\rangle \\
&= 0.
\end{aligned}
$$

Hence $H(x(t; z), p(t; z)) = H(x(T_0; z), p(T_0; z)) = H(z, \varphi(z)) = 1$.

Also, if $(X(\cdot; z), P(\cdot; z))$ is the solution of (3.4.5) with $x(\cdot; z), p(\cdot; z)$ and z replacing $x_0(\cdot), p_0(\cdot)$ and z_0 respectively, then $(X(\cdot; z), P(\cdot; z))$ converges uniformly to $(X_0(\cdot), P_0(\cdot))$ in $[0, T_0]$ as $z \to z_0$.

Since $\operatorname{bdry} \mathcal{K}$ is of class C^2, there exists a C^2- function $\phi : (-\delta, \delta) \to \operatorname{bdry} \mathcal{K}$ such that $\phi(0) = z_0$ and $\phi'(0) = \theta$, where $\delta > 0$.

By Theorem 3.2.2, for some $c_0 \geq 0$ and for every $t \in [0, T_0[$ we can find $R_0 > 0$ such that the following inequality is satisfied for all $R \in (0, R_0)$

$$
\begin{aligned}
-c_0|x(t,\phi(R)) - x(t,\phi(0))|^2 \leq\ & \mathcal{T}(x(t,\phi(R))) - \mathcal{T}(x(t,\phi(0))) \\
& -\langle \nabla\mathcal{T}(x(t,\phi(0))), x(t,\phi(R)) - x(t,\phi(0))\rangle.
\end{aligned}
$$

Let $t \in (t_c, T_0)$ and R_0 be as above. By Theorem 3.4.5, $\mathcal{T}$ is of class C^1 in a neighborhood of $x(t; z_0)$. Therefore, if z is sufficiently close to z_0, then $\mathcal{T}$ is differentiable at $x(t; z)$. This implies that $p(t; z) = \nabla\mathcal{T}(x(t; z))$. Therefore, for all $R \in (0, R_0)$

$$
\begin{aligned}
-c_0|x(t,\phi(R)) - x(t,\phi(0))|^2 & \\
\leq & \int_0^R \langle \nabla \mathcal{T}(x(t,\phi(r))), D_z x(t,\phi(r))\phi'(r)\rangle dr \\
& - \int_0^R \langle \nabla \mathcal{T}(x(t,\phi(0))), D_z x(t,\phi(r))\phi'(r)\rangle dr \\
= & \int_0^R \langle p(t,\phi(r)) - p(t,\phi(0)), D_z x(t,\phi(r))\phi'(r)\rangle dr \\
= & \int_0^R \Big\langle \int_0^r D_z p(t,\phi(s))\phi'(s)ds, D_z x(t,\phi(r))\phi'(r)\Big\rangle dr \\
= & \int_0^R \Big\langle \int_0^r P(t,\phi(s))\phi'(s)ds, X(t,\phi(r))\phi'(r)\Big\rangle dr.
\end{aligned}
$$

Dividing both sides of the latter inequality by R^2 and letting $R \to 0+$, we get

$$
-c_0|X_0(t)\theta|^2 \leq \frac{1}{2}\langle P_0(t)\theta, X_0(t)\theta\rangle. \tag{3.4.6}
$$

We claim that $P_0(t_c)\theta \notin \ker H_{pp}(x_0(t_c), p_0(t_c))$. Indeed, since for any $t \in [0, T_0]$ and z sufficiently close to z_0, we have $H(x(t,z), p(t,z)) = 1$, it follows that,

$$
\langle H_x(x_0(t), p_0(t)), X_0(t)\omega\rangle + \langle H_p(x_0(t), p_0(t)), P_0(t)\omega\rangle = 0,
$$

for all $t \in [0, T_0]$ and $\omega \in \mathbb{R}^N$. In particular, $\langle H_p(x_0(t_c), p_0(t_c)), P_0(t_c)\theta\rangle = 0$.
On the other hand, by the maximum principle,

$$
\langle H_p(x_0(t), p_0(t)), p_0(t)\rangle = 1 \neq 0, \ \forall t \in [0, T_0].
$$

Therefore, $P_0(t_c)\theta \notin \mathbb{R}p_0(t_c) = \ker H_{pp}(x_0(t_c), p_0(t_c))$, proving our claim.

Since $H_{pp}(x,p) \geq 0$ for all $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$, we deduce that

$$
\langle P_0(t_c)\theta, H_{pp}(x_0(t_c), p_0(t_c))P_0(t_c)\theta\rangle > 0. \tag{3.4.7}
$$

On the other hand, since $X_0(\cdot)$ is Lipschitz, for some $k > 0$ and all $t \in [t_c, T_0]$, we have

$$
|X_0(t)\theta| = |X_0(t)\theta - X_0(t_c)\theta| \leq k|t - t_c|,
$$

and

$$\begin{aligned} X_0(t)\theta &= X_0(t_c)\theta + (t-t_c)X_0'(t_c)\theta + o(|t-t_c|) \\ &= -(t-t_c)H_{pp}(x_0(t_c),p_0(t_c))P_0(t_c)\theta + o(|t-t_c|). \end{aligned}$$

Consequently, by (3.4.6), for some $K > 0$ and for all $t \in [t_c, T_0)$ sufficiently close to t_c

$$(t-t_c)\langle P_0(t)\theta, H_{pp}(x_0(t_c),p_0(t_c))P_0(t_c)\theta\rangle \leq K(t-t_c)^2 + o(|t-t_c|).$$

Dividing both sides of the latter inequality by $t - t_c > 0$ and letting $t \to t_c+$, we obtain

$$\langle P_0(t_c)\theta, H_{pp}(x_0(t_c),p_0(t_c))P_0(t_c)\theta\rangle \leq 0.$$

This contradicts (3.4.7). Therefore there is no conjugate time in $[0, T_0]$ for z_0. Then by Theorem 3.4.5, we conclude that $\mathcal{T}$ is of class C^1 in a neighborhood of $\bar{x}$. □ □

Example 3.4.7. *Consider the control system with the dynamics given by*

$$f(x,u) = h(x) + g(x)u,$$

where $h : \mathbb{R}^N \to \mathbb{R}^N$, $g : \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^N;\mathbb{R}^M)$ *and the control set* U *is the closed ball in* $\mathbb{R}^M$ *of center zero and radius* $R > 0$.
Since f *is affine with respect to* u*, assumption (A1) is verified. If* h, g *have locally Lipschitz derivatives and there exists* $k \geq 0$ *such that*

$$|h(x)| + ||g(x)|| \leq k(1+|x|), \quad \forall\, x \in \mathbb{R}^N,$$

then assumptions (A2), (A3) hold and the Hamiltonian

$$\begin{aligned} H(x,p) &= \max_{u\in U}\left(\langle -h(x) - g(x)u, p\rangle\right) \\ &= -\langle h(x), p\rangle + \max_{u\in U}\left(\langle -u, g(x)^T p\rangle\right) \\ &= -\langle h(x), p\rangle + |g(x)^T p| \end{aligned}$$

satisfies assumption (H1) whenever $g(x)$ *is also surjective for all* $x \in \mathbb{R}^N$*. Moreover, if* g, h *are of class* C^2*, then assumption (H2) is satisfied. Furthermore, for all* $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$

$$H_p(x,p) = -h(x) + \frac{1}{|g(x)^T p|}g(x)g(x)^T p$$

and for any $q \in \mathbb{R}^N$,

$$H_{pp}(x,p)(q,q) = \frac{1}{|g(x)^T p|}\ |g(x)^T q|^2 - \frac{1}{|g(x)^T p|^3}\langle g(x)^T p, g(x)^T q\rangle^2.$$

Fix any $q \in$ker $H_{pp}(x,p)$. Then, from the above equality we get

$$|g(x)^T p|^2 |g(x)^T q|^2 = \langle g(x)^T p, g(x)^T q\rangle^2. \tag{3.4.8}$$

On the other hand, if $g(x)^T q \notin \mathbb{R}\left(g(x)^T p\right)$, then

$$|\langle g(x)^T p, g(x)^T q\rangle| < |g(x)^T p||g(x)^T q|.$$

Hence, by (3.4.8), $g(x)^T q \in \mathbb{R} g(x)^T p$. Let $\lambda \in \mathbb{R}$ be such that $g(x)^T q = \lambda g(x)^T p$. Consequently $g(x)^T(q - \lambda p) = 0$. Since $g(x)$ is surjective, we deduce that $q = \lambda p$ and that $q \in \mathbb{R}p$.

Using the inclusion $p \in$ker $H_{pp}(x,p)$, we deduce that $\ker H_{pp}(x,p) = \mathbb{R}p$ *for all $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$, i.e.,*

$$\dim\ker H_{pp}(x,p) = 1, \ \ \forall\,(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}).$$

So, if the target $\mathcal{K}$ is of class C^2 and for any $z \in$bdry $\mathcal{K}$, the classical inward pointing condition

$$\min_{u\in U}\langle n_z, h(z) + g(z)u\rangle < 0$$

holds true, then Theorem 3.4.6 can be applied.

Note that if the drift mapping $h(\cdot)$ is different from zero, then the "Hamiltonian" $(x,p) \to \frac{1}{2}(H(x,p)^2 - 1)$ introduced in [37] may be not of class C^2 and therefore results from [37] can not be used.

Remark 3.4.8. *In Theorem 3.4.6 we assumed that the kernel of $H_{pp}(x,p)$ is one dimensional for all $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$. This assumption is important in our proof and we do not know yet how it can be avoided.*

We now consider the case when the dimension of $\ker H_{pp}(x,p)$ is greater than 1. We first introduce some notations. Let $A \in \mathbb{R}^{N\times N}$ and a^{ij} denote the element of ith-row and jth-column of A, i.e. $A = (a^{ij})_{1\le i,j\le N}$. Then for some $h^{ij} : \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \to \mathbb{R}$, where $i,j = 1,\cdots,N$

$$H_{pp}(x,p) = (h^{ij}(x,p))_{1\le i,j\le N}, \quad \forall\,(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}).$$

Consider a matrix-valued function $G : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^{N\times N}$ defined by

$$G(x,p) = (g^{ij}(x,p))_{1\le i,j\le N}, \qquad x,p \in \mathbb{R}^N,$$

where

$$g^{ij}(x,p) = \langle h_p^{ij}(x,p), H_x(x,p)\rangle - \langle h_x^{ij}(x,p), H_p(x,p)\rangle,$$

for all $i,j = 1,\cdots,N$ and $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$.

Assumption (H):

(i) $H \in C^3(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$;

(ii) For all $x,p \in \mathbb{R}^n$ satisfying $H(x,p) = 1$ and any $0 \neq q \in \ker H_{pp}(x,p)$ such that $\langle H_p(x,p), q\rangle = 0$, we have

$$\langle G(x,p)q,q\rangle > 0.$$

We are now ready to state a general result

Theorem 3.4.9. *Assume that (A1) - (A5) and (H). Let $\bar{x} \in \mathcal{R} \setminus \mathcal{K}$. If $\partial^P \mathcal{T}(\bar{x})$ is nonempty, then $\mathcal{T}$ is of class C^1 in a neighborhood of $\bar{x}$.*

Proof. We progress as the proof of the previous theorem keeping the same notations. It is enough to show that

$$P_0(t_c)\theta \notin \ker H_{pp}(x_0(t_c), p_0(t_c)).$$

For the reader's convenience, we recall from the last proof the following

(a) $X_0(t_c)\theta = 0$ and $P_0(t_c)\theta \neq 0$.

(b) $\langle H_p(x_0(t_c), p_0(t_c)), P_0(t_c)\theta\rangle = 0$.

(c) $H(x_0(t_c), p_0(t_c)) = 1$.

(d) $-c_0|X_0(t)\theta|^2 \le \langle P_0(t)\theta, X_0(t)\theta\rangle/2$ for all $t \in [t_c, T_0[$.

Assume to the contrary that $P_0(t_c)\theta \in \ker H_{pp}(x_0(t_c), p_0(t_c))$. Then by (H),

$$\langle G(x_0(t_c), p_0(t_c))P_0(t_c)\theta, P_0(t_c)\theta\rangle > 0. \tag{3.4.9}$$

For simplicity, let us denote by $[t]$ the couple $(x_0(t), p_0(t))$. By (3.4.5), we have

$$X_0'(t_c)\theta = -(H_{xp}[\cdot])'(t)X_0(t)\theta - H_{pp}[t_c]P_0(t_c)\theta = 0 \tag{3.4.10}$$

and

$$P_0'(t_c)\theta = H_{xx}[t_c]X_0(t_c)\theta + H_{px}[t_c]P_0(t_c)\theta = H_{px}[t_c]P_0(t_c)\theta. \tag{3.4.11}$$

Moreover,

$$X_0''(t)\theta = -H_{xp}[t]X_0'(t)\theta - (H_{pp}[\cdot])'(t)P_0(t)\theta - H_{pp}[t]P_0'(t)\theta. \tag{3.4.12}$$

Since $G[t] = (H_{pp}[\cdot])'(t)$, using (a), (3.4.10) and (3.4.5), we deduce from (3.4.12) that

$$X_0''(t_c)\theta = -G[t_c]P_0(t_c)\theta - H_{pp}[t_c]H_{px}[t_c]P_0(t_c)\theta.$$

Consider the Taylor expansion of $X_0(t)\theta$ at t_c,

$$\begin{aligned} X_0(t)\theta &= X_0(t_c)\theta + X_0'(t_c)\theta(t-t_c) + \frac{1}{2}X_0''(t_c)\theta(t-t_c)^2 + o(|t-t_c|^2) \\ &= -\frac{1}{2}\left(G[t_c]P_0(t_c)\theta + H_{pp}[t_c]H_{px}[t_c]P_0(t_c)\theta\right)(t-t_c)^2 + o(|t-t_c|^2). \end{aligned}$$

Thus there exists $c > 0$ such that

$$|X_0(t)\theta|^2 \le c|t-t_c|^4,$$

for all $t \in [0, T_0]$ sufficiently close to t_c.
Furthermore,

$$\begin{aligned} \langle X_0(t)\theta, P_0(t)\theta\rangle &= -\frac{1}{2}\langle G[t_c]P_0(t_c)\theta + H_{pp}[t_c]H_{px}[t_c]P_0(t_c)\theta, P_0(t)\theta\rangle(t-t_c)^2 \\ &\quad + o(|t-t_c|^2). \end{aligned}$$

Then by (d), there is some $C > 0$ such that, for all $t \in [t_c, T_0)$ sufficiently close to t_c,

$$\langle G[t_c]P_0(t_c)\theta + H_{pp}[t_c]H_{px}[t_c]P_0(t_c)\theta, P_0(t)\theta\rangle(t-t_c)^2 \le C|t-t_c|^4 + o(|t-t_c|^2).$$

Dividing both sides of the latter inequality by $(t-t_c)^2$ and letting $t \to t_c+$, we get

$$\langle G[t_c]P_0(t_c)\theta + H_{pp}[t_c]H_{px}[t_c]P_0(t_c)\theta, P_0(t_c)\theta\rangle \le 0.$$

By the symmetry of H_{pp}, it follows that

$$\langle G[t_c]P_0(t_c)\theta, P_0(t_c)\theta\rangle + \langle H_{px}[t_c]P_0(t_c)\theta, H_{pp}[t_c]P_0(t_c)\theta\rangle \le 0,$$

But $P_0(t_c)\theta \in \ker H_{pp}[t_c]$ and therefore

$$\langle G[t_c]P_0(t_c)\theta, P_0(t_c)\theta\rangle \le 0.$$

This contradicts (3.4.9). Hence $P_0(t_c)\theta \notin \ker H_{pp}[t_c]$. The proof is complete. □ □

Remark 3.4.10. *1) From the proof of Theorem 3.4.6, we can see that the conclusion of Theorem 3.4.6 still holds true under a weaker assumption on the Hamiltionian H. We actually can replace the assumption "the kernel of $H_{pp}(x,p)$ has the dimension equal to 1 for all $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$" by "the kernel of $H_{pp}(x,p)$ has the dimension equal to 1 for all $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$ with $H(x,p) = 1$".*
2) We may also replace the assumptions on H in Theorems 3.4.6 and 3.4.9 by a mixed assumption and get the same result. Assume $H \in C^3(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$ and set

$$\Omega = \left\{(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) : H(x,p) = 1\right\}.$$

Let $\Omega_1 \subset \Omega$ be such that $\dim\ker H_{pp}(x,p) = 1$ for all $(x,p) \in \Omega_1$ and $\Omega_2 \subset \Omega$ be such that $\langle G(x,p)q, q\rangle > 0$ for all $0 \neq q \in \ker H_{pp}(x,p)$ satisfying $\langle H_p(x,p), q\rangle = 0$ and for all $(x,p) \in \Omega_2$. If $\Omega_1 \cup \Omega_2 = \Omega$, then the conclusion of Theorem 3.4.6 (or, equivalently, of Theorem 3.4.9) still holds true.

Chapter 4

Non-Lipschitz singularities, the SBV regularity and the differentiability of the minimum time function

This chapter is devoted to the study of the Hausdorff dimension of the singular set of the minimum time function $\mathcal{T}$ under controllability conditions which do not imply the Lipschitz continuity of $\mathcal{T}$. As consequences, we obtain some regularity properties for $\mathcal{T}$.

4.1 Optimal time to reach the origin

Let us consider again the control system

$$\begin{cases} \dot{y}(t) = f(y(t), u(t)) & \text{a.e.,} \\ u(t) \in U & \text{a.e.,} \\ y(0) = x, & \end{cases} \tag{4.1.1}$$

where the control set $U \subset \mathbb{R}^M$ is nonempty and compact and $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ is continuous and Lipschitz with respect to the state variable x, uniformly with respect to u. Let $\mathcal{U}_{ad}$ be the set of admissible controls, i.e., all measurable functions u such that $u(s) \in U$ for a.e. s. Under our assumptions, for any $u(\cdot) \in \mathcal{U}_{ad}$, there is a unique Carathéodory solution $y(\cdot; x, u)$ of (4.1.1) called the trajectory starting from x corresponding to the control $u(\cdot)$.

For a fixed $x \in \mathbb{R}^N$, we define

$$\theta(x,u) := \min\,\{t \geq 0 \mid y(t;x,u) = 0\}.$$

Of course, $\theta(x,u) \in [0,+\infty]$, and $\theta(x,u)$ is the time taken for the trajectory $y(\cdot;x,u)$ to reach 0, provided $\theta(x,u) < +\infty$. The *minimum time* $\mathcal{T}(x)$ to reach 0 from x is defined by $\mathcal{T}(x) := \inf\,\{\theta(x,u) \mid u(\cdot) \in \mathcal{U}_{\rm ad}\}$ and under standard assumptions the infimum is attained. A minimizing control, say $\bar{u}(\cdot)$, is called an *optimal control*. The trajectory $y^{x,\bar{u}}(\cdot)$ corresponding to $\bar{u}(\cdot)$ is called an *optimal trajectory*.

For $t > 0$, the reachable set $\mathcal{R}(t)$ is the set of points which can be steered to the origin with the control dynamics (4.1.1) within the time t. Then $\mathcal{R}(t)$ is the set of points which can be reached from the origin with the *reversed dynamics*

$$\begin{cases} \dot{x}(t) \;=\; -f(x(t),u(t)) & \text{a.e.,} \\ u(t) \;\in\; U & \text{a.e.,} \\ x(0) \;=\; 0, & \end{cases} \tag{4.1.2}$$

i.e., $\mathcal{R}(t)$ is the sublevel $\{x \in \mathbb{R}^N : T(x) \leq t\}$ of $\mathcal{T}(\cdot)$. If $\bar{u}$ is an admissible control steering x to the origin in the minimum time $\mathcal{T}(x)$, then the Dynamic Programming Principle implies that for all $0 < t < \mathcal{T}(x)$ the point $y(t;x,\bar{u})$ belongs to the boundary of $\mathcal{R}(t)$.

We are going to state Pontryagin's Principle for the problem of reaching the origin in optimal time. Before doing that, we need to introduce the minimized Hamiltonian. We define for every triple $(x,p,u) \in \mathbb{R}^N \times \mathbb{R}^N \times U$, the *Hamiltonian*:

$$\mathcal{H}(x,p,u) = \langle p, f(x,u)\rangle$$

the *minimized Hamiltonian*:

$$h(x,p) = \min_{u\in U} \mathcal{H}(x,p,u).$$

Observe that if $\bar{x}$ is steered to the origin with respect to the system (4.1.1) by the control $\bar{u}(\cdot)$ in the time T, then the origin is steered to $\bar{x}$ with respect to the reversed dynamics (4.1.2) in the same time T by the control $\tilde{u}(t) := \bar{u}(T-t)$. The corresponding trajectory will be denoted by $\bar{y}(t) := y(T-t;\bar{x},\bar{u})$. Then Pontryagin's Principle reads as follows.

