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Cutting Feynman Amplitudes: from Adaptive Integrand Decomposition to Differential Equations on Maximal Cuts

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Abstract

In this thesis we discuss, within the framework of the Standard Model (SM) of particle physics, advanced methods for the computation of scattering amplitudes at higher-order in perturbation theory. We offer a new insight into the role played by the *unitarity* of scattering amplitudes in the theoretical understanding and in the computational simplification of multi-loop calculations, at both the *algebraic* and the *analytical* level.

On the algebraic side, generalized unitarity can be used, within the *integrand reduction method*, to express the integrand associated to a multi-loop amplitude as a sum of fundamental, irreducible contributions, yielding to a decomposition of the amplitude as a linear combination of master integrals. In this framework, we propose an *adaptive* formulation of the integrand decomposition algorithm, which systematically adjusts to the kinematics of the individual integrands the dimensionality of the momentum space, where unitarity cuts are performed. This new formulation makes the integrand decomposition method, which in the past played a key role in streamlining one-loop computations, an efficient tool also at multi-loop level. We provide evidence of the generality of the proposed method by determining a universal parametrization of the integrand basis for two-loop amplitudes in arbitrary kinematics and we illustrate its technical feasibility in the first automated implementation of the analytic integrand decomposition at one- and two-loop level.

On the analytic side, we discuss the role of *maximal-unitarity* for the solution of *differential equations* for dimensionally regulated Feynman integrals. The determination of the analytic expression of the master integrals as a Laurent expansion in the regulating parameter $\epsilon = (4-d)/2$ requires the knowledge of the solutions of the homogeneous part of their differential equations at $\epsilon = 0$. In all cases where Feynman integrals fulfil genuine first-order differential equations with a linear dependence on ϵ, the corresponding homogeneous solutions can be determined through the Magnus exponential expansion. In this work we apply the latter to two-loop corrections to several SM processes such as the Higgs decay to weak vector bosons, $H \to WW$, triple gauge couplings ZWW and $\gamma^* WW$ and to the elastic scattering $\mu e \to \mu e$ in quantum electrodynamics.

In some cases, the inadequacy of the Magnus method hints at the presence of master integrals that obey higher-order differential equations, for which no general theory exists. In this thesis we show that maximal-cuts of Feynman integrals solve, by construction, such homogeneous equations regardless of their order and complexity. Hence, whenever a Feynman integral obeys an irreducible higher-order differential equation, the computation of its maximal-cut along independent contours provides a closed integral representation of the full set of independent homogeneous solutions. We apply this strategy to the two-loop elliptic integrals that appear in heavy-quark mediated corrections to $gg \to gg$ and $gg \to gH$ as well as to the three-loop massive banana graph, which constitute the first example of Feynman integral that obeys a third-order differential equation.

In the light of the results presented in this thesis, generalized unitarity emerges as a powerful tool not only for handling the algebraic complexity of perturbative calculations but also for investigating the nature of new classes of mathematical functions encountered in particle physics.

Publication list

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Journal papers

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- [2] A. Primo and L. Tancredi, *Maximal cuts and differential equations for Feynman integrals. An application to the three-loop massive banana graph*, Nucl. Phys. B **921** (2017) 316, [arXiv:1704.05465 [hep-ph]]

- [3] S. Di Vita, P. Mastrolia, A. Primo and U. Schubert, *Two-loop master integrals for the leading QCD corrections to the Higgs coupling to a W pair and to the triple gauge couplings ZWW and γ^*WW*, JHEP **1704** (2017) 008, [arXiv:1702.07331 [hep-ph]]

- [4] A. Primo and L. Tancredi, *On the maximal cut of Feynman integrals and the solution of their differential equations*, Nucl. Phys. B **916** (2017) 94, [arXiv:1610.08397 [hep-ph]].

- [5] P. Mastrolia, T. Peraro and A. Primo, *Adaptive Integrand Decomposition in parallel and orthogonal space*, JHEP **1608**, 164 (2016), [arXiv:1605.03157 [hep-ph]]

- [6] A. Primo and W. J. Torres Bobadilla, *BCJ Identities and d-Dimensional Generalized Unitarity*, JHEP **1604**, 125 (2016), [arXiv:1602.03161 [hep-ph]]

- [7] P. Mastrolia, A. Primo, U. Schubert and W. J. Torres Bobadilla, *Off-shell currents and color-kinematics duality*, Phys. Lett. B **753**, 242 (2016), [arXiv:1507.07532 [hep-ph]]

Conference proceedings

- [8] W.J. Torres Bobadilla, J. Llanes, P. Mastrolia, T. Peraro, A. Primo, G. Rodrigo, *Interplay of colour kinematics duality and analytic calculation of multi-loop scattering amplitudes: one and two loops one- and two-loops*, to appear in *Proceedings, 13th International Symposium on Radiative Corrections: 25-29 September, 2017, St. Gilgen, Austria*

- [9] P. Mastrolia, T. Peraro, A. Primo and W. J. Torres Bobadilla, *Adaptive Integrand Decomposition,* PoS LL **2016**, 007 (2016), [arXiv:1607.05156 [hep-ph]]

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Chapter 1

Introduction

The ongoing Run II of the Large Hadron Collider (LHC) has been collecting data from scattering events at luminosities and energy scales that had never been reached before in collider experiments. The steadily increasing precision of the resulting experimental measurements demands that correspondingly accurate predictions be given by theoretical physicists.

The theoretical study of the fundamental interactions of nature is based on Quantum Field Theory (QFT) which is a consistent mathematical framework that combines the principles of *Special Relativity* and those of *Quantum Mechanics* and led to the formulation of the *Standard Model* (SM) of particle physics.

The Standard Model unifies under a single guiding principle, i.e. *gauge symmetry*, three of the four known fundamental interactions: the symmetry group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ describes the *strong interaction*, which is associated to the $SU(3)_C$ symmetry of *Quantum Chromodynamics* (QCD), and the *electro-weak* sector $SU(2)_L \otimes U(1)_Y$, which combines Quantum Electrodynamics (QED) and the weak interaction. The symmetry group of QCD is realized in the SM as an exact symmetry, while the $SU(2)_L \otimes U(1)_Y$ symmetry of the electro-weak interaction is spontaneously broken via the so-called *Higgs mechanism*, which is responsible for the masses of all SM particles.

Although open questions (such as the inclusion of the gravitational interaction and the existence of dark matter and dark energy) prevent it from being considered as the ultimate theory of fundamental interactions, the SM encodes our best understanding of the microscopic structure of the universe, and, since its original formulation in the 1970s, it has successfully undergone countless experimental validations, culminating in the discovery of the Higgs boson.

The core of particle physics phenomenology lies in the prediction of *scattering amplitudes*, that is to say the transition probabilities between two configurations of a quantum mechanical system, which constitute the contact point between theoretical models and physical observables. In a QFT such as the SM, the calculation of scattering amplitudes heavily resorts to the *perturbation theory*, i.e. to the possibility of expanding them into powers of the (small) coupling constants of the theory, and of computing each summand of such expansion in terms of Feynman diagrams.

It is by now an indisputable fact that the leading-order (LO) approximation in perturbation theory does not provide reliable predictions, since the magnitude of their theoretical uncertainty is often comparable to the very same value of the computed observables. Therefore, an accurate theoretical description of interaction phenomena requires the computation of next-to-leading-order (NLO) or even higher-terms of the perturbative expansion.

At higher-orders in perturbation theory, the presence of complicated *loop integrals* reveals the quantum effects of the fundamental interactions. The calculation of these integrals is one of the most difficult challenges towards making theoretical predictions.

Within the SM, the computation of even a single Feynman diagram can be made inherently arduous by the presence of many different mass scales. In addition, the high centre-of-mass energy available at the LHC yields a large number of particles in the final state. These particles enhance both the complexity and the number of Feynman diagrams involved in a fixed-order calculation. In this scenario, a direct diagram-by-diagram evaluation of loop-level scattering amplitudes rapidly becomes unfeasible. A strategy must therefore be developed (one as general and automatable as possible) in order to handle this complexity. Such a strategy entails the decomposition of scattering amplitudes into a minimal set of independent functions, on which human computational efforts can be focused.

The standard approach to the calculation of Feynman amplitudes at multi-loop level consists of three main steps:

i) *decomposition* of the entire set of Feynman diagrams contributing to a process in terms of linear combinations of scalar loop integrals[1] ;

ii) reduction of the scalar integrals into a minimal number of independent ones a.k.a. the *master integrals*, which form a *basis* of the space of Feynman integrals for the process;

iii) evaluation of the master integrals.

In this thesis, we will discuss a series of methods for tackling these problems that are originally inspired by the *unitarity* of the scattering matrix.

The unitarity of scattering amplitudes is a reflection of the conservation of probability in Quantum Mechanics. It found its first formulation in the so-called *optical theorem*, which establishes the equivalence between the imaginary part of a scattering amplitude in the forward limit and its total cross section. When the unitarity condition is formulated in terms of the individual Feynman diagrams that contribute to an amplitude, Cutkosky rules [10, 11] state that the imaginary part of a diagram (which corresponds to the discontinuity across a branch-cut determined by a kinematic threshold) is obtained by summing over all possible ways of *cutting* the diagram, i.e. of partitioning it into two connected pieces. This is achieved by enforcing the *on-shellness* of the virtual particles which are traversed by the cut.

Hence, in the perturbative expansion of scattering amplitudes, unitarity can be used, to some extent, in order to retrieve information about higher-order contributions from the knowledge of lower-order ones. For instance the cutting equation can be combined with *dispersion relations* [12, 13] to determine a one-loop amplitude from the computation of the corresponding tree-level cross section.

Starting from this basic idea, and from the intuition of assigning complex values to loop momenta, the standard unitarity techniques were extended by introducing the concept of *generalized unitarity cuts* [14–17]. By cutting an arbitrary number of internal propagators which can satisfy the on-shell conditions simultaneously, Feynman

[1]here, as in the remaing part of the manuscript, we name as *scalar* integral a loop integral whose numerator dependence on the integration momenta is fully expressed in terms of scalar products, i.e. a loop integral with no free Lorentz indices.

diagrams can be grouped according to their multi-particle factorization channels. Such factorization even allows to reconstruct a full loop-level amplitude by assembling on-shell tree-level building blocks.

Passarino and Veltman [18] were the first to demonstrate, on the grounds of Lorentz invariance principles, the possibility of decomposing any one-loop amplitude in terms of a finite basis of scalar loop integrals. This in turn suggested the idea of determining the coefficients of such master integrals by evaluating the amplitude on the multi-pole channels identified by the simultaneous vanishing of all loop denominators associated to a given integral.

Within this framework, a new level of understanding of one-loop scattering amplitudes was brought about by the development of the *integrand decomposition method*. It was originally formulated in four space-time dimensions by Ossola, Papadopoulos and Pittau [19, 20] and later extended to accommodate the effects of dimensional regularization [21–25].

In this perspective, the integrand decomposition consists in the reduction of an integrand into a sum of simpler, irreducible integrands, whose denominators are given by all possible partitions of the initial set of propagators and whose numerators correspond to the *remainders* of the subsequent divisions. Once such integrand reduction is achieved, the corresponding decomposition of the integrated amplitude is obtained by restoring the integration over the loop momenta.

The simplification allowed by the manipulation of integrands comes at the cost of introducing *spurious* contributions, i.e. irreducible integrands that, although present in the integrand-level decomposition, vanish after integration. Nonetheless, at one-loop the identification and elimination of spurious terms follow trivially from Lorentz invariance, and do not constitute a practical problem.

One of the main advantages of the purely algebraic nature of the integrand-level reduction consists in its applicability regardless of the kinematic complexity of the amplitude. This allowed to overcome the issues of the other integral-level unitarity-based method that can be severely limited by the presence of massive particles.

The efficient numerical implementation [26–28] of the integrand decomposition algorithm and its embedding in automated frameworks for one-loop computations [29–38] played a crucial role in the so-called *NLO revolution*. As a consequence of this a deluge of next-to-leading-order predictions for a series of high-multiplicity, multi-scale processes of phenomenological interest for LHC physics was made possible.

The first part of this thesis deals with the extension of the integrand decomposition method at multi-loop level.

The possibility of applying the integrand decomposition algorithm beyond one-loop was first probed in [39, 40] and has been proven, from a rigorous mathematical point of view, through the introduction of algebraic geometry methods. Namely the integrand reduction procedure was formulated in terms of a *multivariate polynomial division* modulo *Gröbner bases* [41, 42]. These theoretical studies, besides finding remarkable applications (such as the first analytic computation of a two-loop five-gluon helicity amplitude [43, 44]), have highlighted the specific features of the multi-loop level problem.

First of all, at multi-loop level, a universal basis of integrals is not known *a priori* and can be determined only after the reduction is actually performed.

Secondly, due to the interplay of different integration momenta, the number of independent scalar products between loop and external momenta becomes, in general, larger than the number of denominators. Thus it is not possible to algebraically express all scalar products in terms of loop denominators. In this sense, the main bounce in complexity is represented by the presence of *irreducible scalar products*, which cause a proliferation of the number of independent integrands appearing in the amplitude decomposition. Although present in the decomposition, a relevant portion of these irreducible integrands is spurious. Beyond one-loop the systematic identification of such spurious terms, which is not entirely straightforward, is a fundamental requirement for a successful application of the integrand decomposition method, in order to avoid a hardly controllable growth of the intermediate expressions generated in the reduction procedure.

Irreducible scalar products can also produce non-spurious integrands. The latter in all respects enter the amplitude decomposition, since they are independent from an integrand-level point of view. However, the existence of integral-level relations between Feynman integrals, such as *integration-by-parts identities* [45–47] and *Lorentz invariance identities* [48], can consistently reduce the number of truly independent integrals.

The determination of a minimal basis of master integrals requires, after the integrand decomposition has been achieved, a further integral-level reduction. Therefore it is important that the outcome of the integrand decomposition be optimized for this purpose.

All these issues have so far prevented the integrand decomposition method from becoming an efficient and fully competitive technique that might be used for analytical and numerical multi-loop computations and that might be adopted in lieu of the more traditional form factor decomposition.

One of the main results of this thesis consists in the formulation of a simplified approach to the multi-loop integrand decomposition, which offers a cleaner theoretical description of the problem and also provides valuable technical solutions to the aforementioned difficulties.

We refer to the newly proposed algorithm as *adaptive integrand decomposition* [5, 8, 9], since it entirely relies on the systematic adaptation of the choice of the loop variables according to external kinematic of each individual integrand topology.

Depending on the number of external legs of a diagram, the d-dimensional space-time can be split into a *longitudinal space* with dimension $d_{||} \leq 4$ determined by the number of independent external momenta, and a complementary *transverse space* with continuous dimension $d_{\perp} = d - d_{||}$ [49–54].

This simple consideration allows to derive a parametrization of dimensionally regulated Feynman integrals in terms of a finite number of integration variables which enjoys remarkable properties. In fact, although Feynman integrands are, in general, *rational functions* of all components of the loop momenta, the $d = d_{||} + d_{\perp}$ parametrization exposes their purely *polynomial* dependence on the transverse components, which are the source of spurious contributions. Hence, since it is always possible to integrate a polynomial through elementary techniques, this parametrization allows a trivial, algorithmic integration over the transverse variables which is able to systematically detect and eliminate all spurious integrands.

Due to the hyperspherical symmetry of the transverse space, an efficient way to compute the transverse integrals, one which provides an alternative to the tensor reduction, consists in performing an expansion of the integrand in terms of *Gegenbauer polynomials* and in reducing all integrations to the iterative application of their orthogonality condition.

After integration over the transverse variables, each integrand is parametrized in terms of the longitudinal components of the loop momenta and of the scalar products between the vectors which represent the transverse part of the loop momenta. This integral representation can be mapped, through a linear change of variables, into the Baikov parametrization [55–57] which, by adopting denominators and irreducible scalar products as integration variables, has recently proved to be a versatile tool for the derivation of integration-by-parts identities, differential equations, and generalized unitarity cuts [58–63].

This last observation has a crucial impact on the integrand decomposition algorithm. Due to the linear dependence of the denominators on the variables which parametrize the integrand in $d = d_{\parallel} + d_{\perp}$, the set of on-shell conditions, which define any multiple-cut at any loop order, are linearized. Therefore, in its adaptive formulation, the multivariate polynomial division turns out to be reduced to the solution of an under-determined linear system, and the very definition of the loop denominators in terms of the adaptive variables can replace the computation of Gröbner bases.

The simplified polynomial division procedure and the systematic integration over the transverse variables can be combined into a new reduction algorithm, which we refer to as *divide-integrate-divide*. The latter will be the subject of a dedicated chapter of this thesis, where it will be employed in the systematic identification of a universal parametrization of the integrand basis for two-loop amplitudes with up to eight external legs and arbitrary kinematics. In addition, the streamlined structure of the new division procedure enables, for the first time, an automated implementation of the *analytic* integrand decomposition of one- and two-loop amplitudes provided by code AIDA (Adaptive Integrand Decomposition Algorithm), which we will discuss in detail. In this respect, we believe that integrand decomposition, in its adaptive formulation, will prove to be a powerful tool able to set the ground for a forthcoming *NNLO revolution*.

The outcome of the proposed division algorithm consists in the decomposition of a multi-loop scattering amplitude which is free of spurious terms and whose irreducible integrands are expressed exclusively in terms of scalar products between loop and external momenta. So, besides being minimal from the integrand-level point of view, the resulting reduction is also in a suitable form for the subsequent integration-by-parts reduction. It is worth mentioning that the adaptive integrand reduction does not resort to any form factor decomposition and thus can be applied to helicity amplitudes as well.

After the adaptive integrand reduction, a scattering amplitude is expressed as a combination of a reduced –but not minimal– set of scalar Feynman integrals. With the systematic generation and solution of symmetry relations, integration-by-parts and Lorentz invariance identities, the full set of integrals can be expressed in terms of a basis of independent master integrals, which can be identified through the Laporta algorithm [64].

Once a decomposition of the amplitude in terms of master integrals has been determined, the ultimate task of their evaluation must be addressed. In the last decades, the rather non-trivial problem of computing loop integrals has been successfully tackled with to different approaches: the *numerical* one and the *analytic* one. These two approaches can be considered, in some sense, complementary.

On the one hand, the search for an algorithm for the efficient numerical evaluation of loop integrals is, indeed, appealing, since it would smooth over the complexity brought about by the presence of many external legs and internal mass scales. The main difficulty related to a numerical approach lies in the presence of divergencies in loop integrals, which must be resolved and regularized before any numerical integration routine can be applied. One effective strategy of regularizing divergencies has been implemented in the *sector decomposition* method [65–67], which has been successfully applied for the numerical evaluation of two-loop virtual amplitudes whose analytic computation is at the present time far beyond our capabilities [68, 69]. Despite these promising results, however, there are severe limitations to a precise numerical evaluation of loop integrals. These limitations are mainly related to the treatment of end-point singularities and can make numerical methods extremely time-consuming and can, in some cases, prevent them from providing reliable results in the whole phase-space.

On the other hand, the analytical evaluation of loop integrals provides, by definition, a complete control of the result at any point of the phase-space. However, at first sight, analytic computations appear greatly limited by our ability to evaluate integrals directly. In fact, although direct integration techniques (such as Feynman parametrization or the *Mellin-Barnes* [70–73] representation) can be effectively applied to one-loop computations and to a number of (mostly massless) multi-loop integrals, when the number of kinematic scales of the problem increases, direct integration rapidly ceases to be a viable option.

Nonetheless, the analytic expression of the master integrals can also be determined through indirect techniques, which resort to the solution of systems of difference [64, 74, 75] and differential [48, 76, 77] equations satisfied by Feynman integrals. The second part of this thesis is devoted to the development of new techniques for the solution of *differential equations* for master integrals.

If master integrals are regarded as analytic functions of the external kinematic invariants and of the internal masses, they can then be computed by solving their first-order differential equations, which can be derived from integration-by-parts identities. The idea of using differential equations for computing master integrals was first introduced by Kotikov [76], for internal masses, and later extended to external invariants by Remiddi [77] and Gehrmann and Remiddi [48] (see [78] for a review on the method). Since then, the differential equations method has become by far the most powerful tool for the analytic evaluation of multi-loop integrals and has led to results which remain out of reach for any other available analytical method. One of the best examples of these results is the evaluation of two-loop five-point integrals [79] obtained through a simplified approach to differential equations [80].

Differential equations for master integrals in d-dimensions are, generally, coupled and exhibit a block-triangular structure, which is naturally inherited from the integration-by-parts identities. The presence of coupled differential equations makes it extremely difficult to evaluate Feynman integrals as exact functions of the space-time dimensions. However, for physical applications, we are usually interested in the expression of loop integrals close to four space-time dimensions, i.e. as a Laurent expansion around small values of the dimensional regulating parameter $\epsilon = (4-d)/2$. If the Laurent expansion is applied directly at the level of the differential equations, the structure of the system can be drastically simplified and, in many cases, it can even assume a complete triangular form, which allows a bottom-up solution of the whole set of differential equations by repeated quadrature.

Given the arbitrariness of the choice of the basis of master integrals, there arises the question about the existence of a particular basis that might make this simplified structure manifest. In [81], it was proposed by Henn to search for a basis, usually referred to as *canonical basis*, which satisfies two different requirements: the ϵ dependence of the corresponding system of differential equations is completely *factorized* from the kinematic one, and the latter is in dlog-*form* (i.e. the total differential of the master integral must be expressible as an exact differential of logarithmic functions).

These two properties allow to decouple the system of differential equations trivially, order-by-order in the ϵ-expansion, and naturally expose the uniform transcendental weight structure of the coefficients of such expansion in terms of *generalized polylogarithms* [82–86] or, more generally, *Chen iterated integrals* [87]. Consequently, the choice of a canonical basis of master integrals makes the derivation of the general solution of the differential equations almost entirely algorithmic, and reduces the whole complexity of the problem to the determination of the boundary constants, which are needed in order to impose physical requirements on the general solution.

A proof of the existence of the canonical basis is still missing – even for master integrals which evaluate to polylogarithmic functions. However, during the last years, several constructive algorithms have been put forward for the determination of a similarity transformation that can bring a non-canonical basis of master integrals to a canonical one, under different kind of assumptions [88–93].

The problem of finding a canonical basis is intimately related to the possibility of explicitly determining a matrix-valued solution of the system of differential equations in the four-dimensional limit, $\epsilon \sim 0$. This connection becomes transparent if we start from an initial set of master integrals which obey ϵ-linear differential equations as it is nearly always the case. In fact, under this assumption, a rotation of the master integrals by a similarity matrix which is an exact solution of the system of equations in $\epsilon = 0$ can absorb the $\mathcal{O}(\epsilon^0)$ term of the differential equations and bring them to an ϵ-factorized form.

The Magnus method [88] determines such a similarity transformation by attempting a solution of the first order differential equations in $\epsilon = 0$ through the Magnus exponential expansion [94]. Whenever the underlying system of differential equations can be triangularized in the four-dimensional limit, the Magnus representation of the solution, generally expressed as an infinite series, terminates after a finite number of terms and, hence, it can be used to algorithmically rotate the system of differential equations to a canonical form.

This method has been successfully applied to several multi-loop, multi-scale problems [1, 3, 88, 95, 96], part of which will be presented in this thesis. In particular, we will discuss the analytic calculation of the two-loop three-point master integrals that enter QCD corrections to the triple gauge couplings ZWW and $\gamma^* WW$, to the Higgs boson decay into a pair of off-shell W-boson, $H \to WW$, and to heavy-quark contributions to massive boson-pair production in the gluon-fusion channel, $gg \to HH$ and $gg \to WW$. In addition, we will compute all planar four-point integrals for the two-loop QED corrections to the elastic scattering $\mu e \to \mu e$. The analytic evaluation of these integrals constitute the first step towards the determination of the full NNLO cross section that is needed as theoretical input for the experimental measurement of the leading hadronic contribution to the muon $g - 2$ to be realized at CERN by the future experiment MUonE [97]. The same integrals are also useful for virtual corrections to $t\bar{t}$-production at colliders [98–101].

The Magnus rotation, as well as the other available methods, fails to find a canonical form when there is no change of basis which triangularizes the system of differential equations in $\epsilon \sim 0$, i.e. when some sector of master integrals obey differential equations which cannot be decoupled, by any means, in four-dimensions. Such coupled systems of first-order equations are actually equivalent to homogeneous *higher-order differential* *equations*, for which no general solving strategy is available. Thus, in these cases, the possibility of determining the analytic expression of the master integrals through the differential equations method has entirely resorted, so far, to our ability to recognize, in a case-by-case analysis, the higher-order-differential equations as belonging to some of the limited classes that are known from the mathematical literature.

This situation can first occur at two-loop level, where a growing number of multi-scale integrals [60, 102–104] are found to obey irreducible second-order differential equations, the most famous of which is the massive two-loop sunrise [102]. In all these cases, the homogeneous solutions of these equations turn out to be expressible in terms of *complete elliptic integrals*. However, while it is unknown whether elliptic integrals can complete the set of functions occurring in the computation of two-loop Feynman integrals, cases of multi-loop master integrals which obey even higher order differential equations are known to exist [2, 105, 106]. Therefore, the development of a general strategy for the determination of the homogeneous solutions is strongly advisable.

The answer to this problem comes, once more, from generalized unitarity. The connection between maximal unitarity and canonical bases was first pointed out in [81], and was based on the conjecture that integrals with *unit leading singularity* (which corresponds to the maximal-cut of the integral computed in a finite number of dimensions) are pure functions with homogeneous transcendental weight [107, 108] and so are expected to obey canonical systems of differential equations.

The underpinning motivation of such conjecture relies on observation that the *maximal-cut* of a Feynman integral satisfies, by construction, the homogeneous part of the differential equations fulfilled by the uncut integral. Therefore, the determination of a basis of master integrals with unit leading singularity, which practically amounts to a redefinition of the master integrals that reabsorbs their non-trivial maximal-cut, corresponds, in all respects, to the solution of the homogeneous system of differential equations in $\epsilon = 0$.

In light of this, the explicit computation of maximal-cuts of Feynman integrals proves to be a powerful and general tool for the determination of a representation of the homogeneous solutions in a closed integral form [4]. This is also true in the case of higher-order differential equations, where the ordinary solution strategies are not applicable. The determination of the complete solution of a n-th order differential equation requires the knowledge of a set of n-independent homogeneous solutions. We will show that, given the integral representation of the maximal-cut of a multi-loop integral, it is possible to identify a set of independent, integration-by-parts compatible, integration contours, each of which provides one independent homogeneous solution. Hence, by defining a *matrix of maximal-cuts*, we can rotate a system of coupled differential equations to an ϵ-factorized form, which allows a recursive determination of the inhomogeneous solutions in terms of iterated integrals over kernels constituted by the homogeneous solutions. In this sense, the resulting basis of master integrals can be considered as a natural extension of the concept of basis with unit leading singularity to the case of higher-order differential equations.

This result is one of the main accomplishments of the work presented in this thesis, and it will be discussed in great detail on a number of explicit examples, which include the determination of the homogeneous solutions of the master integrals for three- and four-point functions involved in the *top*-quark contribution to two-loop corrections to $gg \to gg$ [104] and $gg \to gH$ [103]. Furthermore, a special attention will be dedicated to the application of the proposed method to the solution of the differential equations for the massive three-loop banana graph, which constitutes the first non-trivial example of Feynman integral that obeys an irreducible third-order differential equation [2]. The general applicability of this method is amplified by the developments of efficient algorithms for the computation of generalized unitarity cuts, based on the introduction of the Baikov representation of Feynman integrals.

The complete control of the analytic expression of the master integrals requires, of course, the study of the analytic properties of the homogeneous solutions obtained through maximal-cuts and, possibly, a precise classification of the families of special functions which emerge from the iterative integration of such homogeneous solutions. A vast literature on the possible definition of iterated integrals over elliptic functions has been produced [102, 109–123]. Nonetheless a clear understanding of the underlying algebra of these special functions is still missing and is far from being comparable to the absolute control we have over generalized polylogarithms. In addition, very little is known in the case of differential equations of order higher than two. This subject will definitely constitute an important direction for future investigations in the field of higher-order computations.

This thesis is organized as follows. In chapter 2, we will review the basic definitions and properties of Feynman integrals in dimensional regularization, we will derive a series of equivalent finite dimensional integral representations, and we will introduce the concept of generalised unitarity cut of a Feynman integral.

In chapter 3, we will review the integrand decomposition method for multi-loop scattering amplitudes, by focusing on its algebraic geometry formulation.

In chapter 4, we will present the *adaptive* integrand decomposition algorithm and we apply it to derive a new universal decomposition of one- and two-loop amplitudes in d dimensions. We will discuss our Mathematica implementation of the algorithm Aida, and we will illustrate its features on explicit examples.

In chapter 5, a review of the differential equations method for master integrals will be given and the concept of canonical basis of master integrals will be introduced. We will first discuss the Magnus exponential method for determining such basis, by highlight its role in the determination of a set of master integrals with unit leading singularity, and we will then review the main properties of the class of iterated integrals which appear in the general solution of canonical systems, as well as the most commonly used techniques for the fixing of boundary constants.

In chapters 6-8, we will apply the Magnus method to the analytic calculation of two-loop integrals with both massive internal and external particles: in chapter 6, we will calculate the vertex diagrams needed for the leading QCD corrections to the triple-gauge couplings ZWW and γ^*WW, as well as for the Higgs decay into an off-shell W-boson pair. In chapter 7, we will calculate two-loop three-point functions which enter QCD corrections, with exact dependence on the *top*-quark mass, to the production of a massive boson-pair (HH and ZZ) in the gluon-fusion channel. In chapter 8, we will compute all the planar four-point integrals which are needed for the two-loop QED virtual corrections to the elastic scattering between muons and electrons.

In chapter 9, we will address the problem of solving coupled systems of differential equations, which are obeyed by Feynman integrals that are not expressible in terms of polylogarithmic functions. We will propose a general method, based on maximal-unitarity, for determining the homogeneous solutions of such systems, and define a basis of master integrals which generalizes the concept of unit leading singularity to the case of higher-order differential equations. We will illustrate the proposed method on a series of two-loop cases, which involve homogeneous solutions expressed in terms of complete elliptic integrals, and later apply it to solve the irreducible third-order differential equation satisfied by the three-loop massive banana graph.

We will give our conclusions in chapter 10.

The main body of this thesis is complemented by a series of appendices. In appendix A we will derive all the results needed for the definition of the $d = d_{\parallel} + d_{\perp}$ parametrization of Feynman integrals, presented in chapter 2. In particular, we will discuss the use of multidimensional spherical coordinates in the transverse space, the main properties of Gegenbauer polynomials and the definition of the four-dimensional vector bases which will be used throughout chapter 4. We will provide as well an inventory of one- and two-loop tensor integrals in the transverse space. In appendix B, we will list the dlog-forms of the systems of differential equations discussed in chapters 6-8. In appendix C, we will present the details of the analytic continuation of the master integrals for the three-loop banana graphs discussed in chapter 9. Finally, in appendix D, we will derive a set of identities between complete elliptic integrals and moments of Bessel functions, which will be used in chapter 9 in order to obtain a representation of the homogeneous solutions of the banana graph in terms of known functions.

Chapter 2

Feynman integrals: representations and properties

In this chapter we review the basic definitions and properties of Feynman integrals in dimensional regularization. In particular, we focus on a representation of multi-loop integrals where the integration momenta are parametrized by splitting the d dimensional space-time into a longitudinal space, spanned by the momenta of the external particles, and its complementary, transverse space, $d = d_{\parallel} + d_{\perp}$. Such parametrization will play a fundamental role in the adaptive formulation of the integrand decomposition method, discussed in chapter 4. In addition, we discuss the Baikov parametrization which, by introducing loop denominators and irreducible scalar products as integration variables, allows a clear and efficient definition of the multiple-cuts of Feynman integrals. Part of the content of this chapter is based on the publications [5, 9] and on original research done in collaboration with P. Mastrolia, U.Schubert and W.J.Torres Bobadilla.

2.1 Feynman integrals in dimensional regularization

In a d-dimensional space-time, we define a ℓ-loop Feynman integral with n external legs and m internal propagators as the multivariate integral

$$I^{d\,(\ell,n)}_{a_1\ldots a_m}[\mathcal{N}] = \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{\mathcal{N}(q_i)}{D_1^{a_1} D_2^{a_2} \cdots D_m^{a_m}}\,, \quad a_i \in \mathbb{N}\,, \tag{2.1}$$

where $\mathcal{N}(q_i)$ is an arbitrary tensor numerator depending on the loop momenta q_i and the loop denominators $D_j(q_i)$ are defined as

$$D_j = l_j^2 + m_j^2, \quad \text{with} \quad l_j^\alpha = \sum_i \alpha_{ij} q_i^\alpha + \sum_i \beta_{ij} p_i^\alpha, \tag{2.2}$$

with $\{p_1, \ldots, p_n\}$ being the set of momenta associated to the external particles, which satisfy the conservation rule

$$p_1 + p_2 + \cdots + p_n = 0\,. \tag{2.3}$$

In eq. (2.2), α and β are incidence matrices which entries take values in $\{0, \pm 1\}$. The denominators D_i can be represented as propagators, i.e. internal edges of a connected graph which obeys momentum conservation at each vertex.

When tensor reduction or integrand decomposition is applied to the numerator $\mathcal{N}(q_i)$ (see chapters 3-4), the whole integrand

$$\mathcal{I}_{a_1 \ldots a_m}(q_i) = \frac{\mathcal{N}(q_i)}{D_1^{a_1} D_2^{a_2} \cdots D_m^{a_m}}, \quad a_i \in \mathbb{N}, \tag{2.4}$$

becomes a rational function of the scalar products between loop momenta and external ones, whose total number is given by

$$n_{\mathrm{SP}} = \ell(n-1) + \frac{\ell(\ell+1)}{2} = \frac{\ell(2n+\ell-1)}{2}, \tag{2.5}$$

where the first summand corresponds to the number of scalar products $s_{ij} = q_i \cdot p_j$ and the second one counts all possible scalar products between the loop momenta, $\widetilde{s}_{ij} = q_i \cdot q_j$.

As we will discuss throughly in the next sections, at one loop all scalar products are *reducible* (RSPs), i.e. they can be expressed in terms of denominators, since the number of scalar products always corresponds to the number of propagators appearing in the integrand. Conversely, at multi-loop level, it is generally not possible to rewrite all n_{SP} scalar products in terms of the m denominators, and we have a number

$$r = n_{\mathrm{SP}} - m \tag{2.6}$$

of so-called *irreducible* scalar products (ISPs),

$$S_i(q_j) \in \{s_{ij}\}, = 1, \ldots, r. \tag{2.7}$$

The ISPs are naturally chosen among the scalar products s_{ij} but it is often convenient to parametrize them in terms quadratic polynomials in q_i, by extending the set loop propagators so to include r *auxiliary denominators* $D_{m+1}, \ldots D_{m+r}$,

$$S_i \longrightarrow D_{m+i}, \quad i = 1, \ldots, r. \tag{2.8}$$

In both cases, we can imagine to split the numerator $\mathcal{N}(q_i)$ into a sum of elementary building-blocks, each one corresponding to a different monomial in the ISPs, and define basic Feynman integrals of the type

$$I^{d\,(\ell,n)}(a_1, \ldots, a_m; b_1, \ldots, b_r) = \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{S_1^{-b_1} \cdots S_r^{-b_r}}{D_1^{a_1} \cdots D_m^{a_m}}, \tag{2.9}$$

which are usually referred to as *integral families*. In principle, we can allow the powers a_i and b_i to assume any integer value. Nevertheless, we observe that, if the ISPs are identified with auxiliary denominators, any $b_i > 0$ would not correspond to a graph where momentum is conserved at every vertex. For this reason, it useful to introduce some additional terminology:

- We define a *topology* (or *sector*) as an integral of the type (2.9) where *all* denominators have non-negative powers. Hence, every topology corresponds to a graph which obeys momentum conservation at each vertex;

- Given a topology, its *subtopologies* correspond to integrals where some denominators are raised to zero power. The graph of a subtopology can be obtained from the one of the "parent" topology by pinching (i.e. removing) the corresponding loop propagators. Therefore, the subgraph still satisfies momentum conservation.

Any integral family contains, obviously, infinitely many different integrals, each one corresponding to a particular integer tuple $\{a_1\,, \ldots\,, a_m\,, b_1\,, \ldots\,, b_r\}$. However, only a finite number of such integrals is actually independent, due to the existence of linear relations between Feynman integrals which are a direct consequence of the invariance of eq. (2.9) under Lorentz transformations and re-parametrization of the loop momenta.

2.1.1 Lorentz invariance identities

The integrals defined in eq. (2.9) are Lorentz scalars, i.e. they are invariant under rotation of the external momenta

$$p_i^\alpha \to p_i^\alpha + \delta\omega^{\alpha\beta} p_{i\,\beta}\,, \tag{2.10}$$

where $\delta\omega^{\alpha\beta}$ is an (infinitesimal) antisymmetric tensor $\delta\omega^{\alpha\beta} = -\delta\omega^{\beta\alpha}$. By imposing the invariance of $I^{d\,(\ell\,,n)}(a_1\,, \ldots\,, a_m\,; b_1\,, \ldots\,, b_r)$ under the transformation (2.10), we obtain [48]

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \delta\omega^{\alpha\beta} \left(p_i^\beta \frac{\partial}{\partial p_i^\alpha} \right) I^{d\,(\ell\,,n)}(a_1\,, \ldots\,, a_m\,; b_1\,, \ldots\,, b_r) \\ &= \sum_{i=1}^{n} \left(p_i^{[\beta} \frac{\partial}{\partial p_i^{\alpha]}} \right) I^{d\,(\ell\,,n)}(a_1\,, \ldots\,, a_m\,; b_1\,, \ldots\,, b_r)\,, \end{aligned} \tag{2.11}$$

where we have used the antisymmetry and the arbitrariness of $\delta\omega$.

If we contract eq. (2.11) with all possible antisymmetric tensors built from external momenta, $p_i^{[\alpha} p_j^{\beta]}$, we obtain a set of $n_{\text{LI}} = (n-1)(n-2)/2$ *Lorentz invariance identities* (LIs) between Feynman integrals. In fact, after explicitly computing the derivatives under the integral sign, eq. (2.11) translates into a vanishing linear combination of integrals, whose numerators consist of scalar products between loop momenta and the external momenta. Hence, after expressing all RSPs in terms of denominators, we obtain linear identities between different integrals of the type (2.9). In particular, since differentiation cannot introduce any positive powers of additional denominators, LIs only involve integrals from a same sector as well as their subtopologies.

2.1.2 Integration-by-parts identities

The use of dimensional regularization, allow us to derive another important class of linear relations between Feynman integrals, the *integration-by-parts identities* (IBPs) [45–47]. In fact, provided that the space-time dimension d is treated as a continuous parameter, we can assume the integral (2.9) to be well-defined and, hence, convergent. In order for the integral to be convergent, the integrand must vanish rapidly enough at the boundary of the manifold spanned by the loop momenta. Therefore, when integrating by parts eq. (2.9), no boundary terms is generated.

In other words, in d dimensions, the integral of the total derivative of *any* Feynman integrand must vanish,

$$\int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{\partial}{\partial q_i^\alpha} \left(v^\alpha \frac{S_1^{-b_1} \cdots S_r^{-b_r}}{D_1^{a_1} \cdots D_m^{a_m}} \right) = 0\,, \quad \text{with } i = 1\,, \ldots\,, \ell\,, \tag{2.12}$$

where v^α is an arbitrary vector $v \in \{q_1\,, \ldots\,, q_\ell\,, p_1\,, \ldots\,, p_n\}$. By taking derivatives w.r.t. all loop momenta and by choosing all possible values of the vector v^α, we can produce

$\ell(\ell+n-1)$ IBPs for each integral. As for the case of LIs, differentiation can only produce IBPs which involve integrals belonging to a same sector as well as their subtopologies. We observe that eq. (2.12) offers a clear interpretation of IBPs as a generalization of Gauss theorem to d dimensions. Mathematically speaking, IBPs correspond to exact differential forms, as we will observe in section 2.5.1, when dealing with representations of the integrand in terms of explicit components of the loop momenta.

Alternatively, it can be shown [124] that IBPs descend from the invariance of Feynman integrals under shifts of the loop momenta of the type

$$q_i^\alpha \longrightarrow q_i^\alpha + \delta b_{ij}\, k_j^\alpha \,, \tag{2.13}$$

where $k_j \in \{q_1\,, \dots\,, q_\ell\,, p_1\,, \dots\,, p_n\}$ and δb_{ij} is an infinitesimal parameter. In addition, it has been proven that LIs (2.11) can be expressed as a linear combination of IBPs [125].

2.1.3 Symmetry relations

The shifts of the loop momenta defined in eq. (2.13) can be generalized to a larger symmetry group of Feynman integrals defined by the linear transformations

$$q_i^\alpha \longrightarrow (\mathbb{A})_{ij} q_j^\alpha + (\mathbb{B})_{ij} p_j^\alpha \,, \tag{2.14}$$

where $\mathbb{A}$ in an invertible $\ell\times\ell$ matrix, with $|\det\mathbb{A}|= 1$, and $\mathbb{B}$ is a $\ell\times n$ matrix. A shift of the type (2.14) leaves the value of a Feynman integral unchanged, but it transforms its integrand into a linear combination of different integrands, allowing to determine linear relations between integrals which might even belong to different sectors. In particular, we refer to symmetry relations which map the set of denominators of a sector into itself as *sector symmetries*.

2.2 Feynman integrals in $d = 4 - 2\epsilon$

Up to now, we have made no assumption on the specific dimensional regularization scheme adopted in order to express an amplitude in terms of the Feynman integrals (2.1). Although all schemes share the basic prescription of regularizing divergencies by promoting the four-dimensional integration momenta to arbitrary d-dimensional ones, they may adopt different conventions regarding the treatment of non-divergent elements which appear in Fenyman diagrams, such as γ matrices, metric tensors and, very importantly, vector fields and degrees of freedom associated to the external particles. We refer the reader to [126] for a recent review on the most commonly used regularization schemes.

One customary choice consists in considering momenta and wave functions associated to the external particles as strictly four-dimensional objects. In practice, this amounts to consider a d-dimensional metric tensor $g^{\alpha\beta}$ with a block-diagonal structure of the type

$$g^{\alpha\beta} = \begin{pmatrix} g_{[4]}^{\alpha\beta} & \mathbf{0} \\ \mathbf{0} & g_{[-2\epsilon]}^{\alpha\beta} \end{pmatrix} , \tag{2.15}$$

where $g_{[4]}^{\alpha\beta}$ is the metric tensor of the physical four-dimensional Minkowski space and $g_{[-2\epsilon]}^{\alpha\beta}$ is the Euclidean metric of the infinite-dimensional space which regulates divergencies,

$$g_{[4]}^{\alpha\beta}(g_{[4]})_{\alpha\beta} = 4\,, \quad g_{[-2\epsilon]}^{\alpha\beta}(g_{[-2\epsilon]})_{\alpha\beta} = -2\epsilon\,. \tag{2.16}$$

Consistently, we split all Lorentz vectors v^α (and, in general, all tensors) into a four-dimensional part and a (-2ϵ)-dimensional one,

$$v^\alpha = v^\alpha_{[4]} + v^\alpha_{[-2\epsilon]} \,. \tag{2.17}$$

In particular, if external particles are kept in four-dimensions, we identify external momenta p_i and polarizations ε_i as

$$p^\alpha \equiv p^\alpha_{[4]\,i} \,, \quad \varepsilon^\alpha_i \equiv \varepsilon^\alpha_{[4]\,i} \,. \tag{2.18}$$

Conversely, we decompose the loop momenta q_i as

$$q^\alpha_i = q^\alpha_{[4]\,i} + \mu^\alpha_i \,, \tag{2.19}$$

where we have defined $\mu^\alpha_i \equiv q^\alpha_{[-2\epsilon]\,i}$. By using the metric tensor given in eq. (2.15), we can derive the following scalar product rules between loop momenta and external vectors,

$$\begin{aligned} q_i \cdot p_j &= q^\alpha_i \, g_{\alpha\beta} \, p^\beta_j = q^\alpha_{[4]\,i} \, (g_{[4]})_{\,\alpha\beta} \, p^\beta_j = q_{[4]\,i} \cdot p_j \,, \\ q_i \cdot \varepsilon_j &= q^\alpha_i \, g_{\alpha\beta} \, \varepsilon^\beta_j = q^\alpha_{[4]\,i} \, (g_{[4]})_{\,\alpha\beta} \, \varepsilon^\beta_j = q_{[4]\,i} \cdot \varepsilon_j \,, \\ q_i \cdot q_j &= q^\alpha_i \, g_{\alpha\beta} \, q^\beta_j = q^\alpha_{[4]\,i} \, (g_{[4]})_{\,\alpha\beta} q^\beta_{[4]\,j} + \mu^\alpha_i \, (g_{[-2\epsilon]})_{\,\alpha\beta} \mu^\beta_j = q_{[4]\,i} \cdot q_{[4]\,j} + \mu_{ij} \,, \end{aligned} \tag{2.20}$$

with $\mu_{ij} \equiv \mu_i \cdot \mu_j$. As a consequence of eq. (2.20), the loop denominators defined in eq. (2.2) are rewritten as

$$D_j = l^2_{j[4]} + \sum_{i,k} \alpha_{ij} \alpha_{kj} \, \mu_{ik} + m^2_j, \quad \text{with} \quad l^\alpha_{j[4]} = \sum_i \alpha_{ij} q^\alpha_{i[4]} + \sum_i \beta_{ij} p^\alpha_i. \tag{2.21}$$

For the same reason, the numerator $\mathcal{N}(q_i)$ appearing in (2.1) can depend only on the components of $q^\alpha_{i[4]}$ and on μ_{ij}. Therefore, Feynman integrands in $d = 4 - 2\epsilon$ are completely independent on the individual components of the vectors μ^α_i, which enter the calculation exclusively in terms of the scalar products μ_{ij}. This implies that the integrals over the (-2ϵ)-dimensional Euclidean space can be expressed into spherical coordinates, and that we can integrate away all directions which are orthogonal to the relative orientations μ_{ij} of the vectors μ^α_i.

The resulting parametrization of the Feynman integral (2.1) reads

$$I^{d\,(\ell\,,n)}_{a_1\,\dots\,a_m}[\mathcal{N}] = \Omega^{(\ell)}_d \int \prod_{i=1}^{\ell} \mathrm{d}^4 q_{[4]\,i} \int \prod_{1 \le i \le j \le \ell} \mathrm{d}\mu_{ij} \, [G(\mu_{ij})]^{\frac{d-5-\ell}{2}} \, \frac{\mathcal{N}(q_{[4]\,i}, \mu_{ij})}{\prod_{k=1}^m D^{a_k}_k(q_{[4]\,i}, \mu_{ij})} \,, \tag{2.22}$$

where $G(\mu_{ij})$ is the determinant of the Gram matrix $\mathbb{G}(\mu_1, \dots, \mu_\ell)$ of the μ^α_i vectors,

$$[\mathbb{G}(\mu_1, \dots, \mu_\ell)]_{ij} = \mu_i \cdot \mu_j \,, \tag{2.23}$$

and $\Omega^{(\ell)}_d$ is the product of solid angles

$$\Omega^{(\ell)}_d = \prod_{i=1}^{\ell} \frac{\Omega_{d-4-i}}{2\pi^{\frac{d}{2}}} \,, \qquad \text{with} \quad \Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \,. \tag{2.24}$$

The detailed derivation of eq. (2.22) is described in appendix A.1. It is interesting to observe that, in the r.h.s. of eq. (2.22) the full dependence of the integrand on d is retained in the power of the Gram determinant which defines the integration volume. This implies that tensor numerators which contain additional powers of $G(\mu_{ij})$ can be identified with lower rank numerators of Feynman integrals in *raised* dimensions.

For example, in the case $\mathcal{N}(q_i) = G(\mu_{ij})$ it is easy to verify from eq. (2.22) that

$$I^{d\,(\ell,n)}_{a_1\,\ldots\,a_m}[G(\mu_{ij})] = \frac{1}{2^\ell}\prod_{i=1}^{\ell}(d-3-i)\,I^{d+2\,(\ell,n)}_{a_1\,\ldots\,a_m}[1]\,, \tag{2.25}$$

where the d-dependent coefficient in the r.h.s. originates from the ratio of angular factors $\Omega_d^{(\ell)}/\Omega_{d+2}^{(\ell)}$. Eq. (2.25) generalizes to any number of loops a well-known result at one loop, where $G(\mu_{ij}) \equiv \mu_{11}$, [127].

In practical computations, it is often convenient to decompose the four-dimensional part of the loop momenta $q_{[4]\,i}$ in terms of some basis of four dimensional vectors $\mathcal{E} = \{e_1\,, e_2\,, e_3\,, e_4\}$,

$$q^\alpha_{[4]\,i} = \sum_{j=1}^{4} x_{ji}\, e^\alpha_j, \tag{2.26}$$

and integrate directly on the components x_{ij}. As it is clear from eq. (2.26) one of the advantages of introducing a four-dimensional basis $\mathcal{E}$ consists in trading any tensor numerator in the loop momenta with a numerator in the components x_{ij}, which is, by definition, a scalar. The parametrization of the Feynman integrals in terms of the x_{ij} components can be directly obtained, up to a trivial Jacobian, from eq. (2.22),

$$I^{d\,(\ell,n)}_{a_1\,\ldots\,a_m}[\mathcal{N}] = \Omega_d^{(\ell)}\,\mathcal{J}_{[4]}\int_{-\infty}^{+\infty}\prod_{i=1}^{\ell}\prod_{j=1}^{4}\mathrm{d}x_{j\,i}\int\prod_{1\le i\le j\le \ell}\mathrm{d}\mu_{ij}\,[G(\mu_{ij})]^{\frac{d-5-\ell}{2}}\,\frac{\mathcal{N}(x_{ij},\mu_{ij})}{\prod_{k=1}^{m}D_k^{a_k}(x_{ij},\mu_{ij})}\,, \tag{2.27}$$

where we have defined

$$\mathcal{J}_{[4]} = \prod_{i=1}^{\ell}\sqrt{\left|\det\left(\frac{\partial q^\alpha_{[4]\,i}}{\partial x_{ji}}\frac{\partial q_{[4]\,i\,\alpha}}{\partial x_{ki}}\right)\right|}\,. \tag{2.28}$$

Eq. (2.27) completely specifies a Feynman integral, once appropriate integration boundaries are identified. The four dimensional components of the loop momenta x_{ij} are integrated over the whole real axes, as explicitly indicated in eq. (2.27), whereas the integration region in the μ_{ij}-space is determined by demanding the positivity of both the individual norms $\mu_{ii}^2 \ge 0$ and Gram determinant $G(\mu_{ij}) \ge 0$, which corresponds to the volume of the *parallelotope* described by the vectors μ_i^α. For instance, in a two loop case, $G(\mu_{ij}) = \mu_{11}\mu_{22} - \mu_{12}^2$ and the integration bounds are $\mu_{ii} \ge 0$, $-\sqrt{\mu_{11}\mu_{22}} \le \mu_{12} \le \sqrt{\mu_{11}\mu_{22}}$.

Summarizing, if we adopt a dimensional regularization scheme where external particles propagate in four-dimensions, it is possible to represent a Feynman integral in d continuous dimensions as a multiple integral over a *finite* number $\ell(\ell+9)/2$ of variables

$$\mathbf{z} = \{x_{1\,j}\,, x_{2\,j}\, x_{3\,j}\, x_{4\,j}, \mu_{ij}\}\,, \quad 1 \le i \le j \le \ell\,, \tag{2.29}$$

which correspond to the $4\,\ell$ four-dimensional components of the loop momenta $q_{[4]\,i}$ and the $\ell(\ell+1)/2$ scalar products μ_{ij} between the (-2ϵ)-dimensional vectors μ_i^α.

We observe that, at this level, both numerator and loop denominators depend polynomially on all $\mathbf{z}$, so that the integrand is, generally, a *rational* function of $\ell(\ell+9)/2$ variables. In the next section, we will see how momentum conservation can be used in order to obtain a further simplified representation.

2.3 Feynman integrals in $d = d_{\parallel} + d_{\perp}$

The choice of the four-dimensional basis $\mathcal{E} = \{e_i\}$, which has been used in eq. (2.27) in order to decompose the four-dimensional part of the loop momenta, is completely arbitrary.

A particularly interesting representation of the Feynman integral (2.1) can be obtained by maximazing in $\mathcal{E}$ the number of vectors which are *orthogonal* to the momenta of the external particles. Formally, this amounts to splitting the d-dimensional space-time into a *longitudinal space*, which corresponds to the subspace of Minkowski space spanned by the external momenta, and its orthogonal, complementary part, we can refer to as *transverse space*.

In other words, we consider a d-dimensional metric tensor with the block-diagonal structure

$$g^{\alpha\beta} = \begin{pmatrix} g^{\alpha\beta}_{[d_{\parallel}]} & \mathbf{0} \\ \mathbf{0} & g^{\alpha\beta}_{[d_{\perp}]} \end{pmatrix}, \tag{2.30}$$

where $g_{[d_{\parallel}]}$ and $g_{[d_{\perp}]}$ are, respectively, the metric tensors of the longitudinal and transverse space, which satisfy

$$g^{\alpha\beta}_{[d_{\parallel}]}(g_{[d_{\parallel}]})_{\alpha\beta} = d_{\parallel}\,, \quad g^{\alpha\beta}_{[d_{\perp}]}(g_{[d_{\perp}]})_{\alpha\beta} = d_{\perp}\,. \tag{2.31}$$

In the following, we assume, without loss of generality, the metric of the transverse space to be Euclidean.

Due to momentum conservation (2.3), the dimension of the longitudinal space spanned by the n external legs of a Feynman integral is

$$d_{\parallel} = \min(4, n-1)\,. \tag{2.32}$$

This means that, for $n \leq 4$, the longitudinal space covers only a subspace of the whole Minkowski space and the transverse space, which is still infinite dimensional (as it is needed to regulate divergencies) absorbs a number $(5-n)$ of four-dimensional directions. Obviously, if $n \geq 4$ the transverse space is reduced to the (-2ϵ)-dimensional space introduced in the previous section.

This simple observation is, indeed, not new, since it can traced back at least to [49], and it has been exploited intensively for the direct analytic integrations of one- and two-loop integrals with up to, respectively, three and two external legs [50–54]. In following, based on the discussion of [5], we show how to extend to arbitrary loop order the construction of the $d = d_{\parallel} + d_{\perp}$ parametrization of the integrand.

For a ℓ-loop Feynman integral with $n \leq 4$ external legs, we can introduce in the vector basis $\mathcal{E}$ a maximum of $(4 - d_{\parallel})$ elements lying in the transverse space. For definiteness, we choose the transverse vectors to correspond to $e_n^{\alpha}, \ldots, e_4^{\alpha}$, i.e. we define

$$\begin{aligned} e_i \cdot p_j &= 0, \qquad i > d_{\parallel}, \quad \forall j, \\ e_i \cdot e_j &= \delta_{ij}, \qquad i,j > d_{\parallel}. \end{aligned} \tag{2.33}$$

The normalization $e_i^2 = 1$ of the transverse vectors is purely conventional and it is adopted in order to simplify the discussion. Equivalent results can be obtained by working with non-normalized vectors. The basis defined in eq. (2.33) is closely related to the Van Neerven-Vermaseren basis [128, 129], which played a crucial role in the reduction of one-loop tensor integrals (see, for instance, [25]).

Eq. (2.33) suggests a splitting of the d-dimensional loop momenta of the type

$$q_i^\alpha = q_{\parallel\, i}^\alpha + \lambda_i^\alpha, \tag{2.34}$$

where $q_{\parallel\, i}^\alpha$ is a vector of the $d_\parallel$-dimensional longitudinal space,

$$q_{\parallel\, i}^\alpha = \sum_{j=1}^{d_\parallel} x_{ji}\, e_j^\alpha, \tag{2.35}$$

and $\lambda_i^\alpha \equiv q_{\perp\, i}^\alpha$ is a vector belonging to the $d_\perp$-dimensional transverse space,

$$\lambda_i^\alpha = \sum_{j=d_\parallel+1}^{4} x_{ji}\, e_j^\alpha + \mu_i^\alpha\,. \tag{2.36}$$

The block-diagonal structure (2.30) of the metric tensor ensures that

$$\begin{aligned} q_i \cdot p_j &= q_i^\alpha\, g_{\alpha\beta}\, p_j^\beta = q_{\parallel\, i}^\alpha\, (g_{[d_\parallel]})_{\alpha\beta}\, p_j^\beta = q_{\parallel\, i} \cdot p_j\,, \\ q_i \cdot q_j &= q_i^\alpha\, g_{\alpha\beta}\, q_j^\beta = q_{\parallel\, i}^\alpha\, (g_{[d_\parallel]})_{\alpha\beta} q_{\parallel\, j}^\beta + \lambda_i^\alpha\, (g_{[d_\perp]})_{\alpha\beta} \lambda_j^\beta = q_{\parallel\, i} \cdot q_{\parallel\, j} + \lambda_{ij}\,, \end{aligned} \tag{2.37}$$

where, in the last equality, we have introduced the symbol λ_{ij} to indicate the scalar products between transverse vectors,

$$\lambda_{ij} \equiv \lambda_i \cdot \lambda_j = \sum_{l=d_\parallel+1}^{4} x_{li} x_{lj} + \mu_{i\,j}\,. \tag{2.38}$$

In the following, we will collectively denote by $\mathbf{x}_{\parallel\, i}$ the subset of components of q_i which lie in the longitudinal space and by $\mathbf{x}_{\perp\, i}$ the complementary set of four-dimensional components that belong to the transverse space.

It is evident from the scalar products rules of eq. (2.37) that, when the decomposition (2.34) is applied to the loop denominators (2.2), they become independent of $\mathbf{x}_{\perp\, i}$ and their dependence of the transverse space variables is entirely absorbed into a *linear* dependence on λ_{ij}. In fact, we can write

$$D_j = l_{\parallel\, j}^2 + \sum_{i,l} \alpha_{ij} \alpha_{lj}\, \lambda_{il} + m_j^2\,, \tag{2.39}$$

where $l_{\parallel\, j}^\alpha$ is the longitudinal momentum that flows in the j-th propagator,

$$l_{\parallel\, j}^\alpha = \sum_i \alpha_{ij} q_{\parallel\, i}^\alpha + \sum_i \beta_{ij} p_i^\alpha\,. \tag{2.40}$$

Thus, out of the total number of $\ell(\ell+9)/2$ variables parametrizing a ℓ-loop integrand, the denominators depend on a *reduced* set of $\ell(\ell + 2d_\parallel + 1)/2$ variables, which correspond to the $\ell\, d_\parallel$ components $\mathbf{x}_{\parallel\, i}$ and to $\ell(\ell+1)/2$ scalar products λ_{ij}.

Of course, the numerator of the integrand can still depend on the remaining $\ell(4 - d_\parallel)$ transverse components $\mathbf{x}_{\perp\, i}$, since in general an amplitude can depend on four-dimensional vectors which are orthogonal to the the external momenta. For instance, the contraction of the loop momentum q_i with the polarization vector ε_j associated to a massless external particle of momentum p_j would receive a non-trivial contribution from the transverse components of the loop momentum,

$$q_i \cdot \varepsilon_j \propto \lambda_i \cdot \varepsilon_j = \sum_{k=d_\parallel+1}^{4} x_{k\,i}\,(e_k \cdot \varepsilon_j)\,. \tag{2.41}$$

Nonetheless, any dependence of the integrand on the $\ell(4-d_\parallel)$ transverse components which originates from the numerator function is, by construction, of *polynomial* type. Hence, forasmuch as any polynomial can always be integrated with elementary techniques, the introduction of the $d = d_\parallel + d_\perp$ parametrization allows a straightforward integration over all transverse components $\mathbf{x}_{\perp\,i}$.

In particular, the integral over each transverse variable can be factorized into an *angular integral* involving elementary trigonometric functions and, hence, computed algorithmically. In fact, by applying the recursive orthonormalization procedure described in appendix A.1 to the vector basis $\mathcal{E}$, it can be shown that all $\mathbf{x}_{\perp\,i}$ variables, as well as the scalar products λ_{ij}, can be mapped into $\ell(\ell-2d_\parallel+9)/2$ angular variables θ_{ij}.

More precisely, by introducing the angles

$$\begin{aligned} \mathbf{\Theta}_\Lambda &= \{\theta_{ij}\}, && 1 \le i < j \le \ell, \\ \mathbf{\Theta}_\perp &= \{\theta_{ij}\}, && j \le i \le j+3-d_\parallel,\;\; 1 \le j \le \ell, \end{aligned} \tag{2.42}$$

we can rewrite the ℓ-loop n-point integral of eq. (2.1) as

$$I^{d\,(\ell,n)}_{a_1\ldots a_m}[\mathcal{N}] = \Omega^{(\ell)}_d \int \prod_{i=1}^{\ell} \mathrm{d}^{n-1} q_{\parallel\,i} \int \mathrm{d}^{\frac{\ell(\ell+1)}{2}} \mathbf{\Lambda} \int \mathrm{d}^{(4-d_\parallel)\ell}\mathbf{\Theta}_\perp \frac{\mathcal{N}(q_{i\,\parallel}, \mathbf{\Lambda}, \mathbf{\Theta}_\perp)}{\prod_{j=1}^{m} D_j^{a_j}(q_{\parallel\,i}, \mathbf{\Lambda})}, \tag{2.43}$$

where the integration over the d-dimensional loop momenta q_i has been split in:

- The integral over the $d_\parallel$ components of the longitudinal part of each loop momentum, $q_{\parallel\,i}$, which can be rewritten as an integral over the parallel components $\mathbf{x}_\parallel$,

$$\int \prod_{i=1}^{\ell} \mathrm{d}^{n-1} q_{\parallel\,i} = \mathcal{J}_{[d_\parallel]} \int_{-\infty}^{+\infty} \prod_{i=1}^{\ell} \prod_{j=1}^{d_\parallel} \mathrm{d}x_{j\,i}\,, \tag{2.44}$$

  with

$$\mathcal{J}_{[d_\parallel]} = \prod_{i=1}^{\ell} \sqrt{\left| \det\left(\frac{\partial q^\alpha_{\parallel\,i}}{\partial x_{ji}} \frac{\partial q_{\parallel\,i\,\alpha}}{\partial x_{ki}} \right) \right|}, \qquad j,k \le d_\parallel\,; \tag{2.45}$$

- The integral over the $\ell(\ell+1)/2$ variables λ_{ij},

$$\int \mathrm{d}^{\frac{\ell(\ell+1)}{2}} \mathbf{\Lambda} = \int \prod_{1\le i\le j}^{\ell} \mathrm{d}\lambda_{ij}\,[G(\lambda_{ij})]^{(d_\perp-1-\ell)/2}\,, \tag{2.46}$$

  where $G(\lambda_{ij})$ is the Gram determinant built from the transverse vectors λ_i^α. The integration region of eq. (2.46) is determined, similarly to eq. (2.27), from the positivity conditions $\lambda_{ii}^2 \ge 0$ and of $G(\lambda_{ij}) \ge 0$ which represent, respectively, the square norm of the transverse vectors λ_i^α and the volume of the parallelotope spanned by the latter.

The integral over the scalar products $\lambda_i \cdot \lambda_j = \lambda_{ij}$, $i \neq j$, can be further parametrized in terms of $\ell(\ell-1)/2$ angles $\mathbf{\Theta}_\Lambda$ which describe the relative orientations of the transverse vectors,

$$\int \mathrm{d}^{\frac{\ell(\ell+1)}{2}} \mathbf{\Lambda} = \int_0^\infty \prod_{i=1}^{\ell} \mathrm{d}\lambda_{ii} (\lambda_{ii})^{(d_\perp - 2)/2} \int \mathrm{d}^{\frac{\ell(\ell-1)}{2}} \mathbf{\Theta}_\Lambda \,, \tag{2.47}$$

with

$$\int \mathrm{d}^{\frac{\ell(\ell-1)}{2}} \mathbf{\Theta}_\Lambda = \int_{-1}^{1} \prod_{1 \leq i < j \leq \ell} \mathrm{d}\cos\theta_{ij} (\sin\theta_{ij})^{d_\perp - 2 - i} \,; \tag{2.48}$$

- The integral over the $\ell(4-d_\parallel)/2$ angles $\mathbf{\Theta}_\perp$ which parametrize the four-dimensional transverse components $\mathbf{x}_{\perp\, i}$ of the loop momenta,

$$\int \mathrm{d}^{(4-d_\parallel)\ell} \mathbf{\Theta}_\perp = \int_{-1}^{1} \prod_{i=1}^{4-d_\parallel} \prod_{j=1}^{\ell} \mathrm{d}\cos\theta_{(i+j-1)\, j} (\sin\theta_{(i+j-1)\, j})^{d_\perp - i - j - 1} \,. \tag{2.49}$$

The detailed derivation of eq. (2.43) is presented in appendix A.1 where we also provide the *polynomial transformations* that map λ_{ij} and $\mathbf{x}_\perp$ into the angular variables,

$$\begin{cases} \lambda_{ij} \to & P\,[\lambda_{kk}, \sin[\mathbf{\Theta}_\Lambda], \cos[\mathbf{\Theta}_\Lambda]] \,, \qquad i \neq j, \\ x_{ji} \to & P\,[\lambda_{kk}, \sin[\mathbf{\Theta}_{\perp,\Lambda}], \cos[\mathbf{\Theta}_{\perp,\Lambda}]] \,, \quad j > d_\parallel, \quad k = 1, \dots \ell \,. \end{cases} \tag{2.50}$$

For the present discussion, let us simply remark that:

- The integral defined by eq. (2.43) is dimensionally regularized through the dependence of eqs. (2.48)-(2.49) on $d_\perp = d + 1 - n$;
- The choice of the four-dimensional basis $\mathcal{E}$ (and the consequent definition of the transverse space variables $\mathbf{\Lambda}$ and $\mathbf{\Theta}_\perp$) are determined just by the external kinematics and do not depend on the specific set of denominators which characterizes the integral. Therefore, the integral representation (2.43) can be equally applied to planar and non-planar topologies;
- It is important to observe that, in the case of two-point integrals depending on a light-like external momentum $p^2 = 0$, eq. (2.43) holds for $d_\parallel = 2$, since –in four dimensions– we can only define two directions that are orthogonal to a massless vector.

2.3.1 Angular integration over the transverse space

The most important feature of the integral rapresentation (2.43) is summarized by eq. (2.49), which shows that the integral over each of the transverse components $\mathbf{x}_{\perp\, i}$ can be cast into the standard form

$$\int_{-\infty}^{\infty} \mathrm{d}x_{ij} f(x_{ij}) \sim \int_{-1}^{1} \mathrm{d}\cos\theta (\sin\theta)^a f(\cos\theta, \sin\theta) \,, \quad i > d_\parallel \,, \tag{2.51}$$

where the dependence on d of the exponent a of integration kernel $(\sin\theta)^a$ is fixed by the number of loops and external legs of the topology under consideration. Morevoer, the polynomial dependence on $\mathbf{x}_{\perp\, i}$ of any integrand is translated, via the change of variables (2.50), into a dependence on $\sin[\boldsymbol{\Theta}_\perp]$ and $\cos[\boldsymbol{\Theta}_\perp]$ which is in turn purely polynomial.

This implies that, independently of the particular numerator under consideration, the integration over the transverse components is always reduced to the evaluation of $\ell(4-d_\parallel)$ factorized one-dimensional integrals of the type

$$\int_{-1}^{1} \mathrm{d}\cos\theta (\sin\theta)^a (\cos\theta)^b \,. \tag{2.52}$$

Obviously, the explicit expression of the numerator function determines, together with the number of loops and external momenta, the actual values of the powers a and b. A convenient algorithmic way of mapping any numerator into a product of angular integrals (2.52) is to expand it in terms of *Gegenbauer polynomials* $C_n^{(\alpha)}(\cos\theta)$, a class of orthogonal polynomials over the interval $[-1,1]$ whose orthogonality relation is defined exactly through a weighting function of the type $\omega_a(\theta) = (\sin\theta)^a$,

$$\int_{-1}^{1} \mathrm{d}\cos\theta_{ij}\ (\sin\theta)^{2\alpha-1}\ C_n^{(\alpha)}(\cos\theta)\ C_m^{(\alpha)}(\cos\theta) = \delta_{mn}\frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!\,(n+\alpha)\Gamma^2(\alpha)}. \tag{2.53}$$

We refer the reader to appendix A.4 for a summary of the most important properties of this class of polynomials.

Eq. (2.53) can be used to systematically integrate any numerator over all the transverse components $\mathbf{x}_{\perp\, i}$ in the following way:

- We use the transformation defined in eq. (2.50) in order to map any monomial in the transverse components into a polynomial in $\sin[\boldsymbol{\Theta}_\perp]$ and $\cos[\boldsymbol{\Theta}_\perp]$;
- We expand the resulting polynomial in terms of products of $C_n^{(\alpha)}(\cos\theta)$;
- We integrate each term of the expansion by using the orthogonality relation (2.53).

It is important to remark that we do not need to apply from scratch this procedure to every new numerator. Given the number of loops and external legs, we can simply integrate once and for all every possible monomial in the corresponding transverse variables up to the desired rank and then re-use the result whenever we need it. General results for one- and two-loop transverse integrals, in all external kinematic configurations, are collected in appendices A.2-A.3.
After the outlined procedure has been applied to integrate away all transverse directions, we obtain a representation of the Feynman integral (2.1) of the type

$$\begin{aligned} I_{a_1\dots a_m}^{d\,(\ell,n)}[\mathcal{N}] &= \widetilde{\Omega}_d^{(\ell)} \int \prod_{i=1}^{\ell} \mathrm{d}^{n-1} q_{\parallel\, i} \int \prod_{1\le i\le j}^{\ell} \mathrm{d}\lambda_{ij}\, [G(\lambda_{ij})]^{(d_\perp-1-\ell)/2} \frac{\mathcal{N}(q_{i\,\parallel},\lambda_{ij})}{\prod_{j=1}^m D_j^{a_j}(q_{\parallel\, i},\lambda_{ij})} \\ &= \widetilde{\Omega}_d^{(\ell)} \mathcal{J}_{[d_\parallel]} \int_{-\infty}^{+\infty} \prod_{i=1}^{\ell}\prod_{j=1}^{d_\parallel} \mathrm{d}x_{j\,i} \int \prod_{1\le i\le j}^{\ell} \mathrm{d}\lambda_{ij}\, [G(\lambda_{ij})]^{(d_\perp-1-\ell)/2} \frac{\mathcal{N}(x_{ij},\lambda_{ij})}{\prod_{j=1}^m D_j^{a_j}(x_{ij},\lambda_{ij})}, \end{aligned} \tag{2.54}$$

where $\widetilde{\Omega}_d^{(\ell)}$ absorbs, besides $\Omega_d^{(\ell)}$, the additional prefactor resulting from the angular integrations. For instance, in the case $\mathcal{N}=1$, we have

$$\widetilde{\Omega}_d^{(\ell)} = \Omega_d^{(\ell)} \int \mathrm{d}^{(4-d_\parallel)\ell}\boldsymbol{\Theta}_\perp = \frac{\pi^{-\ell(\ell+2n-3)/4}}{\prod_{i=1}^{\ell}\Gamma\left(\frac{d-n+i}{2}\right)}\,. \tag{2.55}$$

This parametrization of the integral enjoys some remarkable properties.

First of all, we observe that eq. (2.54) represents a d-dimensional Feynman integral as a multiple integral over a number of variables which is equal to the total number of scalar products $n_{\text{SP}} = \ell(2n+\ell-1)/2$.

Secondly, the integration over $\mathbf{x}_{\perp i}$ has removed all vanishing terms, so that the integral is free from any *spurious contribution*. In this perspective, it is interesting to observe that the vanishing of all spurious terms is captured by one single master formula, which corresponds to the orthogonality relation (2.53) of Gegenbauer polynomials.

Finally, once all angular integrations have been performed, the entire dependence of the integrand on the dimensions d is absorbed in the exponent of the Gram determinant $G(\lambda_{ij})$, similarly to the $d = 4-2\epsilon$ case. Therefore, also in $d = d_\parallel + d_\perp$, we are able to identify additional powers of $G(\lambda_{ij})$ in the numerator with higher-dimensional integrals.

For instance, in the case $\mathcal{N}(q_i) = G(\lambda_{ij})$ it is easy to verify from eq. (2.54) that

$$I^{d\,(\ell,n)}_{a_1\,\dots\,a_m}[G(\lambda_{ij})] = \frac{1}{2^\ell}\prod_{i=1}^{\ell}(d_\perp - i + 1) I^{d+2\,(\ell,n)}_{a_1\,\dots\,a_m}[1]\,, \tag{2.56}$$

where the d-dependent coefficient in the r.h.s. originates from the ratio of angular factors $\widetilde{\Omega}^{(\ell)}_d/\widetilde{\Omega}^{(\ell)}_{d+2}$.

As a concluding remark, we would like to comment on the connection between the Gegengebauer polynomial expansion and the tensor integral decomposition in the transverse space [50]. In the $d = d_\parallel + d_\perp$ parametrization, the most general ℓ-loop n-point integral with a non-trivial dependence on the transverse components $\mathbf{x}_{\perp i}$ can be written in the form

$$\int \prod_{i=1}^{\ell} \frac{\mathrm{d}^d q_i}{\pi^{d/2}} f\left(q_{\parallel\, i}, \lambda_{ij}\right) \left(\prod_{j=d_\parallel+1}^{4} \prod_{k=1}^{\ell} (e_j\cdot\lambda_k)^{\alpha_{jk}} \right)\,, \quad \alpha_{jk}\in\mathbb{N}\,, \tag{2.57}$$

where we have collected in $f\left(q_{\parallel\, i}, \lambda_{ij}\right)$ the full dependence of the integrand on the variables λ_{ij} and on the longitudinal components of the loop momenta.

