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Unified Dark Matter in Scalar Field Cosmologies

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Riassunto

In questa tesi si è svolta una ricerca in ambito cosmologico, incentrata sullo studio dei “Modelli Unificati” (UDM) per le componenti oscure (materia ed energia oscura) dell’Universo.

In particolare, la ricerca si è sviluppata nello studio di modelli cosmologici unificati basati su un campo scalare neutro che rendano conto della dinamica media dell’Universo nelle varie epoche (Bertacca, Matarrese, Pietroni 2007).

Successivamente si è fatta un’analisi delle proprietà osservabili delle fluttuazioni di tale campo scalare, in termini di anisotropie del fondo cosmico di radiazione delle micro-onde (CMB) (Bertacca and Bartolo 2007).

In fine ci si è incentrati nella ricerca delle soluzioni statiche a simmetria sferica in tale ambito, che permettano di descrivere la dinamica degli aloni oscuri delle galassie (Bertacca, Bartolo, Matarrese 2007).

Summary

In this thesis I have investigated the possibility that the dynamics of a single scalar field can account for a unified description of the Dark Matter and Dark Energy sectors: Unified Dark Matter (UDM). In particular considering the general Lagrangian of *k-essence* models, I study and classify them through variables connected to the fluid equation of state parameter w_κ . This allows to find solutions around which the scalar field describes a mixture of dark matter and cosmological constant-like dark energy (UDM) (Bertacca, Matarrese, Pietroni 2007).

Subsequently I also perform an analytical study of the Integrated Sachs-Wolfe (ISW) effect within the framework of Unified Dark Matter models based on a scalar field which aim at a unified description of dark energy and dark matter. Computing the temperature power spectrum of the Cosmic Microwave Background anisotropies I am able to isolate those contributions that can potentially lead to strong deviations from the usual ISW effect occurring in a Λ CDM Universe. This helps to highlight the crucial role played by the sound speed in the unified dark matter models. This treatment is completely general in that all the results depend only on the speed of sound of the dark component and thus it can be applied to a variety of unified models, including those which are not described by a scalar field but relies on a single dark fluid (Bertacca and Bartolo 2007).

Finally I also investigated the static and spherically symmetric solutions of Einstein's equations for a scalar field with non-canonical kinetic term (Bertacca, Bartolo, Matarrese 2007).

Chapter 1

Introduction

Gravity¹ remains the most enigmatic of all the fundamental interactions in nature. At the Planck scale, large quantum fluctuations signal a breakdown of the effective field theory description of gravity in terms of general relativity, leading to the expectation that gravity and our notions of space and time must be radically altered at short distances.

Meanwhile, continuing experimental probes of gravity at *large* distances reveal strange phenomena at many length scales, from the flattening of galactic rotation curves to the accelerating Universe (for more details see the Chapters 1 and 3).

Traditionally, these phenomena are explained by invoking new sources of matter and energy, such as dark matter and a tiny cosmological constant [1]. It is however worth contemplating the possibility that gravity itself is changing in the infrared in some way that might address these mysteries (indeed, this approach is further motivated by the cosmological constant problem, which seems to be associated with extreme infrared physics).

In these years alternative routes have been followed, for example *Quintessence* [2, 3, 4, 5, 6, 7, 8, 9] and *k-essence* [10, 11, 12, 13, 14, 15] (a complete list of dark energy models can be found in the recent review [16]).

The *k-essence* is characterized by a Lagrangian with non-canonical kinetic term and it is inspired by earlier studies of k-inflation [14]. In this introduction we will consider the main mathematical and physical property of the *k-essence* fields [12, 13, 14, 15].

Starting from *k-essence*, the main purpose of this thesis is to investigate the dynamics of a single scalar field for a unified description of the Dark Matter and Dark Energy sectors: Unified Dark Matter (UDM).

As we know, in the *k-essence* theories there is a possibility of the presence of superluminal signals on nontrivial backgrounds and, therefore, the breaking of the Lorentz invariance .

In general, one can consider the questions related to the spontaneous breaking of the Lorentz invariance such as superluminal propagation of perturbations in nontrivial backgrounds, attracted a renewed interest among physicists. One of the basic questions here is whether the theories allowing the superluminal velocities possess internal inconsistencies

¹Throughout these preliminary sections, we adopt natural units $c = \hbar = 1$ and have a metric signature $(-, +, +, +)$. We denote the Planck mass as $m_{\text{pl}} = G^{-1/2} = 1.22 \times 10^{19}$ GeV and the reduced Planck mass as $M_{\text{pl}} = (8\pi G)^{-1/2} = 2.44 \times 10^{18}$ GeV. Here G is Newton's gravitational constant.

and, in particular, inevitably lead to the causality violation namely to the appearance of the closed causal curves (CCCs). Concerning this issue there exist two contradicting each other points of view. Some authors (see, for instance, [17, 18, 19, 20, 21, 22, 23]) argue that the subliminally propagation condition should a priori be imposed to make the theory physically acceptable. Indeed, in [17] the authors introduce the “*Postulate of Local Causality*” which excludes the superluminal velocities from the very beginning.

Regarding the general plan, this thesis is organized as follows. In this introduction, first of all, we present a qualitative introduction of the cosmological constant and the scalar-field dark energy models, i.e. *Quintessence* which can act as alternatives to the cosmological constant. Subsequently we discuss the general aspects of causality and propagation of perturbations on a nontrivial background (see for example [15]), determining the “*new aether*” made of *k-essence* fields. In particular, we want to know when the causality is not violated in generic *k-essence* [15] and when there are no causal paradoxes arise in the cases. Then we investigate under which restrictions on the initial conditions the Cauchy problem is well posed for *k-essence* equation of motion [15].

In the second Chapter, considering the general Lagrangian of *k-essence* models, we study and classify them through variables connected to the fluid equation of state parameter w_κ . This allows to find solutions around which the scalar field describes a mixture of dark matter and cosmological constant-like dark energy, an example being the purely kinetic model proposed by Scherrer. Making the stronger assumption that the scalar field Lagrangian is exactly constant along solutions of the equation of motion, we find a general class of *k-essence* models whose classical trajectories directly describe a unified dark matter/dark energy (cosmological constant) fluid. While the simplest case of a scalar field with canonical kinetic term unavoidably leads to an effective sound speed $c_s = 1$, thereby inhibiting the growth of matter inhomogeneities, more general non-canonical *k-essence* models allow for the possibility that $c_s \ll 1$ whenever matter dominates.

In the third Chapter we perform an analytical study of the Integrated Sachs-Wolfe (ISW) effect within the framework of Unified Dark Matter models based on a scalar field which aim at a unified description of dark energy and dark matter. Computing the temperature power spectrum of the Cosmic Microwave Background anisotropies we are able to isolate those contributions that can potentially lead to strong deviations from the usual ISW effect occurring in a Λ CDM Universe. This helps to highlight the crucial role played by the sound speed in the unified dark matter models. Our treatment is completely general in that all the results depend only on the speed of sound of the dark component and thus it can be applied to a variety of unified models, including those which are not described by a scalar field but relies on a single dark fluid.

In the fourth Chapter we investigate the static and spherically symmetric solutions of Einstein’s equations for a scalar field with non-canonical kinetic term, assumed to provide both the dark matter and dark energy components of the Universe. In particular, we give a prescription to obtain solutions (dark halos) whose rotation curve $v_c(r)$ is in good agreement with observational data. We show that there exist suitable scalar field Lagrangians that allow to describe the cosmological background evolution and the static solutions with a single dark fluid.

Finally in the fifth Chapter we give our main conclusions.

1.1 Introduction of the cosmological constant

The Einstein tensor $G^{\mu\nu}$ and the energy momentum tensor $T^{\mu\nu}$ satisfy the Bianchi identities $\nabla_\nu G^{\mu\nu} = 0$ and energy conservation $\nabla_\nu T^{\mu\nu} = 0$. Since the metric $g^{\mu\nu}$ is constant with respect to covariant derivatives ($\nabla_\alpha g^{\mu\nu} = 0$), there is a freedom to add a term $\Lambda g_{\mu\nu}$ in the Einstein equations. Then the modified Einstein equations are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (1.1)$$

By taking a trace of this equation, we find that $-R + 4\Lambda = 8\pi GT$. Combining this relation with Eq. (1.1), we obtain

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} \right). \quad (1.2)$$

Let us consider Newtonian gravity with metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the perturbation around the Minkowski metric $\eta_{\mu\nu}$. If we neglect the time-variation and rotational effect of the metric, R_{00} can be written by a gravitational potential Φ , as $R_{00} \simeq -(1/2)\Delta h_{00} = \Delta\Phi$. Note that g_{00} is given by $g_{00} = -1 - 2\Phi$. In the relativistic limit with $|p| \ll \rho$, we have $T_{00} \simeq -T \simeq \rho$. Then the 00 component of Eq. (1.2) gives

$$\Delta\Phi = 4\pi G\rho - \Lambda. \quad (1.3)$$

In order to reproduce the Poisson equation in Newtonian gravity, we require that $\Lambda = 0$ or Λ is sufficiently small relative to the $4\pi G\rho$ term in Eq. (1.3). Since Λ has dimensions of $[\text{Length}]^{-2}$, the scale corresponding to the cosmological constant needs to be much larger than the scale of stellar objects on which Newtonian gravity works well. In other words the cosmological constant becomes important on very large scales.

In the FRW background the modified Einstein equations (1.1) give

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad (1.4)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \quad (1.5)$$

where $a = a(t)$ is the scale factor. This clearly demonstrates that the cosmological constant contributes negatively to the pressure term and hence exhibits a repulsive effect.

Let us consider a static Universe ($a = \text{const}$) in the absence of Λ . Setting $H = 0$ and $\ddot{a}/a = 0$ in Eqs. (1.4) and (1.5), we find

$$\rho = -3p = \frac{3K}{8\pi Ga^2}. \quad (1.6)$$

Equation (1.6) shows that either ρ or p needs to be negative. When Einstein first tried to construct a static Universe, he considered that the above solution is not physical² and so added the cosmological constant to the original Einstein field equations.

²We note however that the negative pressure can be realized by scalar fields.

Using the modified field equations (1.4) and (1.5) in a dust-dominated Universe ($p = 0$), we find that the static Universe obtained by Einstein corresponds to

$$\rho = \frac{\Lambda}{4\pi G}, \quad \frac{K}{a^2} = \Lambda. \quad (1.7)$$

Since $\rho > 0$ we require that Λ is positive. This means that the static Universe is a closed one ($K = +1$) with a radius $a = 1/\sqrt{\Lambda}$. Equation (1.7) shows that the energy density ρ is determined by Λ .

The requirement of a cosmological constant to achieve a static Universe can be understood by having a look at the Newton's equation of motion

$$\begin{aligned} m\ddot{a} &= -\frac{Gm}{a^2} \left(\frac{4\pi a^3 \rho}{3} \right), \\ \rightarrow \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3} \rho. \end{aligned} \quad (1.8)$$

Since gravity pulls the point particle toward the center of the sphere, we need a repulsive force to realize a situation in which a is constant. This corresponds to adding a cosmological constant term $\Lambda/3$ on the right hand side of Eq. (1.8).

The above description of the static Universe was abandoned with the discovery of the redshift of distant stars, but it is intriguing that such a cosmological constant should return in the 1990's to explain the observed acceleration of the Universe.

1.1.1 Fine tuning problem

If the cosmological constant originates from a vacuum energy density, then this suffers from a severe fine-tuning problem. Observationally we know that Λ is of order the present value of the Hubble parameter H_0 (for the definition of H see chapter 1), that is

$$\Lambda \approx H_0^2 = (2.13h \times 10^{-42} \text{ GeV})^2. \quad (1.9)$$

This corresponds to a critical density ρ_Λ ,

$$\rho_\Lambda = \frac{\Lambda m_{\text{pl}}^2}{8\pi} \approx 10^{-47} \text{ GeV}^4. \quad (1.10)$$

a

Meanwhile the vacuum energy density evaluated by the sum of zero-point energies of quantum fields with mass m is given by

$$\begin{aligned} \rho_{\text{vac}} &= \frac{1}{2} \int_0^\infty \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{k^2 + m^2} \\ &= \frac{1}{4\pi^2} \int_0^\infty dk k^2 \sqrt{k^2 + m^2}. \end{aligned} \quad (1.11)$$

This exhibits an ultraviolet divergence: $\rho_{\text{vac}} \propto k^4$. However we expect that quantum field theory is valid up to some cut-off scale k_{max} in which case the integral (1.11) is finite:

$$\rho_{\text{vac}} \approx \frac{k_{\text{max}}^4}{16\pi^2}. \quad (1.12)$$

For the extreme case of General Relativity we expect it to be valid to just below the Planck scale: $m_{\text{pl}} = 1.22 \times 10^{19}$ GeV. Hence if we pick up $k_{\text{max}} = m_{\text{pl}}$, we find that the vacuum energy density in this case is estimated as

$$\rho_{\text{vac}} \approx 10^{74} \text{ GeV}^4, \quad (1.13)$$

which is about 10^{121} orders of magnitude larger than the observed value given by Eq. (1.10). Even if we take an energy scale of QCD for k_{max} , we obtain $\rho_{\text{vac}} \approx 10^{-3} \text{ GeV}^4$ which is still much larger than ρ_{Λ} .

We note that this contribution is related to the ordering ambiguity of fields and disappears when normal ordering is adopted. Since this procedure of throwing away the vacuum energy is ad hoc, one may try to cancel it by introducing counter terms. However this requires a fine-tuning to adjust ρ_{Λ} to the present energy density of the Universe. Whether or not the zero point energy in field theory is realistic is still a debatable question.

1.2 Quintessence

Quintessence is described by an ordinary scalar field φ minimally coupled to gravity, but as we will see with particular potentials that lead to late time inflation. The action for *Quintessence* is given by

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - V(\varphi) \right], \quad (1.14)$$

where $V(\varphi)$ is the potential of the field. In a flat FRW spacetime (for more details see chapter 1)

$$ds^2 = g_{\mu\nu}^{-1} dx^{\mu} dx^{\nu} = (-dt^2 + a^2(t) d\mathbf{x}^2)$$

where $a(t)$ is the scale factor, the variation of the action (1.14) with respect to φ gives

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \quad (1.15)$$

The energy momentum tensor of the field is derived by varying the action (1.14) in terms of $g^{\mu\nu}$:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_{\mu} \varphi \partial_{\nu} \varphi - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \varphi \partial_{\beta} \varphi + V(\varphi) \right]. \quad (1.16)$$

In the flat Friedmann background we obtain the energy density and pressure density of the scalar field:

$$\rho = -T_0^0 = \frac{1}{2} \dot{\varphi}^2 + V(\varphi), \quad p = T_i^i = \frac{1}{2} \dot{\varphi}^2 - V(\varphi). \quad (1.17)$$

Then by solving the Einstein equations (for more details see chapter 1) we obtain

$$H^2 = \frac{8\pi G}{3} \left[\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right], \quad (1.18)$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3} [\dot{\varphi}^2 - V(\varphi)]. \quad (1.19)$$

We recall that the continuity equation, i.e. the equation of motion for φ Eq. (1.15), is derived by combining these equations.

We find that the Universe accelerates for $\dot{\varphi}^2 < V(\varphi)$. This means that one requires a flat potential to give rise to an accelerated expansion. In the context of inflation the slow-roll parameters

$$\epsilon = \frac{m_{\text{pl}}^2}{16\pi} \left(\frac{1}{V} \frac{dV}{d\varphi} \right)^2, \quad \eta = \frac{m_{\text{pl}}^2}{8\pi} \frac{1}{V} \frac{d^2V}{d\varphi^2}, \quad (1.20)$$

are often used to check the existence of an inflationary solution for the model (1.14) [24]. Inflation occurs if the slow-roll conditions, $\epsilon \ll 1$ and $|\eta| \ll 1$, are satisfied. In the context of dark energy these slow-roll conditions are not completely trustworthy, since there exists dark matter as well as dark energy. However they still provide a good measure to check the existence of a solution with an accelerated expansion. If we define slow-roll parameters in terms of the time-derivatives of H such as $\epsilon = -\dot{H}/H^2$, this is a good measure to check the existence of an accelerated expansion since they implement the contributions of both dark energy and dark matter.

The equation of state for the field φ is given by

$$w_\varphi = \frac{p}{\rho} = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)}. \quad (1.21)$$

In this case the continuity equation Eq. (1.15) can be written in an integrated form:

$$\rho = \rho_0 \exp \left[- \int 3(1 + w_\varphi) \frac{da}{a} \right], \quad (1.22)$$

where ρ_0 is an integration constant. We note that the equation of state for the field φ ranges in the region $-1 \leq w_\varphi \leq 1$. The slow-roll limit, $\dot{\varphi}^2 \ll V(\varphi)$, corresponds to $w_\varphi = -1$, thus giving $\rho = \text{const}$ from Eq. (1.22). In the case of a stiff matter characterized by $\dot{\varphi}^2 \gg V(\varphi)$ we have $w_\varphi = 1$, in which case the energy density evolves as $\rho \propto a^{-6}$ from Eq. (1.22). In other cases the energy density behaves as

$$\rho \propto a^{-m}, \quad 0 < m < 6. \quad (1.23)$$

Since $w_\varphi = -1/3$ is the border of acceleration and deceleration, the Universe exhibits an accelerated expansion for $0 \leq m < 2$.

It is of interest to derive a scalar-field potential that gives rise to a power-law expansion:

$$a(t) \propto t^p. \quad (1.24)$$

The accelerated expansion occurs for $p > 1$. From Eqs. (1.18) and (1.19) we obtain the relation $\dot{H} = -4\pi G \dot{\varphi}^2$. Then we find that $V(\varphi)$ and $\dot{\varphi}$ can be expressed in terms of H and \dot{H} :

$$V = \frac{3H^2}{8\pi G} \left(1 + \frac{\dot{H}}{3H^2} \right), \quad (1.25)$$

$$\varphi = \int dt \left[-\frac{\dot{H}}{4\pi G} \right]^{1/2}. \quad (1.26)$$

Here we chose the positive sign of $\dot{\varphi}$. Hence the potential giving the power-law expansion (1.24) corresponds to

$$V(\varphi) = V_0 \exp\left(-\sqrt{\frac{16\pi}{p}} \frac{\varphi}{m_{\text{pl}}}\right), \quad (1.27)$$

where V_0 is a constant. The field evolves as $\varphi \propto \ln t$. The above result shows that the exponential potential may be used for dark energy provided that $p > 1$.

In addition to the fact that exponential potentials can give rise to an accelerated expansion, they possess cosmological *scaling solutions* [7, 25] in which the field energy density (ρ_φ) is proportional to the fluid energy density (ρ_m). Exponential potentials were used in one of the earliest models which could accommodate a period of acceleration today within it, the loitering Universe [26].

The above discussion shows that scalar-field potentials which are not steep compared to exponential potentials can lead to an accelerated expansion. In fact the original quintessence models [3, 8] are described by the power-law type potential

$$V(\varphi) = \frac{M^{4+\alpha}}{\varphi^\alpha}, \quad (1.28)$$

where α is a positive number (it could actually also be negative [27]) and M is constant. Where does the fine tuning arise in these models? Recall that we need to match the energy density in the quintessence field to the current critical energy density, that is

$$\rho_\varphi^{(0)} \approx m_{\text{pl}}^2 H_0^2 \approx 10^{-47} \text{ GeV}^4. \quad (1.29)$$

The mass squared of the field φ is given by $m_\varphi^2 = \frac{d^2V}{d\varphi^2} \approx \rho_\varphi/\varphi^2$, whereas the Hubble expansion rate is given by $H^2 \approx \rho_\varphi/m_{\text{pl}}^2$. The Universe enters a tracking regime in which the energy density of the field φ catches up that of the background fluid when m_φ^2 decreases to of order H^2 [3, 8]. This shows that the field value at present is of order the Planck mass ($\varphi_0 \sim m_{\text{pl}}$), which is typical of most of the quintessence models. Since $\rho_\varphi^{(0)} \approx V(\varphi_0)$, we obtain the mass scale

$$M = (\rho_\varphi^{(0)} m_{\text{pl}}^\alpha)^{\frac{1}{4+\alpha}}. \quad (1.30)$$

This then constrains the allowed combination of α and M . For example the constraint implies $M = 1 \text{ GeV}$ for $\alpha = 2$ [9]. This energy scale can be compatible with the one in particle physics, which means that the severe fine-tuning problem of the cosmological constant is alleviated. Nevertheless a general problem we always have to tackle is finding such quintessence potentials in particle physics. One of the problems is highlighted in Ref. [5]. The *Quintessence* field must couple to ordinary matter, which even if suppressed by the Planck scale, will lead to long range forces and time dependence of the constants of nature. There are tight constraints on such forces and variations and any successful model must satisfy them.

1.3 *k-essence*

In first of all, hereafter, we use units such that $8\pi G = c^2 = 1$ and signature $(-, +, +, +)$. Let us consider the *k-essence* scalar field φ with the action:

$$S_\varphi = \int d^4x \sqrt{-g} \mathcal{L}(X, \varphi), \quad (1.31)$$

where

$$X = -\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi. \quad (1.32)$$

is the canonical kinetic term and by ∇_μ we always denote the covariant derivative associated with metric $g_{\mu\nu}$. We would like to stress that this action is explicitly generally covariant and Lorentz invariant. The variation of action (1.31) with respect to $g_{\mu\nu}$ gives us the following energy-momentum tensor for the scalar field:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \frac{\partial \mathcal{L}(\varphi, X)}{\partial X} \nabla_\mu \varphi \nabla_\nu \varphi + \mathcal{L}(\varphi, X) g_{\mu\nu}. \quad (1.33)$$

In general there are several classical energy conditions for these field configurations, for example see the Table 1.1 (for a definition of the classical energy conditions, the reader is referred to [17, 28, 29]). The Null Energy Condition (NEC) $T_{\mu\nu} n^\mu n^\nu \geq 0$ (where n^μ is

Weak Energy Condition (WEC)	$T_{\mu\nu} u^\mu u^\nu \geq 0$ where $g_{\mu\nu} u^\mu u^\nu = -1$ & $T_{\mu\nu} n^\mu n^\nu \geq 0$ where $g_{\mu\nu} n^\mu n^\nu = 0$
Null Energy Condition (NEC)	$T_{\mu\nu} n^\mu n^\nu \geq 0$ where $g_{\mu\nu} n^\mu n^\nu = 0$
Dominant Energy Condition (DEC)	$T_{\mu\nu} u^\mu u^\nu \geq T_{\mu\nu} v^\mu v^\nu $ where $g_{\mu\nu} u^\mu u^\nu = -1$, and $g_{\mu\nu} u^\mu v^\nu = 0$ & WEC
Null Dominant Energy Condition (NDEC)	$T_{\mu\nu} u^\mu u^\nu \geq T_{\mu\nu} v^\mu v^\nu $ where $g_{\mu\nu} u^\mu u^\nu = -1$, and $g_{\mu\nu} u^\mu v^\nu = 0$

Table 1.1: Energy conditions.

a null vector: $g_{\mu\nu} n^\mu n^\nu = 0$) is satisfied provided $\partial \mathcal{L} / \partial X \geq 0$. Because violation of this condition would imply the unbounded from below Hamiltonian and hence signifies the inherent instability of the system [30] we consider only the theories with $\partial \mathcal{L} / \partial X \geq 0$. In Particular, in this thesis we consider only fluid that satisfied the Weak Energy Condition (WEC) such that the energy density is positive (or null).

The equation of motion for the scalar field is obtained by variation of action (1.31) with respect to φ i.e. $\delta S / \delta \varphi = 0$,

$$\tilde{G}^{\mu\nu} \nabla_\mu \nabla_\nu \varphi - 2X \frac{\partial^2 \mathcal{L}}{\partial \varphi \partial X} + \frac{\partial \mathcal{L}}{\partial \varphi} = 0, \quad (1.34)$$

where the “effective” metric is given by

$$\tilde{G}^{\mu\nu}(\varphi, \nabla\varphi) \equiv \frac{\partial\mathcal{L}}{\partial X}g^{\mu\nu} - \frac{\partial^2\mathcal{L}}{\partial X^2}\nabla^\mu\varphi\nabla^\nu\varphi. \quad (1.35)$$

First of all, we have to impose that this second order differential equation is hyperbolic that is, $\tilde{G}^{\mu\nu}$ has the Lorentzian signature and hence describes the time evolution of the system provided [21, 31, 32]. If we want to know when hyperbolic condition is satisfied we have to impose an oportune condition to Lagrangian. In this case, for simplicity we can assume that \mathcal{L} only depends on X , though our results also apply if the Lagrangian depends on φ itself. It can be proved (see Theorem 10.1.3. of [33]) that if the metric $\tilde{G}^{\mu\nu}$ is Lorentzian, the field equation admits a well-posed initial value-formulation (at least locally), so all we have to show is that $\tilde{G}^{\mu\nu}$ has a negative determinant.