Theorem 4.1.1 (Pontryagin's Maximum Principle for nonlinear systems)**.** *Fix $T > 0$ and let $\bar{x} \in \mathbb{R}^N$ together with an optimal control steering $\bar{x}$ to the origin in the time T. Then there exists an absolutely continuous function $\lambda : [0, T] \to \mathbb{R}^N$, never vanishing, such that*

(i) $\dot{\lambda}(t) = \lambda(t) D_x f(\bar{y}(t), \bar{u}(T - t))$, *a.e.* $t \in [0, T]$,

(ii) $\mathcal{H}(\bar{y}(t), \lambda(t), \bar{u}(T - t)) = h(\bar{y}(t), \lambda(t))$, *a.e.* $t \in [0, T]$,

(iii) $h(\bar{y}(t), \lambda(t)) =$ *constant, for all* $t \in [0, T]$,

(iv) $\lambda(T) \in N^C_{\mathcal{R}_T}(\bar{x})$.

This formulation of the Maximum principle can be obtained by using the classical one (see, e.g., [45, Theorem 8.7.1]) for the reversed dynamics.

4.2 Properties connected with the minimized Hamiltonian

4.2.1 Minimized Hamiltonian and normals to epi($\mathcal{T}$)

This section is concerned with a relation between normals to the sublevels of $\mathcal{T}$ and normals to epi($\mathcal{T}$), which was one of the main tools used in [21, 23] in order to prove that epi($\mathcal{T}$) has positive reach. We give here a unified and slightly generalized presentation, in order to use it in the sequel.

We recall first that a point $x \in \mathbb{R}^N \setminus \{0\}$ is defined to be an *optimal point* for (4.1.1) if there exist x_1 such that $\mathcal{T}(x_1) > \mathcal{T}(x)$ and a control u with the property that $y(\cdot; x_1, u)$ steers x_1 to the origin in the optimal time $\mathcal{T}(x_1)$ and $y(\mathcal{T}(x_1) - \mathcal{T}(x); x_1, u) = x$, i.e., if there exists an optimal trajectory for (4.1.1) which passes through x.
We are now ready to state our result. The assumptions are indeed strong, but we emphasize the fact that they are all satisfied in the cases we are going to consider in the chapter.

Proposition 4.2.1. *Consider the general control system (4.1.1) with the following assumptions:*

(i) $U \subset \mathbb{R}^M$ *is compact and* $\{f(x, u) : u \in U\}$ *is convex for every* $x \in \mathbb{R}^N$.

(ii) $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ *is continuous and satisfies*

$$|f(x,u) - f(y,u)| \le L|x-y|, \ \forall x, y \in \mathbb{R}^N, u \in U,$$

for a positive constant L. *Moreover, the differential of* f *with respect to the* x *variable,* $D_x f$, *exists everywhere, is continuous with respect to both* x *and* u *and satisfies*

$$||D_x f(x,u) - D_x f(y,u)|| \le L_1|x-y|, \ \forall x, y \in \mathbb{R}^N, u \in U,$$

for a positive constant L_1.

Let $x \in \mathbb{R}^N \setminus \{0\}$ *and let* $\mathcal{T}(x)$ *be the minimum time to reach the origin from* x. *Assume that there exists a neighborhood* $\mathcal{V}$ *of* x *such that*

(1) $\mathcal{T}$ *is finite and continuous in* $\mathcal{V}$,

(2) every $y \in \mathcal{V}$ *is an optimal point,*

(3) for every $y \in \mathcal{V}$ *the optimal control steering* y *to the origin is unique and bang-bang with finitely many switchings,*

(4) there exists $r > \mathcal{T}(x)$ *such that* $\mathcal{R}(t)$ *has positive reach for all* $t < r$.

Let $\zeta \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$. *Then*

(a) $h(x,\zeta) \le 0$.

(b) $(\zeta, h(x,\zeta)) \in N_{\mathrm{epi}(\mathcal{T})}(x, \mathcal{T}(x))$.

Moreover, if $0 \neq \zeta \in \mathbb{R}^N$ *is such that* $(\zeta, h(x,\zeta)) \in N_{\mathrm{epi}(\mathcal{T})}(x, \mathcal{T}(x))$, *then* $\zeta \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$.

Proof. Since $\zeta \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$ and $\mathcal{R}(\mathcal{T}(x))$ has positive reach, there exists a constant $\sigma \ge 0$ such that

$$\langle \zeta, y - x \rangle \le \sigma |\zeta| \, |y-x|^2,$$

for all $y \in \mathcal{R}(\mathcal{T}(x))$.
Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair for x and let $\lambda : [0, \mathcal{T}(x)] \to \mathbb{R}^N$ be absolutely continuous and such that

$$\begin{cases} \dot{\lambda}(t) \ = \ \lambda(t) D_x f(\bar{x}(T(x)-t), \bar{u}(T(x)-t)), \\ \lambda(T(x)) \ = \ \zeta. \end{cases} \tag{4.2.1}$$

and $h(\bar{x}(\mathcal{T}(x)-t), \lambda(t)) = \langle \lambda(t), f(\bar{x}(\mathcal{T}(x)-t), \bar{u}(\mathcal{T}(x)-t)) \rangle = h(x, \zeta)$ for all $t \in [0, \mathcal{T}(x)]$ (see Theorem 3.1 in [9]).

Observing that $\bar{x}(t) \in \mathcal{R}(\mathcal{T}(x))$ for all $0 < t < \mathcal{T}(x)$, we have

$$\langle \zeta, \bar{x}(t) - x \rangle \leq \sigma |\zeta| \, |\bar{x}(t) - x|^2.$$

By using Gronwall's lemma, there is a suitable $M_1 > 0$ such that $|\bar{x}(t) - x| \leq M_1 t$. Since f is Lipschitz with respect to x, we have for some $M_2 > 0$ and for all $0 < t < \mathcal{T}(x)$,

$$\left\langle \zeta, \frac{1}{t} \int_0^t f(x, \bar{u}(s) ds \right\rangle \leq M |\zeta| t.$$

Taking the upper limit, we obtain

$$\limsup_{t \to 0+} \left\langle \zeta, \frac{1}{t} \int_0^t f(x, \bar{u}(s) ds \right\rangle \leq 0,$$

which implies

$$\langle \zeta, f(x, \tilde{u}) \rangle \leq 0,$$

for some $\tilde{u} \in U$, since $\{f(x, u) : u \in U\}$ is convex and U is compact, whence $h(x, \zeta) \leq 0$.

We are now going to show that $(\zeta, h(x, \zeta)) \in N_{\text{epi}(\mathcal{T})}(x, \mathcal{T}(x))$, i.e., there is a $\sigma \geq 0$ such that

$$\langle (\zeta, \vartheta), (y, \beta) - (x, \mathcal{T}(x)) \rangle \leq \sigma (|y - x|^2 + |\mathcal{T}(x) - \beta|^2), \tag{4.2.2}$$

for all (y, β) in a neighborhood of $(x, \mathcal{T}(x))$, say $(y, \beta) \in \mathcal{V} \times [0, r]$, where $\vartheta := h(x, \zeta)$, $\beta \geq \mathcal{T}(y)$, and $r > \mathcal{T}(x)$ is such that $\mathcal{T}(y) < r$ for all $y \in \mathcal{V}$. There are two possible cases:

(i) $\mathcal{T}(y) \leq \mathcal{T}(x)$,

(ii) $\mathcal{T}(y) > \mathcal{T}(x)$.

In the first case, since $y \in \mathcal{R}(\mathcal{T}(x))$ and $\mathcal{R}(\mathcal{T}(x))$ has positive reach, there is a $K_1 \geq 0$ such that

$$\langle \zeta, y - x \rangle \leq K_1 |\zeta| \, |y - x|^2.$$

Since $\vartheta = h(x, \zeta) \leq 0$, if $\beta \geq \mathcal{T}(x)$ then (4.2.2) is satisfied. If instead $\beta < \mathcal{T}(x)$, then we set $x_1 = \bar{x}(\mathcal{T}(x) - \beta)$. By Gronwall's lemma, there is some $K > 0$ such that

$$|x - x_1| = |x - \bar{x}(\mathcal{T}(x) - \beta)| \leq K |\mathcal{T}(x) - \beta|.$$

We have

$$\begin{aligned}\langle \zeta, y-x\rangle =& \ \langle \lambda(\beta), y-x_1\rangle + \langle \lambda(\mathcal{T}(x)) - \lambda(\beta), y-x_1\rangle + \langle \zeta, x_1 - x\rangle \\ =&: (I) + (II) + (III).\end{aligned}$$

We now consider (I). Since $y \in \mathcal{R}(\beta)$ and $\lambda(\beta) \in N^C_{\mathcal{R}(\beta)}(x_1)$ (see Definition 2.3 and Corollary 4.8 in [30]), owing to the fact that $\mathcal{R}(\beta)$ has positive reach, there exist $K_2, K_3 > 0$ such that

$$\begin{aligned}(I) \ &\leq K_2|\lambda(\beta)|\,|y-x_1|^2 \\ &\leq 2K_2|\lambda(\beta)|(|y-x|^2 + |x-x_1|^2) \\ &\leq K_3(|y-x|^2 + |\mathcal{T}(x)-\beta|^2).\end{aligned}$$

Let us now consider (II). We have, for suitable constants $K_4, K_5 > 0$,

$$\begin{aligned}(II) \ &\leq |\lambda(\mathcal{T}(x)) - \lambda(\beta)|\,|y-x_1| \\ &\leq K_4|\mathcal{T}(x)-\beta|(|y-x| + K|\mathcal{T}(x)-\beta|) \\ &\leq K_5(|y-x|^2 + |\mathcal{T}(x)-\beta|^2).\end{aligned}$$

Finally, we have, for a suitable constant $K_6 > 0$,

$$\begin{aligned}(III) &= \int_0^{\mathcal{T}(x)-\beta} \langle \lambda(\mathcal{T}(x)), f(\bar{x}(s), \bar{u}(s))\rangle ds \\ &= \int_0^{\mathcal{T}(x)-\beta} \langle \lambda(\mathcal{T}(x)-s), f(\bar{x}(s), \bar{u}(s))\rangle ds \\ &\quad + \int_0^{\mathcal{T}(x)-\beta} \langle \lambda(\mathcal{T}(x)) - \lambda(\mathcal{T}(x)-s), f(\bar{x}(s), \bar{u}(s))\rangle ds\end{aligned}$$

(since the minimized Hamiltonian is constant)

$$\begin{aligned}&\leq (\mathcal{T}(x)-\beta)h(x,\zeta) + K_6\int_0^{\mathcal{T}(x)-\beta} s\,ds \\ &= (\mathcal{T}(x)-\beta)h(x,\zeta) + \frac{K_6}{2}|\mathcal{T}(x)-\beta|^2.\end{aligned}$$

Putting the estimates together, we obtain

$$\langle \zeta, y-x\rangle \leq (\mathcal{T}(x)-\beta)h(x,\zeta) + K_7(|y-x|^2 + |\mathcal{T}(x)-\beta|^2),$$

for a suitable positive constant K_7. The proof of (4.2.2) is concluded in the case (i).

We are now going to consider the case (ii). Since $\vartheta \leq 0$, it is enough to prove (4.2.2) for $\beta = \mathcal{T}(y)$.

Since x is an optimal point, there exists x_1 such that $\mathcal{T}(x_1) = \mathcal{T}(y)$ together with an optimal pair, still denoted $\bar{x}(\cdot)$ and $\bar{u}(\cdot)$, such that $\bar{x}(\mathcal{T}(y) - \mathcal{T}(x)) = x$. Let $\lambda(\cdot)$ denote the extension up to the time $\mathcal{T}(y)$ of the solution of (4.2.1). Since the optimal control is unique and bang-bang with finitely many switchings, it is easy to prove that $h(\bar{x}(\mathcal{T}(y)-t), \lambda(t)) = \langle \lambda(t), f(\bar{x}(\mathcal{T}(y)-t), \bar{u}(\mathcal{T}(y)-t)) \rangle$ is constant for all $t \in [0, \mathcal{T}(y)]$. Then by using the same argument of the case (i), one can easily show that (4.2.2) holds true.

We are now going to prove the last statement. Since $(\zeta, h(x,\zeta)) \in N_{\text{epi}(\mathcal{T})}(x, \mathcal{T}(x))$, we have $h(x,\zeta) \leq 0$. There are two cases.

Case 1. $h(x,\zeta) < 0$.

We set $\zeta_1 = -\frac{\zeta}{h(x,\zeta)}$. Then $h(x,\zeta_1) = -1$ and $(\zeta_1, -1) \in N_{\text{epi}(\mathcal{T})}(x, \mathcal{T}(x))$. It follows that $\zeta_1 \in \partial^P \mathcal{T}(x)$. From [46], we have

$$\partial^P \mathcal{T}(x) = N_{\mathcal{R}(\mathcal{T}(x))}(x) \cap \{\zeta : h(x,\zeta) = -1\}.$$

Thus $\zeta_1 \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$, i.e., $\zeta \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$.

Case 2. $h(x,\zeta) = 0$.

There is some $C \geq 0$ such that

$$\langle (\zeta, 0), (y, \beta) - (x, \mathcal{T}(x)) \rangle \leq C(|y - x|^2 + |\beta - \mathcal{T}(x)|^2), \tag{4.2.3}$$

for all $\beta \geq \mathcal{T}(y)$. Observe that

$$\text{bdry}\mathcal{R}(\mathcal{T}(x)) = \{y : \mathcal{T}(y) = \mathcal{T}(x)\}.$$

Taking $\beta = \mathcal{T}(y) = \mathcal{T}(x)$ in (4.2.3), we get

$$\langle \zeta, y - x \rangle \leq C|y - x|^2,$$

for all $y \in \text{bdry}\mathcal{R}(\mathcal{T}(x))$. This imples that $\zeta \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$.

The proof is complete. □

A particular feature of the minimum time function, under our assumptions, is that it inherits its regularity from its level sets. More precisely, one has

Proposition 4.2.2. *Under assumptions in Proposition 4.2.1, we have*

$$\dim N_{\mathcal{R}(\mathcal{T}(x))}(x) = \dim N_{epi(\mathcal{T})}(x, \mathcal{T}(x)),$$

for all $x \in \mathcal{R} \setminus \{0\}$.

Proof. Assume that $\dim N_{\mathcal{R}(\mathcal{T}(x))}(x) = n$ and $\dim N_{\mathrm{epi}(\mathcal{T})}(x, \mathcal{T}(x)) = m$.
Let $\zeta_1, \cdots \zeta_n \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$ be linearly independent. Then we have, from Proposition 4.2.1, that $(\zeta_i, h(x, \zeta_i)) \in N_{\mathrm{epi}(\mathcal{T})}(x, \mathcal{T}(x))$ for all $i = 1, \cdots, n$.
Observe that $(\zeta_1, h(x, \zeta_1)), \cdots, (\zeta_n, h(x, \zeta_n))$ are linearly independent. Thus $n \leq m$.

Now let $(\zeta_1, \alpha_1), \cdots, (\zeta_m, \alpha_m) \in N_{\mathrm{epi}(\mathcal{T})}(x, \mathcal{T}(x))$ be linearly independent. Again by Proposition 4.2.1, we have $\zeta_i \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$ and of course $\alpha_i \leq 0$ for all $i = 1, \cdots, m$. Moreover, from [46], we have

$$\partial^P \mathcal{T}(x) = N_{\mathcal{R}(\mathcal{T}(x))}(x) \cap \{\zeta : h(x, \zeta) = -1\}. \tag{4.2.4}$$

One also has (see Proposition 4.2.5)

$$\partial^\infty \mathcal{T}(x) = N_{\mathcal{R}(\mathcal{T}(x))}(x) \cap \{\zeta : h(x, \zeta) = 0\}. \tag{4.2.5}$$

By using (4.2.4) and (4.2.5), one can show that $h(x, \zeta_i) = \alpha_i$ for all $i = 1, \cdots, m$. Since $h(x, \zeta)$ is homogeneous in ζ and $(\zeta_1, \alpha_1), \cdots, (\zeta_m, \alpha_m)$ are linearly independent, we obtain that $\zeta_1, \cdots, \zeta_m$ are linearly independent. Thus $m \leq n$. The proof is complete. □

4.2.2 Minimized Hamiltonian and non-Lipschitz points

This section is devoted to identify points around which the minimum time function $\mathcal{T}$ is not Lipschitz as points where the proximal normal cone to epi($\mathcal{T}$) contains a horizontal vector $\zeta \neq 0$. It will also turn out that if x is a non-Lipschitz point and ζ is such a vector, then $h(x, \zeta) = 0$. A kind of converse statement can also be proved. Note that all results are valid in the domain where epi($\mathcal{T}$) has positive reach.

Definition 4.2.3. *We say that a function* $\mathcal{T} : \mathbb{R}^N \to \mathbb{R}$ *is non-Lipschitz at* x *provided there exist two sequences* $\{x_i\}, \{y_i\}$ *such that* $x_i \neq y_i$ *for all* i, $\{x_i\}, \{y_i\}$ *converge to* x *and*

$$\limsup_{i \to \infty} \frac{|\mathcal{T}(y_i) - \mathcal{T}(x_i)|}{|y_i - x_i|} = +\infty.$$

Observe that the set of non-Lipschitz points is closed.
The first result does not require $\mathcal{T}$ to be a minimum time function.

Proposition 4.2.4. *Let $\Omega \subset \mathbb{R}^N$ be open and let $\mathcal{T}$ be continuous in Ω and such that epi($\mathcal{T}$) has locally positive reach. Let $\bar{x} \in \Omega$. Then $\mathcal{T}$ is non-Lipschitz at $\bar{x}$ if and only if there exists a nonzero vector $\zeta \in \mathbb{R}^N$ such that*

$$(\zeta, 0) \in N_{\text{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x})).$$

Proof. By Theorem 9.13 in [41], $\mathcal{T}$ is non-Lipschitz at $\bar{x}$ if and only if $\partial^\infty \mathcal{T}(\bar{x})$ contains a nonzero vector ζ. This condition is equivalent to $(\zeta, 0) \in N_{\text{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x}))$. □

Now we restrict ourselves to the case where $\mathcal{T}$ is the minimum time function to reach the origin for (4.1.1). We assume that the conditions ensuring that epi($\mathcal{T}$) has positive reach are satisfied.

Proposition 4.2.5. *Under the same assumptions of Theorem 4.2.1, let $\mathcal{T}$ denote the minimum time function to reach the origin for (4.1.1). Let $\bar{x} \neq 0$ and $\delta > 0$ be such that the epigraph of $\mathcal{T}$ restricted to $\bar{B}(\bar{x}, \delta)$ has positive reach. Let $\zeta \in \mathbb{R}^N \setminus \{0\}$. Then*

$$\zeta \in \partial^\infty \mathcal{T}(\bar{x}) \text{ if and only if } h(\bar{x}, \zeta) = 0 \text{ and } \zeta \in N_{\mathcal{R}(\mathcal{T}(\bar{x}))}(\bar{x}).$$

Proof. Recalling Proposition 4.2.4, $\zeta \in \partial^\infty \mathcal{T}(\bar{x})$ if and only if $(\zeta, 0) \in N_{\text{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x}))$, i.e., for a suitable constant $c \geq 0$,

$$\langle \zeta, y - \bar{x} \rangle \leq c|\zeta| \left(|y - \bar{x}|^2 + |\beta - \mathcal{T}(\bar{x})|^2 \right), \tag{4.2.6}$$

for all $y \in \bar{B}(\bar{x}, \delta)$ and for all $\beta \geq \mathcal{T}(y)$.

Let $\zeta \in \partial^\infty \mathcal{T}(\bar{x})$. If $y \in \mathcal{R}(\mathcal{T}(\bar{x}))$, then we can take $\beta = \mathcal{T}(\bar{x})$ in (4.2.6) and so

$$\langle \zeta, y - \bar{x} \rangle \leq c|\zeta| \cdot |y - \bar{x}|^2,$$

i.e., $\zeta \in N_{\mathcal{R}(\mathcal{T}(\bar{x}))}(\bar{x})$.