The integral (2.57) corresponds to a rank $\alpha = \sum_{j\,k}\alpha_{jk}$ tensor integral of the type

$$\int \prod_{i=1}^{\ell} \frac{\mathrm{d}^d q_i}{\pi^{d/2}} f\left(q_{\parallel\, i}, \lambda_{ij}\right) \left(\lambda_1^{\nu_1}\cdots\lambda_\ell^{\nu_\alpha}\right)\,, \tag{2.58}$$

which can be reduced through the standard tensor decomposition technique. Remarkably, the tensor decomposition in the transverse space is significantly simpler than the full d-dimensional tensor reduction, since no external momenta lies in the transverse space and tensor integrals can only be projected onto the $d_\perp$-dimensional metric tensor,

$$\int \prod_{i=1}^{\ell} \frac{\mathrm{d}^d q_i}{\pi^{d/2}} f\left(q_{\parallel\, i}, \lambda_{ij}\right) \left(\lambda_1^{\nu_1}\cdots\lambda_\ell^{\nu_\alpha}\right) = \sum_{\sigma\in S} a_\sigma\, P_\sigma^{\nu_{\sigma(1)}\,\cdots\,\nu_{\sigma(\alpha)}}\,, \tag{2.59}$$

where S is the set of non-equivalent permutations of the Lorentz indexes ν_i and P_σ is the rank-α tensor

$$P_\sigma^{\nu_{\sigma(1)}\,\cdots\,\nu_{\sigma(\alpha)}} = g_{[d_\perp]}^{\nu_{\sigma(1)}\nu_{\sigma(2)}}\cdots g_{[d_\perp]}^{\nu_{\sigma(\alpha-1)}\nu_{\sigma(\alpha)}}\,. \tag{2.60}$$

The coefficients a_σ can be determined in the traditional way, by contracting both sides of the eq. (2.59) with each of the projector (2.60). Such contractions would produce,

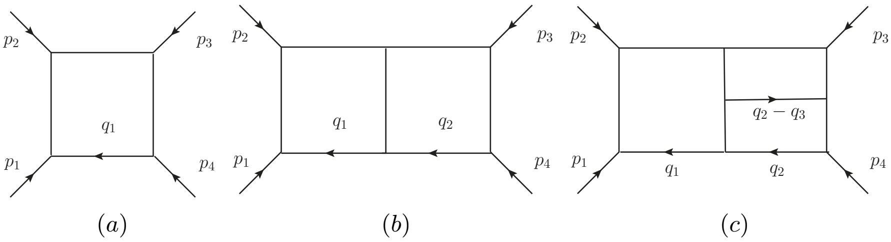


Figure 2.1: One- two- and three-loop four-point integrals.

on the l.h.s., terms proportional to λ_{ij} and, on the r.h.s., a polynomial in d. Therefore, consistently with the result of the Gegenbauer polynomial expansion, the tensor decomposition eq. (2.57) will reduce the dependence on the transverse components of the loop momenta to polynomial numerators in λ_{ij} multiplying $f\left(q_{\parallel\, i}, \lambda_{ij}\right)$.

As in the case of the explicit angular integration, the result of the tensor decomposition in the transverse space solely depends on the number n of external legs and on the powers of transverse loop momenta appearing in the numerator, while it is completely independent of the details of the integral under consideration. Hence transverse integrals can be computed once and for all.

In the following, we use the integral representation (2.43) and apply the integration procedure described above in the case four-point integrals up to three loops. A collection of general results for one- and two- loop integrals in all kinematic configurations, including a list of integrals over the transverse directions, can be found in appendices A.2-A.3.

Example: four-point integrals

Let us consider the one- two- and three-loop four-point topologies depicted in figure 2.1. The following discussion is independent of the one-shell condition of the external momenta p_i and of the presence of internal masses.

Due to momentum conservation $p_1 + p_2 + p_3 + p_4 = 0$, the external momenta span a subspace with dimension $d_\parallel = 3$. Therefore, we can define a four-dimensional vector basis $\mathcal{E}$ which contains one single transverse direction e_4^α,

$$p_i \cdot e_4 = 0 \qquad \forall i = 1, 2, 3. \tag{2.61}$$

Consistently, in all the three cases, we can decompose the d-dimensional loop momenta according to eq. (2.34),

$$q_i^\alpha = q_{\parallel\, i}^\alpha + \lambda_i^\alpha\,, \quad i = 1, \ldots, \ell\,, \tag{2.62}$$

where $q_{\parallel\, i}^\alpha$ is the three-dimensional longitudinal vector defined by

$$q_{\parallel\, i}^\alpha = \sum_{j=1}^{3} x_{ji}\, e_j^\alpha\,, \tag{2.63}$$

and λ_i^α is the transverse vector belonging to the $d_\perp = (d-3)$-dimensional orthogonal space,

$$\lambda_i^\alpha = x_{4i}\, e_4^\alpha + \mu_i^\alpha\,. \tag{2.64}$$

Upon this decomposition, all denominators become independent of the component x_{4i} of all loop momenta. The $d_{\|}+d_{\perp}$ parametrization of the integrals can now be readily obtained by specializing eqs. (2.43)-(2.49) to $d_{\|}=3$ and, according to the case, $\ell = 1,2,3$.

- **One-loop:** For the one-loop integral of figure 2.1(a), we have one single transverse vector, with square norm $\lambda_{11}\equiv\lambda^2$ and one angle θ, which parametrizes the transverse direction x_4,

$$x_4=\sqrt{\lambda^2}\cos\theta\,. \tag{2.65}$$

The $d=d_{\|}+d_{\perp}$ parametrization is

$$I^{d\,(1,4)}_{a_1\ldots a_4}[\mathcal{N}]=\frac{1}{\pi^2\Gamma(\frac{d-4}{2})}\int \mathrm{d}^3q_{\|}\int_0^\infty \mathrm{d}\lambda^2(\lambda^2)^{\frac{d-5}{2}}\int_{-1}^{1}\mathrm{d}\!\cos\theta(\sin\theta)^{d-6}\frac{\mathcal{N}(q)}{\prod_{m=1}^4 D_m^{a_m}(q)}\,. \tag{2.66}$$

Any polynomial dependence of the numerator on x_4, hence on $\cos\theta$, can be simply integrated through the orthogonality relation (2.53). In the case $\mathcal{N}$=1, we obtain

$$I^{d\,(1,4)}_{a_1\ldots a_4}[1]=\frac{1}{\pi^{3/2}\Gamma(\frac{d-3}{2})}\int \mathrm{d}^3q_{\|}\int_0^\infty \mathrm{d}\lambda^2(\lambda^2)^{\frac{d-5}{2}}\frac{1}{\prod_{m=1}^4 D_m^{a_m}(q)}. \tag{2.67}$$

The integrals of odd powers of x_4, which would correspond to products of Gegenbauer polynomials with different indices, vanish by orthogonality. Even powers of x_4, due to eq. (2.65), are reduced to additional λ^2 factors. For instance, let us consider

$$I^{d\,(1,4)}_{a_1\ldots a_4}[\,x_4^2\,]=\frac{1}{\pi^2\Gamma(\frac{d-4}{2})}\int \mathrm{d}^3q_{\|}\int_0^\infty \mathrm{d}\lambda^2(\lambda^2)^{\frac{d-3}{2}}\int_{-1}^{1}\mathrm{d}\!\cos\theta\frac{(\sin\theta)^{d-6}(\cos\theta)^2}{\prod_{m=1}^4 D_m^{a_m}(q)} \tag{2.68}$$

and

$$I^{d\,(1,4)}_{a_1\ldots a_4}[\,x_4^4\,]=\frac{1}{\pi^2\Gamma(\frac{d-4}{2})}\int \mathrm{d}^3q_{\|}\int_0^\infty \mathrm{d}\lambda^2(\lambda^2)^{\frac{d-1}{2}}\int_{-1}^{1}\mathrm{d}\!\cos\theta\frac{(\sin\theta)^{d-6}(\cos\theta)^4}{\prod_{m=1}^4 D_m^{a_m}(q)}\,. \tag{2.69}$$

Powers of $\cos\theta$ are express in terms of Gegenbauer polynomials as (see appendix A.4)

$$(\cos\theta)^2=\frac{1}{(d-5)^2}[C_1^{(\frac{d-5}{2})}(\cos\theta)]^2, \tag{2.70a}$$

$$(\cos\theta)^4=\frac{1}{(d-3)^2}\left[C_0^{(\frac{d-5}{2})}(\cos\theta)+\frac{4}{(d-5)^2}C_2^{(\frac{d-5}{2})}(\cos\theta)\right]^2, \tag{2.70b}$$

where the index $\alpha=(d-5)/2$ has been set in order to match the integration kernel $(\sin\theta)^{d-6}$ of eq. (2.66) with the weight function appearing in the orthogonality relation (2.53). The latter is used to obtain

$$I^{d\,(1,4)}_{a_1\ldots a_4}[\,x_4^2\,]=\frac{1}{d-3}I^{d\,(1,4)}_{a_1\ldots a_4}[\,\lambda^2\,]=\frac{1}{2}I^{d+2\,(1,4)}_{a_1\ldots a_4}[1], \tag{2.71a}$$

$$I^{d\,(1,4)}_{a_1\ldots a_4}[\,x_4^4\,]=\frac{3}{(d-3)(d-1)}I^{d\,(1,4)}_{a_1\ldots a_4}[\,\lambda^4\,]=\frac{3}{4}I^{d+4\,(1,4)}_{a_1\ldots a_4}[1]. \tag{2.71b}$$

As we have already observed, additional powers of $\lambda^2 \equiv G(\lambda^2)$ in the numerator can be identified with higher-dimensional integrals. The rational prefactors arising in the second equality of both eqs. (2.71a) an (2.71b) can be easily derived from the comparison with eq. (2.67), through the shifts $d \to d+2$ and $d \to d+4$, respectively.

- **Two-loop:** The transverse space of the four-point topology shown in figure 2.1(b) is described by the norms λ_{11} and λ_{22}, and by three angles θ_{12}, θ_{11} and θ_{22}, which parametrize, respectively, the relative orientation of the vectors λ_1 and λ_2 and the transverse directions x_{41} and x_{42} of the two loop momenta.

  The mapping between angular variables and the loop components is In this case,(2.50) reads

$$\begin{cases} \lambda_{12} = & \sqrt{\lambda_{11}\lambda_{22}} c_{\theta_{12}} \\ x_{41} = & \sqrt{\lambda_{11}} c_{\theta_{11}} \\ x_{42} = & \sqrt{\lambda_{22}} (c_{\theta_{11}} c_{\theta_{12}} + s_{\theta_{11}} s_{\theta_{12}} c_{\theta_{22}})\,, \end{cases} \tag{2.72}$$

  where, for ease of notation, we have identified $c_{\theta_{ij}} = \cos\theta_{ij}$ and $s_{\theta_{ij}} = \sin\theta_{ij}$. The corresponding $d = d_{\parallel} + d_{\perp}$ parametrization reads

$$I^{d\,(2,4)}_{a_1\ldots a_7}[\mathcal{N}] = \frac{2^{d-6}}{\pi^5 \Gamma(d-5)} \int \mathrm{d}^3 q_{\parallel\,1} \mathrm{d}^3 q_{\parallel\,2} \int_0^\infty \mathrm{d}\lambda_{11} d\lambda_{22} (\lambda_{11})^{\frac{d-5}{2}} (\lambda_{22})^{\frac{d-5}{2}} \times \\ \int_{-1}^1 \mathrm{d}c_{\theta_{12}} \mathrm{d}c_{\theta_{22}} \mathrm{d}c_{\theta_{11}}\, (s_{\theta_{12}})^{d-6}\, (s_{\theta 11})^{d-6} (s_{\theta 22})^{d-7} \frac{\mathcal{N}(q_i)}{\prod_{m=1}^7 D_m^{a_m}(q_i)}\,. \tag{2.73}$$

  In the $\mathcal{N}=1$ case, the integration over the transverse angles θ_{12} and θ_{22} yields to

$$I^{d\,(2,4)}_{a_1\ldots a_7}[\,1\,] = \frac{2^{d-5}}{\pi^4 \Gamma(d-4)} \int \mathrm{d}^3 q_{\parallel\,1} \mathrm{d}^3 q_{\parallel\,2} \int_0^\infty d\lambda_{11} \mathrm{d}\lambda_{22} (\lambda_{11})^{\frac{d-5}{2}} (\lambda_{22})^{\frac{d-5}{2}} \times \\ \int_{-1}^1 \mathrm{d}c_{\theta_{12}}\, (\sin\theta_{12})^{d-6} \frac{1}{\prod_{m=1}^7 D_m^{a_m}(q_i)}\,. \tag{2.74}$$

  Through the change of variables (2.72), any numerator depending on x_{41} and x_{42} is transformed into a polynomial in $c_{\theta_{ij}}$ and $s_{\theta_{ij}}$, with coefficients depending on λ_{ij}, which can be again integrated in terms of Gegenbauer polynomials.

  For instance, rank-2 monomials in x_{41} and x_{42} produce

$$I^{d\,(2,4)}_{a_1\ldots a_7}[x_{4i} x_{4_j}] = \frac{1}{d-3} I^{d\,(2,4)}_{a_1\ldots a_7}[\lambda_{ij}]\,, \quad i,j = 1,2\,. \tag{2.75}$$

  Results for higher rank numerators are listed in appendix A.3.

- **Three-loop:** Finally, we consider the tennis-court topology shown in figure eq. 2.1(c). The transverse space is parametrized in terms of the variables $\mathbf{\Lambda} = \{\lambda_{11}, \lambda_{22}, \lambda_{33}, \theta_{12}, \theta_{13}, \theta_{23}\}$ and $\mathbf{\Theta}_\perp = \{\theta_{11}, \theta_{22}, \theta_{33}\}$, which are related to the components of the loop momenta through the transformation

$$\begin{cases} \lambda_{12} = & \sqrt{\lambda_{11}\lambda_{22}} c_{\theta_{12}} \\ \lambda_{23} = & \sqrt{\lambda_{22}\lambda_{33}} c_{\theta_{13}} \\ \lambda_{13} = & \sqrt{\lambda_{11}\lambda_{33}} (c_{\theta_{12}} c_{\theta_{13}} + s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}}) \\ x_{41} = & \sqrt{\lambda_{11}} c_{\theta_{11}} \\ x_{42} = & \sqrt{\lambda_{22}} (c_{\theta_{11}} c_{\theta_{12}} + s_{\theta_{11}} s_{\theta_{12}} c_{\theta_{22}}) \\ x_{43} = & \sqrt{\lambda_{33}} (c_{\theta_{11}} c_{\theta_{12}} c_{\theta_{13}} + s_{\theta_{11}} s_{\theta_{12}} c_{\theta_{22}} c_{\theta_{13}} - s_{\theta_{11}} s_{\theta_{13}} c_{\theta_{12}} c_{\theta_{22}} c_{\theta_{23}} \\ & + s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{11}} c_{\theta_{23}} + s_{\theta_{11}} s_{\theta_{13}} s_{\theta_{22}} s_{\theta_{23}} c_{\theta_{33}})\,. \end{cases} \tag{2.76}$$

In $d = d_\parallel + d_\perp$, the integral is parametrized as

$$I^{d\,(3,4)}_{a_1\dots a_{10}}[\mathcal{N}] = \frac{2^{d-7}}{\pi^8\Gamma(d-6)\Gamma\left(\frac{d-4}{2}\right)} \int \prod_{i=1}^{3} \mathrm{d}^3 q_{\parallel\, i} \int_0^\infty \prod_{i=1}^{3} \mathrm{d}\lambda_{ii} (\lambda_{ii})^{\frac{d-5}{2}} \times \int_{-1}^{1} \prod_{1\le i\le j\le 3} \mathrm{d}c_{\theta_{ij}} (s_{\theta_{ij}})^{d-5-i} \frac{\mathcal{N}(q_i)}{\prod_{m=0}^{9} D_m^{a_m}(q_i)}\,, \tag{2.77}$$

which, in the $\mathcal{N} = 1$ case, reduces to

$$I^{d\,(3,4)}_{a_1\dots a_{10}}[\,1\,] = \frac{2^{d-5}}{\pi^{13/2}\Gamma(d-4)\Gamma\left(\frac{d-5}{2}\right)} \int \prod_{i=1}^{3} \mathrm{d}^3 q_{\parallel\, i} \int_0^\infty \prod_{i=1}^{3} \mathrm{d}\lambda_{ii} (\lambda_{ii})^{\frac{d-5}{2}} \times \int_{-1}^{1} \prod_{1\le i<j\le 3} dc_{\theta_{ij}} (s_{\theta_{ij}})^{d-5-i} \frac{1}{\prod_{m=1}^{10} D_m^{a_m}(q_i)}\,. \tag{2.78}$$

Higher-rank numerators in the transverse variables x_{4i} can be integrated via the orthogonality relation (2.53). For example, similarly to the previous case, at rank-2 we find

$$I^{d\,(3,4)}_{a_1\dots a_{10}}[x_{4i}x_{4j}] = \frac{1}{d-3} I^{d\,(3,4)}_{a_1\dots a_{10}}[\lambda_{ij}]\,, \qquad i,j = 1,2,3\,. \tag{2.79}$$

2.3.2 Factorized integrals and ladders

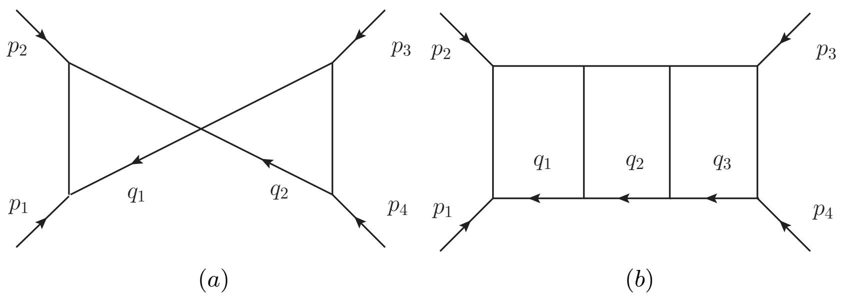


Figure 2.2: Two-loop factorized topology and three-loop ladder topology.

Whenever the set of denominators characterizing a multi-loop integral is independent of a certain number of scalar products λ_{ij}, with $i \neq j$, (i.e. on some angles $\theta_{ij} \in \mathbf{\Theta}_\Lambda$), the $d = d_\parallel + d_\perp$ parametrization defined in eq. (2.43) admits a further simplified form. In fact, in such cases, the dependence of the integrand on λ_{ij} is reduced to a polynomial one, which is then mapped into a polynomial in $s_{\theta ij}$ and $c_{\theta ij}$ by the change of variables (2.50). Therefore, the same integration via expansion in Gegenbauer polynomials applied to the transverse angles $\mathbf{\Theta}_\perp$, can be used also for *any* $\theta_{ij} \in \mathbf{\Theta}_\Lambda$ which does not appear in the denominators.

This simplification occurs for two different types of integral topologies: *factorized* integrals, where there exists one or more loop momenta q_i such that the denominators are independent of $q_i \cdot q_j$, for all $j \neq i$, and *ladder* integrals, where the full set of denominators is independent on at least one scalar product $q_i \cdot q_j$, for some $j \neq i$. In the following, we discuss the simplified $d = d_\parallel + d_\perp$ parametrization for these two classes of integrals, by working on explicit examples.

Factorized integrals

When the loop corresponding to q_i^α is *factorized*, no denominator depends on $q_i \cdot q_j$, with $j \neq i$. In general, whether a factorized integral originates from a Feynman diagram or as a subtopology of a non-factorized integral, the integrand is not necessarily fully factorized, since the numerator could still depend on the $(d-4)$-dimensional part of $q_i \cdot q_j$, corresponding to μ_{ij}.

Nevertheless, it can be shown that, after integrating over μ_{ij}, by means of the orthogonality relation (2.53), the $d = d_\parallel + d_\perp$ parametrization of a factorized integral is given by the product of the $d = d_\parallel + d_\perp$ parametrizations of the integrals corresponding to the subtopologies. Remarkably, the transverse space of the factorized sub-integrals can have a different dimensions, according to the number of external legs attached to them.

As an example, let us consider the two-loop bowtie integral of the type shown in figure 2.2 (a). Before defining the longitudinal and transverse variables associated to this integral, we consider its $d = 4 - 2\epsilon$ parametrization, which according to eq. (2.22), reads

$$I^{d\,(2,4)}_{\text{bowtie}}[\mathcal{N}] = \frac{2^{d-6}}{\pi^5 \Gamma(d-5)} \int \mathrm{d}^4 q_{[4]\,1} d^4 q_{[4]\,2} \int_0^\infty \mathrm{d}\mu_{11} \int_0^\infty \mathrm{d}\mu_{22} \times \int_{-\sqrt{\mu_{11}\mu_{22}}}^{\sqrt{\mu_{11}\mu_{22}}} d\mu_{12} (\mu_{11}\mu_{22} - \mu_{12}^2)^{\frac{d-7}{2}} \frac{\mathcal{N}(q_1, q_2)}{\prod_{i=1}^3 D_i(q_1) \prod_{j=4}^6 D_j(q_2)}. \tag{2.80}$$

As we have explicitly indicated, the loop denominators depend either on q_1 or q_2 and, hence, are independent of μ_{12}. The most general numerator associated to this integral can always be written as a combination of terms of the form

$$\mathcal{N}(q_1, q_2) = (\mu_{12})^a \mathcal{N}_1(q_1) \mathcal{N}_2(q_2), \quad a \in \mathbb{N}. \tag{2.81}$$

Therefore, the integral over μ_{12} can be performed trivially, and the dependence of the full integrand on q_1 and q_2 can be completely factorized. In particular, by introducing the change of variable $\cos\theta = \mu_{12}/\sqrt{\mu_{11}\mu_{22}}$, we can cast the μ_{12} integral in the standard angular integral given in eq. (2.52), which can be evaluated by means of the by-now usual orthogonality relation (2.53),

$$\int_{-1}^{1} d\cos\theta (\sin\theta)^{d-7} (\cos\theta)^a = \begin{cases} 0 & \text{for} \quad a = 2n+1 \\ \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{d-5}{2}\right)}{\Gamma\left(\frac{d+a-4}{2}\right)} & \text{for} \quad a = 2n. \end{cases} \tag{2.82}$$

As one could expect, odd powers of μ_{12} produce vanishing contributions. In non-trvial cases with $a = 2n$, we apply eq. (2.82) to the integral (2.81) and identify

$$I^{d\,(2,4)}_{\text{bowtie}}[(\mu_{12})^a \mathcal{N}_1 \mathcal{N}_2] = (a-1)!! \left(\prod_{k=1}^{a/2} \frac{1}{(d-6+2k)} \right) \int \frac{\mathrm{d}^d q_1}{\pi^{d/2}} \frac{(\mu_{11})^{\frac{a}{2}} \mathcal{N}_1}{\prod_{i=1}^3 D_i(q_1)} \times \int \frac{\mathrm{d}^d q_2}{\pi^{d/2}} \frac{(\mu_{22})^{\frac{a}{2}} \mathcal{N}_2}{\prod_{j=4}^6 D_j(q_2)}. \tag{2.83}$$

where $a!!$ is the double factorial $a!! = a(a-2)\cdots 4 \cdot 2$. The coefficient appearing in the r.h.s. is deduced by comparison with the $d = 4 - 2\epsilon$ parametrization of one-loop integrals, given in eq. (2.22).

Each of two one-loop integrals admits now a $d = d_\parallel + d_\perp$ parametrization (2.43). The decomposition can be implemented by working with two different momentum bases, each one containing two vectors orthogonal to the external legs connected to the corresponding loop. In this case, the factorized graph is obtained from the product of two identical subtopologies (i.e. we have $d_\parallel = 2$ for both integrals) but, in general, factorized subtopologies may have a different number of external legs and, hence, longitudinal space of different dimensions. These results can be, of course, extended to an arbitrary number of loops.

Ladder integrals

Starting from a number of loops $\ell \geq 3$, *ladder* topologies correspond to integrals which denominators depend on a limited number variables λ_{ij}. In these cases, the $d = d_\parallel + d_\perp$ parametrization (2.43) reads exactly as in the general case (2.43) but the integration in terms of Gegenbauer polynomials can be extended to the subsets of angles $\boldsymbol{\Theta}_\Lambda$ corresponding to the λ_{ij} which do not appear in the denominators.

As an example, we consider the three-loop four-point ladder box shown in figure 2.2 (b), for which we introduce the same set of transverse variables as for the three-loop integral discussed in the previous section,

$$\boldsymbol{\Lambda} = \{\lambda_{11}, \lambda_{22}, \lambda_{33}, \theta_{12}, \theta_{13}, \theta_{23}\}\,, \quad \boldsymbol{\Theta}_\perp = \{\theta_{11}, \theta_{22}, \theta_{33}\}\,, \tag{2.84}$$

and we parametrize the integral exactly as in eq. (2.77). The denominators of this ladder integral are independent of $q_1 \cdot q_3$, i.e. on λ_{13}. Hence, as it can be seen from the change of variables given in eq. (2.76), θ_{23} can only appear in the numerator in terms of the type

$$\mathcal{N}(q_1, q_2, q_3) = (\cos\theta_{23})^a\,(\sin\theta_{23})^b \widetilde{\mathcal{N}}(q_1, q_2, q_3), \quad \alpha\,, \beta \in \mathbb{N}. \tag{2.85}$$

with $\widetilde{\mathcal{N}}$ independent of θ_{23}. Therefore, the integration over θ_{23} is reduced to the standard angular form of eq. (2.52), and yields to

$$\int_{-1}^{1} \mathrm{d}\cos\theta_{23}\;(\sin\theta_{23})^{d-7-b}\;(\cos\theta_{23})^a = \begin{cases} 0 & \text{for} \quad a = 2n+1 \\ \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{d-5+b}{2}\right)}{\Gamma\left(\frac{d+a+b-4}{2}\right)} & \text{for} \quad a = 2n. \end{cases} \tag{2.86}$$

For instance, in the $a = b = 0$, this additional integration returns

$$I^{d\,(3,4)}_{4\,\text{ladder}}[\,1\,] = \frac{2^{d-5}}{\pi^6\Gamma(d-4)\Gamma\left(\frac{d-4}{2}\right)} \int \prod_{i=1}^{3} \mathrm{d}^3 q_{\parallel\,i} \int_0^\infty \prod_{i=1}^{3} d\lambda_{ii} (\lambda_{ii})^{\frac{d-5}{2}} \times \\ \int_{-1}^{1} \mathrm{d}c_{\theta_{12}}\;\mathrm{d}c_{\theta_{13}}\;(s_{\theta_{12}})^{d-6}\;(s_{\theta_{13}})^{d-6} \frac{1}{\prod_{m=1}^{10} D_m(q_i)}. \tag{2.87}$$

2.4 Baikov representation

In section 2.3.1 we have shown that, after integrating over the transverse directions, the $d = d_\parallel + d_\perp$ parametrization of Feynman integrals reduces to a multiple integral over the longitudinal components of the loop momenta and the scalar products between the transverse vectors λ_i^α,

$$I^{d\,(\ell,n)}_{a_1\,\dots\,a_m}[\mathcal{N}] = \widetilde{\Omega}^{(\ell)}_d \int \prod_{i=1}^{\ell} \mathrm{d}^{n-1} q_{\parallel\,i} \int \prod_{1 \leq i \leq j}^{\ell} \mathrm{d}\lambda_{ij}\,[G(\lambda_{ij})]^{(d_\perp - 1 - \ell)/2} \frac{\mathcal{N}(q_{i\,\parallel}, \lambda_{ij})}{\prod_{j=1}^{m} D_j^{a_j}(q_{\parallel\,i}, \lambda_{ij})}. \tag{2.88}$$

As we have already observed, the number of integration variables that appear in eq. (2.88) matches the total number $n_{\mathrm{SP}} = \ell(\ell + 2n - 1)/2$ of scalar products between loop momenta and external ones,

$$\begin{aligned} s_{ij} =& q_i \cdot p_j\,, \quad i = 1, \dots, \ell\,, \quad j = 1, \dots, n-1\,, \\ \widetilde{s}_{ij} =& q_i \cdot q_j\,, \quad 1 \leq i \leq j \leq \ell\,. \end{aligned} \tag{2.89}$$

Therefore, we can perform a variables transformation on eq. (2.88) and express a Feynman integral as a multidimensional integral over the scalar products s_{ij} and $\widetilde{s}_{ij}$. Ultimately, because of the existence of an invertible, linear mapping between scalar products, loop denominators and ISPs, it is possible to use as integration variables the loop denominators themselves.

In this section, we derive this alternative parametrization of Feynman integrals, which was first proposed by Baikov [55–57], by starting from eq. (2.88) and we discuss its most relevant features, which emerged from recents studies of its application to IBPs, differential equations and unitarity cuts [58–63]. For an alternative derivation, we refer the reader to [124]. In the following we indicate with $G(v_1, v_2, \dots, v_n) \equiv G(v_i)$ the determinant of the Gram matrix $\mathbb{G}(v_1, v_2, \dots, v_n)$.

First of all, we observe that the volume element of the space spanned by the scalar products s_{ij} can be written as

$$\begin{aligned} \mathrm{d}s_{i1}\mathrm{d}s_{i2}\cdots\mathrm{d}s_{i\,n-1} =& \sqrt{\det\left(\frac{\partial s_{ij}}{\partial q^{\alpha}_{\parallel\,i}}\frac{\partial s_{ik}}{\partial q_{\parallel\,i\,\alpha}}\right)}\,\mathrm{d}^{n-1}q_{\parallel\,i} = \sqrt{\det\left(p_j\cdot p_k\right)}\,\mathrm{d}^{n-1}q_{\parallel\,i} \\ =& \sqrt{G(p_1, p_2, \dots, p_{n-1})}\,\mathrm{d}^{n-1}q_{\parallel\,i}\,. \end{aligned} \tag{2.90}$$

Therefore, we can express the integral over the longitudinal components of the i-th loop momentum as

$$\int \mathrm{d}^{n-1}q_{\parallel\,i} = \int \frac{\mathrm{d}s_{i1}\mathrm{d}s_{i2}\cdots\mathrm{d}s_{i\,n-1}}{\sqrt{G(p_1, p_2, \dots, p_{n-1})}}\,. \tag{2.91}$$

In addition, since $\widetilde{s}_{ij} = q_{\parallel\,i}\cdot q_{\parallel\,j} + \lambda_{ij}$, the mapping between λ_{ij} and $\widetilde{s}_{ij}$ has a trivial Jacobian,

$$\int \prod_{1\leq i\leq j}^{\ell} \mathrm{d}\lambda_{ij} = \int \prod_{1\leq i\leq j}^{\ell} \mathrm{d}\widetilde{s}_{ij}\,. \tag{2.92}$$

The Gram determinant of the transverse vectors λ^{α}_i can be expressed in terms of the scalar products as the ratio of Gram determinants,

$$G(\lambda_{ij}) = \frac{G(q_1, \dots, q_\ell, p_1, \dots, p_{n-1})}{G(p_1, p_2, \dots, p_{n-1})}\,. \tag{2.93}$$

In fact, we observe that any contribution to the Gram determinant of the longitudinal part of the loop mommenta q^{α}_i cancels, since it is linearly dependent on the external kinematics. Hence, we can write

$$G(q_1, \dots, q_\ell, p_1, \dots, p_{n-1}) = G(\lambda_1, \dots, \lambda_\ell, p_1, \dots, p_{n-1})\,. \tag{2.94}$$

In addition, due to the transversality condition $\lambda_i \cdot p_j = 0$, the Gram matrix $\mathbb{G}(\lambda_1, \dots, \lambda_\ell, p_1, \dots, p_{n-1})$ is block-diagonal,

$$\mathbb{G}(\lambda_1, \dots, \lambda_\ell, p_1, \dots, p_{n-1}) = \begin{pmatrix} \mathbb{G}(\lambda_1, \dots, \lambda_\ell) & \mathbf{0} \\ \mathbf{0} & \mathbb{G}(p_1, \dots p_{n-1}) \end{pmatrix}, \tag{2.95}$$

and, thus, by recalling that the determinant of a block-diagonal matrix is equal to the product of the determinants of the individual blocks, we can write

$$\det\left(\mathbb{G}(\lambda_1, \dots, \lambda_\ell, p_1, \dots p_{n-1})\right) = \det\left(\mathbb{G}(\lambda_1, \dots, \lambda_\ell)\right) \det\left(\mathbb{G}(p_1, \dots p_{n-1})\right), \tag{2.96}$$

which, exactly proves eq. (2.93).

We can now combine eqs. (2.91)-(2.93) and rewrite the integration measure of eq. (2.88) as

$$\begin{aligned} &\int \prod_{i=1}^{\ell} \mathrm{d}^{n-1} q_{\parallel\, i} \int \prod_{1 \le i \le j}^{\ell} \mathrm{d}\lambda_{ij} \left[G(\lambda_{ij})\right]^{(d_\perp - 1 - \ell)/2} \\ &= \int \prod_{i=1}^{\ell} \frac{\mathrm{d}s_{i1} \mathrm{d}s_{i2} \cdots \mathrm{d}s_{i\, n-1}}{\sqrt{G(p_1, p_2, \dots, p_{n-1})}} \int \prod_{1 \le i \le j}^{\ell} \mathrm{d}\widetilde{s}_{ij} \left(\frac{G(q_1, \dots, q_\ell, p_1, \dots, p_{n-1})}{G(p_1, p_2, \dots, p_{n-1})} \right)^{(d_\perp - 1 - \ell)/2} \\ &= \left(G(p_i)\right)^{(1-d_\perp)/2} \int \prod_{i=1}^{\ell} \prod_{j=1}^{n-1} \mathrm{d}s_{ij} \int \prod_{1 \le i \le j}^{\ell} \mathrm{d}\widetilde{s}_{ij} \left(G(q_1, \dots, q_\ell, p_1, \dots, p_{n-1})\right)^{(d_\perp - 1 - \ell)/2}, \end{aligned} \tag{2.97}$$

in such a way to obtain a parametrization of the Feynman integral $I^{d\,(\ell,n)}_{a_1 \dots a_m}$ in terms of the scalar products s_{ij} and $\widetilde{s}_{ij}$,

$$\begin{aligned} I^{d\,(\ell,n)}_{a_1 \dots a_m}[\mathcal{N}] =& \widetilde{\Omega}_d^{(\ell)} \left(G(p_i)\right)^{(1-d_\perp)/2} \int \prod_{i=1}^{\ell} \prod_{j=1}^{n-1} \mathrm{d}s_{ij} \times \\ & \int \prod_{1 \le i \le j}^{\ell} \mathrm{d}\widetilde{s}_{ij} \left(G(q_1, \dots, q_\ell, p_1, \dots, p_{n-1})\right)^{(d_\perp - 1 - \ell)/2} \frac{\mathcal{N}(s_{ij}, \widetilde{s}_{ij})}{\prod_{j=1}^{m} D_j^{a_j}(s_{ij}, \widetilde{s}_{ij})}, \end{aligned} \tag{2.98}$$

where $d_\perp = d - n + 1$.

As a final step, we introduce a new set of $n_{\mathrm{SP}} = \ell(\ell + 2n - 1)/2$ variables z_i, which correspond to the denominators D_i and the ISPs S_i that characterize the integral topology,

$$z_i = \begin{cases} D_i\,, & i = 1, \dots, m \\ S_{i-m}\,, & i = m+1, \dots m+r\,. \end{cases} \tag{2.99}$$

As it is obvious for the ISPs and as it can be seen for the denominators from their very definition (2.2), the relation between z_i and the scalar products s_{ij} and $\widetilde{s}_{ij}$ is *linear*, i.e. if we introduce the vectors $\mathbf{z} = \{z_1, \dots, z_{m+r}\}$ and $\mathbf{s} = \{s_{ij}, \widetilde{s}_{ij}\}$, we can write

$$\mathbf{z} = \mathbb{A}\,\mathbf{s} + \mathbf{c}\,, \tag{2.100}$$

with $\mathbb{A}$ being an invertible matrix and $\mathbf{c}$ a vector which depends on the internal masses and kinematic invariants.

Therefore, from eq. (2.98) we can immediately obtain the *Baikov parametrization* of the Feynman integral

$$I^{d\,(\ell,n)}_{a_1\,\ldots\,a_m}[\mathcal{N}] = C^{(\ell,n)}(d)\,(G(p_i))^{(1-d_\perp)/2}\int\prod_{i=1}^{n_{\text{sp}}}\mathrm{d}z_i\,(G(z_j))^{(d_\perp-1-\ell)/2}\,\frac{\mathcal{N}(z_i)}{z_1^{a_1}\cdots z_m^{a_m}}\,, \quad (2.101)$$

where $C^{(\ell,n)}(d)$ a normalization factor, defined by

$$C^{(\ell,n)}(d) = \widetilde{\Omega}^{(\ell)}_d \det\left(\mathbb{A}^{-1}\right)\,, \quad (2.102)$$

and $G(z_i)$, which is often referred to as *Baikov polynomial*, corresponds to the Gram determinant of loop and external momenta, expressed in terms of the z_i variables,

$$G(z_i) = G(q_1,\ldots,q_\ell,p_1,\ldots,p_{n-1})\big|_{\mathbf{s}=\mathbb{A}^{-1}(\mathbf{z}-\mathbf{c})}\,. \quad (2.103)$$

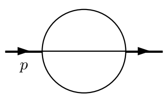
The integration bounds of eq. (2.101) are inherited from the positivity condition $\lambda_{ii}\geq 0$ and $G(\lambda_{ij})\geq 0$ of the $d = d_\parallel + d_\perp$ parametrization of eq. (2.46), once expressed in terms of the z_i.

According to the specific dependence of the integrand on the z_i variables or, equivalently on the scalar products s_{ij} and $\widetilde{s}_{ij}$, the integral representations of eqs. (2.101)-(2.98) can be further simplified. In fact, analogously to the case examined in section 2.3.2, whenever the loop denominators become independent on some variable, the latter can be integrated away with elementary techniques, so to minimize the number of integration variables that parametrize the integrand.

Alternatively to the direct integration of redundant variables, it is possible to obtain a representation in terms of a minimal number variables by recursively applying the Baikov representation to each individual loop momentum [60]. In this way, the integration measure of the Feynman integral is written as product of Gram determinants, where individual loop momenta can play the role external momenta of the other sub-loops. Let us illustrate this construction on a simple two-loop example.

Example 1

Let us consider the massless two-loop sunrise integral family defined by



$$= I^{d\,(2,2)}_{a_1\,a_2\,a_3\,b_1\,b_2} = \int\frac{\mathrm{d}^d q_1}{\pi^{d/2}}\frac{\mathrm{d}^d q_2}{\pi^{d/2}}\frac{S_1^{-b_1}S_2^{-b_2}}{D_1^{a_1}D_2^{a_2}D_3^{a_3}}\,, \quad (2.104)$$

where p is the external momentum ($p^2\neq 0$), the three loop denominators are defined by

$$D_1 = q_1^2\,,\quad D_2 = q_2^2\,,\quad D_3 = (q_2-q_1+p)^2\,, \quad (2.105)$$

and the ISPs are chosen to be the quadratic forms

$$S_1 = (p-q_1)^2\,,\quad S_2 = (q_2+p)^2\,. \quad (2.106)$$

If we identify $z_i = D_i$, for $i=1,2,3$ and $z_4 = S_1$, $z_5 = S_2$, the Baikov representation of the scalar bubble $I^{d\,(2,2)}_{1\,1\,1\,0\,0}$ reads, according to eq. (2.101),

$$I^{d\,(2,2)}_{1\,1\,1\,0\,0} = C^{(2,2)}(d)\,(G(p))^{(2-d)/2}\int\prod_{i=1}^{5}\mathrm{d}z_i\,\frac{(G(z_1,z_2,z_3,z_4,z_5))^{(d-4)/2}}{z_1z_2z_3}\,, \quad (2.107)$$

where $G(z_1, z_2, z_3, z_4, z_5)$ is the Gram determinant $G(q_1, q_2, p)$ expressed in terms of the z_i variables. Hence, we have obtained, consistently with $n_{\rm SP} = 5$, a five-fold integral representation of the sunrise integral. However, we observe that, for $b_1 = b_2 = 0$, the integrand of eq. (2.104) does not depend, separately, on all five scalar products, since $q_2 \cdot p$ and $q_2 \cdot q_1$ only appear in the combination $q_2 \cdot (p - q_1)$ in the definition of D_3. Therefore, it must be possible to obtain a representation of the integrand in terms of four variables only.

Indeed, if we formally rewrite

$$I^{d\,(2,2)}_{1\,1\,1\,0\,0} = \int \frac{\mathrm{d}^d q_1}{\pi^{d/2}} \frac{1}{D_1} \int \frac{\mathrm{d}^d q_2}{\pi^{d/2}} \frac{1}{D_2 D_3}\,, \tag{2.108}$$

we can recognize the q_2 integral as a one-loop bubble with external momentum $(p - q_1)$. Therefore, we can first introduce the Baikov representation (2.101) of the q_2 integral, in terms of an integral over the denominators z_2 and z_3,

$$I^{d\,(2,2)}_{1\,1\,1\,0\,0} = C^{(1,2)}(d) \int \frac{\mathrm{d}^d q_1}{\pi^{d/2}} \frac{1}{D_1} \left(G(p - q_1)\right)^{(2-d)/2} \int \mathrm{d}z_2 \mathrm{d}z_3 \frac{\left(G(z_2, z_3, z_4)\right)^{(d-3)/2}}{z_2 z_3}\,, \tag{2.109}$$

where $G(p - q_1) = (p - q_1)^2 = z_4$ is the external momentum of the q_2 bubble and $G(z_2, z_3, z_4)$ is the the Gram determinant $G(p - q_1, q_2)$ expressed in terms of the variables z_1 , z_2 and z_4.

The remaining q_1 integral depends on q_1^2 and $q_1 \cdot p$, i.e. on z_1 and z_4. Hence, we can parametrize the integral over q_1 again as a one-loop bubble, with external momentum p, and obtain

$$I^{d\,(2,2)}_{1\,1\,1\,0\,0} = C^{(1,2)}(d) C^{(1,2)}(d) \left(G(p)\right)^{(2-d)/2} \times \int \prod_{i=1}^{4} \mathrm{d}z_i \frac{z_4^{(2-d)/2} \left(G(z_2, z_3, z_4)\right)^{(d-3)/2} \left(G(z_1, z_4)\right)^{(d-3)/2}}{z_1 z_2 z_3}\,, \tag{2.110}$$

with $G(z_1, z_4)$ being $G(q_1, p)$ expressed in terms of z_1 and z_4. □

The above example shows that, according to the symmetry of the integrand, it is possible to obtain a representation of a Feynman integral in terms of a minimum number of variables, by applying the Baikov parametrization (2.101) sequentially to each loop. This is, of course, particularly desirable if we need to perform explicitly all integrations, as we will see in chapter 9, where we will examine in more detail the loop-by-loop parametrization of different two- and three-loop integrals. Finally, we observe that, although it is possible to recover eq. (2.110) from the integration over z_5 of the standard Baikov representation (2.107) (see [60] for a detailed discussion), the loop-by-loop approach allows to minimize the number of integration variables in a much simpler way.

2.5 Feynman integral identities in finite dimensional representations

In the previous two sections, we have derived different (but equivalent) representations of d-dimensional Feynman integrals as multiple integrals over a finite number of variables, which corresponds to the total number of scalar products $n_{\rm SP}$ the integral may depend on. In addition, we have seen how to further reduce the number of integration variables in the presence of additional symmetries of the integrand. It is particularly interesting to study how the identities between Feynman integrals, which we discussed in section 2.1 in the usual momentum-space representation, are translated in these finite dimensional representations of the loop integrals.

First of all, we observe that the integral representations of eqs. (2.54)-(2.98) and (2.101) share a common structure. In fact, if we collectively label with $\mathbf{y}$ the kinematic invariants and $\mathbf{x}$ the integration variables, we can write the general expression of the Feynman integral (2.1) as

$$I^{d\,(\ell,n)}_{a_1\ldots a_m}[\mathcal{N}] = C(d,\mathbf{y}) \int \mathrm{d}\mathbf{x}\, G(\mathbf{x},\mathbf{y})^{(d_\perp-1-\ell)/2} R(\mathbf{x},\mathbf{y})\,, \tag{2.111}$$

where, for instance, $\mathbf{x} = \{x_{\parallel\, i}, \lambda_{ij}\}$ in the $d = d_\parallel + d_\perp$ parametrization (2.54), $\mathbf{x} = \{s_{ij}, \widetilde{s}_{ij}\}$ in the scalar products representation (2.98) and $\mathbf{x} = \mathbf{z}$ in the Baikov representation (2.101).

In eq. (2.111), $P(\mathbf{x},\mathbf{y})$ stands for the polynomial that originates from the expression of $G(\lambda_{ij})$ in the corresponding set of variables and $R(\mathbf{x},\mathbf{y})$ is the integrand

$$R(\mathbf{x},\mathbf{y}) = \frac{\mathcal{N}(\mathbf{x},\mathbf{y})}{D_1^{a_1}(\mathbf{x},\mathbf{y})\cdots D_m^{a_m}(\mathbf{x},\mathbf{y})}\,. \tag{2.112}$$

2.5.1 Integration-by-parts

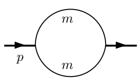
In the finite dimensional representation (2.111), IBPs correspond to exact differential forms, i.e. the most general IBP identity involving $I^{d\,(\ell,n)}_{a_1\ldots a_m}$ can be written as a linear combination of total derivatives w.r.t. *any* of the integration variables $\mathbf{x}$,

$$\int \mathrm{d}\mathbf{x} \sum_{i=1}^{n_{SP}} \frac{\partial}{\partial x_i}\Big(v_i(\mathbf{x},\mathbf{y}) G(\mathbf{x},\mathbf{y})^{(d_\perp-\ell-1)/2} R(\mathbf{x},\mathbf{y})\Big) = 0\,. \tag{2.113}$$

In eq. (2.113) $v_i(\mathbf{x},\mathbf{y})$ are arbitrary polynomials in the loop variables $\mathbf{x}$, which play an analogous role to vectors v^α in the traditional IBPs defined in eq. (2.12). Their choice can be used to select specific classes of IBPs. In order to illustrate this point, let us consider a simple example.

Example 2

We consider a one-loop massive bubble integral with arbitrary external momentum p,

$$I^{d\,(1,2)}_{a_1\,a_2} =$$ 

$$= \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{1}{D_1^{a_1} D_2^{a_2}}\,, \tag{2.114}$$

where the denominators are defined as

$$D_1 = q^2 - m^2\,, \quad D_2 = (q+p)^2 - m^2\,. \tag{2.115}$$

For $p^2 \neq 0$, $d_\parallel = 1$ and, if we decompose the loop momentum as $q^\alpha = q_\parallel^\alpha + \lambda^\alpha$, with $q_\parallel^\alpha = x_1\, p^\alpha$, we can write the $d = d_\parallel + d_\perp$ parametrization (2.54) as

$$I^{d\,(1,2)}_{a_1\,a_2} = \widetilde{\Omega}^{(\ell)}_d \mathcal{J}_{[1]} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \int_0^\infty \mathrm{d}\lambda^2 \frac{(\lambda^2)^{\frac{d-3}{2}}}{D_1^{a_1} D_2^{a_2}}\,, \tag{2.116}$$

with

$$D_1 = x_1^2 p^2 - m^2 + \lambda^2,$$
$$D_2 = (1 + x_1(x_1+2))p^2 - m^2 + \lambda^2. \qquad (2.117)$$

According to eq. (2.113), the most general IBP for $I^{d\,(1,2)}_{a_1 a_2}$ is

$$\begin{aligned} 0 &= \widetilde{\Omega}^{(\ell)}_d \mathcal{J}_{[1]} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \int_0^{\infty} \mathrm{d}\lambda^2 \left[\partial_{x_1} v_{x_1}(x_1,\lambda^2) + \partial_{\lambda^2} v_{\lambda^2}(x_1,\lambda^2)\right] \frac{(\lambda^2)^{\frac{d-3}{2}}}{D_1^{a_1} D_2^{a_2}} \\ &= \widetilde{\Omega}^{(\ell)}_d \mathcal{J}_{[1]} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \int_0^{\infty} \mathrm{d}\lambda^2 \Bigg(\partial_{x_1} v_{x_1} + \partial_{\lambda^2} v_{\lambda^2} - a_1 \frac{v_{x_1}\partial_{x_1} D_1 + v_{\lambda^2}\partial_{\lambda^2} D_1}{D_1} + \\ &\qquad - a_2 \frac{v_{x_1}\partial_{x_1} D_2 + v_{\lambda^2}\partial_{\lambda^2} D_2}{D_2} + \left(\frac{d-3}{2}\right) \frac{v_{\lambda^2}}{\lambda^2} \Bigg) \frac{(\lambda^2)^{\frac{d-3}{2}}}{D_1^{a_1} D_2^{a_2}}\,, \end{aligned} \qquad (2.118)$$

where v_{x_1} and v_{λ^2} are two arbitrary polynomials in x_1 and λ^2. We observe that the last term in eq. (2.118), which is proportional to $1/\lambda^2$, would correspond, according to eq. (2.25), to an integral in $(d-2)$ space-time dimensions. Hence, the above IBP involves, in general, integrals in different dimensions. However, it is clear that by choosing $v_{\lambda^2} \sim \lambda^2$ we are able to select IBPs which involve d-dimensional integrals only. For instance, by picking $v_{x_1} = x_1$ and $v_{\lambda^2} = 2\lambda^2$ and by explicitly computing derivatives, eq. (2.118) we obtain the IBP

$$(d - 2a_1 - a_2) I^{d\,(1,2)}_{1\,1} - a_2 I^{d\,(1,2)}_{0\,2} - 2m^2 a_1 I^{d\,(1,2)}_{2\,1} - (2m^2 - p^2) a_2 I^{d\,(1,2)}_{1\,2} = 0. \qquad (2.119)$$

whose validity can be checked verified from the traditional IBPs (2.12). □

Integration-by-parts in Baikov representation

Eq. (2.111) can be used in order to produce particular classes of IBPs which, for instance, do not enhance the powers of the loop denominators. This possibility is more transparent in the Baikov representation, where the denominators are directly used as integration (and, hence, differentiation) variables. In order to illustrate this point let us specialize eq. (2.111) to the Baikov representation, and absorb any polynomial numerator $\mathcal{N}(z_i)$ into the definition of the IBPs polynomials v_{z_i},

$$C(d)^{(\ell,n)} \left(G(p_i)\right)^{(1-d_\perp)/2} \int \prod_{k=1}^{n_{SP}} \mathrm{d}z_k \left[\sum_{i=1}^{n_{SP}} \partial_i \left(\frac{v_{z_i} \left(G(z_j)\right)^{(d_\perp - 1 - \ell)/2}}{z_1^{a_1} \cdots z_m^{a_m}} \right) = 0\,, \qquad (2.120)$$

where we have indicated $\partial_i \equiv \partial/\partial_{z_i}$. We can now observe that:

- If we restrict, in eq. (2.120), the sum over the partial derivatives w.r.t. the ISPs $z_{m+1}, \ldots z_{n_{\rm SP}}$,

$$C(d)^{(\ell,n)} \left(G(p_i)\right)^{(1-d_\perp)/2} \int \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_{n_{SP}}}{z_1^{a_1} \cdots z_m^{a_m}} \sum_{i=m+1}^{n_{SP}} \partial_i \left(v_{z_i} \left(G(z_j)\right)^{(d_\perp - 1 - \ell)/2} \right) = 0\,, \qquad (2.121)$$

  we can generate IBPs which involve $I^{d\,(\ell,n)}_{a_1 \ldots a_m}$ and its subtopologies but do not produce any term with higher powers of denominators;

- In particular, it is interesting to observe that, since $G(z_j)$ is polynomial, it exists some positive integer k such that $\partial_i^{(k)} G(z_j) = 0$. Thus we can choose $v_i = \partial_i^{(k-1)} G(z_j)$ and $v_{z_j} = 0$, for every $j \neq 0$ and obtain

$$I^{d\,(\ell,n)}_{a_1\cdots a_m}\left[\partial_i^{(k-1)} G(z_j)\,\partial_i G(z_j)\right] = 0\,, \quad i = m+1, \ldots, n_{\text{SP}}\,. \tag{2.122}$$

  In other words, $\mathcal{N}(z_j) = \partial_i^{(k-1)} G(z_j)\,\partial_i G(z_j)$ is a spurious numerator which can produce a non-trivial d-independent identity which does not enhance the powers of the loop denominators. Additional identities, involving d, can be systematically generated by considering mixed derivative of the Baikov polynomial w.r.t. the ISPs;

- A more general analysis, which takes into account the differentiation w.r.t. the loop denominators, has been presented in [58], where it has been shown how to systematically determine the IBP polynomial v_{z_i} in order to select IBPs that do not involve either lower-dimensional integrals or higher powers of denominators. If we expand the derivatives in eq. (2.120),

$$0 = C(d)^{(\ell,n)}\,(G(p_i))^{(1-d_\perp)/2} \int \prod_{k=1}^{n_{SP}} \mathrm{d}z_k \left[\sum_{i=1}^{n_{SP}} \left(\partial_i\, v_{z_i} + \left(\frac{d_\perp - 1 - \ell}{2}\right) \frac{v_{z_i}}{G(z_j)} \partial_i G(z_j)\right) - \sum_{i=1}^{m} a_i \frac{v_{z_i}}{z_i}\right] \frac{(G(z_j))^{(d_\perp - 1 - \ell)/2}}{z_1^{a_1} \cdots z_m^{a_m}}\,, \tag{2.123}$$

  we observe that, as a generalization of the case discussed in the Example 2, the term proportional to $G(z_j)^{-1}$ would correspond to lower-dimensional integrals. Hence, the full d-dimensionality of the IBP can be restored by demanding

$$\sum_{i=1}^{n_{SP}} v_{z_i} \partial_i G(z_j) = v\, G(z_j)\,, \tag{2.124}$$

  where v is some polynomial. In addition, as it is clear from the last term in square brackets, we can avoid the presence of integrals with higher-powers of denominators if we choose $v_{z_i} \sim z_i$ for all z_i which correspond to a denominator,

$$v_{z_i} = \omega_{z_i}\, z_i\,, \quad i = 1, \ldots, m \tag{2.125}$$

  Hence, if we combine eqs. (2.124)-(2.125),

$$\sum_{i=1}^{m} z_i \omega_{z_i} \partial_i G(z_j) + \sum_{i=m+1}^{n_{SP}} v_{z_i} \partial_i G(z_j) - v\, G(z_j) = 0\,, \tag{2.126}$$

  we obtain a single condition which yields to d-dimensional IBPs which leave the powers of the loop denominators unchanged. In eq. (2.126) both $G(z_j)$ and $\partial_i G(z_j)$ are polynomials in the integration variables and the unknowns v, v_{z_i} and ω_{z_i} are polynomials as well. This types of equations are known as *syzygy equations* and they were first introduced in the generation of IBPs without square propagators in [130]. Syzygy solutions can be efficiently determined with several computer algebra systems, such as SINGULAR [131] and MACAULAY2 [132] and an IBPs reduction code, based the construction (2.123), is implemented in the MATHEMATICA package AZURITHE [133].

  A similar analysis on the systematic way to select IBPs by a proper construction of IBP vectors (polynomials), in the framework of numerical unitarity, can be found in [59].

2.5.2 Sector symmetries

In section 2.1.3, we have seen that particular shifts of the loop momenta

$$q_i^\alpha \longrightarrow (\mathbb{A})_{ij} q_j^\alpha + (\mathbb{B})_{ij} p_j^\alpha \,, \tag{2.127}$$

can produce relations between integrals belonging to the same sector. In Baikov representation, the algebraic interpretation of such sector symmetries is particularly clear. In fact, from eq. (2.101) it is evident that a sector symmetry must correspond to a *linear transformation* L of the z_i variables

$$z_i \longrightarrow L(z_i) \tag{2.128}$$

such that

- $L(z_i)$ acts on the denominators as a pure permutation;
- $L(z_i)$ leaves the Baikov polynomial $G(z_i)$ unchanged.

In order to illustrate this two features on a simple example, let us consider again the two-loop massless sunrise discussed in the Example 1.

Example 3

In order to simplify the discussion, we rewrite the Baikov representation of the massless sunrise defined in eq. (2.104),

$$I^{d\,(2,2)}_{1\,1\,1\,0\,0} = C^{(2,2)}(d)\,(G(p))^{(2-d)/2} \int \prod_{i=1}^{5} \mathrm{d}z_i \frac{(G(z_1,z_2,z_3,z_4,z_5))^{(d-4)/2}}{z_1 z_2 z_3}\,, \tag{2.129}$$

by choosing as ISPs the linear forms

$$z_4 = q_1 \cdot p\,, \quad z_5 = q_2 \cdot p\,. \tag{2.130}$$

With this choice, the explicit expression of the Baikov polynomial is

$$\begin{aligned} G(z_1,z_2,z_3,z_4,z_5) =& z_5 z_4 (s + z_1 + z_2 - z_3 - 2z_4 + 2z_5) + \\ & - \frac{1}{4} s (s + z_1 + z_2 - z_3 - 2z_4 + 2z_5)^2 + s z_1 z_2 - z_2 z_4^2 - z_1 z_5^2\,. \end{aligned} \tag{2.131}$$

The scalar sunrise integral $I^{d\,(2,2)}_{1\,1\,1\,0\,0}$ is invariant under a set of 5 different re-parametrizations of the loop momenta

$$\begin{aligned} &\{q_1 \to q_1\,, \quad q_2 \to q_1 - q_2 - p\}\,, \\ &\{q_1 \to q_2 - q_1 + p\,, \quad q_2 \to -q_1\}\,, \\ &\{q_1 \to q_2 - q_1 + p\,, \quad q_2 \to q_2\}\,, \\ &\{q_1 \to -q_2\,, \quad q_2 \to -q_1\}\,, \\ &\{q_1 \to -q_2\,, \quad q_2 \to q_1 - q - 2 - p\}\,, \end{aligned} \tag{2.132}$$

which can be easily generated with automated codes such as Reduze [134]. By starting from the definition of the loop denominators, given in eq. (2.105) and of the ISPs (2.130), each of the above transformation can be mapped into a permutation of the denominators z_1, z_2 and z_3 and a linear transformation of the ISPs z_4 and z_5,

$$\begin{aligned}
&\{z_1, z_2, z_3\} \to \{z_1, z_3, z_2\}, \quad \{z_4, z_5\} \to \{z_4, z_4 - z_5 - p^2\}, \\
&\{z_1, z_2, z_3\} \to \{z_3, z_1, z_1\}, \quad \{z_4, z_5\} \to \{-z4 + z_5 + p^2, -z_4\}, \\
&\{z_1, z_2, z_3\} \to \{z_3, z_2, z_1\}, \quad \{z_4, z_5\} \to \{-z_4 + z_5 + p^2, z_5\}, \\
&\{z_1, z_2, z_3\} \to \{z_2, z_1, z_3\}, \quad \{z_4, z_5\} \to \{-z_5, -z_4\}, \\
&\{z_1, z_2, z_3\} \to \{z_2, z_3, z_1\}, \quad \{z_4, z_5\} \to \{-z_5, z_4 - z_5 - p^2\}.
\end{aligned} \tag{2.133}$$

As expected, it can be checked that eqs. (2.133) correspond to symmetries of the Baikov polynomial (2.131). Hence, by algorithmically applying to any polynomial numerator in the ISPs the transformations (2.133), we can generate d-independent identities internal to a single sector. For the case of the massless sunrise we have, up to rank-3,

$$\begin{aligned}
I^{d\,(2,2)}_{1\,1\,1\,-1\,0} &= \frac{s}{3} I^{d\,(2,2)}_{1\,1\,1\,0\,0}, \\
I^{d\,(2,2)}_{1\,1\,1\,0\,-1} &= -\frac{s}{3} I^{d\,(2,2)}_{1\,1\,1\,0\,0}, \\
I^{d\,(2,2)}_{1\,1\,1\,-2\,0} &= \frac{1}{3}\left(p^4 I^{d\,(2,2)}_{1\,1\,1\,0\,0} + 6 I^{d\,(2,2)}_{1\,1\,1\,-1\,-1}\right), \\
I^{d\,(2,2)}_{1\,1\,1\,0\,-2} &= \frac{1}{3}\left(p^4 I^{d\,(2,2)}_{1\,1\,1\,0\,0} - 6 I^{d\,(2,2)}_{1\,1\,1\,-1\,-1}\right), \\
I^{d\,(2,2)}_{1\,1\,1\,0\,-3} &= -I^{d\,(2,2)}_{1\,1\,1\,-3\,0}, \\
I^{d\,(2,2)}_{1\,1\,1\,-1\,-2} &= \frac{1}{6}\left(p^6 I^{d\,(2,2)}_{1\,1\,1\,0\,0} - 6p^2 I^{d\,(2,2)}_{1\,1\,1\,-1\,-1} + 3 I^{d\,(2,2)}_{1\,1\,1\,-3\,0}\right), \\
I^{d\,(2,2)}_{1\,1\,1\,-2\,-1} &= -\frac{1}{6}\left(p^6 I^{d\,(2,2)}_{1\,1\,1\,0\,0} - 6p^2 I^{d\,(2,2)}_{1\,1\,1\,-1\,-1} + 3 I^{d\,(2,2)}_{1\,1\,1\,-3\,0}\right).
\end{aligned} \tag{2.134}$$

□

2.6 Cut Feynman integrals

In the concluding section of this chapter, we introduce the concept of *generalized cut* of a Feynman integral, which we will ubiquitously encounter in the rest of this thesis.

Roughly speaking, cutting a Feynman integral means to constrain one (or more) virtual particles circulating in the loops to be on the mass-shell by replacing its propagator by a proper δ-function.

The physical meaning of such operation draws its origin from the *unitarity* of the scattering matrix $\mathcal{S}\mathcal{S}^\dagger = \mathbb{1}$, which expresses the conservation of probability in a scattering process of a quantum system.

If we separate the actual interaction by rewriting $\mathcal{S}$ in terms of a transition matrix $\mathcal{T}$,

$$\mathcal{S} = \mathbb{1} + i\,\mathcal{T}, \tag{2.135}$$

we immediately see that the unitarity of $\mathcal{S}$ imposes a relation between the imaginary part of $\mathcal{T}$ and its square modulus,

$$-i(\mathcal{T} - \mathcal{T}^\dagger) = \mathcal{T}\mathcal{T}^\dagger. \tag{2.136}$$

We can relate eq. (2.136) to observable quantities by expressing it in terms of scattering amplitudes, i.e. of the transition probabilities between an initial state p_i and a final one p_f,

$$\mathcal{A}_{i\to f}(s) = i\,\langle p_f|\mathcal{T}|p_i\rangle\,, \tag{2.137}$$

where s collectively labels all the external state variables the amplitude depends on. In fact, if we sandwich both sides of eq. (2.136) between the two states and we insert a completeness relation $\sum|p_x\rangle\langle p_x|= \mathbb{1}$ in the r.h.s., we obtain

$$\mathcal{A}_{p_i\to p_f}(s) - \mathcal{A}_{p_i\to p_f}(s^*) = \sum\!\!\!\!\!\!\int_{p_x} \mathcal{A}_{p_i\to p_x}(s)\mathcal{A}_{p_x\to p_f}(s^*)\,, \tag{2.138}$$

where the integral sum appearing in the r.h.s. stems for the sum over all continuous and discrete degrees of freedom associated to the intermediate states p_x.

Eq. (2.138) establish a relation between the imaginary part of the amplitude $\mathcal{A}_{i\to f}(s)$, i.e. its discontinuity across a branch-cut determined from its physical thresholds, and the sum over all possible states of the product of the partial amplitudes for the production (decay) of the initial (final) states into such intermediate states. If we think of a perturbative expansion of $\mathcal{A}_{i\to f}(s)$, it is clear from simple power counting that the above all-order relation generally connects different orders in perturbation theory, as graphically depicted at one-loop in figure 2.3, for the case of the forward scattering $p_i = p_f$, where eq. (2.138) reduces to the well-known *optical theorem*.

An important observation about eq. (2.138) is that, in order for $\mathcal{A}_{p_i\to p_x}$ to have a physical meaning, all particles of the intermediate state p_x must be *on-shell*. This fact acquires a quantitative meaning when the unitarity equation (2.138) (which holds at the amplitude level) is turned, through the *largest time equation* [135], into a *cutting equation* for the Feynman diagrams that contribute to $\mathcal{A}_{p_i\to p_f}$, which is graphically depicted in figure 2.4.

In the figure, the diagrams lying in the shaded region, i.e. on the right of the dashed lined, must be interpreted as complex conjugated and the (scalar) propagators crossed by the dashed line are *cut* through the Cutkosky rule [10]

$$\frac{1}{D_k} \to 2\pi i\theta(l_k^0)\delta(D_k)\,, \tag{2.139}$$

which imposes the on-shellness of the particle propagating along the k-th line. In eq. (2.139) l_k^0 stems for the time component of the momentum l_k which flows through the propagator D_k (extensions of the Cutkosky rule for particles with spin are possible but they are irrelevant for the present discussion).

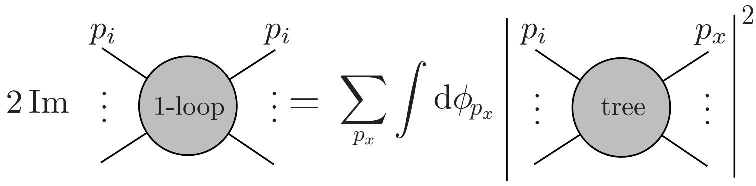


Figure 2.3: Optical theorem.

Given this prescription, we see that the cutting equation states that the real part of a Feynman diagram, which by convention contributes to the imaginary part of the

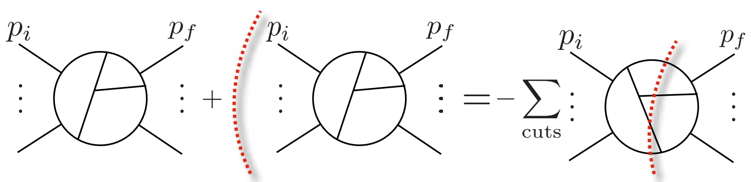


Figure 2.4: Cutting equation for Feynman diagrams. Cut lines represents on-shell particles, and their propagator is replaced by the Cutkosky rule. The diagrams in the shaded region are understood as complex conjugated.

amplitude, can be obtained from the sum over all possible unitarity cuts of the diagram itself. Hence unitarity cuts, i.e. cuts which split Feynman diagrams into two connected pieces, allow a direct determination of the discontinuity of an amplitude across its branch cuts.

This result might suggest that cutting *more* propagators can provide further information about the structure of scattering amplitudes [14, 136, 137]. With this idea in mind, we can extend the Cutkosky rule (2.139) and, given a m-denominator Feynman integral $I^{d\,(\ell,n)}_{1\,\dots\,1}[\mathcal{N}]$ (where we assume all denominators to be raised to power one), we define the *multiple-cut* $D_{k_1} = \dots D_{k_s} = 0$ as the integral

$$\mathrm{Cut}_{k_1\dots k_s}\left[I^{d\,(\ell,n)}_{1\,\dots\,1}[\mathcal{N}]\right] = \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{\mathcal{N}(q_i)}{\prod_{i\neq k_1\,\dots k_s}^{m} D_i} \delta(D_{k_1})\cdots\delta(D_{k_s})\,. \tag{2.140}$$

Although it is intended as a generalization of the standard unitarity cut (2.139), the cutting operation defined by eq. (2.140) is not directly connected to any discontinuity of the Feynman integral, since we gave the θ-function prescriptions which enforce physical requirements on the energy of the cut propagators.

As we will discuss in detail in chapter 9, particular relevance shall be attributed to the *maximal-cut*, i.e. to the simultaneous imposition of the on-shell condition to *all* denominators which characterize the integral,

$$\mathrm{MCut}\left[I^{d\,(\ell,n)}_{1\,\dots\,1}[\mathcal{N}]\right] = \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \mathcal{N}(q_i) \prod_{i=1}^{m} \delta(D_k)\,. \tag{2.141}$$

It some cases, we might have to consider Feynman integrals with higher-powers of denominators and, hence, it is advisable to have an operative definition of the cut of a denominator D_k with $a_k > 1$. One possibility consists in formally modifying the mass the of the k-th propagator, $m_k^2 \to \hat{m}_k^2$ (so that $\hat{m}_k^2$ is different from any other mass the Feynman integral depends on) and observe that, from $\hat{D}_k = l_k^2 + \hat{m}_k^2$, we can immediately write

$$\partial_{\hat{m}_k^2} \hat{I}^{d\,(\ell,n)}_{a_1\,\dots\,a_k\,\dots\,a_m}[\mathcal{N}] = -a_k\, \hat{I}^{d\,(\ell,n)}_{a_1\,\dots\,a_k+1\,\dots\,a_m}[\mathcal{N}]\,, \tag{2.142}$$

where $\hat{I}$ stems for the Feynman integral with modified mass.

If we assume the cutting operation to be insensible to any differentiation, eq. (2.142) allows us to write, recursively, the cut of a Feynman integral with arbitrary power of propagators in terms of derivatives w.r.t. to internal masses of the cut of an analogous integral with denominators raised to power one.

For instance, in the case $a_k = 2$ we can define the cut of the k-th denominator as

$$\mathrm{Cut}_k\left[I^{d\,(\ell,n)}_{1\,\dots\,2\,\dots\,1}[\mathcal{N}]\right] = -\lim_{\hat{m}_k^2\to m_k^2} \partial_{\hat{m}_k^2} \mathrm{Cut}_k\left[\hat{I}^{d\,(\ell,n)}_{1\,\dots\,1\,\dots\,1}[\mathcal{N}]\right]\,. \tag{2.143}$$

At this level, an important observation is in order. As one could verify from direct computation, conversely to the standard unitarity cut obtained through the Cutkosky rules, multiple unitarity cuts do not admit, in general, a solution in terms of real loop momenta, since it is not possible to determine real-valued solutions to the full set of constraints imposed by the on-shell conditions.

In this sense, a more precise definition of generalized unitarity cuts, which naturally assumes the loop momenta as complex variables, consists, rather than substituting a denominator with a δ-function, in deforming the integration contour to a circle enclosing the singularity $D_k = 0$,

$$\int \mathrm{d}^d q \frac{1}{D_k(q)} \to \oint_{D_k=0} \mathrm{d}^d q \frac{1}{D_k} \,. \tag{2.144}$$

Such a definition of generalized cut in terms of a contour integral becomes extremely transparent in the Baikov representation [58–61, 63] introduced in section 2.4, where we define the cut of the k-th propagator, with arbitrary power a_k, as

$$\begin{aligned} \mathrm{Cut}_k \left[I^{d\,(\ell,n)}_{a_1\,\ldots\,a_m}[\mathcal{N}] \right] =& C^{(\ell,n)}(d)\,(G(p_i))^{(1-d_\perp)/2} \int \prod_{i\neq k}^{n\mathrm{sp}} \mathrm{d}z_i \frac{\mathcal{N}(z_i)}{z_1^{a_1}\cdots z_m^{a_m}} \times \\ & \oint_{z_k=0} \mathrm{d}z_k \frac{(G(z_j))^{(d_\perp-1-\ell)/2}}{z_k^{a_k}} \,. \end{aligned} \tag{2.145}$$

In the case of a simple pole in z_k, $a_k = 1$, the above definition corresponds to the naive prescription

$$\frac{1}{z_k} \to 2\pi i\,\delta(z_k) \,, \tag{2.146}$$

while, in the case of higher-order poles, $a_k > 1$, the calculation of the residue of eq. (2.145) in $z_k = 0$ involves, consistently with eq. (2.143), the evaluation derivatives of the Baikov polynomial $G(z_i)$ w.r.t. z_k.

The above definition can be used to compute the generalized unitarity cut of a Feynman integrals corresponding to the vanishing *any* subset of its loop denominators. In particular, in the Baikov representation, the maximal-cut (2.141) corresponds, for arbitrary powers of denominators, to the multiple residue

$$\begin{aligned} \mathrm{MCut} \left[I^{d\,(\ell,n)}_{a_1\,\ldots\,a_m}[\mathcal{N}] \right] =& C^{(\ell,n)}(d)\,(G(p_i))^{(1-d_\perp)/2} \int \prod_{i=m+1}^{n\mathrm{sp}} \mathrm{d}z_i \\ & \prod_{k=1}^{m} \oint_{z_k=0} \mathrm{d}z_k\,(G(z_j))^{(d_\perp-1-\ell)/2} \frac{\mathcal{N}(z_i)}{z_1^{a_1}\cdots z_m^{a_m}} \,. \end{aligned} \tag{2.147}$$

From eq. (2.147) it is easy to see that, if all denominators are raised to power one, the maximal-cut can be almost trivially obtained by evaluating (the integral of) the Baikov polynomomial and the potential numerator $\mathcal{N}$ at the origin $z_1 = \cdots = z_m = 0$ of the hyperplane spanned by the loop denominators

$$\begin{aligned} \mathrm{MCut} \left[I^{d\,(\ell,n)}_{1\,\ldots\,1}[\mathcal{N}] \right] =& (2\pi i)^m C^{(\ell,n)}(d)\,(G(p_i))^{(1-d_\perp)/2} \times \\ & \int \prod_{i=m+1}^{n\mathrm{sp}} \mathrm{d}z_i\,(G(z_j))^{(d_\perp-1-\ell)/2} \mathcal{N}(z_i) \Big|_{z_1=\cdots=z_m=0} \,. \end{aligned} \tag{2.148}$$

In chapter 9 we will examine several explicit examples of maximal-cut of Feynman integrals up to three loops. For the time being, let us conclude with a couple of elementary observations, which will later become of great importance:

- At one-loop level, where the number of integration variables always correspond to the number of denominators, the maximal-cut is, anticipating the discussion of the next chapter, a *maximum-cut*, i.e. it completely localizes the integrand. Conversely, as it is clear from eq. (2.148), at multi-loop level the maximal-cut corresponds to a multiple integral over the ISPs, which are not affected by the cut of the loop denominators, The integration domain of (2.148) corresponds to the patch of the region $\lambda_{ii} \geq 0$, $G(\lambda_{ij}) \geq 0$ identified by the additional conditions $D_1 = \cdots = D_m = 0$. Hence, as for the uncut integral, we can assume the integrand of the maximal-cut to vanish at the boundary of the corresponding integration region. We will will discuss in more detail the features of the integration domain of eq. (2.148) in chapter 9;

- Under the assumption that no boundary term is generated by the cutting operation, generalized unitary cuts are compatible with the IBPs (2.12) satisfied by the uncut integrals. Obviously, when applying the multiple-cut (2.140) to all integrals appearing in an IBP, only the Feynman integrals which contain the whole set of propagators $D_{k_1}, \ldots, D_{k_s}$ will contribute, since the same s-denominator cut cannot be supported by integrals with fewer denominators. In particular, if we apply the maximal-cut corresponding to the highest sector involved in the IBP, we automatically select the *homogeneous* term of the identity.