Let $\tilde{\mathbf{G}}$, \mathbf{g} and \mathbf{n} denote the matrices with elements $\tilde{G}^{\mu\nu}$, $g^{\mu\nu}$ and $\nabla^\mu\varphi\nabla^\nu\varphi$ respectively. We then have

$$\det \tilde{\mathbf{G}} = \det \left(\frac{d\mathcal{L}}{dX}\mathbf{g} - \frac{d^2\mathcal{L}}{dX^2}\mathbf{n} \right) = \left(\frac{d\mathcal{L}}{dX} \right)^4 \cdot \det \mathbf{g} \cdot \det \left(\mathbf{1} - \frac{d^2\mathcal{L}/dX^2}{d\mathcal{L}/dX}\mathbf{g}^{-1}\mathbf{n} \right), \quad (1.36)$$

where \mathbf{g}^{-1} denotes the inverse of \mathbf{g} , and we have assumed $d\mathcal{L}/dX \neq 0$. Using the identity $\det e^{\mathbf{A}} = e^{\text{tr}\mathbf{A}}$ we find

$$\det \left(\mathbf{1} - \frac{d^2\mathcal{L}/dX^2}{d\mathcal{L}/dX}\mathbf{g}^{-1}\mathbf{n} \right) = \exp \left[\text{tr} \ln \left(\mathbf{1} - \frac{d^2\mathcal{L}/dX^2}{d\mathcal{L}/dX}\mathbf{g}^{-1}\mathbf{n} \right) \right], \quad (1.37)$$

and expanding the logarithm in a Taylor series around the identity we obtain

$$\begin{aligned} \text{tr} \ln \left(\mathbf{1} - \frac{d^2\mathcal{L}/dX^2}{d\mathcal{L}/dX}\mathbf{g}^{-1}\mathbf{n} \right) &= \sum_k \frac{(-1)^{k+1}}{k} \left(\frac{d^2\mathcal{L}/dX^2}{d\mathcal{L}/dX} \right)^k \text{tr} (-\mathbf{g}^{-1}\mathbf{n})^k \\ &= \ln \left(1 + 2X \frac{d^2\mathcal{L}/dX^2}{d\mathcal{L}/dX} \right), \end{aligned} \quad (1.38)$$

where we have used $\text{tr}(-\mathbf{g}^{-1}\mathbf{n})^k = (2X)^k$. Finally, substituting equation (1.38) back into equation (1.37) we arrive at our desired result

$$\det \tilde{\mathbf{G}} = \left(\frac{d\mathcal{L}}{dX} \right)^4 \cdot \det \mathbf{g} \cdot \frac{d\mathcal{L}/dX + 2Xd^2\mathcal{L}/dX^2}{d\mathcal{L}/dX}. \quad (1.39)$$

By assumption, the metric of spacetime is Lorentzian, so that $\det \mathbf{g} < 0$. Therefore, $\tilde{\mathbf{G}}$ is Lorentzian if and only if

$$\frac{d\mathcal{L}/dX}{2Xd^2\mathcal{L}/dX^2 + d\mathcal{L}/dX} > 0. \quad (1.40)$$

When this condition holds everywhere the Leray’s theorem (see P. 251 of Ref. [33] and Ref. [34]) states that the perturbations π on given background $\varphi_0(x)$ propagate causally in metric $\tilde{G}^{\mu\nu}(\varphi_0, \nabla\varphi_0)$. moreover, in this case, the effective metric $\tilde{G}^{\mu\nu}$ determines the characteristics (cone of influence) for k-essence, see e.g. [31, 32]. For *nontrivial*

configurations of k-essence field $\partial_\mu\varphi \neq 0$ and the metric $\tilde{G}^{\mu\nu}$ is generally not conformally equivalent to $g^{\mu\nu}$; hence in this case the characteristics do not coincide with those ones for canonical scalar field the Lagrangian of which depends linearly on the kinetic term X . As we will see in the next section starting from the behavior of small perturbations on a given background, the characteristics determine the *local causal structure* of the space time in every point of the manifold. Hence, the *local causal structure* for the k-essence field is generically different from those one defined by metric $g_{\mu\nu}$. Subsequently we will see that for the coupled system of equations for the gravitational field and k-essence the Cauchy problem is well posed only if the initial conditions are posed on the hypersurface which is space-like with respect to both metrics: $g^{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ (see P. 251 of Ref. [33] and Refs. [34, 31, 35] for details). If you return previously to the hyperbolicity condition (1.40), it is convenient to introduce the following function

$$c_s^2 \equiv \frac{\partial\mathcal{L}/\partial X}{2X\partial^2\mathcal{L}/\partial X^2 + \partial\mathcal{L}/\partial X} \quad (1.41)$$

For the case $X > 0$ plays the role of “speed of sound” for small perturbations [14] propagating in the preferred reference frame, where the background is at rest. It is well known that in the case under consideration there exists an equivalent hydrodynamic description of the system (see for example Refs. [36, 37]) and the hyperbolicity condition (1.40) is equivalent to the requirement of the hydrodynamic stability $c_s^2 > 0$.

1.4 Local causal structure for the k-essence field

Let us consider scalar field φ interacting with external source $J(x)$. The equation of motion for the scalar field is

$$\tilde{G}^{\mu\nu}\nabla_\mu\nabla_\nu\varphi - 2X\frac{\partial^2\mathcal{L}}{\partial\varphi\partial X} + \frac{\partial\mathcal{L}}{\partial\varphi} = -J \quad (1.42)$$

where metric $\tilde{G}^{\mu\nu}$ is given by (1.35). Suppose φ_0 is the background solution of (1.42) in the presence of source $J_0(x)$ and gravitational metric $g_{\mu\nu}(x)$. Let us consider a slightly perturbed solution $\varphi = \varphi_0 + \pi$ of (1.42) with the source $J = J_0 + \delta J$ and the original unperturbed metric $g_{\mu\nu}(x)$. The equation of motion for π is

$$\tilde{G}^{\mu\nu}\nabla_\mu\nabla_\nu\pi - V^\mu\nabla_\mu\pi - \tilde{M}^2\pi = -\delta J, \quad (1.43)$$

where

$$V^\mu(x) \equiv \left(2X\frac{\partial^3\mathcal{L}}{\partial\varphi\partial^2X} + \frac{\partial^2\mathcal{L}}{\partial\varphi\partial X} - \frac{\partial\tilde{G}^{\alpha\beta}}{\partial X}\nabla_\alpha\nabla_\beta\varphi_0 \right) \nabla^\mu\varphi_0, \quad (1.44)$$

and

$$\tilde{M}^2(x) \equiv 2X\frac{\partial^3\mathcal{L}}{\partial\varphi^2\partial X} - \frac{\partial^2\mathcal{L}}{\partial^2\varphi} - \frac{\partial\tilde{G}^{\alpha\beta}}{\partial\varphi}\nabla_\alpha\nabla_\beta\varphi_0. \quad (1.45)$$

Considering the eikonal (or short wavelength) approximation [38] we have

$$\pi(x) = A(x) \exp[i\omega S(x)],$$

where ω is a large dimensionless parameter and the amplitude $A(x)$ is a slowly varying function. In the limit $\omega \rightarrow \infty$ the terms containing no second derivatives, $V^\mu(x) \nabla_\mu \pi$ and $\tilde{M}^2(x) \pi$, become unimportant and (1.43) becomes

$$\tilde{G}^{\mu\nu} \partial_\mu S \partial_\nu S = 0. \quad (1.46)$$

The equation of motion in the eikonal approximation (1.46) is conformally invariant. The surfaces of constant eikonal S (constant phase) correspond to the wave front (characteristic surface) in spacetime. Thus the 1-form $\partial_\mu S$ is orthogonal to the characteristic surface. The influence cone at point P is formed by the propagation vectors N^μ tangential to the characteristic surface $N^\mu \partial_\mu S = 0$ and positive projection on the time direction. Using (1.46) one can chose $N^\mu = \tilde{G}^{\mu\nu} \partial_\nu S$ and verify that this vectors are tangential to the characteristic surface. The metric $\tilde{G}^{\mu\nu}$ has an inverse $\tilde{G}_{\mu\nu}^{-1}$ due to the requirement of hyperbolicity (Lorentzian signature of $\tilde{G}^{\mu\nu}$). Therefore $\partial_\nu S = \tilde{G}_{\mu\nu}^{-1} N^\mu$ and we obtain the equation for the influence cone in the form

$$\tilde{G}_{\mu\nu}^{-1} N^\mu N^\nu = 0.$$

Thus the metric $\tilde{G}_{\mu\nu}^{-1}$ governs the division of acoustic spacetime into past, future and inaccessible “spacelike” regions (or in other words this metric yields the notion of causality). Therefore

$$\begin{aligned} g_{\mu\nu} N^\mu N^\nu &= c_s^2 \left(\frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} \right) (\nabla_\mu \varphi N^\mu)^2 \\ &= (1 - c_s^2) \frac{(\nabla_\mu \varphi N^\mu)^2}{2X}, \end{aligned} \quad (1.47)$$

and if $(\partial^2 \mathcal{L} / \partial X^2) / (\partial \mathcal{L} / \partial X)$ is negative, then $g_{\mu\nu} N^\mu N^\nu < 0$, that is, N^μ is spacelike and the cone of influence on this background is larger than the light cone: the wave front (or signal) velocity is larger then the speed of light. Note that this is a coordinate independent statement.

1.5 Action for perturbations

For π in the spacetime of arbitrary dimension $D > 2$ ³ and neglecting the metric perturbations $\delta g_{\mu\nu}$, there exists conformal transformation where it is really possible to rewrite (1.42) in canonical form. In other words there must exist $\Omega(\varphi_0, X_0)$, such that

$$G^{\mu\nu} = \Omega \tilde{G}^{\mu\nu}. \quad (1.48)$$

and the equation of motion for perturbations π takes a canonical (Klein-Gordon) form

$$G^{\mu\nu} D_\mu D_\nu \pi - M_{\text{eff}}^2 \pi = -\delta I, \quad (1.49)$$

³Note that this method makes sense for the dimensions $D > 2$ only. That happens because in $D = 2$ all metrics are conformally equivalent to $\eta_{\mu\nu}$ and the wave equation is conformally invariant, see e.g. Ref. [33], P. 447.

where D_μ is a covariant derivative with associated with the new metric $G^{\mu\nu}$: $D_\mu G^{\alpha\beta} = 0$. Therefore one can rewrite the equation of motion for the scalar field perturbations and for $D = 4$ in the following form

$$\frac{1}{\sqrt{-G}} \partial_\mu \left(\sqrt{-G} G^{\mu\nu} \partial_\nu \pi \right) - M_{\text{eff}}^2 \pi = -\delta I. \quad (1.50)$$

Note that the equations of motion (1.42) and (1.49) should have the same influence cone structure. The equation (1.49) for the perturbations has exactly the same form as equation for the massive Klein-Gordon field in the curved spacetime. Therefore the metric $G^{\mu\nu}$ describes the “emergent” or “analogue” spacetime where the perturbations live. In particular this means that the action from which one can obtain the equation of motion in the canonical Klein-Gordon form (1.49)

$$S_\pi = \frac{1}{2} \int d^4x \sqrt{-G} \left[-G^{\mu\nu} \partial_\mu \pi \partial_\nu \pi - M_{\text{eff}}^2 \pi^2 + 2\pi \delta I \right], \quad (1.51)$$

where the emergent metric $G^{\mu\nu}$ is the conformally transformed eikonal metric $\tilde{G}^{\mu\nu}$, defined in (1.35),

$$G^{\mu\nu} \equiv \frac{c_s}{(\partial\mathcal{L}/\partial X)^2} \tilde{G}^{\mu\nu} = \frac{c_s}{\partial\mathcal{L}/\partial X} \left[g^{\mu\nu} - \left(\frac{\partial^2 \mathcal{L}/\partial X^2}{\partial\mathcal{L}/\partial X} \right) \nabla^\mu \varphi \nabla^\nu \varphi \right]. \quad (1.52)$$

The inverse metric $G_{\mu\nu}^{-1}$ can be easily calculated using the ansatz $G_{\mu\nu}^{-1} = \alpha g_{\mu\nu} + \beta \nabla_\mu \varphi_0 \nabla_\nu \varphi_0$ and the condition

$$G_{\mu\lambda}^{-1} G^{\lambda\nu} = \delta_\mu^\nu. \quad (1.53)$$

Therefore it is given by the formula

$$G_{\mu\nu}^{-1} = \frac{\partial\mathcal{L}/\partial X}{c_s} \left[g_{\mu\nu} + c_s^2 \left(\frac{\partial^2 \mathcal{L}/\partial X^2}{\partial\mathcal{L}/\partial X} \right) \nabla_\mu \varphi \nabla_\nu \varphi \right]. \quad (1.54)$$

Finally the effective mass is

$$M_{\text{eff}}^2 = \frac{c_s}{(\partial\mathcal{L}/\partial X)^2} \left[2X \frac{\partial^3 \mathcal{L}}{\partial\varphi^2 \partial X} - \frac{\partial^2 \mathcal{L}}{\partial^2 \varphi} - \frac{\partial \tilde{G}^{\alpha\beta}}{\partial\varphi} \nabla_\alpha \nabla_\beta \varphi_0 \right], \quad (1.55)$$

and the effective source for perturbations is given by

$$\delta I = \frac{c_s}{(\partial\mathcal{L}/\partial X)^2} \delta J. \quad (1.56)$$

For the reference we also list the formula

$$\sqrt{-G} \equiv \sqrt{-\det G_{\mu\nu}^{-1}} = \sqrt{-g} \frac{(\partial\mathcal{L}/\partial X)^2}{c_s}. \quad (1.57)$$

Now, using the inverse to $G^{\mu\nu}$ matrix one can define the “emergent” interval

$$dS^2 \equiv G_{\mu\nu}^{-1} dx^\mu dx^\nu, \quad (1.58)$$

which determines the influence cone for small perturbations of *k-essence* on a given background⁴. At first glance it looks like the theory under consideration has emergent bimetric structure. However, this theory is inherently different from the bimetric theories of gravity because the emergent metric refers only to the perturbations of *k-essence* and is due to the non-linearity of the theory, while in the bimetric gravity theories both metrics have fundamental origin and are on the same footing.

The derived above form of the action and of the equation of motion for perturbations is very useful. In particular, it simplifies the stability analysis of the background with respect to the perturbations of arbitrary wavelengths, while the hyperbolicity condition (1.40) guarantees this stability only with respect to the short-wavelength perturbations.

It is important to mention that besides of the usual hyperbolicity condition (1.40) one has to require that $\partial\mathcal{L}/\partial X$ is nowhere vanishes or becomes infinite. The points where $\partial\mathcal{L}/\partial X$ vanishes or diverges, generally correspond to the singularities of the emergent geometry. It follows from equations (1.52) and (1.54) that these singularities are of the true nature and cannot be avoided by the change of the coordinate system. Therefore one can argue that before the singularities are formed the curvature of the emergent spacetime becomes large enough for efficient quantum production of the *k-essence* perturbations which will destroy the classical background and therefore $\partial\mathcal{L}/\partial X$ cannot dynamically change its sign. Hence, if one assumes that at some moment of time the *k-essence* satisfies the null energy condition, that is, $\partial\mathcal{L}/\partial X$ everywhere in the space then this condition can be violated only if one finds the way to pass through the singularity in the emergent geometry with taking into account the quantum production of the perturbations. This doubts the possibility of the smooth crossing of the equation of state $w = -1$. The statements above generalize the results obtained in [39] and re-derived later in different ways in next Chapter in cosmological context.

In deriving (1.51) and (1.49) we have assumed that the *k-essence* is sub-dominant component in producing the gravitational field and consequently have neglected the metric perturbations induced by the scalar field. For *k-essence* dark energy [12, 13] action (1.51) can be used only when *k-essence* is a small fraction of the total energy density of the Universe, in particular, this action is applicable during the stage when the speed of sound of a successful *k-essence* has to be larger than the speed of light [40]. During *k-inflation* [41, 42, 43] the geometry $g_{\mu\nu}$ is determined by the scalar field itself and therefore the induced scalar metric perturbations are of the same order of magnitude as the perturbations of the scalar field. For this case the action for cosmological perturbations was derived in [14], see also [44]. One can expect therefore that this emergent geometry $G^{\mu\nu}$ has a much broader range of applicability and determines the causal structure for perturbations also in the case of other backgrounds, where one cannot neglect the induced metric perturbations.

If the hyperbolicity condition (1.40) is satisfied, then at any given point of spacetime the metric $G_{\mu\nu}^{-1}$ can always be brought to the canonical Minkowski form $\text{diag}(-1, 1, 1, 1)$ by the appropriate coordinate transformation. However, the quadratic forms $g_{\mu\nu}$ and $G_{\mu\nu}^{-1}$ are not positively defined and therefore for a general background there

⁴Note that in order to avoid confusion we will be raising and lowering the indices of tensors by gravitational metric $g^{\mu\nu}$ ($g_{\mu\nu}$) throughout this Chapter.

exist no coordinate system where they are both simultaneously diagonal. In some cases both metrics can be nevertheless simultaneously diagonalized at a given point, so that, e.g. gravitational metric $g_{\mu\nu}$ is equal Minkowski metric and the induced metric $G_{\mu\nu}^{-1}$ is proportional to $\text{diag}(-c_s^2, 1, 1, 1)$, where c_s is the speed of sound (3.12). For instance, in isotropic homogeneous Universe both metrics are always diagonal in the Friedmann coordinate frame.

We conclude this section with the following interesting observation. The effective metric (1.54) can be expressed through the energy momentum tensor (1.33) as

$$G_{\mu\nu}^{-1} = \alpha g_{\mu\nu} + \beta T_{\mu\nu} \quad (1.59)$$

where

$$\alpha = \frac{\partial\mathcal{L}/\partial X}{c_s} - \mathcal{L}c_s \frac{\partial^2\mathcal{L}/\partial X^2}{\partial\mathcal{L}/\partial X} \quad \text{and} \quad \beta = c_s \frac{\partial^2\mathcal{L}/\partial X^2}{\partial\mathcal{L}/\partial X}.$$

As we have pointed out the cosmological perturbations propagate in $G_{\mu\nu}^{-1}$ even if the background field determines the dynamics of the Universe. In this case the energy momentum tensor for the scalar field satisfies the Einstein equations and eventually we can rewrite the effective metric in the following form

$$G_{\mu\nu}^{-1} = \left(\alpha - \frac{\beta}{2}R \right) g_{\mu\nu} + \beta R_{\mu\nu}. \quad (1.60)$$

The ‘‘metric redefinition’’ (1.60) (i.e. $g_{\mu\nu} \leftrightarrow G_{\mu\nu}^{-1}$) does not change the light cone and hence the *local causality* only in the Ricci flat $R_{\mu\nu} = 0$ spacetimes. However, neither in the matter dominated Universe nor during inflation the *local causal structures* determined by $g_{\mu\nu}$ and $G_{\mu\nu}^{-1}$ are equivalent.

1.6 Causality conditions on nontrivial backgrounds

The causality paradox is avoided when superluminal signals propagate in the background which breaks the Lorentz symmetry. The observers cannot send a message to themselves in the past.

In this section we discuss the causality issue for superluminal propagation of perturbations on some nontrivial backgrounds, in particular, in Minkowski spacetime with the scalar field, in Friedmann Universe and for black hole surrounded by the accreting scalar field.

First, we would like to recall a well-known paradox sometimes called ‘‘tachyonic anti-telephone’’ [45] arising in the presence of the superluminal hypothetical particles *tachyons* possessing unbounded velocity $c_{tachyon} > 1$. In this case we could send a message to our own past. Indeed, let us consider some observer, who is at rest with respect to his reference frame and sends a tachyon signal to an astronaut in his spacecraft. In turn, after receiving this signal, the astronaut can communicate back sending the tachyon signal. As this signal propagates the astronaut proper time t' grows. However, if the speed of the spacecraft is larger than $1/c_{tachyon}$, then the tachyon signal propagates *backward in time* in the original

rest frame of the observer. Thus, the observers can in principle send information from “their future” to “their past”. It is clear that such situation is unacceptable from the physical point of view.

Now let us turn to the case of the Minkowski spacetime filled with the scalar field, which allows the “superluminal” propagation of perturbations in its background. For simplicity we consider a homogeneous time dependant field $\varphi_0(t)$. Its “velocity” $\partial_\mu\varphi$ is directed along the timelike vector, $u^\mu = (-1, 0, 0, 0)$. Why does the paradox above not arise here? This is because the superluminal propagation of the signals is possible only in the presence of nontrivial background of scalar field which serves as the *aether* for sonic perturbations. The *aether* selects the preferred reference frame and clearly the equation of motion for acoustic perturbations is not invariant under the Lorentz transformations unless $c_s = 1$. In the moving frame of the astronaut the equation for perturbations has more complicated form than in the rest frame and the analysis of its solutions is more involved. However, keeping in mind that *k-essence* signals propagate along the characteristics which are coordinate independent hypersurfaces in the spacetime we can study the propagation of sonic perturbations, caused by the astronaut, in the rest frame of the aether and easily find that the signal propagates always *forward in time* in this frame. Hence no closed causal curves can arise here.

We would like to make a remark concerning the notion of “future-” and “past” directed signals [15]. The notion of past and future is determined by the past and future cones in the spacetime and has nothing to do with a particular choice of coordinates. Thus, the signals, which are future-directed in the rest-frame remain the future-directed also in a fast-moving spacecraft, in spite of the fact that this would correspond to the decreasing time coordinate t' . The confusion arises because of a poor choice of coordinates, when decreasing t' correspond to future-directed signals and vice versa [15].

Another potentially confusing issue is related to the question which particular velocity must be associated with the speed of signal propagation, namely, phase, group or front velocity [15]. It is important remember that neither group nor phase velocities have any direct relation with the causal structure of the spacetime. Indeed the characteristic surfaces of the partial differential equations describe the propagation of the wavefront. This front velocity coincides with the phase velocity only in the limit of the short wavelength perturbations. Generally the wavefront corresponds to the discontinuity of the second derivatives and therefore it moves “off-shell” (a more detailed discussion can be found in e.g. [46]). The group velocity can be less or even larger than the wavefront velocity. One can recall the simple examples of the canonical free scalar field theories: for normal scalar fields the mass squared, $m^2 > 0$, is positive and the phase velocity is larger than c while the group velocity is smaller than c ; on the other hand for tachyons ($m^2 < 0$) the situation is opposite. Thus, if the group velocity were the speed of the signal transfer, one could easily build an opportune time-machine using canonical scalar field with negative mass squared, $m^2 < 0$ [15]. This, however, is impossible because the causal structure in both cases ($m^2 > 0$ and $m^2 < 0$) is governed by the same *light* cones.

To prove the absence of the closed causal curves (CCC) in those known situations where the superluminal propagation is possible, we use the theorem from Ref. [33] (see p. 198): *A spacetime $(\mathcal{M}, g_{\mu\nu})$ is stably causal if and only if there exists a differentiable*

function f on \mathcal{M} such that $\nabla^\mu f$ is a future directed timelike vector field. Here \mathcal{M} is a manifold and $g_{\mu\nu}$ is metric with Lorentzian signature. Note, that the notion of *stable causality* implies that the spacetime $(\mathcal{M}, g_{\mu\nu})$ possesses no CCCs and thus no causal paradoxes can arise in this case. The theorem above has a kinematic origin and does not rely on the dynamical equations. In the case of the effective acoustic geometry the acoustic metric $G_{\mu\nu}^{-1}$ plays the role of $g_{\mu\nu}$ and the function f serves as the “global time function” of the emergent spacetime $(\mathcal{M}, G_{\mu\nu}^{-1})$. For example, in the Minkowski spacetime filled with the scalar field “ether” one can take the Minkowski time t of the rest frame, where this field is homogeneous, as the global time function. Then we have

$$G^{\mu\nu} \partial_\mu t \partial_\nu t = \frac{c_s}{\partial \mathcal{L} / \partial X} g^{00} \left(1 + 2X \frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} \right) = \frac{g^{00}}{(\partial \mathcal{L} / \partial X) c_s}. \quad (1.61)$$

Even for those cases when the speed of perturbations can exceed the speed of light, $c_s > 1$, this expression is positive, provided that $\mathcal{L}_{,X} > 0$, and the hyperbolicity condition (1.40) is satisfied. Thus $\partial_\mu t$ is timelike with respect to the effective metric $G_{\mu\nu}^{-1}$; hence the conditions of the theorem above are met and no CCCs can exist.

Now we consider the Minkowski spacetime with an arbitrary inhomogeneous background $\varphi_0(x)$ and verify under which conditions one can find a global time t for both geometries $g_{\mu\nu}$ and $G_{\mu\nu}^{-1}$ and thus guarantee the absence of CCCs. Let us take the Minkowski t , $\eta^{\mu\nu} \partial_\mu t \partial_\nu t = -1$, and check whether this time can also be used as a global time for $G_{\mu\nu}^{-1}$. We have

$$\begin{aligned} G^{\mu\nu} \partial_\mu t \partial_\nu t &= -\frac{c_s}{\partial \mathcal{L} / \partial X} \left[1 + \left(\frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} \right) (\partial_\mu t \nabla^\mu \varphi_0)^2 \right] \\ &= -\frac{c_s}{\partial \mathcal{L} / \partial X} \left[1 + \left(\frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} \right) \dot{\varphi}_0^2 \right], \end{aligned} \quad (1.62)$$

and assuming that $c_s > 0$, $\mathcal{L}_{,X} > 0$ we arrive to the conclusion that t is a global time for emergent spacetime provided

$$1 + \left(\frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} \right) (\dot{\varphi}_0(x^\mu))^2 > 0, \quad (1.63)$$

holds everywhere on the manifold \mathcal{M} . This inequality is obviously always satisfied in the subluminal case. It can be rewritten in the following form

$$\frac{1}{c_s^2} \left[1 + c_s^2 \left(\frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} \right) \left(\vec{\nabla} \varphi_0(x^\mu) \right)^2 \right] > 0, \quad (1.64)$$

from where it is obvious that, if the spatial derivatives are sufficiently small then this condition can also be satisfied even if $c_s > 1$. Note that the breaking of the above condition for some background field configuration $\varphi_0(x)$ does not automatically mean the appearance of the CCCs. This just tells us that the time coordinate t cannot be used as the global time coordinate. However it does not exclude the possibility that there exists another function serving as the global time. Only, if one can prove that such global time for both metrics does not exist at all, then there arise causal paradoxes.

In the case of the Friedmann Universe with “superluminal” scalar field, one can choose the cosmological time t as the global time function and then we again arrive to (1.61), thus concluding that there exist no CCCs. In particular, the *k-essence* models, where the superluminal propagation is the generic property of the fluctuations during some stage of expansion of the Universe [40], *do not lead to causal paradoxes*.

The absence of the closed causal curves in the Friedmann Universe with *k-essence* can also be seen directly by calculating of the “effective” line element (1.58). Taking into account that the Friedmann metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\mathbf{x}^2, \quad (1.65)$$

we find that the line element (1.58), corresponding to the effective acoustic metric, is

$$dS^2 = G_{\mu\nu}^{-1} dx^\mu dx^\nu = \frac{\partial\mathcal{L}/\partial X}{c_s} (-c_s^2 dt^2 + a^2(t) d\mathbf{x}^2). \quad (1.66)$$

The theory under consideration is generally covariant. After making redefinitions,

$$\sqrt{(\partial\mathcal{L}/\partial X) c_s} dt \rightarrow dt,$$

and,

$$a^2(t) (\partial\mathcal{L}/\partial X) / c_s \rightarrow a^2(t),$$

the line element (1.66) reduces to the interval for the Friedmann Universe (1.65), where obviously no causality violation can occur. Thus we conclude that both the *k-essence* [12, 13] and the “superluminal” inflation with large gravity waves [43] are completely safe and legitimate on the side of causality.

When $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0$ is negative everywhere in the spacetime the background field itself can be used as the global time function. Indeed for general gravitational background $g_{\mu\nu}$ and $c_s > 0$, $(\partial\mathcal{L}/\partial X) > 0$ we have

$$g^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0 < 0 \quad \text{and} \quad G^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0 = -\frac{2X}{(\partial\mathcal{L}/\partial X) c_s} < 0, \quad (1.67)$$

and due to the fact that $X > 0$ the sign in front $\nabla^\mu\varphi_0$ can be chosen so that the vector $\nabla^\mu\varphi_0$ is always future directed on \mathcal{M} . Therefore $\varphi_0(x)$ or $(-\varphi_0(x))$ if necessary) can serve as a global time in both spacetimes $(\mathcal{M}, g_{\mu\nu})$ and $(\mathcal{M}, G_{\mu\nu}^{-1})$, and no causal paradoxes arise.

In all examples above we have considered the “superluminal” acoustic metric. Thus, if there exist no CCCs in $(\mathcal{M}, G_{\mu\nu}^{-1})$ then there are no CCCs with respect to metric $g_{\mu\nu}$ because acoustic cone is larger than the light cone. It may happen that in some cases it is not enough to prove that there no CCCs separately in $(\mathcal{M}, G_{\mu\nu}^{-1})$ and $(\mathcal{M}, g_{\mu\nu})$ and one has to use the maximal cone or introduce an artificial cone [47] encompassing all cones arising in the problem. It is interesting to note that, if the *k-essence* realizes both “superluminal” and subluminal speed of sound in the different regions of the manifold, then there exist hypersurface where the *k-essence* metric is conformally equivalent to the $g_{\mu\nu}$ and one can smoothly glue the maximal cones together everywhere on \mathcal{M} . After

that one can consider a new “artificial metric” $G_{\mu\nu}^{\Sigma}$ as determining the complete causal structure of the manifold.