Recalling (i) in Proposition 4.2.1, if $\zeta \in N_{\mathcal{R}(\mathcal{T}(\bar{x}))}(\bar{x})$ then $h(\bar{x}, \zeta) \leq 0$. Assume by contradiction that $h(\bar{x}, \zeta) < 0$. Then, by using (ii) in Proposition 4.2.1 there exists $\alpha > 0$ such that

$$(\alpha\zeta, -1) \in N_{\text{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x})).$$

Since $N_{\mathrm{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x}))$ is convex, we have, for any $\lambda \in (0,1)$,

$$\begin{aligned} v_\lambda &:= \lambda(\zeta, 0) + (1-\lambda)(\alpha\zeta, -1) \\ &= (\lambda\zeta + (1-\lambda)\alpha\zeta, \lambda - 1) \in N_{\mathrm{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x})). \end{aligned}$$

This implies

$$\frac{v_\lambda}{1-\lambda} = \left(\frac{\lambda\zeta + (1-\lambda)\alpha\zeta}{1-\lambda}, -1\right) \in N_{\mathrm{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x})),$$

i.e.,

$$\frac{v_\lambda}{1-\lambda} \in \partial\mathcal{T}(\bar{x}).$$

By Theorem 5.1(b) in [46], we have

$$h\left(\bar{x}, \frac{\lambda\zeta + (1-\lambda)\alpha\zeta}{1-\lambda}\right) = -1,$$

i.e., $h(\bar{x}, \lambda\zeta + (1-\lambda)\alpha\zeta) = \lambda - 1$, for all $\lambda \in (0,1)$. Letting $\lambda \to 1^-$ in the above equality, we obtain $h(\bar{x}, \zeta) = 0$, which is a contradiction. Thus $h(\bar{x}, \zeta) = 0$.

Conversely, let $\zeta \in N_{\mathcal{R}(\mathcal{T}(\bar{x}))}(\bar{x})$ be a nonzero vector such that $h(\bar{x}, \zeta) = 0$. Applying Proposition 4.2.1, we see that $(\zeta, 0) \in N_{\mathrm{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x}))$, which says exactly that $\zeta \in \partial^\infty\mathcal{T}(\bar{x})$. The proof is concluded.

□

4.3 Non-Lipschitz singularities for linear control systems

4.3.1 Normal linear control systems

Consider the linear control system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) & \text{a.e.}, \\ u(t) \in U & \text{a.e.}, \\ y(0) = x, \end{cases} \tag{4.3.1}$$

where $A \in \mathbb{M}_{N\times N}$, $B \in \mathbb{M}_{N\times M}$ and the control set $U = [-1,1]^M$, $1 \le M \le N$. We will use the notation $B = (b_1, \cdots, b_M)$, where each entry is an N-dimensional column. For any $t > 0$, we denote by $\mathcal{U}_{ad}^t$ the set of all admissible controls on the interval $[0,t]$, i.e. the measure function $u : [0,t] \to U$. For any $u(\cdot) \in \mathcal{U}_{ad}^t$, (4.3.1) has the unique solution denoted by $y(\cdot; x, u)$. One has

$$y(t; x, u) = e^{At}x + \int_0^t e^{A(t-s)}Bu(s)ds. \tag{4.3.2}$$

Note that $\bar{x}$ is reachable by a solution of (4.3.1) at time t if and only if the following (equivalent) conditions hold:

$$\bar{x} = e^{At}x + \int_0^t e^{A(t-s)}Bu(s)\,ds \qquad \text{and} \qquad x = e^{-At}\bar{x} - \int_0^t e^{-As}Bu(s)\,ds, \tag{4.3.3}$$

where $u(\cdot) \in \mathcal{U}_{ad}^t$.
For $t > 0$, the reachable set at time t can be computed explicitly

$$\mathcal{R}(t) = \left\{ \int_0^t e^{-As}Bu(s)ds | u(\cdot) \in \mathcal{U}_{ad}^t \right\}.$$

Thoughout this chapter, we are interested in optimal time for normal linear control systems.

Definition 4.3.1. *The system (4.3.1) is normal if and only if*

$$\text{Rank}\left[b_i, Ab_i, \cdots, A^{N-1}b_i\right] = N \tag{4.3.4}$$

for all $i = 1, 2, \cdots, M$.

Remark 4.3.2. *If the system (4.3.1) is normal then* (A, B) *satisfies the Kalman rank condition. Therefore the minimum time function is everywhere finite and continuous in* $\mathcal{R}$ *(actually Hölder continuous with exponent* $1/N$*, see, e.g., Theorem 17.3 in [34] and Theorem 1.9, Chapter IV, in [3] and references therein).*

We state here a classical results for normal linear systems

Theorem 4.3.3. *The linear control system (4.3.1) is normal if and only if the reachable set* $\mathcal{R}(t)$ *is strictly convex for any* $t > 0$.

Proof. See e.g. [34]. □

It follows from the later Theorem that sublevels of the minimum time function $\mathcal{T}$ for normal systems are strictly convex and so, in particular, they have positive reach. By [21, Theorem 3.7], one also has that $epi(\mathcal{T})$ has positive reach.

We state now the linear version of Pontryagin's Maximum Principle.

Theorem 4.3.4. *Consider the problem* (4.3.1) *under the normality condition (4.3.4). Let* $T > 0$ *and let* $x \in \mathbb{R}^N$*. The following statements are equivalent:*

(i) $x \in \text{bdry}\mathcal{R}(T)$*,*

(ii) there exists an optimal control $\bar{u}$ *steering* x *to the origin in time* T*; in particular,* $\mathcal{T}(x) = T$*;*

(iii) (Pontryagin Maximum Principle) for every $\zeta \in N_{\mathcal{R}(T)}(x), \zeta \neq 0$*, we have*

$$\bar{u}_i(t) = -\text{sign}\left(\langle \zeta, e^{-At} b_i \rangle\right) \ \ a.e.\ t \in [0, T], \tag{4.3.5}$$

for all $i = 1, 2, \ldots, M$*.*

A well known reference for this result is [34, Sections 13 - 15].

It is easy to see, using Pontryagin's Maximum Principle, that for a normal linear system every point is optimal. Indeed, it is enough to extend the adjoint vector and choose a control which maximizes the Hamiltonian (for the reversed dynamics). Notice that all assumptions in Section 4.2 are satisfied by normal linear systems. Therefore, we can apply results in Section 4.2 to the linear case.

4.3.2 Non-Lipschitz points and rectifiability result

In this section, we apply results in Section 4.2 to give a explicit representation of the set of non-Lipschitz points of the minimum time function for a linear control system. We then prove a rectifiability result for the non-Lipschitz set. We assume throughout this section that the linear control system (4.3.1) is normal. We set also

$$k = \text{rank} B. \tag{4.3.6}$$

Of course, $1 \le k \le M$.
We will first characterize the set $\mathcal{S}$ of non-Lipschitz points of $\mathcal{T}$ as

$$\mathcal{S} = \Big\{ x \in \mathbb{R}^N : \text{there exist}\, r > 0 \,\text{and}\, \zeta \in \mathbb{S}^{N-1} \text{ such that} \\ x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \,\text{sign}\left(\langle \zeta, e^{At} b_i \rangle\right) dt \;\text{ and }\; \langle \zeta, b_i \rangle = 0 \;\forall i = 1, \dots, M \Big\}. \tag{4.3.7}$$

If $k = N$, then $\mathcal{S}$ is empty. If $k < N$, $\mathcal{S}$ is nonempty and we will prove also that $\mathcal{S}$ is $(N-k)$-rectifiable.
From now on, we assume

$$k < N. \tag{4.3.8}$$

Remark 4.3.5. *It is easy to see that*

$$\mathcal{S} = \Big\{ x \in \mathbb{R}^N : \textit{there exist } r > 0 \textit{ and } \zeta \in \mathbb{S}^{N-1} \textit{ such that} \\ x = \sum_{i=1}^{M} \int_0^r e^{-At} b_i \,\text{sign}\left(\langle \zeta, e^{-At} b_i \rangle\right) dt, \\ \zeta \in N_{\mathcal{R}(r)}(x) \textit{ and } \langle \zeta, e^{-Ar} b_i \rangle = 0 \;\; \forall i = 1, \dots, M \Big\}. \tag{4.3.9}$$

We deal first with a technical lemma concerning an explicit computation of the minimized Hamiltonian. Before stating it, let us observe that condition (4.3.4) implies that the function $t \mapsto \langle \bar{\zeta}, e^{-At} b_i \rangle$ is not identically 0 for all $i = 1, \dots, M$ and for all $\bar{\zeta} \in \mathbb{S}^{N-1}$.

Lemma 4.3.6. *Let $r > 0$, $\bar{x} \in \mathbb{R}^N$ and $\bar{\zeta} \in \mathbb{S}^{N-1}$ be such that*

$$\bar{x} = \sum_{i=1}^{M} \int_0^r e^{-At} b_i \,\text{sign}\left(\langle \bar{\zeta}, e^{-At} b_i \rangle\right) dt.$$

Then

$$h(\bar{x}, \bar{\zeta}) = -\sum_{i=1}^{M} \left| \langle \bar{\zeta}, e^{-Ar} b_i \rangle \right|. \tag{4.3.10}$$

Proof. We have

$$\begin{aligned}
h(\bar{x},\bar{\zeta}) &= \langle\bar{\zeta},A\bar{x}\rangle + \min_{u\in[-1,1]^M}\langle\bar{\zeta},Bu\rangle \\
&= \langle\bar{\zeta},A\bar{x}\rangle + \min_{\substack{|u_i|\le 1\\ i=1,\dots,M}}\sum_{i=1}^{M}\langle\bar{\zeta},b_iu_i\rangle \\
&= \left\langle\bar{\zeta},\sum_{i=1}^{M}\int_0^r Ae^{-At}b_i\,\mathrm{sign}\left(\langle\bar{\zeta},e^{-At}b_i\rangle\right)dt\right\rangle - \sum_{i=1}^{M}|\langle\bar{\zeta},b_i\rangle| \\
&= \sum_{i=1}^{M}\left(\int_0^r \left\langle\bar{\zeta},Ae^{-At}b_i\,\mathrm{sign}\left(\langle\bar{\zeta},e^{-At}b_i\rangle\right)\right\rangle dt - |\langle\bar{\zeta},b_i\rangle|\right) \\
&=: \sum_{i=1}^{M}h_i(\bar{x},\bar{\zeta}).
\end{aligned}$$

Set $g_i(t) := \langle\bar{\zeta},e^{-At}b_i\rangle$, $t\ge 0$. Then, being not identically zero, g_i vanishes at most finitely many times in $[0,r]$, say at $0\le t_1<\cdots<t_k\le r$. We have, for $i=1,\dots,M$,

$$\begin{aligned}
h_i(\bar{x},\bar{\zeta}) = &\int_0^r -\dot{g}_i(t)\mathrm{sign}(g_i(t))dt - |\langle\bar{\zeta},b_i\rangle| \\
= &-\int_0^{t_1}\dot{g}_i(t)\,\mathrm{sign}(g_i(t))dt - \sum_{j=1}^{k-1}\int_{t_j}^{t_{j+1}}\dot{g}_i(t)\,\mathrm{sign}(g_i(t))dt \\
&-\int_{t_k}^{r}\dot{g}_i(t)\,\mathrm{sign}(g_i(t))dt - |\langle\bar{\zeta},b_i\rangle| \\
= &\ (g_i(0)-g_i(t_1))\,\mathrm{sign}\left(g_i\left(\frac{t_1}{2}\right)\right) + \sum_{j=1}^{k-1}(g_i(t_j)-g_i(t_{j+1}))\,\mathrm{sign}\left(g_i\left(\frac{t_j+t_{j+1}}{2}\right)\right) \\
&+ (g_i(t_k)-g_i(r))\,\mathrm{sign}\left(g_i\left(\frac{t_k+r}{2}\right)\right) - |\langle\bar{\zeta},b_i\rangle| \\
= &\ g_i(0)\mathrm{sign}\left(g_i\left(\frac{t_1}{2}\right)\right) - g_i(r)\mathrm{sign}\left(g_i\left(\frac{t_k+r}{2}\right)\right) - g_i(0)\mathrm{sign}(g_i(0)).
\end{aligned}$$

If $g_i(0)\neq 0$ and $g_i(r)\neq 0$, then $\mathrm{sign}(g_i(0)) = \mathrm{sign}\left(g_i\left(\frac{t_1}{2}\right)\right)$ and $\mathrm{sign}(g_i(r)) = \mathrm{sign}\left(g_i\left(\frac{t_k+r}{2}\right)\right)$. Thus $h_i(\bar{x},\bar{\zeta}) = -|g_i(r)| = -|\langle\bar{\zeta},e^{-Ar}b_i\rangle|$. Analogously, if $g_i(0)=0$ and $g_i(r)=0$, then $h_i(\bar{x},\bar{\zeta}) = 0 = -|g_i(r)|$. If $g_i(0)\neq 0$ and $g_i(r)=0$, then $h_i(\bar{x},\bar{\zeta}) = -|g_i(r)|$. Finally, if $g_i(0)=0$ and $g_i(r)\neq 0$, then $h_i(\bar{x},\bar{\zeta}) = -|g_i(r)|$. In all cases, we have

$$h_i(\bar{x},\bar{\zeta}) = -|g_i(r)| = -|\langle\bar{\zeta},e^{-Ar}b_i\rangle|,$$

and (4.3.10) follows. □

Remark 4.3.7. *The characterization* (4.3.9)*, thanks to Lemma 4.3.6 and Theorem 4.3.4, implies also that*

$$\begin{aligned}\mathcal{S} \;=\; \big\{ & x \in \mathbb{R}^N : \exists r > 0 \;\textit{ such that }\; x \in \mathrm{bdry}\mathcal{R}(r) \\ & \textit{and }\; \zeta \in \mathbb{S}^{N-1} \cap N_{\mathcal{R}(r)}(x) \;\textit{ for which }\; h(x,\zeta) = 0 \big\}.\end{aligned}$$

The computation of the Hamiltonian contained in (4.3.10) permits to prove the following characterization of non-Lispchitz points of $\mathcal{T}$. We recall that, under the assumption (4.3.8), the set $\mathcal{S}$ is nonempty.

Theorem 4.3.8. *Let $\bar{x} \in \mathbb{R}^N \setminus \{0\}$. Then $\mathcal{T}$ is non-Lipschitz at $\bar{x}$ if and only if $\bar{x} \in \mathcal{S}$. Moreover, $\mathcal{S}$ is invariant for optimal trajectories of the reversed dynamics having vanishing Hamiltonian.*

Proof. Let $\bar{x} \neq 0$ be a non-Lipschitz point of $\mathcal{T}$ and set $r = \mathcal{T}(\bar{x}) > 0$. We recall that by Theorem 3.7 in [21] epi($\mathcal{T}$) has positive reach. Therefore, by Propositions 4.2.4 and 4.2.5, there exists $\bar{\zeta} \in \mathbb{S}^{N-1} \cap N_{\mathcal{R}(r)}(\bar{x})$ such that $h(\bar{x}, \bar{\zeta}) = 0$.

Let $\bar{u}(\cdot)$ be the optimal control steering $\bar{x}$ to the origin in the minimum time r. Then $\tilde{u}(t) = \bar{u}(r-t), 0 \leq t \leq r$, steers the origin to $\bar{x}$ in the optimal time r for the reversed dynamics $\dot{x} = -Ax - Bu, u \in [-1,1]^M$, namely

$$\bar{x} = -\int_0^r e^{-A(r-t)} B\tilde{u}(t)dt = -\sum_{i=1}^M \int_0^r e^{-A(r-t)} b_i \tilde{u}_i(t)dt,$$

where b_i are the columns of B and $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_M)$.
By the Maximum Principle,

$$\tilde{u}_i(t) = -\mathrm{sign}\left(\langle \bar{\zeta}, e^{-A(r-t)} b_i \rangle\right), \;\; i = 1, \ldots, M.$$

Therefore,

$$\begin{aligned}\bar{x} \;&=\; \sum_{i=1}^M \int_0^r e^{-A(r-t)} b_i \,\mathrm{sign}\left(\langle \bar{\zeta}, e^{-A(r-t)} b_i \rangle\right) dt \\ &=\; \sum_{i=1}^M \int_0^r e^{-At} b_i \,\mathrm{sign}\left(\langle \bar{\zeta}, e^{-At} b_i \rangle\right) dt.\end{aligned} \tag{4.3.11}$$

Since $h(\bar{x}, \bar{\zeta}) = 0$, (4.3.10) yields

$$\langle \bar{\zeta}, e^{-Ar} b_i \rangle = 0, \quad i = 1, \dots, M,$$

and the proof of $\bar{x} \in \mathcal{S}$ is concluded, recalling (4.3.11) and (4.3.9).

Conversely, let $\bar{x} \in \mathcal{S}$ and set $r = \mathcal{T}(\bar{x}) > 0$. Then by (4.3.9) there exists $\bar{\zeta} \in \mathbb{S}^{N-1} \cap N_{\mathcal{R}(r)}(\bar{x})$ such that $\langle \bar{\zeta}, e^{-Ar} b_i \rangle = 0$ for all $i = 1, \dots, M$, and

$$\bar{x} = \sum_{i=1}^{M} \int_0^r e^{-At} b_i \operatorname{sign}\left(\langle \zeta, e^{-At} b_i \rangle \right) dt.$$

Recalling (4.3.10), $h(\bar{x}, \bar{\zeta})) = 0$. Therefore, by Propositions 4.2.4 and 4.2.5, $\mathcal{T}$ is non-Lipschitz at $\bar{x}$.

The last statement is an immediate consequence of Remark 4.3.5. In fact, the alternative expression of $\mathcal{S}$ given in (4.3.9), together with the Maximum Principle (see Theorem 4.3.4) shows that every $x \in \mathcal{S}$ is the endpoint of a time optimal trajectory for the reversed dynamics with vanishing Hamiltonian, starting from the origin. The proof is concluded. □

We prove now a rectifiability property for $\mathcal{S}$, which is the main result of this section.

Theorem 4.3.9. *Let $\mathcal{S}$ be defined according to (4.3.7) and let k be given by (4.3.6). Then $\mathcal{S}$ is closed and countably $(N-k)$-rectifiable. More precisely, for every $r > 0$ there exist countably many Lipschitz functions $f_j : \mathbb{R}^{N-k-1} \to \mathbb{R}^N$ such that $\mathcal{S} \cap \operatorname{bdry} \mathcal{R}(r) \subseteq \cup_j f_j(\mathbb{R}^{N-k-1})$.*

Proof. The characterization of $\mathcal{S}$ contained in Remark 4.3.7 implies immediately its closedness.

We now deal with the rectifiability of $\mathcal{S}$. Observe that the set

$$Z := \{ \zeta \in \mathbb{S}^{N-1} : \langle \zeta, b_i \rangle = 0, \forall i = 1, \dots, M \} \tag{4.3.12}$$

is a $N - (k+1)$ manifold. We define, for $i = 1, \dots, M$,

$$g_i^+(t, \zeta) = \langle \zeta, e^{At} b_i \rangle, \qquad \zeta \in Z, \ t \geq 0,$$

$$\Phi_i(r, \zeta) = \int_0^r e^{A(t-r)} b_i \operatorname{sign}(g_i^+(t, \zeta)) dt$$

and

$$\Sigma_i = \big\{x \in \mathbb{R}^N : \text{ there exist } r > 0 \text{ and } \zeta \in Z \text{ such that } x = \Phi_i(r,\zeta)\big\}. \tag{4.3.13}$$

We claim now that

$$\text{each } \Sigma_i \text{ is contained in a countable union of Lipschitz graphs of } N-k \text{ variables.} \tag{4.3.14}$$

To this aim, we fix the index i and drop the corresponding subscript for the sake of simplicity.