  In order to clarify this point, let us consider a Feynman integral of the type (2.9) with arbitrary powers of denominators. Through IBPs, it is always possible to express such integral as a linear combination of a certain number N of integrals belonging to the same sector but with all denominators raised to power one, and some subtopology contributions,

$$I^{d\,(\ell,n)}(a_1\,, \ldots\,, a_m; b_1\,, \ldots\,, b_r) = \sum_{i=1}^{N} c_i(d) I^{d\,(\ell,n)}(1\,, \ldots\,, 1; b_1^{(i)}\,, \ldots\,, b_r^{(i)}) + \\ + \text{subtopologies}\,. \tag{2.149}$$

  If we now apply the maximal-cut $D_1 = \cdots = D_m = 0$ to both sides of the equation, all subtopologies drop, since they cannot support the full set of δ-functions, and we are left with the homogenous IBP

$$\text{MCut}\left[I^{d\,(\ell,n)}(a_1\,, \ldots\,, a_m; b_1\,, \ldots\,, b_r)\right] = \\ \sum_{i=1}^{N} c_i(d)\text{MCut}\left[I^{d\,(\ell,n)}(1\,, \ldots\,, 1; b_1^{(i)}\,, \ldots\,, b_r^{(i)})\right]\,. \tag{2.150}$$

  Note that eq. (2.150) provides another operative definition of generalized unitarity cut with higher power of loop denominators, which can be checked to be consistent with both eqs. (2.143) and (2.145).

  The fact that the maximal-cut of a Feynman integral obeys the homogeneous part of the IBPs identities satisfied by the corresponding uncut integral will play a fundamental role in solution of the differential equations fulfilled by the latter, as we will discuss in chapters 5 and 9.

2.7 Conclusions

In this chapter, we have reviewed the most important properties of multi-loop Feynman integrals in dimensional regularization, by focusing on their representation as multiple integrals in a finite number of variables which can be introduced every time the adopted regularization scheme considers purely four-dimensional external particles.

In particular, we have shown how the splitting of the d-dimensional space-time into the longitudinal space spanned by the independent external momenta and its complementary transverse space, $d = d_{||} + d_{\perp}$ [5, 50–54] allows to obtain a representation of Feynman integrals where, after a trivial integration of the transverse directions through an expansion in Gegenbauer polynomials [5], the number of variables parametrizing the integrand matches, at any loop order, the total number of scalar products between loop and external momenta. We have discussed the connection between this representation and the Baikov parametrization [55–57], which introduces loop denominators and irreducible scalar products as integration variables. We analyzed how the linear identities satisfied by Feynman integrals, such as symmetry relations, IBPs and LIs, are rephrased in terms of these finite dimensional representations [58, 60]. Finally, we have given a definition of generalized unitarity cuts of Feynman integrals, which, in the Baikov parametrization, have a clear interpretation as multiple residues at the poles identified by the vanishing of the loop propagators [58–63].

The next chapter will be devoted to the discussion of a general technique, based on the Lorentz invariace and generalized unitarity of scattering amplitudes, to reduce dimensionally regulated amplitudes to a linear combination of Feynman integrals, which goes under the name of *integrand decomposition method.*

Chapter 3

Integrand decomposition of scattering amplitudes

In this chapter we give a brief introduction to the integrand level approach to the decomposition of scattering amplitudes in terms of scalar Feynman integrals. In particular, we focus on the algebraic geometry interpretation of the integrand reduction algorithm, which, by rigorously identifying the partial fractioning of the integrand with a *multivariate polynomial division* modulo *Gröbner basis*, has played a key role in understanding how to extend integrand-level techniques at multi-loop level.

3.1 Introduction

When embarking on higher-order calculations in perturbation theory, the first issue we need to address is the rapid growth of the number of Feynman diagrams with the number of loops and external legs of the amplitude which, in most cases, makes a diagram-by-diagram evaluation prohibitive.

The solution to this problem requires the introduction of a *reduction* procedure, which consists in the initial decomposition of the amplitude in terms of a minimal set of independent functions, i.e. scalar Feynman integrals of the type discussed in chapter 2, and the subsequent evaluation of the latter. In other words, we need a strategy to disentangle the *algebraic* complexity of the problem, dictated by the specific interaction theory under consideration (which determines the number of Feynman diagrams and the tensor properties of their numerators) so to reduce to a minimal set of universal building-blocks the effort needed for the *analytic* computation of loop integrals.

Among the available techniques, in this chapter we discuss the *integrand decomposition algorithm*, which was originally formulated for one-loop amplitudes in four space-time dimensions by Ossola, Papadopoulos and Pittau [19, 20] and subsequently extended to the d-dimensional case [21–25]. The integrand decomposition method combines in a unique framework the basic ideas of the traditional Passarino-Veltman reduction [18], where the amplitude is decomposed in terms of scalar integrals by using Lorentz invariance in order to strip out from the integral sign the tensor structures, and those of generalized unitarity methods [14–17, 138–151], which extracts the rational coefficients of the scalar integrals by sampling the amplitude and multiple unitarity cuts.

The combination of these two driving principles, namely the exploitation of the Lorentz invariance of the amplitude and of its multi-pole factorization properties, can be realized in a very effective way by working at the *integrand* level, where the decomposition algorithm is reduced to a pure *algebraic* manipulation of rational functions, constituted by the integrands associated to the Feynman amplitude.

The main advantage of such an algebraic approach is to level out the decomposition of loop amplitudes, irrespectively of the presence of (internal and external) massive particles, which often constitutes the bottleneck of other reduction methods. Obviously, such simplification does not come without drawbacks, the most relevant being the introduction of *spurious terms*, i.e. rational functions which are present in the decomposition of the integrand but vanish after integration of the loop momenta.

In this chapter, we give a general introduction to the integrand decomposition method in a formulation based on *algebraic geometry* principles [41, 42], which provides a clear picture, on the one hand, of the validity of the method at arbitrary loop order [39, 40, 152–154] and, on the other hand, of the technical issues one needs to circumvent in order to make integrand decomposition a general and systematic framework for multi-loop computations.

The goal of the integrand reduction method consists in recursively decomposing an arbitrary d-dimensional, ℓ-loop amplitude

$$I^{d\,(\ell)}_{i_1\dots i_r} = \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{\mathcal{N}_{i_1\dots i_r}}{D_{i_1}\cdots D_{i_r}}\,, \tag{3.1}$$

in terms of a linear combination of scalar Feynman integrals by exploiting the algebraic properties of associated integrand

$$\mathcal{I}_{i_1\dots i_r}(q_j) \equiv \frac{\mathcal{N}_{i_1\dots i_r}(q_j)}{D_{i_1}(q_j)\cdots D_{i_r}(q_j)}\,. \tag{3.2}$$

The basic observation underlying this method is that, prior to integration, the integrand $\mathcal{I}_{i_1\dots i_r}$ is a purely *rational* function, since both the numerator $\mathcal{N}_{i_1\dots i_r}$ and the loop denominators D_{i_r} are polynomials in the loop momenta q_i. Therefore, one possible way to reduce the full integrated amplitude to a combination of simpler, independent integrals consist in first deriving a decomposition of the integrand (3.2) in terms of simpler rational functions and then reading off the corresponding integral decomposition by integrating back over the loop momenta.

The main advantage of dealing with rational integrands is the possibility of reducing the decomposition of the amplitude to the algebraic partial fractioning of the integrand,

$$\mathcal{I}_{i_1\dots i_r}(q_j) = \sum_{k=0}^{s} \sum_{\{j_1\cdots j_k\}} \frac{\Delta_{j_1\cdots j_k}(q_j)}{D_{j_1}(q_j)\cdots D_{j_k}(q_j)}\,, \quad s \le r\,, \tag{3.3}$$

where the inner sum scans over the possible subsets of denominators D_{i_m} and each $\Delta_{j_1\cdots j_k}$ is, in turn, a polynomial in the loop momenta.

Once such a partial fraction expansion of the integrand has been reached, the corresponding integral decomposition of the Feynman amplitude is obtained by restoring the integration over the loop variables q_j on both sides of eq. (3.3),

$$I^{d\,(\ell)}_{i_1\dots i_r} = \sum_{k=1}^{r} \sum_{\{j_1\cdots j_k\}} \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{\Delta_{j_1\cdots j_k}}{D_{j_1}\cdots D_{j_k}}. \tag{3.4}$$

In this way, we can identify each of the integrals appearing in the r.h.s. of eq. (3.4) with a Feynman amplitude with a reduced number of internal propagators (and of external legs) and, possibly, a simpler numerator. Obviously, whenever an integrand decomposition of the type (3.3) is achieved, the corresponding integral decomposition (3.4) provides a valid representation of the amplitude $I^{d\,(\ell)}_{i_1\ldots i_r}$.

However, since our goal is to reduce the independent integrals to be computed to a minimal number, the effectiveness of the integrand decomposition formula (3.3) will crucially depends on its *minimality* and its *universality*. In other words, we would like each polynomial $\Delta_{j_1\cdots j_k}$ to be *irreducible*, i.e. not to contain any term which can be further rewritten in terms of the denominators $D_{j_1}, \cdots, D_{k_j}$ (and, hence, partial fractioned) and to posses a *universal* polynomial dependence of the loop variables. This latter requirement would make the integrand decomposition method applicable to any amplitude in QFT, regardless of the number of external legs, the presence of multiple, different mass scales and, in general, the complexity of the integrands, which would be confined to the kinematic dependence of the coefficients of $\Delta_{j_1\cdots j_k}$. In order to illustrate how the integrand decomposition formula given in eq. (3.3) can fulfil these minimality and universality requirements, let us first go back to its recursive derivation.

3.2 The integrand recurrence relation

For definiteness, we will hereby make use of the $d = 4-2\epsilon$ representation of Feynman integrals that has been discussed in section 2.2 and parametrize the integrand in terms of the variables $\mathbf{z}$ defined in eq. 2.29, which are in number $\frac{\ell(\ell+9)}{2}$. Therefore we write,

$$\mathcal{N}_{i_1\ldots i_r}(\mathbf{z}), \quad D_{i_s}(\mathbf{z}) \in P[\mathbf{z}] \qquad \mathbf{z} = \{z_1, \ldots, z_{\frac{\ell(\ell+9)}{2}}\}, \tag{3.5}$$

where $P[\mathbf{z}]$ is the ring of all polynomials in the variables $\mathbf{z}$. Moreover, for any set of denominators $D_{i_1}, \ldots, D_{i_s}$ we define the *ideal* $\mathcal{J}_{i_1\cdots i_r}$,

$$\mathcal{J}_{i_1\cdots i_r} \equiv \left\{ \sum_{k=1}^{r} h_k(\mathbf{z}) D_{i_k}(\mathbf{z}) \,:\, h_k(\mathbf{z}) \in P[\mathbf{z}] \right\}, \tag{3.6}$$

as the set of all possible linear combinations of denominators with polynomial coefficients. The partial fractioning of the integrand $\mathcal{I}_{i_1\ldots i_r}$ amounts to the multivariate polynomial division of the numerator

$$\mathcal{N}_{i_1\ldots i_r}(\mathbf{z}) = \mathcal{Q}_{i_1\ldots i_r}(\mathbf{z}) + \Delta_{i_1\ldots i_r}(\mathbf{z}), \tag{3.7}$$

where $\mathcal{Q}_{i_1\ldots i_r}(\mathbf{z})$ is the *quotient* belonging to the ideal $\mathcal{I}_{i_1\ldots i_r}$,

$$\mathcal{Q}_{i_1\ldots i_r}(\mathbf{z}) = \sum_{k=1}^{r} \mathcal{N}_{i_1\cdots i_{k-1}\, i_{k+1}\ldots i_r}(\mathbf{z}) D_{i_k}(\mathbf{z}), \tag{3.8}$$

and $\Delta_{i_1\ldots i_r}(\mathbf{z})$ is the *remainder* of the division, i.e. a polynomial which does not contain any term belonging to the ideal. If we make use eqs. (3.7) and (3.8) to reconstruct the original integrand $\mathcal{I}_{i_1\ldots i_r}$,

$$\begin{aligned}
\mathcal{I}_{i_1\ldots i_r}(\mathbf{z}) &= \sum_{k=1}^{r} \frac{\mathcal{N}_{i_1\cdots i_{k-1}\, i_{k+1}\ldots i_r}(\mathbf{z}) D_{i_k}(\mathbf{z})}{D_{i_1}\cdots D_{i_r}} + \frac{\Delta_{i_1\ldots i_r}(\mathbf{z})}{D_{i_1}\cdots D_{i_r}} \\
&= \sum_{k=1}^{r} \mathcal{I}_{i_1\cdots i_{k-1}\, i_{k+1}\ldots i_r}(\mathbf{z}) + \frac{\Delta_{i_1\ldots i_r}(\mathbf{z})}{D_{i_1}\cdots D_{i_r}},
\end{aligned} \tag{3.9}$$

we can immediately identify the first term of the r.h.s. as a sum of new integrands, each one corresponding to a sub-diagram where the loop propagator D_{i_k} has been removed. This procedure can be iteratively applied to the newly generated integrands, until the final decomposition formula (3.3) is reached.

3.3 Polynomial division and Gröbner bases

Despite the simple logic beneath the recursive algorithm outlined in the previous section, we shall give due consideration to some mathematical subtleties at its basis. First of all, is a well-known fact that, when dealing with the division of the multivariate polynomial $p(\mathbf{z}) \in P[\mathbf{z}]$ by another set of polynomials $p_1(\mathbf{z}), \ldots, p_k(\mathbf{z}) \in P[\mathbf{z}]$, the result of the division, i.e. the identification of the quotients and of the remainder is not unique and depends on two (in principle arbitrary) prescriptions we need to adopt when defining a division algorithm:

1. The *monomial order*, i.e a total order of the monomials in $P[\mathbf{z}]$ that determines which monomials of $p(\mathbf{z})$ must be divided first. In order to motivate the need for a monomial ordering with an example, let us first consider the univariate division of $p(x) = x^2 - 2x$ by $p_1(x) = x + 1$. In the univariate case, we can easily identify the leading term (LT) of both $p(x)$ and $p_1(x)$ as their highest rank monomial, $\mathrm{LT}(p) = x^2$ and $\mathrm{LT}(p_1) = x$. The division is performed by dividing $\mathrm{LT}(p)$ by $\mathrm{LT}(p_1)$ and then by iterating the division on the remainder, until no more divisions are possible:

$$\begin{aligned} \mathrm{LT}(p)/\mathrm{LT}(p_1) &= x & \Rightarrow \qquad & p' = p - x\,p_1 = -3x\,, \\ \mathrm{LT}(p')/\mathrm{LT}(p_1) &= -3 & \Rightarrow \qquad & p'' = p' + 3\,p_1 = 3\,, \end{aligned} \tag{3.10}$$

   and therefore,

$$p = \big(\mathrm{LT}(p)/\mathrm{LT}(p_1) + \mathrm{LT}(p')/\mathrm{LT}(p_1)\big)\,p_1 + p'' = (x-3)p_1 + 3\,. \tag{3.11}$$

   It is clear that, in the case of a single variable the algorithm terminates after a finite number of steps and that the reminder, whose rank is strictly lower than the one of p_1, is unique.

   If we introduce one more variable into the game and we try to divide $p(x) = x^2 - 2y^2$ by $p_1(x) = x + 1$, we immediately see that there is no more obvious definition of $\mathrm{LT}(p)$. In fact, if we assume x^2 to be the leading term we obtain

$$p = (x-1)\,p_1 + 0\,p_2 + (1 - 2y^2)\,, \tag{3.12}$$

   while, if we take $\mathrm{LT}(p) = -2y^2$, we get

$$p = 0\,p_1 + 0\,p_2 + (x^2 - 2y^2)\,. \tag{3.13}$$

   Therefore, we need to introduce a monomial order which can be used to unambiguously compare any pair of monomials. The simplest of such orderings is the so-called *lexicographic* order which, given one ordering between the variables, sorts the monomials by subsequently comparing their rank in each variable. For instance, with lexicographic order $y \prec x$, we have $xy^2 < x^2y$. Obviously, many different monomial orders are admissible, provided that they satisfy the *well-ordering* condition $1 < m$ for any monomial m, in such a way to ensure the converge of the division algorithm. In principle, all these orders are equivalent. However, in practical cases, a proper choice of the monomial order can make the computation significantly more efficient.

2. The *divisor ordering*, i.e. the order used for the division by each of the polynomials $p_1(\mathbf{z}), \ldots, p_k(\mathbf{z})$. Different divisor orderings lead, in general, to different remainders, as we can illustrate again on a simple example.

Consider the two-variable polynomials $p(x,y) = x^2y^3 - 2xy^2$, $p_1(x,y) = x^2y - 2x$ and $p_2(x,y) = y^3 + 4$. Clearly, p belongs to the ideal generated by p_1 and p_2, since, by factoring y^2 we can immediately write

$$p = y^2(x^2y - 2x) = y^2\,p_1\,. \tag{3.14}$$

Indeed the same result can be obtained by choosing the lexicographic order $y \prec x$ and by dividing p first by p_1 and subsequently by p_2. However, if we maintai the same monomial order but we exchange the order of the divisions by p_1 and p_2, we arrive to radically different result,

$$p = x^2\,p_2 - (2xy^2 + 4)\,. \tag{3.15}$$

This simple calculation shows that not only the polynomial division produces different remainders according to the divisor order but that the division for an arbitrary set $p_1(\mathbf{z}),\ldots,p_k(\mathbf{z})$ can even return a non-zero reminder for a polynomial p which belongs to the corresponding ideal.

Going back to the integrand decomposition of Feynman amplitudes, the observation 2 seems to seriously compromise the applicability of the integrand decomposition algorithm, since it implies that, if we perform the polynomial division using as a naive basis of $\mathcal{I}_{i_1\ldots i_r}$ the set of denominators D_{i_k}, we would obtain a far non-unique parametrization of the numerators $\Delta_{j_1\cdots j_k}$. In the ultimate case, we could even fail to identify an integrand which is completely reducible to a sum of sub-amplitudes with fewer loop propagators. Although it turned out not to be the case for one-loop amplitudes, this issue prevented for some time the effective extension of the integrand decomposition algorithm at multi-loop level.

The solution to the problem resorted to the introduction of the *division modulo Gröbner basis* [41, 42].

Definition 1. *Given an ideal $\mathcal{J}$ on a polynomial ring $P[\mathbf{z}]$, a Gröbner basis is a generating set $\mathcal{G}(\mathbf{z}) = g_1(\mathbf{z}),\ldots,g_1(\mathbf{z})$ of $\mathcal{J}$ such that, given a monomial order $\prec$, the multivariate polynomial division of any $p(\mathbf{z}) \in P(\mathbf{z})$ has a unique remainder.*

In other words, a Gröbner basis is a "good" basis of the ideal, in the sense that it allows to define an unambiguous division algorithm and it makes the remainder of the division a univocal identifier of the membership of an arbitrary polynomial to an ideal. Thus, the existence of a Gröbner basis of the ideal generated by the loop propagators $\mathcal{I}_{i_1\ldots i_r}$ allow us to define a consistent decomposition algorithm in the following way:

- Given the numerator $\mathcal{N}_{i_1\ldots i_r}$, and a Gröbner basis $\mathcal{G}_{i_1\cdots i_r}$ of $\mathcal{J}_{i_1\ldots i_r}$ we perform the multivariate polynomial division of $\mathcal{N}_{i_1\ldots i_r}$ modulo $\mathcal{G}_{i_1\cdots i_r}$ and obtain

$$\mathcal{N}_{i_1\ldots i_r}(\mathbf{z}) = \mathcal{Q}_{i_1\ldots i_r}(\mathbf{z}) + \Delta_{i_1\ldots i_r}(\mathbf{z})\,, \tag{3.16}$$

  where, the remainder $\Delta_{i_1\ldots i_r}(\mathbf{z})$ is now uniquely determined and the quotient is written as a combination of the elements of $\mathcal{G}_{i_1\cdots i_r}$,

$$\mathcal{Q}_{i_1\ldots i_r}(\mathbf{z}) = \sum_{k=1}^{m} \Gamma_k(\mathbf{z})g_k(\mathbf{z})\,. \tag{3.17}$$

- Since $\mathcal{G}_{i_1\cdots i_r}$ belongs to $\mathcal{J}_{i_1\ldots i_r}$, it is always possibile to write its element in terms of the loop denominators,

$$g_k(\mathbf{z}) = \sum_{s=1}^{r} h_{s_j}(\mathbf{z})D_{i_k}(\mathbf{z}) \quad \forall k\,, \tag{3.18}$$

so that, by applying this relation to eq. (3.17), we can recover the recurrence relation (3.9) and identify the sub-amplitudes numerators $\mathcal{N}_{i_1\cdots i_{k-1}\cdots i_{k+1}\cdots i_r}$.

The uniqueness of the remainder obtained through this procedure ensure that all the remainders that appear in the integrand decomposition formula(3.3) are *irreducile*, in the sense that they cannot be further decomposed in terms of denominators. Therefore, we can univocally associate each $\Delta_{j_1\cdots j_k}$ to the corrisponding set of loop propagators $D_{j_1}, \cdots, D_{j_k}$ and, for this reason, $\Delta_{j_1\cdots j_k}$ is usually referred to as the *residue* of the multiple-cut $D_{j_1} = \cdots = D_{j_k} = 0$. One last comment on the definition 1 is in order. Although this is a formal definition and it doesn't tell us how to practically calculate $\mathcal{G}$, constructive algorithms for the computation Gröbner bases are available and they have been implemented in several computer programs, such as Singular [131] and Macaulay2 [132], which, besides computing the Gröbner basis of a given ideal, can also provide the relation between its elements and the original generating set of polynomials (3.18).

A fundamental issue related to the integrand decomposition formula (3.3) is the related to the *reducibility* of loop integrands. If we think back to the familiar example of the tensor integral decomposition in $d = 4$, we know that any one-loop amplitude can be decomposed in terms of integrals containing at most four loop propagators. Therefore, in the context of the integrand decomposition, it is natural to wonder whether there exists some condition which ensures that the integrand $\mathcal{I}_{i_1\ldots i_r}$ is *reducibile*, i.e. it has vanishing residue $\Delta_{j_1\cdots j_k}$. The property of ideals over polynomial rings allow to prove [42] the following

Theorem 1. ***Reducibilty criterion.*** *If a multiple-cut $D_{i_1}(\mathbf{z}) = \cdots = D_{i_k}(\mathbf{z}) = 0$ has no solution, any integrand $\mathcal{I}_{i_1\ldots i_r}$ associated to it is reducible.*

It is obvious that any integrand $\mathcal{I}_{i_1\ldots i_r}$ is reducible whenever the ideal $\mathcal{J}_{i_1\ldots i_r}$ generated by the set of loop propagators coincides with the entire polynomial ring $P[\mathbf{z}]$ since, in this case, $\mathcal{I}_{i_1\ldots i_r} \in P[\mathbf{z}]$ independently of its specific expression. The proof of the reducibility criterion, which follows from the so-called *weak nullstellensatz* [155], states that $P[\mathbf{z}] \equiv \mathcal{J}_{i_1\ldots i_r}$ if and only if the set of on-shell conditions $D_{i_1}(\mathbf{z}) = \cdots = D_{i_k}(\mathbf{z}) = 0$ admits no simultaneous solution. This result implies that all the residues that contribute to the in r.h.s. of the decomposition formula (3.3) are associated to systems of polynomial equations in the loop variables which must be at least fully determined. This observation motivates the

Definition 2. *A multiple-cut $D_{i_1}(\mathbf{z}) = \cdots = D_{i_k}(\mathbf{z}) = 0$ is said to be a maximum-cut if the system of on-shell conditions constrains all loop components $\mathbf{z}$ to a finite number n_s of solutions.*

The simplest case of maximum-cut is the four-dimensional quadruple-cut at one-loop. Since one-loop integrands in four dimensions are parametrized in terms of four variables, which correspond to the components of the loop momentum, the quadruple-cut constrains all integration variables and, as it is well known [15], admits $n_s = 2$ two distinct solutions. By contrast, in the case of dimensionally regulated one-loop amplitudes, whose integrands are parametrized in terms of five variables, the maximum-cut corresponds to a pentuple-cut $D_1 = \cdots = D_5 = 0$ which, as it can be verified, admits one single solution, $n_s = 1$. Therefore, at one-loop, the integrand decomposition of any amplitude in d dimensions will receive contributions ranging from five-point integrands down to tadpoles.

It is easy to imagine how this picture extends at higher loops: as a consequence of the reducibility criterion, the integrand decomposition formula will contain contributions from the maximum-cuts (which, at multi-loop level, can correspond to several, distinct integrand topologies) as well as from the multiple-cuts associated to all possible subsets of loop denominators.

A further remarkable result which can be derived from the properties of the ideal $\mathcal{J}_{i_1\dots i_r}$ is the following

Theorem 2. *Maximum-cut.* *The residue of a maximum-cut which admits n_s distinct solution is a polynomial parametrized by n_s coefficients, which admits an univariate representation of degree $(n_s - 1)$.*

This result, which was proven in [42] under the well-motivated assumption that each of the n_s solutions of the maximum-cut has multiplicity one, states that the parametric form of the residues associated to the maximum-cuts can be predicted just by looking at the number of independent solutions of the on-shell conditions. For instance, in four dimensions the residue of the one-loop maximum-cut is parametrized by two coefficients, since the four-dimensional quadruple-cut has $n_s = 2$ and the situation becomes even simpler when moving to d dimensions, where the pentuple-cut admits one single solution and, therefore, the maximum-cut residue will depend on a single coefficient. As we will see in the next chapter, this feature of dimensional regularization remains true at any loop order: the maximal-cuts in d dimensions always corresponds to a fully-determined system of linear equations with $n_s = 1$ solutions. Therefore, the residue of a maximum-cut is always parametrized in terms of one single coefficient. In this respect, the integrand decomposition in d dimensions seems to possess a simpler structure than the one in a finite number of dimensions.

One last comment on the integrand decomposition formula eq. (3.9) is in order. So far we have only considered, for simplicity, integrands where all the loop denominators are raised to power one. However, in multi-loop computations, it is not uncommon to encounter integrands with higher powers of denominators. In can be proven [154] that integrand recurrence relation given in eq. (3.9) is valid, in total generality, for integrands of the type

$$\mathcal{I}_{i_1\dots i_r}(q_j) \equiv \frac{\mathcal{N}_{i_1\dots i_r}(q_j)}{D_{i_1}^{a_1}(q_j)\cdots D_{i_r}^{a_r}(q_j)}\,, \qquad a_i \in \mathbb{N}\,. \tag{3.19}$$

In the case of higher powers of denominators, the polynomial division must be iterated $a_1 \times \dots \times a_r$ times for each ideal $\mathcal{J}_{i_1,\dots i_r}$ in order to determine the residues of all multiple-cuts $D_{i_1} = \cdots = D_{i_r} = 0$ with denominators powers ranging from $\{a_1\,, \dots\,, a_r\}$ down to $\{1\,, \dots\,, 1\}$.

3.4 Example: one-loop integrand decomposition

The most general one-loop amplitude with n external legs,

$$I^{d\,(1)}_{i_1\cdots i_n}[\mathcal{N}_{i_1\cdots i_n}] = \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\mathcal{N}_{i_1\cdots i_n}(q)}{D_{i_1}\dots D_{i_n}}\,, \tag{3.20}$$

is uniquely characterized by the set of denominators $D_{i_1}\,, \dots\,, D_{i_n}$ and by a numerator function $\mathcal{N}_{i_1\cdots i_n}$. The integrand is parametrized in terms of five variables which, in the standard $d = 4 - 2\epsilon$ representation (see section 2.2), are identified with

$$\mathbf{z} = \{z_1, z_2, z_3, z_4, z_5\} \equiv \{x_1, x_2, x_3, x_4, \mu^2\}, \tag{3.21}$$

where x_i are the components of the loop momentum with respect to an arbitrary four-dimensional basis E [19, 27, 28, 156] and μ^2 is the square norm of the loop component along the -2ϵ regulating dimensions. Both the denominators and the numerator are polynomials in $\mathbf{z}$: each of the $D_{i_k}(\mathbf{z})$ is a quadratic polynomial of the form (2.21) while, depending on the specific scattering process, $\mathcal{N}_{i_1\cdots i_n}$ is a general polynomial of the type

$$\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \sum_{\vec{\jmath}\in J} \alpha_{\vec{\jmath}}\, z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} z_5^{j_5}\,, \quad j_k \in \mathbb{N}\,, \tag{3.22}$$

where $\alpha_{\vec{\jmath}}$ are coefficients depending on the external kinematics and polarizations, on the mass of the particles circulating into the loop and on the space-time dimensions d. The sum over the integer 5-tuples $\vec{\jmath} = (j_1, j_2, j_3, j_4, j_5)$ runs over a subset J determined by the renormalization properties of the theory under consideration, which sets limits on the rank of $\mathcal{N}_{i_1\cdots i_n}$. In the following, we will restrict our analysis to the case of renormalizable theories,

$$J \equiv J_5(n) = \{\vec{\jmath} : j_1 + j_2 + j_3 + j_4 + 2j_5 \le n\}\,, \tag{3.23}$$

but higher rank numerators, such as the one appearing in effective theories, can be treated in a similar way, along the lines of [154]. By working on general numerators (3.22) with arbitrary coefficients $\alpha_{\vec{\jmath}}$, we can perform the division modulo Gröbner basis (for which computation we adopt lexicographic ordering $z_1 \prec \ldots \prec z_5$ and we make use the MATHEMATICA built-in Gröbner bases generator) and obtain the well-known parametrization of the one-loop residues [19, 21, 42],

$$\begin{aligned}
\Delta_{ijklm} =& c_0\mu^2,\\
\Delta_{ijkl} =& c_0 + c_1x_4 + c_2\mu^2 + c_3x_4\mu^2 + c_4\mu^4,\\
\Delta_{ijk} =& c_0 + c_1x_4 + c_2x_4^2 + c_3x_4^3 + c_4x_3 + c_5x_3^2 + c_6x_3^3 + c_7\mu^2 + c_8x_4\mu^2\\
&+ c_9x_3\mu^2,\\
\Delta_{ij} =& c_0 + c_1x_1 + c_2x_1^2 + c_3x_4 + c_4x_4^2 + c_5x_3 + c_6x_3^2 + c_7x_1x_4 + c_8x_1x_3\\
&+ c_9\mu^2,\\
\Delta_i =& c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4\,,
\end{aligned} \tag{3.24}$$

where the coeffcients c_k are, in general, rational functions of the kinematics quantities and d. Note that, for sake of simplicity, we have omitted for each coefficient the label $i \ldots j$ of the corresponding multiple-cut. The parametrization (3.24) is universal in the sense that it gives the most general expression of the residues we can obtain in the computation any one-loop amplitude within the Standard Model or any other renormalizable theory. The actual expression of the c_k and, in case, their vanishing depend on the specific process under consideration.

A few comments on eq. (3.24) are in order:

- There is no residue corresponding to integrands with more than six external particle since, in agreement with the *reducibility criterion* (1), they can be fully decomposed in terms lower-point integrands,

$$\frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1}\ldots D_{i_n}} = \sum_{k=0}^{5}\sum_{\{j_1\cdots j_k\}} \frac{\Delta_{j_1\cdots j_k}}{D_{j_1}\cdots D_{j_k}} \quad \forall\, n > 5\,. \tag{3.25}$$

- The residue of the pentuple-cut has a very simple structure which, as we have seen in the previous section, is dictated by the *maximum cut theorem* (2). We observe that the application of the multivariate polynomial division *tout court* leads a constant residue $\Delta_{ijklm} = c_0'$. However, we have used the on-shell conditions of the pentuple-cut to trade the constant residue with a monomial depending on μ^2. The reason for this conventional choice is that, conversely to the case of the scalar five-point integral, in the $d \to 4$ limit the rank-two integral vanishes,

$$\int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\mu^2}{D_{i_1} D_{i_2} D_{i_3} D_{i_4} D_{i_5}} = \mathcal{O}(\epsilon). \tag{3.26}$$

  In this way, the new class of loop integrals introduced by the dimensional regularization of divergencies will not contribute in the four-dimensional limit. (this choice was also shown to improve the stability of numerical implementations).

- By looking at the parametrization of the quadruple-, triple-, double- and single-cuts, we immediately see that the integrand decomposition (3.25) does not match the result of ordinary tensor integral decomposition, which would reduce the amplitude to a combination of rank-zero integrals only, one for every multiple-cut. The reason for this discrepancy is that the residues contain *spurious terms*, i.e. monomials that vanish upon integration,

$$\Delta_{i_1 \cdots i_n}(q) = \Delta^{\text{n-sp}}_{i_1 \cdots i_n}(q) + \Delta^{\text{sp}}_{i_1 \cdots i_n}(q)\,, \tag{3.27}$$

  where

$$\int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\Delta^{\text{sp}}_{i_1 \cdots i_n}(q)}{D_{i_1} \dots D_{i_n}} = 0\,. \tag{3.28}$$

  Eq. (3.28) implies that although the spurious terms $\Delta^{\text{sp}}_{i_1 \cdots i_n}$ are generated during the recursion of the the decomposition algorithm and they are required in order to make the equality (3.25) satisfied at the integrand-level, they will not contribute to the integrated amplitude. It can be easily verified by Lorentz-invariance arguments that all the monomials that depends on the four dimensional loop components x_i are spurious, with the only exception of the rank-one and -two terms of the double-cuts associated to a massless external momentum $p^2 = 0$, Therefore, by inserting eq. (3.24) into the decomposition (3.25) and by integrating over the loop momentum, we obtain an integral decomposition of the one-loop amplitude (3.20),

$$\begin{aligned}
I^{d\,(1)}_{i_1 \cdots i_n}[\mathcal{N}_{i_1 \cdots i_n}] = & \sum_{i \ll m}^{n} c_0^{(ijklm)} I^{d\,(1)}_{ijklm}[\mu^2] \\
& + \sum_{i \ll l}^{n} \Big[c_0^{(ijkl)} I^{d\,(1)}_{ijkl}[1] + c_2^{(ijkl)} I^{d\,(1)}_{ijkl}[\mu^2] \\
& + c_4^{(ijkl)} I^{d\,(1)}_{ijkl}[\mu^4] \Big] + \sum_{i \ll k}^{n} \Big[c_0^{(ijk)} I^{d\,(1)}_{ijk}[1] + c_7^{(ijk)} I^{d\,(1)}_{ijk}[\mu^2] \Big] \\
& + \sum_{i \ll j}^{n} \Big[\delta_{0,p_i^2} \Big(c_1^{(ij)} I^{d\,(1)}_{ij}[(q+p_i)\cdot e_2] + c_2^{(ij)} I^{d\,(1)}_{ij}[((q+p_i)\cdot e_2)^2] \Big) \\
& + c_0^{(ij)} I^{d\,(1)}_{ij}[1] + c_9^{(ij)} I_{ij}[\mu^2] \Big] + \sum_{i}^{n} c_0^{(i)} I^{d\,(1)}_{i}[1]\,.
\end{aligned} \tag{3.29}$$

- After removing the spurious terms, the only higher-rank numerators contributing to the integrated amplitude depend on powers of μ^2 and they can be identified with higher-dimensional integrals (see eq. (2.25)). Integral-level dimensional recurrence relations can then be used in order to reduce each of the μ^2 integrals to the corresponding rank-zero integral. Therefore, the decomposition of the one-loop amplitude in terms of a minimal number of scalar integrals reads

$$
\begin{aligned}
I^{d\,(1)}_{i_1\cdots i_n}[\mathcal{N}_{i_1\cdots i_n}] = & \sum_{i\ll m}^{n} c_0^{(ijklm)} I^{d\,(1)}_{ijklm}[\mu^2] + \\
& \sum_{i\ll l}^{n} c_0^{(ijkl)} I^{d\,(1)}_{ijkl}[1] + \sum_{i\ll k}^{n} c_0^{(ijk)} I^{d\,(1)}_{ijk}[1] \\
& + \sum_{i\ll j}^{n} \Bigg[\delta_{0,p_i^2} \left(c_1^{(ij)} I^{d\,(1)}_{ij}[(q+p_i)\cdot e_2] + c_2^{(ij)} I^{d\,(1)}_{ij}[((q+p_i)\cdot e_2)^2] \right) \\
& + c_2^{(ij)} I^{d\,(1)}_{ij}[((q+p_i)\cdot e_2)^2] \Bigg] + \sum_{i}^{n} c_0^{(i)} I^{d\,(1)}_{i}[1] \,,
\end{aligned} \tag{3.30}
$$

  where the integral coefficients c_j, which are of course different from the one appearing in eq.(3.29), have acquired an additional dependence on the dimensions d, which is inherited from the the dimensional recurrence relations.

3.5 Conclusions

In this chapter we have reviewed the main features of the integrand decomposition method, which allows to decompose multi-loop scattering amplitudes as a combination of scalar Feynman integrals by determining their residues at the singular points identified by a set of multiple-cut conditions $D_{i_1} = \cdots = D_{i_r} = 0$. Since its very first four-dimensional formulation [19, 20] and the subsequent extension to arbitrary dimensions d [21–25], the integrand decomposition algorithm for one-loop amplitudes has been implemented in several public libraries like CUTOOLS [26], SAMURAI [27] and NINJA[28, 157], which played an important role in the developments automatic codes for the numerical evaluation of scattering amplitudes for generic processes at NLO accuracy, as reviewed in [158].

The validity of the algorithm at higher-loop orders [39, 40] has been proven by resorting to the introduction of algebraic geometry methods [41, 42]. In particular, the study of the properties of polynomial ideals and of the Gröbner bases associated to the multiple-cuts of the amplitude allowed to systematize the determination of the residues at any loop order. Implementation of such methods to the determination of parametric integrand basis is provided, for instance, by the public package BASISDET [41]. In addition, the integrand decomposition technique has been successfully applied for the first time to non-trivial two-loop five-point helicity amplitude in [43, 44]. However, despite the tremendous progress in the theoretical understanding of the underpinning algebraic structure (see [159] for a recent review) and the tailored application to specific massless amplitudes, at the present time integrand decomposition has fallen short to become a fully competitive method for the systematic computation of scattering amplitude beyond one-loop, due to a series of technical difficulties which can be summarized as follows:

1. The major source of complexity in the multi-loop extension of the integrand decomposition is the presence of the so-called *irreducible scalar products* (ISPs),

i.e. scalar products involving the loop momenta which cannot be expressed in terms of denominators [39]. The presence of ISPs can introduce both spurious and non-spurious contributions in the residues. The systematic classification of the spurious numerators at multi-loop level is less obvious and their proliferation increments the number irreducible monomials which, although they do not contribute to the final integral decomposition, must be handled through the intermediate steps of the integrand reduction;

2. Conversely to the one-loop case, at multi-loop level the structure of the residues and the algebraic complexity of the multivariate polynomial divisions can be heavily affected by the choice of the variables $\mathbf{z}$ and of the monomial order adopted in the definition of the Gröbner bases;

3. Although in the multi-loop case the number of independent integrals that appear in the amplitude decomposition is not known *a priori*, experience shows that the number of irreducible monomials produced by the integrand integrand decomposition is usually larger than the actual number of master integrals. The minimal basis of independent integrals is, in fact, determined by the existence of integral-level identities, such as IBPs, for which no integrand-level interpretation is available. In this respect, a wise choice of the monomial ordering for the variables $\mathbf{z}$, besides making the computation technically simpler, should also provide a parametrization of the residues suitable for a subsequent integral-level reduction.

These issues can be efficiently addressed with the introduction of an alternative formulation of the integrand decomposition algorithm based on $d = d_{\parallel} + d_{\perp}$ parametrization of Feynman integrals introduced in section (2.3). The latter can be used to simplify the division algorithm by optimizing the choice of loop variables according to kinematic configuration of each integrand. Such formulation, which goes under the name of *adpative* integrand decomposition, will be the main focus of the next chapter.

Chapter 4

Adaptive integrand decomposition

In this chapter we present the *adpative* integrand decomposition of multi-loop Feynman amplitudes and we introduce the *divide-integrate-divide algorithm*, which allows to decompose any amplitude, at the integrand-level, in terms of a reduced set of scalar integrals whose numerator correspond to irreducible scalar products between loop momenta and external momenta. The algorithm applies to scattering amplitudes in any QFT and it can be extended to any order in perturbation theory. As an example, we revisit the one-loop decomposition and we discuss the integrand reduction of two-loop planar and non-planar amplitudes with arbitrary external and internal kinematics, by providing a full classification of the irreducible numerators with up eight external particles. Finally, we discuss the automation of the algorithm through the code Aida, which has been implemented in Mathematica. The content of this chapter is the result of an original research done in collaboration with P. Mastrolia, T. Peraro and W.J. Torres Bobadilla and it is based on the pubblications [5, 9].

4.1 Simplifying the polynomial division

In the previous chapter, we have shown how the algebraic geometry formulation of the integrand decomposition, which relies on the concepts of *ideals*, *Gröbner bases* and *multivariate polynomial division*, had a fundamental role in demonstrating the applicability of the integrand reduction techniques to any loop order.

However, despite being conceptually solved, the issue of practically extending, in a general way, the integrand decomposition method beyond one-loop is still open. In fact, at multi-loop level, the effectiveness of the algorithm is jeopardized by the algebraic complexity of the intermediate expressions generated at each division step. This complexity is originated, on the one hand, by the the proliferation of spurious terms, whose identification might be not straightforward, and, on the other hand, by the presence of ISPs which enlarge the number of monomials appearing in the integrand basis.

In this chapter, we discuss a simplified version of the integrand decomposition algorithm, we refer to as *adaptive integrand decomposition*, which can help in keeping such complexity under control and in making integrand reduction a flexible and automatable tool also at multi-loop level.

The key idea of this simplified approach to integrand decomposition is to *adapt* the choice of the loop variables according to the kinematic of the corresponding multiple-cut at each step of the division algorithm. This amounts to introducing, for each ℓ-loop integrand with $n \leq 4$ external legs, the $d = d_\parallel + d_\perp$ parametrization discussed in sec 2.3, i.e. to defining the $\ell(\ell+9)/2$ variables $\mathbf{z}$ that characterize the integrand as

$$\mathbf{z} = \{\mathbf{x}_{\parallel\, i}, \mathbf{x}_{\perp\, i}, \lambda_{ij}\}, \quad i,j = 1, \ldots \ell, \tag{4.1}$$

where we have split, according to eq. (2.34), the four-dimensional components of the i-th loop momentum $\mathbf{x}_i$ into the subset belonging to the longitudinal space,

$$\mathbf{x}_{\parallel\, i} = \{x_{ji}\}, \quad j \leq d_\parallel, \tag{4.2}$$

and the one lying in the space orthogonal to the external momenta,

$$\mathbf{x}_{\perp\, i} = \{x_{ji}\}, \quad d_\parallel < j \leq 4\,. \tag{4.3}$$

As observed in section 2.3, the loop denominators are independent of $\mathbf{x}_{\perp\, i}$, i.e. they are polynomials in the subset of variables $\boldsymbol{\tau} \subset \mathbf{z}$, defined by

$$\boldsymbol{\tau} = \{\mathbf{x}_\parallel, \lambda_{ij}\}\,. \tag{4.4}$$

In this way, we can rewrite the general loop integrand (3.2) as

$$\mathcal{I}_{i_1 \ldots i_r}(\boldsymbol{\tau}, \mathbf{x}_\perp) \equiv \frac{\mathcal{N}_{i_1\ldots i_r}(\boldsymbol{\tau}, \mathbf{x}_\perp)}{D_{i_1}(\boldsymbol{\tau}) \cdots D_{i_r}(\boldsymbol{\tau})}\,, \tag{4.5}$$

and expose the purely polynomial dependence of $\mathcal{I}_{i_1\ldots i_r}$ on the transverse components $\mathbf{x}_\perp$, which can be integrated away systematically, as discussed in section 2.3. Moreover, the reduction of the number of loop variables appearing in the denominators implies, *per se*, a remarkable simplification of the multivariate polynomial division, since it minimizes the number of variables of the polynomial ring to which each ideal $\mathcal{J}_{i_1\ldots i_r}$ belongs.

As a matter of fact, the adaptive parametrization has even more striking effects on the integrand decomposition algorithm: as we will motivate below, it allows to completely bypass the computation of Gröbner bases and it reduces the polynomial division to the mere application of a set of substitution rules.

In the $d = d_\parallel + d_\perp$ parametrization, it is particularly easy to show that the loop variables $\boldsymbol{\tau}$ (or, beyond one-loop, a subset of them) can be expressed as combinations of denominators just by solving *linear equations*. In fact, starting from the definition of the denominators $D_{i_1}, \ldots, D_{i_r}$ in terms of the loop momenta, we can always build r independent differences of denominators which are linear in $\boldsymbol{\tau}$. More explicitly, at one-loop, where all the denominators can be written in the form

$$D_j = (q + \sum_i \beta_{ij} p_i)^2 + m_j^2\,, \qquad j = 1, \ldots, r\,, \tag{4.6}$$

we can choose one denominator, say D_1, consider the $r-1$ differences $D_j - D_1$, and build a system of equations of the type

$$\begin{cases} D_1 = (q + \sum_i \beta_{i1} p_i)^2 + m_1^2 \\ D_j - D_1 = \sum_i (\beta_{ij} - \beta_{i1}) q \cdot p_i + (\sum_i \beta_{ij} p_i)^2 - (\sum_i \beta_{i1} p_i)^2 + (m_j^2 - m_1^2) \quad j =\neq 1\,. \end{cases} \tag{4.7}$$

By construction, the differences $D_j - D_1$ are linear in the loop momentum (i.e. they are independent of λ^2) and they only depend linearly on $r-1$ of the variables $\mathbf{x}_\parallel$, which are proportional to the scalar products $q \cdot p_i$. Thus, the solution of the corresponding $r-1$ equations allows to express $\mathbf{x}_{\parallel}$ as linear combinations of denominators. Finally, the expression λ^2 in terms of denominators can by directly deduced from the definition of D_1, yielding to

$$\begin{cases} \mathbf{x}_{\parallel} & \to P_1[D_{i_k}] \\ \lambda^2 & \to P_2[D_{i_k}]\,, \quad i=1,2\,. \end{cases} \tag{4.8}$$

where P_i is a general i-rank polynomial.

This strategy can be easily generalized at multi-loop level. We can split the r loop denominators into $\ell(\ell+1)/2$ partitions S_i, identified by the subset of loop momenta each denominator depends on, and consider differences between denominators belonging to the same partition. Such differences will produce a set of linear relations, which can be used to express the longitudinal components $\mathbf{x}_{\parallel\, i}$ in terms of denominators. In general, beyond one-loop, the number $\ell\, d_{\parallel}$ of longitudinal variables is larger than the number k of independent difference of denominators. Therefore, the linear system in $\mathbf{x}_{\parallel\, i}$ is under-determined and its solution will reduce k longitudinal components (which we hereby denote by $\mathbf{x}_{\parallel\, i}^{\mathrm{RSP}}$) in terms of denominators and $(\ell\, d_{\parallel} - k)$ unconstrained $\mathbf{x}_{\parallel\, i}$. These free $(\ell\, d_{\parallel} - k)$ longitudinal components corresponds to the *physical* ISPs, i.e. irreducible scalar products between loop momenta and external momenta, and we will label them as $\mathbf{x}_{\parallel\, i}^{\mathrm{ISP}}$. Finally, having expressed the reducible longitudinal components $\mathbf{x}_{\parallel\, i}^{\mathrm{RSP}}$ in terms of denominators and physical ISPs, we can consider a representative denominator for each partition of denominators and obtain a set of linear relations, which can be solved for the variables λ_{ij}.

For instance, at two loops, we can have at most three partitions S_1, S_2 and S_3, which, respectively, correspond to denominators of the type

$$\begin{aligned} D_j &= (q_1 + \sum_i \beta_{ij} p_i)^2 + m_j^2, \qquad j \in S_1, \\ D_j &= (q_2 + \sum_i \beta_{ij} p_i)^2 + m_j^2, \qquad j \in S_2, \\ D_j &= (q_1 + q_2 + \sum_i \beta_{ij} p_i)^2 + m_j^2, \qquad j \in S_3. \end{aligned} \tag{4.9}$$

Similarly to the one-loop case, we can choose a representative denominator for each partition, say $D_{\bar{\jmath}_i} \in S_i$ for $i=1,2,3$, and observe that, for any $j \in S_i$, the difference $D_j - D_{\bar{\jmath}_i}$ is linear in $\mathbf{x}_{\parallel\, i}$ and independent of λ_{ij}. This allows us to write $r-3$ linear equations which can be solved for a subset of the longitudinal components. The definitions of the three representative denominators provide three complementary relations, which can be solved for λ_{11}, λ_{12} and λ_{22}. Therefore, by solving a linear system, we can obtain the set of substitutions

$$\begin{cases} \mathbf{x}_{\parallel\, i}^{\mathrm{RSP}} & \to P[D_{i_k}, \mathbf{x}_{\parallel\, i}^{\mathrm{ISP}}] \\ \lambda_{ij} & \to P[D_{i_k}, \mathbf{x}_{\parallel\, i}^{\mathrm{ISP}}]\,, \end{cases} \tag{4.10}$$

which express the maximal number of $\boldsymbol{\tau}$ variables in terms of denominators and physical ISPs.

Finally, at multi-loop level, factorized topologies deserve a special attention. In such cases, the denominators are independent of some of the λ_{ij}, with $i \neq j$ or, equivalently, one or more of the partitions S_i are empty. The numerator function associated to a factorized topology can, in principle, still depend on these λ_{ij} but, as we discussed in section 2.3.2, such dependence can be easily integrated away. Hence, after the full integrand has been made independent of these variables, the previous construction can be restricted to the set of non-empty partitions, in such a way to obtain linear relations involving the minimum number of loop variables which parametrize the integrand.

In summary, we have shown that solving the d-dimensional cut constraints needed for the integrand decomposition is never more complex than solving a linear system of equations. The *adaptive* introduction of the $d = d_\parallel + d_\perp$ parametrization of the integrand has the main advantage of removing all the dependence on the orthogonal directions from the denominators, hence from the on-shell conditions.

Indeed, in the perspective of the multivariate polynomial division, this is a remarkable simplification, since it allows us to treat $\mathbf{x}_{\perp\, i}$ as constant parameters rather than variables, i.e. to consider both numerators and denominators as members of polynomial ring $P[\boldsymbol{\tau}]$ and to reduce to a minimal set the number of variables involved in the computation of the Gröbner bases. In particular, if we chose the lexicographic ordering $\lambda_{ij} \prec \mathbf{x}_{\parallel\, i}$ for the variables $\boldsymbol{\tau}$, the polynomials in the Gröbner bases would turn out to be linear in the λ_{ij} and in $\mathbf{x}^{\text{RSP}}_{\parallel\, i}$. On the one hand, we have shown that is is always possible to arrive to analogous linear relations by systematically building differences of denominators and solving linear equations equivalent to the definition of the denominators themselves.

Thus, the polynomial division can be equivalently performed by applying the aforementioned set of linear relations, given by eqs.(4.8)-(4.10), without explicitly computing the corresponding Gröbner basis and performing the multivariate polynomial division.

4.2 The divide-integrate-divide algorithm

The observation made at the end of the previous section suggests a simplified version of the integrand decomposition algorithm, where the multivariate polynomial division of a numerator $\mathcal{N}_{i_1 \cdots_r}$ modulo Gröbner basis of the ideal $\mathcal{J}_{i_1 \ldots i_r}$ is reduced to the application of the substitution rules defined in eqs.(4.8)-(4.10). Moreover, since the $d = d_\parallel + d_\perp$ parametrization exposes the polynomial dependence of the integrand on the transverse components $\mathbf{x}_{\perp\, i}$, the integrand reduction can be performed *in tandem* with the transverse integration via Gegenbauer polynomials expansion, discussed in section 2.3.1, in order the systematically remove spurious terms.

The proposed algorithm is organized in three steps:

1. **Divide:** we parametrize the numerator $\mathcal{N}_{i_1 \ldots i_r}$ in terms of the variables $\mathbf{z} = \{\boldsymbol{\tau}, \mathbf{x}_{\perp\, i}\}$ and we make use of the relations (4.8)-(4.10) in oder to write

$$\mathcal{N}_{i_1 \ldots i_r}(\boldsymbol{\tau}, \mathbf{x}_{\perp\, i}) = \sum_{k=1}^{r} \mathcal{N}_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_r}(\mathbf{x}^{\text{ISP}}_{\parallel\, i}, \mathbf{x}_{\perp\, i}) D_{i_k} + \Delta_{i_1 \ldots i_r}(\mathbf{x}^{\text{ISP}}_{\parallel\, i}, \mathbf{x}_{\perp\, i}). \quad (4.11)$$

   The residue $\Delta_{i_1 \ldots i_r}$ depends on the transverse components $\mathbf{x}_{\perp\, i}$ (which are left untouched by the division) as well as on the physical ISPs $\mathbf{x}^{\text{ISP}}_{\parallel\, i}$, but it is independent of λ_{ij}. As desired, the dependence of the quotient in terms of reducible longitudinal components $x^{\text{RSP}}_{\parallel\, i}$ and of λ_{ij} is entirely expressed in terms of denominators, so that the identification of the lower-cut numerators $\mathcal{N}_{i_1 \ldots i_{k-1} i_{k+1} \ldots i_r}$ is straightforward. This single step, as we argued at the end of the previous section, is equivalent to :

- Compute the Gröbner basis $\mathcal{G}_{i_1\cdots i_r}$ of the ideal $\mathcal{J}_{i_1\cdots i_r}$ with lexicographic ordering $\lambda_{ij} \prec \mathbf{x}_\parallel$;
- Perform the multivariate polynomial division of $\mathcal{N}_{i_1\ldots i_r}$ modulo $\mathcal{G}_{i_1\cdots i_r}$;
- Determine the relation between the elements of $\mathcal{G}_{i_1\cdots i_r}$ and the loop denominators in order to rewrite the quotient of the division in terms of $D_{i_1}, \ldots, D_{i_r}$.

2. **Integrate:** The spurious terms contained in the decomposed numerator can be removed by integrating over the transverse components of the loop momenta. As described in section 2.3.1, this integration can by performed by mapping $\mathbf{x}_{\perp,i}$ to polynomials in $\sin\boldsymbol{\Theta}_\perp$, and $\cos\boldsymbol{\Theta}_\perp$,

$$\mathbf{x}_{\perp\, i} \to P\left[\sqrt{\lambda_{ij}}, \sin\boldsymbol{\Theta}_\perp, \cos\boldsymbol{\Theta}_\perp\right], \tag{4.12}$$

and by applying the orthogonality condition of Gegenbauer polynomials to all integrals in the transverse angles $\boldsymbol{\Theta}_\perp$. In this way, we obtain an integrated residue

$$\begin{aligned}\Delta^{\text{int}}_{i_1\ldots i_r}(\mathbf{x}^{\text{ISP}}_\parallel, \lambda_{ij}) &= \int \prod_{i=1}^{\ell} \mathrm{d}\mathbf{x}_{\perp\, i}\, \Delta_{i_1\ldots i_r}(\mathbf{x}^{\text{ISP}}_\parallel, \mathbf{x}_{\perp i}) \\ &\propto \int \mathrm{d}^{(4-d_\parallel)\ell}\boldsymbol{\Theta}_\perp\, \Delta^{\text{n-sp}}_{i_1\ldots i_r}(\mathbf{x}^{\text{ISP}}, \lambda_{ij}, \boldsymbol{\Theta}_\perp)\,,\end{aligned} \tag{4.13}$$

which is free of spurious terms and whose non-spurious dependence on the transverse variables has been reduced, due to eq. (4.12), to powers of λ_{ij}. It should be noted that the explicit evaluation of the angular integral (4.13) yields to an additional dependence of $\Delta^{\text{int}}_{i_1\ldots i_r}$ on the space-time dimensions d.

3. **Divide:** The full set of residues $\Delta^{\text{int}}_{i_1\ldots i_r}$ obtained by iterating the division and the transverse integration on the numerator of all multiple-cuts provides a representation of the Feynman amplitude which is completely free of spurious terms. However, this representation is non-minimal, since λ_{ij} can be reduced to denominators and ISPs. Therefore, we can perform a further polynomial division of the integrated residue, i.e. we can apply once more the relations (4.8)-(4.10), and obtain

$$\Delta^{\text{int}}_{i_1\ldots i_r}(\mathbf{x}^{\text{ISP}}_\parallel, \lambda_{ij}) = \sum_{k=1}^{r} \mathcal{N}^{\text{int}}_{i_1\ldots i_{k-1} i_{k+1}\ldots i_r}(\mathbf{x}^{\text{ISP}}_\parallel) + \Delta'_{i_1\ldots i_r}(\mathbf{x}^{\text{ISP}}_\parallel)\,, \tag{4.14}$$

where the new residue $\Delta'_{i_1\ldots i_r}(\mathbf{x}^{\text{ISP}}_\parallel)$ depends exclusively on the physical ISPs.

Once the steps 1-3 have been iterated over all multiple-cuts, we obtain the final decomposition

$$I^{d\,(\ell)}_{i_1\ldots i_r} = \sum_{k=0}^{r} \sum_{\{j_1\cdots j_k\}} \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{\Delta'_{j_1\cdots j_k}(\mathbf{x}^{\text{ISP}}_{\parallel\, i})}{D_{j_1}\cdots D_{j_k}}\,, \tag{4.15}$$

which expresses the amplitudes as a sum of integrals whose irreducible numerators are functions of the components of the loop momenta which are parallel to the external kinematics.

At this level, a few comments on the eq. (4.15) are in order:

- In the divide-integrate-divide algorithm, the integration over the transverse directions is performed after the integrand reduction, namely after determining the residues. This means that, as in the standard integrand reduction procedure, the decomposed integrand that arises from the first division contains spurious terms $\Delta^{\rm SP}_{i_1\cdots i_r}$, which are subsequently eliminated by the transverse integration.

  It is clear that, if integration over $\mathbf{x}_{\perp\, i}$ was performed prior to any division, we would obtain a decomposition with no spurious monomials in a single division step. This second option, which we can refer to as *integrate-divide*, is conceptually equivalent to the divide-integrate-divide algorithm but, in view of practical applications, it is indeed preferable, since it allows to thin down the expression of the numerator before the division is performed and, hence, to simplify not only the final decomposition but also the intermediate steps of the calculation.

  Obviously, in order to integrate before the reduction, the dependence of the numerator on the loop momenta must be either known analytically or reconstructed semi-analytically [160, 161]. Such situation may indeed occur when the integrands to be reduced are built from Feynman diagrams or they emerge as quotients of subsequent divisions.

- Although in the discussion we have always referred explicitly to integrands with all denominators raised to power one, the algorithm can be applied with no modifications to cases that involve higher powers of the loop denominators.

  As we argued at the end of section 3.2, in case of integrands characterized by a set of exponents $\{a_1\,, \ldots\,, a_r\}$, like the one defined in eq. (3.19), we simply need to perform the polynomial division modulo the same Gröbner basis (i.e, in the divide-integra-divide approach, apply the same set of substitution rules) a number $a_1 \times \ldots \times a_r$ of times. We will discuss in more detail how to deal with higher power of denominators in section 4.4.

Example

We can illustrate the features of the divide-integrate-divide algorithm on a toy example at one loop. Let us consider the three-point integral

$$I^{d\,(1)}_{1\,2\,3} = \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{(q\cdot p_1)(q\cdot p_2) + 4(q\cdot\varepsilon_{12})(q\cdot\varepsilon_{21})}{D_1\, D_2\, D_3}\,, \tag{4.16}$$

where the denominators are defined as

$$\begin{aligned} D_1 =& q - m^2\,, \\ D_2 =& (q+p_1) - m^2\,, \\ D_3 =& (q+p_1+p_2) - m^2\,, \end{aligned} \tag{4.17}$$

with p_1 and p_2 being massless momenta satisfying $(p_1+p_2)^2 = s$. In the numerator, ε_{ij} indicates the polarization vector $\varepsilon^\alpha_{ij} = \frac{1}{2}\langle i|\gamma^\alpha|j]$, which are orthogonal to both external momenta, $\varepsilon_{ij}\cdot p_k = 0$.

We first introduce the $d = d_\parallel + d_\perp$ parametrization. Being $d_\parallel = 2$, we split the loop momentum as $q^\alpha = q^\alpha_\parallel + \lambda^\alpha$, with

$$q^\alpha_\parallel = x_1 p^\alpha_1 + x_2 p^\alpha_2\,, \qquad \lambda^\alpha = x^\alpha_3 e^\alpha_3 + x^\alpha_4 e^\alpha_4 + \mu^\alpha\,. \tag{4.18}$$

The transverse vectors $e_{3,4}$ are chosen to be $e_{3,4} = \varepsilon_{12} \pm \varepsilon_{21}$. In terms of the variables $\mathbf{z} = \{x_1, x_2, x_3, x_4, \lambda^2\}$, the integral (4.16) becomes

$$I^{d\,(1)}_{1\,2\,3} = s^2 \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{x_1\, x_2 + x_3^2 - x_4^2}{D_1\, D_2\, D_3}\,. \tag{4.19}$$

The three denominators solely depend on the variables $\boldsymbol{\tau} = \{x_1, x_2, \lambda^2\}$,

$$\begin{aligned} D_1(\boldsymbol{\tau}) &= s\, x_1\, x_2 + \lambda^2 - m^2\,, \\ D_2(\boldsymbol{\tau}) &= s\, x_2(x_1+1) + \lambda^2 - m^2\,, \\ D_3(\boldsymbol{\tau}) &= s(x_1\, x_2 + x_1 + x_2 + 1) + \lambda^2 - m^2\,, \end{aligned} \tag{4.20}$$

and, according to the discussion of sec 4.1, we can consider differences of denominators and build a system of three equations which are linear in $\boldsymbol{\tau}$,

$$\begin{cases} D_1 & = s\, x_1\, x_2 + \lambda^2 - m^2 \\ D_2 - D_1 & = s\, x_2 \\ D_3 - D_1 & = s(x_1 + x_2 + 1)\,. \end{cases} \tag{4.21}$$

The solution of this system allows to reduce all $\boldsymbol{\tau}$ variables to combinations of denominators,

$$\begin{cases} x_1 = & \frac{D_3 - D_1 - s}{s} \\ x_2 = & \frac{D_2 - D_1}{s} \\ \lambda^2 = & \frac{D_1^2(D_3 - D_2) + D_2(s + D_2 - D_3) + m^2\, s}{s}\,. \end{cases} \tag{4.22}$$

We can now apply the divide-integra-divide algorithm to (4.16):

1. **Divide:** We substitute the identities (4.22) in the numerator and, by collecting powers of denominators, we obtain

$$\begin{aligned} s^2 \frac{x_1\, x_2 + x_3^2 - x_4^2}{D_1\, D_2\, D_3} =& \frac{1}{D_1} - \frac{1}{D_2} + \frac{1}{D_3} - \frac{s}{D_2 D_3} - \frac{(q+p_1)^2 - m^2 + s}{D_1 D_3} \\ &+ s^2 \frac{x_3^2 - x_4^2}{D_1 D_2 D_3}\,. \end{aligned} \tag{4.23}$$

2. **Integrate:** We integrate over the transverse components x_3 and x_4 (see section 2.52),

$$s^2 \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{x_3^2 - x_4^2}{D_1 D_2 D_3} = -\frac{2s}{(d-2)} \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\lambda^2}{D_1 D_2 D_3}\,. \tag{4.24}$$

   In this case, the transverse integrals are non-spurious and they produce a triple-cut numerator proportional to λ^2.

3. **Divide:** We apply again the identities (4.22) to the integrated residue (4.24) in order to reduce λ^2. By putting everything together we obtain

$$\begin{aligned} I^{d\,(1)}_{1\,2\,3} =& \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \left[\frac{d}{d-2} \left(\frac{1}{D_1} - \frac{1}{D_2} + \frac{1}{D_3} \right) + \frac{s}{D_2 D_3} \right. \\ & \left. - \frac{d}{d-2} \frac{(q+p_1)^2 - m^2 + s}{D_1 D_3} - \frac{2m^2\, s}{D_1 D_2 D_3} \right]. \end{aligned} \tag{4.25}$$

Thus, we have decomposed the rank-two integral $I_{1\,2\,3}^{d\,(1)}$ in terms of a three-point function with constant numerator and a linear combination of lower-point integrals. The full decomposition in terms of rank-zero integrals is obtained by iterating the steps of the division algorithm to the double- and single-cuts (in this simple case, it is enough to apply the division to the double cut $D_1 = D_3 = 0$, since the numerators of all the other cuts have rank-zero already). Let us observe that the application of the divide-integra-divide algorithm to this example was intended as a mere illustration: it is clear that, from the knowledge of the full analytic structure of the integrand, we could have applied the integration (4.24) directly on eq. (4.19) and obtain eq. (4.25) with a single division.□

One-loop adaptive integrand decomposition

In this section, we revisit the one-loop integrand decomposition discussed in section 3.4, in the light of the divide-integrate-divide algorithm.

We start again from the most general one-loop n-point amplitude (3.20), which is now parametrized in terms of the *adaptive* variables $\mathbf{z} = \{\boldsymbol{\tau}, \mathbf{x}_\perp\}$,

$$I_{i_1\cdots i_n}^{d\,(1)}[\mathcal{N}_{i_1\cdots i_n}] = \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\mathcal{N}_{i_1\cdots i_n}(\mathbf{z})}{D_{i_1}(\boldsymbol{\tau})\ldots D_{i_n}(\boldsymbol{\tau})}\,, \tag{4.26}$$

where $\mathcal{N}_{i_1\cdots i_n}(\mathbf{z})$ is the arbitrary polynomial (3.22) satisfying the renormalizability constraint given by eq. (3.23). Such reparametrization doesn't affect integrands with $n \geq 5$ since, in that case, $\lambda^\alpha = \mu^\alpha$ and the longitudinal space correspond to the full four-dimensional space-time. In the first division step, we systematically generate for each multiple-cut the linear system of equations (4.7) and we solve them in order to obtain the substitution rules given in eq. (4.8), which express the longitudinal components $\mathbf{x}_\parallel$ and λ^2 as combinations of denominators. By applying these substitutions to arbitrary parametric numerators, we obtain the set of universal one-loop residues

$$\begin{aligned}
\Delta_{ijklm} =& c_0\mu^2\,,\\
\Delta_{ijkl} =& c_0 + c_1x_4 + c_2x_4^2 + c_3x_4^3 + c_4x_4^4\,,\\
\Delta_{ijk} =& c_0 + c_1x_3 + c_2x_4 + c_3x_3^2 + c_4x_3x_4 + c_5x_4^2 + c_6x_3^3 + c_7x_3^2x_4\\
&+ c_8x_3x_4^2 + c_9x_4^3\,,\\
\Delta_{ij} =& c_0 + c_1x_2 + c_2x_3 + c_3x_4 + c_4x_2^2 + c_5x_2x_3 + c_6x_2x_4 + c_7x_3^2\\
&+ c_8x_3x_4 + c_9x_4^2\,,\\
\Delta_{ij}|_{p^2=0} =& c_0 + c_1x_1 + c_2x_3 + c_3x_4 + c_4x_1^2 + c_5x_1x_3 + c_6x_1x_4 + c_7x_3^2\\
&+ c_8x_3x_4 + c_9x_4^2\,,\\
\Delta_{ij}|_{p^2=0} =& c_0 + c_1x_1 + c_2x_3 + c_3x_4 + c_4x_1^2 + c_5x_1x_3 + c_6x_1x_4 + c_7x_3^2\\
&+ c_8x_3x_4 + c_9x_4^2\,,\\
\Delta_i =& c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4\,,
\end{aligned} \tag{4.27}$$

where we have omitted, for each coefficient c_k, the label $i \ldots j$ of the corresponding multiple-cut.

It is particularly interesting to compare eq. (4.27) with the one-loop residues of eq. (3.24), which were obtained in the $d = 4 - 2\epsilon$ parametrization:

- The *reducibility criterion* (1), which ensures the vanishing of the residues of all $n$ple-cuts with $(n > 5)$, can be easily understood from the construction of the system of the n linear relations (4.7). Whenever $n > 5$, besides expressing all loop variables $\mathbf{z}$ in terms of denominators, eq. (4.7) determines a set $n-5$ additional relations between denominators of form

$$P[D_{i_k}] = 1\,, \tag{4.28}$$

where P is a polynomial which vanishes on the multiple-cut $D_{i_1} = \cdots = D_{i_n} = 0$. Therefore, for any numerator $\mathcal{N}_{i_1\cdots i_n}$, we can use eq. (4.28) to write

$$\mathcal{N}_{i_1\cdots i_n} = P[D_{i_k}]\mathcal{N}_{i_1\cdots i_n}\,, \tag{4.29}$$

from which we immediately see that the integrand is reducible to a combination of lower-cut contributions.

- For $n \leq 5$, the number of variables $\boldsymbol{\tau}$ corresponds exactly to the number of linear relations (4.7). In this sense, at one-loop, *all* multiple-cuts are maximum-cuts with $n_s = 1$ solution, since the on-shell conditions fix all $\boldsymbol{\tau}$ variable to a (unique) constant value. Therefore, according to the *maximum-cut theorem* (2), all residues (4.27), regarded as members of the polynomial ring $P[\boldsymbol{\tau}]$, are rank-zero polynomials in $\boldsymbol{\tau}$, i.e. constants.

  Obviously, each $\Delta_{i_1\cdots i_n}$ can still depend on the transverse variables $\mathbf{x}_\perp$. However, since $\mathbf{x}_\perp$ are left untouched by the division, such dependence is completely fixed by the dependence on $\mathbf{x}_\perp$ of the original numerator. In particular, for the arbitrary numerator (3.22) the residue is just a *complete* polynomial in $\mathbf{x}_\perp$,

$$\Delta_{i_1\cdots i_n}(\mathbf{x}_\perp) = \sum_{\vec{\jmath}\,\in\, J_{4-d_\parallel}(n)} \alpha_{\vec{\jmath}}\, x_{\perp\,1}^{j_1} x_{\perp\,2}^{j_2} \ldots x_{\perp\,4-d_\parallel}^{j_{4-d_\parallel}}\,, \quad n \leq 4, \tag{4.30}$$

  whose rank is determined by the renormalizability condition

$$J_{4-d_\parallel}(n) = \{\vec{\jmath}\,:\, j_1 + j_2 + \cdots + j_{4-d_\parallel} \leq n\}\,. \tag{4.31}$$

  The only exception to eq. (4.30) is the residue of the double-cut with massless external momentum, $\Delta_{ij}|_{p^2=0}$, which depends on the longitudinal component x_1 parallel to p. In fact, due to the reduced dimensions of the transverse space, the denominators depend on three variables $\boldsymbol{\tau} = \{x_1, x_2, \lambda^2\}$ so that in this kinematic configuration the double-cut is not maximum any more.

- Having maximized the dependence of the residues (4.27) on the transverse direction $\mathbf{x}_\perp$, we can easily identify the spurious contributions $\Delta^{\rm sp}_{i_1\cdots i_n}$ by means of eq. (2.52) and reduce all non-spurious terms to powers of λ^2. The resulting decomposition of the one-loop amplitude (4.16) is

$$\begin{aligned}
I^{d\,(1)}_{i_1\cdots i_n}[\mathcal{N}_{i_1\cdots i_n}] = {} & \sum_{i\ll m}^{n} c_0^{(ijklm)} I^{d\,(1)}_{ijklm}[\mu^2] + \sum_{i\ll l}^{n} \Big[c_0^{(ijkl)} I^{d\,(1)}_{ijkl}[1] \\
& + c_2^{(ijkl)} I^{d\,(1)}_{ijkl}[\lambda^2] + c_4^{(ijkl)} I^{d\,(1)}_{ijkl}[\lambda^4] \Big] \\
& + \sum_{i\ll k}^{n} \Big[c_0^{(ijk)} I^{d\,(1)}_{ijk}[1] + c_7^{(ijk)} I^{d\,(1)}_{ijk}[\lambda^2] \Big]
\end{aligned}$$

$$
+ \sum_{i \ll j}^{n} \left[\delta_{0,p_i^2} \left(c_1^{(ij)} I_{ij}^{d\,(1)}[(q+p_i)\cdot e_2] + c_2^{(ij)} I_{ij}^{d\,(1)}[((q+p_i)\cdot e_2)^2] \right) \right.
$$

$$
\left. + c_0^{(ij)} I_{ij}^{d\,(1)}[1] + c_9^{(ij)} I_{ij}^{d\,(1)}[\lambda^2] \right] + \sum_{i}^{k} c_0^{(i)} I_i^{d\,(1)}[1] \,, \tag{4.32}
$$

where the coefficients c_k have now acquired and additional dependence on d, due to the integration over the transverse variables.