We would like to point out that although the theorem on stable causality allowed us to prove that there is no causal paradoxes in those cases we considered above, it is not guaranteed that CCCs cannot arise for some other backgrounds. Indeed, in [23] the authors have found some configurations of fields possessing CCCs: one for the scalar field with non-canonical kinetic term and another for the “wrong”-signed Euler-Heisenberg system. In both cases the small perturbations propagate superluminally on rather non-trivial backgrounds [15].

1.7 Conditions for the well posed Cauchy problem

Using the theorem on stable causality we have proved that the “superluminal” *k-essence* does not lead to any causal paradoxes for cosmological solutions and for accretion onto black hole. However, the consideration above is of a kinematic nature and it does not deal with the question how to pose the Cauchy problem for the background field φ_0 and its perturbations π .

It was pointed out in [23] that in the reference frame of the spacecraft moving with respect to nontrivial background, where $c_s > 1$, with the speed $v = 1/c_s$ the Cauchy problem for small perturbations π is ill posed. This happens because the hypersurface of the constant proper time t' of the astronaut is a null-like with respect to the acoustic metric $G_{\mu\nu}^{-1}$. Hence $t' = \text{const}$ is tangential to the characteristic surface (or sonic cone see section 1.4) and cannot be used to formulate the Cauchy problem for perturbations which “live” in this acoustic metric. Intuitively this happens because the perturbations propagate instantaneously with respect to the hypersurface $t' = \text{const}$. Moreover, for $v > 1/c_s$, the sonic cone deeps below the surface $t' = \text{const}$ and in the spacetimes of dimension $D > 2$ the Cauchy problem is ill posed as well because there always exist two directions along which the perturbations propagate “instantaneously” in time t' . This tells us that not every imaginable configuration of the background can be realized as the result of evolution of the system with the well formulated Cauchy problem and hence not every set of initial conditions for the scalar field is allowed.

In this Section we will find under which restrictions on the initial configuration of the scalar field the Cauchy problem for equation (1.34) is well-posed. For this purpose it is more convenient not to split the scalar field into background and perturbations and consider instead the total value of the field $\varphi = \varphi_0 + \pi$. The *k-essence* field interacts with gravity and therefore for consistency one has to consider the coupled system of equations for the gravitational metric $g_{\mu\nu}$ and the *k-essence* field φ . In this case the Cauchy problem is well posed only if the initial data are set up on a hypersurface Σ which is simultaneously spacelike in both metrics: $g_{\mu\nu}$ and $G_{\mu\nu}^{-1}$ (for details see P. 251 of Ref. [33] and Refs. [34, 31, 35]). We will work in the synchronous coordinate system, where the metric takes the form

$$ds^2 = -dt^2 + \gamma_{ik} dx^i dx^k, \quad (1.68)$$

and select the spacelike in $g_{\mu\nu}$ hypersurface Σ to be a constant time hypersurface $t = t_0$.

The 1-form $\partial_\mu t$ vanishes on any vector R^μ tangential to Σ : $R^\mu \partial_\mu t = 0$. This 1-form is timelike with respect to the gravitational metric $g_{\mu\nu}$, that is $g^{\mu\nu} \partial_\mu t \partial_\nu t < 0$. In case when Lagrangian for *k-essence* depends at maximum on the first derivatives of scalar field the initial conditions which completely specify the unambiguous solution of the equations of motion are the initial field configuration $\varphi(\mathbf{x})$ and its first time derivative $\dot{\varphi}(\mathbf{x}) \equiv (-g^{\mu\nu} \partial_\mu t \partial_\nu \varphi)_\Sigma$. Given these initial conditions one can calculate the metric $G_{\mu\nu}^{-1}$ and consequently the influence cone at every point on Σ . First we have to require that for a given set of initial data the hyperbolicity condition (1.40) is not violated. This imposes the following restriction on the allowed initial values $\varphi(\mathbf{x})$ and $\dot{\varphi}(\mathbf{x})$:

$$c_s^{-2} = 1 + \left[(\dot{\varphi}(\mathbf{x}))^2 - \left(\vec{\nabla} \varphi(\mathbf{x}) \right)^2 \right] \frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} > 0, \quad (1.69)$$

where we have denoted $\left(\vec{\nabla} \varphi(\mathbf{x}) \right)^2 = \gamma^{ik} \partial_i \varphi \partial_k \varphi$. In addition we have to require that the hypersurface Σ is spacelike also with respect to emergent metric $G^{\mu\nu}$, that is, for every vector R^μ , tangential to Σ , we have $G_{\mu\nu}^{-1} R^\mu R^\nu > 0$, or

$$1 + c_s^2 \left(\vec{\nabla} \varphi(\mathbf{x}) \right)^2 \frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} > 0. \quad (1.70)$$

If at some point on Σ the vector R^μ becomes null-like with respect to $G_{\mu\nu}^{-1}$, that is, $G_{\mu\nu}^{-1} R^\mu R^\nu = 0$, the signals propagate instantaneously and one cannot guarantee the continuous dependence on the initial data or even the existence and uniqueness of the solution, see e.g. [48]. Using (3.12) the last inequality can be rewritten as

$$c_s^2 \left(1 + (\dot{\varphi}(\mathbf{x}))^2 \frac{\partial^2 \mathcal{L} / \partial X^2}{\partial \mathcal{L} / \partial X} \right) > 0. \quad (1.71)$$

Therefore, given Lagrangian $\mathcal{L}(\varphi, X)$ and hypersurface Σ one has to restrict the initial data $(\varphi(\mathbf{x}), \dot{\varphi}(\mathbf{x}))$ by inequalities (1.69) and (1.70) (or equivalently (1.71)), to have a well posed Cauchy problem. The condition (1.71) is always satisfied in the subluminal case for which $(\partial^2 \mathcal{L} / \partial X^2) / (\partial \mathcal{L} / \partial X) \geq 0$. In addition, we conclude that, if these conditions are satisfied everywhere on the manifold \mathcal{M} and the selected synchronous frame is nonsingular in \mathcal{M} , then time t plays the role of global time and in accordance with the theorem about stable causality no causal paradoxes arise in this case.

As a concrete application of the conditions derived let us find which restrictions should satisfy the admissible initial conditions for the low energy effective field theory with Lagrangian $\mathcal{L}(X) \simeq X - X^2/\mu^4 + \dots$, where μ is a cut off scale Ref. [23]. In this case (1.71) imply that not only $X \ll \mu^4$, but also $(\dot{\varphi}(\mathbf{x}))^2 \ll \mu^4$ and $\left(\vec{\nabla} \varphi(\mathbf{x}) \right)^2 \ll \mu^4$. Note that these restrictions can be rewritten in the Lorentz invariant way: for example the first condition takes the form $(g^{\mu\nu} \partial_\mu t \partial_\nu \varphi)^2 \ll \mu^4$.

Finally let us note that even well-posed Cauchy problem cannot guarantee the global existence of the unique solution for nonlinear system of the equations of motion: for example, the solution can develop caustics [49] or can become multi-valued [50].

Chapter 2

Unified Dark Matter models in *k-essence* Cosmology

2.1 Introduction

The confidence region of cosmological parameters emerging from the analysis of data from type Ia Supernovae (SNIa), Cosmic Microwave Background (CMB) anisotropies and the large scale structure of the Universe, suggests that two dark components govern the dynamics of the present Universe. These components are the Dark Matter (DM), responsible for structure formation, and an additional Dark Energy (DE) component that drives the cosmic acceleration observed at present (see Fig 2.1).

In this Chapter we focus on Unified models of Dark Matter and dark energy (UDM) that can provide an alternative to our interpretation of the nature of the dark components of our Universe. These models have the advantage over the DM + DE models (e.g. Λ CDM) that one can describe the dynamics of the Universe with a single dark fluid which triggers the accelerated expansion at late times and is also the one which has to cluster in order to produce the structures we see today. However, the viability of UDM models strongly depends on the value of the effective speed of sound c_s [53, 14, 44], which has to be small enough to allow structure formation [54, 55, 56] and to reproduce the observed pattern of CMB temperature anisotropies [53, 57, 54, 58, 59].

Several adiabatic or, equivalently, purely kinetic models have been investigated in the literature. For example, the generalized Chaplygin gas ([60, 61, 62] (see also [63]), the Scherrer [64] and generalized Scherrer [57] solutions, the single dark perfect fluid with a simple 2-parameter barotropic equation of state [65], or the homogeneous scalar field deduced from the galactic halo spacetimes [66].

Moreover, one can build up scalar field models for which the constraint that the Lagrangian is constant along the classical trajectories allows to describe a UDM fluid [57] (see also Ref. [67], for a different approach). Alternative approaches to the unification of DM and DE have been proposed in Ref. [68], in the frame of supersymmetry, and in Ref. [69], in connection with the solution of the strong CP problem.

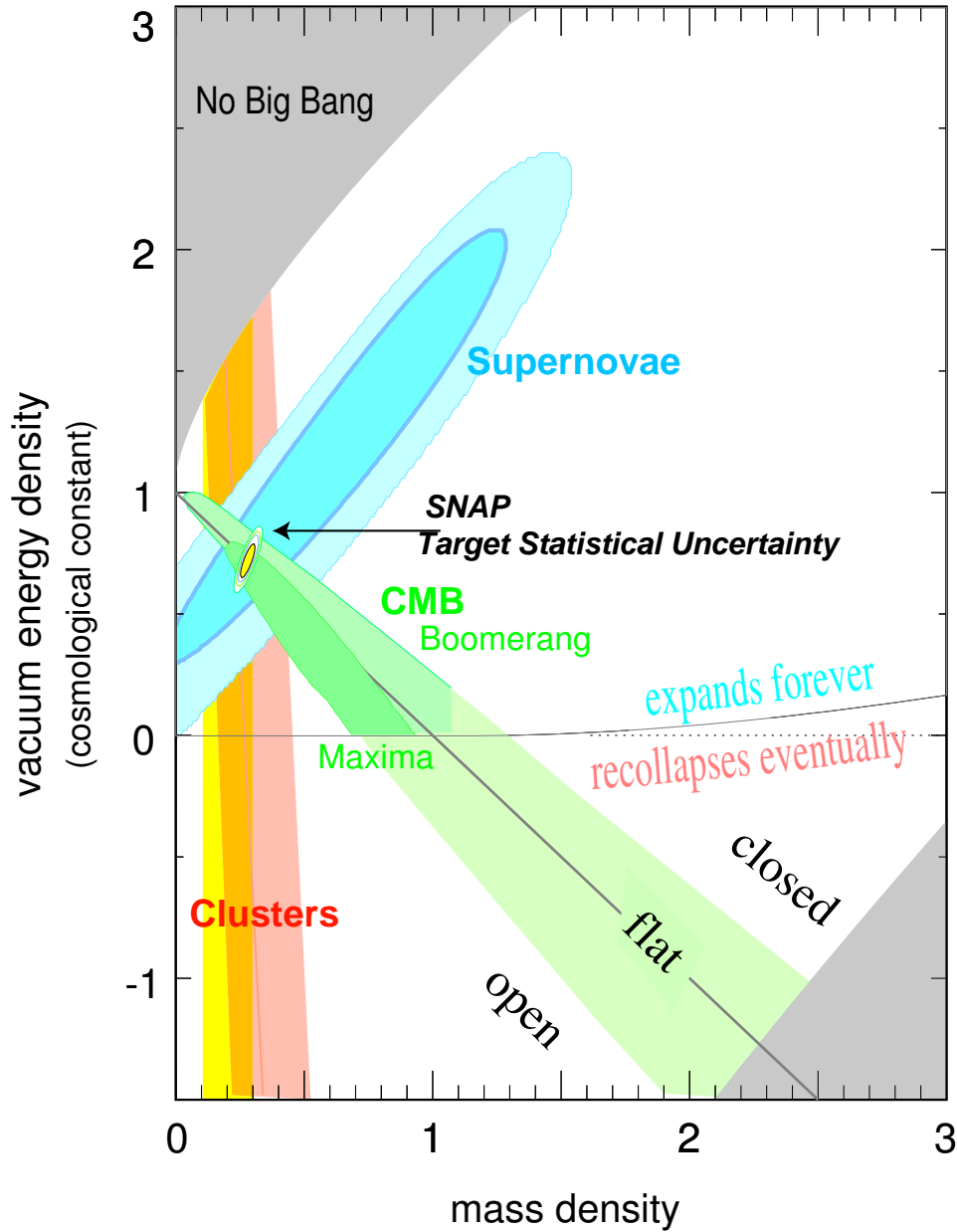


Figure 2.1: The $\Omega_m^{(0)}-\Omega_\Lambda^{(0)}$ confidence regions constrained from the observations of SN Ia, CMB and galaxy clustering. We also show the expected confidence region from a SNAP satellite for a flat Universe with $\Omega_m^{(0)} \approx 0.27$. From Ref. [51] (see also Ref. [52]).

One could easily reinterpret UDM models based on a scalar field Lagrangian in terms of – generally non-adiabatic – fluids [36, 37]. In this Chapter we consider the general Lagrangian of *k-essence* models and classify them through two variables connected to the fluid equation of state parameter w_κ . This allows to find attractor solutions around which the scalar field is able to describe a mixture of dark matter and cosmological constant-like dark energy, an example being Scherrer’s [64] purely kinetic model. Next, we impose that the Lagrangian of the scalar field is constant, i.e. that $p_\kappa = -\Lambda$, where Λ is the cosmological constant, along suitable solutions of the equation of motion, and find a general class of *k-essence* models whose attractors directly describe a unified dark matter/dark energy fluid. While the simplest of such models, based on a neutral scalar field with canonical kinetic term, unavoidably leads to an effective speed of sound c_s which equals the speed of light, thereby inhibiting the growth of matter perturbations, we find a more general class of non-canonical (*k-essence*) models which allow for the possibility that $c_s \ll 1$ whenever matter dominates.

The plan of this Chapter is as follows. In Section 2.2 we introduce the general class of *k-essence* models and we propose a new approach to look for attractor solutions. In Section 2.3 we apply our formalism to obtain the attractors for the purely kinetic case. Moreover for completeness in Section 1.3.5 we provide the spherical collapse top-hat solution for UDM models based on purely kinetic scalar-field Lagrangians, which allow to connect the cosmological solutions to the static configurations (see Chapter 3).

In Section 2.4 we generalize our model giving general prescriptions [Eqs. (2.120) and (2.122)] which allow to obtain unified models where the dark matter and a cosmological constant-like dark energy are described by a single scalar field along its attractor solutions

2.2 Cosmological solutions in *k-essence*

Let us reconsider the following action

$$S = S_G + S_\varphi + S_m = \int d^4x \sqrt{-g} \left[\frac{R}{2} + \mathcal{L}(\varphi, X) \right] + S_m \quad (2.1)$$

where

If X is time-like S_φ describes a perfect fluid with

$$T_{\mu\nu}^\varphi = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} , \quad (2.2)$$

where

$$\mathcal{L} = p(\varphi, X), \quad (2.3)$$

is the pressure,

$$\rho = \rho(\varphi, X) \equiv 2X \frac{\partial p(\varphi, X)}{\partial X} - p(\varphi, X) \quad (2.4)$$

is the energy density and the four-velocity reads

$$u_\mu = \frac{\nabla_\mu \varphi}{\sqrt{2X}}. \quad (2.5)$$

Now we assume a flat, homogeneous Friedmann-Robertson-Walker background metric i.e.

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j = a(\eta)^2 (-d\eta^2 + \delta_{ij} dx^i dx^j), \quad (2.6)$$

where $a(t)$ is the scale factor, δ_{ij} denotes the unit tensor and $d\eta = dt/a$ is the conformal time. In such a case, if we neglect the baryonic matter, the background evolution of the Universe is characterized completely by the following equations

$$H^2 = \frac{1}{3}(\rho + \rho_B), \quad (2.7)$$

$$\dot{H} = -\frac{1}{2}(p + \rho + p_B + \rho_B), \quad (2.8)$$

where the dot denotes differentiation w.r.t. the cosmic time t and where ρ_B and p_B are the energy density and the pressure of the barionic matter and/or the radiation, respectively. On the background $X = \frac{1}{2}\dot{\varphi}^2$ and the equation of motion for the homogeneous mode $\varphi(t)$ becomes

$$\left(\frac{\partial p}{\partial X} + 2X \frac{\partial^2 p}{\partial X^2} \right) \ddot{\varphi} + \frac{\partial p}{\partial X} (3H\dot{\varphi}) + \frac{\partial^2 p}{\partial \varphi \partial X} \dot{\varphi}^2 - \frac{\partial p}{\partial \varphi} = 0. \quad (2.9)$$

As already explained in the previous Chapter, the *k-essence* equation of state $w \equiv p/\rho$ is just

$$w = \frac{p}{2X \frac{\partial p}{\partial X} - p}, \quad (2.10)$$

while the effective speed of sound, which is the quantity relevant for the growth of perturbations, reads [14, 44]

$$c_s^2 \equiv \frac{(\partial p / \partial X)}{(\partial \rho / \partial X)} = \frac{\frac{\partial p}{\partial X}}{\frac{\partial p}{\partial X} + 2X \frac{\partial^2 p}{\partial X^2}}. \quad (2.11)$$

Moreover, one could easily reinterpret *k-essence* models based on a scalar field Lagrangian in terms of – generally non-adiabatic – fluids [36, 37].

2.2.1 Lagrangian of the type $\mathcal{L}(\varphi, X) = f(\varphi)g(X)$

Let us assume that the scalar field Lagrangian depends separately on X and φ , i.e. that it can be written in the form

$$p(\varphi, X) = f(\varphi)g(X), \quad (2.12)$$

then Eq. (4.6) becomes

$$\rho = f(\varphi) \left[2X \frac{dg(X)}{dX} - g(X) \right]. \quad (2.13)$$

Notice that the requirement of having a positive energy density imposes a constraint on the function g , namely,

$$2X \frac{dg}{dX} > g, \quad (2.14)$$

having assumed $f > 0$. Eq. (2.9) can be rewritten as follows

$$\frac{1}{f} \frac{df}{dN} = \lambda(N, X, f) \quad (2.15)$$

$$\left(\frac{\partial g}{\partial X} + 2X \frac{\partial^2 g}{\partial X^2} \right) dX + \left[3 \left(2X \frac{\partial g}{\partial X} \right) + \lambda(N, X, f(N, X)) \left(2X \frac{\partial g}{\partial X} - g \right) \right] dN = 0, \quad (2.16)$$

where $dN = da/a$. Eq. (2.15a) defines the parameter λ as a generic function of N (also through X and f); the same parameter λ in Eq. (2.15b) is considered as a function of both N and X (also through f).

Now we want to find the set of scalar field trajectories where Eq. (2.15b) defines an exact differential form. To this aim, first of all we have to study the differential form

$$P(X, N) dX + Q(X, N) dN = 0 \quad (2.17)$$

One possible way is to search for an integral factor I , explicit function of N , which makes exact the differential form (2.17). In our situation $P(X, N) = P(X)$, thus $I(N)$ is just

$$\frac{dI}{I} = \frac{\frac{\partial Q(X, N)}{\partial X}}{P(X)} dN. \quad (2.18)$$

In this case we have to impose the integrability condition

$$\frac{\partial Q(X, N)}{\partial X} = \alpha(N) P(X) \quad (2.19)$$

so that

$$I(N) = \exp \int^N dN' \alpha(N') \quad (2.20)$$

depends only on N .

Using the explicit expressions of $Q(X, N)$ and $P(X)$ the condition (2.19) becomes

$$3 \frac{\partial \left(2X \frac{\partial g}{\partial X} \right)}{\partial X} + \frac{\partial \lambda}{\partial X} \left(2X \frac{\partial g}{\partial X} - g \right) + (\lambda - \alpha) \frac{\partial \left(2X \frac{\partial g}{\partial X} - g \right)}{\partial X} = 0. \quad (2.21)$$

It is easy to see that $\lambda - \alpha$ is a function of (at least) X ; then, defining $G(X) \equiv \lambda - \alpha$, Eq. (2.21) becomes

$$3 \frac{\partial \left(2X \frac{\partial g}{\partial X} \right)}{\partial X} - \frac{\partial G}{\partial X} \left(2X \frac{\partial g}{\partial X} - g \right) - G \frac{\partial \left(2X \frac{\partial g}{\partial X} - g \right)}{\partial X} = 0 \quad (2.22)$$

which can be trivially integrated to give

$$3 \left(2X \frac{\partial g}{\partial X} \right) + K = G \left(2X \frac{\partial g}{\partial X} - g \right) \quad (2.23)$$

with K a generic constant. Without loss of generality we can set $K = 0$ so that

$$\alpha - \lambda = G = 3(w + 1). \quad (2.24)$$

Replacing Eq. (2.23) in Eq. (2.15b) we find

$$\left(\frac{\partial g}{\partial X} + 2X \frac{\partial^2 g}{\partial X^2} \right) dX + \alpha(N) \left(2X \frac{\partial g}{\partial X} - g \right) dN = 0 \quad (2.25)$$

Multiplying both sides by $I(N)$ we finally obtain

$$d \left[\left(2X \frac{\partial g}{\partial X} - g \right) e^{\int^N dN' \alpha(N')} \right] = 0 \quad (2.26)$$

whose solution is

$$\left(2X \frac{\partial g}{\partial X} - g \right) = \bar{K} e^{-\int^N dN' \alpha(N')} \quad (2.27)$$

where \bar{K} is a new integration constant. Using the general equation (2.27) we can express the energy density as

$$\rho_\kappa = \bar{K} e^{-\int^N dN' (\alpha(N') - \lambda(N'))} = \bar{K} e^{-3 \int^N dN' (w_\kappa(N') + 1)}. \quad (2.28)$$

If $\lambda \rightarrow 0$ and $\alpha \rightarrow 0$ then $w \rightarrow -1$ and $\bar{K} f \rightarrow \text{const.}$. Therefore, in such a case the energy density ρ tends to a constant value

$$\rho_0 \propto e^{-\int^{N_0} dN' \alpha(N')}. \quad (2.29)$$

For $\lambda = 0$ the Lagrangian \mathcal{L} (i.e. the pressure p) depends only on X , that is we are obtaining the equations that describe the purely kinetic model i.e. when $\mathcal{L} = \mathcal{L}(X)$.

It is interesting to note that, if $\alpha \geq 0$ the term $\exp\left(-\int^N dN' \alpha(N')\right)$ determines the value of ρ_0 . In order to have $\rho > 0$ we have to require $\bar{K} > 0$.

Before concluding this section it is worth to make some comments on w and c_s^2 . First of all, if we impose the conditions $1 \leq w$ and $c_s^2 \geq 0$, in terms of α and w , or, equivalently, of α and λ , the effective speed of sound, Eq. (3.12), reads

$$c_s^2 = -\frac{(w+1)}{2\alpha} \frac{d \ln X}{dN} = -\frac{\alpha - \lambda}{6\alpha} \frac{d \ln X}{dN} \geq 0. \quad (2.30)$$

Thus, in purely kinetic models ($\lambda = 0$), we get

$$\frac{1}{\alpha} \frac{d \ln X}{dN} \leq 0. \quad (2.31)$$

Therefore if $\alpha > 0$, X can only decrease in time down to its minimum value.

2.2.2 Study of the solutions for $\alpha = 0$: ”scaling *k-essence*”

If X is constant ($dX/dN = 0$) then, from Eq. (2.21), $\alpha = 0$. In this situation we have that

$$\frac{1}{f} \frac{df}{dN} = \frac{1}{N} \frac{1}{f} \frac{df}{dt} = \lambda = -3(w + 1) \quad (2.32)$$

is constant because w is function only of X .

Now if we consider the case in which the Universe is dominated by a fluid with constant equation of state w_B then

$$H = \dot{N} \sim 2/[3(w_B + 1)t] \quad (2.33)$$

and Eq. (2.32) becomes

$$d \ln f / d \ln t \sim -2(w + 1)/(w_B + 1) . \quad (2.34)$$

Therefore we get

$$f \sim t^{-2\frac{w+1}{w_B+1}} \sim \varphi^{-2\frac{w+1}{w_B+1}} \quad (2.35)$$

because if $\dot{\varphi}$ is constant then

$$\varphi \sim \sqrt{2Xt} . \quad (2.36)$$

In other words, we have recovered, although in a more general way, the result of Ref. [10, 11]. These models have been dubbed ”scaling *k-essence* ” (see also Ref. [70]). In such a case, the equation of state w can be written as

$$w = \beta(w_B + 1)/2 - 1 , \quad (2.37)$$

where $f = \varphi^{-\beta}$.

If $w_B = w$ we have only the *k-essence* as background and we get $\beta = 2$ (see for example Refs. [12, 13] and [70]).

Using the latter approach, we can see that if $f = \varphi^{-\beta}$ and if the background of the Universe is dominated by a fluid with constant equation of state w_B , whenever $\alpha \rightarrow 0$, we have that all the viable functions $g(X)$ have the solutions of motion that converge to the scaling solution (a ”viable $g(X)$ ” is a function that does not have a diverging speed of sound c_s^2).

Otherwise if $\alpha \rightarrow c > 0$ for $N \rightarrow +\infty$ (i.e. when α converges an opportune positive constant), we have $2X \frac{dg}{dX} - g \rightarrow 0$ and we get three possible solutions:

- i) if $g \rightarrow \text{const.}$ we have $w \rightarrow \infty$, which is not acceptable;
- ii) if $g \rightarrow 0$ and $w \rightarrow \text{const.} \neq 0$ we get $w \rightarrow c_s^2 > 0$ [71]; this solution cannot be used to describe an accelerated Universe;
- iii) if $g \rightarrow 0$ and $w \rightarrow 0$ we get $0 \leq c_s^2 \leq 1$; obviously, also in this case the Universe cannot accelerate.

Now, if $\alpha \neq 0$ and $f = \varphi^{-\beta}$ we have that

$$\alpha = 3(w + 1) - \frac{3}{2}\beta(w_B + 1)\frac{\sqrt{2X}t}{\varphi}. \quad (2.38)$$

Therefore, starting from Eq. (2.21), we note that when $\alpha \rightarrow 0$ we have that $2X \frac{dg}{dX} - g \rightarrow \text{const.} \neq 0$ and, consequently, X must be necessarily a constant. Indeed, starting from the equation $2X \frac{dg}{dX} - g = \text{const.}$ we can have two possibilities: either X is constant or we need resolve directly this differential equation. If we solve the differential equation we obtain immediately that $g = b\sqrt{2X} + \text{const.}$. In this case it easy to see that speed of sound c_s^2 is diverging [72].

Therefore, if $g \neq b\sqrt{2X} + \text{const.}$, finally we can conclude that the solution of the equation of motion converges in to the scaling solution.