We set the following definitions. Fix $\tau > 0$. For every $(N-1)$-tuple of nonnegative integers,

$$\mathbf{j} = (j_1, \dots, j_{N-1}) \in \mathbb{N}^{N-1},$$

we define

$$Z_{\mathbf{j}} = \Big\{\zeta \in Z : \quad g^+(t,\zeta) \text{ has in the interval } [0,\tau] \text{ exactly}$$
$$j_1 \text{ zeros of multiplicity } 1,$$
$$\dots$$
$$j_{N-1} \text{ zeros of multiplicity } N-1\Big\}.$$

We set also $|\mathbf{j}| = j_1 + \cdots + j_{N-1}$ and observe that, thanks to (4.3.12), we can consider only $\mathbf{j}'s$ such that $|\mathbf{j}| \geq 1$. Moreover, for any positive integer d and $\mathbf{j} \in \mathbb{N}^{N-1}$ with $|\mathbf{j}| > 1$ we define

$$Z_{\mathbf{j}}^d = \left\{\zeta \in Z_{\mathbf{j}} : \min\{|\tau_1 - \tau_2| : g^+(\tau_1,\zeta) = g^+(\tau_2,\zeta) = 0, \tau_1 \neq \tau_2\} \geq \frac{1}{d}\right\}.$$

Invoking Lemma 3.2 in [23], we obtain that

$$Z = \Big(\bigcup_{|\mathbf{j}|=1} Z_{\mathbf{j}}\Big) \cup \Bigg(\bigcup_{d=1}^{\infty} \bigcup_{\substack{\mathbf{j}\in\mathbb{N}^{N-1}\\ |\mathbf{j}|>1}} Z_{\mathbf{j}}^d\Bigg).$$

We define finally the map

$$\begin{aligned} Y : Z &\to L^1(0,\tau) \\ \zeta &\mapsto \operatorname{sign}(g^+(\cdot,\zeta)), \end{aligned} \tag{4.3.15}$$

and, for all $\mathbf{j} \in \mathbb{N}^{N-1}$, the sets

$$Z_{\mathbf{j}}^{d,\pm} = \Big\{ \zeta \in Z_{\mathbf{j}}^{d} : \lim_{t \to 0+} \operatorname{sign}(g^+(t,\zeta)) = \pm 1 \Big\}.$$

We fix now $\mathbf{j} \in \mathbb{N}^{N-1}$. If $|\mathbf{j}| = 1$, then $Y(\zeta)(t) \equiv \pm 1$ for all $\zeta \in Z_{\mathbf{j}}$, $t \in (0,\tau]$, and so Y is locally Lipschitz in $Z_{\mathbf{j}}$. We claim that Y is locally Lipschitz also in $Z_{\mathbf{j}}^{d,+}$ and in $Z_{\mathbf{j}}^{d,-}$ for each $\mathbf{j} \in \mathbb{N}^{N-1}$. The argument for $Z_{\mathbf{j}}^{d,+}$ and $Z_{\mathbf{j}}^{d,-}$ is the same, so we perform it only for $Z_{\mathbf{j}}^{d,+}$, $|\mathbf{j}| > 1$, $d \geq 1$.

So, fix $|\mathbf{j}| > 1$, $d \geq 1$ and $\zeta_0 \in Z_{\mathbf{j}}^{d,+}$. Let $t_1, \ldots, t_{|\mathbf{j}|}$ be the zeros of $g^+(\cdot,\zeta_0)$ in $[0,\tau]$, each one with multiplicity m_h, $h = 1, \ldots, |\mathbf{j}|$.
By continuity and the implicit function theorem, for each $h = 1, \ldots, |\mathbf{j}|$ there exist a compact neighborhood V_h of ζ_0, a neighborhood I_h of t_h and a $\mathcal{C}^1$-function $\varphi_h : V_h \to I_h$ such that

$$\frac{\partial^{m_h}}{\partial t^{m_h}} g^+(t,\zeta) \neq 0 \qquad \forall (t,\zeta) \in I_h \times V_h \tag{4.3.16}$$

and

$$\Big\{ (\zeta,t) \in V_h \times I_h : \frac{\partial^{m_h - 1}}{\partial t^{m_h - 1}} g^+(t,\zeta) = 0 \Big\} = \operatorname{graph}(\varphi_h). \tag{4.3.17}$$

The neighborhoods I_h can be taken disjoint and satisfying $|I_h| \leq \frac{1}{2d}$. We choose now $V = V(\zeta_0) \subseteq \cap_{h=1}^{|\mathbf{j}|} V_h$ with the further requirement that for all $\zeta \in V$, the set $\{t \in [0,\tau] : g^+(t,\zeta) = 0\}$ is contained in $\bigcup_{h=1}^{|\mathbf{j}|} I_h$. Since $|I_h| \leq \frac{1}{2d}$, the function $g^+(t,\zeta)$ has at most one zero in each I_h.

Set $V_{\mathbf{j}}(\zeta_0) = V \cap Z_{\mathbf{j}}^{d,+}$. Without loss of generality, we may assume that $Z_{\mathbf{j}}^{d,+}$ is contained in a finite union of such $V_{\mathbf{j}}(\cdot)$, say $Z_{\mathbf{j}}^{d,+} = \bigcup_{\ell} V_{\mathbf{j}}(\zeta_\ell)$. We write the functions corresponding to $V_{\mathbf{j}}(\zeta_\ell)$ as $\varphi_h^\ell(\zeta), h = 1, \ldots, |\mathbf{j}|$, and observe that each φ_h^ℓ is Lipschitz continuous on $V_{\mathbf{j}}(\zeta_\ell)$, say with Lipschitz constant L_h^ℓ. We denote also the intervals corresponding to $V_{\mathbf{j}}(\zeta_\ell)$ as I_h^ℓ, $h = 1, \ldots, |\mathbf{j}|$. Of course, some of the $V_{\mathbf{j}}(\zeta_\ell)$'s may be the singleton $\{\zeta_\ell\}$, and in this case everything trivializes. Fix now an index ℓ.

We claim that, for each $\zeta \in V_{\mathbf{j}}(\zeta_\ell)$, $g^+(\cdot,\zeta)$ has a zero of multiplicity m_h exactly at $\varphi_h^\ell(\zeta)$, $h = 1, \ldots, |\mathbf{j}|$, and does not have other zeros in $[0,\tau]$. Indeed, by construction for each $\zeta \in V_{\mathbf{j}}(\zeta_\ell)$ all zeros of $g^+(\cdot,\zeta)$ are contained in $\bigcup_{h=1}^{|\mathbf{j}|} I_h^\ell$. Let κ be the largest index k such that $j_k \neq 0$. Again by construction, for each $\zeta \in V_{\mathbf{j}}(\zeta_\ell)$ the map $t \mapsto g^+(t,\zeta)$ has exactly j_κ zeros of multiplicity κ. Moreover, such j_κ zeros must belong to the same intervals I_h^ℓ to which the j_κ zeros of multiplicity κ of $g^+(\cdot,\zeta_\ell)$ belong, since in all other intervals we have at least one nonvanishing derivative of order $\le \kappa - 1$. Owing to (4.3.17) with $m_h = \kappa$, such zeros must occur at $\varphi_h^\ell(\zeta)$, for the corresponding index h. Let now κ_1 be the largest positive integer $< \kappa$ such that $j_{\kappa_1} > 0$. By definition of $V_{\mathbf{j}}(\zeta_\ell)$, for each $\zeta \in V_{\mathbf{j}}(\zeta_\ell)$ the map $t \mapsto g^+(\cdot, \zeta)$ does not have zeros of order k, with $\kappa_1 < k < \kappa$ and must have exactly $j_{\kappa_1} > 0$ zeros of multiplicity κ_1. Such zeros cannot belong to the intervals to which the κ-zeros of $g^+(\cdot, \zeta)$ belong, since such intervals already contain a zero; on the other hand, by (4.3.16) they must belong to the same intervals I_h^ℓ to which the zeros of multiplicity κ_1 of $g^+(\cdot, \zeta_\ell)$ belong, and therefore they must occur at $\varphi_h^\ell(\zeta)$, for the corresponding index h. An analogous argument can be performed for all further indexes $k < \kappa_1$ such that $j_k \neq 0$. Therefore the claim is proved.

We are now ready to show that Y is Lipschitz on $V_{\mathbf{j}}(\zeta_\ell)$. Indeed, fix the index ℓ and let $\zeta_1, \zeta_2 \in V_{\mathbf{j}}(\zeta_\ell)$. Then

$$\begin{aligned}
\|Y(\zeta_2) - Y(\zeta_1)\|_{L^1(0,\tau)} &\le 2 \sum_{\substack{h=1 \\ m_h \text{ is odd}}}^{|\mathbf{j}|} |\varphi_h^\ell(\zeta_2) - \varphi_h^\ell(\zeta_1)| \\
&\le 2 \sum_{\substack{h=1 \\ m_h \text{ is odd}}}^{|\mathbf{j}|} L_h^\ell \|\zeta_2 - \zeta_1\|,
\end{aligned}$$

which proves the claim.

The Lipschitz continuity of Y on each $V_{\mathbf{j}}(\zeta_\ell)$ implies immediately that, for all fixed $r \in [0, \tau]$, the function $\zeta \mapsto \Phi(r, \zeta)$ is Lipschitz in the same set. On the other hand, the function $r \mapsto \Phi(r, \zeta)$ is immediately seen to be Lipschitz on $[0, \tau]$. Consequently, the set Σ defined in (4.3.13) is contained in a countable union of Lipschitz graphs of $N - k$ variables. The $(N - k)$-rectifiability of $\mathcal{S}$ now follows easily and the proof is concluded. □

4.3.3 Propagation of non - Lipschitz singularities

This subsection deals with a lower estimate of the dimension of $\mathcal{S}$ for the linear case. We show that the $\mathcal{H}^{N-k}$-rectifiability of $\mathcal{S}$ is indeed optimal, in the sense of Theorems 4.3.10 and 4.3.11 below, at least for a small time. Those statements can be seen as propagation results for singularities of non-Lipschitz type for the minimum time function.
We consider the linear system (4.3.1) under the assumptions (4.3.4) and (4.3.6). Let

$N > 2$ and define, for $\tau > 0$, $\mathcal{S}(\tau) = \mathcal{S} \cap \mathrm{bdry}\mathcal{R}(\tau)$. We assume that $k \le N-1$, otherwise $\mathcal{S} = \emptyset$.

Theorem 4.3.10. *There exists $\tilde{\tau} > 0$, depending only on A, B, N, satisfying the following the property: for all $\tau \le \tilde{\tau}$ and $\mathcal{H}^{N-k-1}$-a.e. $x \in \mathcal{S}(\tau)$ and for any neighborhood V of x, we have*

$$\mathcal{H}^{N-1-k}(V \cap \mathcal{S}(\tau)) > 0. \tag{4.3.18}$$

Proof. We divide the proof into some steps.
We will use a result which was proved, e.g., in [6] (see the proof of Lemma 8). The statement is as follows.

There exists $\bar{\tau} > 0$, depending only on A, B, N, such that for every $\zeta \in \mathbb{S}^{N-1}$ the switching function $g(\cdot, \zeta) = \langle \zeta, e^{-A\cdot} b \rangle$ has at most $N-1$ zeros in $[s, s+\bar{\tau}]$ for every $s \ge 0$. (4.3.19)

Claim 1. The statement of the Theorem holds true in the case $B = b$, a vector.
Proof of Claim 1. Let $\bar{\tau}$ be given by (4.3.19). Fix $0 < \tau < \bar{\tau}$ and let $0 < j \le N-2$. We say that $x \in \mathcal{S}(\tau)$ belongs to $\mathcal{S}_j(\tau)$ if there exist times $0 < s_1 < s_2 < \cdots < s_j < \tau$ such that

$$x = \pm \int_0^\tau e^{-As} b \gamma(s) ds,$$

where

$$\gamma(s) = \begin{cases} 1 & \text{if } 0 < s < s_1, \\ -1 & \text{if } s_1 < s < s_2, \\ \cdots & \\ (-1)^j & \text{if } s_j < s < \tau. \end{cases}$$

In other words, the optimal control steering the origin to x for the reversed dynamics has exactly j switchings in the interval $(0, \tau)$.
Step 1. There exists $\tilde{\tau} > 0$, depending only on A, b and N, such that if $0 \le s_1 < s_2 < \cdots < s_j \le \tilde{\tau}$ then

$$\mathrm{rank}\left[e^{-As_1}b, e^{-As_2}b, \cdots, e^{-As_j}b\right] = j. \tag{4.3.20}$$

In order to prove (4.3.20), for $s \ge 0$ and $\zeta \in \mathbb{R}^N$ set $g(s) = \langle \zeta, e^{-As} b \rangle$ and $H = \{\zeta \in \mathbb{R}^N : g(s_i, \zeta) = 0, i = 1, \ldots, j\}$, $0 \le s_1 < s_2 < \cdots < s_j \le \tau$. We claim that $\dim H = N - j$. Indeed, if $g(s_1, \zeta) = 0$, then

$$0 = \langle \zeta, b - As_1 b \rangle + o(\tau),$$

so that $\langle \zeta, b\rangle = 0$ since τ can be chosen small enough. Furthermore, if $j > 1$, there exists $\bar{s}_1 \in (s_1, s_2)$ such that $\frac{\partial}{\partial s} g(\bar{s}_1, \zeta) = 0$, which in turn implies

$$0 = -\langle \zeta, Ab - A^2 \bar{s}_1 b\rangle + o(\tau),$$

so that $\langle \zeta, Ab\rangle = 0$ since τ can be chosen small enough. The same argument provides times $\bar{s}_i$, $i = 2, \dots j-1$ such that

$$0 = \frac{\partial^i g}{\partial s^i}(\bar{s}_i, \zeta) = \langle \zeta, A^i b\rangle + O(\tau),$$

i.e., $\langle \zeta, A^i b\rangle = 0$. The proof is completed by invoking the rank condition (4.3.4).

Step 2. If $0 < j \leq N-2$, then for all $0 < \tau < \tilde{\tau}$ the set $\mathcal{S}_j(\tau)$ is the union of two smooth parametrized j-surfaces. Actually we are going to prove that $\{x \in \mathcal{S}_j(\tau) : x = \int_0^\tau e^{-As} b\gamma(s)ds\}$ is a smooth parametrized j-surface, the other case being entirely analogous.

Indeed, we have

$$x = \int_0^{s_1} e^{-As} bds + \sum_{i=1}^{j-1} (-1)^i \int_{s_i}^{s_{i+1}} e^{-As} bds + (-1)^j \int_{s_j}^{\tau} e^{-As} bds,$$

where $0 < s_1 < \ldots < s_j < \tau$. Observe that $\frac{\partial x}{\partial s_i} = 2(-1)^{i+1} e^{-As_i} b, 1 \leq i \leq j$, and by (4.3.20) the matrix $\left(\frac{\partial x}{\partial s_i}\right)_{i=1,\cdots,j}$ has rank j in the open set $\{(s_1, \ldots, s_j) \in (0,\tau)^j : s_1 < \ldots < s_j\}$. The proof of Step 2 is concluded.

Set now $\mathcal{S}_0(\tau) = \left\{\pm \int_0^\tau e^{-As} bds\right\}$. By the Maximum Principle, owing to (4.3.19) we have that

$$\mathcal{S}(\tau) = \bigcup_{j=0}^{N-2} \mathcal{S}_j(\tau)$$

for all $0 < \tau < \tilde{\tau}$ and the union is disjoint. In particular, Step 2 implies that for all such τ

$$\mathcal{H}^{N-2}\big(\mathcal{S}(\tau) \setminus \mathcal{S}_{N-2}(\tau)\big) = 0 \tag{4.3.21}$$

and that (4.3.18) holds at every point $x \in \mathcal{S}_{N-2}(\tau)$. The proof of Claim 1 is concluded.

Claim 2. The statement of Theorem 4.3.10 holds in the general case.

Proof of Claim 2. Let $0 < \tau < \tilde{\tau}$ be given and fix $x \in \mathcal{S} \cap \operatorname{bdry} \mathcal{R}(\tau)$, together with the optimal control $u = (u_1, \dots, u_M)$ steering the origin to x in time τ by the reversed dynamics. Assume that u_i has exactly $\kappa_i + 1$ zeros, $0 \le \kappa_i \le N-2$, at times

$$0 = s_0^i < s_1^i < \ldots < s_{\kappa_i}^i \le \tau.$$

Then, recalling (4.3.6), we have

$$k \le \operatorname{rank}\big\{b_i, e^{As_1^i}b_i, \dots, e^{As_{\kappa_i}^i}b_i : 1 \le i \le M\big\} \le N-1.$$

For $j = 0, \dots, N-(1+k)$, let $\mathcal{S}_{j+k}(\tau)$ be the set of all $x \in \mathcal{S}(\tau)$ such that

$$\operatorname{rank}\big\{e^{As_1^i}b_i, \dots, e^{As_{\kappa_i}^i}b_i : 1 \le i \le M\big\} = j+1.$$

Observe that

$$\mathcal{S}(\tau) = \bigcup_{j=0}^{N-(1+k)} \mathcal{S}_{j+k}(\tau)$$

and the union is disjoint. Moreover, by arguing exactly as in Steps 1 and 2 in the proof of Claim 1 above, we can see that each $\mathcal{S}_{j+k}(\tau)$ is a union of finitely many disjoint smooth parametrized j-surfaces. Thus,

$$\mathcal{H}^{N-k-1}\big(\mathcal{S}_{j+k}(\tau)\big) = 0 \qquad \forall j = 0, \dots, N-(2+k)$$

and

$$\mathcal{H}^{N-k-1}\big(\mathcal{S}_{N-1}(\tau)\big) > 0.$$

Therefore, for $\mathcal{H}^{N-(k+1)}$-a.e. $x \in \mathcal{S}(\tau)$ there exists a neighborhood $V = V(x)$ such that

$$\mathcal{H}^{N-(k+1)}\big(V \cap \mathcal{S}(\tau)\big) > 0.$$

The proof is now complete. □

By combining the above result with the invariance statement contained in Theorem 4.3.8 we obtain immediately the following

Theorem 4.3.11. *Let $\bar\tau$ be given as in Theorem 4.3.10. Then for $\mathcal{H}^{N-k}$-a.e. $x \in \mathcal{S}$ such that $T(x) < \bar\tau$, we have, for any neighborhood V of x, that*

$$\mathcal{H}^{N-k}(V \cap \mathcal{S}) > 0.$$

Proof. Fix $0 < \tau < \bar{\tau}$ and let E be a subset of $\mathcal{S} \cap \mathrm{bdry}\mathcal{R}(\tau)$ with full $\mathcal{H}^{N-(k+1)}$-measure with the property (4.3.18). Then the optimal trajectories for the reversed dynamics through each point of E from a subset of $\mathcal{S}$ with full $\mathcal{H}^{N-k}$-measure. The proof is complete. □

Remark 4.3.12. *The statement of theorems 4.3.10 and 4.3.11 are somewhat unnatural for linear systems, as they are valid only for small times. The proof for arbitrarily large times requires an analysis of higher order and of linearly dependent zeros of the switching function, which we are not yet able to conclude. The technique used in the proof of Theorem 4.3.10 is similar to the argument presented in [32, Section 3 and proof of Theorem 6].*

4.3.4 The non-Lipschitz set and the switching locus

In this subsection, we study the rectifiability of the switching locus of the control system (4.3.1) and the relationship between the switching locus and the non-Lipschitz set of the minimum time function for (4.3.1).