- We can apply a further-integrand level reduction to the monomials in λ^2 that appear in the decomposition (4.32) and reduce them to rank-zero numerators. In this way, we obtain a final decomposition of the amplitude in terms of a minimal set of integrals,

$$
I_{i_1\cdots i_n}^{d\,(1)}[\mathcal{N}_{i_1\cdots i_n}] = \sum_{i \ll m}^{n} c_0^{(ijklm)} I_{ijklm}^{d\,(1)}[\mu^2]
$$

$$
+ \sum_{i \ll l}^{n} c_0^{(ijkl)} I_{ijkl}^{d\,(1)}[1] + \sum_{i \ll k}^{n} c_0^{(ijk)} I_{ijk}^{d\,(1)}[1]
$$

$$
+ \sum_{i \ll j}^{n} \left[\delta_{0,p_i^2} \left(c_1^{(ij)} I_{ij}^{d\,(1)}[(q+p_i)\cdot e_2] + c_2^{(ij)} I_{ij}^{d\,(1)}[((q+p_i)\cdot e_2)^2] \right) \right.
$$

$$
\left. + c_2^{(ij)} I_{ij}^{d\,(1)}[((q+p_i)\cdot e_2)^2] \right] + \sum_{i}^{n} c_0^{(i)} I_i^{d\,(1)}[1] \,, \tag{4.33}
$$

which is, of course, in agreement with eq. (3.30).

We would like to observe that, at one-loop level, this second division offer a particularly clear interpretation of the dimensional-recurrence relation (2.56) at the *integrand*-level. In fact, according to eq. (2.25), λ^2 numerators can be identified with higher-dimensional integrals and, hence, the integrand reducibility of λ^2 allows to write any higher dimensional integral as a combination of the corresponding integral in d dimensions and integrals with fewer loop denominators.

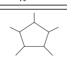
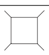
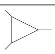
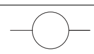
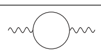
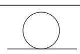
$\mathcal{I}_{i_1\cdots i_n}$	$\boldsymbol{\tau}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\text{int}}_{i_1\cdots i_n}$	$\Delta'_{i_1\cdots i_n}$
$\mathcal{I}_{i_1 i_2 i_3 i_4 i_5}$ 	$\{x_1, x_2, x_3, x_4, \mu^2\}$	1 $\{1\}$	– –	– –
$\mathcal{I}_{i_1 i_2 i_3 i_4}$ 	$\{x_1, x_2, x_3, \lambda^2\}$	5 $\{1, x_4, x_4^2, x_4^3, x_4^4\}$	3 $\{1, \lambda^2, \lambda^4\}$	1 $\{1\}$
$\mathcal{I}_{i_1 i_2 i_3}$ 	$\{x_1, x_2, \lambda^2\}$	10 $\{1, x_3, x_4, x_3^2, x_3 x_4, x_4^2, x_3^3, x_3^2 x_4, x_3 x_4^2, x_4^3\}$	2 $\{1, \lambda^2\}$	1 $\{1\}$
$\mathcal{I}_{i_1 i_2}$ 	$\{x_1, \lambda^2\}$	10 $\{1, x_2, x_3, x_4, x_2^2, x_2 x_3, x_2 x_4, x_3^2, x_3 x_4, x_4^2\}$	2 $\{1, \lambda^2\}$	1 $\{1\}$
$\mathcal{I}_{i_1 i_2}$ 	$\{x_1, x_2, \lambda^2\}$	10 $\{1, x_1, x_3, x_4, x_1^2, x_1 x_3, x_1 x_4, x_3^2, x_3 x_4, x_4^2\}$	4 $\{1, x_1, x_1^2, \lambda^2\}$	3 $\{1, x_1, x_1^2\}$
$\mathcal{I}_{i_1}$ 	$\{\lambda^2\}$	5 $\{1, x_1, x_2, x_3, x_4\}$	1 $\{1\}$	– –

Table 4.1: Irreducible numerators for one-loop topologies. In the first column, $\boldsymbol{\tau}$ labels the variables the denominators depend on. For each of the residues, $\Delta_{i_1\cdots i_n}$, $\Delta^{\text{int}}_{i_1\cdots i_n}$ and $\Delta'_{i_1\cdots i_n}$ we indicate the number of monomials and the list of their variables. In the pictures, wavy lines stand for massless particles and solid ones denote propagators with arbitrary masses.

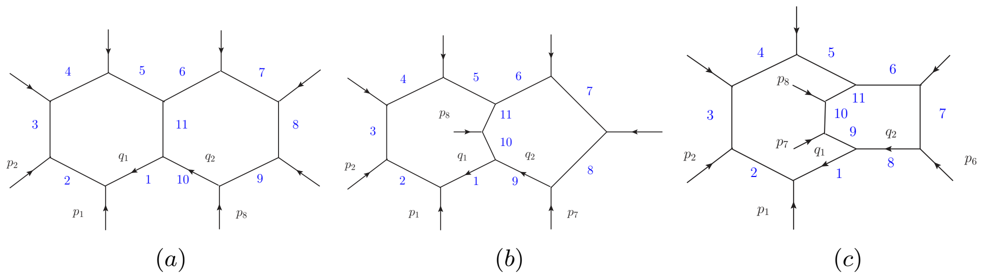


Figure 4.1: Two-loop maximum-cuts.

The whole discussion of the one-loop adaptive integrand decomposition is summarized by Table 4.1, which lists, for each step of the divide-integrate-divide algorithm, the irreducible numerators of all one-loop topologies. In particular, the last column shows that the final result of the algorithm, which doesn't rely on any additional integral-level identity, is completely equivalent to the ordinary tensor integral decomposition. In this regard, we would like to stress once more that the algorithm does not resort to any form factor decomposition and, thus, can be effectively applied also to helicity amplitudes.

4.3 Two-loop adaptive integrand decomposition

In this section, we apply the divide-integra-divide algorithm in order to determine the universal parametric form of the residues appearing in the integrand decomposition of a general r-denominator two-loop amplitude

$$I^{d\,(2)}_{i_1\cdots i_r}[\mathcal{N}_{i_1\cdots i_r}] = \int \frac{\mathrm{d}^d q_1}{\pi^{d/2}} \frac{\mathrm{d}^d q_2}{\pi^{d/2}} \frac{\mathcal{N}_{i_1\cdots i_r}(\mathbf{z})}{D_{i_1}(\boldsymbol{\tau}) \ldots D_{i_r}(\boldsymbol{\tau})} \,. \tag{4.34}$$

The r denominators $D_{i_1}, \ldots, D_{i_r}$ can be grouped into three different partitions, as in eq. (4.9), according to which loop momenta they depend on.

Therefore, we can classify any two-loop integrand topology $\mathcal{I}_{i_1\cdots i_r}$ according to the number r_1 of denominators depending on q_1, the number r_2 of denominators depending on q_2 and the number $r_{12} = r - r_1 - r_2$ of denominators which depend on both loop momenta.

The value of r_{12} allow us to distinguish between three different kind of two-loop integrands: *planar* topologies are identified by $r_{12} = 1$, *non-planar* ones by $r_{12} > 1$ and $r_{12} = 0$ corresponds to two-loop topologies which are *factorized* into the product of two one-loop ones.

As we have argued in section 3.3, the integrand decomposition of the amplitude (4.34) receives, in general, contributions from maximum-cuts, i.e. multiple-cuts which fix all the loop variables, as well as from all other cuts associated to subsets of denominators.

Hence, in order to classify the universal structure of the two-loop residues, we first need to identify the maximum-cut topologies. Provided that two-loop integrands in d dimensions are parametrized in terms of 11 variables, which in $d = 4 - 2\epsilon$ correspond to

$$\mathbf{z} = \{x_{11}, \ldots x_{41}, x_{12}, \ldots, x_{42}, \mu_{11}, \mu_{22}, \mu_{12}\}, \tag{4.35}$$

and that the definition of the loop denominators (4.9) can always be cast in terms of linear equations for $\mathbf{z}$, it is clear that a maximum-cut must have $r = 11$. Additional constraints on r_1 and r_2 must ensure that the system of linear equations (4.9) is determined. If we assume, without loss of generality, $r_1 \geq r_2$, these requirements are satisfied in the following cases:

$$\begin{aligned} &\mathcal{I}^{\mathrm{P}}_{1\,2\ldots\,11} \; : r_1 = 5\,,\; r_2 = 5\,,\; r_{12} = 1\,, \\ &\mathcal{I}^{\mathrm{NP1}}_{1\,2\ldots\,11} \; : r_1 = 5\,,\; r_2 = 4\,,\; r_{12} = 2\,, \\ &\mathcal{I}^{\mathrm{NP2}}_{1\,2\ldots\,11} \; : r_1 = 5\,,\; r_2 = 3\,,\; r_{12} = 3\,, \end{aligned} \tag{4.36}$$

which correspond, respectively, to the three eight-point topologies depicted in figure 4.1. Any other topology which can be obtained from one of the maximum-cuts (4.36) with the addition on any number of loop propagators, is reducible, i.e. it is completely decomposed in terms of the corresponding maximum-cut and its subtopologies.

Therefore, the integrand decomposition of the amplitude $I^{d\,(2)}_{i_1\cdots i_r}$ is of the form

$$\mathcal{I}_{i_1\ldots i_r} \equiv \frac{\mathcal{N}_{i_1\ldots i_r}}{D_{i_1}\cdots D_{i_r}} = \sum_{k=2}^{s} \sum_{\{j_1\cdots j_k\}} \frac{\Delta_{j_1\cdots j_k}}{D_{j_1}\cdots D_{j_k}}\,, \qquad s = \min(r, 11)\,. \tag{4.37}$$

The determination of $\Delta_{j_1\cdots j_k}$ appearing in eq. (4.37) proceeds along the same lines of the one-loop case discussed in section (4.2), namely:

1. For every multiple-cut, starting from the three maximum-cut topologies (4.36) and ranging down to double-cuts with $r_1 = r_2 = 1$, we introduce, according to the number n of external legs of each topology, the *adaptive* parametrization $\mathbf{z} = \{\boldsymbol{\tau}, \mathbf{x}_{\perp\,1}, \mathbf{x}_{\perp\,2}\}$, with

$$\boldsymbol{\tau} = \{\mathbf{x}_{\parallel\,1}, \mathbf{x}_{\parallel\,2}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}\,, \tag{4.38}$$

and we build a system of linear equations in the $\boldsymbol{\tau}$, following the discussion of section 4.1. In general, the number of $\boldsymbol{\tau}$ variables is larger than the number of independent linear equations, so that the solution of such system expresses λ_{ij} and a subset of longitudinal components $\mathbf{x}_{\parallel}$ in terms of denominators and physical ISPs $\mathbf{x}^{\mathrm{ISP}}_{\parallel}$,

$$\begin{cases} \mathbf{x}^{\mathrm{RSP}}_{\parallel\,i} & \to P[D_{i_k}, \mathbf{x}^{\mathrm{ISP}}_{\parallel\,i}] \\ \lambda_{ij} & \to P[D_{i_k}, \mathbf{x}^{\mathrm{ISP}}_{\parallel\,i}]\,, \quad i,j = 1,2\,. \end{cases} \tag{4.39}$$

In the case of factorized topologies $r_{12} = 0$, the structure of (4.39) can be notably simplified. In fact, as we discussed in section 2.3.2, when the loop denominators are independent of $q_1 \cdot q_2$, any possible dependence of the numerator on μ_{12} can be easily integrated away.

Therefore, any factorized integrand can be parametrized in terms of 10 variables only, $\mathbf{z} = \{\mathbf{z}_1, \mathbf{z}_1\}$, each $\mathbf{z}_i$ corresponding to the 5 variables associated to the loop momentum q_i. According to the number n_1 and n_2 of external legs attached, respectively, to each loop, we can introduce the adaptive variables $\mathbf{z}_i = \{\boldsymbol{\tau}_i, \mathbf{x}_{\perp\,i}\}$ *independently* for q_1 and q_2. Since for a factorized topology $n_i = r_1$, this implies that the system of r linear equations built from the definition of the loop denominators allows to write all variables $\boldsymbol{\tau} = \{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$ in terms of denominators and, as in the one-loop case, there is no physical ISPs $\mathbf{x}^{\mathrm{ISP}}_{\parallel}$,

$$\begin{cases} \mathbf{x}_{\parallel\,i} & \to P_1[D_{i_k}] \\ \lambda_{ii} & \to P_2[D_{i_k},]\,, \quad i = 1,2\,. \end{cases} \tag{4.40}$$

2. We make use of the set of substitutions (4.39)-(4.40) and of the transverse integration techniques and apply the divide-integrate-divide algorithm to arbitrary polynomial numerators of the type

$$\mathcal{N}_{i_1\cdots i_r}(\mathbf{z}) = \sum_{\vec{\jmath}\in J} \alpha_{\vec{\jmath}}\, z_1^{j_1} z_2^{j_2} \dots z_{11}^{j_{11}}, \tag{4.41}$$

where $\alpha_{\vec{\jmath}}$ are arbitrary coefficients. The sum over the integer 11-tuples $\vec{\jmath} = \{j_1, j_2, \dots, j_{11}\}$ is restricted to a subset J which is determined by the details of the theory under consideration. In particular, if restrict $N_{i_1\cdots i_r}$ to renormalizable theories, we have $J = J_{11}(s_1, s_2, s)$ with

$$J_{11}(s_1, s_2, s) = \begin{cases} \sum_{i=1}^{4} j_i + 2j_9 + j_{11} \leq s_1 \\ \sum_{i=5}^{8} j_i + 2j_{10} + j_{11} \leq s_2 \\ \sum_{i=1}^{8} j_i + 2(j_9 + j_{10} + j_{11}) \leq s\,, \end{cases} \tag{4.42}$$

with $s_1 = r_1 + r_{12}$, $s_2 = r_2 + r_{12}$ and $s = r_1 + r_2 + r_{12} - 1$. Non-renormalizable theories, which will generally correspond to higher values of s_1, s_2 and s, can be treated in a completely analogous way.

As we discussed above, in the case of factorized topologies, we can assume the full integrand to be independent of μ_{12} (or, equivalently, of λ_{12}). Therefore, we consider the parametric numerators to have the form

$$\mathcal{N}_{i_1\cdots i_r}(\mathbf{z}_1, \mathbf{z}_2) = \mathcal{N}_{i_1\cdots i_{r_1}}(\mathbf{z}_1)\mathcal{N}_{i_{r_1+1}\cdots i_{r_1+r_2}}(\mathbf{z}_2), \tag{4.43}$$

where each of the one-loop numerators appearing in the r.h.s. is parametrized as in eq. (3.22).

The result of this analysis, i.e. the list of residues that contribute to eq. (4.37), is shown in Tables 4.2-4.8, where $\mathcal{I}^{a}_{i_1\dots i_r}$ labels the integrand associated to the subset of denominators $D_{i_1}, \dots, D_{i_r}$ of the maximum-cut $\mathcal{I}^{a}_{1\dots 11}$, with $a \in \{\mathrm{P}, \mathrm{NP1}, \mathrm{NP2}\}$. For each integrand, we provide the number of monomials that appear in the corresponding residue at each step of the divide-integrate-divide algorithm, as well as the loop variables they depend on.

In particular, we would like to point out that:

- The residues of the maximum-cut topologies $\mathcal{I}^{\mathrm{P}}_{1\cdots 11}$, $\mathcal{I}^{\mathrm{NP1}}_{1\cdots 11}$ and $\mathcal{I}^{\mathrm{NP2}}_{1\cdots 11}$ contain one single coefficients, in agreement with the *maximum-cut theorem* 2;

- For all topologies with a number of external legs $5 \leq n \leq 8$, we can apply one single division, since the four-dimensional loop momenta have no transverse components and $\lambda_i^{\alpha} = \mu_i^{\alpha}$. Nonetheless, the use of the set of relations (4.40), which, in the multivariate polynomial division approach, would correspond to the choice of monomial order $\mu_{ij} \prec \mathbf{x}_{\parallel}$, ensure that the residues are written in terms of physical ISPs only, i.e. in terms of scalar products between loop momenta and external ones;

- In the case of $n \leq 4$ external legs, the use of the $d = d_{\parallel} + d_{\perp}$ parametrization maximizes the dependence on the transverse variables $\mathbf{x}_{\perp\, i}$ in the residue $\Delta_{i_1\dots i_r}$ obtained after the first division step. The subsequent integration allows the systematic detection of all spurious-terms, hence a substantial reduction of the number of monomials contributing to the integrated amplitude. Finally, the second division removes the dependence of the residues on λ_{ij}, which is a byproduct the transverse integration of non-spurious terms. Therefore, the final residues $\Delta'_{i_1\ldots i_r}$ depend exclusively on the physical ISPs $\mathbf{x}^{\text{ISP}}_{\parallel i}$ and their polynomial structure reflects the one of the original numerator (4.41), i.e. it is fully determined by the renormalizability properties of the theory. This means that, if we assume the topology $\mathcal{I}_{i_1\ldots i_r}$ to have m_i ISPs in the i-th loop momentum, then its residue is given by

$$\Delta'_{i_1\ldots i_r}(\mathbf{x}^{\text{ISP}}_{\parallel i}) = \sum_{\vec{\jmath}\in J} \alpha_{\vec{\jmath}}\,(x^{\text{ISP}}_{\parallel 11})^{j_1}\ldots(x^{\text{ISP}}_{\parallel m_1 1})^{j_{m_1}}\ldots(x^{\text{ISP}}_{\parallel 12})^{j_{m_1+1}}\ldots(x^{\text{ISP}}_{\parallel m_2 2})^{j_{m_1+m_2}}, \tag{4.44}$$

where $J = J_{m_1+m_2}(s_1, s_2, s)$ is the set of (m_1+m_2)-tuples

$$J_{m_1+m_2}(s_1,s_2,s) = \begin{cases} \sum_{i=1}^{m_1} j_i \le s_1\,, \quad \sum_{i=1}^{m_2} j_{m_1+i} \le s_2 \\ \sum_{i=1}^{m_1+m_2} j_i \le s\,. \end{cases} \tag{4.45}$$

In the case of non-planar topologies, the inspection of the residues $\Delta'_{i_1\ldots i_r}$ reveals an apparent violation of eq. (4.44), since they include monomials which do not fulfil all the above renormalizability conditions. This effect is due to the fact that, whenever $r_{12} > 1$, the linear relations (4.40) produce a mixing of the longitudinal components of the two-loop momenta. Therefore, although they preserve the total rank s as in eq. (4.45), these relations, when applied to a numerator, can enhance the rank in the component of one of the loop momenta above the corresponding s_i. Of course, the full set of the rinormalizability conditions can be restored at the price of expressing the residue in terms of a larger number of variables.

- In the one-loop case, we have seen that the integrand reduction of monomials in λ^2 corresponds to an integrand level implementation of the dimensional recurrence relations between loop integrals. At multi-loop level, as dictated by eq. (2.56), the Gram determinant $G(\lambda_{ij})$ of the transverse vectors λ_i^α (or $G(\mu_{ij})$ in the cases with more than four external legs) acts on any arbitrary numerator as a dimension-rising operator,

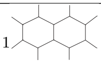
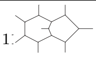
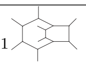
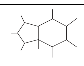
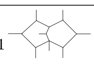
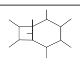
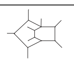
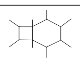
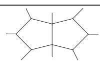
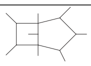
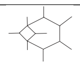
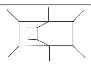
$$\mathcal{I}^{d\,(\ell)}\left[G(\lambda_{ij})\right] \sim \mathcal{I}^{d+2\,(\ell)}\left[1\right]\,. \tag{4.46}$$

Thus, if we apply the substition (4.10) to eq. (4.46), we can relate a combination of d-dimensional integrals with non-trivial numerators to lower rank integrals in $d+2$ dimensions,

$$\mathcal{I}^{d\,(\ell)}\left[G(D_{i_k}, P(\mathbf{x}^{\text{ISP}}_{\parallel i})\right] \sim \mathcal{I}^{d+2\,(\ell)}\left[1\right]\,. \tag{4.47}$$

These kind of relations simplifies the interpretation of the monomials appearing in the residues $\Delta'_{i_1\cdots i_r}(\mathbf{x}_\parallel)$ in terms of a basis of tensor integrals, since they can be used in order to trade some of the higher-rank integrals with lower rank integrals in higher dimensions.

- Table 4.8 shows that the adaptive integrand decomposition of any arbitrary two-loop factorized topology produces a single, constant coefficient. This is a direct consequence of introducing two independent $d = d_\parallel + d_\perp$ parametrizations for each loop momentum, which, as stated in eq. (4.40), allow us to write all loop variables in terms of denominators. In addition, the division of a numerator of the type (4.43) is effectively equivalent to the individual division of each one-loop numerator by the corresponding subset of denominators. Therefore, as expected, the resulting residue is simply given by the product of constant residues associated to the two individual one-loop subtopologies.

$\mathcal{I}_{i_1\cdots i_n}$		$\Delta_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{12345678910\,11}$		1 $\{1\}$
$\mathcal{I}^{\mathrm{NP1}}_{12345678910\,11}$		1 $\{1\}$
$\mathcal{I}^{\mathrm{NP2}}_{12345678910\,11}$		1 $\{1\}$
$\mathcal{I}^{\mathrm{P}}_{2345678910\,11}$		6 $\{1, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2345678910\,11}$		10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{123457891011}$		6 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{123467891011}$		10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{2346789 10\,11}$		15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{2345789 10\,11}$		33 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2345789 10\,11}$		39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{12345691011}$		15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{2346789 10\,11}$		45 $\{1, x_{41}, x_{42}\}$

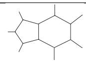
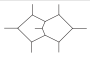
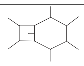
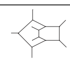
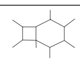
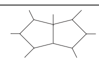
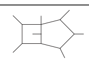
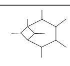
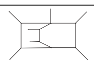
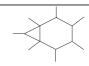
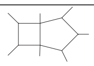
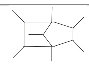
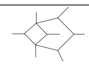
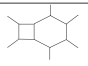
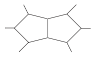
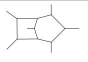
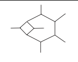
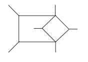
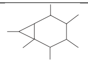
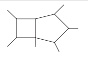
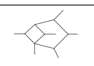
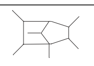
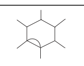
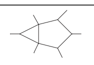
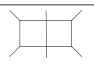
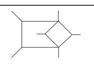
$\mathcal{I}_{i_1\cdots i_r}$		$\Delta_{i_1\cdots i_r}$
$\mathcal{I}^{\mathrm{P}}_{12456789 10\,11}$		6 $\{1, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{12456789 10\,11}$		10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{12345689 10\,11}$		6 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{12456789 10\,11}$		10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{2456789 10\,11}$		15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{1234789 10\,11}$		33 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1245689 10\,11}$		39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1234568 10\,11}$		15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1246789 10\,11}$		45 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{24789 10\,11}$		20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{234789 10\,11}$		76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{245789 10\,11}$		116 $\{1, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{124578 10\,11}$		80 $\{1, x_{31}, x_{41}, x_{42}\}$

Table 4.2: Universal irreducible numerators for two-loop topologies eight- and seven-point topologies. For each residue $\Delta_{i_1\cdots i_n}$ we indicate the number of monomials and the list of their variables.

$\mathcal{I}_{i_1\cdots i_r}$		$\Delta_{i_1\cdots i_r}$
$\mathcal{I}^{\rm P}_{1356789\,10\,11}$		15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\rm P}_{1245679\,10\,11}$		62 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm NP1}_{2356891\,0\,11}$		39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm NP1}_{1234569\,10\,11}$		15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\rm NP2}_{1356789\,10\,11}$		45 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm P}_{25678\,9\,10\,11}$		20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\rm P}_{23568\,9\,10\,11}$		76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm NP1}_{25678\,9\,10\,11}$		80 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm NP1}_{24568\,9\,10\,11}$		116 $\{1, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\rm P}_{3678\,9\,10\,11}$		15 $\{1, x_{11}, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\rm P}_{2578\,9\,10\,11}$		94 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm P}_{2357\,9\,10\,11}$		160 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\rm NP1}_{2457\,9\,10\,11}$		185 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$

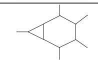
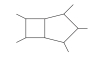
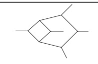
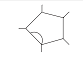
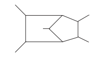
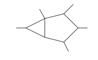
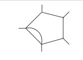
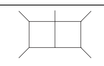
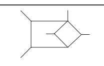
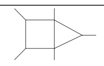
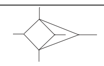
$\mathcal{I}_{i_1\cdots i_r}$		$\Delta_{i_1\cdots i_r}$
$\mathcal{I}^{\rm P}_{15678\,9\,10\,11}$		20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\rm P}_{13567\,9\,10\,11}$		76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm NP1}_{15678\,9\,10\,11}$		80 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm P}_{1678\,9\,10\,11}$		15 $\{1, x_{11}, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\rm NP1}_{13568\,9\,10\,11}$		116 $\{1, x_{31}, x_{32}, x_{42}\}$
$\mathcal{I}^{\rm P}_{1467\,9\,10\,11}$		94 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm P}_{1678\,9\,11}$		66 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\rm P}_{12569\,10\,11}$		160 $\{1, x_{31}, x_{41}, y_{32}, x_{42}\}$
$\mathcal{I}^{\rm NP1}_{13579\,10\,11}$		185 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\rm P}_{12569\,11}$		180 $\{1, x_{11}, x_{31}, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\rm NP1}_{2469\,10\,11}$		246 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$

Table 4.3: Universal irreducible numerators for two-loop six- and five-point topologies. For each residues $\Delta_{i_1\,\cdots\,i_n}$ we indicate the number of monomials and the list of their variables.

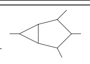
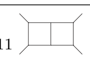
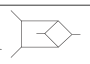
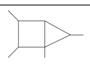
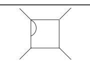
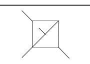
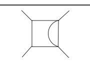
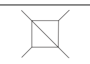
$\mathcal{I}_{i_1\cdots i_r}$		$\Delta_{i_1\cdots i_r}$	$\Delta^{\rm int}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\rm P}_{15679\,10\,11}$		94 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$	53 $\{1, x_{21}, x_{31}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	10 $\{1, x_{21}, x_{31}\}$
$\mathcal{I}^{\rm P}_{12256\,9\,10\,11}$		160 $\{1, x_{31}, x_{41}, x_{32}, x_{42}\}$	93 $\{1, x_{31}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	22 $\{1, x_{31}, x_{32}\}$
$\mathcal{I}^{\rm NP1}_{13569\,10\,11}$		184 $\{1, x_{31}, x_{42}, x_{32}, x_{42}\}$	105 $\{1, x_{31}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	25 $\{1, x_{31}, x_{32}\}$
$\mathcal{I}^{\rm P}_{13568\,11}$		180 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	101 $\{1, x_{31}, x_{22}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	39 $\{1, x_{31}, x_{22}, x_{32}\}$
$\mathcal{I}^{\rm P}_{1689\,10\,11}$		66 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{42}\}$	35 $\{1, x_{11}, x_{21}, x_{31}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	10 $\{1, x_{11}, x_{21}, x_{31}\}$
$\mathcal{I}^{\rm NP1}_{2469\,10\,11}$		245 $\{1, x_{31}, x_{41}, x_{21}, x_{32}, x_{42}\}$	137 $\{1, x_{31}, x_{22}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	55 $\{1, x_{31}, x_{22}, x_{32}\}$
$\mathcal{I}^{\rm P}_{368\,10\,11}$		115 $\{1, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	66 $\{1, x_{31}, x_{12}, x_{22}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	35 $\{1, x_{31}, x_{12}, x_{22}, x_{32}\}$
$\mathcal{I}^{\rm P}_{1368\,11}$		180 $\{1, x_{11}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	103 $\{1, x_{11}, x_{31}, x_{22}, x_{32}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	60 $\{1, x_{11}, x_{31}, x_{22}, x_{32}\}$

Table 4.4: Universal irreducible numerators for two-loop four-point topologies. For each of the residues, $\Delta_{i_1\,\cdots\,i_n}$, $\Delta^{\rm int}_{i_1\,\cdots\,i_n}$ and $\Delta^{'}_{i_1\,\cdots\,i_n}$ we indicate the number of monomials and the list of their variables.

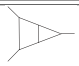
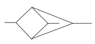
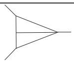
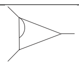
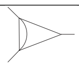
$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\text{int}}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\text{P}}_{1356911}$ 	180 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	22 $\{1, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	4 $\{1, x_{22}\}$
$\mathcal{I}^{\text{NP1}}_{15691011}$ 	240 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	30 $\{1, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{22}\}$
$\mathcal{I}^{\text{P}}_{1571011}$ 	180 $\{1, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	33 $\{1, x_{21}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	13 $\{1, x_{21}, x_{12}\}$
$\mathcal{I}^{\text{P}}_{1691011}$ 	115 $\{1, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	20 $\{1, x_{11}, x_{22}\lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{12}, x_{22}\}$
$\mathcal{I}^{\text{P}}_{361011}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	26 $\{1, x_{11}, x_{21}, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{x_{11}, x_{21}, x_{22}\}$

Table 4.5: Universal irreducible numerators for two-loop three-point topologies. For each of the residues, $\Delta_{i_1\cdots i_n}$, $\Delta^{\text{int}}_{i_1\cdots i_n}$ and $\Delta'_{i_1\cdots i_n}$ we indicate the number of monomials and the list of their variables.

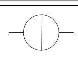
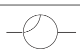
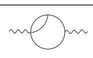
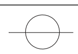
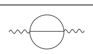
$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\text{int}}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\text{P}}_{561011}$ 	180 $\{1, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	8 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{61011}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_4, x_{22}, y_3, x_{42}\}$	8 $\{1, x_{11}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\text{P}}_{31011}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	26 $\{1, x_{11}, x_{21}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{1, x_{11}, x_{21}, x_{12}\}$
$\mathcal{I}^{\text{P}}_{21011}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	9 $\{1, x_{11}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{11}, x_{12}\}$
$\mathcal{I}^{\text{P}}_{21011}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	18 $\{1, x_{11}, x_{21}, x_{12}, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	15 $\{1, x_{11}, x_{22}, x_{21}, x_{22}\}$

Table 4.6: Universal irreducible numerators for two-loop two-point topologies. For each of the residues, $\Delta_{i_1\cdots i_n}$, $\Delta^{\text{int}}_{i_1\cdots i_n}$ and $\Delta'_{i_1\cdots i_n}$ we indicate the number of monomials and the list of their variables. In the pictures, wavy lines stand for massless particles and solid ones denote propagators with arbitrary masses.

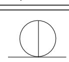
$\mathcal{I}_{i_1\cdots i_r}$	$\Delta_{i_1\cdots i_r}$	$\Delta^{\text{int}}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\text{P}}_{11011}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$

Table 4.7: Irreducible numerators for two-loop one-point topologies. For each of the residues, $\Delta_{i_1\cdots i_n}$, $\Delta^{\text{int}}_{i_1\cdots i_n}$ and $\Delta'_{i_1\cdots i_n}$ we indicate the number of monomials and the list of their variables

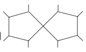
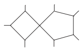
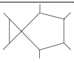
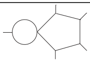
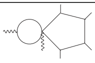
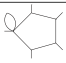
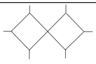
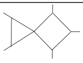
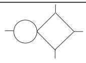
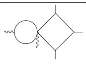
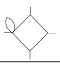
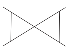
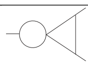
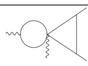
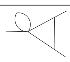
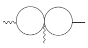
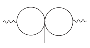
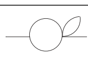
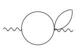
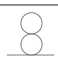
$\mathcal{I}_{i_1\cdots i_r}$		$\Delta_{i_1\cdots i_r}$	$\Delta^{\text{int}}_{i_1\cdots i_r}$	$\Delta'_{i_1\cdots i_r}$
$\mathcal{I}^{\text{P}}_{12345678910}$		1 $\{1\}$	– –	– –
$\mathcal{I}^{\text{P}}_{1245678910}$		5 $\{1, x_{41}\}$	3 $\{1, \lambda_{11}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{125678910}$		10 $\{1, x_{31}, x_{41}\}$	2 $\{1, \lambda_{11}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{15678910}$		10 $\{1, x_{21}, x_{31}, x_{41}\}$	2 $\{1, \lambda_{11}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{12678910}$		10 $\{1, x_{11}, x_{31}, x_{41}\}$	4 $\{1, x_{11}, \lambda_{11}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\text{P}}_{1678910}$		5 $\{1, x_{11}, x_{21}, x_{31}, x_{41}\}$	1 $\{1\}$	– –
$\mathcal{I}^{\text{P}}_{23456789}$		25 $\{1, x_{41}, x_{42}\}$	9 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{2356789}$		50 $\{1, x_{31}, x_{41}, x_{42}\}$	6 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{256789}$		50 $\{1, x_{21}, x_{31}, x_{41}, x_{42}\}$	6 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{236789}$		50 $\{1, x_{11}, x_{31}, x_{41}, x_{42}\}$	12 $\{1, x_{11}, \lambda_{11}, \lambda_{22}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\text{P}}_{26789}$		25 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{42}\}$	3 $\{1, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{245689}$		100 $\{1, x_{31}, x_{42}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{24689}$		100 $\{1, x_{21}, x_{31}, x_{41}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{45689}$		100 $\{1, x_{11}, x_{31}, x_{41}, x_{32}, x_{42}\}$	8 $\{1, x_{11}, \lambda_{11}, \lambda_{22}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\text{P}}_{2689}$		50 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{32}, x_{42}\}$	2 $\{1, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{2569}$		100 $\{1, x_{11}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{4569}$		100 $\{1, x_{11}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	8 $\{1, x_{11}, \lambda_{11}, \lambda_{22}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\text{P}}_{4568}$		100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{32}, x_{42}\}$	16 $\{1, x_{11}, x_{12}, \lambda_{11}, \lambda_{22}\}$	9 $\{1, x_{11}, x_{12}\}$
$\mathcal{I}^{\text{P}}_{269}$		50 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	2 $\{1, \lambda_{22}\}$	1 $\{1\}$
$\mathcal{I}^{\text{P}}_{268}$		50 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	4 $\{1, x_{12}, \lambda_{22}\}$	3 $\{x_{12}\}$
$\mathcal{I}^{\text{P}}_{29}$		25 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	1 $\{1\}$	– –

Table 4.8: Universal irreducible numerators for two-loop factorized topologies. For each of the residues, $\Delta_{i_1\cdots i_n}$, $\Delta^{\text{int}}_{i_1\cdots i_n}$ and $\Delta'_{i_1\cdots i_n}$ we indicate the number of monomials and the list of their variables. In the pictures, wavy lines stand for massless particles and solid ones denote propagators with arbitrary masses.

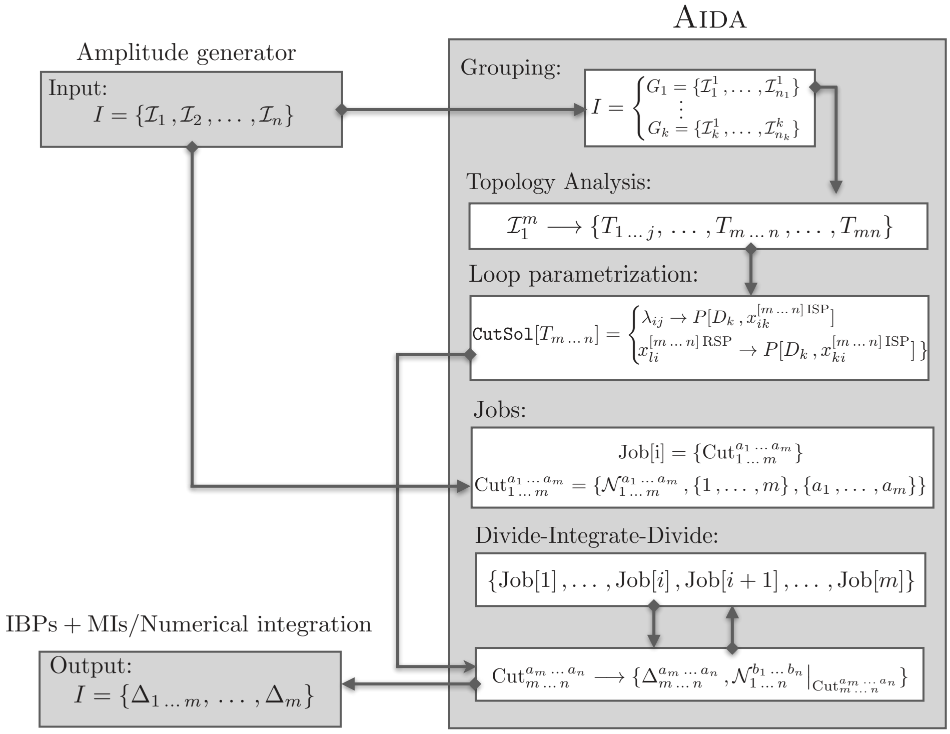


Figure 4.2: AIDA

4.4 AIDA: a MATHEMATICA implementation

In this section, we introduce AIDA (Adaptive Integrand Decomposition Algorithm) a MATHEMATICA implementation of the divide-integrate-divide algorithm proposed in section 4.2. The code performs the integrand decomposition of one- and two-loop amplitudes, both analytically and numerically. In the following, we describe the chain of operations performed by AIDA, which are summurized by the chart in figure 4.2, by referring explicitly to the two-loop case. As a toy example, we illustrate step-by-step the use of this code for the analytic integrand decomposition of the two-loop vacuum polarization in QED.

Input

The input amplitude I of AIDA is generally written as a list of integrands

$$I = \{\mathcal{I}_1\,, \mathcal{I}_2\,, \dots\,, \mathcal{I}_n\}\,, \tag{4.48}$$

where each integrand $\mathcal{I}_k$ in turn consists of a list specifying the numerator function, the set of loop propagators and their corresponding powers,

$$\mathcal{I}_k = \frac{\mathcal{N}_k}{D_1^{a_1}\cdots D_m^{a_m}} \equiv \{\mathcal{N}_k\,, \{D_1\,, \dots\,, D_j\},\, \{a_1\,, \dots\,, a_j\}\}\,, \quad a_i \in \mathbb{N}\,. \tag{4.49}$$

The loop propagators are denoted by

$$D_k = (l_k^2 - m_k^2) \equiv \{\, l_k\,,\, m^2\,\}\,, \tag{4.50}$$

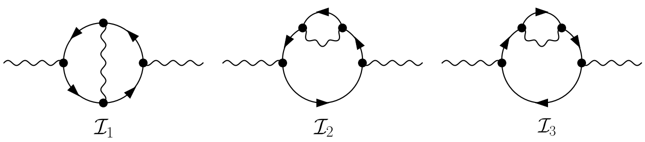


Figure 4.3: Feynman integrals for the two-loop vacuum polarization

where $l_k = \sum_{i=1}^{\ell} \alpha_{ik} q_i + \sum_{i=1}^{n-1} \beta_{ik} p_i$, is the momentum flow in the k-th propagator, which is given by a linear combination of loop momenta q_i and external momenta p_i. If the integrands are built from Feynman diagrams, the input can be easily obtained from the output of commonly used diagrams generators, such as FeynArts [162, 163] or QGraf [164].

Example: The input amplitude for the two-loop vacuum polarization in QED,

$$I = \{\mathcal{I}_1\,, \mathcal{I}_2\,, \mathcal{I}_3\}\,, \tag{4.51}$$

consists of the three Feynman diagrams shown in figure 4.3, which have been produced with FeynArts and algebraically manipulated with the help of the package FeynCalc [165]. For our puposes, it is convenient to consider the contraction of each Feynman diagram with the transverse operator

$$P^{\mu\nu}(p) = g^{\mu\nu} - \frac{p^\mu p^\mu}{p^2}\,. \tag{4.52}$$

In this way, we obtain

```
In[1]:= Int[1]
Out[1]= {Num[1], {{q1, m2}, {p + q1, m2}, {-p + q2, m2}, {q2, m2}, {q1 + q2, 0}}, {1, 1, 1, 1, 1}}

In[2]:= Int[2]
Out[2]= {Num[2], {{q1, m2}, {-p + q1, m2}, {q2, m2}, {p - q1 + q2, 0}}, {1, 2, 1, 1}}

In[3]:= Int[3]
Out[3]= {Num[3], {{q1, m2}, {-p + q1, m2}, {q2, m2}, {p - q1 + q2, 0}}, {1, 2, 1, 1}}
```

with `p` being the external momentum and `m2` the square mass of the electron. For istance, the numerator function of the first integrand is

```
In[1]:= Num[1]
Out[1]= 1/s 4 (-2 (-2 + d) (p.q2)^2 (m2 - q1.q1 + μ1,1) +
      (-2 + d) s p.q2 (-m2 + d m2 - (-3 + d) q1.q1 + 2 q1.q2 - 3 μ1,1 + d μ1,1 - 2 μ1,2) -
      2 (-2 + d) (p.q1)^2 (m2 - q2.q2 + μ2,2) +
      s (2 m2^2 - 3 d m2^2 + d^2 m2^2 - 4 m2 s + 5 d m2 s - d^2 m2 s + 4 (-2 + d) (q1.q2)^2 -
          6 m2 q2.q2 + 5 d m2 q2.q2 - d^2 m2 q2.q2 + 6 m2 μ1,1 - 5 d m2 μ1,1 + d^2 m2 μ1,1 -
          10 q2.q2 μ1,1 + 7 d q2.q2 μ1,1 - d^2 q2.q2 μ1,1 - 4 d m2 μ1,2 + 10 s μ1,2 - 7 d s μ1,2 +
          d^2 s μ1,2 - 8 μ1,2^2 + 4 d μ1,2^2 + q1.q2 (4 d m2 - 10 s + 7 d s - d^2 s - 8 (-2 + d) μ1,2) +
          6 m2 μ2,2 - 5 d m2 μ2,2 + d^2 m2 μ2,2 + 10 μ1,1 μ2,2 - 7 d μ1,1 μ2,2 + d^2 μ1,1 μ2,2 -
          (-2 + d) q1.q1 ((-3 + d) m2 - (-5 + d) q2.q2 + (-5 + d) μ2,2)) -
      p.q1 (4 p.q2 (d m2 - 2 s + d s + (-2 + d) q1.q2 - (-2 + d) μ1,2) +
          (-2 + d) s (-m2 + d m2 + 2 q1.q2 - (-3 + d) q2.q2 - 2 μ1,2 - 3 μ2,2 + d μ2,2)))
```

where `a.b` $= a \cdot b$ indicates four-dimensional scalar products and `s` $= p^2$. In our convetions, $16\pi^2\alpha^2 = 1$. Similar expressions can be generated for $\mathcal{N}_2$ and $\mathcal{N}_3$. □

Grouping

In order to make the reduction more efficient, it is useful to group I into subsets G_k of integrands which can be processed simultaneously,

$$I = \bigcup_k G_k\,, \quad G_k = \{\mathcal{I}_1^{G_k}\,, \mathcal{I}_2^{G_k}\,, \dots\,, \mathcal{I}_m^{G_k}\}\,, \tag{4.53}$$

where, within each G_k, $\mathcal{I}_1^{G_k}$ is chosen to be the integrand whose set of denominators contains as subsets the denominators of all other $\mathcal{I}_j^{G_k}$. In particular, it is convenient to define

$$\mathcal{I}_1^{G_k} = \{\mathcal{N}_1^{G_k}\,, \{D_1\,, \dots\,, D_j\}, \{\max_{G_k}(a_1)\,, \dots\,, \max_{G_k}(a_j)\}\}\,. \tag{4.54}$$

In this way, all members of G_k can be interpreted as subgraphs of $\mathcal{I}_1^{G_k}$, i.e. they can be obtained through the pinching of a certain set of its denominators. For this reason, we can refer to $\mathcal{I}_1^{G_k}$ as the *parent integrand*. Obviously, $\mathcal{I}_1^{G_k}$, as defined in eq. (4.54), might not be present in the original set of integrands I. In such case, $\mathcal{N}_1^{G_k}$ is simply initialized to zero.

Generally, in order to identify an integrand $\mathcal{I}_j$ which belongs to G_k, we need to determine a shift of the loop momenta

$$\begin{cases} q_1 \to \alpha_1 q_1 + \alpha_2 q_2 + \sum_j \eta_j p_j\,, \\ q_2 \to \beta_1 q_1 + \beta_2 q_2 + \sum_j \sigma_j p_j\,, \quad \text{with} \quad \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0\,, \end{cases} \tag{4.55}$$

which maps the denominators of $\mathcal{I}_k$ into (a subset of) of the denominators of $\mathcal{I}_1^{G_k}$. If such a shift exists, eq.(4.55) is applied to $\mathcal{I}_k$ and the integrand is added to G_k. In addition, if the sets of denominators of two or more integrands are mapped exactly into each other through a proper shift of the loop momenta, they can be added to G_k as a single integrand, whose numerator is given by the sum of the corresponding (shifted) numerators.

Example: In the case of the two-loop vacuum polarization, it is easy to see that $\mathcal{I}_2$ and $\mathcal{I}_3$ share the same set of loop denominators and, hence, they can be merged into a single integrand with numerator $\mathcal{N}_2 + \mathcal{N}_3$. Moreover, by performing the shift

$$\begin{cases} q_1 \to -q_1\,, \\ q_2 \to q_2 - p \end{cases} \tag{4.56}$$

the denominators of $\mathcal{I}_{2,3}$ are mapped into a subset of the denominators of $\mathcal{I}_1$. Thus, the full amplitude I corresponds to a single group, defined as

$$G = \{\mathcal{I}_1^G\,, \mathcal{I}_2^G\,, \mathcal{I}_3^G\} \tag{4.57}$$

with

```
In[1]:= IntG[1]
Out[1]= {0, {{q1, m2}, {p + q1, m2}, {-p + q2, m2}, {q2, m2}, {q1 + q2, 0}}, {1, 2, 1, 1, 1}}

In[2]:= IntG[2]
Out[2]= {Num[1], {{q1, m2}, {p + q1, m2}, {-p + q2, m2}, {q2, m2}, {q1 + q2, 0}}, {1, 1, 1, 1, 1}}

In[3]:= IntG[3]
Out[3]= {Num[2] + Num[3],
          {{q1, m2}, {p + q1, m2}, {-p + q2, m2}, {q2, m2}, {q1 + q2, 0}}, {1, 2, 1, 0, 1}}
```

$\mathcal{I}_1^G$ does not correspond to any of the Feynman diagrams of figure 4.3 (since it has five distinct denominators and one them is squared) and, consistently, its numerator has been set to zero. The other two integrands are, respectively, $\mathcal{I}_2^G = \mathcal{I}_1$ and $\mathcal{I}_3^G = \mathcal{I}_2 + \mathcal{I}_3$.□

Topology Analysis

Having determined the set of parent integrands $\mathcal{I}_1^G$ that contribute to I, we can analyze their graph structure in order to extract the kinematic information needed for the definition of the adaptive parametrization.

First of all, the list of denominators of each $\mathcal{I}_1^G$ is translated into a `Graph`, i.e. a list of connected vertices, where each internal edge corresponds to a loop denominator.

By systematically merging the vertices of the parent graph, we can generate the graphs associated to all possible multiple-cuts of $\mathcal{I}_1^G$. Every time two vertices are merged, the use of the `Graph` representation of the integrand allows us to easily reconstruct, by applying momentum conservation, the momentum flow of the corresponding subgraph. After all subgraphs corresponding to $\mathcal{I}_1^G$ have been obtained, each of them is translated into a topology $T_{m\,\ldots\,n}$, i.e. a list containing all the relevant information which will be used in the next steps of the algorithm. This set of operations is performed by the function `FindTopo`,

$$\texttt{FindTopo}[\mathcal{I}_1^{G_k}] = \{T_{1\,\ldots\,j}, \, \ldots \, , T_{m\,\ldots\,n} \, , \ldots \, , T_{mn}\} \, . \tag{4.58}$$

A m-denominator non-factorized topology $T_{1\,\ldots\,m}$ (with m_1 denominators depending on q_1 and m_2 denominators depending on q_2) is identified by the list

$$\begin{aligned} T_{1\,\ldots\,m} = \{ & \mathrm{NF} \, , \, \{q_1 \, , q_2\} \, , \, \{p_1 \, , \ldots \, , \, p_m\}, \\ & \{\{D_1 \, , \ldots \, , D_{m_1}\} \, , \{D_{m_1+1} \, , \ldots \, , D_{m_1+m_2}\} \, , \{D_{m_1+m_2+1} \, , \ldots \, , D_m\}\}\} \, , \end{aligned} \tag{4.59}$$

which collects, in order, the set of loop momenta, the list of external momenta and the set denominators which, in turn, are split into three partitions, according to their dependence, respectively, on q_1, q_2 and $q_1 \pm q_2$.

A m-denominator factorized topology $T_{1\,\ldots\,m}$ (with m_1 denominators depending on q_1 and $m_2 = m - m_1$ denominators depending on q_2) is identified by the two separate lists, collecting loop momentum, external momenta and denominators of each sub-loop,

$$\begin{aligned} T_{1\,\ldots\,m} = \{ & \mathrm{F} \, , \{\{q_1\} \, , \{p_1 \, , .. \, p_{m_1}\}, \{D_1 \, , \ldots \, , D_{m_1}\}\} \, , \\ & \{\{q_2\} \, , \{p_{m_1+1} \, , .. \, p_{m_2}\}, \{D_{m1+1} \, , \ldots \, , D_{m_2}\}\}\} \, . \end{aligned} \tag{4.60}$$

`FindTopo` is applied to all parent integrands $\mathcal{I}_1^{G_k}$. In this way, we can associate a topology to each of the multiple-cuts contributing to I.

Example: The parent integrand $\mathcal{I}_1^G$ of the two-loop vacuum polarization, defines 24 distinct topologies, whose `Graph` representation is depicted in figure 4.4. As an example, we list below the internal representation of the parent topology T_{12345} and of the four-denominator factorized topology T_{1234},

```
In[1]:= T[{{1, 2}, {3, 4}, {5}}]
Out[1]= {NF, {q1, q2}, {p}, {{{q1, m2}, {p + q1, m2}}, {{-p + q2, m2}, {q2, m2}}, {{q1 + q2, 0}}}}

In[2]:= T[{{1, 2}, {3, 4}}]
Out[2]= {F, {{q1}, {p}, {{q1, m2}, {p + q1, m2}}}, {{q2}, {p}, {{-p + q2, m2}, {q2, m2}}}}
```

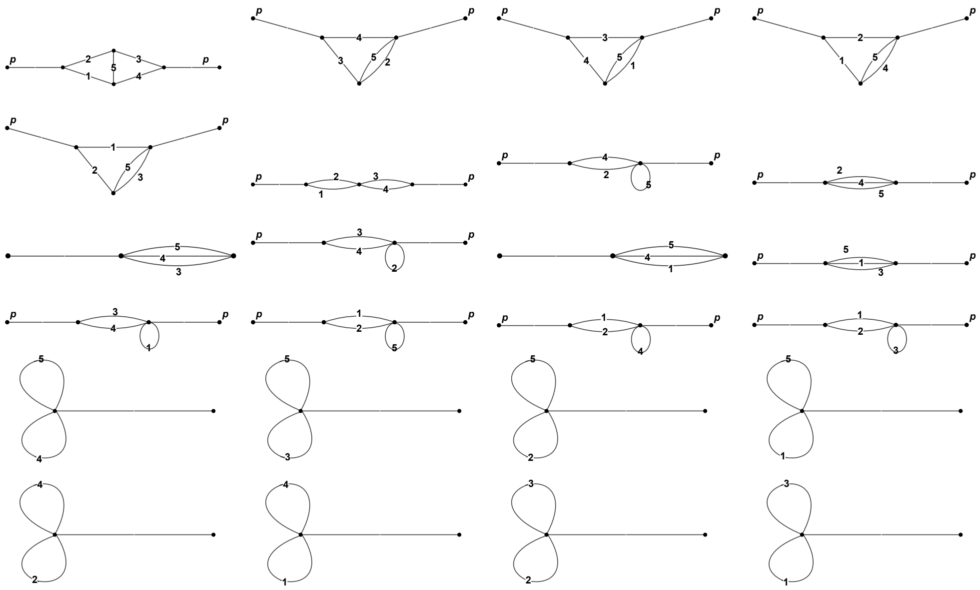


Figure 4.4: Relevant topologies $T_{m\,\ldots\,n}$ for the two-loop vacuum polarization. □

Loop parametrization

The knowledge of the external momenta of each topology allows us to introduce the proper $d = d_\parallel + d_\perp$ parametrization for every $T_{1\,\ldots\,m}$. To this aim, the routine `LoopPar` builds the four-dimensional basis $\mathcal{E} = \{e_1^{[1\,\ldots\,m]}, \ldots, e_4^{[1\,\ldots\,m]}\}$, as defined in eq.(2.33), and produces the decomposition of the loop momenta in terms of the adaptive variables,

$$\texttt{LoopPar}[T_{1\,\ldots\,m}] = \{ q_{i\,\parallel}^{[1\,\ldots\,m]} \to \sum_{j=1}^{d_\parallel} x_{j\,i}^{[1\,\ldots\,m]} e_j^{[1\,\ldots\,m]} \\ \lambda_{ij}^{[1\,\ldots\,m]} \to \sum_{k=d_{\parallel+1}}^{4} x_{k\,i}^{[1\,\ldots\,m]} x_{k\,j}^{[1\,\ldots\,m]} e_k^{[1\,\ldots\,m]} \cdot e_k^{[1\,\ldots\,m]} + \mu_{ij} \} \,. \qquad (4.61)$$

This decomposition is applied by the routine `CutSol` to the definition of the loop denominators, in order to build and solve the system of linear equations expressing λ_{ij} and the reducible physical directions $\mathbf{x}_{\parallel\,i}^{\mathrm{RSP}}$ in terms of denominators and physical ISPs,

$$\texttt{CutSol}[T_{1\,\ldots\,m}] = \{ \lambda_{ij} \to P[D_k \,, x_{ik}^{[1\,\ldots\,m]\,\mathrm{ISP}}] \,,$$

$$x_{li}^{[1\,\dots\,m]\,\text{RSP}} \to P[D_k\,, x_{ki}^{[1\,\dots\,m]\,\text{ISP}}]\,\}\,, \quad i,j=1,2\,. \tag{4.62}$$

In the initialization phase of AIDA, `CutSol` is systematically applied to all $T_{1\,\dots\,m}$ and the resulting substitution rules, which constitute the core of the entire reduction algorithm, are stored for later use in the division step. Inverse substitution rules, which express the physical ISPs $\mathbf{x}_{\parallel\, i}$ in terms of scalar products between loop momenta and external momenta, are generated and stored as well.

Example: Let us consider the topology T_{1235}, whose corresponding `Graph` is shown in figure 4.4. Being $d_\parallel = 1$, the two-loop momenta q_1 and q_2 have one single physical component. Thus, we have

$$\boldsymbol{\tau} = \{x_{11}^{[1235]}, x_{12}^{[1235]}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}\,. \tag{4.63}$$

The system of four linear equations obtained from differences of denominators allows us to express $x_{11}^{[1235]}$ and λ_{ij} as combinations of denominators and $x_{12}^{[1235]}$, which is the only physical ISP. Consistently, `CutSol`$[T_{1235}]$ returns

```
In[1]:= CutSol[T {{1, 2}, {3} {5}}]

Out[1]= {x[1][1, {{1, 2}, {3}, {5}}] → - (s + d[1] - d[2]) / (2 s),
   λ1,1[{{1, 2}, {3}, {5}}] → (4 m2 s - s^2 + 2 s d[1] - d[1]^2 + 2 s d[2] + 2 d[1] d[2] - d[2]^2) / (4 s),
   λ1,2[{{1, 2}, {3}, {5}}] → 1/2 (-2 m2 - d[2] - d[3] + d[5] - s x1,2[{{1, 2}, {3}, {5}}] +
        d[1] x1,2[{{1, 2}, {3}, {5}}] - d[2] x1,2[{{1, 2}, {3}, {5}}]),
   λ2,2[{{1, 2}, {3}, {5}}] → m2 + d[3] - s x1,2[{{1, 2}, {3}, {5}}]^2}
```

where `d[i]` $= D_i$. □

Jobs

Each integrand $\mathcal{I}_j^{G_k}$ in the group G_k corresponds to one of the topologies $T_{m\,\dots n}$ generated from the parent integrand $\mathcal{I}_j^{G_k}$. If in $\mathcal{I}_1^{G_k}$ all denominators are raised to power one, then every $\mathcal{I}_j^{G_k}$ corresponds to a different $T_{m\,\dots n}$. Conversely, when dealing with higher powers of denominators, different integrands (which share the same set of denominators but have different powers) are associated to the same topology. Integrands are distributed to the corresponding topologies through the definition of cuts $\text{Cut}_{1\,\dots\,m}^{a_1\,\dots\,a_m}$,

$$\begin{aligned}\text{Cut}_{1\,\dots\,m}^{a_1\,\dots\,a_m} = \{&\mathcal{N}_{1\,\dots\,m}^{a_1\,\dots\,a_m}\,,\\ &\{\{1\,,\dots\,,m_1\}\,,\{m_1+1\,,\dots\,,m_1+m_2\}\,,\{m_1+m_2+1\,,\dots\,,m\}\}\,,\\ &\{\{a_1\,,\dots\,,a_{m_1}\,,a_{m_1+1}\,,\dots\,,a_{m_1+m_2}\,,a_{m_1+m_2+1}\,,\dots\,,a_m\}\}\,.\end{aligned} \tag{4.64}$$

We assign to each cut $\text{Cut}_{1\,\dots\,m}^{a_1\,\dots\,a_m}$ the numerator of the integrand $\mathcal{I}_j^{G_k}$ with denominators $D_1\,,\dots\,,D_m$ raised to powers $\{a_1\,,\dots\,,a_m\}$. If no such integrand exists, $\mathcal{N}_{1\,\dots\,m}^{a_1\,\dots\,a_m}$ is initialized to zero. Cuts which are not associated to any integrand of G_k might acquire non-trivial numerator during the division algorithm.

The assignments are made by the `JobList` routine,

$$\texttt{JobList}[G_k, \{T_{1\,\dots\,j}\,,\dots\,,T_{m\,\dots\,n}\,,\dots\,,T_{mn}\}] = \{\text{Job}[1]\,,\dots\,,\text{Job}[k]\}\,, \tag{4.65}$$

which defines all cuts $\text{Cut}_{1\,\dots\,m}^{a_1\,\cdots\,a_m}$ and organizes them into sublists, we refer to as Jobs. Each Job[i] consists of the set of cuts $\text{Cut}_{1\,\dots\,m}^{a_1\,\cdots\,a_m}$ which can receive a contribution from the polynomial division of the cuts contained in Job[1] , ... , Job[$i-1$]. Job[1] contains only the cut associated to the parent integrand, since the quotients produced by the division of the numerator of $\mathcal{I}_1^{G_k}$ can, in principle, contribute to all other cuts. By construction, the quotient generated by $\text{Cut}_{1\,\dots\,m}^{a_1\,\cdots\,a_m} \in$ Job[i] does not contain terms belonging to any other cuts in Job[i]. Therefore, the division of all the cuts contained in a given Job can be parallelized.

Example: The integrand decomposition of the two-loop vacuum polarization requires the evaluation of 38 cuts, which are organized in the 8 distinct Jobs:

```
In[1]:= Job[1]
Out[1]= {Cut[{{1, 2}, {3, 4}, {5}}, {1, 2, 1, 1, 1}]}

In[2]:= Job[2]
Out[2]= {Cut[{{1, 2}, {3, 4}, {5}}, {1, 1, 1, 1, 1}]}

In[3]:= Job[3]
Out[3]= {Cut[{{2}, {3, 4}, {5}}, {0, 2, 1, 1, 1}], Cut[{{1, 2}, {4}, {5}}, {1, 2, 0, 1, 1}],
         Cut[{{1, 2}, {3}, {5}}, {1, 2, 1, 0, 1}], Cut[{{1, 2}, {3, 4}}, {1, 2, 1, 1, 0}]}

In[4]:= Job[4]
Out[4]= {Cut[{{2}, {3, 4}, {5}}, {0, 1, 1, 1, 1}],
         Cut[{{1}, {3, 4}, {5}}, {1, 0, 1, 1, 1}], Cut[{{1, 2}, {4}, {5}}, {1, 1, 0, 1, 1}],
         Cut[{{1, 2}, {3}, {5}}, {1, 1, 1, 0, 1}], Cut[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]}

In[5]:= Job[5]
Out[5]= {Cut[{{2}, {4}, {5}}, {0, 2, 0, 1, 1}], Cut[{{2}, {3}, {5}}, {0, 2, 1, 0, 1}],
         Cut[{{2}, {3, 4}}, {0, 2, 1, 1, 0}], Cut[{{1, 2}, {5}}, {1, 2, 0, 0, 1}],
         Cut[{{1, 2}, {4}}, {1, 2, 0, 1, 0}], Cut[{{1, 2}, {3}}, {1, 2, 1, 0, 0}]}

In[6]:= Job[6]
Out[6]= {Cut[{{3, 4}, {5}}, {0, 0, 1, 1, 1}], Cut[{{2}, {4}, {5}}, {0, 1, 0, 1, 1}],
         Cut[{{2}, {3}, {5}}, {0, 1, 1, 0, 1}], Cut[{{2}, {3, 4}}, {0, 1, 1, 1, 0}],
         Cut[{{1}, {4}, {5}}, {1, 0, 0, 1, 1}], Cut[{{1}, {3}, {5}}, {1, 0, 1, 0, 1}],
         Cut[{{1}, {3, 4}}, {1, 0, 1, 1, 0}], Cut[{{1, 2}, {5}}, {1, 1, 0, 0, 1}],
         Cut[{{1, 2}, {4}}, {1, 1, 0, 1, 0}], Cut[{{1, 2}, {3}}, {1, 1, 1, 0, 0}]}

In[7]:= Job[7]
Out[7]= {Cut[{{2}, {5}}, {0, 2, 0, 0, 1}],
         Cut[{{2}, {4}}, {0, 2, 0, 1, 0}], Cut[{{2}, {3}}, {0, 2, 1, 0, 0}]}

In[8]:= Job[8]
Out[8]= {Cut[{{4}, {5}}, {0, 0, 0, 1, 1}], Cut[{{3}, {5}}, {0, 0, 1, 0, 1}],
         Cut[{{2}, {5}}, {0, 1, 0, 0, 1}], Cut[{{2}, {4}}, {0, 1, 0, 1, 0}],
         Cut[{{2}, {3}}, {0, 1, 1, 0, 0}], Cut[{{1}, {5}}, {1, 0, 0, 0, 1}],
         Cut[{{1}, {4}}, {1, 0, 0, 1, 0}], Cut[{{1}, {3}}, {1, 0, 1, 0, 0}]}
```

where `Cut[{m,...,n},{am,...a_n}]` $= \text{Cut}_{n\,\dots\,m}^{a_n\,\cdots\,a_m}$. The cuts

$$\begin{aligned}
\text{Cut}_{12345}^{12111} =& \{0\,,\{\{1,2\}\,,\{3,4\}\,,\{5\}\}\,,\{1,2,1,1,1\}\}\,,\\
\text{Cut}_{12345}^{11111} =& \{\mathcal{N}_1\,,\{\{1,2\}\,,\{3,4\}\,,\{5\}\}\,,\{1,1,1,1,1\}\}\,,\\
\text{Cut}_{1245}^{12101} =& \{\mathcal{N}_2+\mathcal{N}_3\,,\{\{1,2\}\,,\{3\}\,,\{5\}\}\,,\{1,2,1,0,1\}\}\,,
\end{aligned} \tag{4.66}$$

correspond, respectively, to the integrands $\mathcal{I}_1^G$, $\mathcal{I}_2^G$, and $\mathcal{I}_3^G$ of the group G. The numerator of all other cuts are initially set to zero. □

Divide-Integrate-Divide

The Job-organization of all cuts $\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}$ that contribute to the group G_k completes the initialization phase of the algorithm. Having stored the loop parametrization (4.61) and the substitution rules (4.62) for all topologies, we can finally proceed to the division step.

The division is applied sequentially to each Job, starting from Job[1]. When we enter Job[i], the function `BuilNum` redefines the numerator of each $\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots\,a_n} \in \text{Job}[i]$ as follows:

$$\mathcal{N}_{m\,\dots\,n}^{a_m\,\cdots\,a_n} \to \texttt{BuilNum}[\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}] = \mathcal{N}_{m\,\dots\,n}^{a_m\,\cdots\,a_n} + \sum_{\text{Cut}_{k\dots l}^{b_k\dots b_l} \in \text{Job}[j<i]} \mathcal{N}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}\big|_{\text{Cut}_{k\dots l}^{b_k\dots b_l}}\,, \tag{4.67}$$

where $\mathcal{N}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}\big|_{\text{Cut}_{k\dots l}^{b_k\dots b_l}}$ is the contribution to the numerator of $\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}$ originated from the division of the numerator associated to $\text{Cut}_{k\dots l}^{b_k\dots b_l}$. By definition, $\text{Cut}_{k\dots l}^{b_k\dots b_l}$ belongs to a Job[j] with $j < i$. Therefore, in order not to miss any contribution to $\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots,a_n}$, it is important to process Job[i] after all previous Jobs have been completed.

After the numerator of $\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots,a_n}$ has been redefined through eq. (4.67), the adaptive parametrization of eq. (4.61) of the loop momenta is applied. Finally the `Divide` function acts on the numerator $\mathcal{N}_{m\,\dots\,n}^{a_m\,\cdots,a_n}$, plugs in the substitution rules (4.62) of the corresponding topology $T_{m\,\dots\,n}$ and, by collecting the coefficients of the different monomials in $\{D_m\,, \dots\,, D_n\}$, determines the residue $\Delta_{m\,\dots\,n}^{a_m\,\cdots\,a_n}$ and stores the numerators $\mathcal{N}_{k\,\dots\,l}^{b_k\,\dots\,b_l}\big|_{\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}}$ which will be processed in the next Jobs,

$$\texttt{Divide}[\mathcal{N}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}, \texttt{CutSol}[T_{1\,\dots\,m}]] = \{\Delta_{m\,\dots\,n}^{a_m\,\cdots\,a_n}\,, \mathcal{N}_{1\,\dots\,n}^{b_1\,\dots\,b_n}\big|_{\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}}\}\,. \tag{4.68}$$

The integration over the transverse directions $\mathbf{x}_{\perp,i}$ is performed through a set of substitutions rules `PerpIntegrate`[$T_{m\,\dots\,n}$], which selects the result of appendices A.2-A.3 compatible with the $d = d_{\parallel} + d_{\perp}$ parametrization of the topology $T_{m\,\dots\,n}$. These rules can be applied to $\mathcal{N}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}$ at any point, as soon as the loop parametrization of eq. (4.61) has been introduced. If the integration is performed before the `Divide` function is called, the residues obtained from eq. (4.68) will be free of spurious terms, and they will directly correspond to the residues $\Delta_{1\,\dots\,m}^{\prime\, a_1\,\dots\,a_m}$ of the final decomposition of I.

Conversely, if no integration is applied prior to division, we can integrate each $\Delta_{1\,\dots\,m}^{a_1\,\dots\,a_m}$, run again

$$\texttt{Divide}[\Delta_{1\,\dots\,m}^{\text{int}\,a_1\,\dots\,a_m}, \texttt{CutSol}[T_{1\,\dots\,m}]] = \{\Delta_{1\,\dots\,m}^{\prime\, a_1\,\dots\,a_m}\,, \mathcal{N}_{1\,\dots\,n}^{b_1\,\dots\,b_n}\big|_{\text{Cut}_{m\,\dots\,n}^{a_m\,\cdots\,a_n}}\} \tag{4.69}$$

through all Jobs and obtain the final decomposition of I.

Example: Let us consider the factorized cut $\text{Cut}_{1234}^{11110}$, which belongs to Job[4]. The initial value of $\mathcal{N}_{1234}^{1110}$ is zero, since none of the Feynman diagrams of figure 4.3 belongs to the topology T_{1234}. However, additional contributions to $\text{Cut}_{1234}^{11110}$ might come from the division of cuts with a larger number of denominators. Therefore, before the division is performed, `BuildNum` redefines the numerator of $\text{Cut}_{1234}^{11110}$ by summing all possible contributions generated during the reduction of Job[1], Job[2] and Job[3],

```
In[1]:= BuildNum[Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]]

Out[1]= Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}] +
         Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}, {{1, 2}, {3, 4}}, {1, 2, 1, 1, 0}] +
         Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}, {{1, 2}, {3, 4}, {5}}, {1, 1, 1, 1, 1}] +
         Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}, {{1, 2}, {3, 4}, {5}}, {1, 2, 1, 1, 1}]
```

with

```
In[1]:= Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]

Out[1]= 0

In[2]:= Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}, {{1, 2}, {3, 4}}, {1, 2, 1, 1, 0}]

Out[2]= 0

In[3]:= Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}, {{1, 2}, {3, 4}, {5}}, {1, 1, 1, 1, 1}]

Out[3]= -8 (-4 + d) m2 - 2 (-2 + d) ((-8 + d) s - 2 (q1.q1 - 2 q1.q2 + q2.q2) + 2 (μ1,1 + μ2,2))

In[4]:= Num[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}, {{1, 2}, {3, 4}, {5}}, {1, 2, 1, 1, 1}]

Out[4]= 0
```

After introducing the $d = d_{\|} + d_{\perp}$ parametrization of the loop momenta associated to T_{1234} and integrating over the transverse components, the numerator is passed to `Divide`, which returns the residue of $\mathrm{Cut}_{1234}^{11110}$ and distributes the quotients to the numerators of lower cuts,

```
In[11]:= Δ[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]

Out[11]= 2 (8 m2 - (14 - 9 d + d^2) s)

In[12]:= Num[{{1, 2}, {3}}, {1, 1, 1, 0, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]

Out[12]= 2 (-2 + d)

In[13]:= Num[{{1, 2}, {4}}, {1, 1, 0, 1, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]

Out[13]= 2 (-2 + d)

In[14]:= Num[{{1}, {3, 4}}, {1, 0, 1, 1, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]

Out[14]= 2 (-2 + d)

In[15]:= Num[{{2}, {3, 4}}, {0, 1, 1, 1, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]

Out[15]= 2 (-2 + d)
```

```
In[17]:= Num[{{1}, {3}}, {1, 0, 1, 0, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]
```

Out[17]= $\frac{2\,(-2+d)}{s}$

```
In[18]:= Num[{{1}, {4}}, {1, 0, 0, 1, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]
```

Out[18]= $\frac{2\,(2-d)}{s}$

```
In[19]:= Num[{{2}, {3}}, {0, 1, 1, 0, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]
```

Out[19]= $\frac{2\,(2-d)}{s}$

```
In[20]:= Num[{{2}, {4}}, {0, 1, 0, 1, 0}, {{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}]
```

Out[20]= $\frac{2\,(-2+d)}{s}$

This procedure is iterated over all Jobs, one after the other. As a result, we obtain the full set of non vanishing residues which contains a total number of 31 monomials

Δ[{{1, 2}, {3, 4}, {5}}, {1, 1, 1, 1, 1}] == $-4\left(8\,m2^2-8\,m2\,s+2\,d\,m2\,s+2\,s^2-d\,s^2\right)$

Δ[{{1, 2}, {3}, {5}}, {1, 2, 1, 0, 1}] == $-16\left(4\,m2^2-2\,m2\,s+d\,m2\,s\right)$

Δ[{{2}, {3, 4}, {5}}, {0, 1, 1, 1, 1}] == $-4\,(2\,m2-2\,s+d\,s)$

Δ[{{1}, {3, 4}, {5}}, {1, 0, 1, 1, 1}] == $-4\,(2\,m2-2\,s+d\,s)$

Δ[{{1, 2}, {4}, {5}}, {1, 1, 0, 1, 1}] == $-4\,(2\,m2-2\,s+d\,s)$

Δ[{{1, 2}, {3}, {5}}, {1, 1, 1, 0, 1}] ==
$-4\,(10\,m2-6\,s+5\,d\,s-d^2\,s)+8\,(4\,s-4\,d\,s+d^2\,s)\,x_{1,2}$[{{1, 2}, {3}, {5}}]

Δ[{{1, 2}, {3, 4}}, {1, 1, 1, 1, 0}] == $2\,(8\,m2-14\,s+9\,d\,s-d^2\,s)$

Δ[{{2}, {3}, {5}}, {0, 2, 1, 0, 1}] == $16\,(-2\,m2+d\,m2)$

Δ[{{1, 2}, {5}}, {1, 2, 0, 0, 1}] == $4\,(-8\,m2+4\,d\,m2+4\,s-4\,d\,s+d^2\,s)$

Δ[{{1, 2}, {3}}, {1, 2, 1, 0, 0}] == $-4\,(-8\,m2+4\,d\,m2+4\,s-4\,d\,s+d^2\,s)$

Δ[{{3, 4}, {5}}, {0, 0, 1, 1, 1}] == $4\,(-2+d)$

Δ[{{2}, {4}, {5}}, {0, 1, 0, 1, 1}] ==
$\frac{2\,(-4\,m2+6\,s-5\,d\,s+d^2\,s)}{s}+4\,(-2+d)\,x_{1,1}$[{{2}, {4}, {5}}] $+\,4\,(-2+d)\,x_{1,2}$[{{2}, {4}, {5}}]

Δ[{{2}, {3}, {5}}, {0, 1, 1, 0, 1}] == $-\frac{4\,(2\,m2+6\,s-5\,d\,s+d^2\,s)}{s}$

Δ[{{2}, {3, 4}}, {0, 1, 1, 1, 0}] == $-2\,(-2+d)$

Δ[{{1}, {4}, {5}}, {1, 0, 0, 1, 1}] == $\frac{4\,(2\,m2-2\,s+d\,s)}{s}$

Δ[{{1}, {3}, {5}}, {1, 0, 1, 0, 1}] ==
$\frac{2\,(4\,m2+10\,s-7\,d\,s+d^2\,s)}{s}+4\,(-2+d)\,x_{1,1}$[{{1}, {3}, {5}}] $+\,4\,(-2+d)\,x_{1,2}$[{{1}, {3}, {5}}]

Δ[{{1}, {3, 4}}, {1, 0, 1, 1, 0}] == $-2\,(-2+d)$

Δ[{{1, 2}, {5}}, {1, 1, 0, 0, 1}] == $4\,(-2+d)$

Δ[{{1, 2}, {4}}, {1, 1, 0, 1, 0}] == $-2\,(-2+d)$

Δ[{{1, 2}, {3}}, {1, 1, 1, 0, 0}] == $-2\,(-2+d)$

Δ[{{2}, {5}}, {0, 2, 0, 0, 1}] == $-4\,(4-4\,d+d^2)$

Δ[{{2}, {3}}, {0, 2, 1, 0, 0}] == $4\,(4-4\,d+d^2)$

```
Δ[{{2}, {5}}, {0, 1, 0, 0, 1}]==  4 (-2 + d) / s
Δ[{{2}, {3}}, {0, 1, 1, 0, 0}]== - 4 (-2 + d) / s
Δ[{{1}, {5}}, {1, 0, 0, 0, 1}]== - 4 (-2 + d) / s
Δ[{{1}, {3}}, {1, 0, 1, 0, 0}]==  4 (-2 + d) / s
```

In principle, each of these monomials correspond to an integral in the final decomposition. However, the properties of dimensional regularization and the invariance of Feynmam integrals under reparametrization of the loop momenta allow us to reduce the number of irreducible monomials prior to any IBPs reduction.