Obviously it is necessary to use a oportune $g(X)$ such that for $N \rightarrow +\infty$ we have that $\alpha \rightarrow 0$. In Ref [10, 11] the author use properly this property in order to derived a condition for the existence of tracker solutions for the system of matter/radiation and a scalar field with only non-canonical kinetic terms.

Moreover, in literature, there are a subclass of *k-essence* models Ref. [12, 13] feature a tracker behavior during radiation domination, and a cosmological constant behavior shortly after the transition to matter domination. Indeed these models have $\beta = 2$. In particular, as long as this transition seems to occur generically for purely dynamical reasons, these models are claimed to solve the cosmic coincidence problem without fine-tuning. Subsequently, it was realized that the latter models [12, 13] have too small a basin of attraction in the radiation era [73].

2.3 Study of the attractors for UDM with purely kinetic Lagrangians

In this section we want to make a general study of the attractor solutions of our model in the purely kinetic case, i.e. for $\lambda = 0$.

First of all let us stress that starting from the barotropic or adiabatic equation of state $p = p(\rho)$ we can describe through a purely kinetic *k-essence* Lagrangian, obviously if there is the inverse function for $\rho = \rho(p)$. In this case we have to solve the equation

$$\rho(p(X)) = 2X \frac{\partial p(X)}{\partial X} - p(X) \quad (2.39)$$

when X is time-like.

For $\lambda = 0$, from Eq. (2.24) we get $\alpha = 3(w + 1)$ and

$$\left(\frac{\partial g}{\partial X} + 2X \frac{\partial^2 g}{\partial X^2} \right) \frac{dX}{dN} + 3 \left(2X \frac{\partial g}{\partial X} \right) = 0. \quad (2.40)$$

In particular in this section we focus mainly on Unified models of Dark Matter and dark energy (UDM) that can provide an alternative to understand the nature of the dark

components of our Universe. In this case a single fluid behaves both as dark matter and dark energy.

Moreover, as we already explained above, in the $\lambda = 0$ case the Lagrangian \mathcal{L} (i.e. the pressure p) depends only on X and, through Eq. (2.39), it can be described as adiabatic single fluid $p = p(\rho)$. In literature UDM the models described with a adiabatic equation of state $p = p(\rho)$, are also called ‘Quartessence’ (see for example Ref. [64] and [61, 62]).

Now we want to make a general study of the attractor solutions in this case. For $\lambda = 0$, Eq. (2.9) gives the following nodes,

$$1) \quad X = \hat{X} = 0, \quad 2) \quad \left. \frac{dg}{dX} \right|_{\hat{X}} = 0, \quad (2.41)$$

with \hat{X} a constant. Both cases correspond to $w = -1$, as one can read from Eq. (2.10).

In these cases we have either $X = 0$ or $\partial g/\partial X = 0$ on the node. We know from Eq. (2.31) that the worth of X can only decrease in time down to its minimum value. This implies that w , from Eq. (2.10), will tend to -1 for $N \rightarrow \infty$.

At this point we can study the general solution of the differential equation (2.40). For $X \neq 0$ and $\partial g/\partial X \neq 0$ the solution is [64]

$$X \left(\frac{\partial g}{\partial X} \right)^2 = k a^{-6} \quad (2.42)$$

with k a positive constant. This solution was previously derived although in a different form in Ref. [74]. As $N \rightarrow \infty$, X or dg/dX (or both) must tend to zero, which shows that, depending on the specific form of the function $g(X)$, each particular solution will converge toward one of the nodes above. From Eq. (2.42), for $N \rightarrow \infty$, the value of X or $\partial g/\partial X$ (or of both of them) must tend to zero. Then, it is immediate to conclude that $w \rightarrow -1$ is an attractor for $N \rightarrow \infty$ and confirms that each of the above solutions will be an attractor depending on the specific form of the function $g(X)$.

Let us also note that, for $\lambda \neq 0$, if we require $w \rightarrow -1$ (in this case $\alpha = \lambda$) for $N \rightarrow \infty$, the density ρ tends to a constant. Moreover, as far as $w \rightarrow -1$ we have either $X = 0$ (implying $\varphi = \text{const}$) or $\partial g/\partial X = 0$ (which also implies that g is time-independent). Then, from Eq. (2.13) it is easy to see that $f(\varphi)$ should take a positive constant value.

In what follows we will provide some examples of stable node solutions of the equation of motion, some of which have been already studied in the literature. The models below are classified on the basis of the stable node to which they asymptotically converge.

2.3.1 Case 1): Generalized Chaplygin gas

An example of case 1) is provided by the Generalized Chaplygin (GC) model (see e.g. Refs. [60, 61, 62, 58, 59, 75, 76, 54]) whose equation of state has the form

$$p_{GC} = -\rho_* \left(\frac{\rho_{GC}}{-p_*} \right)^{-\gamma}, \quad (2.43)$$

where now $p_{GC} = p$ and $\rho_{GC} = \rho$ and ρ_* and p_* are suitable constants.

Through the equation $\rho = 2X \frac{\partial g(X)}{\partial X} - g(X)$ and the continuity equation $\frac{\rho_{GC}}{dN} + (\rho_{GC} + p_{GC}) = 0$ we can write p_{GC} and ρ_{GC} as functions of either X or a . When the pressure and the energy density are considered as functions of a we have

$$p_{GC} = - \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} [1 + \nu a^{-3(1+\gamma)}]^{-\frac{\gamma}{1+\gamma}} \quad (2.44)$$

$$\rho_{GC} = \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} [1 + \nu a^{-3(1+\gamma)}]^{\frac{1}{1+\gamma}} \quad (2.45)$$

with $\nu = \text{const.}$; when instead the pressure and the energy density are considered as functions of X we have

$$p_{GC} = - \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} \left[1 - \mu X^{\frac{1+\gamma}{2\gamma}} \right]^{\frac{\gamma}{1+\gamma}} \quad (2.46)$$

$$\rho_{GC} = \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} \left[1 - \mu X^{\frac{1+\gamma}{2\gamma}} \right]^{-\frac{1}{1+\gamma}} \quad (2.47)$$

with $\mu = \text{const.}$ To connect μ and ν we have to use Eq. (2.42). We get

$$\nu = \mu^{-1/\gamma} \left(\frac{1}{4k} \right)^{-\frac{\gamma+1}{2}} \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{-1/\gamma} \quad (2.48)$$

It is necessary for our scopes to consider the case $\gamma > 0$ (so that $c_s^2 > 0$) so that $\alpha > 0$ and $d \ln X/dN \leq 0$ (see Eq. (2.31)). Note that $\gamma = -1$ corresponds to the standard ‘‘Chaplygin gas’’ model. Let us also consider $\mu > 0$ and $\nu > 0$.

It easy to see from that for small a (i.e. for $\nu^{\frac{1}{3}} \frac{1}{1+\gamma} a \ll 1$) the expression (2.45) is approximated by

$$\rho_{GC} = \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} \nu^{\frac{1}{1+\gamma}} a^{-3} \quad (2.49)$$

that corresponds to a Universe dominated by dust-like matter. For large values of the cosmological radius a i.e. $\mu^{\frac{2\gamma}{1+\gamma}} X \ll 1$ ($\nu^{\frac{1}{3}} \frac{1}{1+\gamma} a \gg 1$) we can approximate p_{GC} and ρ_{GC} in the following way

$$p_{GC} \simeq - \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} \left[1 - \frac{\gamma}{1+\gamma} \mu X^{\frac{1+\gamma}{2\gamma}} \right] \rightarrow - \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} = -\Lambda \quad (2.50)$$

$$\rho_{GC} \simeq \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} \left[1 + \frac{1}{1+\gamma} \mu X^{\frac{1+\gamma}{2\gamma}} \right] \rightarrow \left(\frac{-p_*}{\rho_*^{-1/\gamma}} \right)^{\gamma/(1+\gamma)} = \Lambda \quad (2.51)$$

which, in turn, corresponds to an empty Universe with a cosmological constant (i.e a de Sitter Universe).

In other words this models explain the acceleration of the Universe via an exotic equation of state causing it to act like dark matter at high density and like dark energy at low density. The model is interesting for phenomenological reasons but can be motivated by a brane-world interpretation. Therefore an attractive feature of the model is that it can explain both dark energy and dark matter in terms of a single component, and has therefore been referred to as “unified dark matter” (UDM) or “quartessence”.

Now, we can see immediately that the speed of sound is strictly connected with the equation of state of this fluid and with the value of γ . Indeed we have that

$$c_s^2 = -\gamma w = \gamma \frac{1}{[1 + \nu a^{-3(1+\gamma)}]} . \tag{2.52}$$

In the generalized Chaplygin gas model, the sound speed is small at early times (when a is small) and becomes larger at late times. Therefore if one study the fluctuations in the Chaplygin gas models find (see next Chapter) that if γ is not very near to 0, these models are ruled out by the matter power spectrum Ref. [54] or the CMB (see for example Ref. [58, 59, 75, 76, 77]). In particular if the sound of speed of the dark matter is very different to zero, its fluctuation are either unstable toward collapse when $c_s^2 < 0$ (since the large sound speed at late times leads to either blow-up of the perturbations) or start to oscillate when $c_s^2 > 0$.

2.3.2 Case 1): The Chimento model

Another model that falls into this class of solution is the one proposed in Ref. [72], in which $g = b\sqrt{2X} - \Lambda$ (with b a suitable constant) which satisfies the constraint that $p = -\Lambda$ along the attractor solution $\hat{X} = 0$. This model, however is well-known to imply a diverging speed of sound.

2.3.3 Case 2): The Scherrer solution model

For the solution of case 1) we want to study the function g around some $X = \hat{X} \neq 0$. In this case we can approximate g as a parabola with $\frac{\partial g}{\partial X} |_{\hat{X}} = 0$

$$g = g_0 + g_2(X - \hat{X})^2 . \tag{2.53}$$

with g_0 and g_2 suitable constants. This solution, with $g_0 < 0$ and $g_2 > 0$, coincides with the model studied by Scherrer in Ref. [64].

It is immediate to see that for $X \rightarrow \hat{X} \neq \infty$ and $N \rightarrow \infty$ the value of dX/dN goes to zero. Replacing this solution into Eq. (2.42) we obtain

$$4g_2^2 X(X - \hat{X})^2 = ka^{-6} , \tag{2.54}$$

while the energy density ρ becomes

$$\rho = -g_0 + 4g_2\hat{X}(X - \hat{X}) + 3g_2(X - \hat{X})^2 . \tag{2.55}$$

Now if we impose that today X is close to \hat{X} so that

$$\epsilon \equiv \frac{X - \hat{X}}{\hat{X}} \ll 1 \quad (2.56)$$

then Eq. (2.54) reduces to

$$X = \hat{X} \left[1 + \left(\frac{a}{a_1} \right)^{-3} \right] \quad (2.57)$$

with $a_1 \ll a$ and with

$$(1/a_1)^{-3} = [1/(2g_2)](k/\hat{X}^3)^{1/2} \quad \text{for} \quad \epsilon \ll 1. \quad (2.58)$$

As a consequence, the energy density becomes

$$\rho = -g_0 + 4g_2\hat{X}^2 \left(\frac{a}{a_1} \right)^{-3}. \quad (2.59)$$

In order for the density to be positive at late times, we need to impose $g_0 < 0$. In this case the speed of sound (3.12) turns out to be

$$c_s^2 = \frac{(X - \hat{X})}{(3X - \hat{X})} = \frac{1}{2} \left(\frac{a}{a_1} \right)^{-3}, \quad (2.60)$$

We notice also that, for $(a/a_1)^{-3} \ll 1$ we have $c_s^2 \ll 1$ for the entire range of validity of this solution.

Thus, Eq. (2.59) tells us that this *k*-fluid behaves like a very low sound-speed with a background energy density that can be written as

$$\rho = \rho_\Lambda + \rho_{\text{DM}}, \quad (2.61)$$

where ρ_Λ behaves like a “dark energy” component ($\rho_\Lambda = \text{const.}$) and ρ_{DM} behaves like a “dark matter” component ($\rho_{\text{DM}} \propto a^{-3}$). Note that, from Eq. (2.59), \hat{X} must be different from zero in order for the matter term to be there. (For this particular case the Hubble parameter H is a function only of this fluid $H^2 = \rho/3$).

This model has been variously referred to as “Unified Dark Matter” (UDM).

If the Lagrangian is strictly quadratic in X we can obtain explicit expressions for the pressure p and the speed of sound c_s in terms of ρ , namely

$$p = \frac{4}{3}g_0 + \frac{8}{9}g_2\hat{X}^2 \left\{ 1 - \left[1 + \frac{3(g_0 + \rho)}{4g_2\hat{X}^2} \right]^{\frac{1}{2}} \right\} + \frac{1}{3}\rho, \quad (2.62)$$

$$c_s^2 = -\frac{1}{3} \left[1 + \frac{3(g_0 + \rho)}{4g_2\hat{X}^2} \right]^{-\frac{1}{2}} + \frac{1}{3}. \quad (2.63)$$

Looking at these equations, we observe that in the early Universe case ($X \gg \hat{X}$ i.e. $\rho \gg (-g_0)$) the *k-essence* behaves like radiation.

Therefore, the k -essence fluid in this case behaves like a low sound-speed fluid with a density which evolves like the sum of a “dark matter” (DM) component with $\rho \propto a^{-3}$ and a “dark energy” (DE) component with $\rho = \text{const.}$. The only difference from a standard Λ CDM model is that in this k -essence model, the dark energy component has $c_s^2 \ll 1$.

Starting from the observational constraints on ρ_{DM} and ρ_{DE} , the value of a_1 is determined by the fact that the k -essence must begin to behave like dark matter prior to the epoch of equal matter and radiation. Therefore, $a_1 < a_{\text{eq}}$, where a_{eq} is the scale factor at the epoch of equal matter and radiation, given by $a_{\text{eq}} = 3 \times 10^{-4}$ (where we have imposed that the value of the scale factor today $a_0 = 1$). At the present time, the component of ρ corresponding to dark energy in equation (2.59) must be roughly twice the component corresponding to dark matter, so

$$-g_0 = 8g_2\hat{X}^2(1/a_1)^{-3}. \tag{2.64}$$

Substituting $a_1 < a_{\text{eq}}$ into this equation, we get [64]

$$\epsilon_0 = \epsilon(a_0 = 1) = \frac{-g_0}{g_2\hat{X}^2} < 8a_{\text{eq}}^3, \tag{2.65}$$

$$\ll 2 \times 10^{-10}. \tag{2.66}$$

In practice, if we assume that $g(X)$ has a local minimum that can be expanded as a quadratic and when Eq. (2.56) is not satisfied (i.e. for $a < a_1$), we can say nothing about the evolution of X and ρ .

The stronger bound $\epsilon_0 \leq 10^{-18}$ is obtained by Giannakis and Hu [55], who considered the small-scale constraint that enough low-mass dark matter halos are produced to reionize the Universe. On the other hand the sound speed can be made arbitrarily small during the epoch of structure formation by decreasing the value of ϵ in equation.

One should also consider the usual constraint imposed by primordial nucleosynthesis on extra radiation degrees of freedom, which however leads to a weaker constraint. Moreover the Scherrer model differs from Λ CDM in the structure of dark matter halos both because of the fact that it behaves as a nearly pressure-less fluid instead of a set of collisionless particles. Analytically we will discuss this problem when we will study the static configuration of the UDM models. Practically we will see that when $X < 0$, the energy density of the Scherrer model is negative. Thus, ρ must depend strongly on the time. In this way, this model will behave necessarily as a fluid and shock, in the static configuration. [55].

2.3.4 Case 2): the Generalized Scherrer solution models

Starting from the condition that we are near the attractor $X = \hat{X} \neq 0$, we can generalize the definition of g , extending the Scherrer model in the following way

$$p = g = g_0 + g_n(X - \hat{X})^n \tag{2.67}$$

with $n \geq 2$ and g_0 and g_n suitable constants.

The density reads

$$\rho = (2n - 1)g_n(X - \hat{X})^n + 2\hat{X}ng_n(X - \hat{X})^{n-1} - g_0 \tag{2.68}$$

If

$$\epsilon^n = [(X - \hat{X})/\hat{X}]^n \ll 1, \quad (2.69)$$

Eq. (2.42) reduces to

$$X = \hat{X} \left[1 + \left(\frac{a}{a_{n-1}} \right)^{-3/(n-1)} \right] \quad (2.70)$$

(where $a_{n-1} \ll a$) and so ρ becomes

$$\rho \simeq 2n\hat{X}^n g_n \left(\frac{a}{a_{n-1}} \right)^{-3} - g_0 \quad (2.71)$$

with

$$(1/a_{n-1})^{-3} = [1/(ng_n)](k/\hat{X}^{2n-1})^{1/2} \quad (2.72)$$

for $\epsilon^n \ll 1$.

We have therefore obtained the important result that this attractor leads exactly to the same terms found in the purely kinetic model of Ref. [64], i.e. a cosmological constant and a matter term. We can therefore extend the constraint of Ref. [64] to this case obtaining

$$(\epsilon_0)^{n-1} = -g_0/(4n\hat{X}^n g_n) \leq 10^{-10}. \quad (2.73)$$

A stronger constraint would clearly also apply to our model by considering the small-scale constraint imposed by the Universe reionization, as in Ref. [55].

If we write the general expressions for w and c_s^2 we have

$$w = - \left[1 + \left(\frac{g_n}{g_0} \right) (X - \hat{X})^n \right] \left[1 - 2n\hat{X} \left(\frac{g_n}{g_0} \right) (X - \hat{X})^{n-1} - (2n-1) \left(\frac{g_n}{g_0} \right) (X - \hat{X})^n \right]^{-1} \quad (2.74)$$

$$c_s^2 = \frac{(X - \hat{X})}{2(n-1)\hat{X} + (2n-1)(X - \hat{X})}. \quad (2.75)$$

For $\epsilon \ll 1$ we obtain a result similar to that of Ref. [64], namely

$$w \simeq -1 + 2n \left(\frac{g_n}{|g_0|} \right) \left(\frac{a}{a_{n-1}} \right)^{-3}, \quad (2.76)$$

$$c_s^2 \simeq \frac{1}{2(n-1)} \epsilon. \quad (2.77)$$

On the contrary, when $X \gg \hat{X}$ we obtain

$$w \simeq c_s^2 \simeq \frac{1}{2n-1} \quad (2.78)$$

In this case we can impose a bound on n so that at early times and/or at high density the *k-essence* evolves like dark matter. In other words, when $n \gg 1$, unlike the purely kinetic case of Ref. [64], the model is well behaved also at high densities.

In the next subsection we study spherical collapse for the generalized Scherrer solution models.

2.3.5 Spherical collapse for generalized Scherrer solution models

Let us assume a flat, homogeneous Friedmann-Robertson-Walker background metric. In such a case, the background evolution of the Universe is characterized completely by the following equations

$$H^2 = \frac{1}{3}\rho, \quad (2.79)$$

$$\dot{H} = -\frac{1}{2}(p + \rho), \quad (2.80)$$

where the dot denotes differentiation w.r.t. the cosmic time t .

Now let us consider a top-hat spherical over-density with the purely kinetic model with the Lagrangian $\mathcal{L} = -\Lambda + g_n(X - \hat{X})^n$ and with $g_n > 0$. For this particular case within the over-dense region we have a single dark fluid undergoing spherical collapse, which is described by the following equation

$$\frac{\ddot{R}}{R} = -\frac{1}{6}(\rho_R + 3p_R) \quad (2.81)$$

where R , ρ_R and p_R are respectively the scale-factor, pressure and energy density of the over-dense region; ρ_R and p_R are defined by the following expressions

$$\rho_R = \Lambda + 2ng_n\hat{X}(X_R - \hat{X})^{n-1} + (2n-1)g_n(X_R - \hat{X})^n \quad (2.82)$$

$$p_R = g_R = -\Lambda + g_n(X_R - \hat{X})^n \quad (2.83)$$

with $X_R = X(R)$ a function of time.

The equation of motion is

$$\left(\frac{\partial g_R}{\partial X_R} + 2X \frac{\partial^2 g_R}{\partial X_R^2} \right) \frac{dX_R}{dN_R} + 3 \left(2X_R \frac{\partial g_R}{\partial X_R} \right) = 0. \quad (2.84)$$

where $dN_R = dR/R$. The solution of Eq. (2.84) (for $\partial g_R/\partial X_R, X_R \neq 0$) is

$$X_R \left(\frac{\partial g_R}{\partial X_R} \right)^2 = k_R R^{-6} \quad (2.85)$$

where we can choose $k_R = R_{ta}^6 \left[X_R \left(\frac{\partial g_R}{\partial X_R} \right)^2 \right]_{ta}$, with R_{ta} the value of R at turnaround.

Replacing Eq. (2.83) in Eq. (2.85) we find

$$X_R \left[ng_n(X_R - \hat{X})^{n-1} \right]^2 = k_R R^{-6} \quad (2.86)$$

Using now the explicit expressions for ρ_R and p_R we arrive at the following set of equations

$$\frac{\ddot{R}}{R} = -\frac{1}{3} \left[-\Lambda + ng_n\hat{X}(X_R - \hat{X})^{n-1} + (n+1)g_n(X_R - \hat{X})^n \right] \quad (2.87)$$

$$(X_R - \hat{X})^{2n-1} + \hat{X}(X_R - \hat{X})^{2(n-1)} = \frac{k_R}{n^2 g_n^2} R^{-6}. \quad (2.88)$$

For $(X_R - \hat{X})/\hat{X} \ll 1$ Eq. (2.87) becomes

$$\frac{\ddot{R}}{R} = -\frac{1}{3} \left\{ -\Lambda + n g_n |X_{R_{ta}} - \hat{X}|^{n-1} (X_{R_{ta}} \hat{X})^{\frac{1}{2}} \left(\frac{R}{R_{ta}} \right)^{-3} \right\} \quad (2.89)$$

We can now write all the equations that describe the spherical collapse

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{1}{3} (\rho_\Lambda + \rho_{DM}) \quad (2.90)$$

$$\rho_\Lambda = \Lambda \quad (2.91)$$

$$\rho_{DM} = 2n g_n |X_{ta} - \hat{X}|^{n-1} (X_{ta} \hat{X})^{\frac{1}{2}} \left(\frac{a}{a_{ta}} \right)^{-3} \quad (2.92)$$

$$\frac{\ddot{R}}{R} = -\frac{1}{6} (\rho_{R_{DM}} - 2\rho_{R_\Lambda}) \quad (2.93)$$

$$\rho_{R_{DM}} = 2n g_n |X_{ta} - \hat{X}|^{n-1} (X_{R_{ta}} \hat{X})^{\frac{1}{2}} \left(\frac{R}{R_{ta}} \right)^{-3} \quad (2.94)$$

where $a_{ta} = a(t_{ta})$.

Following now the same procedure of Ref. [78] we can define x and y

$$x \equiv \frac{a}{a_{ta}} \quad (2.95)$$

$$y \equiv \frac{R}{R_{ta}}. \quad (2.96)$$

In this way we can redefine ρ_{DM} and $\rho_{R_{DM}}$ such that

$$\rho_{DM} = \frac{3H_{ta}^2 \Omega_{DM}(x=1)}{x^3} \quad (2.97)$$

$$\rho_{R_{DM}} = \zeta \frac{3H_{ta}^2 \Omega_{DM}(x=1)}{y^3} \quad (2.98)$$

where Ω_{DM} is the (*k-essence*) dark matter density parameter, and $\zeta = (\rho/\rho_{DM})|_{x=1}$. Then Eqs. (2.90) and (2.93) become

$$\frac{dx}{d\tau} = (x \Omega_{DM}(x))^{-\frac{1}{2}}, \quad (2.99)$$

$$\frac{d^2 y}{d\tau^2} = -\frac{1}{2y^2} [\zeta - 2y^3 K_\Lambda], \quad (2.100)$$

$$\Omega_{DM}(x) = \left(1 - \frac{\Omega_{DM}(x=1)}{\Omega_{DM}(x=1)} x^3 \right)^{-1}, \quad (2.101)$$

where $d\tau = H_{ta} \sqrt{\Omega_{DM}(x=1)}$ and $K_\Lambda = \rho_\Lambda / [3H_{ta}^2 \Omega_{DM}(x=1)]$.

Defining U as the potential energy of the over-density and using energy conservation between virialization and turnaround,

$$\left[U + \frac{R}{2} \frac{\partial U}{\partial R} \right]_{vir} = U_{ta}, \quad (2.102)$$

we obtain

$$(1 + q)y - 2qy^3 = \frac{1}{2} \quad (2.103)$$

where

$$q = \left(\frac{\rho_\Lambda}{\rho} \right)_{y=1} = \frac{K_\Lambda}{\zeta}, \quad (2.104)$$

in full agreement with Ref. [79].

2.4 UDM Scalar Field with canonical kinetic term

Starting from the barotropic equation of state $p = p(\rho)$ we can describe the system either through a purely kinetic *k-essence* Lagrangian, as we already explained in the last section, or through a Lagrangian with canonical kinetic term, as in quintessence-like models. The same problem has been solved in Ref. [80], although with a different procedure and for a different class of models. In the second case we have to solve the two differential equations

$$X - V(\varphi) = p(\varphi, X) \quad (2.105)$$

$$X + V(\varphi) = \rho(\varphi, X) \quad (2.106)$$

where $X = \dot{\varphi}^2/2$ is time-like. In particular if we assume that our model describes a unified dark matter/dark energy fluid we can proceed as follows: starting from $\dot{\rho} = -3H(p + \rho) = -\sqrt{3}\bar{\rho}(p + \rho)$ and $2X = (p + \rho) = (d\varphi/d\rho)^2 \dot{\rho}^2$ we get

$$\varphi = \pm \frac{1}{\sqrt{3}} \int_{\rho_0}^{\rho} \frac{d\rho' / \sqrt{\rho'}}{(p(\rho') + \rho')^{1/2}}, \quad (2.107)$$

up to an additive constant which can be dropped without any loss of generality. Inverting the Eq. (2.107) i.e. writing $\rho = \rho(\varphi)$ we are able to get $V(\varphi) = [\rho(\varphi) - p(\rho(\varphi))]/2$. Now we require that the fluid has constant pressure $p = -\Lambda$, i.e. that the Lagrangian of the scalar field is constant along the classical trajectory corresponding to perfect fluid behavior. In other words one arrives at an exact solution with potential and we get

$$V(\varphi) = \frac{\Lambda}{2} \left[\cosh^2 \left(\frac{\sqrt{3}}{2} \varphi \right) + 1 \right] \quad (2.108)$$

see also Ref. [67]. For large values of φ , $V(\varphi) \propto \exp(\sqrt{3}\varphi)$ (equivalently, for large values of $-\varphi$, $V(\varphi) \propto \exp(-\sqrt{3}\varphi)$) and our scalar field behaves just like a pressureless dark matter fluid. Indeed, this asymptotic form, in the presence of an extra radiation component, allows to recover one of the stable nodes obtained in Ref. [7] for quintessence fields with exponential potentials, where the scalar field mimics a pressureless fluid. Under the latter hypothesis we immediately obtain

$$\varphi(\rho) = \frac{2}{\sqrt{3}} \operatorname{arccosh}(\rho/\Lambda)^{1/2}, \quad (2.109)$$

which can be inverted to give the scalar field potential of Eq. (2.108) as $V(\varphi) = (\rho(\varphi) + \Lambda)/2$. One then obtains

$$\dot{\varphi} = -\sqrt{\Lambda} \sinh\left(\frac{\sqrt{3}}{2}\varphi\right), \quad (2.110)$$

which can be immediately integrated, to give

$$\varphi(t) = \frac{2}{\sqrt{3}} \ln\left(\frac{1+\xi}{1-\xi}\right), \quad \xi \equiv \exp\left[-\frac{\sqrt{3\Lambda}}{2}(t-t_*)\right], \quad (2.111)$$

for $t > t_*$, with t_* such that $\varphi(t \rightarrow t_*) \rightarrow \infty$. Replacing this solution in the expression for the energy density one can easily solve the the Friedmann equation for the scale-factor as a function of cosmic time,

$$a(t) = a_0 \frac{\sinh^{2/3}\left[\frac{\sqrt{3\Lambda}}{2}(t-t_*)\right]}{\sinh^{2/3}\left[\frac{\sqrt{3\Lambda}}{2}(t_0-t_*)\right]}, \quad (2.112)$$

which coincides with the standard expression for a flat, matter plus lambda model [81], with $\Omega_{0\Lambda}/\Omega_{0m} = \sinh^2[\frac{\sqrt{3\Lambda}}{2}(t_0-t_*)]$, $\Omega_{0\Lambda}$ and Ω_{0m} being the cosmological constant and matter density parameters, respectively.