Definition 4.3.13 ([32])**.** *The switching locus of the control system (4.3.1) is the set of all points $x \in \mathbb{R}^N$ satisfying $x = 0$ or $x = x(t)$ for some optimal trajectory $x(\cdot)$ such that $t > 0$ and $\dot{x}(\cdot)$ is discontinuous at t. We denote by Θ the switching locus of (4.3.1).*

Let $x \in \Theta$ and $x \neq 0$. Then there exists a point $x_0 \in \mathcal{R}$ and a time t such that the optimal trajectory $x(\cdot)$ starting at x_0 satisfies $x(t) = x$ and the optimal control $u(\cdot) = (u_1(\cdot), \cdots, u_M(\cdot))^T$ has a component $u_\ell(\cdot)$, $\ell \in \{1, \cdots, M\}$ which changes its sign at the time t, i.e., the corresponding switching function $g_\ell(\zeta, \cdot) = \langle \zeta, e^{-A\cdot} b_\ell \rangle$, $\zeta \in N_{\mathcal{R}(T(x_0))}(x_0)$, has a zero of odd order at t. Thus x can be reached from the origin with the reversed dynamics by the optimal control $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \cdots, \bar{u}_M(\cdot))^T$ with $\bar{u}_i(t) = -\mathrm{sign}(\langle \zeta, e^{-A(T-t)} b_i \rangle)$, $\zeta \in N_{\mathcal{R}(T(x))}(x)$, $t \in [0, T]$ for all $i \in \{1, \cdots, M\}$ such that the function $\langle \zeta, e^{-A(T-t)} b_\ell \rangle$ has a zero of odd order at the time T, i.e., there is some $k \in \mathbb{N}^*$ with $2 \leq 2k \leq N + 1$ such that $\langle \zeta, b_\ell \rangle = \cdots = \langle \zeta, A^{2k-2} b_\ell \rangle = 0$ and $\langle \zeta, A^{2k-1} b_\ell \rangle \neq 0$.
Therefore, $x \in \Theta$ if and only if there exist $\ell \in \{1, \cdots, M\}$, $r \geq 0$, $k \in \mathbb{N}^*, 2 \leq 2k \leq N+1$, $\zeta \in \mathbb{S}^{N-1} \cap N_{\mathcal{R}(r)}(x)$ with

$$\langle \zeta, b_\ell \rangle = \cdots = \langle \zeta, A^{2k-2} b_\ell \rangle = 0 \text{ and } \langle \zeta, A^{2k-1} b_\ell \rangle \neq 0,$$

such that

$$x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \text{sign}\left(\langle \zeta, e^{A(t-r)} b_i \rangle\right) dt$$

or, by changing variables,

$$x = \sum_{i=1}^{M} \int_0^r e^{-At} b_i \text{sign}\left(\langle \zeta, e^{-At} b_i \rangle\right) dt.$$

For each $\ell \in \{1, \cdots, M\}$ and each $k \in \mathbb{N}^*$ with $2 \leq 2k \leq N+1$, we define

$$\begin{aligned} \Theta_k^\ell = \Big\{ x = \int_0^r e^{-At} b_i \,\text{sign}\left(\langle \zeta, e^{-At} b_i \rangle\right) dt : r \geq 0, \zeta \in \mathbb{S}^{N-1} \cap N_{\mathcal{R}(r)}(x), \\ \langle \zeta, b_\ell \rangle = \cdots = \langle \zeta, A^{2k-2} b_\ell \rangle = 0, \langle \zeta, A^{2k-1} b_\ell \rangle \neq 0 \Big\}. \end{aligned} \tag{4.3.22}$$

Θ_k^ℓ can be represented as follows

$$\begin{aligned} \Theta_k^\ell = \Big\{ x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \,\text{sign}\left(\langle \zeta, e^{At} b_i \rangle\right) dt : r \geq 0, \zeta \in \mathbb{S}^{N-1}, \\ \langle \zeta, e^{Ar} b_\ell \rangle = \cdots = \langle \zeta, e^{Ar} A^{2k-2} b_\ell \rangle = 0, \langle \zeta, e^{Ar} A^{2k-1} b_\ell \rangle \neq 0 \Big\}. \end{aligned} \tag{4.3.23}$$

Then

$$\Theta = \bigcup_{\ell=1}^{M} \bigcup_{k=1}^{\left[\frac{N+1}{2}\right]} \Theta_k^\ell.$$

Proposition 4.3.14. Θ *is countably* $\mathcal{H}^{N-1}$*-rectifiable.*

Proof. It follows from Theorem 5.0.1 that Θ_k^ℓ is countably $\mathcal{H}^{N-2k+1}$-rectifiable for each $\ell \in \{1, \cdots, M\}$ and $k \in \mathbb{N}^*$, $2 \leq 2k \leq N+1$. Therefore, Θ is countably $\mathcal{H}^{N-1}$-rectifiable. □

The following example shows that, in general, the switching locus and the non-Lipschitz set of the minimum time function for normal linear control systems are different.

Example 4.3.15. *Consider the linear control system* $\dot{x} = Ax + Bu$ *with* $u \in [-1, 1]$ *and*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can compute explicitly the switching locus and the non-Lipschitz set of the minimum time function.
The switching locus is

$$\begin{aligned}\Theta &= \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0,\ (x_1 - 2k - 1)^2 + x_2^2 = 1, k = 0, 1, 2, \cdots\} \\ &\quad \cup \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geq 0,\ (x_1 + 2k + 1)^2 + x_2^2 = 1, k = 0, 1, 2, \cdots\}\end{aligned}$$

The non-Lipschitz set is

$$\begin{aligned}\mathcal{S} &= \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0,\ (x_1 - 1)^2 + x_2^2 = (2k+1)^2, k = 0, 1, 2, \cdots\} \\ &\quad \cup \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geq 0,\ (x_1 + 1)^2 + x_2^2 = (2k+1)^2, k = 0, 1, 2, \cdots\}\end{aligned}$$

Observe that $\Theta \neq \mathcal{S}$.

However, we will prove that the non-Lipschitz set of the minimum time function is a subset of the switching locus in a neighborhood of the origin if the control system is strictly normal.

Definition 4.3.16 ([47])**.** *The linear control system (4.3.1) is call strictly normal if for any integers $\ell_i \geq 0$ satisfying $\sum_{i=1}^{M} \ell_i = N$, the vectors $A^q b_i$ with $i = 1, \cdots, M$ and $0 \leq q \leq \ell_i - 1$ are linearly independent (if $\ell_i = 0$, there are no terms $A^q b_i$).*

Remark 4.3.17. *(i) A strictly normal system is normal. The inverse is not true in general but true in the case $M = 1$.*

(ii) If (4.3.1) is strictly normal, then $\operatorname{rank} B = M$.

In [47], D. S. Yeung gives a necessary and sufficient condition for a linear control system being strictly normal.

Lemma 4.3.18 ([47])**.** *The control system (4.3.1) is strictly normal if and only if there exists $\varepsilon > 0$ with the following property: for every nonzero vector $\zeta \in \mathbb{R}^N$ and in any interval of length $\leq \varepsilon$, the sum of number of roots, counting multiplicities, of the coordinates of $\zeta^T e^{-As} B$ is less than N.*

From Lemma 4.3.18 the following corollaries follow

Corollary 4.3.19. *Let ε be the constant from Lemma 4.3.18. For any choice of $m \le N$ distinct times $t_1 < \cdots < t_m$ with $t_m - t_1 \le \varepsilon$ and m integers ℓ_q among $1, \cdots, m$ and m integers $m_i > 0$ with $\sum_{i=1}^m = N$, the N vectors*

$$A^{k-1} e^{-At_i} b_{\ell_i}, \quad 1 \le k \le m_i,\, 1 \le i \le m,$$

are linearly independent.

Proof. Assume to the contrary that the N vectors

$$A^{k-1} e^{-At_i} b_{\ell_i}, \quad 1 \le k \le m_i,\, 1 \le i \le m,$$

are linearly dependent. Then there exists a nonzero vector $\zeta \in \mathbb{R}^N$ such that

$$\zeta^T A^{k-1} e^{-At_i} b_{\ell_i} = 0, \quad \forall\; k = 1, \cdots, m_i,\, i = 1, \cdots, m.$$

Therefore the sum of the number of roots , counting multiplicities, in an interval of length $\le \varepsilon$ of the coordinates of $\zeta^T e^{-As} B$ is at least N. This contradicts to Lemma 4.3.18. $\square$

Corollary 4.3.20. *Let ε be the constant from Lemma 4.3.18. Given $m \le N-1$ distinct times $t_1 < \cdots < t_m$ in $(0, \varepsilon)$, there exists $\zeta \in \mathbb{R}^N$ such that the set of simple zeros of the coordinates of $\zeta^T e^{-As} B$ in $(0, \varepsilon)$ is $\{t_1, \cdots, t_m\}$ and the coordinates of $\zeta^T e^{-As} B$ have no other zero in $(0, \varepsilon)$.*

Proof. By Corollary 4.3.19, the $N-1$ vectors

$$b_1, \cdots, A^{N-m-2} b_1, e^{At_1} b_1, \cdots, e^{At_m} b_1$$

are linearly independent. Therefore, there exists a vector $0 \ne \zeta \in \mathbb{R}^N$ perpendicular to all of these vectors. Thus the first coordinate of $\zeta^T e^{At} B$ has roots at m points t_j and also at 0 with multiplicity $N - m - 1$. Then by Lemma 4.3.18, the coordinates of $\zeta^T e^{At} B$ have no other zeros in $[0, \varepsilon]$ and the zeros t_j have multiplicity 1. $\square$

Proposition 4.3.21. *Let ε be the constant from Lemma 4.3.18. If $M = 1$, then*

$$\mathcal{S} \cap \operatorname{Int}\mathcal{R}(\varepsilon) = \Theta \cap \operatorname{Int}\mathcal{R}(\varepsilon).$$

Proof. Let $x \in \mathcal{S} \cap \operatorname{int} \mathcal{R}(\epsilon)$. Then there exists $\theta < \epsilon$ such that $x \in \operatorname{bdry} \mathcal{R}(\theta)$. Since $x \in \mathcal{S}$, there exists $\zeta \in \mathbb{S}^{N-1}$ such that $\langle \zeta, b \rangle = 0$ and

$$x = \int_0^\theta e^{A(t-\theta)} b \operatorname{sign}\left(\langle \zeta, e^{At} b \rangle\right) dt$$

Assume that $\langle \zeta, e^{At} b \rangle$ has the following zeros in $[0, \theta]$: $0 \leq t_1 < t_2 < \cdots < t_m \leq \theta$. By Lemma (4.3.20), we may assume that $t_2, \cdots, t_m$ are single zeros and $m \leq N-1$. If $t_m = \theta$ then $x \in \Theta$. If $t_m < \theta$, then there exist $\bar{\zeta}$ such that $\langle \bar{\zeta}, e^{At} b \rangle$ has single rezos at $t_2, \cdots, t_m, \theta$ and nowhere else in $(0, \theta)$. Let

$$\bar{x} = \int_0^\theta e^{A(t-\theta)} b \operatorname{sign}\left(\langle \bar{\zeta}, e^{At} b \rangle\right) dt$$

then $\bar{x} \in \Theta$. Observe that $\bar{x} \equiv x$. Thus $\mathcal{S} \cap \operatorname{int} \mathcal{R}(\epsilon) \subset \Theta \cap \operatorname{int} \mathcal{R}(\epsilon)$.

Now, let $x \in \Theta \cap \operatorname{int} \mathcal{R}(\epsilon)$. Then there exists $\theta < \epsilon$ such that $x \in \operatorname{bdry} \mathcal{R}(\theta)$. Since $x \in \Theta$, there exists $\zeta \in \mathbb{S}^{N-1}$, $k \in \mathbb{N}^*, 2 \leq 2k \leq N-1$ such that $\langle \zeta, e^{A\theta} b \rangle = \cdots = \langle \zeta, e^{A\theta} A^{2k-1} b \rangle = 0, \langle \zeta, e^{A\theta} A^{2k} b \rangle \neq 0$ and

$$x = \int_0^\theta e^{A(t-\theta)} b \operatorname{sign}\left(\langle \zeta, e^{At} b \rangle\right) dt$$

Let $0 \leq t_1 < \cdots < t_m = \theta$ be all zeros of $\langle \zeta, e^{At} b \rangle$ in $[0, \theta]$. If $t_1 = 0$ then $x \in \mathcal{S}$. If $t_1 > 0$, then there exists $\bar{\zeta} \in \mathbb{S}^{N-1}$ such that $\langle \bar{\zeta}, e^{At} b \rangle$ has simple zeros at t_i, $i \in \{1, \cdots, m-1\}$ which are zeros of odd order of $\langle \zeta, e^{At} b \rangle$ and zero of order p at 0 for some suitable $p \geq 1$ and nowhere else in $[0, \theta]$. Let

$$\bar{x} = \int_0^\theta e^{A(t-\theta)} b \operatorname{sign}\left(\langle \bar{\zeta}, e^{At} b \rangle\right) dt$$

then $\bar{x} \in \mathcal{S}$. Observe that $\bar{x} \equiv x$. Thus $\Theta \cap \operatorname{int} \mathcal{R}(\epsilon) \subset \mathcal{S} \cap \operatorname{int} \mathcal{R}(\epsilon)$. The proof is complete. □

Remark 4.3.22. *From Theorem 20 [35], if $M = 1$ and A has only real eigenvalues then $\varepsilon = \infty$. Hence $\mathcal{S}$ and Θ coincide. In this case, $\mathcal{S}$ and, of course, Θ contains smooth manifolds of $(N-1)$ dimension (see [32]). Therefore the result in Proposition 4.3.14 is sharp.*

Proposition 4.3.23. *Let ε be the constant from Lemma 4.3.18. One has*

$$\mathcal{S} \cap \mathrm{Int}\mathcal{R}(\varepsilon) \subseteq \Theta \cap \mathrm{Int}\mathcal{R}(\varepsilon).$$

Proof. Using the similar arguments to the first part in proof of Proposition 4.3.21. □

One can also prove the following theorem which gives another way to compute the non-Lipschitz set of the minimum time function for a special class of control systems

Theorem 4.3.24. *Assume that $M = 1$ and that A has all real eigenvalues. $x \in \mathcal{S}$ if and only if x can be steered to the origin by the optimal control with $k \leq N-2$ switchings.*

Proof. Let $x \in \mathcal{S}$, then there exists $\zeta \in \mathbb{S}^{N-1}$ such that $\langle \zeta, b\rangle = 0$ and

$$x = \int_0^{T(x)} e^{A(t-T(x))} b \,\mathrm{sign}(\langle \zeta, e^{At}b\rangle)dt.$$

Assume that the optimal control for x has at least $N-1$ switchings, then $\langle \zeta, e^{At}b\rangle$ has at least $N-1$ zeros in $(0, T(x))$. Hence $\langle \zeta, e^{At}b\rangle$ has at least N zeros in $[0, T(x)]$. This is a contradiction.

Now let $x \in \mathcal{R}$ and the optimal control u^* for x has $k \leq N-2$ switchings. There exists $\zeta \in \mathbb{S}^{N-1}$ such that

$$x = \int_0^{T(x)} e^{A(t-T(x))} b \,\mathrm{sign}(\langle \zeta, e^{At}b\rangle)dt.$$

Since u^* has k switchings, $\langle \zeta, e^{At}b\rangle$ has k zeros of odd order in $(0, T(x)$, say $0 < t_1 < \cdots < t_k < T(x)$. It follows from Corollary 4.3.19 that $k+1$ vectors $b, e^{At_1}b, \cdots, e^{At_k}b$ are linearly independent. Then , by Lemma 4.3.18, there exists $\zeta_1 \in \mathbb{S}^{N-1}$ such that $t_1, \cdots, t_k$ are all zeros of $\langle \zeta_1, e^{At}b\rangle$ in $(0, T(x))$ and they are simple zeros. Moreover, 0 is a zero of order $N-k-1$ of $\langle \zeta_1, e^{At}b\rangle$. Observe that

$$x = \int_0^{T(x)} e^{A(t-T(x))} b \,\mathrm{sign}(\langle \zeta_1, e^{At}b\rangle)dt.$$

Since $\langle \zeta_1, b\rangle = 0$, $x \in \mathcal{S}$. □

Example 4.3.25. *Consider the minimum time function $\mathcal{T}$ for a normal linear control system with*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The matrix A has only real eigenvalues. $x \in \mathcal{S}$ if and only if x can be steered to the origin by the optimal control with no switching or one switching.

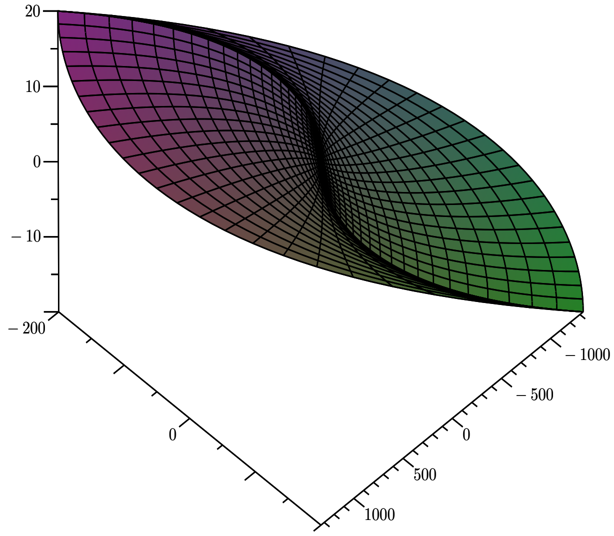

Figure 4.1: The set of non-Lipschitz points of $\mathcal{T}$ within $\mathcal{R}(20)$

Fix $T > 0$, then $x \in \mathcal{S} \cap \text{bdry}\mathcal{R}(T)$ can be steered to the origin by the optimal control of one of the following forms

- $u(s) = 1$ for $0 \leq s \leq T$.
- $u(s) = -1$ for $0 \leq s \leq T$.

- $u(s) = \begin{cases} 1 & \text{if } 0 \le s < \alpha T \\ -1 & \text{if } \alpha T \le s \le T, \end{cases}, \ \alpha \in [0,1].$

- $u(s) = \begin{cases} -1 & \text{if } 0 \le s < \alpha T \\ 1 & \text{if } \alpha T \le s \le T \end{cases}, \ \alpha \in [0,1].$

Then we can compute $\mathcal{S} \cap \mathcal{R}(T)$. Figure 4.1 is the set $\mathcal{S} \cap \mathcal{R}(20)$.

4.4 Non-Lipschitz singularities for nonlinear systems in $\mathbb{R}^2$

This section is devoted to the study of non-Lipschitz points of $\mathcal{T}$ for the nonlinear system

$$\begin{cases} \dot{x}(t) = F(x(t)) + G(x(t))u(t), \\ u(t) \in [-1,1]^M, \\ x(0) = x, \end{cases} \tag{4.4.1}$$

where the state x is in $\mathbb{R}^2$ and M is either 1 or 2.

The assumptions are the following:

1) $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{M}_{2\times M}$ are of class $\mathcal{C}^{1,1}$ and all partial derivatives are Lipschitz with constant L;

2) $F(0) = 0$;

3) $\operatorname{rank}[G_i(0), DF(0)G_i(0)] = 2$ for $i = 1, \dots, M$, where we mean $G = G_1$ if $M = 1$ and $G = (G_1, G_2)$ if $M = 2$;

4) $DG(0) = 0$.

Theorems 5.1, 6.2 and 6.5 in [23] yield that there exists $T > 0$, depending only on L, $DF(0)$, and $G(0)$, such that for all $0 < \tau < T$,

a) $\mathcal{R}(\tau)$ is strictly convex and for all $x \in \operatorname{bdry}\mathcal{R}(\tau)$ there exists a unique optimal control $u(\cdot)$ steering x to the origin in the minimum time τ, and $u(\cdot)$ is bang-bang with finitely many switchings,

b) every $x \in \mathcal{R}(\tau)$ is optimal (the definition of optimal point was recalled in Section 4.2.1),

c) epi$(\mathcal{T})$ has locally positive reach.

We prove here a result which is the nonlinear two dimensional analogue of Theorem 4.3.9. Fix $0 < \tau < T$ and define

$$S = \left\{x \in \mathcal{R}(\tau) : \exists \zeta \in \mathbb{S}^1 \cap N_{\mathcal{R}(\mathcal{T}(x))}(x) \text{ such that } h(x,\zeta) = 0\right\}. \tag{4.4.2}$$

Recalling Propositions 4.2.4 and 4.2.5, S is exactly the set of non-Lipschitz points of $\mathcal{T}$ within $\mathcal{R}(\tau)$. We show first that S is invariant for a class of optimal trajectories and then that it is countably $\mathcal{H}^1$-rectifiable.

Proposition 4.4.1. *Let S be defined according to* (4.4.2) *and let F, G satisfy the assumptions 1) – 4). Then S is invariant for optimal trajectories.*

Proof. We wish to prove that if $\bar{x} \in S$ and $x(\cdot)$ is the optimal trajectory steering $\bar{x}$ to the origin in the minimum time $\mathcal{T}(\bar{x})$ then $x(t) \in S$ for all $0 \le t \le \mathcal{T}(\bar{x})$. In fact, let $\bar{u}(\cdot)$ be the corresponding optimal control and set $\tilde{u}(t) = \bar{u}(\mathcal{T}(\bar{x}) - t), 0 \le t \le \mathcal{T}(\bar{x})$. Let $\tilde{x}(\cdot)$ be the solution of the system

$$\begin{cases} \dot{x}(t) &= -F(x(t)) - G(x(t))\tilde{u}(t), \\ x(0) &= 0 \end{cases} \tag{4.4.3}$$

and let $\bar{\zeta} \in \mathbb{S}^1 \cap N_{\mathcal{R}(\tau)}(\bar{x})$ be such that $h(\bar{x}, \bar{\zeta}) = 0$.
Claim. The solution $\tilde{\lambda}(t)$, $t \in [0, \mathcal{T}(\bar{x})]$ of the adjoint system

$$\begin{cases} \dot{\lambda}(t) &= \lambda(t)\big(DF(\tilde{x}(t)) + DG(\tilde{x}(t))\tilde{u}(t)\big) \\ \lambda(\mathcal{T}(\bar{x})) &= \bar{\zeta}, \end{cases} \tag{4.4.4}$$

satisfies the following properties:

(i) $\tilde{u}_i(t) = \text{sign}\big(\langle \tilde{\lambda}(t), -G_i(\tilde{x}(t))\rangle\big)$ for a.e. $t \in [0, \mathcal{T}(\bar{x})]$, $i = 1, \ldots, M$,

(ii) $0 = h(\tilde{x}(t), \tilde{\lambda}(t)) = \langle F(\tilde{x}(t)), \tilde{\lambda}(t)\rangle - \sum_{i=1}^{M} |\langle G_i(\tilde{x}(t)), \tilde{\lambda}(t)\rangle|$ for all $t \in [0, \mathcal{T}(\bar{x})]$,

(iii) $0 \ne \tilde{\lambda}(t) \in N_{\mathcal{R}(\mathcal{T}(\tilde{x}(t)))}(\tilde{x}(t))$, for all $t \in [0, \mathcal{T}(\bar{x})]$.