First of all, we observe that the integrals associated to Δ_{15}^{10001}, Δ_{25}^{01001}, Δ_{25}^{02001}, Δ_{125}^{11001}, Δ_{125}^{12001}, Δ_{345}^{00111} vanish in d-dimensions, since they are all proportional to a massless tapdole. Moreover, by performing shifts of the loop momenta of the type (4.55), we can determine mappings between the set of denominators of different cuts and, hence, combine together their residues.

For instance, we have

$$\int \frac{\mathrm{d}^d q_1}{\pi^{d/2}} \frac{\mathrm{d}^d q_2}{\pi^{d/2}} \left[\frac{\Delta_{2345}}{D_2 D_3 D_4 D_5} + \frac{\Delta_{1345}}{D_1 D_3 D_4 D_5} + \frac{\Delta_{1245}}{D_1 D_2 D_4 D_5} + \frac{\Delta_{1235}}{D_1 D_2 D_3 D_5} \right] = \int \frac{\mathrm{d}^d q_1}{\pi^{d/2}} \frac{\mathrm{d}^d q_2}{\pi^{d/2}} \frac{\Delta_{2345} + \Delta_{1345} + \Delta_{1245} \Delta_{1235}}{D_1 D_2 D_3 D_5}, \tag{4.70}$$

where, in the absence of square propagators, we have suppressed the label $a_m \ldots a_n$ of $\Delta_{m \ldots n}^{a_m \ldots a_n}$.

In this way, we can reduce down to 11 the total number integrals appearing in the decomposition of the two-loop vacuum polarization, which reads

$$\begin{aligned}
\int \frac{\mathrm{d}^d q_1}{\pi^{d/2}} \frac{\mathrm{d}^d q_2}{\pi^{d/2}} [\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3] = \int \frac{\mathrm{d}^d q_1}{\pi^{d/2}} \frac{\mathrm{d}^d q_2}{\pi^{d/2}} \Bigg[&+ \frac{\Delta_{23}^{21}}{D_2^2 D_3} + \frac{\Delta_{123}}{D_1 D_2 D_3} + \frac{\Delta_{123}^{121}}{D_1 D_2^2 D_3} \\
&+ \frac{\Delta_{235}^{211}}{D_2^2 D_3 D_5} + \frac{\Delta_{235}}{D_2 D_3 D_5} + \frac{\Delta_{135}}{D_1 D_3 D_5} \\
&+ \frac{\Delta_{1234}}{D_1 D_2 D_3 D_4} + \frac{\Delta_{1235}}{D_1 D_2 D_3 D_5} \\
&+ \frac{\Delta_{1235}^{1211}}{D_1 D_2^2 D_3 D_5} + \frac{\Delta_{12345}}{D_1 D_2 D_3 D_4 D_5} \Bigg],
\end{aligned} \tag{4.71}$$

with

$$\begin{aligned}
\Delta_{23}^{21} =& 4(d-2)^2 \\
\Delta_{235}^{211} =& 16m^2(d-2), \\
\Delta_{123}^{121} =& 4(2-d)(4m^2 + s(d-2)), \\
\Delta_{123} =& 8(2-d), \\
\Delta_{235} =& 4(2-d)(d-4), \\
\Delta_{135} =& 4(8 - 6d + d^2) \qquad\qquad , \\
\Delta_{1234} =& 16m^2 - 2(d-7)(d-2)s, \\
\Delta_{1235} =& -40m^2 - 2(10 - 11d + 3d^2)s + 8(d-2)^2 q_2 \cdot p,
\end{aligned}$$

$$\Delta_{1235}^{1211} = -16m^2(4m^2 + s(d-2)),$$
$$\Delta_{12345} = 4(s-2m^2)(4m^2 + s(d-2)). \tag{4.72}$$

From the point of view of the integrand decomposition, the integrals occurring in eq. (4.71) are to be considered as independent. A further IBPs reduction would bring down to 5 the final number of master integrals.

4.4.1 An application: muon-electron scattering

As an example of the analytic integrand-level decomposition obtained through AIDA, we consider the NLO and NNLO virtual QED corrections to the muon-electron elastic scattering

$$\mu^-(p_1) + e^-(p_2) \to e^-(p_3) + \mu^-(p_4). \tag{4.73}$$

The analytic computation of the two-loop master integrals related to this process, which will be discussed in chapter 8, is possible, at the present time, only in the limit of vanishing electron mass. Therefore, although the integrand reduction can be also obtained by retaining the full dependence on the masses of both leptons, we will assume from the very beginning $m_e^2 = 0$, in order keep our results more compact. In such limit, the kinematics of the process is defined by

$$s = (p_1 + p_2)^2, \quad t = (p_2 - p_3)^2, \quad u = -s - t + 2m^2, \tag{4.74}$$

with $p_1^2 = p_4^2 = m^2$ and $p_2^2 = p_3^2 = 0$.

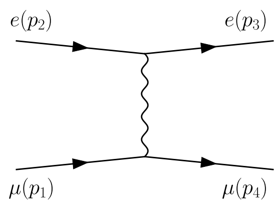


Figure 4.5: μe scattering at tree-level in QED.

The QED crossection for μe scattering is expanded in the fine structure constant $\alpha = e^2/4\pi$ as

$$\sigma = \sigma_{\text{LO}} + \sigma_{\text{NLO}} + \sigma_{\text{NNLO}} + \dots \tag{4.75}$$

where, schematically,

$$\begin{aligned}
\sigma_{\text{LO}} &= \int \mathrm{d}\phi_2 |\mathcal{M}^{(0)}|^2, \\
\sigma_{\text{NLO}} &= \int \mathrm{d}\phi_2 2Re\ \mathcal{M}^{(0)*}\mathcal{M}^{(1)} + \int \mathrm{d}\phi_3 |\mathcal{M}_\gamma^{(0)}|^2, \\
\sigma_{\text{NNLO}} &= \int \mathrm{d}\phi_2 \left(2Re\ \mathcal{M}^{(0)*}\mathcal{M}^{(2)} + |\mathcal{M}_\gamma^{(1)}|^2\right) + \int \mathrm{d}\phi_3 Re\ \mathcal{M}_\gamma^{(0)*}\mathcal{M}_\gamma^{(1)} + \\
&\quad + \int \mathrm{d}\phi_4 |\mathcal{M}_{\gamma\gamma}^{(0)}|^2,
\end{aligned} \tag{4.76}$$

with $\mathrm{d}\phi_n$ being the n-body phase-space. In eq. (4.76), $\mathcal{M}^{(\ell)}$ indicates the ℓ-loop virtual contribution to the μe scattering amplitude, which is $\mathcal{O}(\alpha^{\ell+1})$, whereas $\mathcal{M}_\gamma^{(\ell)}$ and $\mathcal{M}_{\gamma\gamma}^{(\ell)}$

are, respectively, the amplitudes for the real emission of one and two unresolved photons.

At LO $\mathcal{M}^{(0)}$ receives contribution from one single t-channel diagram, which is depicted in figure 4.5,

$$\mathcal{M}^{(0)} = ie^2 \frac{\bar{u}(p_3)\gamma^\mu u(p_2)\,\bar{u}(p_4)\gamma^\mu u(p_1)}{t}\,. \tag{4.77}$$

In the following we apply the adaptive integrand decomposition to the NLO virtual contribution, which is given by the interference between the one-loop amplitude with eq. (4.77), and we present some preliminary results for the NNLO interference with the two-loop virtual diagrams, whose computation is the goal of an ongoing project (for further details, see chapter 8). All results have been obtained through AIDA by starting from the integrands generated with the help of FEYNARTS and FEYNCALC. In view of the full NNLO computation, we stress that the same reduction algorithm can be applied to the decomposition of the square one-loop amplitude $|\mathcal{M}^{(1)}|^2$ as well as the single emission contribution $\mathcal{M}_\gamma^{(0)\,*}\mathcal{M}_\gamma^{(1)}$.

Muon-electron scattering at one loop

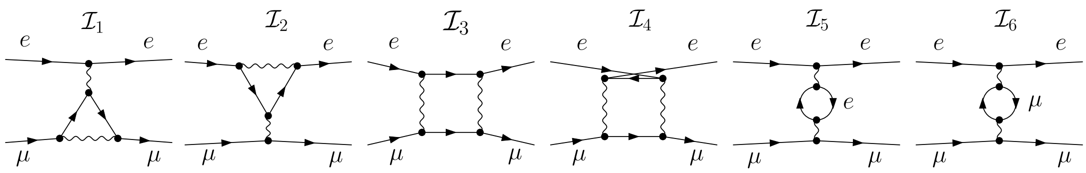


Figure 4.6: Feynman diagrams for μe scattering at one loop.

The one-loop amplitude $\mathcal{M}^{(1)}$ for μe scattering receives contribution from the six Feynman diagrams depicted in figure 4.11. We are interested in the decomposition of the virtual contribution to σ_{NLO} which, according to eq. (4.76), is given by

$$2Re\,\mathcal{M}^{(0)\,*}\mathcal{M}^{(1)} = \left(\frac{\alpha}{\pi}\right)^3 \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \sum_{j=1}^{6} \mathcal{I}_j, \tag{4.78}$$

where the numerator of each integrand $\mathcal{I}_j$ is obtained from the contraction of the corresponding diagram of figure 4.11 with the (conjugate) tree-level amplitude (4.77).

The input integrands for AIDA,

$$I = \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6\}\,, \tag{4.79}$$

can be organized into three groups

$$I = \bigcup_{k=1}^{3} G_k\,, \tag{4.80}$$

with

$$G_1 = \{\mathcal{I}_1, \mathcal{I}_6\}\,, \quad G_1 = \{\mathcal{I}_3, \mathcal{I}_2, \mathcal{I}_5\}\,, \quad G_2 = \{\mathcal{I}_4\}\,. \tag{4.81}$$

As we can see from eq. (4.81), in the one-loop case, due to the absence of higher powers of denominators, the parent integrand of each group always corresponds to one of the elements of I.
We choose the denominators of the parent integrands of each G_k as

$$
\begin{aligned}
G_1: \quad & D_1 = q^2 - m^2\,,\\
& D_2 = (q+p_1)^2\,,\\
& D_3 = (q+p_1-p_4)^2 - m^2\,,\\
G_2: \quad & D_1 = q^2\,,\\
& D_2 = (q+p_1)^2 - m^2\,,\\
& D_3 = (q+p_1-p_4)^2\,,\\
& D_4 = (q+p_1-p_4-p_3)^2\,,\\
G_3: \quad & D_1 = q^2\,,\\
& D_2 = (q+p_2)^2\,,\\
& D_3 = (q+p_2-p_4)^2 - m^2\,,\\
& D_4 = (q+p_1+p_2-p_4)^2\,.
\end{aligned}
\tag{4.82}
$$

The integrand reduction is performed for every group individually and the integration over the transverse components is applied prior to the division. The list of non-vanishing residues for G_1, G_2 and G_3 is given, respectively, in figures 4.7-4.8. We observe that, as expected from the general discussion of the adaptive integrand decompostion at one-loop (see section 4.2), all residues are constant w.r.t. the loop momentum.

By applying proper shifts (4.55) of the loop momentum, we can detect symmetries between the 30 different sectors emerging from the reduction and reduce down to 13 the number of independent rank-zero integrals that appear in the final decomposition of eq. (4.78),

$$
\begin{aligned}
2Re\,\mathcal{M}^{(0)\,*}\mathcal{M}^{(1)} = \left(\frac{\alpha}{\pi}\right)^3 \int \frac{\mathrm{d}^d q}{\pi^{d/2}} & \left[\frac{c_1}{D_1D_3} + \frac{c_2}{D_1D_2D_3}\right]_{G_1} + \left[\frac{c_3}{D_2} + \frac{c_4}{D_1D_2} + \frac{c_5}{D_2D_4}\right.\\
& \left. + \frac{c_6}{D_1D_3} + \frac{c_7}{D_1D_3D_4} + \frac{c_8}{D_2D_3D_4} + \frac{c_9}{D_1D_2D_3} + \frac{c_{10}}{D_1D_2D_3D_4}\right]_{G_2}\\
& + \left[\frac{c_{11}}{D_2D_4} + \frac{c_{12}}{D_1D_3D_4} + \frac{c_{13}}{D_1D_2D_3D_4}\right]_{G_3},
\end{aligned}
\tag{4.83}
$$

where the subscript G_i identifies the sets of denominators defined in eq. (4.82). The coefficients c_i, which depend on s, t and m^2, as well as on the space-time dimensions d, are given by

$$
\begin{aligned}
c_1 =& \frac{8}{(d-1)t^3(4m^2-t)}(4m^4(\left(d^2-2d-5\right)t^2 - 32(d-3)st + 32s^2)\\
& + \left(d^2-10d+11\right)t^2\left((d-2)t^2+4s^2+4st\right)\\
& - 4m^2t(2\left(d^2-18d+27\right)st + \left(d^3-12d^2+33d-26\right)t^2\\
& - 16(d-2)s^2) - 64m^6(4s-(d-2)t) + 128m^8)\,,\\
c_2 =& -\frac{16}{t^2}(2m^2-t)\left((d-2)t^2+4m^4-8m^2s+4s^2+4st\right)\,,\\
c_3 =& -\frac{32(d-2)}{(d-1)t^3(4m^2-t)}\times
\end{aligned}
$$

$$
\begin{aligned}
&(m^4(2(d-3)t-32s)4m^2\left(-(d-7)st+(d-2)t^2+4s^2\right)\\
&+t\left(2(d-3)s^2+2(d-3)st-(d-2)t^2\right)+16m^6)\,,\\
c_4=&-\frac{16}{t^2(m^2-s)(4m^2-t)(m^2-s-t)}(-8m^8(8s-(d-5)t)\\
&+4m^6\left(-6(d-6)st+(5d-12)t^2+24s^2\right)\\
&-4m^4\left((44-6d)s^2t+4dst^2+3(2d-5)t^3+16s^3\right)\\
&+m^2(-8(d-10)s^3t-4(d-16)s^2t^2\\
&+2(7d-12)st^3+(5d-12)t^4+16s^4)\\
&-2st(s+t)\left((d-2)t^2+4s^2+4st\right)+16m^{10})\,,\\
c_5=&\frac{8}{t(m^2-s)}((2(d-2)m^4+m^2((3d-8)t-4(d-2)s)\\
&+s(2(d-2)s+(3d-8)t)))\,,\\
c_6=&-\frac{8}{(d-1)t^2(4m^2-t)}\times\\
&(16\left(d^2-10d+11\right)m^6-4\left(d^2-10d+11\right)m^4(8s+t)\\
&+4m^2(4\left(d^2-10d+11\right)s^2+6\left(d^2-10d+11\right)st+\\
&\left(d^3-13d^2+34d-24\right)t^2)-t(4\left(d^2-10d+11\right)s^2\\
&+\left(d^3-14d^2+37d-26\right)t^2-4(7d-9)st))\,,\\
c_7=&\frac{16}{t}\left((d-4)t^2+4m^4+m^2(4t-8s)+4s^2\right)\,,\\
c_8=&\frac{8}{t}(m^2-s)(dt+8s)\,,\\
c_9=&-\frac{32}{t(4m^2-t)}\left(-(d+4)m^2t+8m^4+t^2\right)\left(2m^2-2s-t\right),\\
c_{10}=&\frac{4}{t}(m^2-s)\left((3d-8)t^2+16m^4-32m^2s+16s^2+8st\right)\,,\\
c_{11}=&-\frac{8}{t(-m^2+s+t)}(2(d-2)m^4\\
&+m^2((5d-16)t-4(d-2)s)+(s+t)(2(d-2)s-(d-4)t))\,,\\
c_{12}=&-\frac{8}{t}(-m^2+s+t)((d-8)t+16m^2-8s)\,,\\
c_{13}=&\frac{4}{t}(m^2-s-t)\left(3dt^2+16m^4-16m^2(2s+t)+16s^2+24st\right)\,,
\end{aligned}
\tag{4.84}
$$

and, together with the analytic expression of the one-loop master integrals, which will derived in chapter 8 through the differential equations method, completely specifies the (unrenormalized) NLO virtual contribution to μe scattering cross section.

$$\triangle[\{\{1,2,3\}\},\{1,1,1,0\}] == -\frac{8\,(2\,\mathrm{m2}-t)\,\left(4\,\mathrm{m2}^2-8\,\mathrm{m2}\,s+4\,s^2+4\,s\,t+(-2+d)\,t^2\right)}{t^2}$$

$$\triangle[\{\{2,3\}\},\{0,1,1,0\}] == -\frac{8\,\left(8\,\mathrm{m2}^3-4\,\mathrm{m2}^2\,(4\,s+t)+4\,\mathrm{m2}\,(2\,s^2+4\,s\,t+(-2+d)\,t^2)-t\,(4\,s^2+4\,s\,t+(-2+d)\,t^2)\right)}{(4\,\mathrm{m2}-t)\,t^2}$$

$$\triangle[\{\{1,3\}\},\{1,0,1,0\}] == -\frac{1}{(-1+d)\,(4\,\mathrm{m2}-t)\,t^3} 4\,\Big(-128\,\mathrm{m2}^4+64\,\mathrm{m2}^3\,(4\,s-(-2+d)\,t)-(11-10\,d+d^2)\,t^2\,(4\,s^2+4\,s\,t+(-2+d)\,t^2)- 4\,\mathrm{m2}^2\,(32\,s^2-32\,(-3+d)\,s\,t+(-5-2\,d+d^2)\,t^2)+ 4\,\mathrm{m2}\,t\,(-16\,(-2+d)\,s^2+2\,(27-18\,d+d^2)\,s\,t+(-26+33\,d-12\,d^2+d^3)\,t^2)\Big)$$

$$\triangle[\{\{1,2\}\},\{1,1,0,0\}] == -\frac{8\,\left(8\,\mathrm{m2}^3-4\,\mathrm{m2}^2\,(4\,s+t)+4\,\mathrm{m2}\,(2\,s^2+4\,s\,t+(-2+d)\,t^2)-t\,(4\,s^2+4\,s\,t+(-2+d)\,t^2)\right)}{(4\,\mathrm{m2}-t)\,t^2}$$

$$\triangle[\{\{3\}\},\{0,0,1,0\}] == -\frac{1}{(-1+d)\,(4\,\mathrm{m2}-t)\,t^3} 8\,(-2+d)\,\Big(16\,\mathrm{m2}^3+\mathrm{m2}^2\,(-32\,s+2\,(-3+d)\,t)+t\,(2\,(-3+d)\,s^2+2\,(-3+d)\,s\,t-(-2+d)\,t^2)+ 4\,\mathrm{m2}\,(4\,s^2-(-7+d)\,s\,t+(-2+d)\,t^2)\Big)$$

$$\triangle[\{\{2\}\},\{0,1,0,0\}] == \frac{32\,(-2+d)\,\left(\mathrm{m2}^2-2\,\mathrm{m2}\,s+s\,(s+t)\right)}{(4\,\mathrm{m2}-t)\,t^2}$$

$$\triangle[\{\{1\}\},\{1,0,0,0\}] == -\frac{1}{(-1+d)\,(4\,\mathrm{m2}-t)\,t^3} 8\,(-2+d)\,\Big(16\,\mathrm{m2}^3+\mathrm{m2}^2\,(-32\,s+2\,(-3+d)\,t)+t\,(2\,(-3+d)\,s^2+2\,(-3+d)\,s\,t-(-2+d)\,t^2)+ 4\,\mathrm{m2}\,(4\,s^2-(-7+d)\,s\,t+(-2+d)\,t^2)\Big)$$

Figure 4.7: Residues for G_1.

$$\triangle[\{\{1,2,3,4\}\},\{1,1,1,1\}] == \frac{2\,(\mathrm{m2}-s-t)\,\left(16\,\mathrm{m2}^2+16\,s^2+24\,s\,t+3\,d\,t^2-16\,\mathrm{m2}\,(2\,s+t)\right)}{t}$$

$$\triangle[\{\{2,3,4\}\},\{0,1,1,1\}] == \frac{1}{(4\,\mathrm{m2}-t)\,t} 2\,\Big(-128\,\mathrm{m2}^3+8\,\mathrm{m2}^2\,(24\,s+(18+d)\,t)+t\,(16\,s^2+24\,s\,t+3\,d\,t^2)- 8\,\mathrm{m2}\,(8\,s^2+(16+d)\,s\,t+(3+2\,d)\,t^2)\Big)$$

$$\triangle[\{\{1,3,4\}\},\{1,0,1,1\}] == \frac{2\,(\mathrm{m2}-s-t)\,(16\,\mathrm{m2}-8\,s+(-8+d)\,t)}{t}$$

$$\triangle[\{\{1,2,4\}\},\{1,1,0,1\}] == -\frac{2\,\left(16\,\mathrm{m2}^2+16\,s^2+24\,s\,t+3\,d\,t^2-16\,\mathrm{m2}\,(2\,s+t)\right)}{t}$$

$$\triangle[\{\{1,2,3\}\},\{1,1,1,0\}] == \frac{2\,(\mathrm{m2}-s-t)\,(16\,\mathrm{m2}-8\,s+(-8+d)\,t)}{t}$$

$$\triangle[\{\{3,4\}\},\{0,0,1,1\}] == -\frac{1}{(4\,\mathrm{m2}-t)\,(\mathrm{m2}-s-t)\,t} 4\,\mathrm{m2} \Big(4\,(-2+d)\,\mathrm{m2}^2+4\,(-2+d)\,s^2+6\,(-2+d)\,s\,t-(-4+d)\,t^2+\mathrm{m2}\,(-8\,(-2+d)\,s+2\,(-10+3\,d)\,t)\Big)$$

$$\triangle[\{\{2,4\}\},\{0,1,0,1\}] == -\frac{4\,((-20+6\,d)\,\mathrm{m2}+2\,(-2+d)\,s-(-4+d)\,t)}{4\,\mathrm{m2}-t}$$

$$\triangle[\{\{2,3\}\},\{0,1,1,0\}] == -\frac{1}{(4\,\mathrm{m2}-t)\,(\mathrm{m2}-s-t)\,t} 4\,\mathrm{m2} \Big(4\,(-2+d)\,\mathrm{m2}^2+4\,(-2+d)\,s^2+6\,(-2+d)\,s\,t-(-4+d)\,t^2+\mathrm{m2}\,(-8\,(-2+d)\,s+2\,(-10+3\,d)\,t)\Big)$$

$$\triangle[\{\{1,3\}\},\{1,0,1,0\}] == -\frac{4\,\left(2\,(-2+d)\,\mathrm{m2}^2+(s+t)\,(2\,(-2+d)\,s-(-4+d)\,t)+\mathrm{m2}\,(-4\,(-2+d)\,s+(-16+5\,d)\,t)\right)}{t\,(-\mathrm{m2}+s+t)}$$

Figure 4.8: Residues for G_3.

$$\Delta[\{\{1,2,3,4\}\},\{1,1,1,1\}]==\frac{2\,(\mathrm{m2}-s)\left(16\,\mathrm{m2}^2-32\,\mathrm{m2}\,s+16\,s^2+8\,s\,t+(-8+3\,d)\,t^2\right)}{t}$$

$$\Delta[\{\{2,3,4\}\},\{0,1,1,1\}]==\frac{2\,(\mathrm{m2}-s)\,(8\,s+d\,t)}{t}$$

$$\Delta[\{\{1,3,4\}\},\{1,0,1,1\}]==\frac{2\left(32\,\mathrm{m2}^2-64\,\mathrm{m2}\,s+32\,s^2+24\,s\,t+(-16+7\,d)\,t^2\right)}{t}$$

$$\Delta[\{\{1,2,4\}\},\{1,1,0,1\}]==\frac{2\,(\mathrm{m2}-s)\,(8\,s+d\,t)}{t}$$

$$\Delta[\{\{1,2,3\}\},\{1,1,1,0\}]==\frac{1}{(4\,\mathrm{m2}-t)\,t}\,2\left(-8\,\mathrm{m2}^2\,(8\,s-(-2+d)\,t)+8\,\mathrm{m2}\,(8\,s^2-(-8+d)\,s\,t+(-3+d)\,t^2)-t\,(16\,s^2+8\,s\,t+(-8+3\,d)\,t^2)\right)$$

$$\Delta[\{\{3,4\}\},\{0,0,1,1\}]==-\frac{8\left(4\,\mathrm{m2}^2-8\,\mathrm{m2}\,s+4\,s^2+4\,s\,t+(-2+d)\,t^2\right)}{t^2}$$

$$\Delta[\{\{2,4\}\},\{0,1,0,1\}]==\frac{4\left(2\,(-2+d)\,\mathrm{m2}^2+\mathrm{m2}\,(-4\,(-2+d)\,s+(-8+3\,d)\,t)+s\,(2\,(-2+d)\,s+(-8+3\,d)\,t)\right)}{(\mathrm{m2}-s)\,t}$$

$$\Delta[\{\{2,3\}\},\{0,1,1,0\}]==-\frac{4\,\mathrm{m2}\left(4\,(-2+d)\,\mathrm{m2}^2+4\,(-2+d)\,s^2+2\,(-2+d)\,s\,t+(8-3\,d)\,t^2-2\,\mathrm{m2}\,(4\,(-2+d)\,s+(14-5\,d)\,t)\right)}{(\mathrm{m2}-s)\,(4\,\mathrm{m2}-t)\,t}$$

$$\Delta[\{\{1,4\}\},\{1,0,0,1\}]==-\frac{8\left(4\,\mathrm{m2}^2-8\,\mathrm{m2}\,s+4\,s^2+4\,s\,t+(-2+d)\,t^2\right)}{t^2}$$

$$\begin{aligned}\Delta[\{\{1,3\}\},\{1,0,1,0\}]==&-\frac{1}{(-1+d)\,(4\,\mathrm{m2}-t)\,t^2}4\left(16\,(11-10\,d+d^2)\,\mathrm{m2}^3-4\,(11-10\,d+d^2)\,\mathrm{m2}^2\,(8\,s+t)-\right.\\&t\,(4\,(11-10\,d+d^2)\,s^2+2\,(20-17\,d+d^2)\,s\,t+(-30+42\,d-15\,d^2+d^3)\,t^2)+\\&\left.2\,\mathrm{m2}\,(8\,(11-10\,d+d^2)\,s^2+12\,(11-10\,d+d^2)\,s\,t+(-58+81\,d-29\,d^2+2\,d^3)\,t^2)\right)\end{aligned}$$

$$\Delta[\{\{1,2\}\},\{1,1,0,0\}]==-\frac{4\,\mathrm{m2}\left(4\,(-2+d)\,\mathrm{m2}^2+4\,(-2+d)\,s^2+2\,(-2+d)\,s\,t+(8-3\,d)\,t^2-2\,\mathrm{m2}\,(4\,(-2+d)\,s+(14-5\,d)\,t)\right)}{(\mathrm{m2}-s)\,(4\,\mathrm{m2}-t)\,t}$$

$$\Delta[\{\{4\}\},\{0,0,0,1\}]==\frac{8\left(4\,\mathrm{m2}^2-8\,\mathrm{m2}\,s+4\,s^2+4\,s\,t+(-2+d)\,t^2\right)}{t^3}$$

$$\Delta[\{\{3\}\},\{0,0,1,0\}]==-\frac{4\,(-5+3\,d)\left(4\,\mathrm{m2}^2-8\,\mathrm{m2}\,s+4\,s^2+4\,s\,t+(-2+d)\,t^2\right)}{(-1+d)\,t^3}$$

$$\Delta[\{\{1\}\},\{1,0,0,0\}]==-\frac{4\,(-5+3\,d)\left(4\,\mathrm{m2}^2-8\,\mathrm{m2}\,s+4\,s^2+4\,s\,t+(-2+d)\,t^2\right)}{(-1+d)\,t^3}$$

Figure 4.9: Residues for G_2.

Muon-electron scattering at two loops

The same procedure can be applied to the NNLO virtual corrections to μe scattering that originate from the interference term between the tree-level amplitude (4.77) and the two-loop virtual diagrams

$$2Re\,\mathcal{M}^{(0)\,*}\mathcal{M}^{(2)}=\left(\frac{\alpha}{\pi}\right)^4\int\frac{\mathrm{d}^dq_1\mathrm{d}^dq_2}{\pi^d}\sum_{j=1}^{69}\mathcal{I}_j\,. \qquad (4.85)$$

As explicitly indicated in eq. (4.85), $\mathcal{M}^{(2)}$ receives contribution from 69 distinct diagrams. By applying shifts of the loop momenta of the type (4.55), the full set of diagrams can be organized (in a non-unique way) in 26 different groups G_i, whose parent topologies are depicted in figure 4.10.

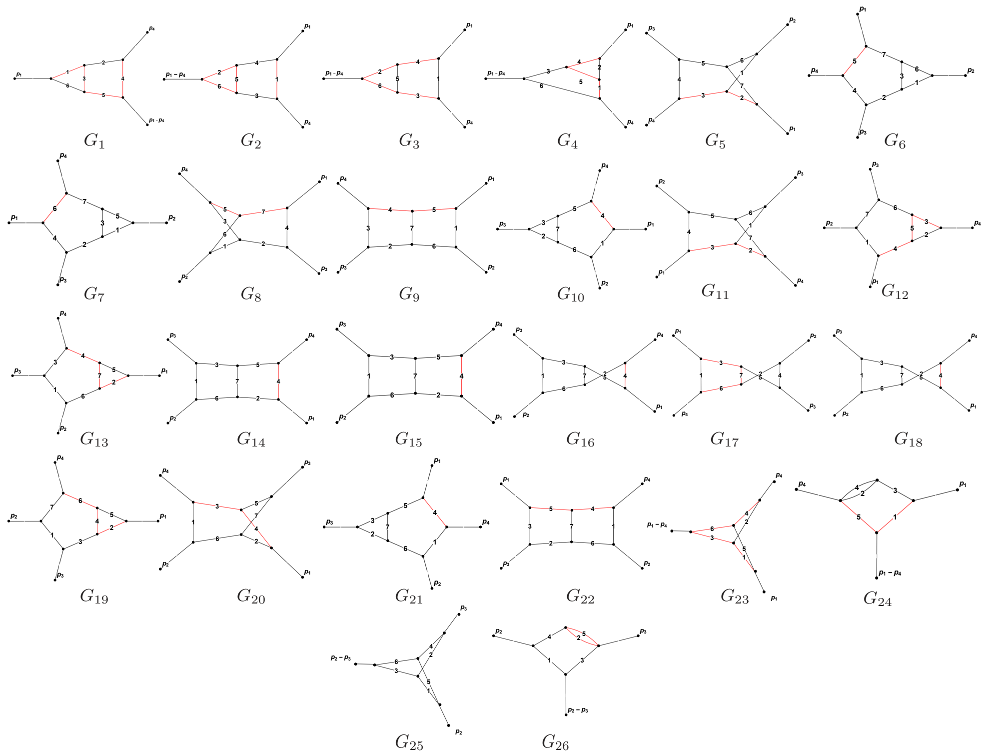


Figure 4.10: Parent topologies of the groups $G_{1,2,\ldots 26}$ for two-loop μe scattering. Red lines represents massive propagators

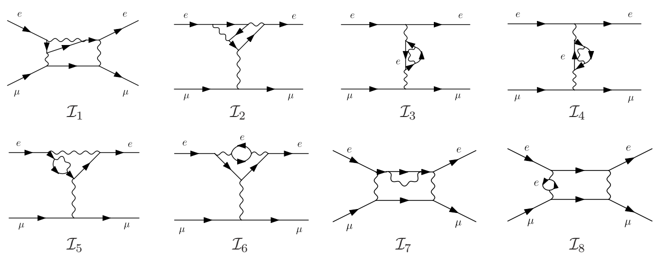


Figure 4.11: Two-loop Feynman diagrams belonging to G_6.

As an example, we consider G_6,

$$G_6 = \{\mathcal{I}_1^{G_6}, \mathcal{I}_2^{G_6}, \ldots, \mathcal{I}_8^{G_6}\}, \tag{4.86}$$

which groups the 8 Feynman diagrams shown in figure 4.11. The integrands belonging to G_6 are given by

$$\begin{aligned}
\mathcal{I}_1^{G_6} =& \{0\,, \{D_1, D_2, D_3, D_4, D_5, D_6, D_7\},\ \{1,2,1,1,1,2,1\}\}\,,\\
\mathcal{I}_2^{G_6} =& \{\mathcal{N}_1\,, \{D_1, D_2, D_3, D_4, D_5, D_6, D_7\},\ \{1,1,1,1,1,1,1\}\}\,,\\
\mathcal{I}_3^{G_6} =& \{\mathcal{N}_2\,, \{D_1, D_2, D_4, D_5, D_6, D_7\},\ \{1,1,0,1,1,1,1\}\}\,,\\
\mathcal{I}_4^{G_6} =& \{\mathcal{N}_3 + \mathcal{N}_4\,, \{D_1, D_2, D_4, D_5\},\ \{1,2,0,1,1,0,0\}\}\,,\\
\mathcal{I}_5^{G_6} =& \{\mathcal{N}_5\,, \{D_1, D_2, D_4, D_5, D_6\},\ \{1,2,0,1,1,1,0\}\}\,,\\
\mathcal{I}_6^{G_6} =& \{\mathcal{N}_6\,, \{D_2, D_4, D_5, D_6, D_7\},\ \{0,1,0,1,1,2,1\}\}\,,\\
\mathcal{I}_7^{G_6} =& \{\mathcal{N}_7\,, \{D_2, D_3, D_4, D_5, D_6, D_7\},\ \{0,1,1,1,1,2,1\}\}\,,
\end{aligned}$$

$$\mathcal{I}_8^{G_6} = \{\mathcal{N}_8\,, \{D_1, D_2, D_3, D_4, D_5, D_6\},\ \{1,2,1,1,1,1,0\}\}\,. \tag{4.87}$$

with $\mathcal{N}_i$ being the numerator of the integrand $\mathcal{I}_i$, according to the labelling of figure 4.11. The loop denominators D_i are defined by

$$D_1 = (q_1 - p_1 + p_3 + p_4)^2, \quad D_2 = (q_2 + p_1 - p_3 - p_4)^2, \quad D_3 = (q_2 - p_3 - p_4)^2 - m^2,$$
$$D_4 = (q_2 - p_3)^2, \quad D_5 = (q_1 + q_2)^2, \quad D_6 = q_2^2, \quad D_7 = q_1^2\,. \tag{4.88}$$

Let us remark that, according to the general discussion of section (4.4), at multi-loop level the parent topology of each group $\mathcal{I}_1^{G_i}$ might not correspond to any Feynman diagram, due to the presence of higher-powers of denominators. For this reason, in eq. (4.87) the numerator of $\mathcal{I}_1^{G_6}$ has been initialized to zero. In addition, we observe that $\mathcal{I}_3$ and $\mathcal{I}_4$ have been grouped together into $\mathcal{I}_4^{G_6}$, since they clearly correspond to the same topology.

The integrand reduction is performed for every group individually and the integration over the transverse components is applied prior to the division. As an example, we show in figure 4.12 the residues corresponding to the integrands $\mathcal{I}_i^{G_6}$. Similar results are have been obtained for their subtopologies, as well as for the integrands belonging to the other groups.
As expected, at two-loop the residues contain, besides constant terms, monomials in the ISPs $\mathbf{x}_{\parallel}^{\rm ISP}$. The number of independent integrals appearing in the decomposition can be reduced by applying symmetry relations, IBPs and LIs. In this way, eq. (4.85) will be expressed as a linear combination of master integrals, whose computation will be discussed in chapter 8.

```
△[{{1, 7}, {2, 3, 4, 6}, {5}}, {1, 1, 1, 1, 1, 1, 1}] == 0

△[{{1, 7}, {2, 4, 6}, {5}}, {1, 1, 0, 1, 1, 1, 1}] == 0

△[{{1}, {2, 4}, {5}}, {1, 2, 0, 1, 1, 0, 0}] == - 1/((4 m2 - t) t^2)
  32 x{1,1}[{{1}, {2, 4}, {5}}] (-8 m2^3 + 4 m2^2 (4 s + t) - 4 m2 (2 s^2 + 4 s t + (-2 + d) t^2) +
     t (4 s^2 + 4 s t + (-2 + d) t^2) + (-4 + d) (4 m2 - t) t^2 x{1,1}[{{1}, {2, 4}, {5}}])

△[{{1}, {2, 4, 6}, {5}}, {1, 2, 0, 1, 1, 1, 0}] == 1/t^2
 8 x{2,1}[{{1}, {2, 4, 6}, {5}}] (-4 m2^2 + 8 m2 s - 4 s^2 - 4 s t + 2 t^2 -
    d t^2 + (-8 m2^2 + 16 m2 s - 8 s^2 + d t^2) x{1,1}[{{1}, {2, 4, 6}, {5}}] +
    (8 m2^2 + 8 s^2 + 8 s t + d t^2 - 8 m2 (2 s + t)) x{2,1}[{{1}, {2, 4, 6}, {5}}])

△[{{7}, {2, 4, 6}, {5}}, {0, 1, 0, 1, 1, 2, 1}] ==
 - 1/t^2 32 (4 m2^2 - 8 m2 s + 4 s^2 + 4 s t + (-2 + d) t^2)
   x{1,1}[{{7}, {2, 4, 6}, {5}}] x{2,1}[{{7}, {2, 4, 6}, {5}}]

△[{{7}, {2, 3, 4, 6}, {5}}, {0, 1, 1, 1, 1, 2, 1}] == 0

△[{{1}, {2, 3, 4, 6}, {5}}, {1, 2, 1, 1, 1, 1, 0}] ==
 - 1/t^4 8 (16 m2^2 - 32 m2 s + 16 s^2 + 8 s t + (-4 + d) t^2)
   ((m2 - s) x{1,1}[{{1}, {2, 3, 4, 6}, {5}}] + (-m2 + s + t)
      x{2,1}[{{1}, {2, 3, 4, 6}, {5}}] + (2 m2 - t) x{3,1}[{{1}, {2, 3, 4, 6}, {5}}])
   (t x{2,1}[{{1}, {2, 3, 4, 6}, {5}}] + (m2 - s - t) x{3,1}[{{1}, {2, 3, 4, 6}, {5}}])
```

Figure 4.12: Representative residues for G_6.

4.5 Conclusions

In this chapter, we have presented the adaptive integrand decomposition algorithm, a simplified version of the multi-loop integrand reduction method which leads to a decomposition of dimensionally regulated amplitudes in terms of non-spurious integrals, whose tensor structures strictly correspond to irreducible scalar products between loop momenta and external momenta. The algorithm does not rely on the usual projection of the amplitude into form factors and, hence, it is suitable for application to helicity amplitudes, which may generally enjoy better symmetry properties than the latter.

Beyond one-loop, the number of integrals emerging from the integrand reduction is not minimal, due to the existence of symmetry relations, Lorentz invariance and integration-by-parts identities between integrals. Nonetheless, the parametrization in terms of tensor integrals involving physical irreducible scalar products is particularly well-suited for the subsequent integral-level reduction.

The adaptive integrand decomposition is formulated in terms of the divide-integrate-divide algorithm, which exploits the properties of $d = d_{\parallel} + d_{\perp}$ representation of Feynman integrals in order to eliminate the dependence of the integrand on the transverse directions and to simplify the polynomial division procedure, by eliminating the need for the explicit computation of Gröbner bases.

These ideas have been put together in Aida, a Mathematica code which implements the adaptive integrand decomposition of one- and two-loop amplitudes. The code can perform both numerical and analytical computations. As it stands, Aida can deal with the numerical reduction independently from the number of different internal masses and external particles. Although the efficiency of the analytic reduction is obviously affected by the number of scales of the process under consideration, our preliminary results suggest a wide applicability to two-loop multi-scale problems [8]. At the present time, the optimization of the analytic reduction for multi-scale amplitudes through finite fields techniques [166] is under study. The future application of the prosed decomposition method to the computation of scattering amplitudes at NNLO will require the systematic organization of a complete reduction chain. To this aim, we are taking steps towards interfacing Aida, on the input side, with the available automatic amplitude generators [162–164] and, on the output one, with IBPs reduction codes [167–169] and, eventually, with libraries containing the analytic expressions of the master integrals (see, for instance [170]). A valuable option, in the absence of the latter, will be the interface with codes for the numerical evaluation of Feynman integrals [171, 172].

In this perspective, we believe that the integrand decomposition method, in its adaptive formulation, will play an important in the multi-loop extension of the existing automatic frameworks for perturbative calculations, such as [38].

Having addressed the problem of reducing scattering amplitudes to linear combinations of scalar Feynman integrals, we will now move our attention to the available techniques for the analytic computation of the latter, which will be the main focus of the remaining parts of this thesis.

Chapter 5

Differential equations for Feynman integrals

In this chapter we give an introduction to the differential equations method for master integrals. Dimensionally regulated Feynman integrals obey first-order coupled differential equations in the kinematic invariants which can be solved as an expansion for small values of the regulating parameter ϵ in terms of iterated integrals. We discuss the advantages of determining a *canonical* basis of master integrals, which fulfil ϵ-factorized systems of differential equations in dlog-*form*, and we review the Magnus exponential method, which can be used to find a canonical basis starting from an ϵ-linear systems. Finally, we recall the main properties of the iterated integrals that appear in the general solution of such systems and we show how boundary conditions can be determined from the singularity structure of the differential equations.

5.1 Reduction to master integrals

The final outcome of the integrand reduction or, alternatively, of the standard form factor decomposition, consists in the expression of a scattering amplitude as a linear combination of scalar Feynman integrals of the type defined in eq. (2.9),

$$I^{d\,(\ell\,,n)}(a_1\,,\ldots\,,a_m\,;b_1\,,\ldots\,,b_r) = \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{S_1^{-b_1}\,\cdots\,S_r^{-b_r}}{D_1^{a_1}\,\cdots\,D_m^{a_m}}\,. \tag{5.1}$$

As we throughly discussed in chapters 3-4, a convenient choice of the variables used to parametrize the integrands can significantly help in the minimization and optimization of the *integrand basis* appearing in the decomposition. However, the corresponding *integral basis* is usually not minimal, due to the presence of additional identities, introduced by the integration over the loop momenta, such as symmetry relations, IBPs and LIs, which have been discussed in chapter 2. Therefore, given the large number of independent integrands (which, for a typical two-loop computation, can easily reach $\mathcal{O}(1000)$), it is mandatory, before embarking the non-trivial computation of Feynman integrals, to determine a minimal basis of independent *master integrals* (MIs) which spans the full space of integrals that are needed for the specific process under consideration.

To this aim, we must systematically generate and solve such sets of integral-level identities. As it is clear from the definition of the most general IBP, given in eq. (2.12), by modifying the arbitrary values of the powers of denominators and ISPs, we can generate an infinite number of linear equations for an infinite number of integrals even within a single topology. It turns out that the number of equations grows faster than the number of integrals they involve, so that the thousands, or even millions, of equations we can generate are mostly redundant [47]. In addition, besides empirical evidence, it has been proven that it is always possible to solve these systems of equations in terms of a finite number of MIs [173]. Therefore, the problem of determining a basis of MIs and expressing all elements of a given space of Feynman integrals in terms of such basis is conceptually solved, since it reduces to the solution, by Gaussian elimination, of a system of linear equations that leaves a finite number of free unknowns. However, the inversion of such huge linear systems and the complexity of the intermediate expressions generated by the Gaussian substitution, demand for an extensive use of computer algebra resources and can encounter practical limitations.

First of all, it is obvious that only a (large but) finite number of identities can be practically be generated, usually by fixing a total power of denominators $R = \sum_{i=1}^{m} a_i$ and a total power of ISPs in the numerator, $S = \sum_{i=1}^{r} b_i$. Hence, it is essential to find a correct balance in the ranges of R and S so to allow, on the one hand, the correct identification of a minimal basis of MIs and, on the other hand, the feasibility of the algebraic solution of the generated system. In this respect, we should take into account that, by restricting the generation of IBPs up to some R and S, we are implicitly assuming that no IBP with higher total powers can further reduce the number of MIs, which have been determined from the selected subsystem. In addition, as we have already observed, LIs are not needed in order to identify the minimal set of MIs, since they are not linear independent from IBPs. However, in many concrete cases, the use of LIs can consistently speed up the solution of the system, since obtaining them from IBPs would require the expensive generation of much larger systems of equations.

Once the selected identities have been generated and a basis of MIs has been determined, the last technically relevant issue consists in choosing the order in which the equations must be solved. In fact, although the final expression of all integrals in terms of a basis must be independent of such ordering, the complexity of the intermediate expressions can be heavily affected by one choice or another. The Laporta algorithm [47, 64] provides a set of guiding principles that allow to choose, step-by-step in the solution, which equations must be inverted in order to keep intermediate expressions as compact as possible. This algorithm has been implemented and optimized in several public computer codes such as AIR [174], REDUZE [167], FIRE [175], LITERED [176] and KIRA [169].

Finally let us stress that, given the fixed number of MIs determined from the integral reduction, the choice of the basis of MIs is by no means unique, since we can pick as a basis any set of integrals which are linearly independent under symmetry relations, IBPs and LIs. Of course, different choices must lead to completely equivalent results but some specific choices can produce huge simplifications both in the complexity of the reduction identities and in the determination of the analytic expression of the MIs themselves. We will come back to discuss about the optimal choice of the basis of MIs once we will have introduced the differential equations method for the computation of Feynman integrals.

5.2 The differential equations method

Provided that the the solution of the system of IBPs, LIs and symmetry relations has been successful and all integrals originated from the integrand decomposition of the amplitude have been expressed in terms of a chosen basis of MIs, the last remaining step consists in the explicit evaluation of the latter.

The most natural possibility to analytically computed the MIs consists in performing suitable manipulations of the integrand in such a way to allow the direct evaluation of the loop integrals. Examples of such direct integration methods are the *Feynman parameters* and the *Mellin-Barnes* [70–73] representations of loop integrals, which have been extensively applied to one-loop computations as well as to a number of multi-loop cases with a restrained number of scales. However, as soon as the number of external legs and internal masses increases, direct integration methods swiftly becomes unfeasible.

The intrinsic difficulties of explicit integration can be obviated by resorting to indirect integration techniques, such as *difference equations* methods, which determine functional relations between MIs that differ from a discrete shift of the space-time dimensions [75] or of the exponents associated to the loop denominators [64, 74].

Alternatively, the analyticity of Feynman integrals in the kinematic invariants can be used in order to derive sets of *differential equations* (DEQs) satisfied by the MIs, whose solution can be significantly easier to obtain than the direct evaluation of loop integrals. The DEQs method was first was first proposed by Kotikov in [76], where Feynman integrals were differentiated w.r.t. the internal masses, and subsequently generalized to external invariants by Remiddi [77] and Gehrmann and Remiddi [48]. Since then, DEQs have proven to be an essential tool for the analytic calculation of multi-loop, multi-scale Feynman integrals.

In order to illustrate the main features of the DEQs method, let us start by considering internal mass derivatives. In the following we will denote by $\mathrm{F}_i(\epsilon, \vec{x}, m_j^2)$, $i = 1, 2, \ldots, N$ the N MIs of a basis which spans the whole space of Feynman integrals for a particular process. As explicitly indicated, each $\mathrm{F}_i(\epsilon, \vec{s}, m_j^2)$ depends on the space-time dimensions, on the external invariants (we hav collectively labelled with $\vec{x}$), and on the internal masses.

Each MI is a Feynman integral of the type defined in eq. (5.1) (we hereby omit all labels referring to the number of loops and external legs, which do not affect the present discussion). As already observed in section 2.6, if we pick one of the MIs and we assume that the mass m_i^2 of its i-th internal propagator is non-degenerate, i.e. it is different from any other internal mass, we can differentiate the MI with respect to this mass and obtain

$$\partial_{m_i^2} I(a_1\,, \ldots\,, a_i\,, \ldots\,, a_m\,; b_1\,, \ldots\,, b_r) = a_i I(a_1\,, \ldots\,, a_i + 1\,, \ldots\,, a_m\,; b_1\,, \ldots\,, b_r)\,, \tag{5.2}$$

as it trivially follows from the definition of the loop denominator $D_i = l_i^2 - m_i^2$ (note that an overall sign would change in the case of Euclidean signature). If m_i^2 is degenerate, the r.h.s. would include more contributions, according to the number of denominators which depend on the mass m_i^2.

The integral generated by differentiation belongs, by construction, to the same sector as the starting integral, hence it still lies in the space spanned by the chosen basis of MIs. Therefore, we can use IBPs in order to express the r.h.s. of eq. (5.2) as a linear combination of MIs, which include $I(a_1\,, \ldots\,, a_m\,; b_1\,, \ldots\,, b_r)$ and its subtopologies,

$$\begin{aligned}\partial_{m_i^2} I(a_1\,, \ldots\,, a_m\,; b_1\,, \ldots\,, b_r) =& A^2_{m_i}(\epsilon, \vec{x}) I(a_1\,, \ldots\,, a_m\,; b_1\,, \ldots\,, b_r) \\ &+ \text{Subtopologies}\,, \end{aligned} \tag{5.3}$$

where the coefficient $A^2_{m_i}$, as well as the coefficients multiplying the subtopology contributions, is a rational function of the kinematic invariants and of the space-time dimensions, as it is inherited from IBPs.

This construction can be systematically applied to all MIs of the basis, which we will hereby denote with the vector $\mathbf{F}(\epsilon, \vec{s}, m_j^2)$. The resulting DEQs can be collected into a system of coupled first-order DEQs, represented by the matrix equation

$$\partial_{m_i^2}\mathbf{F}(\epsilon, \vec{x}, m_j^2) = \mathbb{A}_{m_i^2}(\epsilon, \vec{x}, m_j^2)\mathbf{F}(\epsilon, \vec{x}, m_j^2)\,, \tag{5.4}$$

where $\mathbb{A}_{m_i^2}$ is a $N \times N$ matrix with rational entries. If the master integrals $\mathbf{F}(\epsilon, \vec{x}, m_j^2)$ are ordered by growing number of loop denominators, $\mathbb{A}_{m_i^2}$ acquires a natural block-triangular structure, whose size of the non-trivial blocks is determined by the number of MIs belonging to a same sector.

Analogous systems of DEQs can be generated by differentiating w.r.t. the external invariants. First of all, we observe that, given a Feynman integral with n external legs, momentum conservation allows us to define $n_{\text{ext}} = n(n-1)/2$ distinct external invariants, which we can identify with the scalar products $s_{ij} = p_i \cdot p_j$,

$$\vec{x} = \{x_1\,, x_2\,, \ldots\,, x_{n_{\text{ext}}}\} = \{s_{11}\,, \ldots\,, s_{1\,(n-1)}\,, \ldots s_{(n-1)\,(n-1)}\}\,. \tag{5.5}$$

In the momentum-space representation (5.1), Feynman integrands depend, rather than on the kinematic invariants $\vec{x}$, on the external momenta p_i^α. Therefore, in order to compute the derivates of the MIs w.r.t. $\vec{x}$, we need to invert the chain-rule which expresses the derivatives w.r.t. to the $(n-1)$ independent momenta in terms of the invariants,

$$\frac{\partial}{\partial p_i^\alpha} = \sum_{j=1}^{n_{\text{ext}}} \frac{\partial x_j}{\partial p_i^\alpha}\frac{\partial}{\partial x_j}\,, \quad i = 1, 2, \ldots, n-1\,. \tag{5.6}$$

Eq. (5.6) can be contracted with all possible p_k^α in order to obtain $(n-1)^2$ scalar relations,

$$p_k^\alpha \frac{\partial}{\partial p_i^\alpha} = p_k^\alpha \sum_{j=1}^{n_{\text{ext}}} \frac{\partial x_j}{\partial p_i^\alpha}\frac{\partial}{\partial x_j}\,, \quad i, k = 1, 2, \ldots, n-1\,. \tag{5.7}$$

We observe that, in the case $n > 2$, not all derivatives w.r.t. external momenta are independent (when they are applied to scalar Feynman integrals) due to the existence of $n_{\text{LI}} = (n-1)(n-2)/2$ LIs. The number of independent equations of the type (5.6) is, hence, reduced to

$$(n-1)^2 - n_{\text{LI}} = n_{\text{ext}}\,, \tag{5.8}$$

which exactly matches the number of external invariants $\vec{x}$. Once a set of independent relations has been chosen, they can be inverted in order to express the derivatives w.r.t. each x_i in terms of a combination of derivatives w.r.t. the external momenta.

As for the case of mass derivatives, by acting with such differential operators on the integral representation (5.1), we can express the derivatives of each MI as a linear combination of integrals belonging its sector and subtopology contributions, which can be further reduced to a combination of MIs by applying IBPs.

Therefore, if we include the internal masses into the definition of the vector $\vec{x}$, we can write, in total generality, the system of DEQs satisfied by the MIs in *any* of the kinematic variables $x_i \in \vec{x}$ as

$$\partial_{x_i}\mathbf{F}(\epsilon, \vec{x}) = \mathbb{A}_{x_i}(\epsilon, \vec{x})\mathbf{F}(\epsilon, \vec{x})\,, \tag{5.9}$$

where $\mathbb{A}_{x_i}$ are $N \times N$ block-triangular matrices with rational entries in $\vec{x}$ and in the space-time dimensions.

At this level, a few observations on eq. (5.9) are in order:

- It is often convenient to express the dimensionful invariants $\vec{x}$ in terms of dimensionless variables, which can be obtained, for instance, by choosing one of the variables $\vec{x}$, say x_1 (which, typically, corresponds to an internal mass) and by building the ratios

  $$\hat{x}_i = \frac{x_i}{x_1}\,, \quad i \neq 1\,. \tag{5.10}$$

  The systems of DEQs (5.9) are then totally equivalent to considering the DEQs in x_1 and the systems of DEQs in the parameters $\hat{x}_i$,

  $$\partial_{\hat{x}_i}\mathbf{F}(\epsilon, x_1, \vec{\hat{x}}) = \mathbb{A}_{\hat{x}_i}(\epsilon, \vec{\hat{x}})\mathbf{F}(\epsilon, x_1, \vec{\hat{x}})\,, \tag{5.11}$$

  In this way, the DEQs in the single dimensionful invariant x_1 can be solved trivially and produces the mass scaling of the MIs, which can also be determined by direct power counting. The dependence of the MIs on x_1 can then be removed by a proper rescaling of the MIs, in such a way to deal with a basis of dimensionless integrals $\mathbf{F}(\epsilon, \vec{\hat{x}})$, whose non-trivial kinematic dependence is fully encoded in the system of DEQs in the dimensionless variables $\vec{\hat{x}}$;

- The systems of DEQs in $\hat{x}_i$ can be solved sequentially, i.e. by integrating in one variable at a time and by adding to the resulting solution an integration constant that depends on the kinematic variables corresponding to the unsolved DEQs. The continuity of $\mathbf{F}(\epsilon, \vec{\hat{x}})$ as a function of the kinematic invariants, which implies the *Schwarz integrability condition*

  $$\partial_{\hat{x}_i}\partial_{\hat{x}_j}\mathbf{F}(\epsilon, \vec{\hat{x}}) = \partial_{\hat{x}_j}\partial_{\hat{x}_i}\mathbf{F}(\epsilon, \vec{x})\,, \tag{5.12}$$

  ensures the convergence of the integration procedure. In fact, if impose (5.12) on eq. (5.9), we obtain

  $$\partial_{\hat{x}_i}\mathbb{A}_{\hat{x}_j} - \partial_{\hat{x}_j}\mathbb{A}_{\hat{x}_i} + [\mathbb{A}_{\hat{x}_j}\,, \mathbb{A}_{\hat{x}_i}] = 0\,, \tag{5.13}$$

  which states that all non-factorizable terms, i.e. terms that depend on several invariants, are common to all the corresponding DEQs;

- After the systems of DEQs in all variables have been solved, the residual integration constant is independent of the kinematics and, hence, must be fixed by imposing suitable *boundary conditions*. The latter correspond to the analytic values of the MIs at some specific kinematic point. Thus, their determination through independent integration methods can be a rather difficult task.

  However, it is often possible to fix the boundary constants by imposing physical requirements on the solution obtained from the integration of the DEQs, such as the regularity or the finiteness of the MIs at kinematic pseudo-thresholds. In such cases, as we will discuss later on in this chapter, quantitative relations between the boundary constants can be inferred from the qualitative information about the analyticity properties of the MIs, due to the singularity structure of the very same DEQs. In this way, the amount of independent input information that is needed for the computation of MIs trough the DEQs method, is consistently reduced;

- According to the above discussion, it seems not possible to apply the DEQs method to MIs which depend on a single dimensionful kinematic invariant. In fact, in this case, the solution of the system of DEQs would simply provide the mass dimensions of each MIs and all relevant information would be enclosed in the boundary constants, which have to be determined independently. Nonetheless, in some cases (see, for example, [177–179]) it is possible to evaluate single-scale MIs through the DEQs method, by introducing an additional auxiliary kinematical invariant, so to produce a non-trivial system of DEQs. After these DEQs have been solved, the MIs can be obtained by properly taking the limit of the auxiliary invariant (which restores the original kinematic configuration) on the resulting solution. The introduction of auxiliary invariants can also be useful for the fixing of boundary constants, as we will explicitly see in chapter 8.

- Auxiliary parameters have also been used in [80], where it has been proposed a complementary method to derive a DEQs for MIs, based on the introduction of a smart rescaling of the external momenta. The differentiation w.r.t. the scaling parameter allows to obtain a simplified system of DEQs, whose solution provides the expression of the MIs when the auxiliary parameter is set back to one. This method has been successfully applied to the computation of the planar five-point two-loop massless MIs with one off-shell leg [79].

Differential equations in Baikov parametrization

The derivation of the systems of DEQs (5.9) resorted to the expression of the derivatives w.r.t. kinematic invariants in terms of partial derivatives w.r.t. the external momenta, which constitute the natural differentiation variables for Feynman integrals in momentum-space representation (5.1). It has been recently suggested in [60] an alternative way to derived DEQs for MIs based on the Baikov representation of Feynman integrals, which we introduced in section 2.4.

From the Baikov representation of $I(a_1, \dots, a_m; b_1, \dots, b_r)$ given in eq. (2.101),

$$I(a_1, \dots, a_m; b_1, \dots, b_r) = C^{(\ell,n)}(d)\,(G(p_j))^{(1-d_\perp)/2}\int\prod_{i=1}^{n_{\text{sp}}}\mathrm{d}z_i\,(G(z_j))^{(d_\perp-1-\ell)/2}\times \frac{z_{m+1}^{-b1}\cdots z_{m+1}^{-b_{m+1}}}{z_1^{a_1}\cdots z_m^{a_m}}, \tag{5.14}$$

we can easily realize that the differentiation w.r.t. any kinematic invariant x_i only affects the two Gram determinants $G(p_j)$ and $G(z_j)$ and yields to

$$\begin{aligned}\partial_{x_i} I(a_1, \dots, a_m; b_1, \dots, b_r) = \frac{1-d_\perp}{2}\left(\frac{\partial_{x_i}G(p_j)}{G(p_j)}\right) I(a_1, \dots, a_m; b_1, \dots, b_r) + \\ C^{(\ell,n)}(d)\,(G(p_j))^{(1-d_\perp)/2}\int\prod_{i=1}^{n_{\text{sp}}}\mathrm{d}z_i\,(G(z_j))^{(d_\perp-1-\ell)/2}\,\frac{d_\perp-1-\ell}{2}\left(\frac{\partial_{x_i}G(z_j)}{G(z_j)}\right)\times \\ \frac{z_{m+1}^{-b1}\cdots z_{m+1}^{-b_{m+1}}}{z_1^{a_1}\cdots z_m^{a_m}},\end{aligned} \tag{5.15}$$

where we have immediately recognized that the derivative of $G(p_j)$ produces a term proportional to the original integral. In order to cast the second term on the r.h.s. back to a Feynman integral of the type $I(a'_1, \dots, a'_m; b'_1, \dots, b'_r)$, we observe that $G(z_j)$ is a polynomial both in the integration variables and in the kinematic invariants and so are its derivatives. Hence, we can write down a *syzygy equation*

$$v\partial_{x_i}G(z_j) = \sum_k v_k \partial z_k G(z_j)\,, \tag{5.16}$$

whose solutions v and v_k allows us to trade the derivatives w.r.t. the external invariant x_i with derivatives w.r.t. the integration variables z_i,

$$\begin{aligned}
&\int \prod_{i=1}^{n_{\rm sp}} \mathrm{d}z_i\,(G(z_j))^{(d_\perp-1-\ell)/2}\,\frac{d_\perp-1-\ell}{2}\left(\frac{\partial_{x_i}G(z_j)}{G(z_j)}\right)\frac{z_{m+1}^{-b1}\cdots z_{m+1}^{-b_{m+1}}}{z_1^{a_1}\cdots z_m^{a_m}}\\
&\quad= -\int \prod_{i=1}^{n_{\rm sp}} \mathrm{d}z_i\,(G(z_j))^{(d_\perp-1-\ell)/2}\,\frac{d_\perp-1-\ell}{2}\left(\sum_k\frac{v_k}{v}\frac{\partial z_k G(z_j)}{G(z_j)}\right)\frac{z_{m+1}^{-b1}\cdots z_{m+1}^{-b_{m+1}}}{z_1^{a_1}\cdots z_m^{a_m}}\\
&\quad= \sum_k\int \prod_{i=1}^{n_{\rm sp}} \mathrm{d}z_i\,(G(z_j))^{(d_\perp-1-\ell)/2}\,\partial_{z_k}\left(\frac{v_k}{v}\frac{z_{m+1}^{-b1}\cdots z_{m+1}^{-b_{m+1}}}{z_1^{a_1}\cdots z_m^{a_m}}\right),
\end{aligned} \tag{5.17}$$

where, in the second equality, we have integrated by parts and neglected the vanishing boundary terms. If we assume v to be independent of z_i, the explicit differentiation of eq. (5.17) will now originate Feynman integrals belonging to the same space as $I(a_1\,,\ldots\,,a_m\,;b_1\,,\ldots\,,b_r)$, so that the subsequent application of IBPs would produce a DEQs consistent with eq. (5.9).

Similar techniques, restricted to the homogeneous part of the DEQs, have been applied to the Baikov representation (5.14) in [61], in order to define DEQs polynomials $v_k\partial_{z_k}$ (similar to the IBPs polynomials discussed in section 2.5) which avoid the generation of integrals with higher power of denominators, in favour of higher-rank numerators in the ISPs.

5.3 Solution of the differential equations

The block-triangular structure of the systems of DEQs (5.9) suggests a natural bottom-up solving strategy, where the sectors with lower number of denominators are solved first. Once their analytic expression is known, the latter is used as input for the inhomogeneous part of the DEQs for higher sectors. In order to illustrate this procedure we will restrict to the case of N master integrals $\mathbf{F}(\epsilon,x)$ depending on a single (dimensionless) kinematic invariant x that obey the system of first-order coupled DEQs,

$$\partial_x\mathbf{F}(\epsilon,x) = \mathbb{A}(\epsilon,x)\mathbf{F}(\epsilon,x)\,. \tag{5.18}$$

In the presence of more than one kinematic invariant, the following discussion is extended according to the method outlined in the previous section, i.e. by solving the DEQs in one variable at a time and by fixing the integration constants (that might depend on the other invariants) by demanding that the resulting solution satisfies all other systems of DEQs.

Let us now consider the DEQs for one single sector, containing n MIs. We can distinguish two cases:

- $\mathbf{n=1}$ **:** If a sector contains one single MI, $\mathrm{F}_i(\epsilon,x)$, the latter can be immediately obtained by applying Euler variations of constants to the first-order inhomogeneous DEQ

$$\partial_x\mathrm{F}_i(\epsilon,x) = \mathbb{A}_{ii}(\epsilon,x)\mathrm{F}_i(\epsilon,x) + \mathbf{S}(\epsilon,x)\,, \tag{5.19}$$

where the inhomogeneous term $\mathbf{S}(\epsilon, x)$ collects the contributions from subtopologies, which we assume to be known. The solution of eq. (5.19) can be written by first determining the homogeneous solution $H_i(\epsilon, x)$, which is simply given by the integration factor

$$H_i(\epsilon, x) = \exp\left(\int_{x_0}^{x} \mathrm{d}t\ \mathbb{A}_{ii}(\epsilon, t)\right), \tag{5.20}$$

and by substituting the *ansatz* $\mathrm{F}_i(\epsilon, x) = f_i(\epsilon, x) H_i(\epsilon, x)$ into the complete equation, in order to obtain a DEQ for the arbitrary function $f_i(\epsilon, x)$. The solution of the latter yields to

$$\mathrm{F}_i(\epsilon, x) = H_i(\epsilon, x)\left[1 + \int_{x_0}^{x} \mathrm{d}t\ \frac{\mathbf{S}(\epsilon, t)}{H_i(\epsilon, t)}\right] c_i(\epsilon)\,, \tag{5.21}$$

where c_i is an integration constant, which can have a residual dependence on ϵ, to be fixed by imposing suitable boundary conditions. We remark once more that, in the multivariate case, the constant c_i should be promoted to a function of the other kinematic variables, and determined by requiring that eq. (5.21) be a solution of the systems of DEQs in the other invariants. Eq. (5.21) provides, at least formally, a close form representation of MI $\mathrm{F}_i(\epsilon, x)$ as a one-fold integral which retains full dependence in the dimensional regulator ϵ. Although in many cases this one-fold integral can be evaluated explicitly in terms of known class of functions, such as hypergeometric functions, the determination of the exact dependence of $\mathrm{F}_i(\epsilon, x)$ on ϵ is an unnecessary complication, since we are usually interested in the expansion of the MIs around four-dimensions, i.e. for $\epsilon \sim 0$. Therefore, we can expand $\mathrm{F}_i(\epsilon, x)$ as a Laurent series in ϵ,

$$\mathrm{F}_i(\epsilon, x) = \sum_{k=k_{\text{min}}}^{\infty} \mathrm{F}_i^{(k)}(x)\epsilon^k\,, \tag{5.22}$$

and substitute such expansion into (5.19), in order to obtain directly a set of *chained* DEQs for the Laurent coefficients $\mathrm{F}_k^{(k)}$. These DEQs can be solved sequentially starting from the leading pole $\mathrm{F}_i^{(k_{\text{min}})}$. The DEQs for the Taylor coefficients are usually much simpler, since the dependence on ϵ is completely factorized from the integration over the kinematic variable x.

Finally, let us point out that, in the Laurent expansion (5.22), the generally negative power k_{min} is determined by the convergence properties of $\mathrm{F}_i(\epsilon, x)$. However, it is always possible to define a proper normalization of the MIs,such that they become finite in the $\epsilon \to 0$ limit and, hence, they can be Taylor expanded in ϵ. Therefore, from now on, we will always assume to work with a basis of finite MIs.

- $\mathbf{n > 1}$ **:** If a sector contains more than one MIs, we must consider the full set of n coupled first-oder DEQs simultaneously,

$$\partial_x \mathbf{F}(\epsilon, x) = \mathbb{A}_n(\epsilon, x)\mathbf{F}(\epsilon, x) + \mathbf{S}(\epsilon, x)\,, \tag{5.23}$$

where $\mathbf{F}(\epsilon, x)$ is the vector of the n MIs of the sector, $\mathbb{A}_n(\epsilon, x)$ is the $n \times n$ submatrix of the DEQs matrix $\mathbb{A}(\epsilon, x)$ that corresponds to the homogeneous part of the DEQs for $\mathbf{F}(\epsilon, x)$ and, again, $\mathbf{S}(\epsilon, x)$ stems for known subtopology contributions.

Obviously, if $\mathbb{A}_n(\epsilon, x)$ happens to be triangular, the solution of eq. (5.23) can be obtained, analogously to the case $n = 1$, integrating by quadrature one MI at a time. If this is not case for the chosen set of MIs, we could try to find basis transformation which brings $\mathbb{A}_n(\epsilon, x)$ to a triangular form. The determination of such basis transformation, which (as we will discuss later on) is related to the solution of a n-th order homogeneous DEQ, is a highly non trivial task, if the full dependence on the space-time dimensions is retained. Conversely, the Taylor expansion of the MIs around $\epsilon \sim 0$ can significantly simplify the structure of the DEQs and, in most applications, it has been verified that it is possible to find, by simple trail and error, a basis of MIs, which triangularizes the DEQs in $\epsilon = 0$. Hence, in such cases, the expression of the Taylor coefficients of the MIs can be obtained in closed form by repeated quadrature, as discussed above. In the following, we will assume this triangularization to be possible, and we postpone the discussion of the solution of systems of DEQs which remain coupled even in $\epsilon = 0$ to chapter 9.

5.4 Canonical systems of differential equations

In the previous section we showed that, if the systems of DEQs (5.9) for the MIs $\mathbf{F}(\epsilon, \vec{x})$ become triangular in the $\epsilon \to 0$ limit, we are able to determine the coefficients $\mathbf{F}^{(k)}(\vec{x})$ of their Taylor expansion around $\epsilon = 0$ by simple quadrature. If the system of DEQs for $\mathbf{F}(\epsilon, \vec{x})$ is non-triangular, we can try to perform a change of basis, i.e. to find a similarity transformation $\mathbb{B}(\epsilon, x)$ such that the new set of MIs $\mathbf{I}(\epsilon, \vec{x})$, defined by

$$\mathbf{F}(\epsilon, \vec{x}) = \mathbb{B}(\epsilon, x)\mathbf{I}(\epsilon, \vec{x}) \,, \tag{5.24}$$

obey systems of DEQs

$$\partial_{x_i}\mathbf{I}(\epsilon, \vec{x}) = \hat{\mathbb{A}}_{x_i}(\epsilon, \vec{x})\mathbf{I}(\epsilon, \vec{x}) \,, \tag{5.25}$$

with

$$\hat{\mathbb{A}}_{x_i}(\epsilon, \vec{x}) = \mathbb{B}^{-1}(\epsilon, x)[\mathbb{A}_{x_i}(\epsilon, \vec{x}) - \partial_{x_i}\mathbb{B}(\epsilon, x)] \,, \tag{5.26}$$

such that the transformed matrices $\hat{\mathbb{A}}(\epsilon, \vec{x})$ become triangular in the $\epsilon \to 0$ limit.

Once such basis has been found, the *chained* systems of DEQs for the Taylor coefficients $\mathbf{F}^{(k)}(\vec{x})$ can be solved bottom-up, starting from leading term $\mathcal{O}(\epsilon^0)$. At every order in the ϵ, the solution can be represented by a one-dimensional integral over the previous terms in the expansion.

Of course, besides writing the solution as a formal one-fold integral, we must be able to evaluate it in terms of known special functions. The *recursive* structure of the Taylor coefficients of the MIs suggests a natural representation of the solution in terms *generalized polylogarithms* (GPLs) [82–86] or, more generally, *Chen iterated integrals* [87].

Before discussing the properties of this class of special functions, we would like to observe that, as it was first suggested by Henn in [81], the representation of the solution in terms of iterated integrals is streamlined if we are able to determine a rotation matrix $\mathbb{B}(\epsilon, \vec{x})$ such that ϵ-dependence the coefficient matrices $\mathbb{A}_{x_i}(\vec{x})$ of the new systems of DEQs is completely *factorized* from the kinematics,

$$\partial_{x_i}\mathbf{I}(\epsilon, \vec{x}) = \epsilon\, \mathbb{A}_{x_i}(\vec{x})\mathbf{I}(\epsilon, \vec{x}) \,. \tag{5.27}$$

In this case, the order-by-order solution of eq. (5.27) is almost trivialized, since all the DEQs are completely decoupled as $\epsilon \to 0$. This implies that, at order zero in the ϵ-expansion, the MIs are pure constants,

$$\partial_{x_i}\mathbf{I}^{(0)}(\epsilon,\vec{x}) = 0 \quad \longrightarrow \quad \mathbf{I}^{(0)}(\epsilon,\vec{x}) \equiv \mathbf{I}^{(0)}(\epsilon,\vec{x_0})\,, \tag{5.28}$$

and, while in the case of a triangular system of DEQs the k-th order coefficient is generally written as an integral involving all the previously determined orders, in the ϵ-factorized form of eq. (5.27), $\mathbf{I}^{(k)}(\epsilon,\vec{x})$ is directly determined as an integral over the $(k-1)$-order term only, since

$$\partial_{x_i}\mathbf{I}^{(k)}(\epsilon,\vec{x}) = \mathbb{A}_{x_i}(\vec{x})\mathbf{I}^{(k-1)}(\epsilon,\vec{x})\,. \tag{5.29}$$

The simplified structure of ϵ-factorized systems of DEQs is also reflected in the decoupling of the integrability condition (5.13), which splits into two equations,

$$\partial_{x_i}\mathbb{A}_{x_j} - \partial_{x_j}\mathbb{A}_{x_i} = 0\,, \qquad [\mathbb{A}_{x_j}\,,\mathbb{A}_{x_i}] = 0\,, \tag{5.30}$$

that must be individually satisfied.