Using standard criteria (e.g. Ref. [16]) it is immediate to verify that the above trajectory corresponds to a stable node even in the presence of an extra-fluid (e.g. radiation) with equation of state $w_{\text{fluid}} \equiv p_{\text{fluid}}/\rho_{\text{fluid}} > 0$, where p_{fluid} and ρ_{fluid} are the fluid pressure and energy density, respectively. Along the above attractor trajectory our scalar field behaves precisely like a mixture of pressureless matter and cosmological constant. Using the expressions for the energy density and the pressure we immediately find, for the matter energy density

$$\rho_m = \rho - \Lambda = \Lambda \sinh^2\left(\frac{\sqrt{3}}{2}\varphi\right) \propto a^{-3}. \quad (2.113)$$

The peculiarity of this model is that the matter component appears as a simple consequence of having assumed the constancy of the Lagrangian.

A closely related solution was found by Salopek & Stewart [82], using the Hamiltonian formalism.

To conclude this section, we should stress that, like any scalar field with canonical kinetic term [83], our UDM model predicts $c_s^2 = 1$, as it is clear from Eq. (3.12), which inhibits the growth of matter inhomogeneities. In summary, we have obtained a ‘‘quartessence’’ model which behaves exactly like a mixture of dark matter and dark energy along the attractor solution, whose matter sector, however is unable to cluster on sub-horizon scales (at least as long as linear perturbations are considered).

2.5 UDM Scalar Field with non-canonical kinetic term

We can summarize our findings so far by stating that purely kinetic *k-essence* cannot produce a model which *exactly* describes a unified fluid of dark matter and cosmological

constant, while scalar field models with canonical kinetic term, while containing such an exact description, unavoidably lead to $c_s^2 = 1$, in conflict with cosmological structure formation. In order to find an exact UDM model with acceptable speed of sound we consider more general scalar field Lagrangians.

2.5.1 Lagrangians of the type $\mathcal{L}(\varphi, X) = g(X) - V(\varphi)$

Let us consider Lagrangians with non-canonical kinetic term and a potential term, in the form

$$\mathcal{L}(\varphi, X) = g(X) - V(\varphi) . \quad (2.114)$$

The energy density then reads

$$\rho = 2X \frac{dg(X)}{dX} - g(X) + V(\varphi) , \quad (2.115)$$

while the speed of sound keeps the form of Eq. (3.12). The equation of motion for the homogeneous mode reads

$$\left(\frac{dg}{dX} + 2X \frac{d^2g}{dX^2} \right) \frac{dX}{dN} + 3 \left(2X \frac{dg}{dX} \right) = - \frac{dV}{dN} . \quad (2.116)$$

One immediately finds

$$p + \rho = 2X \frac{dg(X)}{dX} \equiv 2\mathcal{F}(X) . \quad (2.117)$$

We can rewrite the equation of motion, Eq. (2.116), in the form

$$\left[2X \frac{d\mathcal{F}}{dX} - \mathcal{F} \right] \frac{dX}{dN} + X \left(6\mathcal{F} + \frac{dV}{dN} \right) = 0 . \quad (2.118)$$

It is easy to see that this equation admits 2 nodes, namely:

- 1) $dg/dX|_{\hat{X}} = 0$ and
- 2) $\hat{X} = 0$.

In all cases, for $N \rightarrow \infty$, the potential V should tend to a constant, while the kinetic term can be written around the attractor in the form

$$g(X) = M^4 \left(\frac{X - \hat{X}}{M^4} \right)^n \quad n \geq 2 , \quad (2.119)$$

with M a suitable mass-scale and the constant \hat{X} can be either zero or non-zero. The trivial case $g(X) = X$ obviously reduces to the one of Section 4.

Following the same procedure adopted in the previous section we impose the constraint $p = -\Lambda$, which yields the general solution $\rho_m = 2\mathcal{F}(X)$.

This allows to define $\varphi = \varphi(\rho_m)$ as a solution of the differential equation

$$\rho_m = 2\mathcal{F} \left[\frac{3}{2} (\rho_m + \Lambda) \rho_m^2 \left(\frac{d\varphi}{d\rho_m} \right)^2 \right]. \quad (2.120)$$

As found in the case of *k-essence*, the most interesting behavior corresponds to the limit of large n and $\widehat{X} = 0$ in Eq. (2.119), for which we obtain

$$\rho_m \approx \Lambda \sinh^{-2} \left[\left(\frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right], \quad (2.121)$$

leading to $V(\varphi) \approx \rho_m/2n - \Lambda$, and $c_s^2 = 1/(2n - 1) \approx 0$. The Lagrangian of this model is similar to that analyzed in Ref. [66].

2.5.2 Lagrangians of the type $\mathcal{L}(\varphi, X) = f(\varphi)g(X)$

Let us now consider Lagrangians with a non-canonical kinetic term of the form of Eq. (2.12), namely $\mathcal{L}(\varphi, X) = f(\varphi)g(X)$.

Imposing the constraint $p = -\Lambda$, we obtain $f(\varphi) = -\Lambda/g(X)$, which inserted in the equation of motion yields the general solution

$$X \frac{d \ln |g|}{dX} = -\frac{\rho_m}{2\Lambda}. \quad (2.122)$$

The latter equation, together with Eq. (2.120) define our general prescription to get UDM models describing both DM and cosmological constant-like DE.

As an example of the general law in Eq. (2.122) let us consider an explicit solution. Assuming that the kinetic term is of Born-Infeld type, as in Refs. [84, 85, 67],

$$g(X) = -\sqrt{1 - 2X/M^4}, \quad (2.123)$$

with M a suitable mass-scale, which implies $\rho = f(\varphi)/\sqrt{1 - 2X/M^4}$, we get

$$X(a) = \frac{M^4}{2} \frac{\bar{k}a^{-3}}{1 + \bar{k}a^{-3}}, \quad (2.124)$$

where $\bar{k} = \rho_m(a_*)a_*^3/\Lambda$ and a_* is the scale-factor at a generic time t_* . In order to obtain an expression for $\varphi(a)$, we impose that the Universe is dominated by our UDM fluid, i.e. $H^2 = \rho/3$. This gives

$$\varphi(a) = \frac{2M^2}{\sqrt{3\Lambda}} \left\{ \arctan \left[(\bar{k}a^{-3})^{-1/2} \right] - \frac{\pi}{2} \right\}, \quad (2.125)$$

which, replaced in our initial ansatz $p = -\Lambda$ allows to obtain the expression (see also Ref. [67])

$$f(\varphi) = \frac{\Lambda}{\left| \cos \left[\left(\frac{3\Lambda}{4M^4} \right)^{1/2} \varphi \right] \right|}. \quad (2.126)$$

If we expand $f(\varphi)$ around $\varphi = 0$, and $X/M^4 \ll 1$ we get the approximate Lagrangian

$$\mathcal{L} \approx \frac{\Lambda}{2M^4} \dot{\varphi}^2 - \Lambda \left[1 + \frac{3\Lambda}{8M^4} \varphi^2 \right]. \quad (2.127)$$

Note that our Lagrangian depends only on the combination φ/M^2 , so that one is free to reabsorb a change of the mass-scale in the definition of the field variable. Without any loss of generality we can then set $M = \Lambda^{1/4}$, so that the kinetic term takes the canonical form in the limit $X \ll 1$. We can then rewrite our Lagrangian as

$$\mathcal{L} = -\Lambda \frac{\sqrt{1 - 2X/\Lambda}}{\left| \cos\left(\frac{\sqrt{3}}{2}\varphi\right) \right|}. \quad (2.128)$$

This model implies that for values of $\sqrt{3}\varphi \approx -\pi$ and $2X/\Lambda \approx 1$,

$$\cos\left(\frac{\sqrt{3}}{2}\varphi\right) \propto a^{3/2}, \quad \sqrt{1 - 2X/\Lambda} \propto a^{-3/2}, \quad (2.129)$$

the scalar field mimics a dark matter fluid. In this regime the effective speed of sound is $c_s^2 = 1 - 2X/\Lambda \approx 0$, as desired.

To understand whether our scalar field model gives rise to a cosmologically viable UDM solution, we need to check if in a Universe filled with a scalar field with Lagrangian (2.128), plus a background fluid of e.g. radiation, the system displays the desired solution where the scalar field mimics both the DM and DE components. Notice that the model does not contain any free parameter to specify the present content of the Universe. This implies that the relative amounts of DM and DE that characterize the present Universe are fully determined by the value of $\varphi_0 \equiv \varphi(t_0)$. In other words, to reproduce the present Universe, one has to tune the value of $f(\varphi)$ in the early Universe. However, a numerical analysis shows that, once the initial value of φ is fixed, there is still a large basin of attraction in terms of the initial value of $d\varphi/dt$, which can take any value such that $2X/\Lambda \ll 1$.

The results of a numerical integration of our system including scalar field and radiation are shown in Figures 1 - 3. Figure 1 shows the density parameter, Ω_{UDM} as a function of redshift, having chosen the initial value of φ so that today the scalar field reproduces the observed values $\Omega_{\text{DM}} \approx 0.268$, $\Omega_{\text{DE}} \approx 0.732$ [52]. Notice that the time evolution of the scalar field energy density is practically indistinguishable from that of a standard DM plus Lambda (Λ CDM) model with the same relative abundances today. Figure 2 shows the evolution equation of state parameter w_{UDM} ; once again the behavior of our model is almost identical to that of a standard Λ CDM model for $1 + z < 10^4$. Notice that, since $c_s^2 = -w_{\text{UDM}}$, the effective speed of sound of our model is close to zero, as long as matter dominates, as required. In Figure 3 we finally show the redshift evolution of the scalar field variables $X = \dot{\varphi}^2/2$ and φ : one can easily check that the evolution of both quantities is accurately described by the analytical solutions above, Eqs. (2.124) and (2.125), respectively (the latter being obviously valid only after the epoch of matter-radiation equality).

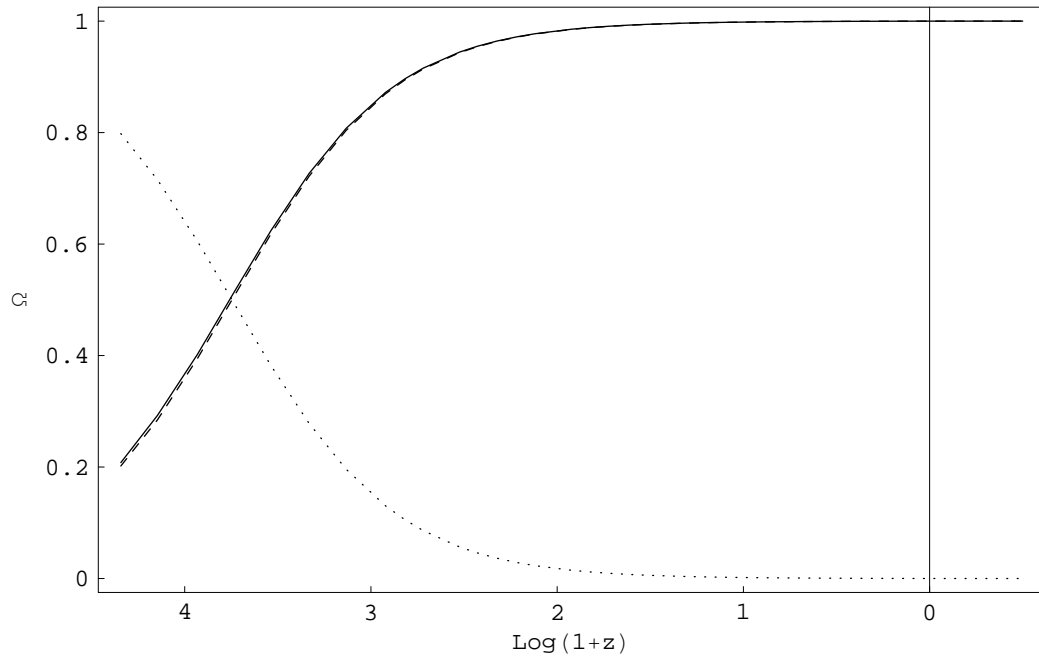


Figure 2.2: Evolution of the scalar field density parameter vs. redshift. The continuous line shows the UDM density parameter; the dashed line is the density parameter of the DM + DE components in a standard Λ CDM model; the dotted line is the radiation density parameter.

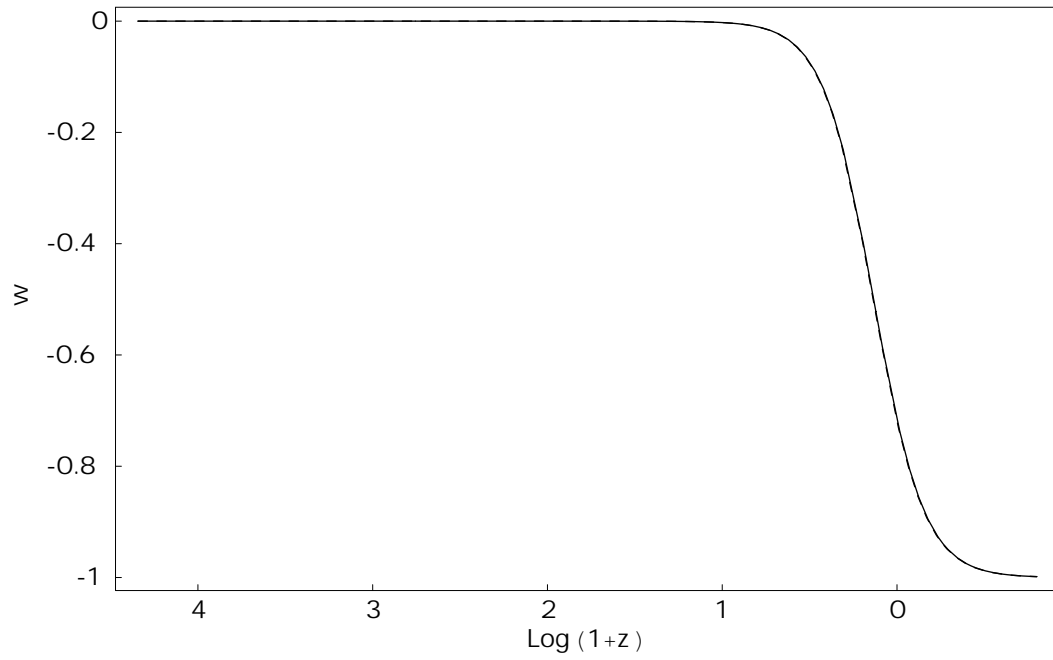


Figure 2.3: The redshift evolution of the scalar field equation of state parameter w_{UDM} (continuous line) is compared with that of the sum of the DM + DE components in a standard Λ CDM model (dashed line).

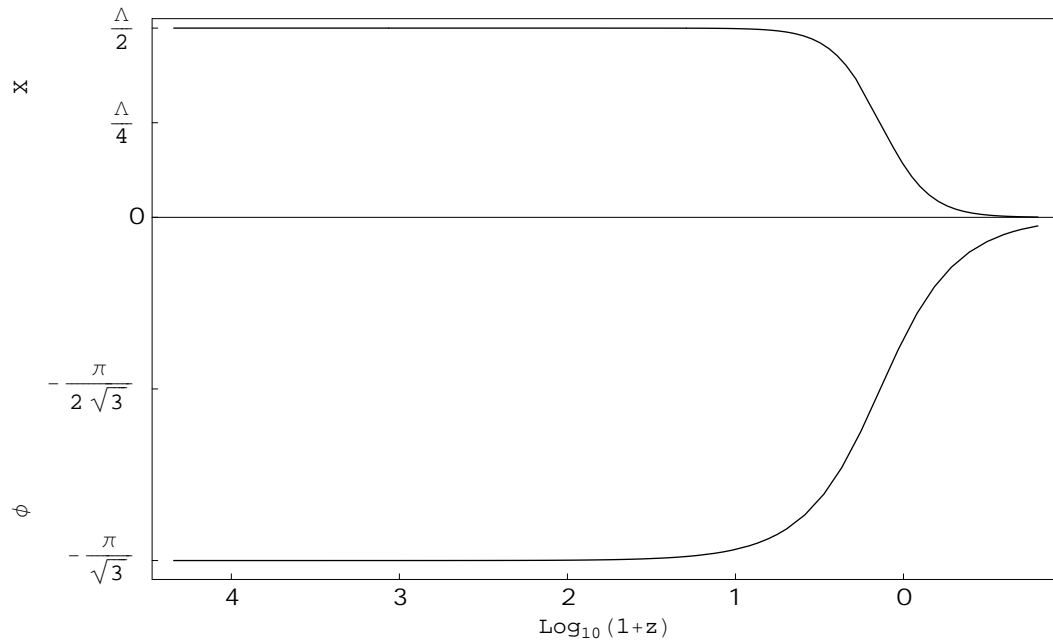


Figure 2.4: Redshift evolution of the scalar field of the scalar field variables $X = \dot{\phi}^2/2$ (top) and ϕ (bottom).

Chapter 3

ISW effect in Unified Dark Matter Scalar Field Cosmologies: an analytical approach

3.1 Introduction

The recent attitude in analyzing the observational consequences of the DE models has been that of considering not only the background equation of state and its evolution with time, but also to focus on the sound speed which regulates the growth of the dark energy fluid perturbations on different cosmological scales. In this case the sound speed has been often treated as a completely independent parameter in order to explore the consequences on the CMB anisotropies and its effects on the low ℓ multipoles [86, 87, 77]. The efficiency of this method relies on the observation that, for a single scalar field with canonical kinetic term, the speed of sound is equal to the speed of light, and thus it can cluster only on scales of the horizon size, while for other models it can be lower than unity, implying the possibility of clustering on smaller scales [53]. Another important issue is whether the dark matter clustering is influenced by the dark energy and, for the unified models, it becomes especially relevant in view of this approach. In the GCG model (both as dark energy and unified dark matter) strong constraints come from the CMB anisotropies [76, 58, 59] and the analysis of the mass power spectrum [54]. In the Scherrer solution the parameters of the model have to be fine-tuned in order for the model not to exhibit finite pressure effects in the non-linear stages of structure formation [55].

In this Chapter we consider cosmological models where dark matter and dark energy are manifestations of a single scalar field, and we focus on the contribution to the large-scale CMB anisotropies which is due to the evolution in time of the gravitational potential from the epoch of last scattering up now, the so called late Integrated Sachs-Wolfe (ISW) effect [88]. Through an analytical approach we point out the crucial role of the speed of sound in the unified dark matter models in determining strong deviations from the usual standard ISW occurring in the Λ CDM models. Our treatment is completely general in that all the results depend only on the speed of sound of the dark component and thus it can be applied to a variety of models, including those which are not described by a

scalar field but relies on a single perfect dark fluid. In the case of Λ CDM models the ISW is dictated by the background evolution, which causes the late time decay of the gravitational potential when the cosmological constant starts to dominate [89]. In the case of the unified models there are two simple but important aspects: first, the fluid which triggers the accelerated expansion at late times is also the one which has to cluster in order to produce the structures we see today. Second, from the last scattering to the present epoch, the energy density of the Universe is dominated by a single dark fluid, and therefore the gravitational potential evolution is determined by the background and perturbation evolution of just such a fluid. As a result the general trend is that the possible appearance of a sound speed significantly different from zero at late times corresponds to the appearance of a Jeans length (or a sound horizon) under which the dark fluid does not cluster any more, causing a strong evolution in time of the gravitational potential (which starts to oscillate and decay) and thus a strong ISW effect. Our results show explicitly that the CMB temperature power spectrum C_ℓ for the ISW effect contains some terms depending on the speed of sound which give a high contribution along a wide range of multipoles ℓ . As the most straightforward way to avoid these critical terms one can require the sound speed to be always very close to zero (see Sec. 3.3.2 for a more detailed discussion on this point). Moreover we find that such strong imprints from the ISW effect comes primarily from the evolution of the dark component perturbations, rather than from the background expansion history. The Chapter is organized as follows. In Sec. 3.2 we obtain the evolution equation for the gravitational potential. In Sec. 3.3 we start the analytical analysis of the ISW effect, dividing the resulting expression for the angular CMB power spectrum according to three relevant regions: those perturbation modes that enter the horizon after the acceleration of the Universe becomes relevant, and perturbation modes that are inside or outside the sound horizon of the dark fluid. In Sec. 3.3.2 we point out those contributions to the ISW effect that are triggered by the sound speed and that are responsible for a strong ISW imprint. Sec. 3.4 contains our conclusions and a discussion of our results applied to various unified dark matter models.

3.2 Linear perturbations in scalar field unified dark matter models

Reconsidering the action that describes most of the dark matter unified models within the framework of *k-essence*

$$S = S_G + S_\varphi = \int d^4x \sqrt{-g} \left[\frac{R}{2} + \mathcal{L}(\varphi, X) \right] \quad (3.1)$$

As already explained in the previous Chapter, if X is time-like, S_φ describes a perfect fluid with $T_{\mu\nu}^\varphi = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$, with pressure

$$\mathcal{L} = p(\varphi, X), \quad (3.2)$$

and energy density

$$\rho = \rho(\varphi, X) = 2X \frac{\partial p(\varphi, X)}{\partial X} - p(\varphi, X) \quad (3.3)$$

where

$$u_\mu = \frac{\nabla_\mu \varphi}{\sqrt{2X}}. \quad (3.4)$$

the four-velocity.

Now, using the conformal time like a time variable, and assuming a flat, homogeneous Friedmann-Robertson-Walker background metric we have

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j = a(\eta)^2 (-d\eta^2 + \delta_{ij} dx^i dx^j), \quad (3.5)$$

where $a(t)$ is the scale factor, δ_{ij} denotes the unit tensor and $d\eta = dt/a$ is the conformal time. In such a case, the background evolution of the Universe is characterized completely by the following equations

$$\mathcal{H}^2 = a^2 H^2 = \frac{1}{3} a^2 \rho, \quad (3.6)$$

$$\mathcal{H}' - \mathcal{H}^2 = a^2 \dot{H} = -\frac{1}{2} a^2 (p + \rho), \quad (3.7)$$

where $\mathcal{H} = a'/a$, the dot denotes differentiation w.r.t. the cosmic time t and a prime w.r.t. the conformal time η . On the background $X = \frac{1}{2} \dot{\varphi}^2 = \varphi'^2/(2a^2)$ and the equation of motion for the homogeneous mode $\varphi(t)$, the equation of state $w \equiv p/\rho$, are already considered in the first Chapter.

On the other hand we will focus on the other relevant physical quantity, the speed of sound, which enters in governing the evolution of the scalar field perturbations. Considering small inhomogeneities of the scalar field

$$\varphi(t, \mathbf{x}) = \varphi_0(t) + \delta\varphi(t, \mathbf{x}), \quad (3.8)$$

we can write the metric in the longitudinal gauge as

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)a(t)^2 \delta_{ij} dx^i dx^j \quad (3.9)$$

since $\delta T_i^j = 0$ for $i \neq j$ [90].

From the linearized (0-0) and (0-i) Einstein equation one obtains (see Ref. [14] and Ref. [44])

$$\nabla^2 \Phi = \frac{1}{2} \frac{a^2 (p + \rho)}{c_s^2 \mathcal{H}} \left(\mathcal{H} \frac{\delta\varphi}{\varphi_0'} + \Phi \right)', \quad (3.10)$$

$$\left(a^2 \frac{\Phi}{\mathcal{H}} \right)' = \frac{1}{2} \frac{a^2 (p + \rho)}{\mathcal{H}^2} \left(\mathcal{H} \frac{\delta\varphi}{\varphi_0'} + \Phi \right), \quad (3.11)$$

where one defines a “speed of sound” c_s^2 relative to the pressure and energy density fluctuation of the kinetic term [14] as

$$c_s^2 \equiv \frac{(\partial p / \partial X)}{(\partial \rho / \partial X)} = \frac{\frac{\partial p}{\partial X}}{\frac{\partial p}{\partial X} + 2X \frac{\partial^2 p}{\partial X^2}}. \quad (3.12)$$

Eqs. (3.10) and (3.11) are sufficient to determine the gravitational potential Φ and the perturbation of the scalar field. Defining two new variables

$$u \equiv 2 \frac{\Phi}{(p + \rho)^{1/2}}, \quad v \equiv z \left(\mathcal{H} \frac{\delta\varphi}{\varphi'_0} + \Phi \right), \quad (3.13)$$

where

$$z = \frac{a^2(p + \rho)^{1/2}}{c_s \mathcal{H}},$$

we can recast (3.10) and (3.11) in terms of u and v [44]

$$c_s \Delta u = z \left(\frac{v}{z} \right)', \quad c_s v = \theta \left(\frac{u}{\theta} \right)' \quad (3.14)$$

where

$$\theta = \frac{1}{c_s z} = \frac{(1 + p/\rho)^{-1/2}}{\sqrt{3} a}.$$

Starting from (3.14) we arrive at the following second order differential equations for u [44]

$$u'' - c_s^2 \nabla^2 u - \frac{\theta''}{\theta} u = 0. \quad (3.15)$$

Notice that this equation can also be used to describe any perfect fluid with equation of state $p = p(\rho)$, up to a redefinition of c_s . In this case $c_s^2 = p'/\rho'$ corresponds to the usual adiabatic sound speed. In this way, with the same equation (3.15), we can also describe the Λ CDM model. Also pure kinetic Lagrangian (3.2) $\mathcal{L}(X)$ models (see for example Ref. [57]), can be described as a perfect fluid with the pressure p uniquely determined by the energy density, since they both depend on a single degree of freedom, the kinetic term X .