Proof of the Claim. We recall that under our assumptions $\mathcal{R}(\mathcal{T}(\bar{x}))$ is strictly convex. In particular, $N_{\mathcal{R}(\mathcal{T}(\bar{x}))}(\bar{x})$ is the convex hull of its exposed rays (see [40, p.163] and [40, Corollary 18.7.1, p. 169]). Therefore let $\zeta \neq 0$ belonging to an exposed ray of $N_{\mathcal{R}(\mathcal{T}(\bar{x}))}(\bar{x})$. Recalling (b) in Proposition 4.2.1, there exists $\sigma \leq 0$ such that $\frac{(\zeta,\sigma)}{\sqrt{|\zeta|^2+\sigma^2}}$ belongs to an exposed ray of $N_{\mathrm{epi}(\mathcal{T})}(\bar{x}, \mathcal{T}(\bar{x}))$.
By Theorem 4.9 in [20], there exists a sequence $\{x_n\} \subset \mathrm{dom}(D\mathcal{T})$ such that $x_n \to \bar{x}$ and

$$\lim_{n\to\infty} \frac{(D\mathcal{T}(x_n), -1)}{\sqrt{|D\mathcal{T}(x_n)|^2 + 1}} = \frac{(\zeta, \sigma)}{\sqrt{|\zeta|^2 + \sigma^2}}$$

Let $u_n = (u_{1,n}, \ldots, u_{M,n})$ be the optimal control steering the origin to x_n. Since $N_{\mathcal{R}(\mathcal{T}(x_n))}(x_n)$ is the half ray $\mathbb{R}^+ D\mathcal{T}(x_n)$, for n large enough, then Pontryagin's Maximum Principle yields that

$$u_{i,n}(t) = \mathrm{sign}\left(\langle \lambda_n(t), -G_i(x_n(t))\rangle\right), \text{ a.e. } t \in [0, \mathcal{T}(\bar{x})],\ i = 1, \ldots, M, \tag{4.4.5}$$

where $x_n(\cdot)$ is the solution of

$$\begin{cases} \dot{y} &= -F(y) - G(y)u_n \\ y(0) &= 0, \end{cases}$$

and λ_n is the solution of

$$\begin{cases} \dot{\lambda}(t) &= \lambda(t)\big(DF(x_n(t)) + DG(x_n(t))u_n(t)\big), \quad a.e. \\ \lambda(\mathcal{T}(x_n)) &= \zeta_n \in \mathbb{R}^+ D\mathcal{T}(x_n). \end{cases}$$

Since all controls u_n are bang-bang with a finite number of switchings independent of n, up to a subsequence we can assume that $u_n(\cdot)$ (where we have put $u_n(t) \equiv 0$ for $t \in (0, \mathcal{T}(\bar{x}) - \mathcal{T}(x_n))$ if $\mathcal{T}(x_n) < \mathcal{T}(\bar{x})$) converges pointwise a.e. to some admissible $u_0 : [0, \mathcal{T}(\bar{x})] \to [-1, 1]^M$. Let $x_0(\cdot)$ be the solution of (4.4.3) with u_0 in place of $\tilde{u}$. Since obviously $x_0(\mathcal{T}(\bar{x})) = \bar{x}$, by the uniqueness of the optimal control we have that $u_0(t) = \tilde{u}(t)$ a.e. on $[0, \mathcal{T}(\bar{x})]$. Up to another subsequence, we can assume that $x_n(\cdot)$ converges uniformly to $\tilde{x}(\cdot)$ on $[0, \mathcal{T}(\bar{x})]$, and $\lambda_n(\cdot)$ converges uniformly to $\lambda(\cdot)$ on $[0, \mathcal{T}(\bar{x})]$. Then $\lambda(\cdot)$ is the solution of (4.4.4) with ζ in place of $\bar{\zeta}$. Recalling (4.4.5), the above convergence properties imply that

$$\tilde{u}_i(t) = \mathrm{sign}\left(\langle \lambda(t), -G_i(\tilde{x}(t))\rangle\right), \text{ a.e. } t \in [0, \mathcal{T}(\bar{x})],\ i = 1, \ldots, M. \tag{4.4.6}$$

Let now $\bar{\zeta}_1, \bar{\zeta}_2 \in \mathbb{S}^1$ belong to exposed rays of $N_{\mathcal{R}(\mathcal{T}(\bar{x}))}(\bar{x})$ and let $\alpha, \beta \geq 0$ be such that $\bar{\zeta} = \alpha\bar{\zeta}_1 + \beta\bar{\zeta}_2$. Let $\tilde{\lambda}_1(\cdot)$ (resp., $\tilde{\lambda}_2(\cdot)$) be the solutions of (4.4.4) with $\bar{\zeta}_1$ (resp., $\bar{\zeta}_2$) in place of $\bar{\zeta}$. By (4.4.6), we have, for a.e. $t \in [0, \mathcal{T}(\bar{x})]$, that

$$\tilde{u}_i(t) = \text{sign}\left(\langle\tilde{\lambda}_1(t), -G_i(\tilde{x}(t))\rangle\right) = \text{sign}\left(\langle\tilde{\lambda}_2(t), -G_i(\tilde{x}(t))\rangle\right), \ i = 1, \dots, M.$$

Therefore, for a.e $t \in [0, \mathcal{T}(\bar{x})]$,

$$\begin{aligned}\tilde{u}_i(t) &= \text{sign}\left(\langle \alpha\tilde{\lambda}_1(t) + \beta\lambda_2(t), -G_i(\tilde{x}(t))\rangle\right)\\ &= \text{sign}\big(\langle\tilde{\lambda}(t), -G_i(\tilde{x}(t))\rangle\big), \ i = 1, \dots, M,\end{aligned}$$

which proves (i).
To prove (ii), observe that the fact that $h(\tilde{x}(t), \tilde{\lambda}(t))$ is constant follows in a standard way from the maximization property (i) (see, e.g., Corollary 6.4 in [23]). Since $h(\tilde{x}(\mathcal{T}(\bar{x})), \tilde{\lambda}(\mathcal{T}(\bar{x}))) = h(\bar{x}, \bar{\zeta}) = 0$, (ii) is proved.
Statement (iii) again follows from the maximization property (i) (see, e.g., Remark 5.2 in [23]), and the proof of the Claim is concluded.

We now complete the proof that S is invariant for optimal trajectories. To this aim, fix $\bar{x} \in S$, together with $\bar{\zeta} \in \mathbb{S}^1 \cap N_{\mathcal{R}(\mathcal{T}(\bar{x}))}$ such that $h(\bar{x}, \bar{\zeta}) = 0$. By the above claim, the never vanishing adjoint vector $\tilde{\lambda}(\cdot)$ which is the solution of (4.4.4) is such that $h(\tilde{x}(t), \tilde{\lambda}(t)) = 0$ and $\tilde{\lambda}(t) \in N_{\mathcal{R}(t)}(\tilde{x}(t))$ for all $t \in [0, \mathcal{T}(\bar{x})]$, which shows that each point $\tilde{x}(t)$ of the optimal trajectory $\tilde{x}(\cdot)$ steering the origin to $\bar{x}$ belongs to S. The prove of the invariance of S is complete. □

Theorem 4.4.2. *Under the same assumptions of Proposition 4.4.1, the set S is countably $\mathcal{H}^1$-rectifiable. Moreover, for all $\bar{x} \in S$ there exists $\delta > 0$ such that*

$$\mathcal{H}^1\big(S \cap B(\bar{x}, \delta)\big) > 0. \tag{4.4.7}$$

Proof. In order to prove the rectifiability property of S, it is enough to show that, if S is nonempty, then it consists exactly of two optimal trajectories of the reversed dynamics

$$\begin{cases} \dot{x}(t) = -F(x(t)) - G(x(t))u(t), \ u \in [-1, 1]^M, t \in [0, \tau], \\ x(0) = 0. \end{cases} \tag{4.4.8}$$

Let $\bar{x} \in S$ together with $\bar{\zeta} \in \mathbb{S}^1 \cap N_{\mathcal{R}(\mathcal{T}(\bar{x}))}$ be such that $h(\bar{x}, \bar{\zeta}) = 0$. Let $\tilde{u}(\cdot)$ be the optimal control steering the origin to $\bar{x}$ and let $\tilde{x}(\cdot)$ (resp., $\tilde{\lambda}(\cdot)$) be the corresponding optimal trajectory (resp., adjoint vector, the solution of (4.4.4)). Set $\zeta_0 = \tilde{\lambda}(0) \neq 0$.

We assume now that $M = 1$, i.e., the control is scalar. Since the Hamiltonian is constant along the optimal trajectory $\tilde{x}$, we have that

$$|\langle G(0), \zeta_0\rangle| = h(0, \zeta_0) = 0 \; (= h(0, -\zeta_0)).$$

We now prove that each one of the vectors ζ_0 and $-\zeta_0$ determines uniquely an optimal trajectory of (4.4.8) contained in S. In fact, for every optimal trajectory $x(\cdot)$ of (4.4.8), with a corresponding adjoint vector $\lambda(\cdot)$, we can define the switching function

$$g^+_{x,\lambda}(t) = \langle -G(x(t)), \lambda(t)\rangle.$$

Of course, $g^+_{x,\lambda}(0) = \langle -G(0), \pm\zeta_0\rangle = 0$ and $\dot{g}^+_{x,\lambda}(0) = \mp\langle DF(0)G(0), \zeta_0\rangle$. The last expression is nonzero, due to the assumption 3), so that in a neighborhood of $t = 0$, the sign of $g^+_{x,\lambda}(\cdot)$ is uniquely determined by $\pm\zeta_0$. Therefore, in a neighborhood of $t = 0$ the optimal control is uniquely determined by $\operatorname{sign}(g^+_{x,\lambda}(\cdot))$, by the Maximum Principle, and so there are exactly two optimal trajectories of (4.4.8) which belong to S in a neighborhood of $t = 0$. Since at every zero of $g^+_{x,\lambda}(\cdot)$ the derivative $\dot{g}^+_{x,\lambda}(\cdot)$ is non-vanishing (see, [23, Sections 3.2 and 5]), the optimal control can be uniquely extended up to the time $t = \tau$. The proof is now complete for the case of a single input.

To conclude the proof of the rectifiability, let $M = 2$. The condition $h(0, \zeta_0) = 0$ means that the system of equations

$$\begin{cases} \langle G_1(0), \zeta_0\rangle &= 0, \\ \langle G_2(0), \zeta_0\rangle &= 0. \end{cases}$$

has nontrivial solutions. So, if $G_1(0)$ and $G_2(0)$ are linearly independent, then S is empty. Otherwise, both components of the optimal controls are uniquely determined by the sign of the corresponding switching functions, exactly as for the single input case.

The propagation property (4.4.7) is an immediate consequence of Proposition 4.4.1. The proof is concluded. □

4.5 The SBV regularity of $\mathcal{T}$

As a consequence of the results contained in Sections 4.3 and 4.4 we prove the SBV regularity of the minimum time function $\mathcal{T}$. We recall first some properties of functions with bounded variation, and next we collect some known results on functions having epigraph with positive reach. As it was proved in [21] and in [23], the minimum time function has this property under the assumptions taken in Section 4.3 or in Section 4.4.

Let $\Omega \subset \mathbb{R}^N$ be open. We say that a function $f \in L^1_{\mathrm{loc}}(\Omega)$ has locally bounded variation, and we denote this fact by $f \in BV_{\mathrm{loc}}(\Omega)$, if for every ball $\Delta \subset \Omega$ the distributional derivative of f in Δ is a finite Radon measure (see, e.g., [1, Definition 3.1]), which we denote by Df. We write $Df = D^a f + D^s f$, where $D^a f$ is absolutely continuous with respect to Lebesgue measure, and $D^s f$ is singular. The singular part $D^s f$ can also be decomposed into the jump part, $D^j f$, and the Cantor part, $D^c f$ (see, [1, Section 3.9]). In the case where f is continuous, like in the case $f = \mathcal{T}$ under our assumptions, the jump part obviously vanishes.

Definition 4.5.1. *(see, e.g., [1, Section 4.1]) We say that $f \in BV_{\mathrm{loc}}(\Omega)$ is a special function of locally bounded variation, $f \in SBV_{\mathrm{loc}}(\Omega)$, if the Cantor part of its derivative $D^c f$ vanishes.*

It is our aim, in this section, to prove that under the assumptions of Section 4.3 and 4.4, the Cantor part $D^c\mathcal{T}$ vanishes, and so $\mathcal{T}$ is a special function of locally bounded variation.

We state some further results.

Proposition 4.5.2. *(see [1, Proposition 4.2]) Let $f \in BV(\Omega)$. Then $f \in SBV(\Omega)$ if and only if $D^s f$ is concentrated on a Borel set σ-finite with respect to $\mathcal{H}^{N-1}$, in particular, if it vanishes outside a countably $\mathcal{H}^{N-1}$-rectifiable set.*

Recalling the definition of non-Lipschitz points given in Section 4.2 (see Definition 4.2.3), we obtain the following result. The notation $\mu_{\lfloor E}$ means the restriction of the measure μ to the set E.

Proposition 4.5.3. *Let $\Omega \subset \mathbb{R}^N$ be open and let $f \in BV_{\mathrm{loc}}(\Omega)$. Let*

$$K = \{x \in \Omega : f \text{ is non-Lipschitz at } x\}.$$

Then $D^s f_{\lfloor \Omega \setminus K} = 0$.

Proof. By definition, f is locally Lipschitz in the open set $\Omega \setminus K$. Therefore Df is absolutely continuous with respect to Lebesgue measure $\mathcal{L}^N$ in $\Omega \setminus K$ (see, e.g., [1, Proposition 2.13]), i.e., $D^s f_{\lfloor \Omega \setminus K} = 0$. □

Consequently, by putting together the two previous Propositions, we obtain

Corollary 4.5.4. *Let $\Omega \in \mathbb{R}^N$ be open and let $f \in BV_{\rm loc}(\Omega)$. Assume that the set of non-Lipschitz points of f be countably $\mathcal{H}^{N-1}$-rectifiable. Then $f \in SBV_{\rm loc}(\Omega)$.*

We are now ready for the main results of this section.

Theorem 4.5.5. *Consider the linear control system (4.3.1) under the assumption (4.3.4). Then the minimum time function $\mathcal{T}$ to reach the origin satisfies $\mathcal{T} \in SBV_{\rm loc}(\mathbb{R}^N)$.*

Corollary 4.5.6. *Under the assumptions of Theorem 4.5.5, we have*

$$\mathcal{T} \in W^{1,1}_{\rm loc}(\mathbb{R}^N).$$

Theorem 4.5.7. *Consider the nonlinear system (4.4.1) under the assumptions 1) – 4) stated in Section 5. Then there exists $\mathcal{T} > 0$ depending only on $G(0)$, $DF(0)$, and on the Lipschitz constant L of DF and DG, such that $\mathcal{T} \in SBV_{\rm loc}(\mathrm{int}(\mathcal{R}_{\mathcal{T}}))$.*

Proof of Theorem 4.5.5 and 4.5.7. The statements follow immediately by putting together Theorem 2.2.2 and Theorem 4.3.9 (resp., Theorem 4.4.2) and Corollary 4.5.4. □

Proof of Corollary 4.5.6. Since $\mathcal{T}$ is continuous and belongs to $SBV_{\rm loc}(\mathbb{R}^N)$ then its distributional derivative $D\mathcal{T}$ is a locally summable function. Moreover, it is well known that $\mathcal{T}$ is Hölder continuous with exponent $1/J$ (see, e.g., [3, Theorem IV.1.9] and references therein). The statement then follows by applying standard results on Sobolev spaces (see, e.g., Theorem 3, p. 277, in [27]). □

4.6 The differentiability of the minimum time function

This section is devoted to the differentiability of the minimum time function for normal linear control systems. We first give some technical results

Lemma 4.6.1. *Let $\{r_n\} \subset \mathbb{R}_+, \{\zeta_n\} \subset \mathbb{S}^{N-1}, \{x_n\} \subset \mathbb{R}^N$ be such that*

$$x_n = \sum_{i=1}^{M} \int_0^{r_n} Q_i(r_n, t)\, \mathrm{sign}\, g_i(\zeta_n, t) dt,$$

where $Q_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^N, g_i : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, i = 1, \cdots, M,$ *are smooth functions. If* $r_n \to r, \zeta_n \to \zeta, x_n \to x$ *as* $n \to \infty$ *for some* $r \in \mathbb{R}, \zeta \in \mathbb{R}^N, x \in \mathbb{R}^N$ *then*

$$x = \sum_{i=1}^{M} \int_0^r Q_i(r,t) \operatorname{sign} g_i(\zeta, t) dt.$$

Proof. Obvious. □

Let $\mathcal{S}$ be the set of non-Lipschitz points of the minimum time function. Now we are going to define some more exceptional sets which will be useful in the sequel and study their rectifiability. All results are based on the general rectifiability statement proved in Appendix - Theorem 5.0.1.

We fix $\ell \in \{1, \cdots, M\}$ and define

$$\Sigma^\ell = \{\zeta \in \mathbb{S}^{N-1} : \exists t \in [0,\infty) \text{ such that } \langle \zeta, e^{At} b_\ell \rangle = \langle \zeta, e^{At} A b_\ell \rangle = 0\}$$

and

$$\mathcal{S}_0^\ell = \left\{ x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}(\langle \zeta, e^{At} b_i \rangle) dt : r \geq 0, \zeta \in \Sigma^\ell \right\}$$

Proposition 4.6.2. $\mathcal{S}_0^\ell$ *is countably* $\mathcal{H}^{N-1}$ *- rectifiable.*

Proof. By Theorem 5.0.1 it is enough to show that Σ^ℓ is countably $\mathcal{H}^{N-2}$ - rectifiable. Set

$$\Sigma_1^\ell = \{\zeta \in \mathbb{R}^N : \text{there exists } t \in [0,\infty) \text{ such that } \langle \zeta, e^{At} b_\ell \rangle = \langle \zeta, e^{At} A b_\ell \rangle = 0\}$$

and

$$\Sigma_2^\ell = \{(t,\zeta) \in \mathbb{R} \times \mathbb{R}^N : \langle \zeta, e^{At} b_\ell \rangle = \langle \zeta, e^{At} A b_\ell \rangle = 0\}.$$

Consider the function $G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^2$ defined by

$$G(t,\zeta) = \left(\langle \zeta, e^{At} b_\ell \rangle, \langle \zeta, e^{At} A b_\ell \rangle \right), \quad \forall (t,\zeta) \in \mathbb{R} \times \mathbb{R}^N.$$

Then G is smooth and we have, for all (t,ζ),

$$DG(t,\zeta) = \begin{pmatrix} \langle \zeta, e^{At} A b_\ell \rangle & e^{At} b_\ell \\ \langle \zeta, e^{At} A^2 b_\ell \rangle & e^{At} A b_\ell \end{pmatrix}$$

Since $e^{At} b_\ell$ and $e^{At} A b_\ell$ are linearly independent for all $t \in \mathbb{R}$, we have $\text{rank} DG(t,\zeta) = 2$, for all $(t,\zeta) \in \mathbb{R} \times \mathbb{R}^N$. Then by Theorem 2.3.4, $\Sigma_2^\ell = G^{-1}(0,0)$ is countably $\mathcal{H}^{N-1}$ - rectifiable. Thus, Σ_1^ℓ is countably $\mathcal{H}^{N-1}$ - rectifiable. It follows that Σ^ℓ is countably $\mathcal{H}^{N-2}$ - rectifiable. This ends the proof. □

Now, let us define

$$\mathcal{S}_1^\ell = \Big\{x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}(\langle \zeta, e^{At} b_i \rangle) dt : r \geq 0, \zeta \in \mathbb{S}^{N-1},$$
$$\text{and } \langle \zeta, e^{At} b_\ell \rangle = 0 \text{ has zeros of order } \geq 2 \text{ in } [0, r]\Big\}.$$

Proposition 4.6.3. *$\mathcal{S}_1^\ell$ is closed and countably $\mathcal{H}^{N-1}$ - rectifiable.*

Proof. The rectifiability of $\mathcal{S}_1^\ell$ follows the rectifiability of $\mathcal{S}_0^\ell$ and the fact that $\mathcal{S}_1^\ell \subset \mathcal{S}_0^\ell$.