In multi-scale problems, is often convenient to combine the systems of partial DEQs into a total differential

$$\mathrm{d}\mathbf{I}(\epsilon,\vec{x}) = \epsilon\mathrm{d}\mathbb{A}(\vec{x})\mathbf{I}(\epsilon,\vec{x})\,, \tag{5.31}$$

where the matrix $\mathbb{A}(\vec{x})$ is defined by

$$\partial_{x_i}\mathbb{A}(\vec{x}) = \mathbb{A}_{x_i}(\vec{x})\,. \tag{5.32}$$

When the MIs are expanded around $\epsilon = 0$, eq. (5.31) is reduced to the set of chained DEQs for the Taylor coefficients,

$$\mathrm{d}\mathbf{I}^{(k)}(\vec{x}) = \mathrm{d}\mathbb{A}(\vec{x})\mathbf{I}^{(k-1)}(\vec{x})\,, \tag{5.33}$$

which can be naturally solved by quadrature as

$$\mathbf{I}^{(k)}(\vec{x}) = \int_\gamma \mathrm{d}\mathbb{A}\,\mathbf{I}^{(k-1)}(\vec{x})\,, \tag{5.34}$$

or, equivalently,

$$\mathbf{I}^{(k)}(\vec{x}) = \int_\gamma \underbrace{\mathrm{d}\mathbb{A}\ldots\mathrm{d}\mathbb{A}}_{\text{k times}}\mathbf{I}^{(0)}(\vec{x_0})\,. \tag{5.35}$$

We will provide a more formal definition of the integration along the *path* γ in the kinematic space introduced in eq. (5.34) in section 5.6. For the moment, let us just stress that, if the system is in ϵ-factorized form, the bottom-up solving strategy discussed in section 5.3, which proceeds one block after the other, is streamlined to a pure matrix multiplication, as dictated by eq. (5.35).

Although it is has been obtained in a straightforward way, the integral representation of eq. (5.35) is still a purely formal one. However, its evaluation to known iterated integrals becomes likewise algorithmic if the systems of DEQs (5.31) is in dlog-*form*, i.e. if the coefficient matrix $\mathbb{A}(\vec{x})$ can be written as

$$\mathbb{A}(\vec{x}) = \sum_{i=1}^{k} \mathbb{M}_i \log \eta_k(\vec{x}) \,, \tag{5.36}$$

with $\mathbb{M}_i$ being completely constant matrices. In such case, the full kinematic dependence of the DEQs is enclosed in exact differentials of the logarithms of the so-called *letters* $\eta_k(\vec{x})$ of the DEQs. We refer to the full set of letters appearing in eq. (5.36) as the *alphabet* of the problem. As we will see in section 5.6, eq. (5.36) ensures that the MIs can be expressed, order-by-order in ϵ, in terms of Chen iterated integrals. In addition, the specific functional dependence of the alphabet on the kinematic variables $\vec{x}$ can reduce the set of iterated integrals appearing in the solutions to particular classes of functions, such as GPLs.

In summary, the determination of the general solution of a system of DEQs in terms of iterated integrals, is rendered almost entirely algorithmic if :

1. The ϵ-dependence is completely *factorized* from the kinematics, as in eq. (5.31);
2. The coefficient matrix $\mathbb{A}(\vec{x})$ is in dlog-form, as defined by eq. (5.36).

In fact, the ϵ-factorization allows to decouple, order-by-order, the DEQs for the Taylor coefficients $\mathbf{I}^{(k)}(\vec{x})$ and the dlog-form of the kinematic matrix provides a straightforward representation of such coefficients in terms of Chen iterated integrals. For such reasons, a system of DEQs which satisfies these two requirements is commonly referred to as *canonical* system of DEQs.

In light of the above discussion, it is clear that the main difficulty in the computation of the MIs through the DEQs method lies in the determination of a basis of MIs which fulfil canonical systems of equations. In this respect, an important observation is that canonical MIs are pure functions of *uniform transcendentality*, i.e. order-by-order in the ϵ-expansion, each MI is expressed as a combination of transcendental functions of *uniform weight* [81]. In fact, if we define the weight of a trascendental function as the number of repeated integrations over a dlog which define it, it is evident from eq. (5.35) that $\mathbf{I}^{(k)}(\vec{x})$ has uniform weight k (provided that the trascendentality properties are not spoiled by the boundary constants).

Therefore, if we could determine *a priori* whether an integral is a function of uniform weight, we could have a practical criterion to construct a basis of putative canonical MIs. One possibility to test the trascendental weight of a MIs consists in considering its integral representation and, by performing suitable manipulation of the integrand, bring it to a form where its trascendentality properties are manifest. For instance, in relatively simple cases, it is possible to cast the Feynman parameter representation of a MI (which, *a posteriori* turns out to be canonical), into an explicit dlog-form [180].

Morevoer, it has been conjectured that integrals with a *unit leading singularity* are pure functions with *uniform transcendentality* [81, 107, 108]. Therefore, the computation of the leading singularity of an integral, which can be extracted by evaluating its maximal-cut in a finite number of dimensions, can be used to check whether the integral has uniform weight and, eventually, to practically define canonical MIs by absorbing the non-constant terms of the leading singularity. This last point is strictly related to the possibility of reaching a canonical form by rotating the MIs by the *homogeneous* *solution* the DEQs in $\epsilon = 0$. This point will be throughly discussed in the next section.

Despite the presence of these guiding criteria, the issue of the existence and, hence, of the determination, of a canonical basis for a general multi-loop case still remains an open problem, especially as far as regards its extension to the case of MIs which cannot be expressed in terms polylogarithmic functions (see chapter 9 for a detailed discussion). Nonetheless, in the last few years several constructive methods to determine a similarity transformation $\mathbb{B}(\epsilon, \vec{x})$ that brings a basis MIs $\mathbf{F}(\epsilon, \vec{x})$ to a canonical form have been put forward, under a number of different assumptions on the dependence of the DEQs both on ϵ and $\vec{x}$:

- The Magnus exponential method proposed [88] can be used to bring to canonical form systems of DEQs, in an arbitrary number of kinematic variables, with a *linear* dependence on ϵ and it is based on the Magnus expansion representation of the solution of the DEQs at $\epsilon = 0$;
- In [89] the ϵ-linearity requirement is restricted to the homogeneous part of DEQs for each sector and a bottom-up approach to the construction of a canonical basis, based on the solution of the homogeneous system at $\epsilon = 0$ for each sector in terms of *rational functions*, is proposed. The recursive construction of similarity transformation to a canonical basis through rational *ansatzes* has been systematized in [92] and implemented in the code Canonica [181];
- The algorithm proposed in [90, 91], which can be applied to systems of DEQs in a single variable x, employs *balance transformations* in order to bring the singularities of the DEQs in x to a fuchsian form and to factor out ϵ from its coefficient matrices. This algorithm has been implemented in the public codes Fuchsia [182] and epsilon [183];
- Within the study of the DEQs for MIs which evaluates to elliptic functions (see chapter 9), in [93] it has been proposed a method, based on the properties of the Picard-Fuchs operator, to reduce to a minimum size the blocks of a system of DEQs (admitting expansion in positive powers of ϵ) which remain coupled in the $\epsilon \to 0$ limit. In the case of MIs that evaluate to GPLs, the complete reduction to blocks of unit size can be used to bring the DEQs in ϵ-factorized form.

In the next section, we describe in some detail the Magnus exponential method, which has successfully applied to several multi-loop computations [1, 3, 88, 95, 96](that involve up to three different dimensionless kinematic variables) and that will be used throughout chapters 6-8.

5.5 Magnus exponential method

Let us start by considering a set of m MIs $\mathbf{F}(\epsilon, x)$ that depend on a single kinematic variable x. We assume $\mathbf{F}(\epsilon, x)$ to fulfil a system of DEQs with a *linear* dependence on ϵ,

$$\partial_x \mathbf{F}(\epsilon, x) = [\mathbb{A}_0(x) + \epsilon \mathbb{A}_1(x)] \, \mathbf{F}(\epsilon, x) \, , \tag{5.37}$$

The entries of $\mathbb{A}_{0,1}(x)$ are, in general, rational functions of x. Under this assumption, it is clear from the definition of the similarity transformation given in eq.(5.24) that, if $\mathbb{B}(x)$ is a $m \times m$ *matrix valued solution* of the DEQs at $\epsilon = 0$,

$$\partial_x \mathbb{B}(x) = \mathbb{A}_0(x) \, \mathbb{B}(x) \, , \tag{5.38}$$

than the new set of MIs $\mathbf{I}(\epsilon, x)$, defined through

$$\mathbf{F}(\epsilon, x) = \mathbb{B}(x)\mathbf{I}(\epsilon, x)\,, \tag{5.39}$$

fulfils an ϵ-factorized system of DEQs

$$\partial_x \mathbf{I}(\epsilon, x) = \epsilon\hat{\mathbb{A}}_1(x)\mathbf{F}(\epsilon, x)\,, \tag{5.40}$$

where the new coefficient matrix $\hat{\mathbb{A}}_1(x)$ is given by

$$\hat{\mathbb{A}}_1(x) = \mathbb{B}^{-1}(x)\mathbb{A}_1(x)\mathbb{B}(x)\,. \tag{5.41}$$

Therefore, by starting from ϵ-linear DEQs, the determination of a change of basis which brings the system to an ϵ-factorized form is rephrased in terms of the solution of the first-order homogeneous DEQ (5.38) for the linear operator $\mathbb{B}$. In general cases, the solution of eq. (5.38) is far from being trivial. In cite [88] it has been proposed to determine the of solution eq. (5.38) of by means of the Magnus exponential [94, 184]

$$\mathbb{B}(x) = e^{\Omega[\mathbb{A}_0](x)} \equiv \mathbb{1} + \Omega[\mathbb{A}_0](x) + \frac{1}{2!}\Omega[\mathbb{A}_0](x)\,\Omega[\mathbb{A}_0](x) + \ldots\,, \tag{5.42}$$

where $\Omega[\mathbb{A}_0]$ is a linear operator defined by the infinite series

$$\Omega[\mathbb{A}_0](x) = \sum_{n=0}^{\infty} \Omega_n[\mathbb{A}_0](x)\,, \tag{5.43}$$

whose summands are built from iterated integrals of the nested commutators of the kernel matrix $\mathbb{A}_0(x)$. For instance, the first three terms of eq. (5.43) are given by

$$\begin{aligned}
\Omega_1[\mathbb{A}_0](x) =& \int \mathrm{d}x_1\, \mathbb{A}_0(x_1),\\
\Omega_2[\mathbb{A}_0](x) =& \frac{1}{2}\int \mathrm{d}x_1 \int \mathrm{d}x_2\, [\mathbb{A}_0(x_1), \mathbb{A}_0(x_2)],\\
\Omega_3[\mathbb{A}_0](x) =& \frac{1}{6}\int \mathrm{d}x_1 \int \mathrm{d}x_2 \int \mathrm{d}x_3\, [\mathbb{A}_0(x_1), [\mathbb{A}_0(x_2), \mathbb{A}_0(x_3)]] + [\mathbb{A}_0(x_3), [\mathbb{A}_0(x_2), \mathbb{A}_0(x_1)]].
\end{aligned} \tag{5.44}$$

Such representation of $\mathbb{B}(x)$ is useful whenever the Magnus series terminates after a certain number of terms, i.e. if there exist some finite $n_{\max}$ such that $\Omega_n[\mathbb{A}_0](x) = 0$, for $n > n_{max}$. In such cases, eq. (5.42) provides an *exact* solution of the DEQ (5.38) and it can be used to rotate $\mathbf{F}(\epsilon, x)$ to the canonical MIs $\mathbf{I}(\epsilon, \vec{x})$. We observe that the Magnus series terminates at $n_{\max} = 1$ for every matrix which commutes with its integral but, in more general cases, nested commutators appearing in eq. (5.45) can become vanishing after a larger number of steps. For later convenience, we notice that, if we split $\mathbb{A}_0(x)$ into diagonal and off-diagonal parts,

$$\mathbb{A}_0(x) = \mathbb{D}_0(x) + \mathbb{N}_0(x)\,, \tag{5.45}$$

the Magnus exponential with kernel $\mathbb{D}_0(x)$ is simply given by

$$e^{\Omega[\mathbb{D}_0](x)} = e^{\int \mathrm{d}x_1\, \mathbb{B}_0(x_1)}\,, \tag{5.46}$$

and $\Omega[\mathbb{N}_0](x)$ can be equivalently computed by means of the Dyson series [185],

$$\Omega[\mathbb{N}_0](x) = \mathbb{1} + \sum_{n=1}^{\infty} Y_n[\mathbb{N}_0](x)\,, \tag{5.47}$$

with

$$Y_n[\mathbb{N}_0](x) = \int \mathrm{d}x_1 \int \mathrm{d}x_2 \, \dots \int \mathrm{d}x_n \mathbb{N}_0(x_1)\mathbb{N}_0(x_1) \, \dots \mathbb{N}_0(x_n) \, . \tag{5.48}$$

The correspondence between the summands of the two series is (we give the same kernel function as understood)

$$\begin{aligned} Y_1 =& \Omega_1 \, , \\ Y_2 =& \Omega_2 + \frac{1}{2!}\Omega_1^2 \, , \\ Y_3 =& \Omega_3 + \frac{1}{2!}(\Omega_1\Omega_2 + \Omega_2\Omega_1) + \frac{1}{3!}\Omega_1^3 \, , \\ & \dots \end{aligned} \tag{5.49}$$

The Magnus algorithm for differential equations in many variables

The Magnus exponential expansion can be used to bring to an ϵ-factorized form also systems of DEQs for MIs that depend on more than one kinematic variable.

Let us consider a set m of MIs $\mathbf{F}(\epsilon, \vec{x})$ depending on n variables $\vec{x} = (x_1 \, , x_2 \, , \dots x_n)$ which fulfils systems of DEQs of the type

$$\partial_a \mathbf{F}(\epsilon, \vec{x}) = \mathbb{A}_a^{[0]}(\epsilon, \vec{x})\mathbf{F}(\epsilon, \vec{x}) \, , \quad a = 1 \, , 2 \, , \dots \, , n \, , \tag{5.50}$$

where we have defined $\partial_a \equiv \partial / \partial x_a$. If $\mathbb{A}_a^{[0]}$ exhibits a linear dependence on ϵ,

$$\mathbb{A}_a^{[0]}(\epsilon, \vec{x}) = \mathbb{A}_{0\,,a}^{[0]}(\vec{x}) + \epsilon \mathbb{A}_{1\,,a}^{[0]}(\vec{x}) \, , \tag{5.51}$$

we can reach an ϵ-factorized form by applying a chain of Magnus rotations built on the kernel matrices $\mathbb{A}_{0\,,a}^{[0]}$. In particular, we can proceed as follows (for ease of notation we herby give as understood the dependence of all matrices on $\vec{x}$):

1. We begin by splitting $\mathbb{A}_{0\,,1}^{[0]}$ into diagonal and off-diagonal parts,

$$\mathbb{A}_{0\,,1}^{[0]} = \mathbb{D}_{0\,,1}^{[0]} + N_{0\,,1}^{[0]} \, , \tag{5.52}$$

and we perform a first change of basis $B^{[1]}$, given by the Magnus exponential of $\mathbb{D}_{0\,,1}^{[0]}$,

$$B^{[1]} = e^{\Omega[\mathbb{D}_{0\,,1}^{[0]}]} \quad \longrightarrow \quad \mathbb{A}_a^{[1]} = \left(\mathbb{B}^{[1]}\right)^{-1} (\mathbb{A}_a^{[0]}\mathbb{B}^{[1]} - \partial_a \mathbb{B}^{[1]}) \, . \tag{5.53}$$

This transformation absorbs the diagonal part of $\mathbb{A}_{0\,,1}^{[1]}$,

$$\mathbb{A}_{0\,,1}^{[1]} = N_{0\,,1}^{[1]} \, . \tag{5.54}$$

2. We iterate the same procedure for all the diagonal parts of the n systems of DEQs. After n transformations

$$\mathbb{B}^{[i]} = e^{\Omega[\mathbb{D}_{0\,,i}^{[i-1]}]} \, , \quad i = 1 \, , 2 \, , \dots \, , n, \tag{5.55}$$

we arrive to a set of matrices

$$\mathbb{A}_a^{[n]} = \mathbb{A}_{0\,,a}^{[n]} + \epsilon \mathbb{A}_{1\,,a}^{[n]} \, , \tag{5.56}$$

whose diagonal parts are completely ϵ-factorized, i.e.

$$\mathbb{A}_{0\,,a}^{[n]} = N_{0\,,a}^{[n]} \, , \quad \forall a = 1 \, , 2 \, , \dots \, , n \, . \tag{5.57}$$

3. We rotate away the ϵ independent part of $\mathbb{A}_1^{[n]}$ with the Magnus transformation

$$\mathbb{B}^{[n+1]} = e^{\Omega[\mathbb{N}_{0,1}^{[n]}]} \qquad \longrightarrow \qquad \mathbb{A}_a^{[n+1]} = \left(\mathbb{B}^{[n+1]}\right)^{-1} (\mathbb{A}_a^{[n]}\mathbb{B}^{[n+1]} - \partial_a \mathbb{B}^{[n+1]})\,, \tag{5.58}$$

which can be obtained equivalently through the Dyson series

$$\mathbb{B}^{[n+1]} = \mathbb{1} + \sum_{n=1}^{\infty} \int \mathrm{d}x_1 \int \mathrm{d}x_2 \ldots \int \mathrm{d}x_n \mathbb{N}_{0,1}^{[n]} \mathbb{N}_{0,1}^{[n]} \ldots \mathbb{N}_{0,1}^{[n]}\,. \tag{5.59}$$

In this way $\mathbb{A}_1^{[n]}$ is completely ϵ-factorized.

4. We repeat the previous steps for all n systems of DEQs, by performing n change of basis

$$\mathbb{B}^{[n+i]} = e^{\Omega[\mathbb{N}_{0,i}^{[n+1-i]}]}\,, \quad i = 1\,,2\,,\ldots\,,n, \tag{5.60}$$

which yields to a final set of matrices of the type

$$\mathbb{A}_a^{[2n]} = \epsilon \mathbb{A}_{1\,,a}^{[2n]}\,, \tag{5.61}$$

Therefore, if we rotate the original set of MIs $\mathbf{F}(\epsilon, \vec{x})$ as

$$\mathbf{F}(\epsilon, \vec{x}) = \mathbb{B}(x)\mathbf{I}(\epsilon, \vec{x})\,, \quad \text{with} \quad \mathbb{B}^{[1]}\,\mathbb{B}^{[2]} \cdots \mathbb{B}^{[2n]}\,, \tag{5.62}$$

we arrive to a new of DEQs for $\mathbf{I}(\epsilon, \vec{x})$ in the desired form,

$$\partial_a \mathbf{I}(\epsilon, \vec{x}) = \epsilon \mathbb{A}_{1\,,a}^{[2n]}(x)\mathbf{F}(\epsilon, \vec{x})\,, \tag{5.63}$$

since, by construction,

$$\partial_a \mathbb{B}(\vec{x}) = \mathbb{A}_{0\,,a}^{[0]}(\vec{x})\mathbb{B}(\vec{x})\,, \quad a = 1\,,2\,,\ldots\,,n\,. \tag{5.64}$$

A few important comments on the proposed algorithm are in order:

- The algorithm is based on the assumption that the starting set of MIs $\mathbf{F}$ obeys ϵ-linear system of DEQs and it does not provide any criterion to find such kind of basis. Hence, $\mathbf{F}$ must be determined from IBPs reduction based on experience. Nonetheless, it should be said that finding an ϵ-linear basis is considerably easier than directly determining an ϵ-factorized form by trial and error, and it has proven to be possible in most of the known cases;

- Besides being ϵ-factorized, the system of DEQs obtained in eq. (5.63) are empirically found to be in the dlog-*form* of eq. (5.36), so that the resulting systems of DEQs are in truly canonical form;

- The homogeneous solutions at $\epsilon = 0$ might involve irrational functions. Hence, the transformation to a canonical form can introduce irrational terms in the DEQs. Provided that the system of DEQs is in dlog-form, the presence of irrational letters in the alphabet does not prevent us from writing the solution in terms of Chen iterated integrals. However, especially in view of the analytic continuation and of a fast and precise numerical evaluation of the MIs, it is advisable, whenever possible, to perform a change of variables which rationalizes the whole alphabet, in such a way to be able to express the solution in terms of GPLs;

- In the above description, we have always assumed that each Magnus exponential $\mathbb{B}^{[i]}$ is convergent, i.e. that the defining series of eq. (5.43) terminates after a finite number of steps. This is obviously the case for $i \leq n$, since the Magnus exponential of a diagonal matrix $\mathbb{D}$ stops at $\Omega_1[\mathbb{D}]$, but it is not generally true for arbitrary non-diagonal matrices. If this occurs for any of the $\mathbb{B}^{[i]}$ defined in eq.(5.58), then algorithm cannot be used to find the canonical basis, since the Magnus exponential does not provide an exact solution of the system of DEQs at $\epsilon = 0$.

  The (up to now) few cases where the discussed algorithm is known to be not applicable are typically associated to systems of DEQs that are non-triangularizable in $\epsilon \to 0$ limit, i.e where there exists some subset of MIs which obey, at $\epsilon = 0$, an *irreducible higher order* DEQs. In chapter 9 we will address this issue by proposing a general method, based on the extension of the concept of leading singularity to irreducible systems of coupled DEQs, which can be used in order to bring to an ϵ-factorized form also DEQs where the Magnus algorithm, as well as the other methods discussed in section 5.4 are not applicable.

In chapters 6-8 we will apply the Magnus exponential algorithm in order to determine a canonical basis for classes of two-loop MIs depending on up to three different variables. Before discussing these applications, in the next sections we describe the main features of the general solution of a canonical system of DEQs and the properties of the transcendental functions that appear in it.

5.6 General solution of canonical systems

We now go back to the formal definition of the general solution of the ϵ-factorized system of DEQs (5.31) in terms of iterated integrals, which was introduced in eq. (5.34).

Given a set of MIs $\mathbf{I}(\epsilon, \vec{x})$ with total differential

$$d\mathbf{I}(\epsilon, \vec{x}) = \epsilon d\mathbb{A}(\vec{x})\mathbf{I}(\epsilon, \vec{x}) \,, \tag{5.65}$$

we can express the solution of their systems of DEQs as

$$\mathbf{I}(\epsilon, \vec{x}) = \mathcal{P} \exp\left\{\epsilon \int_\gamma d\mathbb{A}\right\} \mathbf{I}(\epsilon, \vec{x}_0) \,, \tag{5.66}$$

where $\mathbf{I}(\epsilon, \vec{x}_0)$ is a vector of integration constants depending on ϵ only and the *path-ordered exponential* is defined as the series

$$\mathcal{P} \exp\left\{\epsilon \int_\gamma d\mathbb{A}\right\} = \mathbb{1} + \epsilon \int_\gamma d\mathbb{A} + \epsilon^2 \int_\gamma d\mathbb{A}\, d\mathbb{A} + \epsilon^3 \int_\gamma d\mathbb{A}\, d\mathbb{A}\, d\mathbb{A} + \ldots \,, \tag{5.67}$$

whose k-th coefficient corresponds to a line integral of the product of k matrix-valued $d\mathbb{A}$ along some piece-wise smooth path γ connecting the point $\vec{x}_0$ and $\vec{x}$,

$$\begin{cases} \gamma : [0,1] \ni t \mapsto \gamma(t) = (x_1(t)\,, x_2(t)\,, \ldots\,, x_n(t)) \\ \gamma(0) = \vec{x}_0 \\ \gamma(1) = \vec{x}. \end{cases} \tag{5.68}$$

The general solution defined in eq. (5.66) must be specialized to the physically meaningful MIs through a proper choice of the integration constants. In this perspective, we observe that, in the limit $\vec{x} \to \vec{x}_0$, all the line integrals appearing in eq. (5.67) vanish (since the integration path γ shrinks to a point) and, consequently, $\mathbf{I}(\epsilon, \vec{x}) \to \mathbf{I}(\epsilon, \vec{x}_0)$. Therefore, the integration constants $\mathbf{I}(\epsilon, \vec{x}_0)$ have a clear interpretation in terms of *boundary constants* representing the initial values of the MIs at $\vec{x}_0$, which then evolve to the arbitrary point $\vec{x}$ under the action of the path-ordered exponential.

By choosing a proper normalization, we can always assume the canonical MIs to be finite in the $\epsilon \to 0$ limit, in such a way that $\mathbf{I}(\vec{x})$ admits a Taylor expansion in ϵ,

$$\mathbf{I}(\epsilon, \vec{x}) = \mathbf{I}^{(0)}(\vec{x}) + \epsilon\, \mathbf{I}^{(1)}(\vec{x}) + \epsilon^2 \mathbf{I}^{(2)}(\vec{x}) + \dots\,. \tag{5.69}$$

In particular, since we have identified $\mathbf{I}(\epsilon, \vec{x}_0)$ with the value of the MIs at $\vec{x}_0$, we can assume the boundary constants to be Taylor-expanded as well,

$$\mathbf{I}(\epsilon, \vec{x}_0) = \mathbf{I}^{(0)}(\vec{x}_0) + \epsilon\, \mathbf{I}^{(1)}(\vec{x}_0) + \epsilon^2 \mathbf{I}^{(2)}(\vec{x}_0) + \dots\,. \tag{5.70}$$

Therefore, if we combine the definition of the path-ordered exponential given in eq. (5.67) with eq. (5.70), we immediately see that the k-th order coefficient of the Taylor expansion of the MIs is given by

$$\mathbf{I}^{(k)}(\vec{x}) = \sum_{i=0}^{k} \Delta_\gamma^{(k-i)}[\mathrm{d}\mathbb{A}]\, \mathbf{I}^{(i)}(\vec{x}_0), \tag{5.71}$$

where $\Delta_\gamma^{(k)}$ is the weight-k integral operator

$$\begin{aligned} \Delta_\gamma^{(0)}[\mathrm{d}\mathbb{A}] &= 1\,, \\ \Delta_\gamma^{(k)}[\mathrm{d}\mathbb{A}] &= \int_\gamma \underbrace{\mathrm{d}\mathbb{A} \dots \mathrm{d}\mathbb{A}}_{\text{k times}}\,, \end{aligned} \tag{5.72}$$

which iterates k ordered integration of $\mathrm{d}\mathbb{A}$ along the path γ.

Formally, the previous discussion holds for any kind of dependence of the matrix $\mathbb{A}(\vec{x})$ on the kinematic variables $\vec{x}$. In particular, if $\mathbb{A}(\vec{x})$ is in the dlog-form (5.36), we see that each entry of $\Delta^{(k)}$ is a linear combination of iterated integrals of the type

$$\mathcal{C}^{[\gamma]}_{i_k,\dots,i_1} = \int_\gamma \mathrm{dlog}\, \eta_{i_k} \dots \mathrm{dlog}\, \eta_{i_1}\,. \tag{5.73}$$

As we have explicitly indicated in (5.73), each individual iterated integral is, in general, a *functional* of the path. However, we observe that the full combinations of integrals appearing in the entries of $\Delta^{(k)}$ must be independent of the particular choice of γ, since they correspond to integrals of the *total* differential (5.65).

The theory of the iterated integrals defined in eq. (5.73) was originally formulated by Chen [87]. In the next section, we give a brief summary of their most relevant properties.

5.7 Chen iterated integrals

For definiteness, we define the Chen iterated integrals introduced in eq. (5.73) as

$$\mathcal{C}^{[\gamma]}_{i_k,\dots,i_1} = \int_{0\le t_1 \le \dots \le t_k \le 1} g^\gamma_{i_k}(t_k) \dots g^\gamma_{i_1}(t_1)\, \mathrm{d}t_1 \dots \mathrm{d}t_k\,, \tag{5.74}$$

where γ is the piecewise-smooth path connecting $\vec{x}_0$ to $\vec{x}$ given in eq. (5.68) and

$$g_i^\gamma(t) = \frac{d}{dt} \log \eta_i(\gamma(t)) \,. \tag{5.75}$$

We refer to the number k of iterated integrations involved in eq. (5.74) as the *weight* of $\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1}$. At weight one, the line integral of one dlog produces, as expected,

$$\int_\gamma \mathrm{dlog}\,\eta = \int_{0\leq t\leq 1} \frac{\mathrm{dlog}\,\eta(\gamma(t))}{dt} \mathrm{d}t = \log \eta(\vec{x}) - \log \eta(\vec{x}_0) \,, \tag{5.76}$$

which depends on the end-points $\vec{x}_0$ and $\vec{x}$. At higher weights, eq. (5.74) can be rewritten in a recursive way,

$$\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} = \int_0^1 g^\gamma_{i_k}(s)\, \mathcal{C}^{[\gamma_s]}_{i_{k-1},\ldots,i_1} \mathrm{d}s \,, \tag{5.77}$$

where $\mathcal{C}^{[\gamma_s]}_{i_{k-1},\ldots,i_1}$ is a weight $k-1$ iterated integral along the path

$$\gamma_s : [0,1] \ni t \mapsto \vec{x} = (\gamma^1(s\,t)\,, \gamma^2(s\,t)\,, \ldots\, \gamma^n(s\,t)) \,. \tag{5.78}$$

In addition, the integral representation of $\mathcal{C}^{[\gamma_s]}_{i_k,\ldots,i_1}$ immediately allows to obtain the derivative identity

$$\frac{d}{ds} \mathcal{C}^{[\gamma_s]}_{i_k,\ldots,i_1} = g^\gamma_{i_k}(s)\, \mathcal{C}^{[\gamma_s]}_{i_{k-1},\ldots,i_1} \,. \tag{5.79}$$

Chen iterated integrals obey a number of properties:

- *Invariance under path reparametrization:* the integral $\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1}$ does not depend on the way one chooses to parametrize the path γ.
- *Reverse path formula:* if γ^{-1} is the path γ traversed in the opposite direction, then

$$\mathcal{C}^{[\gamma^{-1}]}_{i_k,\ldots,i_1} = (-1)^k \mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} \,. \tag{5.80}$$

- *Shuffle algebra:* Chen iterated integrals fulfil shuffle algebra relations. If $\vec{m} = (m_M, \ldots, m_1)$ and $\vec{n} = (n_N, \ldots, n_1)$, with M and N natural numbers, we have

$$\mathcal{C}^{[\gamma]}_{\vec{m}}\, \mathcal{C}^{[\gamma]}_{\vec{n}} = \sum_{\text{shuffles } \sigma} \mathcal{C}^{[\gamma]}_{\sigma(m_M),\ldots,\sigma(m_1),\sigma(n_N),\ldots,\sigma(n_1)} \,, \tag{5.81}$$

  where the sum runs over all the permutations σ that preserve the internal order of $\vec{m}$ and $\vec{n}$.

- *Path composition formula:* if $\alpha, \beta : [0,1] \to \mathcal{M}$ are two paths such that $\alpha(0) = \vec{x}_0$, $\alpha(1) = \beta(0)$, and $\beta(1) = \vec{x}$, then the iterated integral along the composed path $\gamma = \alpha\beta$, obtained by first traversing α and then β, satisfies

$$\mathcal{C}^{[\alpha\beta]}_{i_k,\ldots,i_1} = \sum_{p=0}^{k} \mathcal{C}^{[\beta]}_{i_k,\ldots,i_{p+1}}\, \mathcal{C}^{[\alpha]}_{i_p,\ldots,i_1} \,. \tag{5.82}$$

  The composition of an arbitrary number of paths can be obtained by recursively applying eq. (5.82).

- *Integration-by-parts formula:* the computation of eq. (5.74) requires, in principle, the evaluation of k nested integrals. Nevertheless, we observe that the innermost integration is always reduced to (5.76), so that we have $k-1$ actual integrations to perform. For instance, at weight $k=2$, we have

$$\begin{aligned}\mathcal{C}^{[\gamma]}_{m,n} &= \int_0^1 g_m(t)\, \mathcal{C}^{[\gamma_t]}_n \, \mathrm{d}t \\ &= \int_0^1 g_m(t)(\log \eta_n(\vec{x}(t)) - \log \eta_n(\vec{x}_0))\, \mathrm{d}t\,, \end{aligned} \tag{5.83}$$

  and we are left with a single integral to be evaluated, either analytically or numerically.

  Moreover, we can show that the integration involving the outermost weight g_k can be performed by parts and gives

$$\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} = \log \eta_{i_k}(\vec{x})\, \mathcal{C}^{[\gamma]}_{i_{k-1},\ldots,i_1} - \int_0^1 \log \eta_{i_k}(\vec{x}(t))\, g_{i_{k-1}}(t)\, \mathcal{C}^{[\gamma_t]}_{i_{k-2},\ldots,i_1} \mathrm{d}t\,. \tag{5.84}$$

  The combined use of eqs. (5.83) and (5.84) allows a remarkable simplification in the numerical evaluation of weight $k \geq 3$ iterated integrals, since the analytic calculation of the inner- and outermost integrals leaves only $k-2$ integrations to be performed via numerical methods.

5.8 Generalized polylogarithms

The solution of a canonical system of DEQs in terms of the path-order exponential (5.66) leads automatically to a representation of the MIs in terms of Chen iterated integrals. However, if the alphabet is *rational* and it is possible to determine explicitly its algebraic roots, we can directly integrate the canonical DEQs in terms of *generalized polylogarithms* (GPLs) [82–86].

In this case, in fact, we can factor (in general over $\mathbb{C}$) the letters of the dlog-form w.r.t. each kinematic variable x_i,

$$\eta_k(\vec{x}) = \prod_{j_k=1}^{m_k} (x_i - \omega_{j_k})\,, \tag{5.85}$$

where the weight ω_{j_k} can depend on all the other $\vec{x}$ variables. In this way, the coefficient matrix of the system of DEQs w.r.t. x_i can be written in the form

$$\mathbb{A}_{x_i}(\vec{x}) = \sum_{j=1}^{m} \frac{\mathbb{M}_j}{x_i - \omega_j}\,, \tag{5.86}$$

where $\mathbb{M}_j$ are constant matrices. Therefore, at any order k in the Taylor expansion of the MIs, we can solve by quadrature the systems DEQs and obtain, by starting from x_1,

$$\mathbf{I}^{(k)}(\vec{x}) = \sum_{j=1}^{m} \int_0^{x_1} \mathrm{d}t \frac{\mathbb{M}_j}{t-\omega_j} \mathbf{I}^{(k-1)}(\vec{x}) + C(x_2 \ldots x_n)\,, \tag{5.87}$$

where $C(x_2 \ldots x_n)$, which is independent of x_1, has to be determined by recursively substituting the solution (5.87) into the remaining systems of DEQs and integrating the latter in a similar way.

Eq. (5.87) can be directly evaluated in terms of GPLs, which are defined in terms of iterated integrals as

$$\begin{aligned} G(\vec{\omega}_n; x) &= \int_0^x \mathrm{d}t \frac{1}{t-\omega_1} G(\vec{\omega}_{n-1}; t)\,, \quad n>0\,, \\ G(\vec{0}_n; x) &= \frac{1}{n!}\log^n x\,, \end{aligned} \tag{5.88}$$

where x is a complex variable and $\vec{\omega}_n = (\omega_1\,, \omega_2\,, \ldots\,, \omega_n)$ is vector with n complex indices. The length of $\vec{\omega}_n$, which we refer to as the *weight* of a GPL, corresponds to the number of nested integrations which define $G(\vec{\omega}_n; x)$. An alternative definition can be given in terms of the derivates,

$$\frac{\partial}{\partial x} G(\vec{\omega}_n; x) = \frac{1}{x-w_1} G(\vec{\omega}_{n-1}; x)\,, \tag{5.89}$$

which is clearly equivalent to eq. (5.88).

GPLs, which in principle can depend on an arbitrary number of variables, constitute a wide class of transcendental functions and contain, as special cases, several families of polylogarithmic functions occurring in loop computations, including;

- The ordinary logarithm and the classical polylogarithms, which are given in terms of GPLs as

$$\begin{aligned} G(\vec{\omega}_n; x) &= \frac{1}{n!}\log^n\left(1-\frac{x}{\omega}\right)\,, \quad \text{for } \vec{\omega}_n = \underbrace{(\omega\,, \ldots\,, \omega)}_{\text{n times}}\,, \\ G(\vec{0}_{n-1}, 1; x) &= -\,\mathrm{Li}_n(x)\,. \end{aligned} \tag{5.90}$$

- The harmonic polylogarithms (HPLs) [83, 84, 186], which correspond to one-dimensional GPLs with $\omega_i \in \{0, \pm 1\}$. Due to different conventions in their definition, the correspondence between GPLs and HPLs is

$$H(\vec{\omega}; x) = (-1)^p G(\vec{\omega}; x)\,, \tag{5.91}$$

  where p is the number of indices $+1$ contained in $\vec{\omega}$.

- Two-dimensional harmonic polylogarithms (2dHPLs)[187, 188], which correspond to GPLs with $\omega_i \in \{0, 1, -y, -1-y\}$ and $y \in \mathbb{C}$.

GPLs satisfy a series of fundamental properties, which are inherited from their iterative structure:

- *Shuffle algebra:* GPLs fulfil a shuffle algebra relation of the form

$$G(\vec{m}; x)\, G(\vec{n}; x) = G(\vec{m}; x) \sqcup\!\sqcup\, G(\vec{n}; x) = \sum_{\vec{p}=\vec{m}\sqcup\!\sqcup\vec{n}} G(\vec{\omega}; x)\,, \tag{5.92}$$

  where the shuffle product $\vec{m} \sqcup\!\sqcup \vec{n}$ denotes all possible merges of $\vec{m}$ and $\vec{n}$ which preserve their internal ordering.

- *Rescale invariance*: if the rightmost index ω_n of $\vec{\omega}$ is different from zero, then $G(\vec{\omega}_{n-1}; x)$ is invariant under the rescaling of all its argument by a factor $z \in \mathbb{C}^*$,

$$G(\vec{\omega}_n; x) = G(z\,\vec{\omega}_n; z\,x)\,, \quad \omega_n \neq 0\,. \tag{5.93}$$

- *Holder convolution:* for $x = 1$, we have

$$G(\omega_1\,, \ldots\,, \omega_n; 1) = (-1)^n G(1-\omega_n\,, \ldots 1-\omega_1; 1)\,, \tag{5.94}$$

  which is a special case of the *Holder convolution*.

Besides these basic properties, the linearity of the rational functions which constitute the integration kernels of the GPLs allows to derive, for instance by manipulating the integral representation (5.88), a series of additional identities between GPLs of different argument, series expansion and limiting values.

Finally, we would like to mention that GPLs, equipped with the shuffle algebra (5.92) can be proven to form a *Hopf algebra* [189, 190] graded by weight. Such algebra allows to define the *symbol* [191] and *coproduct* [190] maps, which encode important information about GPLs, such as their behaviour under differentiation and their discontinuity across branch cuts and they can be used to systematically determine functional relations between GPLs.

Connection between GPLs and Chen iterated integrals

If the letters of the dlog-form are rational, it is possible, of course, to establish a connection between the representation of the solution in terms of Chen iterated integrals and GPLs. In particular, we can convert a Chen integral with rational letters into a combination of GPLs evaluated at 1.

Such conversion to GPLs is straightforward in the case of an iterated integral whose letters depend *linearly* on a single variable x, i.e. if eq. (5.85) simply reads

$$\eta_i = x - \omega_i\,, \quad \omega_i \in \mathbb{C}\,. \tag{5.95}$$

In fact, in such case, the integral representations given in eqs. (5.74) and (5.88) can be used to derive the conversion formula

$$\int_\gamma d\log(x-w_k)\ldots d\log(x-w_1) = G\left(\frac{x_0-w_k}{x_0-x_1}, \ldots, \frac{x_0-w_1}{x_0-x_1}; 1\right), \tag{5.96}$$

where x_0 and x_1 indicate the integration end-points. In order to convert a GPLs with rational letters depending on an arbitrary number of variables, we can proceed as follows:

1. We connect $\vec{x}_0 = (x_{1,0},\, x_{2,0},\, \ldots\,, x_{n,0})$ to $\vec{x}_1 = (x_{1,1},\, x_{2,1},\, \ldots\,, x_{n,1})$ through a piecewise linear path $\gamma \equiv \gamma_1\,\gamma_2\,\cdots\,\gamma_n$, with

$$\begin{cases} \gamma_1(t):\ (x_{1,0} + t(x_{1,1} - x_{1,0}),\, x_{2,0},\, \ldots\,, x_{n,0}) \\ \gamma_2(t):\ (x_{1,1}\,, x_{2,0} + t(x_{2,0} - x_{2,0}),\, \ldots x_{n,0}) \\ \quad \ldots \\ \gamma_n(t):\ (x_{1,1},\, x_{2,1},\, \ldots\,, x_{n,0} + t(x_{n,1} - x_{n,0}))\,. \end{cases} \tag{5.97}$$

2. We use the *path composition formula* given in eq. (5.82) and we rewrite $\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1}$ as a combination of iterated integrals $\mathcal{C}^{[\gamma_i]}_{i_m,\ldots,i_n}$, where there linear path γ_i affects the variable x_i and leaves all the others unchanged;

3. For each $\mathcal{C}^{[\gamma_i]}_{i_m,\dots,i_n}$, we factor over $\mathbb{C}$ all the $\text{dlog}\,\eta_k$ w.r.t. the variable which changes along the path γ_i;
4. We apply the conversion formula (5.96). In the multivariate case, the weights w_j must be evaluated at the constant value of the variables which are left unaffected by the path γ_i.

Example 1

Let us illustrate the above conversion procedure on a weight two Chen iterated integral depending on three variables $\vec{x} = (v\,, z\,, \bar{z})$,

$$\mathcal{C}^{[\gamma]}_{m\,,n} = \int_\gamma \text{dlog}\;\eta_m\;\text{dlog}\;\eta_k \tag{5.98}$$

with

$$\eta_m = v + (1-v)^2 z\bar{z}\,, \quad \eta_n = v\,, \tag{5.99}$$

and where γ is a path connecting $\vec{x}_0 = (1\,,1\,,\bar{1})$ to $\vec{x}_1 = (v_1\,,z_1\,,\bar{z}_1)$. This particular integral appears in the computation of the two-loop master integrals for the leading QCD corrections to the coupling X^0WW (with $X^0 = H, \gamma^*, Z$), which will be discussed in chapter 7.

The path γ can be split into three linear paths γ_i, $i = 1, 2, 3$ which affect, respectively $\bar{z}$, v and z, and leave the other two variables unchanged,

$$\begin{cases} \gamma_1(t):\; (1,\,1,\,1+t(\bar{z}_1-1)) \\ \gamma_2(t):\; (1+t(v_1-1),\,1,\,\bar{z}_1) \\ \gamma_3(t):\; (v_1,\,1+t(z_1-1),\,\bar{z}_1)\;. \end{cases} \tag{5.100}$$

According to the *path composition formula* given in eq. (5.82), we can write $\mathcal{C}^{[\gamma]}_{m\,,n}$ as

$$\mathcal{C}^{[\gamma]}_{m\,,n} = \mathcal{C}^{[\gamma_3]}_{m\,,n} + \mathcal{C}^{[\gamma_2]}_{m\,,n} + \mathcal{C}^{[\gamma_1]}_{m\,,n} + \mathcal{C}^{[\gamma_3]}_{m}\left(\mathcal{C}^{[\gamma_2]}_{n} + \mathcal{C}^{[\gamma_1]}_{n}\right) + \mathcal{C}^{[\gamma_2]}_{m}\mathcal{C}^{[\gamma_1]}_{n}\,. \tag{5.101}$$

We immediately see that $\mathcal{C}^{[\gamma_1]}_{n} = 0$, since $\text{dlog}\,\eta_n = 0$ along γ_1. For similar reasons, $\mathcal{C}^{[\gamma_1]}_{m\,,n}$ and $\mathcal{C}^{[\gamma_2]}_{m\,,n}$ vanish as well. Therefore, we are left with

$$\mathcal{C}^{[\gamma]}_{m\,,n} = \mathcal{C}^{[\gamma_2]}_{m\,,n} + \mathcal{C}^{[\gamma_3]}_{m}\;\mathcal{C}^{[\gamma_2]}_{n}\,. \tag{5.102}$$

We can now apply the conversion formula of eq. (5.96) to the Chen iterated integrals appearing on the r.h.s, by factoring η_m and η_n w.r.t. the variable affected by γ_i and by setting the other two variables to the corresponding constant value. In this way, we have, for the weight one integrals,

$$\begin{aligned} \mathcal{C}^{[\gamma_2]}_{n} &= \int_{\gamma_2} \text{dlog}\;v = G\left(\frac{1}{1-v_1};1\right)\,, \\ \mathcal{C}^{[\gamma_3]}_{m} &= \int_{\gamma_2} \text{dlog}\left(z + \frac{v_1}{\bar{z}_1(v_1-1)^2}\right) = G\left(\frac{1+\frac{v_1}{\bar{z}_1(v_1-1)^2}}{1-z_1};1\right)\,, \end{aligned} \tag{5.103}$$

and, for the weight two integral,

$$\mathcal{C}^{[\gamma_2]}_{m\,,n} = \int_{\gamma_2} \text{dlog}\left(v - \frac{2\bar{z}_1 - 1 + \sqrt{1-4\bar{z}_1}}{2\bar{z}_1}\right)\text{dlog}\;v +$$

$$+\int_{\gamma_2} \mathrm{dlog}\left(v - \frac{2\bar{z}_1 - 1 - \sqrt{1-4\bar{z}_1}}{2\bar{z}_1}\right) \mathrm{dlog}\, v$$
$$= G\left(\frac{1 - \frac{2\bar{z}_1-1+\sqrt{1-4\bar{z}_1}}{2\bar{z}_1}}{1-v_1}, \frac{1}{1-v_1}; 1\right) + G\left(\frac{1 - \frac{2\bar{z}_1-1-\sqrt{1-4\bar{z}_1}}{2\bar{z}_1}}{1-v_1}, \frac{1}{1-v_1}; 1\right). \tag{5.104}$$

By combining these results, we obtain the conversion of $\mathcal{C}^{[\gamma]}_{m,n}$ to GPLs,

$$\mathcal{C}^{[\gamma]}_{m,n} = G\left(\frac{1 - \frac{2\bar{z}_1-1+\sqrt{1-4\bar{z}_1}}{2\bar{z}_1}}{1-v_1}, \frac{1}{1-v_1}; 1\right) + G\left(\frac{1 - \frac{2\bar{z}_1-1-\sqrt{1-4\bar{z}_1}}{2\bar{z}_1}}{1-v_1}, \frac{1}{1-v_1}; 1\right)$$
$$+ G\left(\frac{1}{1-v_1}; 1\right) G\left(\frac{1 + \frac{v_1}{\bar{z}_1(v_1-1)^2}}{1-z_1}; 1\right). \tag{5.105}$$

A similar procedure can be applied to convert any Chen iterated integral with rational letters (provided that their algebraic root are known) in terms of GPLs evaluated at 1. □

Boundary conditions

Once a representation of the general solution of a system of DEQs has been obtained in terms of iterated integrals, the full determination of the analytic expression of the MIs requires the specification of suitable boundary conditions. As we have already stressed, the boundary constants $\mathbf{I}(\epsilon, \vec{x}_0)$ correspond to the value of the MIs at the integration base-point $\vec{x}_0$. Therefore one possible (but generally difficult) way of fixing the boundary conditions consists in calculating the boundary values through some independent integration method. Alternatively, the simple knowledge of the analyticity properties of the MIs often allows to determine the boundary constants directly from the singularity structure of the system od DEQs. In this perspective, having a system in canonical form, can make the boundary fixing even more straightforward.

Let us suppose that the canonical set of MIs $\mathbf{I}(\epsilon, \vec{x})$ is regular at the unphysical threshold $\eta_i(\vec{x})$, which is part of the alphabet of the dlog form (5.36). In this case, boundary constants can be fixed by exploiting the relation

$$\lim_{\eta_i \to 0} \mathbb{M}_i \mathbf{I}(\epsilon, \vec{x}) = 0\,, \tag{5.106}$$

which holds order-by-order in the ϵ-expansion. The validity of eq. (5.106) can be easily proved by induction, based on the iterative structure of the Taylor coefficients $\mathbf{I}^{(k)}(\vec{x})$:

- Since at order zero $\mathbf{I}^{(0)}$ is a pure constant and, hence, it is finite under every limit, we start by considering the $\mathcal{O}(\epsilon)$ term, which is given, according to eq. (5.71), by

$$\mathbf{I}^{(1)}(\vec{x}) = \int_\gamma \mathrm{d}\mathbb{A}\mathbf{I}^{(0)}(\vec{x}_0) + \mathbf{I}^{(1)}(\vec{x}_0)\,. \tag{5.107}$$

  In the $\eta_i(\vec{x}) \to 0$ limit, the divergent behaviour of $\mathbf{I}^{(1)}(\vec{x})$ is given by

$$\lim_{\eta_i \to 0} \mathbf{I}^{(1)}(\vec{x}) \sim \lim_{\eta_i \to 0} \int_\gamma \mathrm{dlog}\, \eta_i\, \mathbb{M}_i \mathbf{I}^{(0)}(\vec{x}_0)\,. \tag{5.108}$$

Therefore, $\mathbf{I}^{(1)}(\vec{x})$ is finite only if the coefficients of the logarithmic singularity is set to zero,

$$\lim_{\eta_i \to 0} \mathbb{M}_i \mathbf{I}^{(0)}(\vec{x}_0) = 0\,, \tag{5.109}$$

which exactly corresponds to the zeroth-order of eq. (5.106) and provides a set of relations between the boundary constants $\mathbf{I}^{(0)}(\vec{x}_0)$.

- With similar arguments, if we suppose that regularity has been imposed on the solution up to $\mathcal{O}(\epsilon^{k-2})$, we see from eq. (5.71) that the potentially singular term of $\mathbf{I}^{(k)}(\vec{x})$ is given by

$$\lim_{\eta_i \to 0} \mathbf{I}^{(k)}(\vec{x}) \sim \lim_{\eta_i \to 0} \int_\gamma \mathrm{dlog}\, \eta_i\, \mathbb{M}_i \mathbf{I}^{(k-1)}(\vec{x})\,. \tag{5.110}$$

Hence, the finiteness of $\mathbf{I}^{(k)}(\vec{x})$ is ensured by

$$\lim_{\eta_i \to 0} \mathbb{M}_i \mathbf{I}^{(k-1)}(\vec{x}) = 0\,, \tag{5.111}$$

which corresponds to the $\mathcal{O}(\epsilon^{k-1})$ term of eq. (5.106)

Besides its simple derivation, the regularity condition (5.106) are, in general, easier to solve than the regularity conditions imposed on non-canonical MIs, since they involve, at each order in ϵ, only functions of uniform weight.

Finally, we observe that Feynman integrals can have, besides logarithmic divergencies, also power divergencies. In a canonical basis, power divergencies are encoded in the kinematic prefactors which relate $\mathbf{I}(\epsilon, \vec{x})$ to the original basis of MIs $\mathbf{F}(\epsilon, \vec{x})$. This observation provides, sometimes, a very simple condition which can be used in order to fix boundary constants at all order in ϵ. In fact, if there is kinematic limit where $\mathbf{F}(\epsilon, \vec{x})$ are regular and the prefactors expressing $\mathbf{I}(\epsilon, \vec{x})$ in terms of $\mathbf{F}(\epsilon, \vec{x})$ are vanishing, then the canonical MIs must vanish in this limit.

5.9 Conclusions

In this chapter, we have given an overview of the differential equations method [48, 76, 77] for master integrals, which so far has proven to be the most effective technique for the analytic evaluation of multi-loop, multi-scale Feynman integrals.

Given a basis of MIs which span the space of Feynman integrals related to a particular process, IBPs allows to derive systems of first-order coupled differential equations in the kinematic invariants for such MIs. In most cases, when the system of DEQs is expanded around some integer number of space-time dimensions, i.e. around $\epsilon \sim 0$, the substantial decoupling of the DEQs allows to determine the coefficients of the series expansion in ϵ of the MIs in terms of iterated integrals.

In particular, if the DEQs in ϵ can be cast into a triangular form the formal iterated integrals can be evaluated in terms of Chen iterated integrals [87], which include as a special (but, indeed, very common) case the well-studied generalized polylogarithms [82–86].

In this respect, we have discussed the advantages of finding a basis of MIs which obey *canonical* DEQs [81], where the dependence on ϵ is completely factorized from the kinematics and the total differential of each integral can be written as an exact dlog-form. This two features of the canonical DEQs render the evaluation of the MIs in terms of Chen iterated integrals (of course whenever a representation in terms of such functions is possible) almost entirely algorithmic and leave only the fixing of the boundary conditions to a case-by-case analysis.

Among the several methods which have been proposed for the determination of canonical bases of MIs, we have examined in some details the Magnus exponential method [88], which can be used in order to bring to canonical form triagularizable systems of DEQs with a *linear* dependence on ϵ.

In chapters 6-8, we will apply the Magnus method for the calculation of two-loop three- and four-point integrals depending on up to three dimensionless kinematic variables. In all cases, the discussion of the solution of the DEQs for the resulting MIs will be structured as follows:

- We first define the integral topologies to be reduced to MIs,

$$\int \prod_{i=1}^{\ell} \widetilde{\mathrm{d}^d k_i} \, \frac{1}{D_1^{n_1} \ldots D_m^{n_m}} \,, \quad n_i \in \mathbb{Z} \,, \tag{5.112}$$

  where, in our conventions, the integration measure is defined as

$$\widetilde{\mathrm{d}^d k_i} = \frac{\mathrm{d}^d k_i}{(2\pi)^d} \left(\frac{i\, S_\epsilon}{16\pi^2} \right)^{-1} \left(\frac{m^2}{\mu^2} \right)^{\epsilon} , \tag{5.113}$$

  with μ being the 't Hooft scale of dimensional regularization and

$$S_\epsilon = (4\pi)^\epsilon \, \Gamma(1+\epsilon) \, ; \tag{5.114}$$

- We identify, through IBPs reduction, an initial set of MIs $\mathbf{F}(\epsilon, \vec{x})$ which fulfils ϵ-linear systems of DEQs in the kinematic invariants $\vec{x}$;
- We use the Magnus exponential method in order to rotate $\mathbf{F}(\epsilon, \vec{x})$ to a canonical basis of MIs $\mathbf{I}(\epsilon, \vec{x})$;
- We determine the general solution of the canonical systems in terms of iterated integrals (either GPLs or Chen iterated integrals, according to the alphabet of the problem under consideration);
- We discuss the fixing of boundary conditions.

The resulting analytical expressions of the MIs have been evaluated numerically either with the help of the computer code GiNaC [192], in the case of GPLs, or with an in-house implementation of the numerical integration of Chen iterated integrals, and then checked with the numerical values of the MIs provided by the code SecDec [171].

The problem of the evaluation of MIs obeying systems of DEQs that remain coupled in the $\epsilon = 0$ limit will be addressed in chapter 9.

Chapter 6

Master integrals for QCD corrections to HWW and gauge couplings ZWW-γ^*WW

In this chapter, we use the differential equations method to analytically evaluate the two-loop master integrals required for the leading QCD corrections to the interaction vertex of a massive neutral boson X^0, e.g. H, Z or γ^*, with a pair of W bosons, mediated by a $SU(2)_L$ quark doublet composed of one massive and one massless flavor. All the external legs are allowed to have arbitrary invariant masses. The Magnus exponential is employed to identify a set of master integrals that obey canonical systems of differential equations. In their expansion around four space-time dimensions, the master integrals are expressed in terms of generalized polylogarithms. In the context of the Standard Model, the considered integrals are relevant for the mixed EW-QCD corrections to the Higgs decay to a W pair, as well as to the production channels obtained by crossing, and to the triple gauge boson vertices ZWW and γ^*WW. The content of this chapter is the result of an original research done in collaboration with S. Di Vita, P. Mastrolia and U. Schubert, and it is based on the pubblication [3].

6.1 Introduction

In this chapter, we consider the three-particle reaction

$$X^0(q) \to W^+(p_1) + W^-(p_2)\,, \tag{6.1}$$

where $W^\pm$ are the charged electroweak gauge bosons and X^0 is a neutral boson which, in the Standard Model, can correspond both to the Higgs boson or to the vector bosons Z and γ. We assume the momenta of all three particles to be off-shell, so that the kinematics of the process reads

$$s = q^2 = (p_1+p_2)^2 \qquad \text{and} \qquad p_1^2 \neq p_2^2 \neq 0\;. \tag{6.2}$$

In the Standard Model, the coupling between two W bosons and one neutral boson $X^0 = H, Z, \gamma^*$ is present in the tree-level Lagrangian. At one-loop, the $X^0\,W^+W^-$ interaction receives electro-weak (EW) corrections, either *via* bosonic- or *via* fermionic-loop. Strong (QCD) corrections must proceed through a closed quark-loop so that they can first occur at the two-loop level.

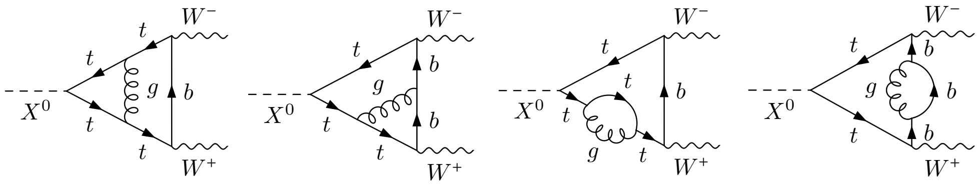


Figure 6.1: Representative two-loop Feynman diagrams contributing to the $X^0\, W^+W^-$ interaction, where $X^0 = H, Z, \gamma^*$. Similar diagrams where t and b quarks are exchanged are also taken into account.

Two-loop corrections involving massless partons propagating in the loops has been studied in [193–195], while effects due to finite quark mass are still unknown. In this chapter, we present the calculation of the two-loop three-point integrals required for the determination of the leading QCD corrections to the interaction vertex between a neutral boson X^0 with arbitrary mass and a pair of W bosons of arbitrary squared four-momenta ($X^0\, W^+W^-$), mediated by a fermion loop of a $SU(2)_L$ quark doublet, with one massive and one massless flavors. In what follows, we refer to the massive flavor as to the *top* ($m_t = m$), and to the massless one as to the *bottom* ($m_b = 0$). Representative Feynman graphs for the considered integrals are shown in figure 9.3. The corresponding four integral families T_i, which are depicted in figure 6.2 can be distinguished in two sets, according to the flavor that couples to the X^0 boson, i.e. either the massless (T_1 and T_2) or the massive one (T_3 and T_4),

The present calculation provides the full set of MIs needed for computation of the $\mathcal{O}(\alpha\alpha_s)$ corrections to the Higgs decay into a pair of W bosons, and to the triple gauge boson processes Z^*WW and γ^*WW, with leptonic final states, at e^+e^- colliders. As for the latter process with semi-leptonic or hadronic final states, these MIs would only be a subset of the needed integrals. Likewise, the computed MIs constitute a subset of the integrals for the computation of the two-loop mixed EW-QCD to crossing-related processes, such as Higgs production in WW-fusion or in association with a W boson, as well as to to WW production in higher multiplicity processes. In addition, since fermionic one-loop diagrams always involve a fermionic doublet with a (nearly) massless flavor (with the only exception of the approximately degenerate first generation), this set of MIs can also be used in NNLO EW corrections, where the exchanged gluon is replaced by a photon. Finally, we observe that several motivated extensions of the Standard Model feature an extended Higgs sector with Yukawa couplings to the $SU(2)$ fermion doublets. Although we do not refer explicitly to this possibility, our results is also applicable to the case $X^0 = S^0$, with S^0 being any neutral (pseudo) scalar present in the spectrum of the extended theory.

In the next section we describe the general features of the systems of DEQs obeyed by the MIs belonging to the integral topologies of figure 6.2 and the properties of the corresponding solutions. The details of the calculations of the MIs for $X^0\, W^+W^-$ are discussed in the section 6.3.

6.2 System of differential equations

The two-loop Feynman diagrams contributing to $X^0\, W^+W^-$ can be reduced to four integral topologies, which are depicted in figure 6.2. The integrals belonging to these topologies depend on the three external invariants p_1^2, p_2^2 and s and on the internal mass m^2. These four variables can be combined into three independent dimensionless parameters, $\vec{x} = (x_1, x_2, x_3)$, whose explicit definition will be later specified according

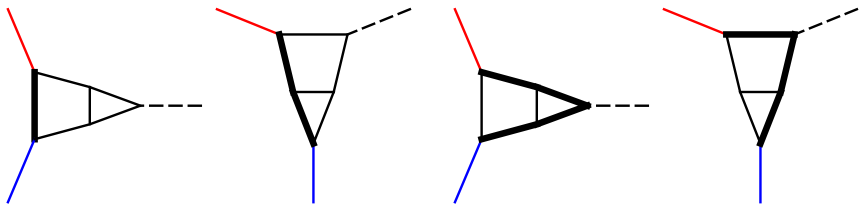


Figure 6.2: Two-loop topologies for $X^0\,W^+W^-$ interactions. Thin lines represent massless propagators and thick lines stand for massive ones. The dashed external line corresponds to the off-shell leg with squared momentum equal to s. The red and blue lines respectively represent the two external vector bosons with off-shell momenta p_1^2 and p_2^2.

to the topology under consideration. We evaluate the MIs by solving their systems of DEQs in the variables $\vec{x}$. First, by means of IBPs reduction, we identify a set of MIs $\mathbf{F}$ that fulfils a system of DEQs of the type

$$\frac{\partial \mathbf{F}}{\partial x_a} = (\mathbb{A}_{0\,x_a}(\vec{x}) + \epsilon \mathbb{A}_{1\,x_a}(\vec{x}))\mathbf{F}\,, \quad \text{with } a = 1, 2, 3\,. \tag{6.3}$$

Subsequently, with the help of Magnus exponential matrix, we perform a change of basis to a new set of of MIs $\mathbf{I}$ obeying a canonical systems of DEQs,

$$\frac{\partial \mathbf{I}}{\partial x_a} = \epsilon \hat{\mathbb{A}}_{x_a}(x_1, x_2, x_3)\mathbf{I}\,. \tag{6.4}$$

After combining the three systems of DEQs into a single total differential, we arrive at the following canonical form

$$d\mathbf{I} = \epsilon \mathrm{d}\mathbb{A}\mathbf{I}\,, \qquad \mathrm{d}\mathbb{A} \equiv \hat{\mathbb{A}}_{x_a}\mathrm{d}x^a \tag{6.5}$$

with a total differential matrix of the form

$$\mathrm{d}\mathbb{A} = \sum_{i=1}^{n} \mathbb{M}_i \,\mathrm{dlog}(\eta_i)\,, \tag{6.6}$$

with $\mathbb{M}_i$ being constant matrices.

For all the considered MIs, the alphabet contains only rational letters with algebraic roots, so that the general solution of eq. (6.5) can be written in terms of GPLs. The boundary constants, which are needed in order to completely specify the solution, are fixed by exploiting either the known expression of the MIs at specific kinematic point or by demanding their regularity at pseudo-thresholds contained in the DEQs. To this aim, we find convenient, according to the integral topology under consideration, either to integrate eq. (6.5) directly in terms of GPLs or to first derive a compact representation of the solution in terms of Chen iterated integrals and then convert it to GPLs (following the discussion of section 5.8), after the boundary constants have been fixed. In both cases the MIs are evaluated in the region where all letters η_i are real and positive. In the final part of this chapter, we will discuss in detail the required prescriptions for the the analytic continuation of the MIs to the physical region, which can be then obtained through standard techniques.

6.3 Two-loop master integrals

In this section, we present the solution of the system of DEQs for the MIs associated to the four integral topologies T_i depicted in figure 6.2. Since the four topologies can be mapped into two two distinct integral families, we discuss their evaluation separately.

6.3.1 First integral familiy

The topologies T_1 and T_2 belong to the integral family

$$\int \widetilde{\mathrm{d}^d k_1} \widetilde{\mathrm{d}^d k_2} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7}}, \quad n_i \in \mathbb{Z}, \tag{6.7}$$

identified by the set of denominators

$$\begin{aligned} D_1 = k_1^2, \quad D_2 = k_2^2, \quad D_3 = (k_1 - p_2)^2 - m^2, \quad D_4 = (k_2 - p_2)^2 - m^2, \\ D_5 = (k_1 - p_1 - p_2)^2, \quad D_6 = (k_2 - p_1 - p_2)^2, \quad D_7 = (k_1 - k_2)^2 . \end{aligned} \tag{6.8}$$

The integrals belonging to this family can be reduced to a set of 29 MIs which are conveniently expressed in terms of the dimensionless variables $\vec{x} = \{u, z, \bar{z}\}$ defined by

$$-\frac{s}{m^2} = u, \quad \frac{p_1^2}{s} = z\,\bar{z}, \quad \frac{p_2^2}{s} = (1-z)(1-\bar{z}). \tag{6.9}$$

The same parametrization for p_1^2 and p_2^2 was used also for the massless triangles considered in [195]. The following set of MIs obeys an ϵ-linear system of DEQs:

$$\begin{aligned}
&\mathrm{F}_1 = \epsilon^2\,\mathcal{T}_1, &&\mathrm{F}_2 = \epsilon^2\,\mathcal{T}_2, &&\mathrm{F}_3 = \epsilon^2\,\mathcal{T}_3,\\
&\mathrm{F}_4 = \epsilon^2\,\mathcal{T}_4, &&\mathrm{F}_5 = \epsilon^2\,\mathcal{T}_5, &&\mathrm{F}_6 = \epsilon^2\,\mathcal{T}_6,\\
&\mathrm{F}_7 = \epsilon^2\,\mathcal{T}_7, &&\mathrm{F}_8 = \epsilon^2\,\mathcal{T}_8, &&\mathrm{F}_9 = \epsilon^2\,\mathcal{T}_9,\\
&\mathrm{F}_{10} = \epsilon^3\,\mathcal{T}_{10}, &&\mathrm{F}_{11} = \epsilon^2\,\mathcal{T}_{11}, &&\mathrm{F}_{12} = \epsilon^2\,\mathcal{T}_{12},\\
&\mathrm{F}_{13} = \epsilon^2\,\mathcal{T}_{13}, &&\mathrm{F}_{14} = \epsilon^2\,\mathcal{T}_{14}, &&\mathrm{F}_{15} = \epsilon^2\,\mathcal{T}_{15},\\
&\mathrm{F}_{16} = \epsilon^3\,\mathcal{T}_{16}, &&\mathrm{F}_{17} = \epsilon^2\,\mathcal{T}_{17}, &&\mathrm{F}_{18} = \epsilon^3\,\mathcal{T}_{18},\\
&\mathrm{F}_{19} = \epsilon^3\,\mathcal{T}_{19}, &&\mathrm{F}_{20} = \epsilon^2\,\mathcal{T}_{20}, &&\mathrm{F}_{21} = \epsilon^3\,\mathcal{T}_{21},\\
&\mathrm{F}_{22} = \epsilon^2\,\mathcal{T}_{22}, &&\mathrm{F}_{23} = \epsilon^3\,\mathcal{T}_{23}, &&\mathrm{F}_{24} = \epsilon^3\,\mathcal{T}_{24},\\
&\mathrm{F}_{25} = \epsilon^4\,\mathcal{T}_{25}, &&\mathrm{F}_{26} = \epsilon^4\,\mathcal{T}_{26}, &&\mathrm{F}_{27} = \epsilon^3\,\mathcal{T}_{27},\\
&\mathrm{F}_{28} = \epsilon^3\,\mathcal{T}_{28}, &&\mathrm{F}_{29} = \epsilon^2\,\mathcal{T}_{29},
\end{aligned} \tag{6.10}$$

where the $\mathcal{T}_i$ are depicted in figure 6.3. We observe that some of integrals $\mathcal{T}_i$ are trivially related by $p_1^2 \leftrightarrow p_2^2$ symmetry,

$$\mathcal{T}_4 \leftrightarrow \mathcal{T}_2, \quad \mathcal{T}_8 \leftrightarrow \mathcal{T}_5, \quad \mathcal{T}_9 \leftrightarrow \mathcal{T}_6, \quad \mathcal{T}_{14} \leftrightarrow \mathcal{T}_{12}, \quad \mathcal{T}_{21} \leftrightarrow \mathcal{T}_{16}, \quad \mathcal{T}_{22} \leftrightarrow \mathcal{T}_{17}, \tag{6.11}$$

so that the actual number of independent integrals is reduced to 23. However, in order to determine the solution of the DEQs by simultaneously integrating the whole system of equations, one has to consider the full set of integrals given in eq. (6.10).