Unfortunately we do not know the exact solution for a generic Lagrangian. However we can consider the asymptotic solutions i.e. long-wavelength and short-wavelength perturbations, depending whether $c_s^2 k^2 \ll |\theta''/\theta|$ or $c_s^2 k^2 \gg |\theta''/\theta|$, respectively. This means to consider perturbations on scale much larger or much smaller than the effective Jeans length for the gravitational potential

$$\lambda_J^2 = c_s^2 |\theta/\theta''|.$$

For a plane wave perturbation $u \propto u_k(\eta) \exp(i\mathbf{k}\mathbf{x})$ in the short-wavelength limit ($c_s^2 k^2 \gg |\theta''/\theta|$) we obtain

$$u_k = C_k(\bar{\eta}) \cos \left(k \int_{\bar{\eta}}^{\eta} c_s d\tilde{\eta}, \right) \quad (3.16)$$

where C_k is a constant of integration. Instead, neglecting the decaying mode, the long-wavelength solution ($c_s^2 k^2 \ll |\theta''/\theta|$) is

$$u_k = A_k(\bar{\eta}) \theta \int_{\bar{\eta}}^{\eta} \frac{d\tilde{\eta}}{\theta^2}, \quad (3.17)$$

where A_k is a constant of integration .

Once u is computed we can obtain the value of the gravitational potential Φ through Eq. (3.13) and the perturbation of the scalar field from Eq. (3.11)

$$\delta\varphi = 2\sqrt{2X} \frac{(\Phi' + \mathcal{H}\Phi)}{a(p + \rho)} . \quad (3.18)$$

3.3 Analytical approach to the ISW effect

Let us now focus on the ISW effect. The ISW contribution to the CMB power spectrum is given by

$$\frac{2l+1}{4\pi} C_l^{\text{ISW}} = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} k^3 \frac{|\Theta_l^{\text{ISW}}(\eta_0, k)|^2}{2l+1} , \quad (3.19)$$

where Θ_l^{ISW} is the fractional temperature perturbation due to ISW effect

$$\frac{\Theta_l^{\text{ISW}}(\eta_0, k)}{2l+1} = 2 \int_{\eta_*}^{\eta_0} \Phi'(\tilde{\eta}, k) j_l[k(\eta_0 - \tilde{\eta})] d\tilde{\eta} , \quad (3.20)$$

with η_0 and η_* the present and the last scattering conformal times respectively and j_l are the spherical Bessel functions.

We now evaluate analytically the power spectrum (3.19). As a first step, following the same procedure of Ref. [89], we notice that, when the acceleration of the Universe begins to be important, the expansion time scale $\eta_{1/2} = \eta(w = -1/2)$ sets a critical wavelength corresponding to $k\eta_{1/2} = 1$. It is easy to see that if we consider the Λ CDM model then $\eta_{1/2} = \eta_\Lambda$ i.e. when $a_\Lambda/a_0 = (\Omega_0/\Omega_\Lambda)^{1/3}$ [89]. Thus at this critical point we can break the integral (3.19) in two parts [89]

$$\frac{2l+1}{4\pi} C_l^{\text{ISW}} = \frac{1}{2\pi^2} [I_{\Theta_l}(k\eta_{1/2} < 1) + I_{\Theta_l}(k\eta_{1/2} > 1)] , \quad (3.21)$$

where

$$I_{\Theta_l}(k\eta_{1/2} < 1) \equiv \int_0^{1/\eta_{1/2}} \frac{dk}{k} k^3 \frac{|\Theta_l^{\text{ISW}}(\eta_0, k)|^2}{2l+1} , \quad (3.22)$$

and

$$I_{\Theta_l}(k\eta_{1/2} > 1) \equiv \int_{1/\eta_{1/2}}^\infty \frac{dk}{k} k^3 \frac{|\Theta_l^{\text{ISW}}(\eta_0, k)|^2}{2l+1} . \quad (3.23)$$

As explained in Ref. [89] the ISW integrals (3.20) takes on different forms in these two regimes

$$\frac{\Theta_l^{\text{ISW}}(\eta_0, k)}{2l+1} = \begin{cases} 2\Delta\Phi_k j_l[k(\eta_0 - \eta_{1/2})] & k\eta_{1/2} \ll 1 \\ 2\Phi'_k(\eta_k) I_l/k & k\eta_{1/2} \gg 1 \end{cases} \quad (3.24)$$

where $\Delta\Phi_k$ is the change in the potential from the matter-dominated (for example at recombination) to the present epoch η_0 and

$$\eta_k \simeq \eta_0 - (l + 1/2)/k$$

is the conformal time when a given k-mode contributes maximally to the angle that this scale subtends on the sky, obtained at the peak of the Bessel function j_ℓ . The first limit in Eq. (3.24) is obtained by approximating the Bessel function as a constant evaluated at the critical epoch $\eta_{1/2}$. Since it comes from perturbations of wavelengths longer than the distance a photon can travel during the the time $\eta_{1/2}$, a kick ($2\Delta\Phi_k$) to the photons is the main result, and it will corresponds to very low multipoles, since $\eta_{1/2}$ is very close to the present epoch η_0 . It thus appears similar to a Sachs-wolfe effect (or also to the early ISW contribution). The second limit in Eq. (3.24) is achieved by considering the strong oscillations of the Bessel functions in this regime, and thus evaluating the time derivative of the potentials out of the integral at the peak of the Bessel function, leaving the integral [89]

$$I_l \equiv \int_0^\infty j_l(y)dy = \frac{\sqrt{\pi} \Gamma[(l+1)/2]}{2 \Gamma[(l+2)/2]} . \tag{3.25}$$

With this procedure, replacing (3.24a) in (3.22) and (3.24b) in (3.23) we can obtain the ISW contribution to the CMB anisotropies power spectrum (3.19).

Now we have to calculate, through Eqs. (3.16)-(3.17) and (3.13), the value of $\Phi(k, \eta)$ for $k\eta_{1/2} \ll 1$ and $k\eta_{1/2} \gg 1$. As we will see that main differences (and the main difficulties) of the unified dark matter models with respect to the Λ CDM case will appear from the second regime of Eq. (3.24).

3.3.1 Derivation of I_{Θ_l} for modes $k\eta_{1/2} < 1$

In the UDM models when $k\eta_{1/2} \ll 1$ then $c_s^2 k^2 \ll |\theta''/\theta|$ is always satisfied. This is due to the fact that before the dark fluid start to be relevant as a cosmological constant, for $\eta < \eta_{1/2}$, its sound speed generically is very close to zero in order to guarantee enough structure formation, and moreover the limit $k\eta_{1/2} \ll 1$ involves very large scales (since $\eta_{1/2}$ is very close to the present epoch). For the standard Λ CDM model the condition is clearly satisfied. In this situation we can use the relation (3.17) and Φ_k becomes

$$\Phi_k = A_k \left(1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)} \int_{\eta_i}^\eta a^2(\tilde{\eta}) d\tilde{\eta} \right) . \tag{3.26}$$

We immediately see that $A_k = \Phi_k(0)$, the large scale gravitational potential during the radiation dominated epoch. The integral in Eq. (3.26) may be written as follows

$$\int_{\eta_i}^\eta a^2(\tilde{\eta}) d\tilde{\eta} = I_R + \int_{\eta_R}^\eta a^2(\tilde{\eta}) d\tilde{\eta} , \tag{3.27}$$

where $I_R = \int_{\eta_i}^{\eta_R} a^2(\tilde{\eta}) d\tilde{\eta}$ and η_R is the conformal time at recombination. When $\eta_i < \eta < \eta_R$ the UDM Models behave as dark matter ¹. In this temporal range the Universe is

¹In fact the Scherrer [64] and generalized Scherrer solutions [57] in the very early Universe, much before the equality epoch, have $c_s \neq 0$ and $w > 0$. However at these times the dark fluid contribution is sub-dominant with respect to the radiation energy density and thus there is no substantial effect on the following equations.

dominated by a mixture of “matter” and radiation and I_R becomes

$$I_R = \eta_* a_{eq} \left(\frac{\xi_R^5}{5} + \xi_R^4 + \frac{4\xi_R^3}{3} \right) \quad (3.28)$$

where a_{eq} is the value of the scalar factor at matter-radiation equality, $\xi = \eta/\eta_*$ and $\eta_* = (\rho_{eq} a_{eq}^2/24)^{-1/2} = \eta_{eq}/(\sqrt{2}-1)$. With these definitions it is easy to see that $a_R = a_{eq}(\xi_R^2 + 2\xi_R)$. Notice that Eq. (3.26) is obtained in the case of adiabatic perturbations. Since we are dealing with unified dark matter models based on a scalar field, there will always be an intrinsic non-adiabatic pressure (or entropic) perturbation. However for the very long wavelengths, $k\eta_{1/2} \ll 1$ under consideration here such an intrinsic perturbation turns out to be negligible [14]. For adiabatic perturbations $\Phi_k(\eta_R) \cong (9/10)\Phi_k(0)$ [90] and accounting for the primordial power spectrum,

$$k^3 |\Phi_k(0)|^2 = Bk^{n-1},$$

where n is the scalar spectral index, we get from Eq. (3.24a)

$$\begin{aligned} I_{\Theta_l}(k\eta_{1/2} < 1) &\approx 4(2l+1)B \int_0^{1/\eta_{1/2}} \frac{dk}{k} k^{n-1} j_l^2[k(\eta_0 - \eta_{1/2})] \\ &\times \left| \frac{1}{10} - \frac{\mathcal{H}(\eta_0)}{a^2(\eta_0)} \left[\int_{\eta_R}^{\eta_0} a^2(\tilde{\eta}) d\tilde{\eta} \right] \right|^2, \end{aligned} \quad (3.29)$$

where we have neglected I_R since it gives a negligible contribution.

A first comment is in order here. There is a vast class of unified dark matter models that are able to reproduce exactly the same background expansion history of the Universe as the Λ CDM model (at least from the recombination epoch on wards). For example this is the case of the the Scherrer and generalized Scherrer unified models [64] [57], the generalized Chaplygin gas [60, 61, 62] for the parameter α which tends to zero, the models proposed in Ref. [57] and [67] where one impose the langrangian (i.e. the pressure) to be a constant, and also the model of a single dark perfect fluid proposed in Ref. [65]. For such cases it is clear that the low ℓ contribution (3.29) to the ISW effect will be the same that is predicted by the Λ CDM model. This is easily explained considering that for such long wavelength perturbations the sound speed in fact plays no role at all.

3.3.2 Derivation of I_{Θ_l} for modes $k\eta_{1/2} > 1$

As we have already mentioned in the previous section, in general a viable UDM must have a sound speed very close to zero for $\eta < \eta_{1/2}$ in order to behave as dark matter also at the perturbed level to form the structures we see today, and thus the gravitational potential will start to change in time for $\eta > \eta_{1/2}$. Therefore for the modes $k\eta_{1/2} > 1$, in order to evaluate Eq. (3.24b) into Eq. (3.23) we can impose that $\eta_k > \eta_{1/2}$ which, from the definition of $\eta_k \simeq \eta_0 - (l+1/2)/k$, moves the lower limit of Eq. (3.23) to $(l+1/2)/(\eta_0 - \eta_{1/2})$. Moreover we have that $\eta_{1/2} \sim \eta_0$. We can use this property to estimate any observable at the value of η_k . Defining

$$\chi = \frac{\eta}{\eta_{1/2}}, \quad \text{and} \quad \kappa = k\eta_{1/2}, \quad (3.30)$$

we have

$$a_k = a(\eta_k) = a(\chi_k) = a_0 + \frac{da}{d\chi} \Big|_{\chi_0} \delta\chi_k = 1 - \eta_{1/2} \mathcal{H}_0 \frac{l+1/2}{\kappa}, \quad (3.31)$$

taking $a_0 = 1$, and

$$\frac{d\Phi_k}{d\chi}(\chi_k) = \eta_{1/2} \Phi'(\eta_k) = \frac{d\Phi_k}{d\chi} \Big|_{\chi_0} - \frac{d^2\Phi_k}{d\chi^2} \Big|_{\chi_0} \left(\frac{l+1/2}{\kappa} \right), \quad (3.32)$$

where

$$\delta\chi_k = \chi_k - \chi_0 = \frac{\eta_k - \eta_0}{\eta_{1/2}} = -\frac{l+1/2}{\kappa}.$$

Notice that the expansion (3.32) is fully justified, since as already mentioned above, the minimum value of κ in Eq. (3.23) moves to $(l+1/2)/(\eta_0/\eta_{1/2} - 1)$, making $\delta\chi_k$ much less than 1. Therefore we can write

$$\begin{aligned} \frac{|\Theta_{l \text{ ISW}}(\eta_0, k)|^2}{(2l+1)^2} &= 4 \left| \frac{\Phi'_k(\eta_k) I_l}{k} \right|^2 = \frac{4I_l^2}{\kappa^2} \left| \frac{d\Phi_k}{d\chi}(\chi_k) \right|^2 = \\ &= \frac{4I_l^2}{\kappa^2} \left[\left| \frac{d\Phi_k}{d\chi}(\chi_0) \right|^2 - 2 \frac{d\Phi_k}{d\chi}(\chi_0) \frac{d^2\Phi_k}{d\chi^2}(\chi_0) \left(\frac{l+1/2}{\kappa} \right) + \right. \\ &\quad \left. + \left| \frac{d^2\Phi_k}{d\chi^2}(\chi_0) \right|^2 \left(\frac{l+1/2}{\kappa} \right)^2 \right]. \end{aligned} \quad (3.33)$$

In this case, during $\eta_{1/2} < \eta < \eta_0$, there will be perturbation modes whose wavelength stays bigger than the Jeans length or smaller than it, i.e. we have to consider both the possibilities $c_s^2 k^2 \ll |\theta''/\theta|$ and $c_s^2 k^2 \gg |\theta''/\theta|$. In general the sound speed can vary with time, and in particular it might become significantly different from zero at late times. However, just as a first approximation, we exclude the intermediate situation because usually $\eta_{1/2}$ is very close to η_0 (this situation will be briefly analyzed later).

Perturbation modes on scales bigger than the Jeans length.

When $c_s^2 k^2 \ll |\theta''/\theta|$ the value of $\Phi'(\eta_k)$ can be written from Eq.(3.26) as

$$\Phi'(\eta_k) = \Phi_k(0) \tilde{\Phi}'_k(\eta_k) = \Phi_k(0) a(\eta_k) \left[\frac{d^2}{dt^2} \left(\frac{1}{a} \int_{t_i}^t a(\tilde{t}) d\tilde{t} \right) \right]_{t=t(\eta_k)}. \quad (3.34)$$

Now, using this expression in Eq. (3.33), with the primordial power spectrum $k^3|\Phi_k(0)|^2 = Bk^{n-1}$, the value of (3.23) may be written as

$$\begin{aligned}
\frac{I_{\Theta_l}(k\eta_{1/2} > 1)}{2l+1} &= 4I_l^2 B\eta_{1/2}^{n-1} \left[\int_{\frac{\chi_0-1}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^3} \kappa^{n-1} \left| \frac{d\tilde{\Phi}_k}{d\chi}(\chi_k) \right|^2 \right] = \\
&= 4I_l^2 B\eta_{1/2}^{n-1} \left[\frac{1}{3-n} \left(\frac{\chi_0-1}{l+1/2} \right)^{3-n} \left| \frac{d\tilde{\Phi}_k}{d\chi}(\chi_0) \right|^2 + \right. \\
&\quad - \frac{2(l+1/2)}{4-n} \left(\frac{\chi_0-1}{l+1/2} \right)^{4-n} \frac{d\tilde{\Phi}_k}{d\chi}(\chi_0) \frac{d^2\tilde{\Phi}_k}{d\chi^2}(\chi_0) + \\
&\quad \left. + \frac{(l+1/2)^2}{5-n} \left(\frac{\chi_0-1}{l+1/2} \right)^{5-n} \left| \frac{d^2\tilde{\Phi}_k}{d\chi^2}(\chi_0) \right|^2 \right] \quad (3.35)
\end{aligned}$$

with $(d\tilde{\Phi}_k/d\chi)_{\chi_0} = \eta_{1/2}\tilde{\Phi}'_k(\eta_0)$ and with $(d^2\tilde{\Phi}_k/d\chi^2)_{\chi_0} = \eta_{1/2}^2\tilde{\Phi}''_k(\eta_0)$.

A second relevant comment follows from the fact that $I_l^2 \sim 1/l$ for $l \gg 1$. We thus see that for $n = 1$ and for $l \gg 1$ the contribution to the angular power spectrum from the modes under consideration is

$$\frac{l(l+1)}{4\pi} C_l^{ISW} = l(l+1) \frac{I_{\Theta_l}(k\eta_{1/2} > 1)}{2\pi^2(2l+1)} \sim \frac{1}{l}.$$

In other words we find a similar slope as in [89, 91] found in the Λ CDM model. Recalling the results of the previous section, this means that in the unified dark matter models the contribution to the ISW effect from those perturbations that are outside the Jeans length is very similar to the one produced in a Λ CDM model. The main difference on these scales will be present if the background evolution is different from the one in the Λ CDM model, but for the models where the background evolution is the same, as those proposed in Refs. [64, 57, 67, 65] no difference at all can be observable.

Perturbation modes on scales smaller than the Jeans length.

When $c_s^2 k^2 \gg |\theta''/\theta|$ we must use the solution (3.16) and through the relation (3.13a) the gravitational potential is given by

$$\Phi_k(\eta) = \frac{1}{2}(p+\rho)^{1/2}(\eta) C_k(\eta_{1/2}) \cos \left(k \int_{\eta_{1/2}}^{\eta} c_s(\tilde{\eta}) d\tilde{\eta} \right). \quad (3.36)$$

In Eq. (3.36) $C_k(\eta_{1/2}) = \Phi_k(0) C_{1/2}$ is a constant of integration where

$$C_{1/2} = 2 \frac{\left[1 - \frac{\mathcal{H}(\eta_{1/2})}{a^2(\eta_{1/2})} \left(I_R + \int_{\eta_R}^{\eta_{1/2}} a^2(\tilde{\eta}) d\tilde{\eta} \right) \right]}{(P+\rho)^{1/2}(\eta_{1/2})}, \quad (3.37)$$

and it is obtained under the approximation that for $\eta < \eta_{1/2}$ one can use the longwavelength solution (3.26), since for these epochs the sound speed must be very close to zero.

Notice that Eq. (3.36) shows clearly that the gravitational potential is oscillating and decaying in time.

Defining for simplicity $\bar{C}^2 = C_{1/2}^2(P + \rho)(\eta_0)/4$, we take the time derivative of the gravitational potential appearing in Eq. (3.24b) by employing the expansion of Eq.(3.33). We thus find that Eq. (3.23) yields

$$\begin{aligned}
 \frac{I_{\Theta_l}(k\eta_{1/2} > 1)}{2l + 1} &= 4\bar{C}^2 B I_l^2 \eta_{1/2}^{n-1} \left\{ \mathcal{C}_{\{k5,l2,c^2\}}(l + 1/2)^2 \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^5} \kappa^{n-1} \cos^2(D_0\kappa) \right] + \right. \\
 &+ \mathcal{C}_{\{k4,l1,c^2\}}(l + 1/2) \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^4} \kappa^{n-1} \cos^2(D_0\kappa) \right] + \\
 &+ \mathcal{C}_{\{k4,l2,sc\}}(l + 1/2)^2 \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^4} \kappa^{n-1} \cos(D_0\kappa) \sin(D_0\kappa) \right] + \\
 &+ [\mathcal{C}_{\{k3,l0,c^2\}} + \mathcal{C}_{\{k3,l2,c^2\}}(l + 1/2)^2] \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^3} \kappa^{n-1} \cos^2(D_0\kappa) \right] + \\
 &+ \mathcal{C}_{\{k3,l2,s^2\}}(l + 1/2)^2 \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^3} \kappa^{n-1} \sin^2(D_0\kappa) \right] \\
 &+ \mathcal{C}_{\{k3,l1,sc\}}(l + 1/2) \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^3} \kappa^{n-1} \cos(D_0\kappa) \sin(D_0\kappa) \right] + \\
 &+ \mathcal{C}_{\{k2,l1,c^2\}}(l + 1/2) \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^2} \kappa^{n-1} \cos^2(D_0\kappa) \right] + \\
 &+ \mathcal{C}_{\{k2,l1,s^2\}}(l + 1/2) \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^2} \kappa^{n-1} \sin^2(D_0\kappa) \right] + \\
 &+ [\mathcal{C}_{\{k2,l0,sc\}} + \mathcal{C}_{\{k2,l2,sc\}}(l + 1/2)^2] \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa^2} \kappa^{n-1} \cos(D_0\kappa) \sin(D_0\kappa) \right] + \\
 &+ \mathcal{C}_{\{k1,l0,s^2\}} \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa} \kappa^{n-1} \sin^2(D_0\kappa) \right] + \\
 &+ \mathcal{C}_{\{k1,l2,c^2\}}(l + 1/2)^2 \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa} \kappa^{n-1} \cos^2(D_0\kappa) \right] + \\
 &+ \left. \mathcal{C}_{\{k1,l1,sc\}}(l + 1/2) \left[\int_{\frac{l+1/2}{\chi_0^{-1}}}^{\infty} \frac{d\kappa}{\kappa} \kappa^{n-1} \cos(D_0\kappa) \sin(D_0\kappa) \right] \right\} \quad (3.38)
 \end{aligned}$$

with $D_0 = \int_1^{\chi_0} c_s(\tilde{\chi}) d\tilde{\chi}$ and where

$$\begin{aligned}
\mathcal{C}_{\{k5,l2,c^2\}} &= \left\{ \frac{(p+\rho)_{,\chi\chi}}{(p+\rho)} - \frac{1}{2} \left[\frac{(p+\rho)_{,\chi}}{(p+\rho)} \right]^2 \right\}^2 \Big|_{\chi_0}, \\
\mathcal{C}_{\{k4,l1,c^2\}} &= -2 \left\{ \left[\frac{(p+\rho)_{,\chi}}{(p+\rho)} \right]^2 \left[\frac{(p+\rho)_{,\chi\chi}}{(p+\rho)} - \frac{1}{2} \frac{(p+\rho)_{,\chi}}{(p+\rho)} \right] \right\} \Big|_{\chi_0}, \\
\mathcal{C}_{\{k4,l2,sc\}} &= 4 \left\{ c_s \left[\frac{(p+\rho)_{,\chi\chi}}{(p+\rho)} - \frac{1}{2} \left(\frac{(p+\rho)_{,\chi}}{(p+\rho)} \right)^2 \right] \left[\frac{(p+\rho)_{,\chi}}{(p+\rho)} + \frac{c_{s,\chi}}{c_s} \right] \right\} \Big|_{\chi_0}, \\
\mathcal{C}_{\{k3,l0,c^2\}} &= \left[\frac{(p+\rho)_{,\chi}}{(p+\rho)} \right]^2 \Big|_{\chi_0}, \quad \mathcal{C}_{\{k3,l2,s^2\}} = 4 \left\{ c_s \left[\frac{(p+\rho)_{,\chi}}{(p+\rho)} + \frac{c_{s,\chi}}{c_s} \right] \right\}^2 \Big|_{\chi_0}, \\
\mathcal{C}_{\{k3,l2,c^2\}} &= 4 \left\{ c_s^2 \left[\frac{(p+\rho)_{,\chi\chi}}{(p+\rho)} - \frac{1}{2} \left(\frac{(p+\rho)_{,\chi}}{(p+\rho)} \right)^2 \right] \right\} \Big|_{\chi_0}, \\
\mathcal{C}_{\{k3,l1,sc\}} &= 4 \left\{ c_s \left[\frac{(p+\rho)_{,\chi\chi}}{(p+\rho)} - \frac{1}{2} \left(\frac{(p+\rho)_{,\chi}}{(p+\rho)} \right)^2 \right] - c_s \frac{(p+\rho)_{,\chi}}{(p+\rho)} \left[\frac{(p+\rho)_{,\chi}}{(p+\rho)} + \frac{c_{s,\chi}}{c_s} \right] \right\} \Big|_{\chi_0}, \\
\mathcal{C}_{\{k2,l1,c^2\}} &= -4 \left[c_s^2 \frac{(p+\rho)_{,\chi}}{(p+\rho)} \right] \Big|_{\chi_0}, \quad \mathcal{C}_{\{k2,l1,s^2\}} = 8 \left\{ c_s^2 \left[\frac{(p+\rho)_{,\chi}}{(p+\rho)} + \frac{c_{s,\chi}}{c_s} \right] \right\} \Big|_{\chi_0}, \\
\mathcal{C}_{\{k2,l0,sc\}} &= -4 \left[c_s \frac{(p+\rho)_{,\chi}}{(p+\rho)} \right] \Big|_{\chi_0}, \quad \mathcal{C}_{\{k2,l2,sc\}} = 8 \left\{ c_s^3 \left[\frac{(p+\rho)_{,\chi}}{(p+\rho)} + \frac{c_{s,\chi}}{c_s} \right] \right\} \Big|_{\chi_0}, \\
\mathcal{C}_{\{k1,l0,s^2\}} &= 4c_s^2 \Big|_{\chi_0}, \quad \mathcal{C}_{\{k1,l2,c^2\}} = 4c_s^4 \Big|_{\chi_0}, \quad \mathcal{C}_{\{k1,l1,sc\}} = 8c_s^3 \Big|_{\chi_0}.
\end{aligned} \tag{3.39}$$

In this case we have defined $(\cdot)_{,\chi} \equiv d(\cdot)/d\chi$ and we recall that the dimensionless variables χ and κ are defined in Eq. (3.30). We have indicated the coefficients $\mathcal{C}_{\{k[j],l[i],[sc]\}}$ in such a way to signal that they multiply an integral in κ of κ^{n-1}/κ^j times $\sin(D_0\kappa) \cos(D_0\kappa)$ and the overall multipole coefficient is $(l+1/2)^i$. We can infer from (3.38) that for $n < 1$ all integrals are convergent. Notice that a natural cut-off in the various integrals is introduced for those modes that enter the horizon during the radiation dominated epoch, due to the Meszaros effect that the matter fluctuations will suffer until the full matter domination epoch. Such a cut-off will show up in the gravitational potential and in the various integrals of Eq. (3.38) as a $(k_{eq}/k)^4$ factor, where k_{eq} is the wavenumber of the Hubble radius at the equality epoch.

A simple inspection of Eq. (3.38) shows one of our main results. The terms of Eq. (3.38) where the coefficients \mathcal{C} turn out to be proportional to the sound speed c_s cause the growth of $l(l+1)I_{\Theta_i}(k\eta_{1/2} > 1)/(2l+1)$, (and hence of the power spectrum $l(l+1)C_l$ through Eq. (3.21)), as l increases. This means that, if the sound speed of the unified dark matter fluid starts to differ significantly from zero at late times, the consequence is to produce a very strong ISW effect, and clearly this does not happen in a Λ CDM Universe since

$c_s^2 = 0$ always. This effect is easily explained by considering that the energy density of the Universe in the unified models is dominated at late time by a just single fluid. Therefore an eventual appearance of a Jeans length (i.e. a departure of the sound speed from zero) makes the oscillating behavior of the dark fluid perturbations under the Jeans length immediately visible through a strong time dependence of the gravitational potential. In fact one can verify that the scalar field fluctuations (3.18) are oscillating and decaying in time as $\delta\varphi \sim (kc_s/a) \sin(k \int_{\eta_{1/2}}^{\eta} c_s d\eta)$. Similar results have been discussed in the case of the GCG model in Refs. [58, 59].