Let $x_n \in \mathcal{S}_1^\ell$ be such that $x_n \to x$ as $n \to \infty$ for some x. Since $x_n \in \mathcal{S}_1^\ell$, there exist $r_n \geq 0, t_n \in [0, r_n], \zeta_n \in \mathbb{S}^{N-1}$ such that

$$x_n = \sum_{i=1}^{M} \int_0^{r_n} e^{A(t-r_n)} b_i \operatorname{sign}(\langle \zeta_n, e^{At} b_i \rangle) dt$$

and

$$\langle \zeta_n, e^{At_n} b_\ell \rangle = \langle \zeta_n, e^{At_n} A b_\ell \rangle = 0.$$

Since $x_n \to x$, $r_n := \mathcal{T}(x_n) \to r := \mathcal{T}(x)$. By passing to subsequences, we may assume that $t_n \to \bar{t} \in [0, r], \zeta_n \to \zeta \in \mathbb{S}^{N-1}$. By Lemma 4.6.1, we have

$$x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}(\langle \zeta, e^{At} b_i \rangle) dt.$$

Moreover, we have $\langle \zeta, e^{A\bar{t}} b_\ell \rangle = \langle \zeta, e^{A\bar{t}} A b_\ell \rangle = 0$. This means that $x \in \mathcal{S}_1^\ell$. □

We now set

$$\mathcal{S}_2^\ell = \left\{ x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}(\langle \zeta, e^{At} b_i \rangle) dt : r \geq 0, \zeta \in \mathbb{S}^{N-1}, \langle \zeta, b_\ell \rangle = 0 \right\}.$$

We can see that when $M = 1$, the set $\mathcal{S}_2^\ell$ is actually the non-Lipschitz set of $\mathcal{T}$. One also can show that

Proposition 4.6.4. *$\mathcal{S}_2^\ell$ is closed and countably $\mathcal{H}^{N-1}$-rectifiable.*

Proof. The rectifiability of $\mathcal{S}_2^\ell$ follows Theorem 5.0.1, while the closedness can be proved in the same way of the proof of Lemma 4.6.3. □

Set

$$\mathcal{S}_3^\ell = \left\{ x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}(\langle \zeta, e^{At} b_i \rangle) dt : r \geq 0, \zeta \in \mathbb{S}^{N-1}, \langle \zeta, e^{Ar} b_\ell \rangle = 0 \right\}.$$

One can prove the following

Proposition 4.6.5. *$\mathcal{S}_3^\ell$ is closed and countably $\mathcal{H}^{N-1}$-rectifiable.*

We now define the set

$$\mathcal{S}_4 = \left\{ x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}(\langle \zeta, e^{At} b_i \rangle) dt : r \geq 0, \zeta \in \mathbb{S}^{N-1}, \dim N_{\mathcal{R}(r)}(x) \geq 2 \right\}.$$

Observe that $\mathcal{S}_4$ is the set of points $x \in \mathcal{R}$ where $\mathcal{R}(\mathcal{T}(x))$ is not smooth at x. We have

$$\mathcal{S}_4 = \left\{ x \in \mathcal{R} : \dim N_{\mathcal{R}(\mathcal{T}(x))}(x) \geq 2 \right\}.$$

Proposition 4.6.6. *$\mathcal{S}_4$ is countably $\mathcal{H}^{N-1}$-rectifiable.*

Proof. Set $\Gamma = \{(x, \mathcal{T}(x)) \in \mathbb{R}^{N+1} : \dim N_{\mathrm{epi}(\mathcal{T})}(x, \mathcal{T}(x)) \geq 2\}$. Since $\mathrm{epi}(\mathcal{T})$ has positive reach [21], the set Γ is countably $\mathcal{H}^{N-1}$ - rectifiable [29].

Let $\alpha : \mathrm{epi}(\mathcal{T}) \to \mathbb{R}^N$ be a mapping defined by $\alpha(x, \mathcal{T}(x)) = x$. Then α is Lipschitz. From Proposition 4.2.2, we observe that $\mathcal{S}_4 \subset \alpha(\Gamma)$. It follows that $\mathcal{S}_4$ is countably $\mathcal{H}^{N-1}$ - rectifiable. □

We set

$$\Lambda = \mathcal{S} \cup \bigcup_{\ell=1}^{M} \mathcal{S}_1^\ell \cup \bigcup_{\ell=1}^{M} \mathcal{S}_2^\ell \cup \bigcup_{\ell=1}^{M} \mathcal{S}_3^\ell \cup \mathcal{S}_4.$$

Then, by the above results, Λ is countably $\mathcal{H}^{N-1}$ - rectifiable. Notice that the set $\mathcal{S}_4$ may not be closed. However we will show that Λ is closed. Finally, set

$$\Omega = \mathcal{R} \setminus \Lambda.$$

By the definition of Ω, we see that $\mathcal{T}$ is Lipschitz on Ω and that for each $x \in \Omega$, the normal cone $N_{\mathcal{R}(\mathcal{T}(x))}(x)$ has only one unit vector. Therefore, by Corollary 1 in [5], $\mathcal{T}$ is differentiable on Ω. It follows from [33] that $\nabla \mathcal{T}$ is continuous on Ω.

Observe that Ω has the following representation

$$\Omega = \Big\{ x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}\big(\langle \zeta, e^{At} b_i\rangle\big)\, dt : r > 0, \zeta \in \mathbb{S}^{N-1}, \langle \zeta, b_\ell\rangle \neq 0, \langle \zeta, e^{Ar} b_\ell\rangle \neq 0$$
$$\langle \zeta, e^{At} b_\ell\rangle \text{ has only simple zeros in } [0,r],\ \forall \ell \in \{1, \cdots, M\} \text{ and } \dim N_{\mathcal{R}(r)}(x) = 1 \Big\}.$$

We have also the following characterization of Ω.

Lemma 4.6.7. *A point x of the form*

$$x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}(\langle \zeta, e^{At} b_i\rangle) dt,$$

for some $r > 0$ and $\zeta \in \mathbb{S}^{N-1}$, belongs to Ω if and only if for each $i \in \{1, \cdots, M\}$, there exists $k_i \geq 0$ such that $\sum_{i=1}^{M} k_i \geq N-1$ and $g_i(\zeta, t) := \langle \zeta, e^{At} b_i\rangle$ has $k_i \geq 0$ simple zeros in $(0, r)$, say $t_j^i, j = 1, \cdots, k_i$ (if $k_i > 0$), $i = 1, \cdots, M$ satisfying

$$\operatorname{rank} \Big\{ e^{At_j^i} b_i : j = 1, \cdots, k_i, i = 1, \cdots, M \Big\} = N - 1. \tag{4.6.1}$$

and $g_i(\zeta, t)$ has no more zeros in $[0, r]$.

Proof. Assume that

$$x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \operatorname{sign}(\langle \zeta, e^{At} b_i\rangle) dt,$$

for some $r > 0$ and $\zeta \in \mathbb{S}^{N-1}$ and that $x \in \Omega$. Assume also that $g_i(\zeta, \cdot) = \langle \zeta, e^{A\cdot} b_i\rangle$ has $k_i \geq 0$ zeros in $[0, r]$. Since $x \in \Omega$, there exists $\ell \in \{1, \cdots, M\}$ such that $k_\ell > 0$ and $g_i(\zeta, \cdot)$ has only k_i simple zeros in $(0, r)$ and has no more zeros in $[0, r]$.

Set $\mathcal{I} = \{i \in \{1, \cdots, M\} : k_i > 0\}$. For each $i \in \mathcal{I}$, let $0 < t_1^i < \cdots < t_{k_i}^i < r$ be zeros of $g_i(\zeta, \cdot)$ in $[0, r]$. Since t_j^i is simple zero of $g_i(\zeta, \cdot)$, we have $\langle \zeta, e^{At_j^i} A b_i\rangle \neq 0$ for all $i \in \mathcal{I}$ and $j = 1, \cdots, k_i$. Then for each $i \in \mathcal{I}$, there exist open neighborhoods V in $\mathbb{S}^{N-1}$ of ζ and I_j^i in $[0, r]$ of t_j^i such that $I_j^i \cap I_m^i = \emptyset$ if $j \neq m$ and $\langle \eta, e^{At} A b_i\rangle \neq 0$ for all $\eta \in V$ and $t \in I_j^i, j = 1, \cdots, k_i$.

Set $I^i = [0, r] \setminus \cup_{j=1}^{k_i} I_j^i$ for each $i \in \mathcal{I}$. Then I^i is closed. Set $\sigma^i = \min_{t \in I^i} |\langle \zeta, e^{At} b_i\rangle|$ and $\sigma = \min\{\sigma_i : i \in \mathcal{I}\}$. Observe that $\sigma > 0$. By the continuity, we can choose V and I_j^i such that

$$|\langle \eta, e^{At} b_i\rangle| \geq \frac{\sigma}{2} > 0, \ \ \forall \eta \in V, \forall t \in I^i.$$

Suppose that (4.6.1) fails i.e., $\operatorname{rank}\{e^{At^i_j}b_i : 1 \le j \le k_i, i \in \mathcal{I}\} \le N-2$. Then there exists $\bar{\zeta} \in \mathbb{S}^{N-1}$ such that $\bar{\zeta}, \zeta$ are linearly independent and $\langle \bar{\zeta}, e^{At^i_j}b_i\rangle = 0$ for all $1 \le j \le k_i$ and for all $i \in \mathcal{I}$. Choose $\lambda > 0$ sufficiently small such that $\zeta_1 := \zeta + \lambda\bar{\zeta} \in V$. Then ζ and ζ_1 are linearly independent and $\langle \zeta_1, e^{At^i_j}b_i\rangle = 0$ for all $j \in \{1, \cdots, k_i\}$ and $|\langle \zeta_1, e^{At}b_i\rangle| > 0$, for all $t \in I^i$, $i \in \mathcal{I}$. Since $\langle \zeta_1, e^{At}Ab_i\rangle \ne 0$ for all $t \in I^i_j$, we observe that t^i_j are all zeros of $\langle \zeta_1, e^{A\cdot}b_i\rangle$ in $[0, r]$ and they are simple zeros. Therefore

$$x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)}b_i \operatorname{sign}(\langle \zeta, e^{At}b_i\rangle)dt = \sum_{i=1}^{M} \int_0^r e^{A(t-r)}b_i \operatorname{sign}(\langle \zeta_1, e^{At}b_i\rangle)dt.$$

Since ζ, ζ_1 are linearly independent, $\dim N_{\mathcal{R}(r)}(x) \ge 2$. This leads to a contradiction with $x \in \Omega$. The other implication is obvious. The proof is complete. □

Proposition 4.6.8. Ω *is open.*

Proof. Assume to the contrary that Ω is not open. Then there exist a point $x \in \Omega$ and a sequence $\{x_k\} \subset \Lambda$ such that

$$x_k \to x \quad \text{as} \quad k \to \infty.$$

For each k, there exists $\zeta_k \in \mathbb{S}^{N-1}$ such that

$$x_k = \sum_{i=1}^{M} \int_0^{r_k} e^{A(t-r_k)}b_i \operatorname{sign}(\langle \zeta_k, e^{At}b_i\rangle)dt,$$

where $r_k = T(x_k)$.

Since $\zeta_k \in \mathbb{S}^{N-1}$, we may assume that $\zeta_k \to \zeta \in \mathbb{S}^{N-1}$ as $k \to \infty$. On the other hand, since $x_k \to x$, $r_k \to r := T(x)$. Therefore

$$x = \sum_{i=1}^{M} \int_0^{r} e^{A(t-r)}b_i \operatorname{sign}(\langle \zeta, e^{At}b_i\rangle)dt.$$

By Lemma 4.6.7, $\langle \zeta, e^{A\cdot}b_i\rangle$ has only $k_i \ge 0$ simple zeros in $(0, r)$ and has no other zeros in $[0, r]$. We denote by $\mathcal{A}$ the set of $i \in \{1, \cdots, M\}$ such that $k_i \ge 1$. For $i \in \mathcal{A}$, let $0 < t^i_1 < \cdots, t^i_{k_i} < r$ be all zeros of $\langle \zeta, e^{A\cdot}b_i\rangle$ in $[0, r]$, then one has

$$\sum_{i \in \mathcal{A}} k_i \ge N-1 \text{ and } \operatorname{rank}\left\{e^{At^i_j}b_i : 1 \le j \le k_i, i \in \mathcal{A}\right\} = N-1.$$

Since $t^i_1, \cdots, t^i_{k_i}$ are simple zeros of $\langle \zeta, e^{A\cdot} b_i \rangle$ in $[0, r]$, we can find open neighborhoods V in $\mathbb{S}^{N-1}$ of ζ and I^i_j in $[0, r]$ of t^i_j, $1 \le j \le k_i$ and I^0 of r such that $I^i_j \cap I^i_\ell = \emptyset$ for $j \neq \ell$ and $I^i_j \cap I^0 = \emptyset$ and for all $\eta \in V$ the equation $\langle \eta, e^{At} b_i \rangle = 0$ has only one simple zero in I^i_j and has no more zeros in $[0, \bar{r}]$, for all $\bar{r} \in I^0$ with $i \in \mathcal{A}$ and further the equation $\langle \eta, e^{At} b_m \rangle = 0$ has no zero in $[0, \bar{r}]$ for all $\bar{r} \in I^0$ with $m \in \{1, \cdots, M\} \setminus \mathcal{A}$.

For any $\eta \in V$, let $s^i_j \in I^i_j$ be the zeros of $\langle \eta, e^{At} b_i \rangle$, $1 \le j \le k_i$, $i \in \mathcal{A}$, then by choosing V, I^i_j, I^0 small enough, we obtain that

$$\operatorname{rank}\left\{ e^{As^i_j} b_i : 1 \le j \le k_i, i \in \mathcal{A} \right\} = N - 1.$$

It follows from Lemma 4.6.7 that

$$\bar{x} := \sum_{i=1}^{M} \int_0^{\bar{r}} e^{A(t-\bar{r})} b_i \operatorname{sign}\left(\langle \eta, e^{At} b_i \rangle\right) dt \in \Omega,$$

for all $\eta \in V$ and $\bar{r} \in I^0$. Therefore $x_k \in \Omega$ for k sufficiently large. This contradiction concludes that Ω is open. □

From the above results, we observe that

Theorem 4.6.9. *The minimum time function is of class C^1 in an open set Ω whose complement in the reachable set is countably $\mathcal{H}^{N-1}$ rectifiable.*

We end this section with a proposition which can be seen as a propagation property of the differentiability of the minimum time function along optimal trajectories

Proposition 4.6.10. *Let $x \neq 0$ and $y(\cdot)$ be the optimal trajectory for x. Let $r \in (0, \mathcal{T}(x))$ be such that $\mathcal{T}$ is differentiable at $y(r)$. Then $\mathcal{T}$ is differentiable at $y(s)$ for all $s \in [0, r]$.*

Proof. Since $\mathcal{T}$ is differentiable at $y(r)$, the normal cone $N_{\mathcal{R}(\mathcal{T}(y(r)))}(y(r))$ has only one unit vector. Hence $N_{\mathcal{R}(\mathcal{T}(y(s)))}(y(s))$ also has only one unit vector (see, e.g., exersice 15.1 [34]). Observe that $\mathcal{T}$ is Lipschitz at $y(s)$ for all $s \in [0, r]$. Indeed, if $\mathcal{T}$ is not Lipschitz at $y(s)$ for some $s \in [0, r]$. Then $\mathcal{T}$ is not Lipschitz at $y(t)$ for all $t \in [s, \mathcal{T}(x)]$. Hence $\mathcal{T}$ is not Lipschitz at $y(r)$ which is a contradiction. Thus, by Corollary 1 [5], $\mathcal{T}$ is differentiable at $y(s)$ for all $s \in [0, r]$. □

Remark 4.6.11. *Notice that in Proposition 4.6.10, $\mathcal{T}$ may not be differentiable at $y(s)$ for some $s \in (r, \mathcal{T}(x)]$.*

4.7 The analyticity of the minimum time function

This section is devoted to the analyticity of the minimum time function for normal linear control systems.

Let Ω be the open set as in Theorem 4.6.9. Given $x_0 \in \Omega$. Let $r_0 > 0$ and $\zeta_0 \in \mathbb{S}^{N-1}$ be such that

$$x_0 = \sum_{i=1}^{M} \int_0^{r_0} e^{A(t-r_0)} b_i \text{sign}(\langle \zeta_0, e^{At} b_i \rangle) dt.$$

Set

$$\mathcal{A} = \{i \in \{1, \cdots, M\} : \langle \zeta_0, e^{At} b_i \rangle = 0 \text{ has zeros in } [0, r_0]\}.$$

For each $i \in \mathcal{A}$, assume that $\langle \zeta_0, e^{At} b_i \rangle = 0$ has $k_i \geq 1$ zeros in $[0, r_0]$ and let $0 < t_1^i < \cdots < t_{k_i}^i < r_0$ be all zeros of $\langle \zeta_0, e^{At} b_i \rangle = 0$ in that interval. Then they are all simple zeros and satisfy

$$\text{rank} \left\{ e^{At_j^i} b_i : 1 \leq j \leq k_i,\ i \in \mathcal{A} \right\} = N - 1.$$

We now fix $i \in \mathcal{A}$. Since $t_1^i, \cdots, t_{k_i}^i$ are simple zeros of $\langle \zeta_0, e^{At} b_i \rangle = 0$, by implicit function theorem, there exist open neighborhoods V_ℓ^i in $\mathbb{S}^{N-1}$ of ζ_0, I_ℓ^i in $[0, r_0]$ of t_ℓ^i and analytic functions $\varphi_\ell^i : V_\ell^i \to I_\ell^i$ such that

$$\langle \zeta, e^{At} A b_i \rangle \neq 0, \ \ \forall (\zeta, t) \in V_\ell^i \times I_\ell^i,$$

and

$$\left\{ (\zeta, t) \in V_\ell^i \times I_\ell^i : \langle \zeta, e^{At} b_i \rangle = 0 \right\} = \text{graph}(\varphi_\ell^i),$$

for all $\ell = 1, \cdots, k_i$.
Set

$$V_0 = \bigcap_{i \in \mathcal{I}} \bigcap_{\ell=1}^{k_i} V_\ell^i.$$

Then V_0 is open (in $\mathbb{S}^{N-1}$). We can choose V_ℓ^i, I_ℓ^i and an sufficently small open neighborhood I_0 of r_0 such that $I_1^i, \cdots, I_{k_i}^i, I_0$ are disjoint. Moreover, for each $\zeta \in V_0$ and $r \in I_0$ the equation $\langle \zeta, e^{At} b_i \rangle = 0$ has only zero $\varphi_\ell^i(\zeta)$ in I_ℓ^i and no more zero in $[0, r]$ for $i \in \mathcal{A}$ and $\langle \zeta, e^{At} b_i \rangle = 0$ still has no zero in $[0, r]$ for $i \notin \mathcal{A}$.