The Magnus exponential allow us to obtain a set of canonical MIs obeying a system of equations of the form (6.5)

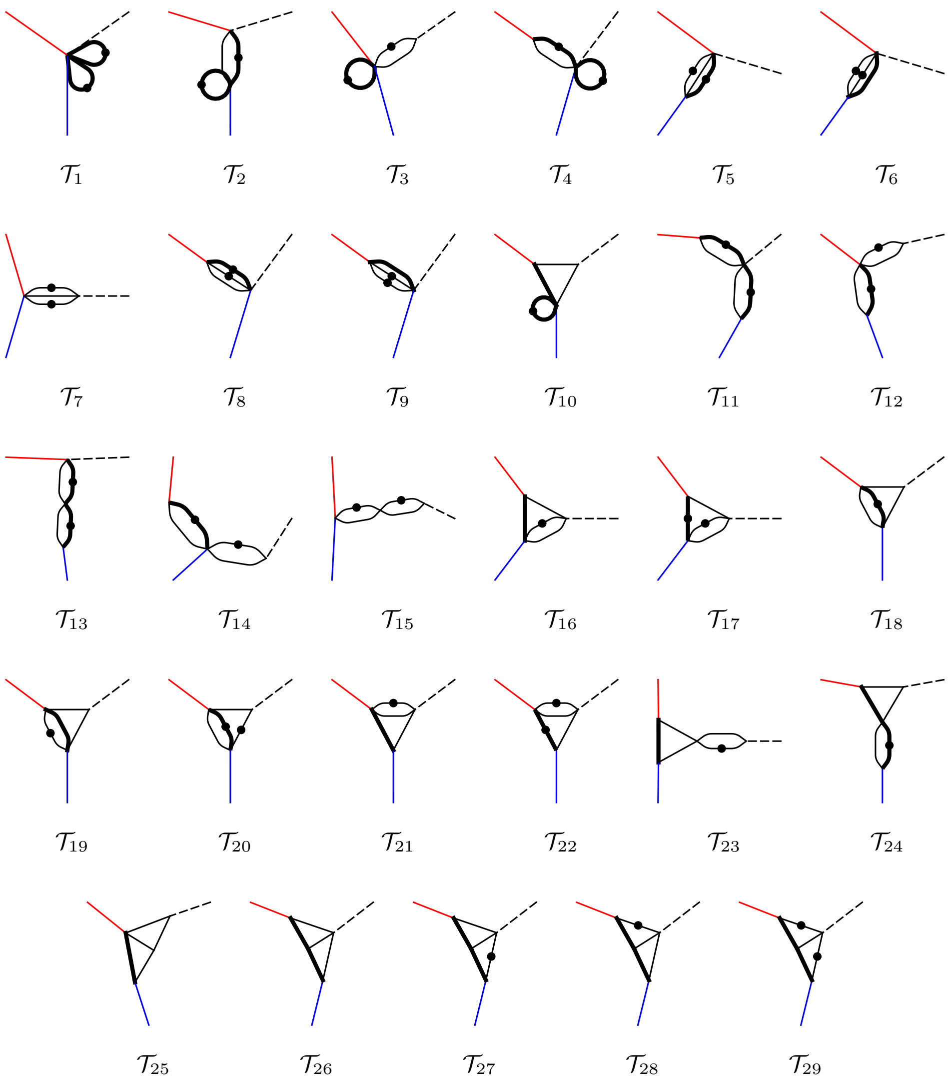


Figure 6.3: Two-loop MIs $\mathcal{T}_{1,\dots,29}$ for topologies T_1 and T_2. Graphical conventions are the same as in figure 6.2. Dots indicate squared propagators.

$$
\begin{aligned}
\mathrm{I}_1 &= \mathrm{F}_1\,, & \mathrm{I}_2 &= -p_2^2\,\mathrm{F}_2\,,\\
\mathrm{I}_3 &= -s\,\mathrm{F}_3\,, & \mathrm{I}_4 &= -p_1^2\mathrm{F}_4\,,\\
\mathrm{I}_5 &= -p_2^2\,\mathrm{F}_5\,, & \mathrm{I}_6 &= 2m^2\,\mathrm{F}_5 + (m^2-p_2^2)\,\mathrm{F}_6\,,\\
\mathrm{I}_7 &= -s\,\mathrm{F}_7\,, & \mathrm{I}_8 &= -p_1^2\mathrm{F}_8\,,\\
\mathrm{I}_9 &= 2m^2\,\mathrm{F}_8 + (m^2-p_1^2)\,\mathrm{F}_9\,, & \mathrm{I}_{10} &= -\sqrt{\lambda}\,\mathrm{F}_{10}\,,\\
\mathrm{I}_{11} &= p_1^2\,p_2^2\,\mathrm{F}_{11}\,, & \mathrm{I}_{12} &= p_1^2\,s\,\mathrm{F}_{12}\,,\\
\mathrm{I}_{13} &= p_2^4\,\mathrm{F}_{13}\,, & \mathrm{I}_{14} &= p_1^2\,s\,\mathrm{F}_{14}\,,\\
\mathrm{I}_{15} &= s^2\,\mathrm{F}_{15}\,, & \mathrm{I}_{16} &= -\sqrt{\lambda}\,\mathrm{F}_{16}\,,\\
\mathrm{I}_{17} &= c_{16,17}\,\mathrm{F}_{16} + c_{17,17}\,\mathrm{F}_{17}\,, & \mathrm{I}_{18} &= -\sqrt{\lambda}\,\mathrm{F}_{18}\,,\\
\mathrm{I}_{19} &= -\sqrt{\lambda}\,\mathrm{F}_{19}\,, & \mathrm{I}_{20} &= c_{18,20}\,\mathrm{F}_{18} + c_{19,20}\,\mathrm{F}_{19} + c_{20,20}\,\mathrm{F}_{20}\,,\\
\mathrm{I}_{21} &= -\sqrt{\lambda}\,\mathrm{F}_{21}\,, & \mathrm{I}_{22} &= c_{21,22}\,\mathrm{F}_{21} + c_{22,22}\,\mathrm{F}_{22}\,,
\end{aligned}
$$

$$\begin{aligned} \mathrm{I}_{23} &= s\sqrt{\lambda}\,\mathrm{F}_{23}\,, & \mathrm{I}_{24} &= p_2^2\sqrt{\lambda}\,\mathrm{F}_{24}\,,\\ \mathrm{I}_{25} &= -\sqrt{\lambda}\,\mathrm{F}_{25}\,, & \mathrm{I}_{26} &= -\sqrt{\lambda}\,\mathrm{F}_{26}\,,\\ \mathrm{I}_{27} &= (p_2^2-m^2)\sqrt{\lambda}\,\mathrm{F}_{27}\,, & \mathrm{I}_{28} &= (p_1^2-m^2)\sqrt{\lambda}\,\mathrm{F}_{28}\,, \end{aligned}$$
$$\mathrm{I}_{29} = c_{1,29}\,\mathrm{F}_1 + c_{2,29}\,\mathrm{F}_2 + c_{4,29}\,\mathrm{F}_4 + c_{11,29}\,\mathrm{F}_{11} + c_{27,29}\,\mathrm{F}_{27} + c_{28,29}\,\mathrm{F}_{28} + c_{29,29}\,\mathrm{F}_{29}\,, \tag{6.12}$$

where λ is the Källén function related to the external kinematics,

$$\lambda \equiv \lambda(s,p_1^2,p_2^2) = (s-p_1^2-p_2^2)^2 - 4\,p_1^2\,p_2^2. \tag{6.13}$$

The explicit expressions for the coefficients $c_{i,j}$ for $\mathrm{I}_{20,22,29}$ are

$$\begin{aligned} c_{16,17} &= \frac{3}{2}\left(\sqrt{\lambda}+s-p_1^2-p_2^2+2m^2\right)\,,\\ c_{17,17} &= p_1^2p_2^2-(p_1^2+p_2^2)m^2+m^4+m^2s\,,\\ c_{21,22} &= \frac{3}{2}\left(\sqrt{\lambda}+s-p_1^2-p_2^2+2m^2\right)\,,\\ c_{22,22} &= p_1^2p_2^2-(p_1^2+p_2^2)m^2+m^4+m^2s\,,\\ c_{1,29} &= \frac{m^2s}{(p_1^2-m^2)(p_2^2-m^2)}\,,\\ c_{2,29} &= -\frac{m^2sp_2^2}{(p_1^2-m^2)(p_2^2-m^2)}\,,\\ c_{4,29} &= -\frac{m^2sp_1^2}{(p_1^2-m^2)(p_2^2-m^2)}\,,\\ c_{27,29} &= p_1^2\left(p_2^2-m^2\right)+p_2^2\left(\sqrt{\lambda}+m^2+s-p_2^2\right)+m^2\left(s-\sqrt{\lambda}\right)\,,\\ c_{28,29} &= p_1^2\left(p_2^2-\sqrt{\lambda}+m^2+s-p_1^2\right)+m^2(\sqrt{\lambda}+s-p_2^2)\,,\\ c_{29,29} &= -\,s\left(p_1^2(p_2^2-m^2)+m^2(s+m^2-p_2^2)\right)\,. \end{aligned} \tag{6.14}$$

The alphabet of the corresponding dlog-form contains the following 10 letters:

$$\begin{aligned} \eta_1 &= u\,, & \eta_2 &= z\,,\\ \eta_3 &= 1-z\,, & \eta_4 &= \bar{z}\,,\\ \eta_5 &= 1-\bar{z}\,, & \eta_6 &= z-\bar{z}\,,\\ \eta_7 &= 1+u\,z\,\bar{z}\,, & \eta_8 &= 1-u\,z(1-\bar{z}),\\ \eta_9 &= 1-u\,\bar{z}(1-z)\,, & \eta_{10} &= 1+u\,(1-z)(1-\bar{z})\,. \end{aligned} \tag{6.15}$$

The coefficient matrices $\mathbb{M}_i$ are collected in the appendix B.1. All letters are real and positive in the region

$$0<z<1\,,\quad 0<\bar{z}<z\,,\quad 0<u<\frac{1}{z(1-\bar{z})}\,, \tag{6.16}$$

which, for $m^2>0$, corresponds to a patch of the Euclidean region, $s\,,p_1^2\,,p_2^2<0$, defined by

$$\begin{aligned} &\sqrt{-p_1^2}\sqrt{-p_2^2} > m^2\,,\\ -&\frac{\left(p_1^2-m^2\right)\left(p_2^2-m^2\right)}{m^2} < s < p_1^2+p_2^2-2\sqrt{-p_1^2}\sqrt{-p_2^2}\,. \end{aligned} \tag{6.17}$$

In the region defined by eq. (6.16), the general solution of the DEQs is expressed directly in terms of GPLs, with argument depending on the kinematics variables u, z and $\bar{z}$. Imposing the regularity of our solutions at the unphysical thresholds, $z, \bar{z} = 0$ (corresponding to $p_1^2 = 0$) and $z, \bar{z} = 1$ (corresponding to $p_2^2 = 0$) entails relations between the boundary constants. These relations allow us to derive all boundary constants from five simpler integrals $\mathrm{I}_{1,3,6,7,15}$, which are obtained in the following way:

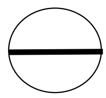
- I_1 is a constant to be determined by direct integration and, due to the normalization of the integration measure (5.113), it is simply set to

$$\mathrm{I}_1(\epsilon, \vec{x}) = 1. \tag{6.18}$$

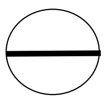
- I_3 can be obtained by direct integration

$$\mathrm{I}_3(\epsilon, \vec{x}) = \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} u^{-\epsilon} \,. \tag{6.19}$$

- Besides being regular in the massless kinematic limit $z \to 1$ ($p_2^2 \to 0$), I_6 is reduced, through IBPs, to a two-loop vacuum diagram,

$$\mathrm{I}_6(\epsilon, z = 1) = -\frac{2\epsilon^2(1-\epsilon)(1-2\epsilon)}{m^2}$$  $. \quad (6.20)$

Therefore, by using as an input the analytic expression of the two-loop vacuum graph,

 $$= -\frac{m^2\Gamma(-\epsilon)\Gamma(-1+2\epsilon)}{(1-\epsilon)\Gamma(1+\epsilon)}, \tag{6.21}$$

we can fix the boundary constants by matching the $z \to 1$ limit of the expression of I_6 obtained from the solution of the DE against the ϵ-expansion of eq. (6.20),

$$\mathrm{I}_6(\epsilon, z = 1) = -1 - 2\zeta_2\epsilon^2 + 2\zeta_3\epsilon^3 - 9\zeta_4\pi^4\epsilon^4 + \mathcal{O}(\epsilon^5). \tag{6.22}$$

- I_7 and I_{15} can be directly integrated

$$\mathrm{I}_7(\epsilon, \vec{x}) = \; -u^{-2\epsilon}\left(1 - 2\zeta_2\epsilon^2 - 10\zeta_3\epsilon^3 - 11\zeta_4\epsilon^4 + \mathcal{O}(\epsilon^5)\right) , \tag{6.23}$$

$$\mathrm{I}_{15}(\epsilon, \vec{x}) = u^{-2\epsilon}\left(1 - \zeta_2\epsilon^2 - 2\zeta_3\epsilon^3 - \frac{9}{4}\zeta 4\epsilon^4 + \mathcal{O}(\epsilon^5)\right) . \tag{6.24}$$

The prescriptions for the analytic continuation to the other patches of the Euclidean region ($s, p_1^2, p_2^2 < 0$) and to the physical regions are given in section 6.4. The results have been numerically checked, in both the Euclidean and the physical regions, with the help of the computer codes GiNaC and SecDec.

6.3.2 Second integral family

The topologies T_3 and T_4 belong to the integral family

$$\int \widetilde{\mathrm{d}^d k_1}\widetilde{\mathrm{d}^d k_2} \, \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7}} \,, \quad n_i \in \mathbb{Z} \,, \tag{6.25}$$

defined by the set of denominators

$$D_1 = k_1^2 - m^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_1 - p_2)^2, \quad D_4 = (k_2 - p_2)^2,$$
$$D_5 = (k_1 - p_1 - p_2)^2 - m^2, \quad D_6 = (k_2 - p_1 - p_2)^2 - m^2, \quad D_7 = (k_1 - k_2)^2 . \quad (6.26)$$

The integrals belonging to this family can be reduced to a set of 31 MIs which are conveniently expressed in terms of the variables $\vec{x} = \{v, z, \bar{z}\}$, defined by

$$-\frac{s}{m^2} = \frac{(1-v)^2}{v}, \qquad \frac{p_1^2}{s} = z\bar{z}, \qquad \frac{p_2^2}{s} = (1-z)(1-\bar{z}). \quad (6.27)$$

The following set of MIs obeys a system of DEQs which is linear in ϵ :

$$\begin{aligned}
&\mathrm{F}_1 = \epsilon^2\,\mathcal{T}_1\,, && \mathrm{F}_2 = \epsilon^2\,\mathcal{T}_2\,, && \mathrm{F}_3 = \epsilon^2\,\mathcal{T}_3\,,\\
&\mathrm{F}_4 = \epsilon^2\,\mathcal{T}_4\,, && \mathrm{F}_5 = \epsilon^2\,\mathcal{T}_5\,, && \mathrm{F}_6 = \epsilon^2\,\mathcal{T}_6\,,\\
&\mathrm{F}_7 = \epsilon^2\,\mathcal{T}_7\,, && \mathrm{F}_8 = \epsilon^2\,\mathcal{T}_8\,, && \mathrm{F}_9 = \epsilon^2\,\mathcal{T}_9\,,\\
&\mathrm{F}_{10} = \epsilon^2\,\mathcal{T}_{10}\,, && \mathrm{F}_{11} = \epsilon^2\,\mathcal{T}_{11}\,, && \mathrm{F}_{12} = \epsilon^2\,\mathcal{T}_{12}\,,\\
&\mathrm{F}_{13} = \epsilon^2\,\mathcal{T}_{13}\,, && \mathrm{F}_{14} = \epsilon^2\,\mathcal{T}_{14}\,, && \mathrm{F}_{15} = \epsilon^2\,\mathcal{T}_{15}\,,\\
&\mathrm{F}_{16} = \epsilon^2\,\mathcal{T}_{16}\,, && \mathrm{F}_{17} = \epsilon^3\,\mathcal{T}_{17}\,, && \mathrm{F}_{18} = \epsilon^3\,\mathcal{T}_{18}\,,\\
&\mathrm{F}_{19} = \epsilon^2\,\mathcal{T}_{19}\,, && \mathrm{F}_{20} = \epsilon^3\,\mathcal{T}_{20}\,, && \mathrm{F}_{21} = -\epsilon^2(1-2\epsilon)\,\mathcal{T}_{21}\,,\\
&\mathrm{F}_{22} = \epsilon^3\,\mathcal{T}_{22}\,, && \mathrm{F}_{23} = \epsilon^3\,\mathcal{T}_{34}\,, && \mathrm{F}_{24} = \epsilon^2\,\mathcal{T}_{24}\,,\\
&\mathrm{F}_{25} = \epsilon^2\,\mathcal{T}_{25}\,, && \mathrm{F}_{26} = \epsilon^2\,\mathcal{T}_{26}\,, && \mathrm{F}_{27} = \epsilon^4\,\mathcal{T}_{27}\,,\\
&\mathrm{F}_{28} = \epsilon^3\,\mathcal{T}_{28}\,, && \mathrm{F}_{29} = \epsilon^3\,\mathcal{T}_{29}\,, && \mathrm{F}_{30} = \epsilon^2\,\mathcal{T}_{30}\,,\\
&\mathrm{F}_{31} = \epsilon^4\,\mathcal{T}_{31}\,, && && \qquad (6.28)
\end{aligned}$$

where the $\mathcal{T}_i$ are depicted in figure 6.4. As for the first integral family, some of the integrals $\mathcal{T}_i$ are related by $p_1^2 \leftrightarrow p_2^2$,

$$\mathcal{T}_4 \leftrightarrow \mathcal{T}_2\,, \quad \mathcal{T}_9 \leftrightarrow \mathcal{T}_5\,, \quad \mathcal{T}_{10} \leftrightarrow \mathcal{T}_6\,, \quad \mathcal{T}_{15} \leftrightarrow \mathcal{T}_{12}\,, \quad \mathcal{T}_{22} \leftrightarrow \mathcal{T}_{17}\,, \quad \mathcal{T}_{23} \leftrightarrow \mathcal{T}_{18}\,, \quad \mathcal{T}_{24} \leftrightarrow \mathcal{T}_{19}\,, \quad (6.29)$$

so that the total number of independent integrals is 24. Nonetheless, as discussed already after eq. (6.11), we work with the complete set of integrals given in eq. (6.28).

With the help of the Magnus exponential, we obtain a set of canonical MIs

$$\begin{aligned}
&\mathrm{I}_1 = \mathrm{F}_1\,, && \mathrm{I}_2 = -p_2^2\,\mathrm{F}_2\,,\\
&\mathrm{I}_3 = \rho\,\mathrm{F}_3\,, && \mathrm{I}_4 = -p_1^2\mathrm{F}_4\,,\\
&\mathrm{I}_5 = (m^2 - p_2^2)\,\mathrm{F}_5 + 2m^2\,\mathrm{F}_6\,, && \mathrm{I}_6 = -p_2^2\,\mathrm{F}_6\\
&\mathrm{I}_7 = \rho\,\mathrm{F}_7 + \frac{1}{2}(\rho - s)\,\mathrm{F}_8\,, && \mathrm{I}_8 = -s\mathrm{F}_8\,,\\
&\mathrm{I}_9 = -p_1^2\,\mathrm{F}_9\,, && \mathrm{I}_{10} = 2m^2\,\mathrm{F}_9 + (m^2 - p_1^2)\,\mathrm{F}_{10}\,,\\
&\mathrm{I}_{11} = p_2^4\,\mathrm{F}_{11}\,, && \mathrm{I}_{12} = -p_2^2\rho\,\mathrm{F}_{12}\,,\\
&\mathrm{I}_{13} = p_1^2p_2^2\,\mathrm{F}_{13} && \mathrm{I}_{14} = \rho^2\,\mathrm{F}_{14}\,,\\
&\mathrm{I}_{15} = -p_1^2\rho\,\mathrm{F}_{15}\,,
\end{aligned}$$

$$\mathrm{I}_{16} = c_{2,16}\,\mathrm{F}_2 + c_{3,16}\,\mathrm{F}_3 + c_{4,16}\,\mathrm{F}_4 + c_{16,16}\,\mathrm{F}_{16}\,,$$

$$\begin{aligned}
&\mathrm{I}_{17} = -\sqrt{\lambda}\,\mathrm{F}_{17}\,, && \mathrm{I}_{18} = -\sqrt{\lambda}\,\mathrm{F}_{18}\,,\\
&\mathrm{I}_{19} = c_{17,19}\,\mathrm{F}_{17} + c_{18,19}\,\mathrm{F}_{18} + c_{19,19}\,\mathrm{F}_{19}\,, && \mathrm{I}_{20} = -\sqrt{\lambda}\,\mathrm{F}_{20}\,,
\end{aligned}$$

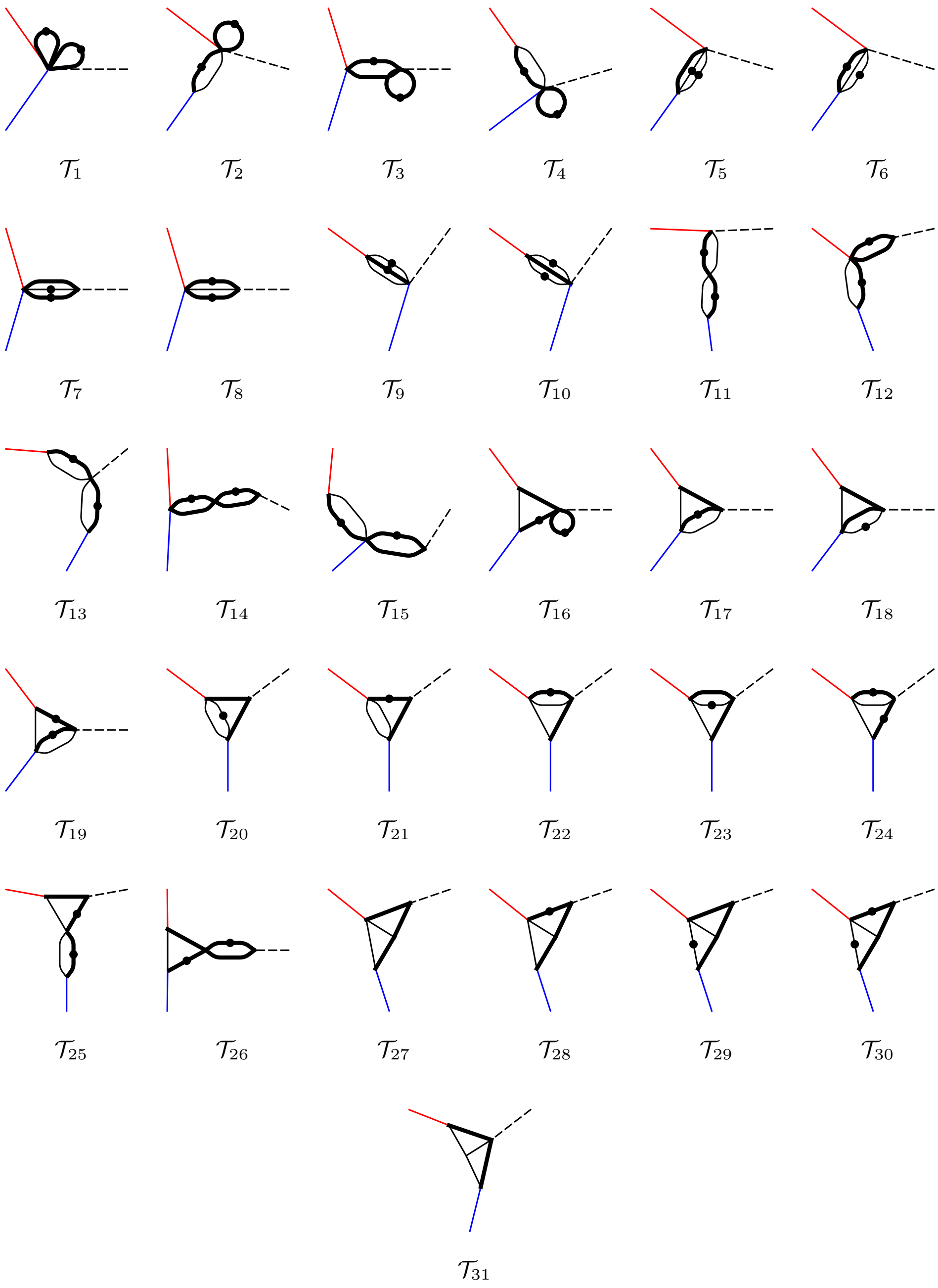


Figure 6.4: Two-loop MIs $\mathcal{T}_{1,\ldots,31}$ for the topologies T_3 and T_4. Graphical conventions are the same as in figure 6.2. Dots indicate squared propagators.

$$
\begin{aligned}
\mathrm{I}_{21} &= c_{5,21}\,\mathrm{F}_5 + c_{6,21}\,\mathrm{F}_6 + c_{9,21}\,\mathrm{F}_9 + c_{10,21}\,\mathrm{F}_{10} + c_{20,21}\,\mathrm{F}_{20} + c_{21,21}\,\mathrm{F}_{21}\,,\\
\mathrm{I}_{22} &= -\sqrt{\lambda}\,\mathrm{F}_{22}\,, \qquad \mathrm{I}_{23} = -\sqrt{\lambda}\,\mathrm{F}_{23}\,,\\
\mathrm{I}_{24} &= c_{22,24}\,\mathrm{F}_{22} + c_{23,24}\,\mathrm{F}_{23} + c_{24,24}\,\mathrm{F}_{24}\,,\\
\mathrm{I}_{25} &= c_{11,25}\,\mathrm{F}_{11} + c_{12,25}\,\mathrm{F}_{12} + c_{13,25}\,\mathrm{F}_{13} + c_{25,25}\,\mathrm{F}_{25}\,,\\
\mathrm{I}_{26} &= c_{12,26}\,\mathrm{F}_{12} + c_{14,26}\,\mathrm{F}_{14} + c_{15,26}\,\mathrm{F}_{15} + c_{26,26}\,\mathrm{F}_{26}\,,\\
\mathrm{I}_{27} &= -\sqrt{\lambda}\,\mathrm{F}_{27}\,, \qquad \mathrm{I}_{28} = -\rho\sqrt{\lambda}\,\mathrm{F}_{28}\,,\\
\mathrm{I}_{29} &= (p_2^2 - m^2)\sqrt{\lambda}\,\mathrm{F}_{29}\,,\\
\mathrm{I}_{30} &= c_{3,30}\,\mathrm{F}_3 + c_{12,30}\,\mathrm{F}_{12} + c_{28,30}\,\mathrm{F}_{28} + c_{29,30}\,\mathrm{F}_{29} + c_{30,30}\,\mathrm{F}_{30}\,,\\
\mathrm{I}_{31} &= -\sqrt{\lambda}\,\mathrm{F}_{31}\,,
\end{aligned}
\tag{6.30}
$$

which obey a system of equations of the form (6.5). In the above definition, λ is given as in eq. (6.13) and we have set $\rho \equiv \sqrt{-s}\sqrt{4m^2-s}$. The expression of the coefficients $c_{i,j}$ for $\mathrm{I}_{16,19,21,24,25,26,30}$ are

$$
\begin{aligned}
c_{2,16} &= -\frac{p_2^2\left(-m^2\left(p_2^2 - s + \sqrt{\lambda}\right) - p_1^2\left(p_2^2 - p_1^2 + s - m^2 - \sqrt{\lambda}\right)\right)}{(p_1^2 - p_2^2)(p_1^2 + m^2) - s(p_1^2 - m^2)},\\
c_{3,16} &= \frac{\sqrt{\lambda}\left(p_1^2(2m^2 - s) - m^2(2p_2^2 - s)\right)}{(p_1^2 - p_2^2)(p_1^2 + m^2) - (p_1^2 - m^2)s} - \rho,\\
c_{4,16} &= -\frac{p_1^2\left(m^2\left(p_2^2 - s + \sqrt{\lambda}\right) + p_1^2\left(p_2^2 - p_1^2 + s - m^2 - \sqrt{\lambda}\right)\right)}{(p_1^2 - p_2^2)(p_1^2 + m^2) - (p_1^2 - m^2)s},\\
c_{16,16} &= \frac{\sqrt{\lambda}\left(m^2(p_1^2 - p_2^2)^2 + s(p_1^2 - m^2)(p_2^2 - m^2)\right)}{(p_1^2 - p_2^2)(p_1^2 + m^2) - (p_1^2 - m^2)s},\\
c_{17,19} &= \left(p_2^2 - p_1^2 + 2m^2 - s - \sqrt{\lambda}\right),\\
c_{18,19} &= \frac{1}{2}\left(p_2^2 - p_1^2 + 2m^2 - s - \sqrt{\lambda}\right),\\
c_{19,19} &= m^2(p_2^2 + m^2 - s) + p_1^2(s - m^2),\\
c_{5,21} &= \frac{m^2(p_2^2 - m^2)}{s + \rho},\\
c_{6,21} &= -\frac{2m^2(p_2^2 + m^2)}{s + \rho},\\
c_{9,21} &= \frac{2m^2(p_1^2 + m^2)}{s + \rho},\\
c_{10,21} &= -\frac{m^2(p_1^2 - m^2)}{s + \rho},\\
c_{20,21} &= -\frac{1}{2}\sqrt{\lambda} + \frac{1}{2s}(s + p_2^2 - p_1^2)\rho,\\
c_{21,21} &= \rho,\\
c_{22,24} &= p_1^2 - p_2^2 - s + 2m^2 + \sqrt{\lambda},\\
c_{23,24} &= \frac{1}{2}(p_1^2 - p_2^2 - s + 2m^2 + \sqrt{\lambda}),\\
c_{24,24} &= m^2 p_1^2 - (p_2^2 - m^2)(m^2 - s),\\
c_{11,25} &= -\frac{p_2^2\left(m^2\left(p_2^2 - s + \sqrt{\lambda}\right) + p_1^2\left(p_2^2 - p_1^2 + s - m^2 - \sqrt{\lambda}\right)\right)}{(p_1^2 - p_2^2)(p_1^2 + m^2) - s(p_1^2 - m^2)},
\end{aligned}
$$

$$
\begin{aligned}
c_{12,25} &= \frac{p_2^2\sqrt{\lambda}\left(s(p_1^2-m^2)-2m^2(p_1^2-p_2^2)\right)}{(p_1^2-p_2^2)(p_1^2+m^2)-(p_1^2-m^2)s}+p_2^2\rho,\\
c_{13,25} &= \frac{p_1^2p_2^2\left(p_1^2\left(p_2^2-p_1^2+s-m^2-\sqrt{\lambda}\right)+m^2\left(p_2^2-s+\sqrt{\lambda}\right)\right)}{(p_1^2-p_2^2)(p_1^2+m^2)-(p_1^2-m^2)s},\\
c_{25,25} &= -\frac{p_2^2\sqrt{\lambda}\left(m^2(p_1^2-p_2^2)^2+s(p_1^2-m^2)(p_2^2-m^2)\right)}{(p_1^2-p_2^2)(p_1^2+m^2)-(p_1^2-m^2)s},\\
c_{12,26} &= \frac{p_2^2\rho\left(p_1^2\left(p_2^2-p_1^2+s-m^2-\sqrt{\lambda}\right)+m^2\left(p_2^2-s+\sqrt{\lambda}\right)\right)}{(p_1^2-p_2^2)(p_1^2+m^2)-(p_1^2-m^2)s},\\
c_{14,26} &= \rho^2+\frac{\rho\sqrt{\lambda}\left(p_1^2\left(2m^2-s\right)+m^2\left(s-2p_2^2\right)\right)}{(p_1^2-p_2^2)(p_1^2+m^2)-(p_1^2-m^2)s},\\
c_{15,26} &= -\frac{p_1^2\rho\left(p_1^2\left(p_2^2-p_1^2-s+\sqrt{\lambda}\right)+m^2\left(p_2^2-s+\sqrt{\lambda}\right)\right)}{(p_1^2-p_2^2)(p_1^2+m^2)-(p_1^2-m^2)s},\\
c_{26,26} &= \frac{8\sqrt{\lambda}\left(m^2(p_1^2-p_2^2)^2+s(p_1^2-m^2)(p_2^2-m^2)\right)}{\left((p_1^2-p_2^2)(p_1^2+m^2)-(p_1^2-m^2)s\right)\left(s-2m^2+\rho\right)^4}\times\\
&\quad\times\Big(s^4\left(s+\rho\right)+2m^8\left(8s+\rho\right)-m^2s^3\left(5s+4\rho\right)-4m^6s\left(11s+4\rho\right)\\
&\quad+2m^4s^2\left(17s+10\rho\right)\Big),\\
c_{3,30} &= 2(p_1^2-p_2^2)-\frac{2p_2^2(p_1^2-p_2^2)}{p_2^1-m^2}-s+\rho\,,\\
c_{12,30} &= 2p_2^2\left(-2(p_1^2-p_2^2)+\frac{2p_2^2(p_1^2-p_2^2)}{p_2^2-m^2}+s-\rho\right),\\
c_{28,30} &= s(p_1^2+p_2^2+2m^2-s)-\rho\sqrt{\lambda},\\
c_{29,30} &= \left(p_2^2(s-m^2-p_2^2)+p_1^2(p_2^2-m^2)-m^2s\right)+(p_2^2-m^2)\sqrt{\lambda}\,,\\
c_{30,30} &= -m^2(p_1^2-p_2^2)^2-(p_1^2-m^2)(p_2^2-m^2)s\,,
\end{aligned}
\tag{6.31}
$$

The alphabet of the corresponding dlog-form contains the following 18 letters

$$
\begin{aligned}
\eta_1 &= v\,, & \eta_2 &= 1-v\,,\\
\eta_3 &= 1+v\,, & \eta_4 &= z\,,\\
\eta_5 &= 1-z\,, & \eta_6 &= \bar{z}\,,\\
\eta_7 &= 1-\bar{z}\,, & \eta_8 &= z-\bar{z}\,,\\
\eta_9 &= z+v(1-z)\,, & \eta_{10} &= 1-z(1-v)\,,\\
\eta_{11} &= \bar{z}+v(1-\bar{z})\,, & \eta_{12} &= 1-\bar{z}(1-v)\,,\\
\eta_{13} &= v+z\bar{z}(1-v)^2\,, & \eta_{14} &= v+(1-z-\bar{z}+z\bar{z})(1-v)^2\,,\\
\eta_{15} &= v+z(1-v)^2\,, & \eta_{16} &= v+(1-z)(1-v)^2\,,\\
\eta_{17} &= v+\bar{z}(1-v)^2\,, & \eta_{18} &= v+(1-\bar{z})(1-v)^2\,,
\end{aligned}
\tag{6.32}
$$

and the coefficient matrices $\mathbb{M}_i$ are collected in the appendix B.1. In this case, all the letters are real and positive in the region

$$
0<v<1\,,\quad 0<z<1\,,\quad 0<\bar{z}<z\,,
\tag{6.33}
$$

which, being $m^2 > 0$, corresponds to a patch of the Euclidean region, $s\,, p_1^2\,, p_2^2 < 0$, defined by the following constraint

$$s < -\left(\sqrt{-p_1^2}+\sqrt{-p_2^2}\right)^2 < 0\,. \tag{6.34}$$

Since all letters are rational and have algebraic roots, the solution of the DEQs can be expressed in terms of GPLs. However, for the purpose of fixing the boundary constants, we find it convenient to first derive a straightforward representation of the solution in terms Chen iterated integrals and then convert it to GPLs of argument 1 and kinematic-dependent weights, by following the discussion of sec. 5.8. The boundary constants can be determined by demanding the regularity of the basis (6.28) for vanishing external momenta, $s = p_1^2 = p_2^2 = 0$. In particular, if we choose as a base-point for the integration

$$\vec{x}_0 = (1, 1, 1), \tag{6.35}$$

then the prefactors appearing in the definitions (6.30) of the canonical MIs $\mathbf{I}$ vanish, with the only exceptions of $\mathrm{I}_{1,5,10,19,21,24}$. Therefore, the boundaries of the former MIs are determined by demanding their vanishing at $\vec{x} \to \vec{x}_0$,

$$\mathrm{I}_i(\epsilon, \vec{x}_0) = 0, \qquad i \neq 1, 5, 10, 19, 21, 24. \tag{6.36}$$

The integrals $\mathrm{I}_{1,\,5,\,10}$ correspond to $\mathrm{I}_{1,\,6,\,9}$ of the first integral family, whereas $\mathrm{I}_{19,\,21,\,24}$ are fixed as follows:

- The boundary constants for I_{19} and I_{24} can be determined by imposing regularity at the pseudothresholds $v \to 1$ ($s = p_1^2 = p_2^2 = 0$) and, respectively, $z \to 1$, $\bar{z} \to 1$ (both corresponding to $p_2^2 = 0$),

$$\mathrm{I}_{19,\,24}(\epsilon, \vec{x}_0) = \frac{1}{6}\pi^2\epsilon^2 - \zeta_3\epsilon^3 + \frac{1}{20}\pi^4\epsilon^4 + \mathcal{O}(\epsilon^5). \tag{6.37}$$

- The boundary constants for I_{21} can be fixed by observing that, from (6.30), we can derive

$$\mathrm{F}_{21}(\epsilon,\, \vec{x}_0) = \lim_{\vec{x}\to\vec{x}_0} \frac{v}{m^2(1-v^2)}\mathrm{I}_{21}(\epsilon,\, \vec{x}_0). \tag{6.38}$$

  Therefore, in order for $\mathrm{F}_{21}(\epsilon,\, \vec{x}_0)$ to be regular we must demand

$$\mathrm{I}_{21}(\epsilon, \vec{x}_0) = 0\,. \tag{6.39}$$

The prescriptions for the analytic continuation to the other patches of the Euclidean region and to the physical regions are given in section 6.4. All results have been numerically checked, in both the Euclidean and the physical regions, with the help of the computer codes GiNaC and SecDec.

6.4 Analytic continuation

In this section, we discuss in detail the variables used to parametrize the dependence of the MIs on the kinematic invariants. In particular, we elaborate on the prescriptions to analytically continue our results to arbitrary values of s, p_1^2 an p_2^2. Both integral families feature two independent kinematic structures:

1. the off-shell external legs are responsible for the presence in the DEQs of the square root of the Källén function, $\sqrt{\lambda(s,p_1^2,p_2^2)}$;

2. the presence of massive internal lines can generate square roots in the DEQs, as in the case of topologies T_3 and T_4, where one has also $\sqrt{-s}\sqrt{4m^2-s}$.

In the following we separately discuss the variable changes that rationalize the two types of square roots.

6.4.1 Off-shell external legs: the $z, \bar{z}$ variables

To deal with the square root of the Källén function, we begin by choosing one of the external legs as reference, s, and trading the other squared momenta for dimensionless ratios

$$\tau_{1,2} = \frac{p_{1,2}^2}{s}\,. \tag{6.40}$$

In the (s,τ_1,τ_2) variables, the square root of the Källén function is proportional to

$$\sqrt{\lambda(1,\tau_1,\tau_2)} = \sqrt{(1-\tau_1-\tau_2)^2-4\tau_1\tau_2} \tag{6.41}$$

and is rationalized by the following change of variables [195]

$$\tau_1 = z\bar{z}\,, \tag{6.42}$$
$$\tau_2 = (1-z)(1-\bar{z})\,, \tag{6.43}$$

(see eqs. (6.9) and (6.27)), that leads to

$$\lambda\left(1,\tau_1(z,\bar{z}),\tau_2(z,\bar{z})\right) = (z-\bar{z})^2\,. \tag{6.44}$$

Without loss of generality, we choose the following root of eq. (6.43)

$$z = \frac{1}{2}\left(1+\tau_1-\tau_2+\sqrt{\lambda(1,\tau_1,\tau_2)}\right)\,, \tag{6.45}$$
$$\bar{z} = \frac{1}{2}\left(1+\tau_1-\tau_2-\sqrt{\lambda(1,\tau_1,\tau_2)}\right)\,. \tag{6.46}$$

Varying the pair (τ_1,τ_2) in the real plane, we identify the following possibilities for $z,\bar{z}$

$$\begin{cases} \bar{z}=z^* & \lambda(1,\tau_1,\tau_2)<0\,, \quad \tau_1,\tau_2>0 & \text{(region I)}\\ 0<\bar{z}<z<1 & \sqrt{\tau_1}+\sqrt{\tau_2}<1\,, \quad 0<\tau_1\,,\tau_2<1 & \text{(region II)}\\ \bar{z}<z<0 & \sqrt{\tau_2}>1+\sqrt{\tau_1}\,, \quad \tau_1>0 & \text{(region III)}\\ z>\bar{z}>1 & \sqrt{\tau_1}>1+\sqrt{\tau_2}\,, \quad \tau_2>0 & \text{(region IV)}\\ z=\bar{z}=\pm\sqrt{\tau_1} & \tau_2=\left(1\pm\sqrt{\tau_1}\right)^2\,, \quad \tau_1,\tau_2>0 & \text{(region V)}\\ z>1\,, \quad \bar{z}<0 & \tau_1,\tau_2<0 & \text{(region VI)}\\ 0<z<1\,, \quad \bar{z}<0 & \tau_1<0\,, \quad \tau_2>0 & \text{(region VII)}\\ z>1\,, \quad 0<\bar{z}<1 & \tau_1>0\,, \quad \tau_2<0 & \text{(region VIII)} \end{cases} \tag{6.47}$$

where the first five regions were discussed also in [195]. A graphical representation of these eight regions in the $\tau_1\tau_2$-plane is shown in figure 6.5.

The variables $z,\bar{z}$ are complex conjugates in region I, where $\lambda(1,\tau_1,\tau_2)<0$, and real in all the other regions. In regions I-V one has $\tau_{1,2}>0$, which requires that either $s,p_1^2,p_2^2<0$ or $s,p_1^2,p_2^2>0$. The former case defines the Euclidean region. The latter case, for $\lambda(1,\tau_1,\tau_2)>0$, describes $1\to2$ or $2\to1$ processes involving three timelike

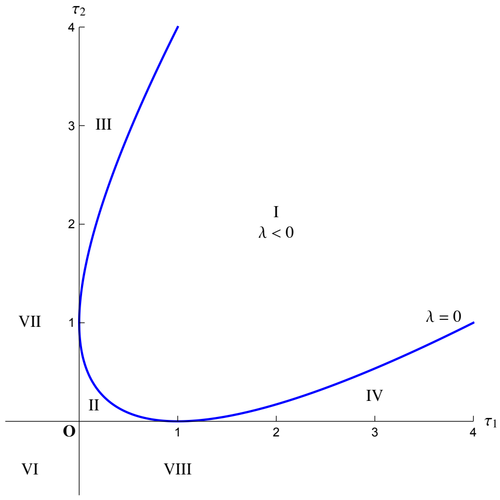


Figure 6.5: Regions of the (τ_1, τ_2)-plane classified in eq (6.47). Region V, which is identified by the condition $\lambda(1, \tau_1, \tau_2) = 0$, corresponds to the blue curve.

particles. Region V is where $\lambda(1, \tau_1, \tau_2) = 0$, so that $z = \bar{z}$. Since our expressions are obtained in general for $z \neq \bar{z}$, the limit $\bar{z} \to z$ has to be taken carefully. Regions VI-VIII have at least one of the $\tau_i < 0$, which requires either two external legs to be spacelike and the remaining one to be timelike, or vice versa. The former configuration, in the $2 \to 1$ kinematics, describes the vertex entering the production of a timelike particle via the "fusion" of two spacelike particles.

In regions other than II, the variables $z, \bar{z}$ are not in the half of the unit square where all the letters are real, therefore analytic continuation is required. A consistent physical prescription is inherited in regions VI-VIII from the Feynman prescription on the kinematic invariants, and it is naturally extended to the other regions, as we argue below. For the moment we hold $s < 0$, and we will discuss later the case $s > 0$.
In region VI, $s < 0$ and $p_1^2, p_2^2 > 0$, then

$$\tau_i \to -|\tau_i| - i\varepsilon\,, \tag{6.48}$$

so that the vanishing imaginary parts outside the square root in eq (6.46) cancel against each other, and only the one stemming from the square root is left:

$$z \to z + i\varepsilon\,, \qquad \bar{z} \to \bar{z} - i\varepsilon\,. \tag{6.49}$$

In region VII, $p_1^2 > 0$ and $s, p_2^2 < 0$, then

$$\tau_1 \to -|\tau_1| - i\varepsilon\,, \qquad \tau_2 \to |\tau_2|\,, \tag{6.50}$$

so that

$$\begin{aligned} z &\to z + i\,\frac{\varepsilon}{2}\left(\frac{1 + |\tau_1| + \tau_2}{\sqrt{\lambda(1, -|\tau_1|, \tau_2)}} - 1\right) \simeq z + i\varepsilon\,, \\ \bar{z} &\to \bar{z} - i\,\frac{\varepsilon}{2}\left(\frac{1 + |\tau_1| + \tau_2}{\sqrt{\lambda(1, -|\tau_1|, \tau_2)}} + 1\right) \simeq \bar{z} - i\varepsilon\,, \end{aligned} \tag{6.51}$$

where the approximate equalities are allowed because the factor in the bracket is always positive, and a redefinition of ε is understood.

In region VIII, $p_2^2 > 0$ and $s, p_1^2 < 0$, then

$$\tau_1 \to |\tau_1|\,, \qquad \tau_2 \to -|\tau_2| - i\varepsilon\,, \tag{6.52}$$

so that

$$\begin{aligned} z &\to z + i\,\frac{\varepsilon}{2}\left(\frac{1+\tau_1+|\tau_2|}{\sqrt{\lambda(1,\tau_1,-|\tau_2|)}}+1\right) \simeq z + i\varepsilon\,, \\ \bar{z} &\to \bar{z} - i\,\frac{\varepsilon}{2}\left(\frac{1+\tau_1+|\tau_2|}{\sqrt{\lambda(1,\tau_1,-|\tau_2|)}}-1\right) \simeq \bar{z} - i\varepsilon\,, \end{aligned} \tag{6.53}$$

where again the approximate equalities are allowed because the factor in the bracket is always positive, and a redefinition of ε is understood.
We have so far only discussed the case in which $s < 0$. It is easy to see that, if instead $s > 0$, the prescription on $z, \bar{z}$ is the opposite.

In regions I-V there is no physical prescription for the analytic continuation of $z, \bar{z}$. Indeed, if $s, p_1^2, p_2^2 > 0$, then the vanishing imaginary parts of the Feynman prescription cancel out in the ratios τ_1 and τ_2:

$$\tau_i \to \frac{p_i^2\,(1+i\varepsilon)}{s\,(1+i\varepsilon)} = \tau_i\,. \tag{6.54}$$

This cancellation affects also region I, where $\sqrt{\lambda(1,\tau_1,\tau_2)} < 0$ and $z^* = \bar{z}$. Indeed, while this condition fixes the relative sign of their imaginary parts, the sign of $\mathrm{Im}\,z$ depends on the choice of the branch of the square root in eq. (6.46), which is not fixed. This last statement holds true also in the Euclidean region.

This ambiguity is resolved by the definite $i\varepsilon$ prescription in regions VI-VIII discussed above. In order to have a smooth analytic continuation in the Euclidean region, in region I we choose the branch of the square root that gives $\mathrm{Im}\sqrt{\lambda(1,\tau_1,\tau_2)} > 0$, and in regions III-IV we assign vanishing imaginary parts for $z, \bar{z}$ according to the previous discussion. The opposite prescription should be used if the three external legs are timelike.

Summarizing, according to the sign of s, we choose the following analytic continuation prescriptions for $z, \bar{z}$ in the whole real (p_1^2, p_2^2) plane

$$z \to z + i\varepsilon\,, \quad \bar{z} \to \bar{z} - i\varepsilon \quad s < 0\,, \tag{6.55}$$

$$z \to z - i\varepsilon\,, \quad \bar{z} \to \bar{z} + i\varepsilon \quad s > 0\,. \tag{6.56}$$

6.4.2 Internal massive lines: the u, v variables

For the first integral family, i.e. topologies T_1 and T_3, the change of variables eq. (6.43) is actually enough to rationalize the DEQs completely. In eq. (6.9) we simply rescale s by the internal mass ($m^2 > 0$), $-s/m^2 = u$, to deal with a dimensionless variable. If $s < 0$, $u > 0$. If $s > 0$, the Feynman prescription $s \to s + i\varepsilon$ fixes the analytic continuation for u

$$u \to -u' - i\varepsilon\,, \quad \text{with } u' > 0\,. \tag{6.57}$$

In the case of the second integral family, which collects topologies T_3 and T_4, the DEQs still contain the square roots related to the s-channel threshold at $s = 4m^2$. They are rationalized by the usual variable change (see eq. (6.27)),

$$-\frac{s}{m^2} = \frac{(1-v)^2}{v}\,, \tag{6.58}$$

of which we choose the following root

$$v = \frac{\sqrt{4m^2 - s} - \sqrt{-s}}{\sqrt{4m^2 - s} + \sqrt{-s}}\,. \tag{6.59}$$

For completeness, we discuss how v varies with s. Holding $m^2 > 0$, and keeping in mind the Feynman prescription for $s > 0$, one finds the following cases

- For $s < 0$, v is on the unit interval, $0 \le v \le 1$;
- For $0 \le s \le 4m^2$, v is a pure phase, $v = e^{i\phi}$, with $0 < \phi < \pi$;
- For $s > 4m^2$, v is on the negative unit interval, and one must replace

$$v \to -v' + i\varepsilon\,, \quad 0 \le v' \le 1\,. \tag{6.60}$$

6.4.3 Analytic continuation of the master integrals

As discussed in section 6.3, for all the topologies we start in the patch of the Euclidean region where the alphabet is real and positive (see eqs. (6.16) and (6.33)), and we solve the DEQs there. As far as the variables $z, \bar{z}$ are concerned, the conditions of positivity of the alphabet are the same for all our topologies,

$$0 < z < 1\,, \quad 0 < \bar{z} < z\,, \tag{6.61}$$

i.e. we start from region II (see eq. (6.47)). Regarding the condition on the variables associated to s, i.e. u and v (respectively for topologies $T_{1,2}$ and $T_{3,4}$), we require

$$0 < u < \frac{1}{z(1-\bar{z})}\,, \quad 0 < v < 1\,, \tag{6.62}$$

It is clear from eq. (6.47) that, if these conditions are satisfied, one does not have access even to the full Euclidean region. Results in the remaining patches of the latter, as well as in the physical regions, are obtained by analytic continuation using the prescriptions described in sections 6.4.1 and 6.4.1.

We performed the analytic continuation numerically, i.e. we assigned to $u, v, z, \bar{z}$ the vanishing imaginary parts discussed above choosing sufficiently small numerical values. For convenience, we summarize the analytic continuation prescription for the physically interesting cases.

- $X^0 \to WW$: In this region a particle of mass $s > 0$ decays in two (possibly off-shell) particles with invariant masses $p_1^2 > 0$ and $p_2^2 > 0$, so that

$$\sqrt{s} \ge \sqrt{p_1^2} + \sqrt{p_2^2}\,. \tag{6.63}$$

  Regarding $z, \bar{z}$, this corresponds to region II (see eq. (6.47)), therefore no analytic continuation is needed. Furthermore, for topologies T_1 and T_2 one must replace

$$u \to -u' - i\varepsilon$$

  irrespectively of the value of s, according to eq. (6.57). Instead, for topologies T_3 and T_4, if $0 < s < 4m^2$ then v is on the unit circle in the complex plane, while if $s > 4m^2$, one has to replace

$$v \to -v + i\varepsilon\,,$$

  according to eq. (6.60).

- $W \to WX^0$: This is again a $1 \to 2$ process involving timelike particles, the only difference being that now

$$\sqrt{p_1^2} \geq \sqrt{s} + \sqrt{p_2^2} > 0\,, \tag{6.64}$$

or

$$\sqrt{p_2^2} \geq \sqrt{s} + \sqrt{p_1^2} > 0\,. \tag{6.65}$$

The former case corresponds to region IV, the latter to region III (see eq. (6.47)). Therefore, in addition to the analytic continuation in u, v discussed already for the X^0-decay, one must further use the replacement (6.56)

$$z \to z - i\varepsilon\,, \quad \bar{z} \to \bar{z} + i\varepsilon\,. \tag{6.66}$$

- $WW \to X^0$: Here

$$p_1^2, p_2^2 < 0\,, \quad s > 0\,, \tag{6.67}$$

corresponding to region VI, so that $z, \bar{z}$ inherit the analytic continuation prescription eq. (6.56) from $s \to s + i\varepsilon$

$$z \to z - i\varepsilon\,, \quad \bar{z} \to \bar{z} + i\varepsilon\,.$$

Concerning u, v, the discussion is the same as for X^0-decay.

- $X^0\,W \to W$: Here

$$p_1^2, s < 0\,, \quad p_2^2 > 0\,, \tag{6.68}$$

or

$$p_2^2, s < 0\,, \quad p_1^2 > 0\,, \tag{6.69}$$

corresponding to region VII and VIII respectively, so that $z, \bar{z}$ inherit from $p_i^2 \to p_i^2 + i\varepsilon$ the prescription (6.55)

$$z \to z + i\varepsilon\,, \quad \bar{z} \to \bar{z} - i\varepsilon\,.$$

Since $s < 0$, no continuation is due on u, v.

6.5 Conclusions

In this chapter, we have computed the two-loop master integrals required for the leading QCD corrections to the interaction vertex between a massive neutral boson X^0, such as H, Z or γ^*, and pair of W bosons, mediated by a $SU(2)_L$ quark doublet composed of one massive and one massless flavor. We considered external legs with arbitrary invariant masses. The MIs were computed by means of the differential equation method, by using the Magnus exponential in order to define set of canonical master integrals. The master integrals have been expressed as Taylor series in $\epsilon = (4-d)/2$, up to order four, with coefficients written as combinations of GPLs.

In the context of the Standard Model, these results are relevant for the computation of mixed EW-QCD virtual corrections to the Higgs decay to a W pair, which we plan to address through the adaptive integrand decomposition algorithm Aida, as well as the production channels obtained by crossing, and to the triple gauge boson vertices ZWW and $\gamma^* WW$.

Chapter 7

Master integrals for QCD corrections to massive boson-pair production

In this chapter we present the computation of a class two-loop master integrals that appear in the virtual corrections to the production of a pair of massive neutral bosons mediated by a heavy-quark loop. The master integrals are evaluated through the differential equations method, by using the Magnus exponential in order to identify a set of canonical integrals. The results, which retain full dependence on the *top*-mass, are given as a Taylor series in $\epsilon = (4-d)/2$, with coefficients expressed in terms of Chen iterated integrals. In the context of LHC physics, the considered master integrals are relevant for the study of NLO QCD corrections to $gg \to HH$ and $gg \to ZZ$. The content of this chapter is the result of an original research done in collaboration with P. Mastrolia and U. Schubert.

7.1 Introduction

In this chapter, we consider another class of two-loop three-point integrals associated to the kinematics $Q \to p_1 + p_2$, with

$$Q^2 = (p_1 + p_2)^2 \equiv s \,, \qquad p_1^2 = p_2^2 = m_B^2, \tag{7.1}$$

and to the presence of a closed massive loop. The corresponding integral topologies are depicted in figure 7.1

In this case, two external momenta p_1 and p_2 share the same on-shell condition, so that these integrals depend on one kinematic variables less w.r.t. the cases considered in chapter 6. Nonetheless, as we will see in the following, the presence of a large number of massive internal propagators makes the computation of the associated MIs more involved.

The considered class of three-point integrals has a phenomenological interest, as they enter as subtopologies of the four-point Feynman diagrams for the NLO QCD corrections to $gg \to ZZ$ and $gg \to HH$ (representative Feynman diagrams for the latter process are shown in figure 7.2).

Both processes provide fundamental tests of the Standard Model. On the one hand, the production of a pair of Higgs bosons [68, 69, 196–208] would probe the Higgs self-coupling and, hence, the intimate mechanism of the electroweak symmetry breaking.

Figure 7.1: Two-loop vertex topologies for massive boson-pair production. The thin line represent a massless propagator and thick lines stand for massive ones. Dashed external lines represent legs with on-shell squared momentum equal to m_B^2 whereas the blue line indicates the external leg with off-shell momentum equal to s.

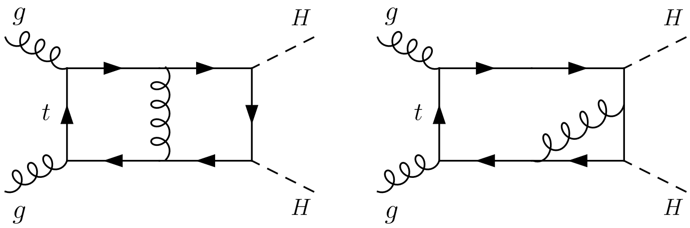


Figure 7.2: Representative two-loop Feynman diagrams for Higgs-pair production, which includes the consider massive thee-point function as subdiagrams.

On the other hand, the resonant production of a ZZ-pair [209–214] can be used to test anomalous gauge couplings, while their off-shell production [89, 215–230] plays an important role in Higgs phenomenology, both in the discrimination of signal from background and in the study of anomalous Higgs coupling. In addition, a precise theoretical prediction of ZZ-production can help in constraining the total decay width of the Higgs boson [231, 232].

As far as regards QCD virtual corrections at two-loop level for both $gg \to HH$ and $gg \to VV$, the coexistence of off-shell external legs and massive internal lines, associated to a *top*-quark circulating in the loops, pushes the available technical capability (both in the solution of IBPs and in the analytic evaluation of the MIs) beyond its limit. The available results rely either on some approximation, like an expansion in $1/m_t^2$, on the use of Effective Field Theory, or on a fully numerical approach to the evaluation of Feynman integrals. For the case ZZ production, the effect of m_t on the total cross section has been estimated at the per mille level [214], but asymptotic expansions suggest the possibility of a larger impact [233, 234], which makes the investigation of the exact m_t dependence a challenging and interesting problem.

With the present computation, we take a first step towards the analytic evaluation of two-loop integrals with exact m_t dependence, by the determining the analytic expression of the MIs associated to the topologies of figure 7.1.

7.2 System of differential equations

The integral topologies depicted in figure 7.1 belong to a single integral family,

$$\int \widetilde{\mathrm{d}^d k_1} \widetilde{\mathrm{d}^d k_2} \, \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7}} \,, \quad n_i \in \mathbb{Z} \,, \tag{7.2}$$

whose 7 denominators are defined by

$$D_1 = k_1^2 - m^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_1 - p_2)^2 - m^2, \quad D_4 = (k_2 - p_2)^2 - m^2,$$
$$D_5 = (k_1 - p_1 - p_2)^2 - m^2, \quad D_6 = (k_2 - p_1 - p_2)^2 - m^2, \quad D_7 = (k_1 - k_2)^2, \quad (7.3)$$

The two topologies correspond, respectively, to $n_{3,6} \leq 0$. The MIs are functions of three kinematic scales, s, the boson mass m_B^2 and the heavy-quark mass m^2. These three variables can be combined into the dimensionless ratios

$$x = -\frac{s}{m^2}\,, \qquad y = -\frac{m_B^2}{m^2}\,, \tag{7.4}$$

and the MIs can be evaluated by studying their DEQs in x and y. According to the solving strategy discussed in section5.5, we first use IBPs identities in order to identify an initial set of 27 MIs $\mathbf{F}$ that fulfils systems of DEQs which coefficients have a linear dependence on ϵ

$$\begin{aligned}
\partial_x \mathbf{F}(\epsilon, x, y) =& (\mathbb{A}_{0\,x}(x,y) + \epsilon \mathbb{A}_{1\,x}(x,y))\mathbf{F}(\epsilon, x, y)\,, \\
\partial_y \mathbf{F}(\epsilon, x, y) =& (\mathbb{A}_{0\,y}(x,y) + \epsilon \mathbb{A}_{1\,y}(x,y))\mathbf{F}(\epsilon, x, y)\,.
\end{aligned} \tag{7.5}$$

Subsequently, by means of the Magnus exponential matrix, we perform a change of basis to a set of MIs $\mathbf{I}$ that obey the canonical systems of DEQs,

$$\partial_x \mathbf{I}(\epsilon, x, y) = \epsilon \hat{\mathbb{A}}_x(x,y)\mathbf{I}(\epsilon, x, y)\;, \qquad \partial_y \mathbf{F}(\epsilon, x, y) = \epsilon \hat{\mathbb{A}}_y(x,y)\mathbf{I}(\epsilon, x, y)\,, \tag{7.6}$$

which can be combined in a single total differential

$$\mathbf{I} = \epsilon \mathrm{d}\mathbb{A}\mathbf{I}\,, \qquad \mathrm{d}\mathbb{A} \equiv \hat{\mathbb{A}}_x \mathrm{d}x + \hat{\mathbb{A}}_y \mathrm{d}y\,. \tag{7.7}$$

For the problem under consideration the total differential matrix $\mathrm{d}\mathbb{A}$ reads

$$\mathrm{d}\mathbb{A} = \sum_{i=1}^{18} \mathbb{M}_i \,\mathrm{dlog}(\eta_i)\,, \tag{7.8}$$

where $\mathbb{M}_i$ are constant matrices and the 18 letters η_i are

$$\begin{aligned}
\eta_1 &= x\,, & \eta_2 &= 4 + x\,, \\
\eta_3 &= \sqrt{x} + \sqrt{4+x}\,, & \eta_4 &= y\,, \\
\eta_5 &= 1 + y\,, & \eta_6 &= 4 + y\,, \\
\eta_7 &= \sqrt{y} + \sqrt{4+y}\,, & \eta_8 &= x - 4y\,, \\
\eta_9 &= x - y(4+y)\,, & \eta_{10} &= x^2 - y - xy(3+y)\,, \\
\eta_{11} &= \sqrt{x} + \sqrt{x-4y}\,, & \eta_{12} &= \sqrt{4+x} + \sqrt{x-4y}\,, \\
\eta_{13} &= \frac{2x - y(y+3) - (y+1)\lambda_y}{2x - y(y+3) + (y+1)\lambda_y}\,, & \eta_{14} &= \frac{\sqrt{x}(y+2) - \sqrt{x+4}\lambda_y}{\sqrt{x}(y+2) + \sqrt{x+4}\lambda_y}\,, \\
\eta_{15} &= \frac{-\sqrt{xy} + \sqrt{(y+4)(x-4y)}}{\sqrt{xy} + \sqrt{(y+4)(x-4y)}}\,, & \eta_{16} &= -\frac{\sqrt{x}\,(\lambda_y - 1 - y) + \sqrt{x-4y}}{\sqrt{x}\,(\lambda_y - 1 - y) - \sqrt{x-4y}}\,, \\
\eta_{17} &= \frac{\sqrt{x}\,(\lambda_y + 1 + y) - \sqrt{x-4y}}{\sqrt{x}\,(\lambda_y + 1 + y) + \sqrt{x-4y}}\,,
\end{aligned}$$

$$\eta_{18} = \frac{\left(\sqrt{x-4y}+y\right)\left(y\left(\sqrt{x-4y}+\sqrt{x+4}+4\right)+2\left(\sqrt{x+4}+2\right)\right)}{\left(\sqrt{x-4y}-y\right)\left(y\left(-\sqrt{x-4y}+\sqrt{x+4}+4\right)+2\left(\sqrt{x+4}+2\right)\right)}\,, \tag{7.9}$$

where we have introduced the abbreviations

$$\lambda_x = \sqrt{x}\sqrt{4+x}\,, \qquad \lambda_y = \sqrt{y}\sqrt{4+y}\,. \tag{7.10}$$

All letters are real positive in the region identified by

$$\begin{aligned} 0 <& x \leq 1\,, \\ 0 <& y < \frac{1}{2x}\left((1+x)\sqrt{1+4x}-1-3x\right)\,, \end{aligned} \tag{7.11}$$

The alphabet defined in eq. (7.2) is non-rational and we haven't been able to determine a change of variables capable of removing all square roots simultaneously. Therefore, provided that any left-over square root would anyway prevent us from expressing the complete solution in terms of GPLs, we decided, for the time being, to work in variables given in eq. (7.4) and write the general solution, in the positivity region defined by eq. (7.11), in terms of Chen iterated integrals.

Although the resulting representation of the MIs proved to be more than adequate for numerical evaluation, the possibility of determining a different change of variables yielding to a more efficient representation is, indeed, to be explored, in particular in view of the analytic continuation to the physical region and of the use of these results in the computation of the still unknown four-point integrals for virtual two-loop corrections to massive boson pair production.

7.3 Two-loop master integrals

In this section, we discuss the details of the solution of the system of DEQs given in eq. (7.7). The set of 27 MIs that fulfil an ϵ-linear system of DEQs is identified, through IBPs reductions, as

$$\begin{aligned}
&\mathrm{F}_1 = \epsilon^2\,\mathcal{T}_1\,, && \mathrm{F}_2 = \epsilon^2\,\mathcal{T}_2\,, && \mathrm{F}_3 = \epsilon^2\,\mathcal{T}_3\,,\\
&\mathrm{F}_4 = \epsilon^2\,\mathcal{T}_4\,, && \mathrm{F}_5 = \epsilon^2\,\mathcal{T}_5\,, && \mathrm{F}_6 = \epsilon^2\,\mathcal{T}_6\,,\\
&\mathrm{F}_7 = \epsilon^2\,\mathcal{T}_7\,, && \mathrm{F}_8 = \epsilon^2\,\mathcal{T}_8\,, && \mathrm{F}_9 = \epsilon^9\,\mathcal{T}_9\,,\\
&\mathrm{F}_{10} = \epsilon^2\,\mathcal{T}_{10}\,, && \mathrm{F}_{11} = \epsilon^2\,\mathcal{T}_{11}\,, && \mathrm{F}_{12} = \epsilon^2\,\mathcal{T}_{12}\,,\\
&\mathrm{F}_{13} = \epsilon^3\,\mathcal{T}_{13}\,, && \mathrm{F}_{14} = \epsilon^2\,\mathcal{T}_{14}\,, && \mathrm{F}_{15} = \epsilon^2\,\mathcal{T}_{15}\,,\\
&\mathrm{F}_{16} = \epsilon^3\,\mathcal{T}_{16}\,, && \mathrm{F}_{17} = \epsilon^2\,\mathcal{T}_{17}\,, && \mathrm{F}_{18} = \epsilon^2\,\mathcal{T}_{18}\,,\\
&\mathrm{F}_{19} = \epsilon^3\,\mathcal{T}_{19}\,, && \mathrm{F}_{20} = \epsilon^3\,\mathcal{T}_{20}\,, && \mathrm{F}_{21} = \epsilon^4\,\mathcal{T}_{21}\,,\\
&\mathrm{F}_{22} = \epsilon^3\,\mathcal{T}_{22}\,, && \mathrm{F}_{23} = \epsilon^3\,\mathcal{T}_{23}\,, && \mathrm{F}_{24} = \epsilon^4\,\mathcal{T}_{24}\,,\\
&\mathrm{F}_{25} = \epsilon^3\,\mathcal{T}_{21}\,, && \mathrm{F}_{26} = \epsilon^3\,\mathcal{T}_{22}\,, && \mathrm{F}_{27} = \epsilon^3\,\mathcal{T}_{23}\,,
\end{aligned} \tag{7.12}$$

where the $\mathcal{T}_i$ are depicted in figure 7.3.

With the help of the Manus exponential, we rotate this set MIs to a canonical basis, which is given by

$$
\begin{aligned}
&\mathrm{I}_1 = \mathrm{F}_1\,, && \mathrm{I}_2 = m^2\lambda_y\,\mathrm{F}_2\\
&\mathrm{I}_3 = m^2\lambda_x\,\mathrm{F}_3\,, && \mathrm{I}_4 = m^2 y\mathrm{F}_4\\
&\mathrm{I}_5 = m^2\frac{\lambda_y}{2}(\,\mathrm{F}_4 + 2\,\mathrm{F}_5)\,, && \mathrm{I}_6 = m^2 x\,\mathrm{F}_6\\
&\mathrm{I}_7 = m^2\frac{\lambda_x}{2}(\,\mathrm{F}_6 + 2\,\mathrm{F}_7)\,, && \mathrm{I}_8 = m^4\lambda_y^2\mathrm{F}_8\\
&\mathrm{I}_9 = m^2\lambda_{xy}\mathrm{F}_9\,, && \mathrm{I}_{10} = m^4\lambda_x\lambda_y\mathrm{F}_{10}\\
&\mathrm{I}_{11} = m^4\lambda_y^2\,\mathrm{F}_{11}\,, && \mathrm{I}_{12} = m^4\lambda_x^2\,\mathrm{F}_{12}\,,\\
&\mathrm{I}_{13} = m^2\lambda_{xy}\,\mathrm{F}_{13}\,, && \mathrm{I}_{14} = m^2\lambda_{xy}\,\mathrm{F}_{14}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{I}_{15} = \frac{3}{4}m^2\frac{\lambda_y}{y - x(2+y)}\,&((1+x)(y\,\mathrm{F}_4 - x\,\mathrm{F}_6) + 2x(1-x+2y)\,\mathrm{F}_{13}) +\\
+ m^4\frac{\lambda_y}{y - x(2+y)}\,&\big((x^2 - x + 2xy)\,\mathrm{F}_{14} - (x^2 - y - xy(3+y))\,\mathrm{F}_{15}\big)\,,
\end{aligned}
$$

$$
\mathrm{I}_{16} = m^2\lambda_{xy}\,\mathrm{F}_{16}\,, \qquad \mathrm{I}_{17} = m^4\lambda_{xy}\,\mathrm{F}_{17}\,,
$$

$$
\mathrm{I}_{18} = \frac{3}{2}m^2\lambda_x\,\mathrm{F}_{13} + m^4\lambda_x\,(\,\mathrm{F}_{17} + (1+y)\,\mathrm{F}_{18})\,,
$$

$$
\begin{aligned}
&\mathrm{I}_{19} = m^4\lambda_{xy}\lambda_x\,\mathrm{F}_{19}\,, && \mathrm{I}_{20} = m^4\lambda_{xy}\lambda_x\,\mathrm{F}_{20}\,,\\
&\mathrm{I}_{21} = m^2\lambda_{xy}\,\mathrm{F}_{21}\,, && \mathrm{I}_{22} = m^4\lambda_{xy}\lambda_y\,\mathrm{F}_{22}\,,
\end{aligned}
$$

$$
\mathrm{I}_{23} = \frac{m^2}{2}\,(-x\,\mathrm{F}_{23} + 2(x-2y)\,\mathrm{F}_{21} + 2y\,\mathrm{F}_{23}) + \frac{m^4}{2}xy\,\mathrm{F}_{23}\,,
$$

$$
\mathrm{I}_{24} = m^2\lambda_{xy}\,\mathrm{F}_{24}\,, \qquad \mathrm{I}_{25} = m^4\lambda_{xy}\lambda_x\,\mathrm{F}_{25}\,,
$$

$$
\mathrm{I}_{26} = -m^4 x\,((x-2y)(\,\mathrm{F}_{20} +\,\mathrm{F}_{25}) + y\,\mathrm{F}_{27}) - m^6 x\,(x - y(4+y))\,\mathrm{F}_{26}\,,
$$

$$
\mathrm{I}_{27} = m^4\lambda_{xy}\lambda_y\,\mathrm{F}_{27}\,, \tag{7.13}
$$

where we have introduced the additional abbreviation

$$
\lambda_{xy} = \sqrt{x}\sqrt{x-4y}\,. \tag{7.14}
$$

The canonical MIs satisfy a DEQ of the form given in eq. (7.7). The corresponding coefficient matrices $\mathbb{M}_i$ are collected in appendix B.2. The general solution of such system is expressed, in the positivity region defined in eq. (7.11), in terms of Chen iterated integrals. In order to completely determine the MIs, we need to specify, for each of them, a suitable boundary condition.

To this aim, we observe that all MIs of the ϵ-linear basis $\mathbf{F}$ defined in eq. (7.12) are regular when both s and m_B^2 vanish, i.e. at end-point $x = y = 0$ of the region (7.11). Moreover, all kinematic prefactors that appear in the definition of the canonical basis given in eq. (7.13) vanish in this limit, with the only exception of I_1 (which is nonetheless normalized to 1).

Therefore, the canonical MIs $\mathrm{I}_{2\cdots27}$ vanish when $x \to 0$ and $y \to 0$ and the boundary constants are trivially given by

$$
\mathrm{I}_i(\epsilon, \vec{x}_0) = \delta_{i\,1}, \qquad \vec{x}_0 = (0,0). \tag{7.15}
$$

Through a private implementation of the numerical evaluation of Chen iterated integrals, we have checked the numerical values of all MIs against SecDec.

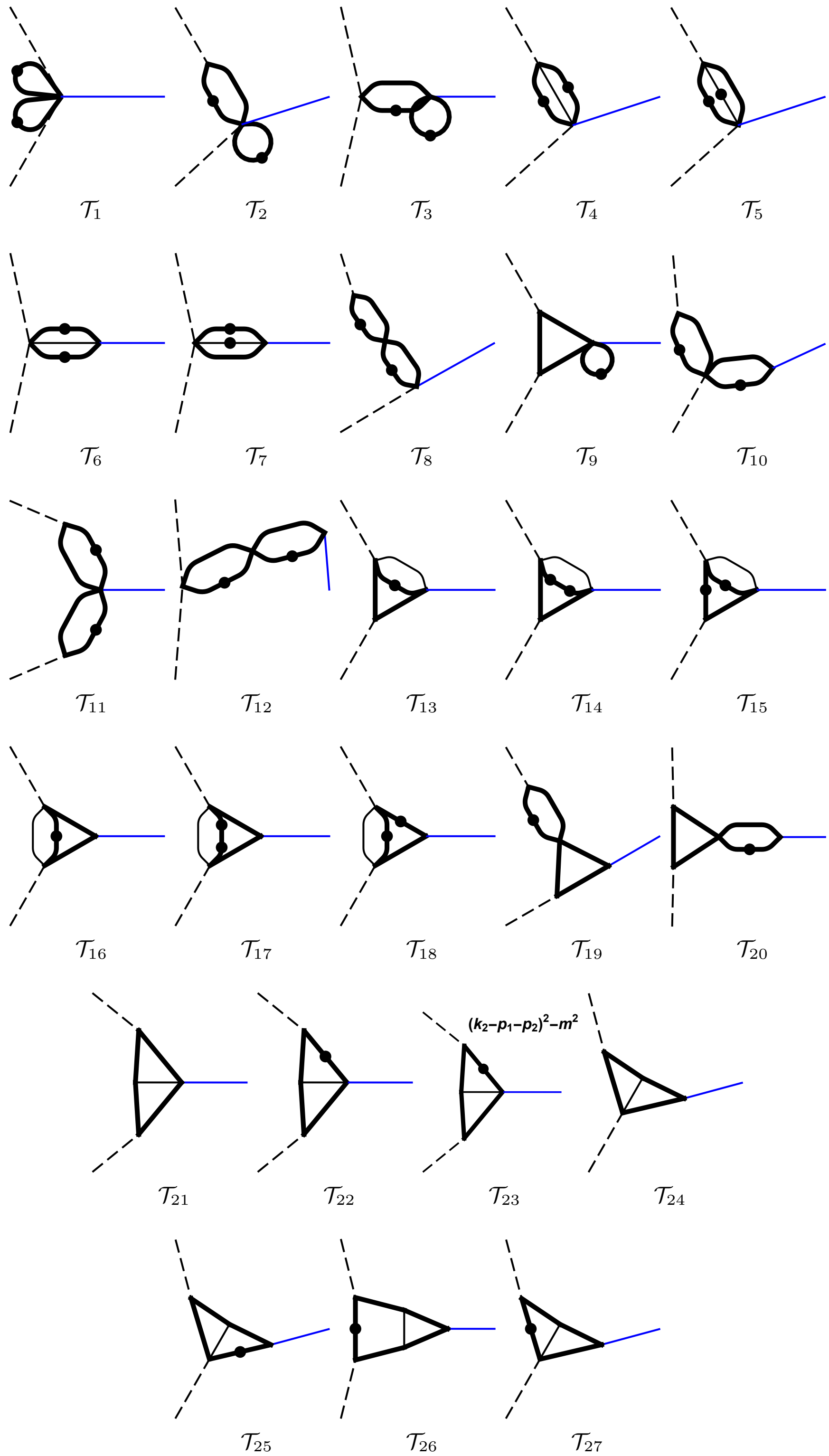


Figure 7.3: Two-loop MIs $\mathcal{T}_{1,\ldots,27}$. Graphical conventions are the same as in figure 7.1. Dots indicate squared propagators.

7.4 Conclusions

In this chapter, we have discussed the computation of a set of two-loop three-point integrals which enters as sub-diagram contributions to the QCD virtual corrections to the production of a pair of massive neutral bosons, mediated by a heavy-quark loop. The evaluation of the considered integrals is a first step towards the analytic study of the finite m_t^2 effects to the NLO corrections to the production of Higgs- and Z-boson pairs in the gluon-fusion channel, which are presently unknown. The MIs have been evaluated through the differential equations method, by using the Magnus exponential in order to identify a set of canonical integrals. The results, which retains full dependence on the top-quark mass, have been expressed as a a Taylor series in ϵ, with coefficients given in terms of Chen iterated integrals.

Chapter 8

Master integrals for NNLO corrections to μe scattering

In this chapter we evaluate the master integrals for the two-loop planar box-diagrams contributing to NNLO virtual QED corrections to muon-electron scattering, which will be part of the theoretical prediction required by the future experiment MUonE. We adopt the method of differential equations and the Magnus exponential series to determine a canonical set of integrals. The results are given as a Taylor series in $\epsilon = (4-d)/2$, with coefficients expressed in terms of generalized polylogarithms. Besides μe scattering, the considered master integrals are relevant also for crossing-related processes such as muon-pair production at e^+e^--colliders, as well as for the QCD corrections to *top*-pair production at hadron colliders. The content of this chapter is the result of an original research done in collaboration with P. Mastrolia, M. Passera and U. Schubert, and it is based on the pubblication [1].

8.1 Introduction

In this chapter, we consider the elastic scattering

$$\mu^+(p_1) + e^-(p_2) \to e^-(p_3) + \mu^+(p_4)\,. \tag{8.1}$$

which we have already discussed in section 4.4.1, in the framework of the adaptive integrand decomposition algorithm.

At LO in the Standard Model, the process receives contribution from the exchange of both a photon and a Z-boson. The corresponding differential cross section is given by

$$\frac{d\sigma_0}{dt} = -4\pi\alpha^2 \frac{\left(m^2+m_e^2\right)^2 - su + t^2/2}{t^2\lambda\left(s,m^2,m_e^2\right)}(1+\delta_Z)\,, \tag{8.2}$$

where α is the fine structure constant, $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$ is the Källen function and s, t and u are the Mandelstam invariants

$$s = (p_1+p_2)^2, \quad t = (p_2-p_3)^2, \quad u = (p_1-p_3)^2, \tag{8.3}$$

which satisfy $s+t+u = 2m^2+2m_e^2$. In eq. (8.2) δ_Z represents the contribution from the weak-boson exchange

$$\delta_Z = -\frac{G_F\, t}{4\pi\alpha\sqrt{2}}\left[(4s_\theta^2-1)^2 - \frac{(s-u)t}{2\left((m^2+m_e^2)^2 - su + t^2/2\right)}\right]\,, \tag{8.4}$$

where G_F is the Fermi constant and s_θ the sine of the weak-mixing angle. At δ_Z=0, eq. (8.2) reduces to the pure QED contribution.