We point out that the potentially most dangerous term in Eq. (3.38) is the one identified by the coefficient $\mathcal{C}_{\{k1,l2,c^2\}}$

$$4c_s^4|_{\chi_0} (l+1/2)^2 \left[\int_{\frac{l+1/2}{\chi_0-1}}^{\infty} \frac{d\kappa}{\kappa} \kappa^{n-1} \cos^2(D_0\kappa) \right]. \quad (3.40)$$

Such a term makes the power spectrum $l(l+1)C_l$ to scale as l^3 until $l \approx 25$. This angular scale is obtained by considering the peak of the Bessel functions in correspondence of the cut-off scale k_{eq} , $l \approx k_{eq}(\eta_0 - \eta_{1/2})$. In fact, for smaller scales, the integral identified by the coefficient $\mathcal{C}_{\{k1,l2,c^2\}}$ will decrease as $1/\ell$.

Intermediate case.

Now we shall briefly discuss the intermediate case that corresponds to perturbation modes that initially are outside the Jeans length and then, due to a time variation of the sound speed, they fall inside them. This corresponds to consider the range $[(l+1/2)/(\chi_0-1)]^2 < \kappa_J^2 = |\theta_{,\chi\chi}/\theta|/c_s^2 < \kappa_{eq}^2$. In this case $\kappa > (l+1/2)/(\chi_0-1)$ and so we can use the same procedure described before. Indeed when $k \sim k_J$ i.e. $c_s^2 k_J^2 \sim |\theta''/\theta|$ it can be written as follows

$$\left\{ [c_s^2 |\theta_{,\chi\chi}/\theta|^{-1}] \Big|_{\chi_0} - [c_s^2 |\theta_{,\chi\chi}/\theta|^{-1}]_{,\chi} \Big|_{\chi_0} \left(\frac{l+1/2}{\kappa_J} \right) \right\} \kappa_J^2 = 1 \quad (3.41)$$

i.e.

$$\kappa_J = \kappa_J(l) = B_l + \sqrt{B_l^2 + A} \quad (3.42)$$

where $B_l \equiv \{[\ln(c_s^2)]_{,\chi} - [\ln|\theta_{,\chi\chi}/\theta|]_{,\chi}\} \Big|_{\chi_0} (2l+1)/4$ and $A \equiv [|\theta_{,\chi\chi}/\theta|/(2c_s^2)] \Big|_{\chi_0}$. We immediately see that $I_{\Theta_i}(k\eta_{1/2} > 1)$ can be divided into two parts. The first part is identical to (3.35) except for the upper limit of the integral. Indeed now the upper limit is $\kappa_J(l)$. In order to derive the second part we note that now the lower limit of the integral in κ is $\kappa_J(l)$ and that $u_k(\eta_k) = C_k(\eta_J) \cos\left(k \int_{\eta_J}^{\eta_k} c_s(\tilde{\eta}) d\tilde{\eta}\right)$ where η_J is the conformal time when $c_s^2|_{\eta_J} k^2 \sim |\theta''/\theta|_{\eta_J}$ (η_J is function of k) and

$$C_k(\eta_J) = 2\Phi_k(0) \frac{\left[1 - \frac{\mathcal{H}(\eta_J)}{a^2(\eta_J)} \left(I_R + \int_{\eta_R}^{\eta_J} a^2(\tilde{\eta}) d\tilde{\eta}\right)\right]}{(P + \rho)^{1/2}(\eta_J)}. \quad (3.43)$$

3.4 Discussion of some examples

In most of the UDM models there are several properties in common. It is easy to see that in Eq.(3.27) I_R is negligible because of the low value of a_{eq} .

Moreover in the various models usually we have that strong differences with respect to the ISW effect in the Λ CDM case can be produced from those scales that are inside the Jeans length as the photons pass through them. For these scales (which depend on the particular model) the perturbations of the unified dark matter fluid play the main role. On larger scales instead we find that they play no role and ISW signatures different from the Λ CDM case can come only from the different background expansion history. We have found that when $k^2 \gg k_J^2 = c_s^{-2} |\theta''/\theta|$ (see (3.15)) one must take care of the term in Eq. (3.38) proportional to $\mathcal{C}_{\{k1,l2,c^2\}}$. Indeed this term grows faster than the other integrals contained in (3.38) when l increases up to $l \approx 25$. It is responsible for a strong ISW effect and hence it will cause in the CMB power spectrum $l(l+1)C_l/(2\pi)$ a decrease in the peak to plateau ratio (once the CMB power spectrum is normalized). In order to avoid this effect, a sufficient (but not necessary) condition is that all the models have to satisfy $c_s^2 k^2 < |\theta''/\theta|$ for the scales of interest. The maximum constraint is found in correspondence of the scale at which the contribution Eq. (3.38) proportional to $\mathcal{C}_{\{k1,l2,c^2\}}$ takes its maximum value, that is $k \approx k_{eq}$. For example in the Generalized Chaplygin Gas model (GCG), i.e when $p = -\Lambda^{1/(1+\alpha)}/\rho^\alpha$ and $c_s^2 = -\alpha w$, we deduce that $|\alpha| < 10^{-4}$ (see Refs. [62] [58] [59] and [63]). This is also in accordance with [54] which performs an analysis on the mass power spectrum and gravitational lensing constraints thus finding a more stringent constraint.

As far as the generalized Scherrer solution models [57] are concerned, in these models the pressure of the unified dark matter fluid is given by $p = g_n(X - X_0)^n - \Lambda$, where g_n is a suitable constant and $n > 1$. The case $n = 2$ corresponds to unified model proposed by Scherrer [64]. In this case we find that imposing the constraint $c_s^2 k^2 < |\theta''/\theta|$ for the scales of interest we get that $\epsilon = (X - X_0)/X_0 < (n - 1) 10^{-4}$.

If we want to study in greater detail what happens in the GCG model when $c_s^2 k^2 \gg |\theta''/\theta|$ we discover the following things:

- for $10^{-4} < \alpha \leq 5 \times 10^{-3}$, where we are in the ‘‘Intermediate case’’. Now $c_s^2 = -\alpha w$ is very small and the background of the cosmic expansion history of the Universe is very similar to the Λ CDM model. In this situation the pathologies, described before, are completely negligible.
- When treating $6 \times 10^{-3} < \alpha \leq 1$ a very strong ISW effect will be produced and we have estimated the same orders of magnitude for the decrease of the peak to plateau ratio in the anisotropy spectrum $l(l+1)C_l/(2\pi)$ (once it is normalized) that can be inferred from the authors of [58] obtained in the numerical simulations (having assumed that the production of the peaks during the acoustic oscillations at recombination is similar to what happens in a Λ CDM model, since at recombination the effects of the sound speed will be negligible).

An important observation arises when considering those UDM models that reproduce the same cosmic expansion history of the Universe as the Λ CDM model. Among these models one can impose the condition $w = -c_s^2$ which, for example, is predicted by UDM models with a kinetic term of the Born-Infeld type [57] [67] [92]. In this case, computing the integral in Eq. (3.38) proportional to $\mathcal{C}_{\{k_1, l_2, c^2\}}$ which give the main contribution to the ISW effect we have estimated that the corresponding decrease of peak to plateau ratio is about one third with respect to what we have in the GCG when the value of α is equal to 1. The special case $\alpha = 1$ is called ‘‘Chaplygin Gas’’ (see for example [61]) and it is characterized by a background equation of state w which evolves in a different way to the standard Λ CDM case.

From these considerations we deduce that this specific effect stems only in part from the background of the cosmic expansion history of the Universe and that the most relevant contribution to the ISW effect is due to the value of the speed of sound c_s^2 .

Chapter 4

Halos of Unified Dark Matter Scalar Field

4.1 Introduction

A complete analysis of UDM models should necessarily include the study of static solutions of Einstein's field equations. This is complementary to the study of cosmological background solutions and would allow to impose further constraints to the Lagrangian of UDM models. The authors of Refs. [32] and [66] have studied spherically symmetric and static configuration for *k-essence* models. In particular, they studied models where the rotation velocity becomes flat (at least) at large radii of the halo. In these models the scalar field pressure is not small compared to the mass-energy density, similarly to what found in the study of general fluids in Refs. [93, 94, 95, 96], and the Einstein's equations of motion do not reduce to the equations of Newtonian gravity. Further alternative models have been considered, even with a canonical kinetic term in the Lagrangian, that describe dark matter halos in terms of bosonic scalar fields, see e.g. Refs. [97, 98, 99, 100, 101, 102, 103]. In this Chapter we assume that our scalar field configurations only depend on the radial direction. Three main results are achieved. First, we are able to find a purely kinetic Lagrangian which allows simultaneously to provide a flat rotation curve and to realize a unified model of dark matter and dark energy on cosmological scales. Second, we have found an invariance property of the expression for the halo rotation curve. This allows to find purely kinetic Lagrangians that reproduce the same rotation curves that are obtained starting from a given density profile within the standard Cold Dark Matter (CDM) paradigm. Finally, we consider a more general class of models with non-purely kinetic Lagrangians. In this case we extend to the static and spherically symmetric spacetime metric the procedure used in Ref. [57] to find UDM solutions in a cosmological setting. Such a procedure requires that the Lagrangian is constant along the classical trajectories; we are thus able to provide the conditions to obtain reasonable rotation curves within a UDM model of the type discussed in Ref. [57].

The plan of the Chapter is as follows. In Section 4.2 we provide the general framework for the study of static spherically symmetric solutions in UDM models. Section 4.3 is devoted to the general analysis of purely kinetic Lagrangians, while in Section 4.4 we

analyze more general models with non-canonical kinetic term.

4.2 Static solutions in Unified Dark Matter

We consider the following action

$$S = S_G + S_\varphi + S_b = \int d^4x \sqrt{-g} \left[\frac{R}{2} + \mathcal{L}(\varphi, X) \right] + S_b \quad (4.1)$$

where S_b describes the baryonic matter and we redefine

$$X = -\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi. \quad (4.2)$$

As in the previous chapters, we use units such that $8\pi G = c = 1$ and signature $(-, +, +, +)$ (greek indices run over spacetime dimensions, while latin indices label spatial coordinates). Obviously, also in this case, the energy-momentum tensor of the scalar field φ is defined in the following way

$$T_{\mu\nu}^\varphi = -\frac{2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \frac{\partial \mathcal{L}(\varphi, X)}{\partial X} \nabla_\mu \varphi \nabla_\nu \varphi + \mathcal{L}(\varphi, X) g_{\mu\nu}, \quad (4.3)$$

and its equation of motion reads

$$\nabla^\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right] = \frac{\partial \mathcal{L}}{\partial \varphi}. \quad (4.4)$$

We consider a scalar field which is static and spatially inhomogeneous, i.e. such that $X < 0$. In this situation the energy-momentum tensor is not described by a perfect fluid and its stress energy-momentum tensor reads

$$T_{\mu\nu}^\varphi = (p_{\parallel} + \rho) n_\mu n_\nu - \rho g_{\mu\nu} \quad (4.5)$$

where

$$\rho = -p_{\perp} = -\mathcal{L}, \quad (4.6)$$

$n_\mu = \nabla_\mu \varphi / \sqrt{-2X}$ and $p_{\parallel} = \mathcal{L} - 2X \partial \mathcal{L} / \partial X$. In particular, p_{\parallel} is the pressure in the direction parallel to n_μ whereas p_{\perp} is the pressure in the direction orthogonal to n_μ . It is simpler to work with a new definition of X . Indeed, defining $X = -\chi$ we have

$$n_\mu = \nabla_\mu \varphi / (2\chi)^{1/2} \quad (4.7)$$

$$p_{\parallel} = 2\chi \frac{\partial \rho}{\partial \chi} - \rho. \quad (4.8)$$

Let us consider for simplicity the general static spherically symmetric spacetime metric i.e.

$$ds^2 = -\exp(2\alpha(r)) dt^2 + \exp(2\beta(r)) dr^2 + r^2 d\Omega^2, \quad (4.9)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and α and β are two functions that only depend upon r . As the authors of Refs. [32, 66] have shown, it is easy to see that the non-diagonal term T^{rt} vanishes. Therefore φ could be either strictly static or depend only on time. In this Chapter we study the solutions where φ depends on the radius only.

In the following we will consider some cases where the baryonic content is not negligible in the halo. In this case we will assume that most of the baryons are concentrated within a radius r_b . If we define M_* as the entire mass of the baryonic component then for $r > r_b$ we can simply assume that M_* is concentrated in the center of the halo.

Considering, therefore, the halo for $r > r_b$, starting from the Einstein's equations and the covariant conservation of the stress-energy (or from the equation of motion of the scalar field, Eq. (4.4)), we obtain

$$\frac{1}{r^2} \{1 - [r \exp(-2\beta)]'\} = \rho \iff \frac{dM}{dr} = 4\pi\rho r^2, \quad (4.10)$$

$$\frac{1}{r^2} \{ \exp[-2(\alpha + \beta)] [r \exp(2\alpha)]' - 1 \} = p_{\parallel} \iff \alpha' = \frac{\frac{M+M_*}{8\pi} + \frac{p_{\parallel}r^3}{2}}{r^2 \left[1 - \frac{M+M_*}{4\pi r}\right]}, \quad (4.11)$$

$$\frac{\exp[-(\alpha + 2\beta)]}{r} \{ [r \exp \alpha]' \beta' - [r (\exp \alpha)']' \} = \rho, \quad (4.12)$$

$$\frac{dp_{\parallel}}{dR} = -(p_{\parallel} + \rho) \quad (4.13)$$

(which are the 00, rr and $\theta\theta$ components of Einstein's equations and the r component of the continuity equation respectively) where

$$\exp(-2\beta(r)) = 1 - (M + M_*)/(4\pi r)$$

and

$$R = \ln[r^2 \exp(\alpha(r))]$$

. Here a prime indicates differentiation with respect to the radius r . A first comment is in order here. If *i*) $\beta' = 0$ and *ii*) $[r (\exp \alpha)]' > 0$, then we can immediately see that $\rho < 0$. These conditions must therefore be avoided when trying to find a reasonable rotation curve. For example, neglecting the baryonic mass, the special case of $\rho = A/r^2$ and $\exp(\alpha) \sim r^m$, where A and m are constants, fall into this case. We thus recover the *no-go* theorem derived in Ref. [66] under the assumption that the rotation curve $v_c \ll 1$ is constant for all r .

The value of the circular velocity v_c is determined by the assumption that a massive test particle is also located at $\theta = \pi/2$. We define as massive test particle the object that sends out a luminous signal to the observer who is considered to be stationary and far away from the halo. Considering the motion of massive test particle, say a star, in a such a halo, its trajectory is then described by a curve $x^\mu = (t, r, \theta, \phi)$ parameterized by some affine parameter; here we use its proper time τ . Its four velocity is then simply $u^\mu \equiv dx^\mu/d\tau$. Due to spherical symmetry, we can assume without loss of generality that

the star's ecliptic is located in the $\theta = \pi/2$ plane. Since the star is a massive particle, its norm is $u_\mu u^\mu = -1$, which becomes the constraint equation

$$\exp(2\alpha)\dot{t}^2 - \exp(2\beta)\dot{r}^2 - r^2\dot{\phi}^2 = 1, \quad (4.14)$$

where a dot denotes a derivative with respect to proper time τ . Since the metric does not explicitly depend on θ , the star's angular momentum l is conserved,

$$l = r^2\dot{\phi}. \quad (4.15)$$

Similarly, the metric does not explicitly depend on t , and there is a conserved energy E ,

$$E = \exp(2\alpha)\dot{t}. \quad (4.16)$$

Substituting equations (4.16) and (4.15) into equation (4.14), we find a first integral for the motion of the star,

$$\frac{1}{2}\dot{r}^2 + \mathcal{V}(r) = 0, \quad (4.17)$$

where its effective potential is

$$\mathcal{V}(r) = \frac{1}{2} \exp(-2\beta) \left(1 + \frac{l^2}{r^2}\right) - \frac{1}{2} E^2 \exp[-2(\alpha + \beta)]. \quad (4.18)$$

Note that the potential explicitly depends on the energy. Stationary orbits at radius r exist if \mathcal{V} and $d\mathcal{V}/dr$ vanish at that radius. The former condition yields

$$1 + \frac{l^2}{r^2} = E^2 \exp[-2\alpha(r)], \quad (4.19)$$

whereas the latter gives us

$$-\beta' \left(1 + \frac{l^2}{r^2}\right) - \frac{l^2}{r^3} + E^2 (\alpha' + \beta') \exp[-2\alpha(r)] = 0. \quad (4.20)$$

Substituting equation (4.19) into equation (4.20) and using the Eqs. (4.10), (4.11) and (4.4), we get the following equation

$$\frac{l^2/r^2}{1 + l^2/r^2} = \frac{(M + M_*)(r)}{8\pi r} + r^2 \frac{p_{\parallel}(r)}{2}, \quad (4.21)$$

which directly relates the angular momentum l to the density profile of the halo.

In this case, through the definition of the star's angular momentum l and the Eq. (4.21), the value of $v_c \equiv l/r$ can be rewritten as

$$v_c^2 = \frac{p_{\parallel} r^2/2 + (M + M_*)/(8\pi r)}{1 - [p_{\parallel} r^2/2 + (M + M_*)/(8\pi r)]}, \quad (4.22)$$

but when we consider the weak-field limit condition $(M + M_*)/(8\pi r) \ll 1$ and since the rotation velocities of the halo of a spiral galaxy are typically non-relativistic, $v_c \ll 1$, Eq. (4.22) simplifies to [32]

$$v_c^2 \approx \frac{M + M_*}{8\pi r} + \frac{p_{\parallel} r^2}{2}. \quad (4.23)$$

A second comment follows from the fact that the pressure is not small compared to the mass-energy density. In other words we do not require that general relativity reduces to Newtonian gravity (see also Refs. [93, 94, 95, 96]). Notice also that in the region where $v_c \approx \text{const.} \ll 1$ it is easy to see that in general $\exp(\alpha) \approx \text{const.}$ since from Eqs. (4.11) and (4.23) one obtains $r\alpha' \approx v_c^2$.

Finally, let us point out one of our main results. We can see that the relation (4.23) is invariant under the following transformation

$$\rho \longrightarrow \tilde{\rho} = \rho + \sigma(r) \quad p_{\parallel} \longrightarrow \tilde{p}_{\parallel} = p_{\parallel} + q(r) \quad (4.24)$$

if

$$3q(r) + rq(r)' = -\sigma(r), \quad (4.25)$$

up to a proper choice of some integration constants. Thanks to this transformation we can consider an ensemble of solutions that have the same rotation curve. We will come back to this point in more detail in the next section. Obviously, these solutions have to satisfy the Einstein's equations (4.10), (4.11) and (4.12), and the covariant conservation of the stress-energy (4.13). Moreover, we will require the validity of the weak energy conditions, $\rho \geq 0$ and $p_{\parallel} + \rho \geq 0$, i.e.

$$2 \frac{\exp(-2\beta)}{r} (\alpha' + \beta') = 2\chi \frac{\partial \rho}{\partial \chi} \geq 0. \quad (4.26)$$

In the following sections we will consider first a purely kinetic Lagrangian $\mathcal{L}(X)$ and then two Lagrangians $\mathcal{L} = f(\varphi)g(X)$ and $\mathcal{L} = g(X) - V(\varphi)$.

4.3 Unified Dark Matter models with purely kinetic Lagrangians

Let us consider a scalar field Lagrangian \mathcal{L} with a non-canonical kinetic term that depends only on X or χ . Moreover, in this section we assume that $M_* = 0$ (or $M \gg M_*$).

First of all we must impose that \mathcal{L} is negative when $X < 0$, so that the energy density is positive. Therefore, we define a new positive function

$$g_s(\chi) \equiv -\mathcal{L}(X). \quad (4.27)$$

As already explained in Ref. [32], when the equation of state $p_{\parallel} = p_{\parallel}(\rho)$ is known, one can write the purely kinetic Lagrangian that describes this dark fluid with the help of Eqs. (4.6) and (4.8). Alternatively, using (4.13), one can connect p_{\parallel} and ρ in terms of r through the variable R . Moreover, it is easy to see that starting from the field equation of

motion (4.4), there exists another relation that connects χ (i.e. X) with r . This relation is

$$\chi \left[\frac{dg_s(\chi)}{d\chi} \right]^2 = \frac{k}{[r^2 \exp \alpha(r)]^2} \quad (4.28)$$

with k a positive constant. If we add an additive constant to $g_s(\chi)$, the solution (4.28) remains unchanged. One can see this also through the Eq. (4.13). Indeed, using Eqs. (4.6) and (4.8) one immediately finds that Eq. (4.13) is invariant under the transformation $\rho \rightarrow \rho + K$ $p_{\parallel} \rightarrow p_{\parallel} - K$. In this way we can add the cosmological constant $K = \Lambda$ to the Lagrangian and we can describe the dark matter and the cosmological constant-like dark energy as a single dark fluid i.e. as Unified Dark Matter (UDM).

Let us notice that one can adopt two approaches to find reasonable rotation curves $v_c(r)$. A static solution can be studied in two possible ways:

- i) The first approach consists simply in adopting directly a Lagrangian that provides a viable cosmological UDM model and exploring what are the conditions under which it can give a static solution with a rotation curve that is flat at large radii. This prescription has been already applied, for example, in Ref. [32].
- ii) A second approach consists in exploiting the invariance property of Eq. (4.23), with respect to the transformation (4.24) (when the condition (4.25) is satisfied). Usually in the literature one reduces the problem to the Newtonian gravity limit, because one makes use of a CDM density profile, i.e. one assumes that in Eq. (4.23), $p_{\parallel} \ll M/(4\pi r^3)$. We can therefore use Eqs. (4.24) and (4.25) to obtain energy density and pressure profiles $\rho(r)$ and $p_{\parallel}(r)$ that reproduce the same rotation curve in a model with non-negligible pressure. Next, we find an acceptable equation of state $p_{\parallel} = p_{\parallel}(\rho)$ such that we can reconstruct, through Eqs. (4.6) and (4.8), the expression for the Lagrangian \mathcal{L} . Such a procedure establishes a mapping between UDM and CDM solutions that predict the same halo rotation curve $v_c(r)$. As a starting point we could, of course, use very different CDM density profiles to this aim, such as the modified isothermal-law profile [104], the Burkert profile [105], the Moore profile [106], the Navarro-Frenk-White profile [107, 108] or the profile proposed by Salucci et al. (see for example [109]).

As we have already mentioned, the possible solutions one finds in this way have to satisfy the Einstein equations (4.10), (4.11) and (4.12), the conservation of stress-energy (4.13) and the weak energy conditions. Moreover, the resulting UDM scalar field Lagrangian must be able to provide cosmological solutions that yield an acceptable description of the cosmological background (see, e.g., Ref. [57]) and low effective speed of sound (see for example Refs. [14, 44, 56]) so that cosmic structure formation successfully takes place and CMB anisotropies fit the observed pattern [54, 55, 58, 59].

Below, using approach i), we provide a worked example of a UDM model with purely kinetic Lagrangian which is able to describe a flat halo rotation curve and then, using approach ii), we give a general systematic procedure to obtain a possible Lagrangian of UDM model starting from a given CDM density profile.

4.3.1 Approach i): The generalized Scherrer solution

Let us consider the generalized Scherrer solution models obtained in Ref. [57]. These models are described by the following Lagrangian

$$\mathcal{L} = -\Lambda + g_n \left(X - \hat{X} \right)^n \quad (4.29)$$

where $g_n > 0$ is a suitable constant and $n > 1$. The case $n = 2$ corresponds to the unified model proposed by Scherrer [64]. If we impose that today $[(X - \hat{X})/\hat{X}]^n \ll 1$, the background energy density can be written as

$$\rho(a(t)) = \rho_\Lambda + \rho_{\text{DM}} , \quad (4.30)$$

where ρ_Λ behaves like a “dark energy” component ($\rho_\Lambda = \text{const.}$) and ρ_{DM} behaves like a “dark matter” component i.e. $\rho_{\text{DM}} \propto a^{-3}$, with $a(t)$ the scale factor.

A static solution for the generalized Scherrer model can be obtained in two possible ways:

- 1) Starting from the analysis of Ref. [66], in the case of a barotropic Lagrangian for the homogeneous field. The authors of Ref. [66] indeed concluded that for $n \gg 1$ flat halo rotation curves can be obtained. In particular they studied spherically symmetric solutions with the following metric,

$$ds^2 = - \left(\frac{r}{r_\star} \right)^b dt^2 + N(r) dr^2 + r^2 d\Omega^2. \quad (4.31)$$

where r_\star is a suitable length-scale and $b = 2v_c^2$. In the trivial case where $N(r)$ is constant they find $\mathcal{L}(X) \propto X^{2/b}$ with $b \ll 1$. For $X \gg \hat{X}$ the Lagrangian $\mathcal{L} = -\Lambda + g_n(X - \hat{X})^n$ takes precisely this form.

- 2) In the analysis of Ref. [32], solutions where φ is only a function of the radius are considered. When the Lagrangian has the form $\mathcal{L} \propto X^n$, with $n \sim 10^6$ the halo rotation curve becomes flat at large radii. In this case n must be an odd natural number, such that the energy density is positive. Our model is able to reproduce this situation when the matter density is large, i.e. when $|X| \gg \hat{X}$.

Alternatively, if we wish to avoid large n (c.f. case 2) above) we can start from the following Lagrangian

$$\mathcal{L} = -\Lambda + \epsilon_X g_n \left(|X| - \hat{X} \right)^n \quad (4.32)$$

where ϵ_X is some differentiable function of X that is 1 when $X \geq \hat{X}$ and -1 when $X \leq -\hat{X} < 0$. In this way when $X > \hat{X} > 0$ we recover the Lagrangian of the generalized Scherrer solutions. When $X < 0$ and $\chi = -X > \hat{X}$ we get

$$\mathcal{L} = -\Lambda - g_n \left(\chi - \hat{X} \right)^n \quad (4.33)$$

and, with the help of Eqs. (4.6) and (4.8), we obtain

$$\rho = -p_\perp = -\mathcal{L} , \quad p_\parallel = (2n - 1)g_n \left(\chi - \hat{X} \right)^n + 2ng_n\hat{X} \left(\chi - \hat{X} \right)^{n-1} - \Lambda . \quad (4.34)$$

Now, requiring that χ be close to \hat{X} (i.e. $(\chi - \hat{X}) \ll \hat{X}$) and $2ng_n\hat{X}(\chi - \hat{X})^{n-1} \gg O(\Lambda)$, and starting from the relation (4.28) that connects χ with r , we get

$$(\chi - \hat{X})^{n-1} = \frac{k^{1/2}}{ng_n\hat{X}^{1/2}} \frac{1}{r^2 \exp(\alpha(r))}. \quad (4.35)$$

Consistency with our approximations implies that we have to consider the following expressions for radial configurations with r bigger than a minimum radius r_{min} . In this case $p_{||}$ and ρ become

$$p_{||} = \frac{A}{r^2 \exp(\alpha(r))}, \quad \rho = \frac{B}{[r^2 \exp(\alpha(r))]^{n/(n-1)}} \quad (4.36)$$

where $A = 2(k\hat{X})^{1/2}$ and $B = g_n \left[k^{1/2}/(ng_n\hat{X}^{1/2}) \right]^{n/(n-1)}$.