Set

$$\Omega_0 = \left\{ x = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \,\text{sign}\,(\langle \zeta, e^{At} b_i \rangle) dt : \zeta \in V_0, r \in I_0 \right\},$$

and consider the function $F : V_0 \times I_0 \to \Omega_0$ defined by

$$F(\zeta, r) = \sum_{i=1}^{M} \int_0^r e^{A(t-r)} b_i \,\mathrm{sign}\,(\langle \zeta, e^{At} b_i \rangle) dt.$$

Lemma 4.7.1. *F is a homeomorphism. Moreover, F is analytic.*

Proof. Set $\sigma_i = \mathrm{sign}(\langle \zeta_0, b_i \rangle)$. Then F can be written in the following

$$\begin{aligned} F(\zeta, r) &= \sum_{i \in \mathcal{A}} \sigma_i \left(2 \int_0^{\varphi_1^i(\zeta)} e^{A(t-r)} b_i dt + \cdots + 2(-1)^{k_i - 1} \int_0^{\varphi_{k_i}^i(\zeta)} e^{A(t-r)} b_i dt \right. \\ &\quad \left. + (-1)^{k_i} \int_0^r e^{A(t-r)} b_i dt \right) + \sum_{i \notin \mathcal{A}} \sigma_i \int_0^r e^{A(t-r)} b_i dt \end{aligned}$$

Since φ_ℓ^i is analytic for all $\ell = 1, \cdots, k_i, i \in \mathcal{A}$, F is also analytic.
It is clear that F is bijective. Moreover F^{-1} is continuous. Indeed, suppose $x_n \to x$ in Ω_0 and $x_n = F(\zeta_n), x = F(\zeta)$ for some $\zeta_n, \zeta \in V_0$. Since $\|\zeta_n\| = 1$, we may assume that $\zeta_n \to \eta \in \bar{V}_0$. By the continuity of F, $F(\zeta_n) \to F(\eta)$. The uniqueness of limit implies that $F(\zeta) = x = F(\eta)$. By the injection of F, we have $\zeta = \eta$. Thus F^{-1} is continuous. This ends the proof. $\square$

Since $V_0 \subset \mathbb{S}^{N-1}$ is sufficiently small, there exist an open bounded subset U_0 of $\mathbb{R}^{N-1}$ and an analytic function ψ on U_0 such that $\psi(U_0) = V_0$ and $\mathrm{rank} D\psi(y) = N - 1$ for all $y \in U_0$.

Set $\phi_\ell^i = \varphi_\ell^i \circ \psi$ for $\ell = 1, \cdots, k_i, i \in \mathcal{A}$ and consider the function $G : U_0 \times I_0 \to \Omega_0$ defined by $G(y, r) = F(\phi(y), r), \forall\, (y, t) \in U_0 \times I_0$. Then

$$\begin{aligned} G(y, r) &= \sum_{i \in \mathcal{A}} \sigma_i \left(2 \int_0^{\phi_1^i(y)} e^{A(t-r)} b_i dt + \cdots + 2(-1)^{k_i - 1} \int_0^{\phi_{k_i}^i(y)} e^{A(t-r)} b_i dt \right. \\ &\quad \left. + (-1)^{k_i} \int_0^r e^{A(t-r)} b_i dt \right) + \sum_{i \notin \mathcal{A}} \sigma_i \int_0^r e^{A(t-r)} b_i dt \end{aligned}$$

Observe that G is still analytic and homeomorphism on $U_0 \times I_0$.
Set now $\mathcal{W} = \{(y, r) \in U_0 \times I_0 : \det DG(y, r) = 0\}$. Thanks to Theorem 2.3.5, $\mathcal{W}$ is closed in $U_0 \times I_0$ and is countably $\mathcal{H}^{N-1}$-rectifiable.

Since G is analytic, $G(\mathcal{W})$ is countably $\mathcal{H}^{N-1}$ - rectifiable. Moreover, $G(\mathcal{W})$ is closed. We have $\det DG(y,r) \neq 0$ for all $(y,r) \in (U_0 \times I_0) \setminus \mathcal{W}$. Therefore G is a diffeomorphism from $(U_0 \times I_0) \setminus \mathcal{W}$ into $\Omega_0 \setminus G(\mathcal{W})$. Furthermore, G^{-1} is analytic on $\Omega_0 \setminus G(\mathcal{W})$.

Proposition 4.7.2. *T is analytic on $\Omega_0 \setminus G(\mathcal{W})$.*

Proof. By the injection of G on $(U_0 \times I_0) \setminus \mathcal{W}$, for each $x \in \Omega_0 \setminus G(\mathcal{W})$, there is a unique point $(y,r) \in (U_0 \times I_0) \setminus \mathcal{W}$ such that $G(y,r) = x$. By the definition of G, we have $\mathcal{T}(x) = r$. Consider the function $P : U_0 \times I_0 \to \mathbb{R}$ defined by $P(y,r) = r$. Then P is analytic. We have $\mathcal{T}(x) = P(G^{-1}(x))$ for all $x \in \Omega_0 \setminus G(\mathcal{W})$. It follows that $\mathcal{T}$ is analytic on $\Omega_0 \setminus G(\mathcal{W})$. □

Since Ω can be covered by countably many open neighborhoods of its points, by rectifiability of $G(\mathcal{W})$, it follows from Proposition 4.7.2 the following

Theorem 4.7.3. *The minimum time function $\mathcal{T}$ is analytic on an open set $\tilde{\Omega}$ whose complement in the reachable set is countably $\mathcal{H}^{N-1}$ - rectifiable.*

Remark 4.7.4. *Theorem 4.7.3 has several similarities with Theorem 5 in [44]. That result implies that the minimum time function to reach the origin for a normal linear system has the following property: the reachable set $\mathcal{R}$ admits a countable partition $\mathcal{P}$ such that each $\mathcal{R}(T)$ with $T > 0$ meets finitely many members of $\mathcal{P}$ and that each $P \in \mathcal{P}$ is a connected analytic submanifold which is a subanalytic set and $\mathcal{T}$ is analytic on P. Consequently, $\mathcal{T}$ is analytic on an open set whose complement is countably $\mathcal{H}^{N-1}$ - rectifiable. Our contribution is, first, identifying the exceptional sets in term of the Hamiltonian and of the normals to reachable sets, and, second, giving a completely different proof.*

4.8 A partial differential equation for normal vectors to reachable sets

Current numerical methods for approximating level sets of $\mathcal{T}$ do not provide enough information on normal vectors. Such vectors are relevant, as final conditions of the costate, in order to apply Pontryagin's Maximum Principle to a given trajectory. The regularity we have proved permits to write a systems of PDE's satisfied by $\nabla\mathcal{T}$ on the open set $\tilde{\Omega}$

together with some boundary conditions. More precisely, $\mathcal{T}$ is a classical solution of the Hamilton - Jacobi equation

$$h(x, \nabla\mathcal{T}(x)) + 1 = 0 \tag{4.8.1}$$

on $\tilde{\Omega}$ with the boundary condition $\mathcal{T}(0) = 0$ and $\lim_{x\to x_0\in \mathrm{bdry}\mathcal{R}} \mathcal{T}(x) = +\infty$.

For $x \in \tilde{\Omega}$, we have $\nabla\mathcal{T}(x) \in N_{\mathcal{R}(\mathcal{T}(x))}(x)$ (see, e.g., [33]). By the definition of $\tilde{\Omega}$, $\langle \nabla\mathcal{T}(x), b_i\rangle \neq 0$ for all $x \in \tilde{\Omega}$ and for all $i \in \{1, \cdots, M\}$. Therefore we can differentiate the Hamilton - Jacobi equation (4.8.1) on $\tilde{\Omega}$ and obtain a first order system of PDE's in the unknown $\zeta(x) = \nabla\mathcal{T}(x)$:

$$A\nabla\mathcal{T}(x) + \left(Ax - \sum_{i=1}^{M} \mathrm{sign}(\langle \nabla\mathcal{T}(x), b_i\rangle) b_i\right)^T \nabla^2\mathcal{T}(x) = 0 \tag{4.8.2}$$

on $\tilde{\Omega}$ satisfying the boundary conditions: for $x \notin \tilde{\Omega}$,

(BC1) if there exists $\{x_n\} \subset \tilde{\Omega}$ such that $x_n \to x$, $\frac{\nabla\mathcal{T}(x_n)}{|\nabla\mathcal{T}(x_n)|} \to \zeta \in \mathbb{S}^{N-1}$ and $|\nabla\mathcal{T}(x_n)| \to +\infty$ as $n \to \infty$, then $h(x, \zeta) = 0$.

(BC2) if there exists $\{x_n\} \subset \tilde{\Omega}$ such that $x_n \to x$ and $|\nabla\mathcal{T}(x_n)| \to \zeta$ as $n \to \infty$, then, for $i \in \{1, \cdots, M\}$,

either $\langle \zeta, e^{A\mathcal{T}(x)} b_i\rangle = 0$,

or $\langle \zeta, e^{A\cdot} b_i\rangle$ has multiple zeros in $[0, \mathcal{T}(x)]$,

or there exists ζ_1 linearly independent from ζ such that

$$\mathrm{sign}(\langle \zeta_1, e^{At} b_i\rangle) = \mathrm{sign}(\langle \zeta, e^{At} b_i\rangle), \quad \text{a.e. in } [0, \mathcal{T}(x)].$$

Chapter 5

Appendix

In this chapter, we state and prove a rectifiability result which is a generalization of Theorem 4.3.9.

Theorem 5.0.1. *Let $M \subset \mathbb{R}^N$ be a countably $\mathcal{H}^m$ - rectifiable set. Define*

$$E = \left\{ x = \int_0^r Q(t,r)\,\mathrm{sign}\,P(\zeta,t)dt : \; r \geq 0, \zeta \in M \right\}$$

where $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^N$ is a smooth function and $P : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is an analytic function. Then E is countably $\mathcal{H}^{m+1}$ - rectifiable.

Proof. We consider the function $F : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}^N$ which is defined by

$$F(\zeta, r) = \int_0^r Q(t,r)\,\mathrm{sign}\,P(\zeta,t)dt$$

for all $(\zeta, r) \in \mathbb{R}^N \times [0, \infty)$. Then we have $E = F(M \times [0, \infty))$.

Fix $\tau > 0$. For each $k \in \mathbb{N}$, we define

$$M^k = \{\zeta \in M : P(\zeta,t) \,\text{has zeros of order}\, k \,\text{and has no zero of order} \, > k \,\text{in}\; [0,\tau]\}\,.$$

Then

$$M = \bigcup_{k=0}^{\infty} M^k.$$

Fix $k \geq 1$. For every k- tuple of nonnegative integers $\mathbf{j} = (j_1, \cdots, j_k) \in \mathbb{N}^k$, we define

$$M_{\mathbf{j}}^k = \Big\{\zeta \in M^k : \quad P(\zeta, t) \text{ has in the interval } [0, \tau] \text{ exactly}$$
$$j_1 \text{ zeros of multiplicity } 1,$$
$$\dots$$
$$j_k \text{ zeros of multiplicity } k\Big\}.$$

Set $|\mathbf{j}| = j_1 + \cdot + j_k$. For each $\mathbf{j} \in \mathbb{N}^k$, we define

$$M_{\mathbf{j}}^{k,\pm} = \left\{\zeta \in M_{\mathbf{j}}^k : \lim_{t \to 0^+} \operatorname{sign} P(\zeta, t) = \pm 1\right\}.$$

Let $\mathbf{j} \in \mathbb{N}^k$. If $|\mathbf{j}| = 1$, we set $M_1^{k,\pm} := M_{\mathbf{j}}^{k,\pm}$ and if $|\mathbf{j} > 1$, we define, for any positive integer d,

$$M_{\mathbf{j},d}^{k,\pm} = \left\{\zeta \in M_{\mathbf{j}}^{k,\pm} : \min\{|\tau_1 - \tau_2| : P(\zeta, \tau_1) = P(\zeta, \tau_2) = 0, \tau_1 \neq \tau_2\} \geq \frac{1}{d}\right\}.$$

Then we have

$$M = M^{0,+} \cup M^{0,-} \cup \left(\bigcup_{k=1}^{\infty} M_1^{k,+}\right) \cup \left(\bigcup_{k=1}^{\infty} M_1^{k,-}\right) \cup \left(\bigcup_{k=1}^{\infty} \bigcup_{d=1}^{\infty} \bigcup_{\substack{\mathbf{j} \in \mathbb{N}^k \\ |\mathbf{j}| > 1}} M_{\mathbf{j},d}^{k,+}\right) \cup \left(\bigcup_{k=1}^{\infty} \bigcup_{d=1}^{\infty} \bigcup_{\substack{\mathbf{j} \in \mathbb{N}^k \\ |\mathbf{j}| > 1}} M_{\mathbf{j},d}^{k,-}\right).$$

We now consider the mapping $Y : M \to L^1(0, \tau)$ which is given by

$$Y(\zeta)(\cdot) = \operatorname{sign} P(\zeta, \cdot)$$

We have $Y(\zeta)(t) = 1$ for all $\zeta \in M^{0,+}$, $t \in [0, \tau]$ and $Y(\zeta)(t) = -1$ for all $\zeta \in M^{0,-}$, $t \in [0, \tau]$, so Y is Lipschitz in $M^{0,+}$ and $M^{0,-}$. Fix $k \geq 1$, we are going to show that Y is locally Lipschitz in $M_{\mathbf{j},d}^{k,+}$ and $M_{\mathbf{j},d}^{k,-}$ for $\mathbf{j} \in \mathbb{N}^k, |\mathbf{j}| \geq 1$ and $d \in \mathbb{N}$, here, for simplicity, $d = 0$ if and only if $|\mathbf{j}| = 1$. The argument for $M_{\mathbf{j},d}^{k,+}$ and $M_{\mathbf{j},d}^{k,-}$ is the same, so we do it only for $M_{\mathbf{j},d}^{k,+}$ with $\mathbf{j} \in \mathbb{N}^k, |\mathbf{j}| \geq 1$ and $d \geq 0$.

We fix now $d \geq 0$ and fix $\mathbf{j} \in \mathbb{N}^k$ with $|\mathbf{j}| \geq 1$. Let $\zeta_0 \in M_{\mathbf{j},d}^{k,+}$ and let $t_1, \cdots, t_{|\mathbf{j}|}$ be the zeros of $P(\zeta_0, \cdot)$ in $[0, \tau]$ and each one with multiplicity m_h, $h = 1, \cdots, |\mathbf{j}|$. Since

$$\frac{\partial^{m_h - 1}}{\partial t^{m_h - 1}} P(\zeta_0, t_h) = 0, \text{ and } \frac{\partial^{m_h}}{\partial t^{m_h}} P(\zeta_0, t_h) \neq 0,$$

for $h=1,\cdots,|\mathbf{j}|$, by the continuity and implicit function theorem, there exist a compact neighborhood V_h of ζ_0 and a neighborhood I_h of t_h and a smooth function $\varphi_h : V_h \to I_h$ such that

$$\frac{\partial^{m_h}}{\partial t^{m_h}} P(\zeta, t) \neq 0, \text{ for all } (\zeta, t) \in V_h \times I_h \tag{5.0.1}$$

and

$$\left\{ (\zeta, t) \in V_h \times I_h : \frac{\partial^{m_h - 1}}{\partial t^{m_h - 1}} P(\zeta, t) = 0 \right\} = \text{graph}(\varphi_h) \tag{5.0.2}$$

We can take $I_h, h = 1, \cdots, |\mathbf{j}|$ small enough such that they are disjoint and satisfy $|I_h| \leq \frac{1}{2d}$ when $|\mathbf{j}| > 1$. We can choose

$$V = V(\zeta_0) \subseteq \bigcap_{h=1}^{|\mathbf{j}|} V_h$$

with an additional requirement that for each $\zeta \in V$, the set $\{t \in [0, \tau] : P(\zeta, t) = 0\}$ is contained in $\bigcup_{h=1}^{|\mathbf{j}|} I_h$. If $|\mathbf{j}| = 1$ then $P(\zeta, t)$ has only one zero in I_1 and nowhere else in $[0, \tau]$ for each $\zeta \in V$. If $|\mathbf{j}| > 1$, since $|I_h| \leq \frac{1}{2d}$, the function $P(\zeta, t)$ has at most one zero in each interval I_h for each $\zeta \in V$.

Set $V_{\mathbf{j}} = V \cap M^{k,+}_{\mathbf{j},d}$. We may assume, without lost of generality, that $M^{k,+}_{\mathbf{j},d}$ is contained in a finite union of such $V_{\mathbf{j}}(\cdot)$, say $M^{k,+}_{\mathbf{j},d} = \bigcup_\ell V_{\mathbf{j}}(\zeta_\ell)$. The function corresponding to $V_{\mathbf{j}}(\zeta_\ell)$ is written as $\varphi^\ell_h(\cdot)$, $h = 1, \cdots, |\mathbf{j}|$ and of course each $\varphi^\ell_h(\cdot)$ is Lipschitz continuous on $h = 1, \cdots, |\mathbf{j}|$ with Lipschitz constant, say, L^ℓ_h. We also denote by I^ℓ_h the interval corresponding to $V_{\mathbf{j}}(\zeta_\ell)$, $h = 1, \cdots, |\mathbf{j}|$. It may happen that $V_{\mathbf{j}}(\zeta_\ell)$ is the singleton $\{\zeta_\ell\}$ for some $\mathbf{j}$ and some ℓ. In that case everything is trivial. We now fix an index ℓ.

We claim that for each $\zeta \in V_{\mathbf{j}}(\zeta_\ell)$ the function $P(\zeta, \cdot)$ has a zero of multiplicity m_h exactly at $\varphi^\ell_h(\zeta)$, $h = 1, \cdots, |\mathbf{j}|$ and does not have other zero in $[0, \tau]$. Indeed, by our construction for each $\zeta \in V_{\mathbf{j}}(\zeta_\ell)$, all zeros of $P(\zeta, \cdot)$ belong to $\bigcup_{h=1}^{|\mathbf{j}|} I_h$. We have $j_k \neq 0$. Then by the construction, for each $\zeta \in V_{\mathbf{j}}(\zeta_\ell)$, $P(\zeta, \cdot)$ has exactly j_k zeros of order k and such zeros must be in the same intervals I^ℓ_h where the j_k zeros of multiplicity k of $P(\zeta, \cdot)$ are in, since in all other intervals, we have at leat one non-vanishing derivative of order less than or equals $k-1$. Thanks to (5.0.2) with $m_h = k$, such zeros must occur at $\varphi^\ell_h(\zeta)$ with the corresponding index h.

Now let p be the largest positive integer $< k$ such that $j_p \neq 0$. By definition of $V_{\mathbf{j}}(\zeta_\ell)$, for each $\zeta \in V_{\mathbf{j}}(\zeta_\ell)$, the function $P(\zeta, \cdot)$ does not have any zero of multiplicity q with $p < q < k$ and it must have exactly $j_p > 0$ zeros of order p. Such zeros cannot belong to the intervals to which the zeros of multiplicity k of $P(\zeta,\cdot)$ belong to, since such interval already contains a zero; on the other hand, by (5.0.2), they must belong to the same intervals I_h^ℓ to which the zeros of multiplicity p of $P(\zeta,\cdot)$ belong to and they must occur at $\varphi_h^\ell(\zeta)$ for the corresponding index h. Using the analogous argument we can perform for all further index $q < p$ such that $j_q > 0$. Thus the claim is proved.

We claim that Y is Lipschitz on $V_{\mathbf{j}}(\zeta_\ell)$. Indeed, we fixe the index ℓ and take $\zeta_1, \zeta_2 \in V_{\mathbf{j}}(\zeta_\ell)$. The we have

$$\begin{aligned} \|Y(\zeta_2) - Y(\zeta_1)\|_{L^1(0,\tau)} &\leq 2 \sum_{\substack{h=1 \\ m_h \text{ is odd}}}^{|\mathbf{j}|} |\varphi_h^\ell(\zeta_2) - \varphi_h^\ell(\zeta_1)| \\ &\leq 2 \sum_{\substack{h=1 \\ m_h \text{ is odd}}}^{|\mathbf{j}|} L_h^\ell \|\zeta_2 - \zeta_1\|, \end{aligned}$$

which proves the claim.

The Lipschitz continuity of Y on each $V_{\mathbf{j}}(\zeta_\ell)$ implies immediately that for all fixed $r \in [0,\tau]$, the function $\zeta \mapsto F(\zeta, r)$ is Lipschitz on the same set. On the other hand, it is easy to see that the function $r \mapsto F(\zeta, r)$ is Lipschitz on $[0,\tau]$ for each $\zeta \in M$. Therefore the function F is Lipschitz on each $V_{\mathbf{j}}(\zeta_\ell) \times [0,\tau]$. It follows that the set E is contained in a countable union of Lipschitz images of countably $\mathcal{H}^{m+1}$ - rectifiable sets. The proof is complete. □

As a consequence of Theorem 5.0.1, we have

Theorem 5.0.2. *Given $T > 0$. Let $M \subset \mathbb{R}^N$ be a countably $\mathcal{H}^m$ - rectifiable set. Define*

$$D = \left\{ x = \int_0^T Q(t) \operatorname{sign} P(\zeta, t) dt : \ \zeta \in M \right\}$$

where $Q : \mathbb{R} \to \mathbb{R}^N$ is a smooth function and $P : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is an analytic function. Then D is countably $\mathcal{H}^m$ - rectifiable.

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