Using the Eqs. (4.10) and (4.11), we be able to calculate the values of the metric terms $\exp(\alpha)$ and $\exp(\beta)$ and, thus the value of ρ and $p_{||}$. Alternatively we know that when $v_c \approx \text{const.} \ll 1$ at large radii, in a first approximation, we can set $\exp(\alpha(r)) \approx C = \text{const.}$ Therefore for $n \neq 3$, we can write the function M as

$$M(r) \approx \frac{4\pi B}{C^{n/(n-1)}} \left(\frac{n-1}{n-3} r^{\frac{n-3}{n-1}} + D \right) \quad (4.37)$$

where we could also set $D = 0$ for $n > 3$. Instead, when $1 < n < 3$, the second term has to be larger than the first one.

In these cases v_c^2 becomes

$$v_c^2(r) \approx \frac{A}{2C} + \frac{B}{2C^{n/(n-1)}} \left(\frac{n-1}{n-3} \frac{1}{r^{2/(n-1)}} + \frac{D}{r} \right). \quad (4.38)$$

For $n = 3$ we have

$$M(r) \approx \frac{4\pi B}{C^{3/2}} \ln\left(\frac{r}{\bar{r}}\right) + M(\bar{r}) \quad (4.39)$$

where $r > \bar{r}$ and

$$v_c^2 \approx \frac{A}{2C} + \frac{B}{2C^{3/2}} \frac{1}{r} \ln\left(\frac{r}{\bar{r}}\right) + \frac{M(\bar{r})}{8\pi r}. \quad (4.40)$$

In other words we see that the circular velocity becomes approximately constant for sufficiently large r .

However, let us stress that $\exp(\alpha(r))$ cannot be strictly constant, and that it should be chosen in such a way that the positivity of Eq. (4.12) is ensured.

This example can be generalized also to $M_* \neq 0$. Obviously, in such a case we have to assume that $r > r_b \geq r_{min}$. In this case k , r_{min} , A , B (through $\exp(\beta(r))$) and C depend on M_* .

The spherical top-hat solution for this model, which provides the link with the cosmological initial conditions, is described in the second Chapter.

4.3.2 Approach ii): A general prescription to obtain UDM Lagrangians starting from a profile of an energy density distribution of CDM

Defining the energy density distribution of CDM as $\rho_{\text{CDM}}(r)$ (with $p_{\text{CDM}} = 0$), the transformation (4.24) becomes

$$\rho(r) = \rho_{\text{CDM}}(r) + \sigma(r), \quad p_{\parallel}(r) = q(r). \quad (4.41)$$

Now, starting from a given CDM density profile, through Eqs. (4.10), (4.11), (4.13) and (4.25) we can determine $\exp(\alpha)$, $\exp(\beta)$, ρ and p_{\parallel} . In a second step we provide the conditions to ensure that the energy density is positive¹. In this case, after some simple but lengthy calculations, we find

$$\begin{aligned} \mathcal{Q}'(r) \left(r \frac{M_{\text{CDM}}(r)}{4\pi} - 2r\mathcal{Q}(r) \right) - 2\mathcal{Q}^2(r) \\ + \mathcal{Q}(r) \left(4r + 3 \frac{M_{\text{CDM}}(r)}{4\pi} + 4r^3 \rho_{\text{CDM}} \right) &= \frac{r M_{\text{CDM}}(r)}{4\pi} (4 + 3r^2 \rho_{\text{CDM}}), \end{aligned} \quad (4.42)$$

$$\mathcal{B}(r) = \mathcal{Q}(r) - \frac{M_{\text{CDM}}(r)}{4\pi}, \quad (4.43)$$

$$\mathcal{A}(r) = \frac{\mathcal{Q}(r) + \mathcal{B}(r)}{2\mathcal{B}(r)}, \quad (4.44)$$

$$\sigma(r) = \frac{1 - \mathcal{Q}'(r)}{r^2} \quad (4.45)$$

where $\mathcal{Q}(r) = r(r^2 q + 1)$, $\mathcal{B}(r) = r \exp(-2\beta)$ and $\mathcal{A}(r) = (r\alpha' + 1)$. Here we define $M_{\text{CDM}}(r) = 4\pi \int_0^r \tilde{r}^2 \rho_{\text{CDM}}(\tilde{r}) d\tilde{r}$. At this point it is easy to see that Eq. (4.42) does not admit a simple analytical solution for a generic ρ_{CDM} . On the other hand we know that, through ρ_{CDM} , all these functions depend on the velocity rotation curve $v_c(r)$. Moreover $v_c^2(r) \ll 1$. Therefore, defining \bar{v}_c as the value that v_c assumes when the rotation curve is flat at large radii or the maximum value of v_c with a particular profile of ρ_{CDM} , we can expand \mathcal{Q} , \mathcal{A} and \mathcal{B} as

$$\begin{aligned} \mathcal{Q}(r) &= \mathcal{Q}_{(0)}(r) + \bar{v}_c^2 \mathcal{Q}_{(1)}(r) + \frac{(\bar{v}_c^2)^2}{2!} \mathcal{Q}_{(2)}(r) + \dots, \\ \mathcal{A}(r) &= \mathcal{A}_{(0)}(r) + \bar{v}_c^2 \mathcal{A}_{(1)}(r) + \frac{(\bar{v}_c^2)^2}{2!} \mathcal{A}_{(2)}(r) + \dots, \\ \mathcal{B}(r) &= \mathcal{B}_{(0)}(r) + \bar{v}_c^2 \mathcal{B}_{(1)}(r) + \frac{(\bar{v}_c^2)^2}{2!} \mathcal{B}_{(2)}(r) + \dots \end{aligned} \quad (4.46)$$

Following this procedure we can determine ρ and p_{\parallel} in a perturbation way, i.e.

$$\rho(r) = \rho_{(0)}(r) + \bar{v}_c^2 \rho_{(1)}(r) + \frac{(\bar{v}_c^2)^2}{2!} \rho_{(2)}(r) + \dots, \quad (4.47)$$

¹Thanks to this condition, through Einstein's Eq. (4.12), we can evade the *no-go* theorem derived in Ref. [66].

$$p_{\parallel}(r) = p_{\parallel(0)}(r) + \bar{v}_c^2 p_{\parallel(1)}(r) + \frac{(\bar{v}_c^2)^2}{2!} p_{\parallel(2)}(r) + \dots \quad (4.48)$$

Now, looking at the various CDM density profiles which have been proposed in the literature [104, 105, 106, 107, 108, 109], we see that we can always take ρ_{CDM} as

$$\rho_{\text{CDM}}(r) = \bar{v}_c^2 \rho_{\text{CDM}(1)}(r), \quad (4.49)$$

then

$$M_{\text{CDM}}(r) = \bar{v}_c^2 M_{\text{CDM}(1)}(r) = 4\pi \bar{v}_c^2 \int_0^r \tilde{r}^2 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r}. \quad (4.50)$$

For the zeroth-order terms we immediately obtain

$$\begin{aligned} \mathcal{Q}_{(0)} &= r, \\ \mathcal{A}_{(0)} &= 1, \\ \mathcal{B}_{(0)} &= r. \end{aligned} \quad (4.51)$$

At the first order we get

$$\begin{aligned} \mathcal{Q}_{(1)} &= \frac{2}{r} \int_0^r \tilde{r}^3 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r}, \\ \mathcal{A}_{(1)} &= \frac{1}{2r} \frac{M_{\text{CDM}(1)}(r)}{4\pi}, \\ \mathcal{B}_{(1)} &= \frac{2}{r} \int_0^r \tilde{r}^3 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r} - \frac{M_{\text{CDM}(1)}(r)}{4\pi}. \end{aligned} \quad (4.52)$$

For completeness we write also the second order for \mathcal{Q}

$$\mathcal{Q}_{(2)} = \frac{1}{r} \int_0^r d\tilde{r} \frac{M_{\text{CDM}(1)}(\tilde{r})}{4\pi} \left[\frac{2}{\tilde{r}} \mathcal{Q}_{(1)}(\tilde{r}) - \tilde{r}^2 \rho_{\text{CDM}(1)}(\tilde{r}) \right]. \quad (4.53)$$

Let us stress that if one considers also terms $O(\bar{v}_c^4)$, Eq. (4.22) instead of Eq. (4.23) should be used. In such a case, v_c slightly changes with respect to the velocity rotation curve that one obtains using a CDM density profile.

For our purposes we can consider only the zeroth and the first-order terms. At this point, we can finally calculate the value of ρ and p_{\parallel} . We get

$$\rho(r) = \rho_{\text{CDM}}(r) + \frac{1 - \mathcal{Q}'(r)}{r^2} = \bar{v}_c^2 \left(\frac{2}{r^4} \int_0^r \tilde{r}^3 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r} - \rho_{\text{CDM}(1)}(r) \right), \quad (4.54)$$

$$p_{\parallel}(r) = \frac{\mathcal{Q}(r) - r}{r^3} = \bar{v}_c^2 \frac{2}{r^4} \int_0^r \tilde{r}^3 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r}. \quad (4.55)$$

As far as the values of the metric terms $\exp(\alpha)$ and $\exp(\beta)$ are concerned, we obtain the following expressions

$$\exp(2\alpha) = \exp(2\alpha(\hat{r})) \exp \left[\bar{v}_c^2 \int_{\hat{r}}^r \frac{1}{\tilde{r}^2} \frac{M_{\text{CDM}(1)}(\tilde{r})}{4\pi} d\tilde{r} \right] \quad (4.56)$$

$$\exp(-2\beta) = 1 + \frac{\bar{v}_c^2}{r^2} \left(2 \int_0^r \tilde{r}^3 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r} - r \frac{M_{\text{CDM}(1)}(r)}{4\pi} \right). \quad (4.57)$$

Now, it is immediate to see that if we want a positive energy density we have to impose $2 \int_0^r \tilde{r}^3 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r} \geq r^4 \rho_{\text{CDM}(1)}(r)$. From Eq. (4.10) we know that $M(r) = 4\pi \int_{\hat{r}(0)}^r \tilde{r}^2 \rho(\tilde{r}) + M(\hat{r}(0))$ and $M_{\text{CDM}}(r) = 4\pi \int_{\bar{r}}^r \tilde{r}^2 \rho_{\text{CDM}}(\tilde{r}) d\tilde{r} + M_{\text{CDM}}(\bar{r})$. Therefore we need to know what is the relation between \bar{r} and $\hat{r}(0)$. This condition is easily obtained if we make use of Eq. (4.23). Indeed, we get

$$\frac{M_{(1)}(\hat{r}(0)) - M_{\text{CDM}(1)}(\bar{r})}{4\pi} + \frac{2}{\hat{r}(0)} \int_0^{\hat{r}(0)} \tilde{r}^3 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r} = \int_{\bar{r}}^{\hat{r}(0)} \tilde{r}^2 \rho_{\text{CDM}(1)}(\tilde{r}) d\tilde{r}, \quad (4.58)$$

which finally guarantees the invariance of the rotation velocity with respect to the transformation in Eqs. (4.24) and (4.25).

Let us, to a first approximation, parametrize the various CDM density profiles, at very large radii (i.e. when we can completely neglect the baryonic component) as

$$\rho_{\text{CDM}} = \frac{\kappa \bar{v}_c^2}{r^n} \quad (4.59)$$

where κ is a proper positive constant which depends on the particular profile that is chosen [104, 105, 106, 107, 108, 109]. For example for many of the density profiles the slope is $n = 3$ for large radii [105, 106, 107, 108, 109].

In this case a positive energy density $\rho > 0$ requires $n \geq 2$. At this point let us focus on the case where $2 \leq n < 4$, since this gives rise to the typical slope of most of the density profiles studied in the literature. Therefore we obtain for $\rho(r)$ and $p_{\parallel}(r)$:

$$\rho(r) = \bar{v}_c^2 \kappa \frac{n-2}{4-n} \frac{1}{r^n}, \quad p_{\parallel}(r) = \bar{v}_c^2 \kappa \frac{2}{4-n} \frac{1}{r^n}. \quad (4.60)$$

In particular,

- 1) for $n = 2$, we get

$$\rho(r) = 0, \quad p_{\parallel}(r) = \rho_{\text{CDM}} = \bar{v}_c^2 \kappa \frac{1}{r^2}, \quad (4.61)$$

and for the relation between $\hat{r}(0)$ and \bar{r} one can choose, for example, $\hat{r}(0) = \bar{r} = 0$. In other words, for large radii we have that $\rho(r) \ll p_{\parallel}(r)$.

- 2) Also for $2 < n < 3$ one can choose $\hat{r}(0) = \bar{r} = 0$.

- 3) For $n = 3$

$$\rho(r) = \rho_{\text{CDM}}, \quad p_{\parallel}(r) = \bar{v}_c^2 \kappa \frac{2}{r^3}, \quad (4.62)$$

and, through Eq. (4.58), we have to impose that

$$\frac{M_{(1)}(\hat{r}(0)) - M_{\text{CDM}(1)}(\bar{r})}{4\pi} = \ln \left(\frac{\hat{r}(0)}{\bar{r}} \right) - 2. \quad (4.63)$$

Notice that the energy density profile is the same as the CDM one only for large radii so that $M_{(1)}(r)$ differs from $M_{\text{CDM}(1)}(r)$.

4) In addition, for $3 < n < 4$, also through Eq. (4.58), we have to impose that

$$\frac{M_{(1)}(\hat{r}_{(0)}) - M_{\text{CDM}(1)}(\bar{r})}{4\pi} = \frac{\bar{r}^{3-n}}{n-3} - \frac{(n-2)}{(4-n)(n-3)} \hat{r}_{(0)}^{3-n}. \quad (4.64)$$

Now let us focus where $2 < n < 4$. Starting from Eq. (4.60) to express $p_{\parallel} = p_{\parallel}(\rho)$ we solve Eq. (4.8) to recover the Lagrangian for the scalar field

$$\rho(\chi) = -\mathcal{L} = k\chi^{\frac{n}{2(n-2)}}, \quad p(\chi) = \frac{2k}{(n-2)}\chi^{\frac{n}{2(n-2)}} \quad (4.65)$$

where k is a positive integration constant. We can see that, for this range of n , the exponent is larger than 1; thus there are no problems with a possible instability of the Lagrangian (see Refs. [32, 15, 31]). Therefore, through the transformation $\rho \rightarrow \rho + \Lambda$ $p_{\parallel} \rightarrow p_{\parallel} - \Lambda$, this Lagrangian can be extended to describe a unified model of dark matter and dark energy. Indeed, starting from the Lagrangian of the type (4.32), when $|X| \gg \hat{X}$ and if $k = g_n$, \mathcal{L} takes precisely the form (4.65).

Finally, we want to stress that this prescription does not apply only to the case of an adiabatic fluid, such as the one provided by scalar field with a purely kinetic Lagrangian, but it can be also used for more general Lagrangians $\mathcal{L}(\varphi, X)$.

4.4 Unified Dark Matter models with non-purely kinetic Lagrangians

Let us consider more general Lagrangians of type $\mathcal{L} = \mathcal{L}(\varphi, X)$, with a non-canonical kinetic term, in order to find a UDM model with acceptable cosmological speed of sound. In this case we have one more degree of freedom: the scalar field configuration itself. Therefore, we have to impose a new condition to the solutions of the equation of motion. Ref. [57] required that the Lagrangian of the scalar field is constant along the classical trajectories. We want to know whether such a condition could be applied to the static spherically symmetric spacetime metric. We would also like to know what the behavior of the rotation velocity v_c in the halo of a spiral galaxy is like for this class of models. In the next subsections we will consider first a Lagrangian of the form $\mathcal{L} = f(\varphi)g(X)$ and then a Lagrangian of the form $\mathcal{L} = g(X) - V(\varphi)$. In these cases, for simplicity, we will assume that $f(\varphi)$ and $V(\varphi)$ are positive.

4.4.1 Lagrangian of the type $\mathcal{L} = f(\varphi)g(X)$

Let us write the Lagrangian in the form $\mathcal{L} = f(\varphi)g(X) = -f(\varphi)g_s(\chi)$. Immediately we notice that the requirement of having a positive energy density imposes that $g_s(\chi)$ is positive. In this particular case the equation of motion (4.4) becomes

$$\frac{d}{dR} \left\{ \ln \left| 2\chi \frac{dg_s(\chi)}{d\chi} - g_s(\chi) \right| \right\} + 2\chi \frac{dg_s(\chi)}{d\chi} \left[2\chi \frac{dg_s(\chi)}{d\chi} - g_s(\chi) \right]^{-1} = -\frac{d \ln f(\varphi)}{dR}. \quad (4.66)$$

Moreover from Eq. (4.8) we obtain for p_{\parallel}

$$p_{\parallel} = f(\varphi)g_s(\chi) \left\{ 2\chi \frac{d \ln [g_s(\chi)]}{d\chi} - 1 \right\} . \quad (4.67)$$

Following the procedure previously explained we impose the constraint $\mathcal{L} = -\rho = -\Lambda$, i.e.

$$f(\varphi) = \frac{\Lambda}{g_s(\chi)} , \quad (4.68)$$

which, inserted in the equation of motion (4.66), allows to find the following general solution

$$\chi \frac{d \ln [g_s(\chi)]}{d\chi} = \frac{k/2}{r^2 \exp(\alpha)} , \quad (4.69)$$

where k is a constant of integration. Now, inserting Eqs. (4.68) and (4.69) into the relation (4.67), we obtain

$$p_{\parallel} = \frac{\Lambda k}{r^2 \exp(\alpha)} - \Lambda . \quad (4.70)$$

Using this expression and considering the halo for $r > r_b$ and $M \gg M_*$, we are finally able to get the expression for the rotation velocity

$$v_c^2 \simeq \frac{\Lambda k/2}{\exp(\alpha)} - \frac{\Lambda r^2}{3} . \quad (4.71)$$

If $\exp(\alpha) \approx \text{const.}$ this expression leads to a flat rotation curve for all radii $r < r_{\text{max}}$ such that $r_{\text{max}}^2 \ll 3k/(2 \exp(\alpha))$ and provided that the constant k is positive. Therefore, in the future we will always neglect the second term in Eq. (4.71).

It is important to stress that the results outlined in Eqs. (4.68)-(4.71) give an efficient recipe to obtain a flat halo rotation curve within the UDM scenario. Once a Lagrangian (i.e. $g(X)$) leading to a viable UDM model on cosmological scales is obtained by imposing the constraint $\mathcal{L} = -\Lambda$ (see Ref. [57]), a flat rotation curve is guaranteed through Eq.(4.71). There are however two important requirements that have to be satisfied. The function $g_s(\chi)$ must allow for a positive integration constant k through Eq.(4.69), and the Lagrangian must satisfy the stability conditions discussed in Refs. [32, 15, 31]), which require $\partial\mathcal{L}/\partial X > 0$ and $\partial\mathcal{L}/\partial X + 2X\partial^2\mathcal{L}/\partial X^2 > 0$ (so that the speed of sound is positive both in the cosmological setting and for the static solution).

In the second part of this subsection we will consider first a situation where $M_* = 0$, in other words when our halo is composed only of the dark fluid, and then a situation where there is a non-negligible baryon contribution in the inner part of the halo.

Case $M_* = 0$: halo composed only of the dark fluid

Starting from $\rho = \Lambda$ and $p_{\parallel} = \Lambda k/(r^2 \exp(\alpha)) - \Lambda$ we can explicitly calculate the value of $\exp(\alpha)$ and $\exp(\beta)$ through Eqs. (4.10) and (4.11). Therefore, for $M_* = 0$ we get

$$\exp(-2\beta) = 1 - \frac{\Lambda r^2}{3} , \quad (4.72)$$

$$\exp(\alpha) = \frac{\Lambda k}{2} \left\{ \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2} \left[\frac{2\kappa}{\Lambda k} - \ln \frac{\left(\frac{\Lambda}{3}\right)^{1/2} r}{1 - \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2}} \right] + 1 \right\}, \quad (4.73)$$

where κ is a suitable positive integration constant. In particular, the value of κ should be such that the term on the RHS of Eq. (4.73) is positive, i.e.

$$\left(\frac{\Lambda}{3}\right)^{1/2} r > \left[\cosh \left(\frac{2\kappa}{\Lambda k} + 1 \right) \right]^{-1}. \quad (4.74)$$

It is very important to stress that, in this case, the weak energy conditions are satisfied. In other words, through this prescription, we are able to evade the *no-go* theorem derived in Ref. [66].

Using Eq. (4.73) and Eq. (4.71) we can obtain the following expression for the circular velocity

$$v_c^2(r) = \left\{ \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2} \left[\frac{2\kappa}{\Lambda k} - \ln \frac{\left(\frac{\Lambda}{3}\right)^{1/2} r}{1 - \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2}} \right] + 1 \right\}^{-1}. \quad (4.75)$$

In order to have values of $v_c \sim 10^{-3}$ we must impose that $2\kappa/(\Lambda k) \sim 10^6 \ll 3/(\Lambda r_{max}^2)$. Imposing this condition we can obtain an approximately flat halo rotation curve.

A simple inspection of Eqs. (4.73), (4.74) and (4.75) shows an interesting property of our result. There is a minimum radius $r_{min} \approx (\Lambda/3)^{-1/2} [\cosh(2\kappa/(\Lambda k))]^{-1}$ required for the validity of (4.74). Obviously it is necessary that $r_{min} \ll r_{gal} (\ll r_{max})$ where r_{gal} is the typical radius of our halo.

Case $M_* \neq 0$: non-negligible baryonic component in the center of the halo

In this subsection we assume that $r > r_b$. If $M \gg M_*$ we recover the same result of the previous subsection; if $M_* \gg O(\Lambda r^3)$, using Eqs. (4.10) and (4.11), we obtain

$$\exp(-2\beta) \approx 1 - \frac{M_*}{4\pi r} \quad (4.76)$$

$$\exp(\alpha) \approx \Lambda k \left\{ \left(1 - \frac{M_*}{4\pi r}\right)^{1/2} \left[\frac{\kappa_*}{\Lambda k} + \cosh^{-1} \left(\frac{4\pi r}{M_*} \right)^{1/2} \right] - 1 \right\}. \quad (4.77)$$

where κ_* is a suitable positive integration constant. In particular it easy to see that κ_* and k (through $\exp(\beta(r))$) depend also on the value of M_* , since one is considering $r > r_b$. Obviously, these functions exist only for $r > r_* > M_*/(4\pi)$, having defined r_* as the value of the radius for which $\exp(\alpha(r_*)) = 0$. In this case, using the approximate relation (4.23), v_c reads

$$v_c^2 \approx \frac{1}{2} \left\{ \left(1 - \frac{M_*}{4\pi r}\right)^{1/2} \left[\frac{\kappa_*}{\Lambda k} + \cosh^{-1} \left(\frac{4\pi r}{M_*} \right)^{1/2} \right] - 1 \right\}^{-1} + \frac{M_*}{8\pi r}. \quad (4.78)$$

To have halo rotation velocities $v_c \sim 10^{-3}$ for $r \gg r_*$, we need to impose $2\kappa_*/(\Lambda k) \sim 10^6$. One can see that this condition leads to $r_* \approx M_*/(4\pi)$. Moreover, also in this case we have a minimum radius r_{min} such that $v_c^2(r_{min}) = 1$. Starting from Eq. (4.22) we get

$$r_{min} \approx r_* . \quad (4.79)$$

4.4.2 Lagrangian of the type $\mathcal{L} = g(X) - V(\varphi)$

In this subsection we briefly discuss Lagrangians of the type $\mathcal{L} = g(X) - V(\varphi)$. Let us rewrite \mathcal{L} as $\mathcal{L} = -[g_s(\chi) + V(\varphi)]$. In order to have $\rho > 0$ we impose that $g_s(\chi) > 0$. For these Lagrangians the equation of motion (4.4) becomes

$$\chi \frac{dg_s(\chi)}{d\chi} \left\{ \frac{d \ln \left[\chi \left(\frac{dg_s(\chi)}{d\chi} \right)^2 \right]}{dR} + 2 \right\} = \frac{d \ln V(\varphi)}{dR} . \quad (4.80)$$

Requiring that the Lagrangian of the scalar field is constant along the classical trajectory, i.e

$$V(\varphi) = -g_s(\chi) + \Lambda , \quad (4.81)$$

from the Eq. (4.80) we get

$$\chi \frac{dg_s(\chi)}{d\chi} = \frac{k/2}{r^2 \exp \alpha} \quad (4.82)$$

where k is a positive constant.

Now, inserting Eq. (4.82) into Eq. (4.8) we obtain the same expressions for $p_{||}$ and v_c that we obtained in the last subsection i.e. Eqs. (4.70) and (4.71), respectively.

Chapter 5

Conclusions

In this thesis we have investigated the possibility that the dynamics of a single scalar field can account for a unified description of the dark matter and dark energy sectors. In particular in the the first Chapter we have studied the case of purely kinetic k-essence, showing that these models have only one late-time attractor with equation of state $w_\kappa = -1$ (cosmological constant). Studying all possible solutions near the attractor we have found a generalization of the Scherrer model [64], which describes a unified dark matter fluid.

Subsequently, we have generalized our analysis to the case where the Lagrangian is not purely kinetic and we have given general prescriptions [Eqs. (2.120) and (2.122)] to obtain unified models where the dark matter and a cosmological constant-like dark energy are described by a single scalar field along its attractor solutions.

In the second Chapter we perform an analytical study of the Integrated Sachs-Wolfe (ISW) effect within the framework of Unified Dark Matter models based on a scalar field. Our treatment is completely general in that all the results depend only on the speed of sound of the dark component and thus it can be applied to a variety of unified models, including those which are not described by a scalar field but relies on a single dark fluid.

In the third Chapter we have investigated static spherically symmetric solutions (“dark halos”) of Einstein’s equations for a scalar field with non-canonical kinetic term. Assuming that the scalar field depends only on the radius, we studied Unified Dark Matter models with purely kinetic Lagrangians. In particular, we obtained a purely kinetic Lagrangian which allows simultaneously to produce flat halo rotation curves and to realize a unified model of dark matter and dark energy on cosmological scales. Moreover, we gave a prescription to obtain UDM model solutions that have the same rotation curve $v_c(r)$ as a CDM model with a specified density profile. Next, we considered a more general class of Lagrangians with non-canonical kinetic term. In this case we have one more degree of freedom (the scalar field configuration itself) and we need to impose one more constraint. To this aim, we required that the Lagrangian is constant, $\mathcal{L} = -\Lambda$ along the solutions of the equation of motion. We have studied whether this condition can be applied to the static spherically symmetric space-time metric and what the behavior of $v_c(r)$ is for this class of models. Let us finally stress that these solutions allow for the possibility to find suitable Lagrangians that describe with a single fluid viable cosmological and static

solutions.

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