Despite being one the simplest and cleanest processes in particle physics, μe scattering has so far received little attention, both on the experimental and theoretical side. The available measurements are scarse and go back mainly to the 60s, when μe collisions were studied in accelerator experiments at CERN and Brookhaven [235–238], as well as in cosmic-ray experiments [239–242]. The scattering of muons off polarized electrons was then proposed as a polarimeter for high-energy muon beams in the late 80s [243] and measured by the SMC collaboration at CERN a few years later [244]. On the theory side, the few existing studies are mainly focused on NLO QED corrections [245–251] and tests of the Standard Model [252–254].

Recently, the proposal of the new experiment MUonE, to be realized at CERN, renewed the interest in μe scattering. MUonE aims at measuring the differential cross section of the elastic scattering of high-energy ($\sim 150\,\text{GeV}$) muons on fixed-target electrons as a function of the space-like (negative) squared momentum transfer [97]. This measurement will provide the running of the effective electromagnetic coupling in the space-like region and, as a result, a new and independent determination of the leading hadronic contribution to the muon g-2 [97, 255]. In order for this new determination to be competitive with the present dispersive one, which is obtained via time-like data, the μe differential cross section must be measured with statistical and systematic uncertainties of the order of 10ppm. This high experimental precision demands an analogous accuracy in the theoretical prediction. In this respect, while Z-boson effects are estimated to be negligible, the NNLO QED corrections will have crucial role in the interpretation of the high-precision data of future experiments like MUonE. Although some of the two-loop corrections which were computed for Bhabha scattering in QED [256, 257], for the heavy-to-light quark decay [258–262] and the $t\bar{t}$ production [98–101] in QCD can be applied to elastic μe scattering as well, the full NNLO QED corrections are still unknown.

The genuine two-loop $2 \to 2$ integral topologies T_i contributing to μe scattering are represented by the diagrams depicted in figure 8.1. In this chapter we take a first step towards the calculation of the full NNLO QED corrections to μe scattering by evaluating the MIs occurring in the decomposition of the planar box-diagrams, namely all the two-loop four-point topologies for μe scattering except for the crossed double box diagram T_6. Given the small value of the electron mass m_e when compared to the muon one m, we work in the approximation $m_e = 0$.

It is important to observe that the MIs of the QED corrections to $\mu e \to \mu e$ scattering are related by crossing to the MIs of the QCD corrections to the $t\bar{t}$-pair production at hadron colliders. The analytic evaluation of the MIs for the leading-color corrections to $pp \to t\bar{t}$, due to planar diagrams only, was already considered in refs. [98–101]. They correspond to the MIs appearing in the evaluation of the Feynman graphs associated to the topologies T_i with $i \in \{1, 2, 3, 7, 8, 9, 10\}$ in figure 8.1, which we recompute independently. The MIs for the planar topology T_4 and T_5, instead, would correspond to the MIs of subleading-color contributions to $t\bar{t}$-pair production, and were not considered previously.

In the following section we describe the computation of the MIs through the DEQs method, by deserving special attention to the fixing of the boundary constants which, for some MIs, constituted the most challenging part of the computation. In some cases considered in refs. [98–101], the direct integration of the MIs in special kinematic configura-

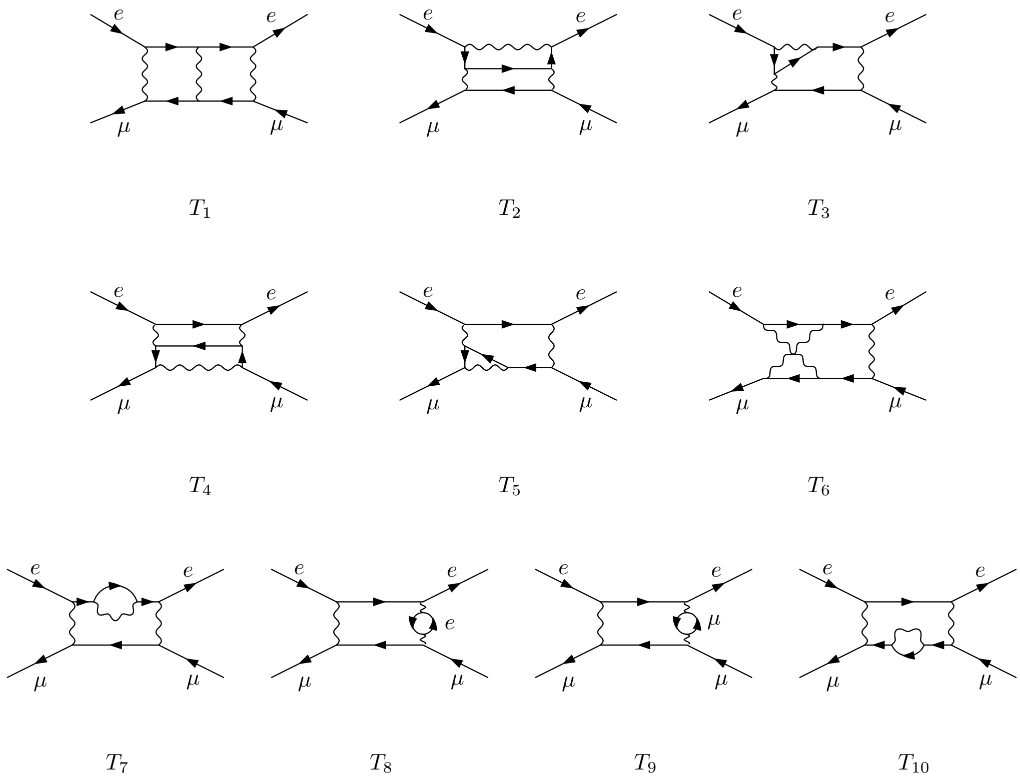


Figure 8.1: Two-loop four-point topologies for μe scattering

tions was addressed by using techniques based on Mellin-Barnes representations [70, 71]. Alternatively, here we exploit either the regularity conditions at pseudo-thresholds or the expression of the integrals at well-behaved kinematic points. The latter might be obtained by solving simpler auxiliary systems of DEQs, hence limiting the use of direct integration only to a simple set of input integrals. Our preliminary studies make us believe that the adopted strategy can be applied to the non-planar graphs as well. In particular, anticipating the computation of the non-planar topology T_6, we show its application to the determination of the MIs for the non-planar vertex graph [258–262].

8.2 System of differential equations

In this section, we summarize the properties of the systems of DEQs satisfied by the MIs that appear in the integral topologies T_i, $i \in \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$ of figure (8.1) and we describe the adopted solving strategy.

In order to compute the MIs, we first derive their DEQs in the dimensionless variables $-s/m^2$ and $-t/m^2$. Upon the change of variables

$$-\frac{s}{m^2} = x, \quad -\frac{t}{m^2} = \frac{(1-y)^2}{y}\,, \tag{8.5}$$

the coefficients of the DEQs become rational functions of x and y. By means of IBPs reduction, we identify an initial set of MIs $\mathbf{F}$ that fulfil systems of DEQs of the type

$$\frac{\partial \mathbf{F}}{\partial x} = (\mathbb{A}_{0\,x}(x,y) + \mathbb{A}_{1\,x}(x,y))\epsilon \mathbf{F}\ , \qquad \frac{\partial \mathbf{F}}{\partial y} = (\mathbb{A}_{0\,y}(x,y) + \mathbb{A}_{1\,y}(x,y))\mathbf{F}\,. \tag{8.6}$$

With the help of the Magnus exponential matrix, we transform such basis of integrals into a new set of MIs $\mathbf{I}$ which obey canonical systems of DEQs,

$$\frac{\partial \mathbf{I}}{\partial x} = \epsilon \hat{\mathbb{A}}_x(x,y)\mathbf{I} \,, \qquad \frac{\partial \mathbf{I}}{\partial y} = \epsilon \hat{\mathbb{A}}_y(x,y)\mathbf{I} \,. \tag{8.7}$$

After combining both systems of DEQs into a single total differential, we arrive at the following canonical form

$$d\mathbf{I} = \epsilon \mathrm{d}\mathbb{A}\mathbf{I} \,, \qquad \mathrm{d}\mathbb{A} \equiv \hat{\mathbb{A}}_x \mathrm{d}x + \hat{\mathbb{A}}_y \mathrm{d}y \,, \tag{8.8}$$

where the total differential matrix for the considered MIs reads as

$$\mathrm{d}\mathbb{A} = \sum_{i=1}^{9} \mathbb{M}_i \, \mathrm{dlog}(\eta_i) \,, \tag{8.9}$$

with $\mathbb{M}_i$ being constant matrices. The alphabet of the problem consists of the following 9 letters:

$$\begin{aligned} &\eta_1 = x \,, && \eta_2 = 1 + x \,, \\ &\eta_3 = 1 - x \,, && \eta_4 = y \,, \\ &\eta_5 = 1 + y \,, && \eta_6 = 1 - y, \,, \\ &\eta_7 = x + y \,, && \eta_8 = 1 + x\, y, \\ &\eta_9 = 1 - y\,(1 - x - y) \,. \end{aligned} \tag{8.10}$$

Since the alphabet is rational and has only algebraic roots, the general solution of eq. (8.8) can be directly expressed in terms of GPLs. The expression of all MIs is derived in the kinematic region where all letters are real and positive,

$$x > 0 \,, \quad 0 < y < 1 \,, \tag{8.11}$$

which corresponds to the Euclidean region $s < 0$, $t < 0$. The analytic continuation of the MIs to the physical region can be obtained through by-now standard techniques.

Constant GPLs

The boundary constants of most of the considered MIs have been determined by taking special kinematics limits on the general solution of the DEQs written in terms of GPLs. Through this procedure, the boundary constants are expressed as combinations of constant GPLs of argument 1, with weights drawn from six different sets:

- $\{-1, 0, 1, 3, -(-1)^{\frac{1}{3}}, (-1)^{\frac{2}{3}}\}$,
- $\{-\frac{1}{2}, -\frac{2}{7}, 0, \frac{1}{7}, \frac{1}{2}, \frac{4}{7}, 1, -\frac{1}{2}(-1)^{\frac{1}{3}}, \frac{1}{2}(-1)^{\frac{2}{3}}\}$,
- $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2, 3, 4, \frac{1}{2}(-1)^{\frac{1}{3}}, -\frac{1}{2}(-1)^{\frac{2}{3}}\}$,
- $\{-1, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, 1, \frac{7}{4}, \frac{1}{2}(-1)^{\frac{1}{3}}, -\frac{1}{2}(-1)^{\frac{2}{3}}\}$,
- $\{-2, -\frac{1}{2}, 0, \frac{1}{2}, 1, 4, 7, -\frac{1}{2}(-1)^{\frac{1}{3}}, \frac{1}{2}(-1)^{\frac{2}{3}}\}$,
- $\{-2, -1, -\frac{1}{2}, 0, \frac{1}{4}, 1, \frac{7}{4}, 2, -2(-1)^{\frac{1}{3}}, 2(-1)^{\frac{2}{3}}\}$.

Each set arises from a different kinematic limit imposed on the alphabet given in eq. (8.10). We used GiNaC to numerically verify that, at each order in ϵ^k, the corresponding combination of constant GPLs is proportional to Riemann ζ_k. Examples of the found identities are

$$\zeta_2 = -\frac{1}{2}G(-1;1)^2 + G(0,-2;1) + G(0,-\frac{1}{2};1)\,, \tag{8.12}$$

$$\begin{aligned}-59\zeta_4 = {} & \pi^2\left(G(-1;1)^2 - 2\,G(0,-(-1)^{\frac{1}{3}};1) - 2\,G(0,(-1)^{\frac{2}{3}};1)\right) - 21\,\zeta_3\,G(-1;1) \\ & -G(-1;1)^4 - 18\,G(0,0,0,-(-1)^{\frac{1}{3}};1) - 18\,G(0,0,0,(-1)^{\frac{2}{3}};1) \\ & +12\,G(0,0,-(-1)^{\frac{1}{3}},-1;1) + 12\,G(0,0,(-1)^{\frac{2}{3}},-1;1) \\ & +12\,G(0,-(-1)^{\frac{1}{3}},-1,-1;1) + 12\,G(0,(-1)^{\frac{2}{3}},-1,-1;1) + 24\,G(0,0,0,2;1)\,.\end{aligned} \tag{8.13}$$

For related studies we refer the reader to [263–265].

8.3 One-loop master integrals

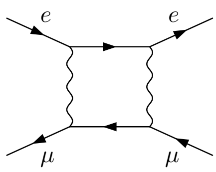


Figure 8.2: One-loop four-point topology for μe scattering

Before entering the details of the computation of the two-loop MIs, we briefly discuss the computation of the MIs for the relevant one-loop four-point topology,

$$\int \widetilde{\mathrm{d}^d k_1}\,\frac{1}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}}\,, \qquad n_i \geq 0\,, \tag{8.14}$$

where the loop denominators are defined to be

$$\begin{aligned} D_1 = k_1^2 - m^2\,, \quad & D_2 = (k_1+p_1)^2\,, \\ D_3 = (k_1+p_1+p_2)^2\,, \quad & D_4 = (k_1+p_4)^2\,. \end{aligned} \tag{8.15}$$

The corresponding Feynman diagram is depicted in figure 8.2. From the IBPs reduction, we determine a set of 5 MIs which satisfy ϵ-linear DEQs,

$$\mathrm{F}_1 = \epsilon\,\mathcal{T}_1\,, \qquad \mathrm{F}_2 = \epsilon\,\mathcal{T}_2\,, \qquad \mathrm{F}_3 = \epsilon\,\mathcal{T}_3\,, \qquad \mathrm{F}_4 = \epsilon^2\,\mathcal{T}_4\,, \qquad \mathrm{F}_5 = \epsilon^2\,\mathcal{T}_5\,, \tag{8.16}$$

where the $\mathcal{T}_i$ are depicted in figure 8.3. With the help the Magnus algorithm, we identify the corresponding canonical basis, which is given by

$$\begin{aligned} \mathrm{I}_1 &= \mathrm{F}_1\,, & \mathrm{I}_2 &= -s\,\mathrm{F}_2\,, \\ \mathrm{I}_3 &= -t\mathrm{F}_3\,, & \mathrm{I}_4 &= \lambda_t\,\mathrm{F}_4\,, \\ \mathrm{I}_5 &= (s-m^2)t\,\mathrm{F}_5\,. \end{aligned} \tag{8.17}$$

with $\lambda_t = \sqrt{-t}\sqrt{4m^2-t}$.

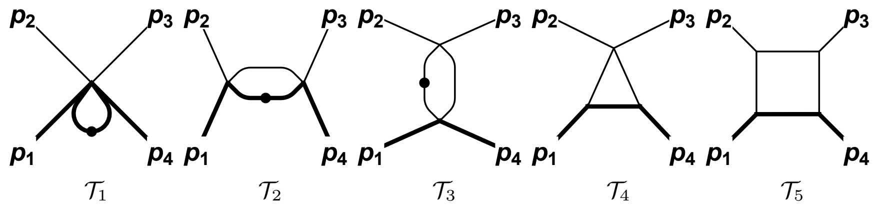


Figure 8.3: One-loop MIs $\mathcal{T}_{1,\ldots,5}$.

This set of MIs satisfies canonical DEQs of the form given in eq. (8.8), whose coefficient matrices read (in this case $\mathbb{M}_3$ and $\mathbb{M}_9$ vanish),

$$\mathbb{M}_1 = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \quad \mathbb{M}_2 = \begin{pmatrix} 0&0&0&0&0\\ -1&-2&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 2&4&0&0&-2 \end{pmatrix}, \quad \mathbb{M}_4 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 1&0&-1&0&0\\ 1&2&0&0&0 \end{pmatrix},$$
$$\mathbb{M}_5 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&2&0\\ 0&0&0&0&0 \end{pmatrix}, \quad \mathbb{M}_6 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&-2&0&0\\ 0&0&0&-2&0\\ 0&0&2&0&-2 \end{pmatrix}, \quad \mathbb{M}_7 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -1&-2&-1&-1&1 \end{pmatrix},$$
$$\mathbb{M}_8 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -1&-2&-1&1&1 \end{pmatrix}. \tag{8.18}$$

The integration of the DEQs in terms of GPLs as well as the fixing of boundary constants is straightforward. $\mathrm{I}_{1,3}$ are input integrals, obtained by direct integration,

$$\mathrm{I}_1(\epsilon) = 1\,, \qquad \mathrm{I}_3(\epsilon, y) = \left(\frac{(1-y)^2}{y}\right)^{-\epsilon} \left(1 - \zeta_2\epsilon^2 - 2\zeta_3\epsilon^3 + \mathcal{O}\left(\epsilon^4\right)\right)\,, \tag{8.19}$$

whereas the boundary constants for I_2, I_4 and I_5 can be fixed by demanding regularity, respectively, at pseudothresholds $s \to 0$, $t \to 4m^2$, and $s = -t \to m^2/2$. The final expressions of these MIs are,

$$\mathrm{I}_i(\epsilon, x, y) = \sum_{k=0}^{2} \mathrm{I}_i^{(k)}(x, y)\epsilon^k + \mathcal{O}(\epsilon^3)\,, \tag{8.20}$$

with

$$\begin{aligned} \mathrm{I}_2^{(0)}(x) =& 0\,,\\ \mathrm{I}_2^{(1)}(x) =& -G(-1;x)\,,\\ \mathrm{I}_2^{(2)}(x) =& 2G(-1,-1;x) - G(0,-1;x)\,, \end{aligned} \tag{8.21}$$

$$\begin{aligned} \mathrm{I}_4^{(0)}(y) =& 0\,,\\ \mathrm{I}_4^{(1)}(y) =& 0\,,\\ \mathrm{I}_4^{(2)}(y) =& -4\zeta_2 - G(0,0;y) + 2G(0,1;y)\,, \end{aligned} \tag{8.22}$$

$$\begin{aligned} \mathrm{I}_5^{(0)}(x,y) =&\ \ 2\,,\\ \mathrm{I}_5^{(1)}(x,y) =& -2G(-1;x) + G(0;y) - 2G(1;y)\,, \end{aligned}$$

$$\mathrm{I}_5^{(2)}(x,y) = -\,5\zeta_2 + 2G(-1;x)\,(2G(1;y) - G(0;y))\;. \tag{8.23}$$

These MIs and their related crossings, which can be obtained from the above analytic expressions through a suitable permutations of the Mandelstam invariants s, t, and u can be used in the decomposition of the one-loop amplitude derived in eq. (4.83) with the integrand reduction method, in order to obtain the analytic expression of the (unrenormalized) one-loop virtual QED correction to μe scattering. For completeness, we observe that the knowledge of the full analytic expression of eq. (4.83) requires the evaluation of additional two- and three-point integrals with two massive internal propagators, which are not included in the integral family defined by eq. (8.14). Although we do not evaluate the missing integrals explicitly, their expression can be easily found in the literature, or it can be anyway obtained with the same technique discussed above.

8.4 Two-loop master integrals

In this section, we discuss the details of the evaluation of the planar the two-loop MIs contributing to the NNLO corrections to μe scattering. The 9 topologies corresponding to the planar diagrams of figure 8.1 can be mapped into two distinct integral families, which group, respectively, T_1, T_2, T_3, T_7, T_8 and T_4, T_5, T_9, T_{10}. We describe the computation of the MIs for each integral family separately.

8.4.1 The first integral family

The first two-loop integral family, which includes the topologies T_1, T_2, T_3, T_7 and T_8 of figure 8.1, is defined as

$$\int \widetilde{\mathrm{d}^d k_1}\widetilde{\mathrm{d}^d k_2}\,\frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8} D_9^{n_9}}\,, \quad n_i \in \mathbb{Z}\,, \tag{8.24}$$

where the 9 denominators are chosen to be

$$\begin{gathered} D_1 = k_1^2 - m^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_1+p_1)^2, \quad D_4 = (k_2+p_1)^2, \\ D_5 = (k_1+p_1+p_2)^2, \quad D_6 = (k_2+p_1+p_2)^2, \quad D_7 = (k_1-k_2)^2, \\ D_8 = (k_1+p_4)^2, \quad D_9 = (k_2+p_4)^2 \quad . \end{gathered} \tag{8.25}$$

Each of the topologies T_i corresponds to a particular choice of the ISPs, i.e. of the set of negative exponents n_k. The IBPs reduction returns 34 MIs and it is possible to determine a basis which fulfils an ϵ-linear system of DEQs,

$$\begin{aligned}
&\mathrm{F}_1 = \epsilon^2\,\mathcal{T}_1\,, && \mathrm{F}_2 = \epsilon^2\,\mathcal{T}_2\,, && \mathrm{F}_3 = \epsilon^2\,\mathcal{T}_3\,,\\
&\mathrm{F}_4 = \epsilon^2\,\mathcal{T}_4\,, && \mathrm{F}_5 = \epsilon^2\,\mathcal{T}_5\,, && \mathrm{F}_6 = \epsilon^2\,\mathcal{T}_6\,,\\
&\mathrm{F}_7 = \epsilon^2\,\mathcal{T}_7\,, && \mathrm{F}_8 = \epsilon^2\,\mathcal{T}_8\,, && \mathrm{F}_9 = \epsilon^2\,\mathcal{T}_9\,,\\
&\mathrm{F}_{10} = \epsilon^3\,\mathcal{T}_{10}\,, && \mathrm{F}_{11} = \epsilon^3\,\mathcal{T}_{11}\,, && \mathrm{F}_{12} = \epsilon^3\,\mathcal{T}_{12}\,,\\
&\mathrm{F}_{13} = \epsilon^3\,\mathcal{T}_{13}\,, && \mathrm{F}_{14} = \epsilon^2\,\mathcal{T}_{14}\,, && \mathrm{F}_{15} = \epsilon^2\,\mathcal{T}_{15}\,,\\
&\mathrm{F}_{16} = \epsilon^3\,\mathcal{T}_{16}\,, && \mathrm{F}_{17} = \epsilon^4\,\mathcal{T}_{17}\,, && \mathrm{F}_{18} = \epsilon^3\,\mathcal{T}_{18}\,,\\
&\mathrm{F}_{19} = \epsilon^4\,\mathcal{T}_{19}\,, && \mathrm{F}_{20} = \epsilon^2(1+2\epsilon)\,\mathcal{T}_{20}\,, && \mathrm{F}_{21} = \epsilon^4\,\mathcal{T}_{21}\,,\\
&\mathrm{F}_{22} = \epsilon^3\,\mathcal{T}_{22}\,, && \mathrm{F}_{23} = \epsilon^3\,\mathcal{T}_{23}\,, && \mathrm{F}_{24} = \epsilon^2\,\mathcal{T}_{24}\,,\\
&\mathrm{F}_{25} = \epsilon^3\,\mathcal{T}_{25}\,, && \mathrm{F}_{26} = \epsilon^3(1-2\epsilon)\,\mathcal{T}_{26}\,, && \mathrm{F}_{27} = \epsilon^3\,\mathcal{T}_{27}\,,
\end{aligned}$$

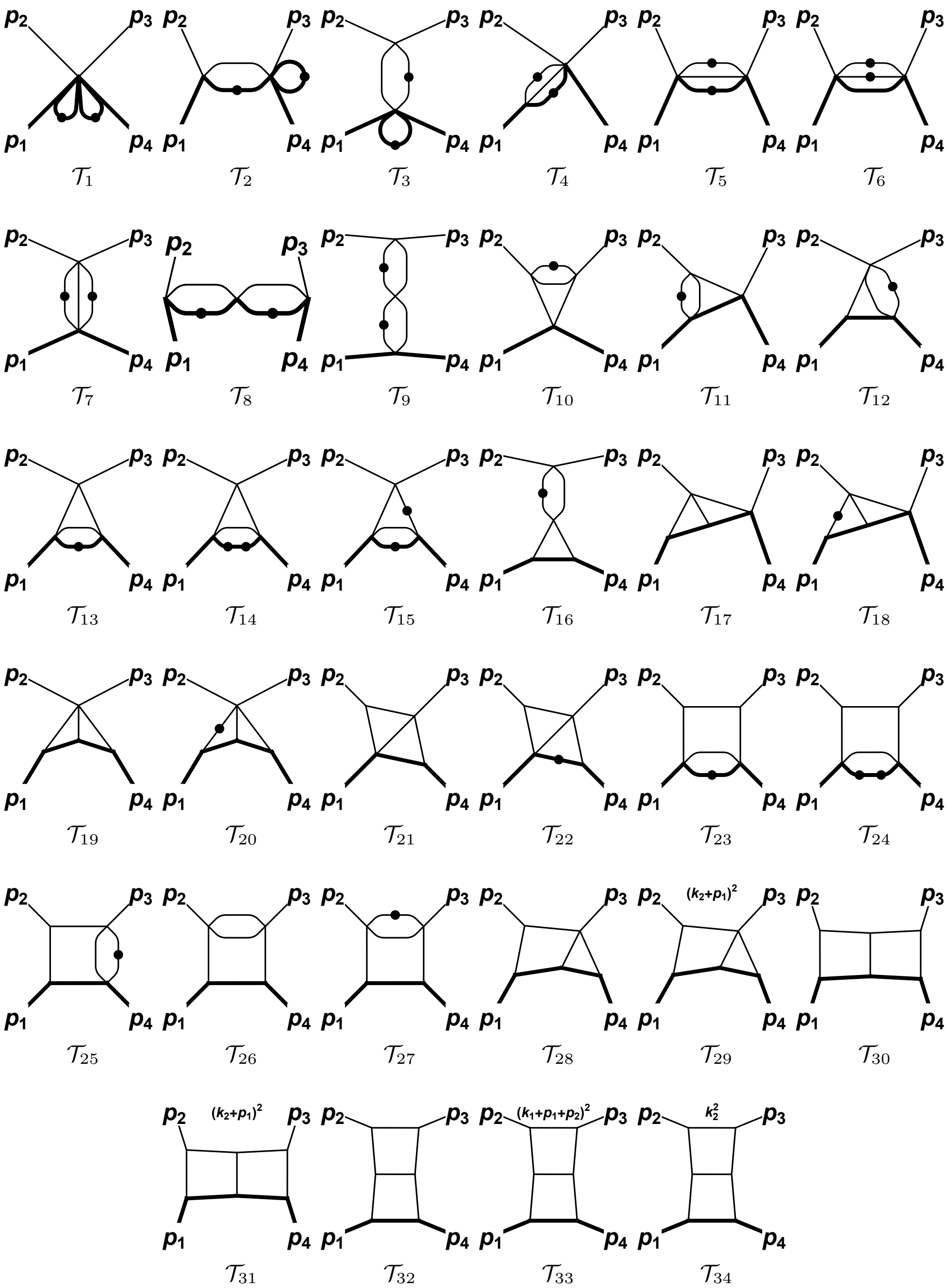


Figure 8.4: Two-loop MIs $\mathcal{T}_{1,\ldots,34}$ for the first integral family.

$$
\begin{aligned}
&\mathrm{F}_{28} = \epsilon^4\,\mathcal{T}_{28}\,, && \mathrm{F}_{29} = \epsilon^3(1-2\epsilon)\,\mathcal{T}_{29}\,, && \mathrm{F}_{30} = \epsilon^4\,\mathcal{T}_{30}\,,\\
&\mathrm{F}_{31} = \epsilon^4\,\mathcal{T}_{31}\,, && \mathrm{F}_{32} = \epsilon^4\,\mathcal{T}_{32}\,, && \mathrm{F}_{33} = \epsilon^4\,\mathcal{T}_{33}\,,\\
&\mathrm{F}_{34} = \epsilon^4\,\mathcal{T}_{34}\,,
\end{aligned}
\tag{8.26}
$$

where the $\mathcal{T}_i$ are depicted in figure 8.4.

Through the Magnus exponential, we rotate this set of integrals to the canonical basis

$$
\begin{aligned}
&\mathrm{I}_1 = \mathrm{F}_1\,, && \mathrm{I}_2 = -s\,\mathrm{F}_2\,,\\
&\mathrm{I}_3 = -t\mathrm{F}_3\,, && \mathrm{I}_4 = m^2\,\mathrm{F}_4\,,\\
&\mathrm{I}_5 = -s\mathrm{F}_5\,, && \mathrm{I}_6 = 2m^2\,\mathrm{F}_5 + (m^2-s)\,\mathrm{F}_6\,,\\
&\mathrm{I}_7 = -t\mathrm{F}_7\,, && \mathrm{I}_8 = s^2\,\mathrm{F}_8\,,\\
&\mathrm{I}_9 = t^2\mathrm{F}_9\,, && \mathrm{I}_{10} = -t\,\mathrm{F}_{10}\,,\\
&\mathrm{I}_{11} = (m^2-s)\,\mathrm{F}_{11}\,, && \mathrm{I}_{12} = \lambda_t\,\mathrm{F}_{12}\,,\\
&\mathrm{I}_{13} = \lambda_t\,\mathrm{F}_{13}\,, && \mathrm{I}_{14} = \lambda_t\,m^2\,\mathrm{F}_{14}\,,\\
&\mathrm{I}_{15} = (t-\lambda_t)\left(\frac{3}{2}\mathrm{F}_{13} + m^2\mathrm{F}_{14}\right) - m^2\,t\,\mathrm{F}_{15}\,, && \mathrm{I}_{16} = -t\,\lambda_t\,\mathrm{F}_{16}\,,\\
&\mathrm{I}_{17} = (m^2-s)\,\mathrm{F}_{17}\,, && \mathrm{I}_{18} = m^2(m^2-s)\,\mathrm{F}_{18}\,,\\
&\mathrm{I}_{19} = \lambda_t\,\mathrm{F}_{19}\,, && \mathrm{I}_{20} = \frac{\lambda_t - t}{2}\,(\mathrm{F}_{12} - 4\,\mathrm{F}_{19}) +\\
& && \quad - m^2 t\,\mathrm{F}_{20}\,,\\
&\mathrm{I}_{21} = (m^2-s-t)\,\mathrm{F}_{21}\,, && \mathrm{I}_{22} = -m^2\,t\,\mathrm{F}_{22}\,,\\
&\mathrm{I}_{23} = s\,t\,\mathrm{F}_{23}\,, && \mathrm{I}_{24} = -m^2\,t\,\mathrm{F}_{23} +\\
& && \quad + (s-m^2)\,m^2\,t\,\mathrm{F}_{24}\,,\\
&\mathrm{I}_{25} = -(m^2-s)\,t\,\mathrm{F}_{25}\,, && \mathrm{I}_{26} = \lambda_t\,\mathrm{F}_{26}\,,\\
&\mathrm{I}_{27} = -(m^2-s)\,t\,\mathrm{F}_{27}\,, && \mathrm{I}_{28} = (m^2-s)\,\lambda_t\,\mathrm{F}_{28}\,,\\
&\mathrm{I}_{29} = -2t\,\mathrm{F}_{21} - (m^2-s)(2(\lambda_t - t)\mathrm{F}_{28} - \mathrm{F}_{29})\,, && \mathrm{I}_{30} = -(m^2-s)^2 t\,\mathrm{F}_{30}\,,\\
&\mathrm{I}_{31} = (m^2-s)^2\,\mathrm{F}_{31}\,, && \mathrm{I}_{32} = (m^2-s)\,t^2\,\mathrm{F}_{32}\,,\\
&\mathrm{I}_{33} = -\lambda_t\,t\,\mathrm{F}_{33}\,, && \mathrm{I}_{34} = -m^2\,t^2\,\mathrm{F}_{32} + t^2\,\mathrm{F}_{34}\,.
\end{aligned}
\tag{8.27}
$$

This set of MIs $\mathbf{I}$ satisfies a system of DEQs of the form given in eq.(8.8). The corresponding 34×34 coefficient matrices are collected in appendix B.3. The general solution of such the system of DEQs is written in terms of GPLs and the boundary constants are fixed either by using as an input the values of the MIs at some special kinematic point or by demanding the absence of unphysical thresholds that appear in the alphabet defined in eq. (8.10).

More specifically, the boundary constants are determined as follows:

- The boundary values of $\mathrm{I}_{1,3,4,7,9}$ are obtained by direct integration,

$$
\mathrm{I}_1(\epsilon) = 1, \tag{8.28}
$$

$$
\mathrm{I}_3(\epsilon) = \left(\frac{(1-y)^2}{y}\right)^{-\epsilon}\left(1 - \zeta_2\epsilon^2 - 2\zeta_3\epsilon^3 - \frac{9}{4}\zeta_4\epsilon^4 + \mathcal{O}(\epsilon^5)\right), \tag{8.29}
$$

$$
\mathrm{I}_4(\epsilon) = -\frac{1}{4} - \zeta_2\epsilon^2 - 2\zeta_3\epsilon^3 - 16\zeta_4\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)\,, \tag{8.30}
$$

$$\mathrm{I}_7(\epsilon, y) = \left(\frac{(1-y)^2}{y}\right)^{-2\epsilon} \left(-1 + 2\zeta_2\epsilon^2 + 10\zeta_3\epsilon^3 + 11\zeta_4\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)\right) , \tag{8.31}$$

$$\mathrm{I}_9(\epsilon) = \left(\frac{(1-y)^2}{y}\right)^{-2\epsilon} \left(1 - 2\,\zeta_2\,\epsilon^2 - 4\,\zeta_3\,\epsilon^3 - 2\,\zeta_4\,\epsilon^4 + \mathcal{O}(\epsilon^5)\right) , \tag{8.32}$$

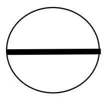
$$\mathrm{I}_{10}(\epsilon) = \left(\frac{(1-y)^2}{y}\right)^{-2\epsilon} \left(\frac{1}{4} - 2\,\zeta_3\,\epsilon^3 - 3\,\zeta_4\,\epsilon^4 + \mathcal{O}(\epsilon^5)\right) . \tag{8.33}$$

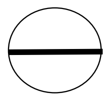
- The boundary constants of $\mathrm{I}_{2,8,11,23}$ are fixed by demanding finiteness in the limit $s \to 0$.

- In the regular limit $s \to 0$, I_5 and I_6 become, respectively,

$$\begin{aligned} \mathrm{I}_5(\epsilon, 0) &= 0\,, \\ \mathrm{I}_6(\epsilon, 0) &= \epsilon^2 m^2 \left(2\mathrm{F}_5(\epsilon, 0) + \mathrm{F}_6(\epsilon, 0)\right) . \end{aligned} \tag{8.34}$$

$\mathrm{F}_5(\epsilon, 0)$ and $\mathrm{F}_6(\epsilon, 0)$ correspond to two-loop vacuum diagrams which can be reduced via IBPs to a single integral that can be analytically computed,

$$\mathrm{F}_5(\epsilon, 0) = \frac{2\epsilon(2\epsilon - 1)}{m^4}$$  ,

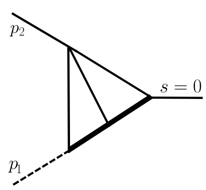
$$\mathrm{F}_6(\epsilon, 0) = -\frac{2(\epsilon + 1)(2\epsilon - 1)}{m^4}$$  . (8.35)

In this way, we obtain the boundary values

$$\mathrm{I}_5(\epsilon, 0) = 0\,, \quad \mathrm{I}_6(\epsilon, 0) = -1 - 2\zeta_2\epsilon^2 + 2\zeta_3\epsilon^3 - 9\zeta_4\epsilon^4 + \mathcal{O}\left(\epsilon^5\right) . \tag{8.36}$$

- The integration constants of $\mathrm{I}_{12\ldots16,19,20,26,27,32,33}$ are fixed by demanding finiteness in the $t \to 4m^2$ limit and by demanding that the resulting boundary constants are real.

- The integrals I_{17} and I_{18} are regular in the $s \to 0$ limit. By imposing the regularity on their DEQs we can only fix the constant of one of them, say I_{18}. The boundary constants of I_{17} must be then computed in an independent way. We observe that the value of $\mathrm{I}_{17}(\epsilon, 0)$ can be obtained in the limit $p_1^2 \to m^2$ of a similar vertex integral with off-shell momentum p_1^2 and $s \equiv (p_1 + p_2)^2 = p_2^2 = 0$,

$$\mathrm{I}_{17}(\epsilon, 0) = \epsilon^4 m^2 \lim_{p_1^2 \to m^2}$$ 

. (8.37)

As we discuss at the end of this section, the limit appearing in the r.h.s of eq. (8.37) is smooth and gives,

$$\mathrm{I}_{17}(\epsilon, 0) = -\frac{27}{4}\zeta_4\epsilon^4 + \mathcal{O}\left(\epsilon^5\right) . \tag{8.38}$$

- The regularity of the four-point integrals $\mathrm{I}_{21,22,25,28\ldots,31}$ in either $s \to 0$ or $t \to 4m^2$ provides two boundary conditions, which can be complemented with additional relations obtained by imposing the regularity of the integrals at $s = -t = m^2/2$.
- The boundary constants of integral I_{24} are determined by demanding regularity in the limit $s \to -m^2$ and $t \to 4m^2$.
- The boundary constants of I_{34} are found by demanding finiteness in the limit $u \to \infty$.

All results have been numerically checked with the help of the computer codes GiNaC and SecDec.

Auxiliary vertex integral for eq. (8.37)

We conclude this section by showing how the boundary value of I_{17} at $s = 0$ can be extracted from the solution of the system of DEQs for the auxiliary vertex integrals

$$\int \widetilde{\mathrm{d}^d k_1} \widetilde{\mathrm{d}^d k_2} \, \frac{D_6^{n_6} D_7^{n_7}}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} \,, \quad n_i \geq 0 \,, \tag{8.39}$$

identified by the set of denominators

$$D_1 = k_1^2 - m^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_1 + p_1)^2, \quad D_4 = (k_2 + p_1 + p_2)^2,$$
$$D_5 = (k_1 - k_2)^2, \quad D_6 = (k_2 + p_1)^2, \quad D_7 = (k_1 + p_1 + p_2)^2, \tag{8.40}$$

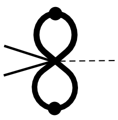
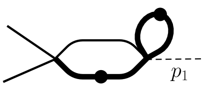
and by external momenta p_1, p_2 and p_3 satisfying

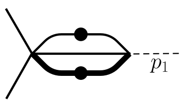
$$p_2^2 = 0 \,, \quad p_3^2 = (p_1 + p_2)^2 = 0 \,. \tag{8.41}$$

All integrals belonging to this family can be reduced to a set of 8 MIs, whose dependence on p_1^2 is parametrized in terms of the dimensionless variable

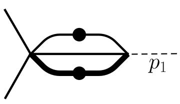
$$x = -\frac{p_1^2}{m^2} \,. \tag{8.42}$$

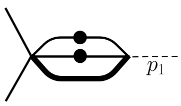
The basis of integrals

$\mathrm{I}_1 = \epsilon^2$  , $\quad \mathrm{I}_2 = -\epsilon^2 p_1^2$ 

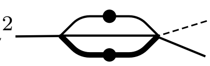
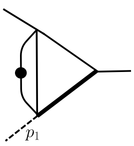
, $\quad \mathrm{I}_3 = -\epsilon^2 p_1^2$ 

,

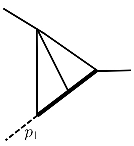
$\mathrm{I}_4 = \epsilon^2 \, 2m^2$ 

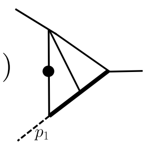
$+ \; \epsilon^2 (m^2 - p_1^2)$ 

,

$\mathrm{I}_5 = \epsilon(1-\epsilon) m^2$  , $\quad \mathrm{I}_6 = -\epsilon^3 p_1^2$ 

,

$\mathrm{I}_7 = -\epsilon^4 \, p_1^2$ 

, $\quad \mathrm{I}_8 = \epsilon^3 \, p_1^2 (p_1^2 - m^2)$ 

(8.43)

fulfils a canonical system of DEQs,

$$d\mathbf{I} = \epsilon \, \mathrm{d}\mathbb{A} \, \mathbf{I} \,, \tag{8.44}$$

where

$$\mathrm{d}\mathbb{A} = \mathbb{M}_1 \, \mathrm{dlog}\, x + \mathbb{M}_2 \, \mathrm{dlog}(1+x) \,, \tag{8.45}$$

with

$$\mathbb{M}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & -\frac{1}{2} & -1 & 0 & 0 & \frac{1}{2} & -1 & \frac{5}{2} \end{pmatrix}, \quad \mathbb{M}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -\frac{1}{2} & -3 & 0 & -4 \end{pmatrix}. \tag{8.46}$$

In the Euclidean region $x > 0$, the general solution of this system can be expressed in terms of harmonic polylogarithms (HPLs) [83] and the boundary constants of all MIs, with the only exception of $\mathrm{I}_1 = 1$ and

$$\mathrm{I}_5(\epsilon) = 1 + 2\zeta_2\epsilon^2 - 2\zeta_3\epsilon^3 + 9\zeta_4\epsilon^4 + \mathcal{O}\left(\epsilon^5\right) \,, \tag{8.47}$$

can be fixed by demanding their regularity at $x \to 0$. In particular, for the $\mathrm{I}_7(\epsilon, x)$ we obtain

$$\mathrm{I}_7(\epsilon, x) = \left(-2\zeta_2 H(0,-1;x) - H(0,-1,-1,-1;x) + H(0,-1,0,-1;x)\right)\epsilon^4 + \mathcal{O}(\epsilon^5). \tag{8.48}$$

This expression, when it is analytically continued to the region $x < 0$, has a smooth limit for $x \to -1$ ($p_1^2 = m^2$),

$$\mathrm{I}_7(\epsilon, -1) = \frac{27}{4}\zeta_4\epsilon^4 + \mathcal{O}(\epsilon^5) \,, \tag{8.49}$$

which has been used in eq. (8.38) (the overall minus sign is due to the different kinematic factors relating the canonical MIs I_i to the corresponding F_i, as defined by eq. (8.27) and eq. (8.43)).

8.4.2 The second integral family

The second two-loop integral family, which includes the topologies T_4, T_5, T_9 and T_{10} of figure 8.1, is defined as

$$\int \widetilde{\mathrm{d}^d k_1} \widetilde{\mathrm{d}^d k_2} \, \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7} D_8^{n_8} D_9^{n_9}} \,, \quad n_i \in \mathbb{Z} \,, \tag{8.50}$$

where the 9 denominators are

$$\begin{aligned} D_1 &= k_1^2 - m^2, \quad D_2 = k_2^2, \quad D_3 = (k_2+p_2)^2, \quad D_4 = (k_1+p_2)^2, \\ D_5 &= (k_2+p_2-p_3)^2, \quad D_6 = (k_1+p_2-p_3)^2 - m^2, \quad D_7 = (k_1-p_1)^2, \\ D_8 &= (k_2-p_1)^2 - m^2, \quad D_9 = (k_1-k_2)^2 - m^2 \,. \end{aligned} \tag{8.51}$$

As in the previous case, different topologies T_i corresponds to different choices of the set of negative exponents n_k. From IBPs reduction, we select a set of 42 MIs which satisfy an ϵ-linear system of DEQs:

$$\mathrm{F}_1 = \epsilon^2\, \mathcal{T}_1 \,, \qquad \mathrm{F}_2 = \epsilon^2\, \mathcal{T}_2 \,, \qquad \mathrm{F}_3 = \epsilon^2\, \mathcal{T}_3 \,,$$

$$
\begin{aligned}
&\mathrm{F}_4 = \epsilon^2\,\mathcal{T}_4\,, && \mathrm{F}_5 = \epsilon^2\,\mathcal{T}_5\,, && \mathrm{F}_6 = \epsilon^2\,\mathcal{T}_6\,,\\
&\mathrm{F}_7 = \epsilon^2\,\mathcal{T}_7\,, && \mathrm{F}_8 = \epsilon^2\,\mathcal{T}_8\,, && \mathrm{F}_9 = \epsilon^2\,\mathcal{T}_9\,,\\
&\mathrm{F}_{10} = \epsilon^2\,\mathcal{T}_{10}\,, && \mathrm{F}_{11} = \epsilon^2\,\mathcal{T}_{11}\,, && \mathrm{F}_{12} = \epsilon^3\,\mathcal{T}_{12}\,,\\
&\mathrm{F}_{13} = \epsilon^2\,\mathcal{T}_{13}\,, && \mathrm{F}_{14} = \epsilon^2\,\mathcal{T}_{14}\,, && \mathrm{F}_{15} = \epsilon^3\,\mathcal{T}_{15}\,,\\
&\mathrm{F}_{16} = \epsilon^2\,\mathcal{T}_{16}\,, && \mathrm{F}_{17} = \epsilon^2\,\mathcal{T}_{17}\,, && \mathrm{F}_{18} = \epsilon^3\,\mathcal{T}_{18}\,,\\
&\mathrm{F}_{19} = \epsilon^3\,\mathcal{T}_{19}\,, && \mathrm{F}_{20} = \epsilon^2\,\mathcal{T}_{20}\,, && \mathrm{F}_{21} = \epsilon^3\,\mathcal{T}_{21}\,,\\
&\mathrm{F}_{22} = \epsilon^2\,\mathcal{T}_{22}\,, && \mathrm{F}_{23} = \epsilon^3\,\mathcal{T}_{23}\,, && \mathrm{F}_{24} = \epsilon^2\,\mathcal{T}_{24}\,,\\
&\mathrm{F}_{25} = \epsilon^3\,\mathcal{T}_{25}\,, && \mathrm{F}_{26} = (1-2\epsilon)\epsilon^3\,\mathcal{T}_{26}\,, && \mathrm{F}_{27} = \epsilon^3\,\mathcal{T}_{27}\,,\\
&\mathrm{F}_{28} = \epsilon^2\,\mathcal{T}_{28}\,, && \mathrm{F}_{29} = \epsilon^3\,\mathcal{T}_{29}\,, && \mathrm{F}_{30} = \epsilon^2\,\mathcal{T}_{30}\,,\\
&\mathrm{F}_{31} = (1-2\epsilon)\epsilon^3\,\mathcal{T}_{31}\,, && \mathrm{F}_{32} = \epsilon^3\,\mathcal{T}_{32}\,, && \mathrm{F}_{33} = \epsilon^4\,\mathcal{T}_{33}\,,\\
&\mathrm{F}_{34} = \epsilon^3\,\mathcal{T}_{34}\,, && \mathrm{F}_{35} = \epsilon^3\,\mathcal{T}_{35}\,, && \mathrm{F}_{36} = \epsilon^4\,\mathcal{T}_{36}\,,\\
&\mathrm{F}_{37} = \epsilon^4\,\mathcal{T}_{37}\,, && \mathrm{F}_{38} = \epsilon^3\,\mathcal{T}_{38}\,, && \mathrm{F}_{39} = \epsilon^4\,\mathcal{T}_{39}\,,\\
&\mathrm{F}_{40} = \epsilon^4\,\mathcal{T}_{40}\,, && \mathrm{F}_{41} = \epsilon^4\,\mathcal{T}_{41}\,, && \mathrm{F}_{42} = \epsilon^4\,\left(\mathcal{T}_{26}+\mathcal{T}_{42}\right)\,,
\end{aligned} \tag{8.52}
$$

where the $\mathcal{T}_i$ are depicted in figure 8.5. Through the Magnus exponential, we identify the corresponding canonical basis:

$$
\begin{aligned}
&\mathrm{I}_1 = \mathrm{F}_1\,, && \mathrm{I}_2 = -t\,\mathrm{F}_2\,,\\
&\mathrm{I}_3 = \lambda_t \mathrm{F}_3\,, && \mathrm{I}_4 = -t\,\mathrm{F}_4\,,\\
&\mathrm{I}_5 = \frac{1}{2}(\lambda_t - t)\,\mathrm{F}_4 - \lambda_t\,\mathrm{F}_5\,, && \mathrm{I}_6 = -s\,\mathrm{F}_6\,,\\
&\mathrm{I}_7 = 2m^2\,\mathrm{F}_6 + (m^2-s)\,\mathrm{F}_7\,, && \mathrm{I}_8 = m^2\,\mathrm{F}_8\,,\\
&\mathrm{I}_9 = m^2 \mathrm{F}_9\,, && \mathrm{I}_{10} = -s\,\mathrm{F}_{10}\,,\\
&\mathrm{I}_{11} = -t\,\lambda_t\,\mathrm{F}_{11}\,, && \mathrm{I}_{12} = -t\,\mathrm{F}_{12}\,,\\
&\mathrm{I}_{13} = -t\,m^2\,\mathrm{F}_{13}\,, && \mathrm{I}_{14} = -m^2(\lambda_t - t)\left(\frac{3}{2}\mathrm{F}_{12} + \mathrm{F}_{13}\right) +\\
& && \qquad - m^2\,\lambda_t\,\mathrm{F}_{14}\,,\\
&\mathrm{I}_{15} = \lambda_t\,\mathrm{F}_{15}\,, && \mathrm{I}_{16} = m^2\,\lambda_t\,\mathrm{F}_{16}\,,\\
&\mathrm{I}_{17} = m^2(t-\lambda_t)\left(\frac{3}{2}\mathrm{F}_{15} + \mathrm{F}_{16}\right) && \\
&\qquad - m^2\,t\,\mathrm{F}_{17}\,, && \mathrm{I}_{18} = \lambda_t\,\mathrm{F}_{18}\,,\\
&\mathrm{I}_{19} = (m^2-s)\,\mathrm{F}_{19}\,, && \mathrm{I}_{20} = m^2\,(m^2-s)\,\mathrm{F}_{20}\,,\\
&\mathrm{I}_{21} = (m^2-s)\,\mathrm{F}_{21}\,, && \mathrm{I}_{22} = -\frac{3}{2}s\,\mathrm{F}_9 + (s^2-m^4)\,\mathrm{F}_{22}\,,\\
&\mathrm{I}_{23} = \lambda_t\,\mathrm{F}_{23}\,, && \mathrm{I}_{24} = \frac{1}{4}\left(4m^2 - t + \lambda_t\right)(\mathrm{F}_4 + 2\mathrm{F}_5) +\\
& && \qquad + m^2(4m^2-t)\,\mathrm{F}_{24}\,,\\
&\mathrm{I}_{25} = \lambda_t\,\mathrm{F}_{25}\,, && \mathrm{I}_{26} = -t\,\mathrm{F}_{26}\,,\\
&\mathrm{I}_{27} = s\,t\,\mathrm{F}_{27}\,, && \mathrm{I}_{28} = -m^4\,t\,\mathrm{F}_{27} - m^2(m^2-s)\,t\,\mathrm{F}_{28}\,,\\
&\mathrm{I}_{29} = -s\,\lambda_t\,\mathrm{F}_{29}\,, && \mathrm{I}_{30} = m^4\lambda_t\,\mathrm{F}_{29} + m^2\,(m^2-s)\,\lambda_t\,\mathrm{F}_{30}\,,\\
&\mathrm{I}_{31} = -(m^2-s)\mathrm{F}_{31} + && \\
&\qquad - (m^2-s)\,(4m^2 - t + \lambda_t)\,\mathrm{F}_{32}\,, && \mathrm{I}_{32} = (m^2-s)\lambda_t \mathrm{F}_{32}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{I}_{33} = (m^2 - s - t)\,\mathrm{F}_{33}\,, && \mathrm{I}_{34} = (m^2 - s)\,\lambda_t\,\mathrm{F}_{34}\,,\\
&\mathrm{I}_{35} = 2\frac{m^4(m^2-s)}{2m^2-t-\lambda_t}\,\mathrm{F}_{34} + m^2\,(m^2-s)\mathrm{F}_{35}\,, && \mathrm{I}_{36} = \lambda_t\,\mathrm{F}_{36}\,,\\
&\mathrm{I}_{37} = -t\,(4m^2 - t)\,\mathrm{F}_{37}\,, && \mathrm{I}_{38} = -(m^2 - s)\,t\,\mathrm{F}_{38}\,,\\
&\mathrm{I}_{39} = -(m^2 - s)\,t\mathrm{F}_{39}\,, && \mathrm{I}_{40} = -(m^2 - s)\,t\,\lambda_t\,\mathrm{F}_{40}\,,\\
&\mathrm{I}_{41} = t\lambda_t\,(\mathrm{F}_{40} - \mathrm{F}_{41})\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{I}_{42} = {}& (m^2 - t + \lambda_t)\times\\
&\bigg(\frac{2}{3}\mathrm{F}_3 + \frac{1}{4}\mathrm{F}_4 + \frac{1}{2}\mathrm{F}_5 - \frac{1}{2}\,t\,\mathrm{F}_{11} + \frac{5}{2}\mathrm{F}_{12} + \frac{5}{3}m^2\mathrm{F}_{13} + \frac{5}{3}m^2\mathrm{F}_{14}\\
&\qquad +2\mathrm{F}_{36} - \frac{1}{2}(m^2+s)\mathrm{F}_{40} + t\,\mathrm{F}_{41}\bigg) +\\
&+ m^2\left(\frac{1}{3}\mathrm{F}_3 - \frac{1}{2}t\mathrm{F}_{11} + \frac{1}{2}\mathrm{F}_{12} + \frac{1}{3}m^2\mathrm{F}_{13} + \frac{1}{3}m^2\mathrm{F}_{14} + \frac{1}{2}\mathrm{F}_{18} - \frac{1}{2}\mathrm{F}_{40}\right) +\\
&- t\,(m^2 - s)\,\mathrm{F}_{11} - 2\frac{m^4}{2m^2-t-\lambda_t}\mathrm{F}_{15} + t\,\mathrm{F}_{26} + \frac{m^2(m^2-s)(t+\lambda_t)}{2m^2-t-\lambda_t}\left(\frac{2}{3}\mathrm{F}_{29} - \mathrm{F}_{34}\right) +\\
&- \frac{2}{3}\frac{m^2\,s\,(t-\lambda_t)}{2m^2-t-\lambda_t}\mathrm{F}_{29} + 2t\,\mathrm{F}_{33} + \frac{4}{3}tm^4\frac{m^2-s}{\lambda_t+t}\mathrm{F}_{30} - t\,\mathrm{F}_{42}\,, \qquad (8.53)
\end{aligned}
$$

which satisfies a system of DEQs of the form given in eq. (8.8). The corresponding 42×42 matrices are collected in appendix B.3. We observe that $\mathrm{I}_{1,2,6,7,8,10,15,16,17,27,28}$ correspond, respectively, to $\mathrm{I}_{1,3,5,6,4,2,13,14,15,23,24}$ of integral family (8.24) previously discussed. The boundary constants of the remaining integrals can be fixed in the following way:

- The integration constants of $\mathrm{I}_{3,4,5,11,\ldots,14,18,23,24,26,29\ldots,35}$ are determined by demanding regularity in the limit $t \to 0$.
- The boundary values of I_9 can be obtained by direct integration and is given by

$$
\begin{aligned}
\mathrm{I}_9(\epsilon) = {}& -\frac{\zeta_2}{2}\epsilon^2 + \frac{1}{4}\,(12\zeta_2\log(2) - 7\zeta_3)\,\epsilon^3\\
&+ \left(-12\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{31}{40}\zeta_4 - \frac{\log^4(2)}{2} - 6\zeta_2\log^2(2)\right)\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)\,. \qquad (8.54)
\end{aligned}
$$

- The boundary constants of $\mathrm{I}_{19,21}$ can be fixed by demanding regularity when $s \to 0$.
- The boundary constants of $\mathrm{I}_{20,22,25}$ are determined by demanding regularity, respectively, when $s \to -\frac{1}{2}\left(2m^2 - t - \lambda_t\right)$, $s \to -m^2$, and $t \to 4m^2$.
- Finally, the boundary constants of $\mathrm{I}_{36\ldots42}$ can be determined by demanding regularity in the simultaneous limits $t \to \frac{9}{2}m^2$ and $s \to -2m^2$.

All results have been numerically checked with the help of the computer codes GiNaC and SecDec.

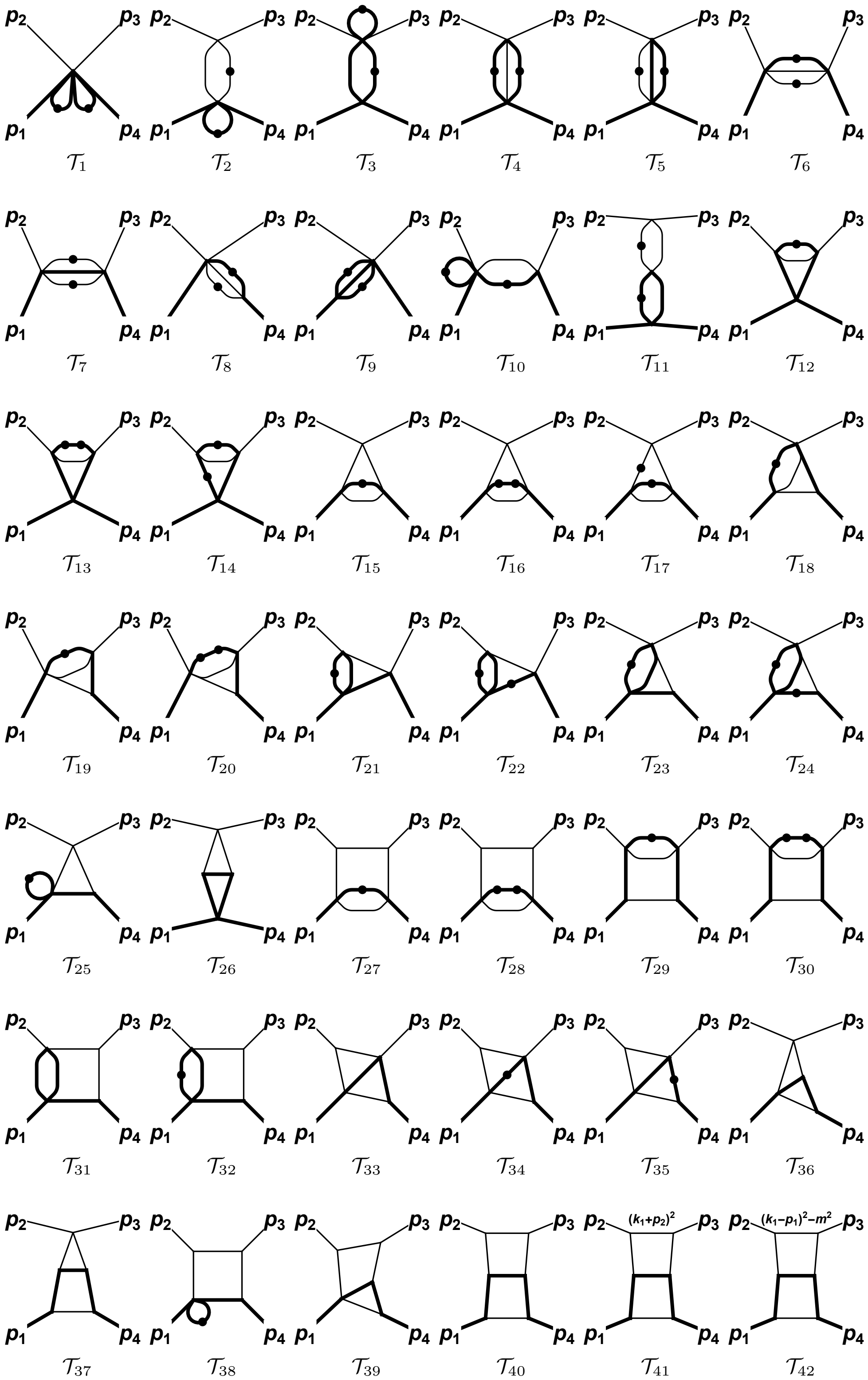


Figure 8.5: Two-loop MIs $\mathcal{T}_{1,\ldots,42}$ for the second integral family.

8.4.3 The non-planar vertex integral

The complete computation of the NNLO virtual QED corrections to μe scattering requires the evaluation of one last missing four-point topology, which corresponds to the non-planar diagram T_6 of figure 8.1. In view of future studies dedicated to this last class of integrals, we hereby show how the previously adopted strategy, based on differential equations, Magnus exponential and regularity conditions, can be efficiently applied to compute the MIs of a simpler vertex integral belonging to same family.

We consider the non-planar vertex depicted in fig. 8.6, whose integral family is defined as

$$\int \widetilde{\mathrm{d}^d k_1} \widetilde{\mathrm{d}^d k_2} \, \frac{D_4^{n_4}}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_5^{n_5} D_6^{n_6} D_7^{n_7}} \,, \quad n_i \geq 0 \,, \tag{8.55}$$

where the loop propagators are chosen to be

$$D_1 = k_1^2 - m^2 \,, \quad D_2 = k_2^2 - m^2 \,, \quad D_3 = (k_1 + p_1 + p_2)^2 \,, \quad D_4 = (k_2 + p_1 + p_2)^2 \,,$$
$$D_5 = (k_1 - k_2 + p_3)^2 \,, \quad D_6 = (k_2 + p_4)^2 \,, \quad D_7 = (k_1 - k_2)^2 \,. \tag{8.56}$$

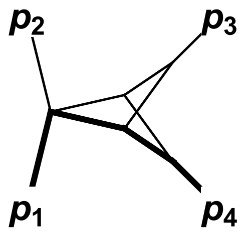


Figure 8.6: Non-planar two-loop three-point topology.

The MIs belonging to this integral family, which will be part of the full set of MIs needed for the computation of T_6, have been already considered in the literature [258–262]. In all previous computations, the determination of the boundary constants resorted either to the fitting of numerical values to trascendental constants [258–260] or to Mellin-Barnes techniques [262]. With the present calculation, we show that they can be fixed equivalently by imposing the regularity of the solution at specific kinematic pseudo-thresholds and by matching a particular linear combination of integrals to their massless counterpart.

In order to determine the MIs belonging to the integral family (8.55), we derive their DEQs in the dimensionless variable x. We identify a set of 14 MIs which fulfil an ϵ-linear system of DEQs,

$$\begin{aligned}
&\mathrm{F}_1 = \epsilon^2\, \mathcal{T}_1 \,, && \mathrm{F}_2 = \epsilon^2\, \mathcal{T}_2 \,, && \mathrm{F}_3 = \epsilon^2\, \mathcal{T}_3 \,, \\
&\mathrm{F}_4 = \epsilon^2\, \mathcal{T}_4 \,, && \mathrm{F}_5 = \epsilon^2\, \mathcal{T}_5 \,, && \mathrm{F}_6 = \epsilon^3\, \mathcal{T}_6 \,, \\
&\mathrm{F}_7 = \epsilon^2\, \mathcal{T}_7 \,, && \mathrm{F}_8 = \epsilon^3\, \mathcal{T}_8 \,, && \mathrm{F}_9 = \epsilon^2\, \mathcal{T}_9 \,, \\
&\mathrm{F}_{10} = \epsilon^2\, \mathcal{T}_{10} \,, && \mathrm{F}_{11} = \epsilon^2 (2\epsilon - 1)\, \mathcal{T}_{11} \,, && \mathrm{F}_{12} = \epsilon^4\, \mathcal{T}_{12} \,, \\
&\mathrm{F}_{13} = \epsilon^3\, \mathcal{T}_{13} \,, && \mathrm{F}_{14} = \epsilon^4\, \mathcal{T}_{14} \,,
\end{aligned} \tag{8.57}$$

where the $\mathcal{T}_i$ are depicted in figure 8.7. By making use of the Magnus exponential, we

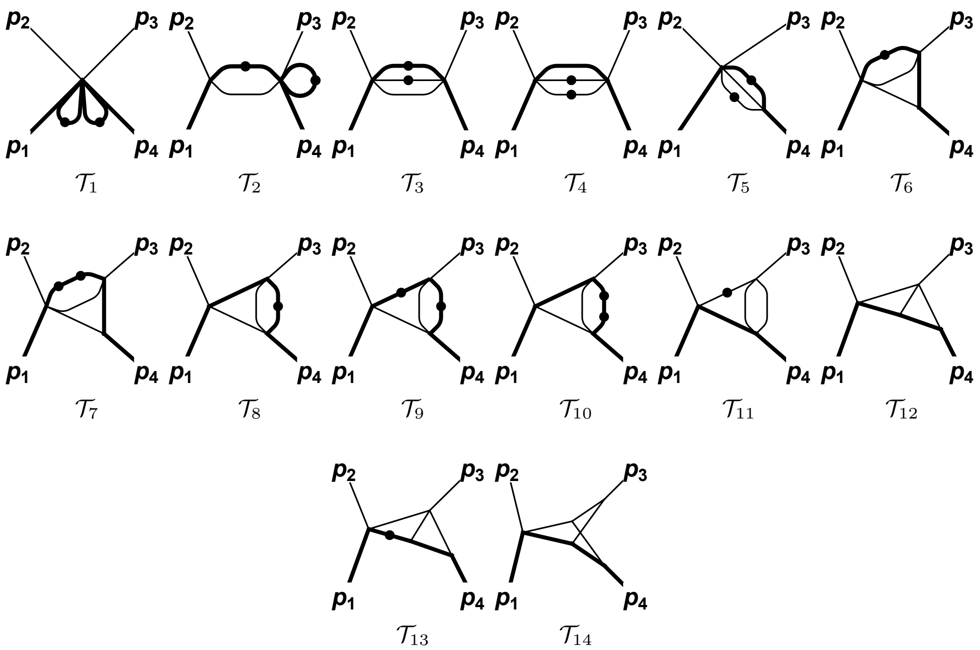


Figure 8.7: Two-loop MIs $\mathcal{T}_{1,\ldots,14}$ for the non-planar vertex.

can transform these MIs into the canonical basis

$$\begin{aligned}
&\mathrm{I}_1 = \mathrm{F}_1\,, && \mathrm{I}_2 = -s\,\mathrm{F}_2\,,\\
&\mathrm{I}_3 = -s\mathrm{F}_3\,, && \mathrm{I}_4 = 2m^2\,\mathrm{F}_3 + (m^2-s)\,\mathrm{F}_4\,,\\
&\mathrm{I}_5 = m^2\mathrm{F}_5\,, && \mathrm{I}_6 = (m^2-s)\,\mathrm{F}_6\,,\\
&\mathrm{I}_7 = m^2(m^2-s)\,\mathrm{F}_7\,, && \mathrm{I}_8 = (m^2-s)\,\mathrm{F}_8\,,\\
&\mathrm{I}_9 = m^2\left(3\,\mathrm{F}_8 + (2m^2-s)\,\mathrm{F}_9 + 2m^2\,\mathrm{F}_{10}\right)\,, && \mathrm{I}_{10} = m^2(m^2-s)\mathrm{F}_{10}\,,\\
&\mathrm{I}_{11} = \frac{1}{(s+m^2)}\left(-2m^4\,\mathrm{F}_5 + (s-m^2)s\,\mathrm{F}_{11}\right)\,, && \mathrm{I}_{12} = (m^2-s)\,\mathrm{F}_{12}\,,\\
&\mathrm{I}_{13} = m^2(m^2-s)\,\mathrm{F}_{13}\,, && \mathrm{I}_{14} = (m^2-s)^2\,\mathrm{F}_{14}\,,
\end{aligned} \tag{8.58}$$

which satisfies a system of DEQs of the form,

$$d\mathbf{I} = \epsilon\left(\mathbb{M}_1\,\mathrm{dlog}(x) + \mathbb{M}_2\,\mathrm{dlog}(1+x) + \mathbb{M}_3\,\mathrm{dlog}(2+x)\right)\mathbf{I}\,, \tag{8.59}$$

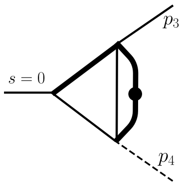
where $\mathbb{M}_i$ are the constant matrices

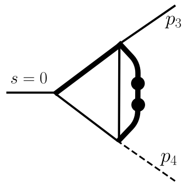
$$\mathbb{M}_1 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\frac{1}{2}&0&1&\frac{1}{2}&0&3&2&0&0&0&0&0&0&0\\
\frac{1}{2}&0&\frac{1}{2}&\frac{1}{2}&0&3&2&0&0&0&0&0&0&0\\
\frac{1}{2}&0&0&0&2&0&0&1&1&0&0&0&0&0\\
\frac{3}{2}&1&0&0&6&0&0&3&3&0&0&0&0&0\\
0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&2&0&0&0\\
\frac{1}{4}&0&0&\frac{1}{4}&1&0&0&0&0&0&\frac{1}{2}&3&2&0\\
\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0&0&0&0&0&0&0&6&4&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}, \quad
\mathbb{M}_2 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&1&0&0&0&0&0&0&0&0&0&0\\
0&0&4&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&3&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
\frac{3}{2}&1&0&0&6&0&0&6&4&2&0&0&0&0\\
0&0&0&0&0&0&0&3&0&2&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&6&4&0\\
1&1&\frac{4}{3}&\frac{2}{3}&4&3&4&0&2&2&1&2&2&2
\end{pmatrix},$$

$$\mathbb{M}_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&3&1&2&0&0&0&0\\ 0&0&0&0&0&0&0&3&1&2&0&0&0&0\\ 0&0&0&0&0&0&0&3&1&2&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}. \tag{8.60}$$

The 3 letters are real and positive in the range $x > 0$, which corresponds to the Euclidean region $s < 0$. The general solution of the system can be written in terms of one-dimensional GPLs. In order to completely determine the solution of the DEQs, we fix the boundary constants as follows:

- $\mathrm{I}_{1,2,3,4,5}$ correspond, respectively, to $\mathrm{I}_{1,2,5,6,4}$ of the first integral family, defined in eq. (8.24).
- $\mathrm{I}_{6,7}$ correspond, respectively, to $\mathrm{I}_{19,20}$ of the second integral family, defined in eq. (8.50).
- The regularity at $s \to 0$ of $\mathrm{I}_{8,9,10}$ can be used to fix the boundary constants of one single MIs, which we choose to be I_9. The boundary values $\mathrm{I}_8(\epsilon, 0)$ and $\mathrm{I}_{10}(\epsilon, 0)$ can be obtained in the limit $p_4^2 \to m^2$ of similar vertex integrals with off-shell momentum p_4^2 and $s \equiv (p_1+p_2)^2 = p_3^2 = 0$,

$\mathrm{I}_8(\epsilon, 0) = \epsilon^3 m^2 \lim_{p_4^2 \to m^2}$ 

, $\mathrm{I}_{10}(\epsilon, 0) = \epsilon^2 m^4 \lim_{p_4^2 \to m^2}$ 

(8.61)

The computation of the auxiliary vertex integrals is discussed at the end of this section and leads to

$$\begin{aligned} \mathrm{I}_8(\epsilon, 0) =& \left(\frac{5\zeta_3}{4} - 3\zeta_2 \log(2)\right)\epsilon^3 + \left(8\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{33}{8}\zeta_4 + \frac{\log^4(2)}{3} - 2\zeta_2\log^2(2)\right)\epsilon^4, \\ \mathrm{I}_{10}(\epsilon, 0) =& \frac{\zeta_2}{2}\epsilon^2 + \left(\frac{\zeta_3}{4} + 3\zeta_2\log(2)\right)\epsilon^3 \\ &+ \left(-8\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{65}{4}\zeta_4 - \frac{\log^4(2)}{3} + 2\zeta_2\log^2(2)\right)\epsilon^4 + \mathcal{O}\left(\epsilon^5\right). \end{aligned} \tag{8.62}$$

- The boundary constant of I_{11} is determined by imposing regularity when $s \to 0$.
- I_{12} corresponds to I_{17} of the integral family (8.24) and the boundary constant of I_{13} can be fixed by demanding regularity when $s \to 0$.
- The boundary condition for I_{14} is determined from the $m \to 0$, or equivalently $s \to \infty$ behaviour of the solutions, where all the internal lines of the diagrams become *massless*. In this regime, we search for a combination of integrals which behaves as

$$\lim_{z\to 0} \sum_i c_i \mathrm{I}_i = z^{a\epsilon} F(\epsilon), \quad a \in \mathbb{Z}, \tag{8.63}$$

where $F(\epsilon)$ is finite as $z \to 0$.

Following the ideas outlined in [266], we begin by performing a change of variables $z = 1/x = (-m^2/s)$ that yields to a total differential of the form,

$$d\mathbf{I} = \epsilon\Big(\mathbb{M}_1 \,\mathrm{dlog}(z) + \mathbb{M}_2 \,\mathrm{dlog}(1+z) + \mathbb{M}_3 \,\mathrm{dlog}(1+2z)\Big)\mathbf{I}\,, \tag{8.64}$$

where $\mathbb{M}_i$ are the constant matrices

$$\mathbb{M}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & -\frac{1}{4} & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 2 & 0 \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -\frac{4}{3} & -\frac{2}{3} & -4 & -3 & -4 & 0 & -2 & 2 & -1 & -2 & -2 & 2
\end{pmatrix}, \quad
\mathbb{M}_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2} & -1 & 0 & 0 & -6 & 0 & 0 & -6 & -4 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -4 & 0 \\
1 & 1 & \frac{4}{3} & \frac{2}{3} & 4 & 3 & 4 & 0 & 2 & -2 & 1 & 2 & 2 & -2
\end{pmatrix},$$

$$\mathbb{M}_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{8.65}$$

Around the $z = 0$ singularity, the system reduces to

$$d\mathbf{I} \approx \epsilon \; \mathbb{M}_1 \,\mathrm{dlog}(z)\mathbf{I}\,. \tag{8.66}$$

We perform a Jordan decomposition of $\mathbb{M}_1$, identifying the matrices $\mathbb{J}$ and $\mathbb{S}$, related by $\mathbb{J} \equiv \mathbb{S}\mathbb{M}_1\mathbb{S}^{-1}$,

$$\mathbb{J} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}, \quad
\mathbb{S} = \begin{pmatrix}
\frac{5}{2} & 1 & 0 & 0 & 3 & 0 & -1 & 0 & \frac{3}{2} & -1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{5}{4} & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{2} & 3 & -\frac{8}{3} & \frac{25}{6} & 0 & 3 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & \frac{1}{4} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{2} & \frac{3}{4} & \frac{3}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 1 & 0 \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2} & -2 & \frac{2}{3} & -\frac{8}{3} & -1 & -3 & -5 & 0 & -\frac{3}{2} & 1 & -1 & -2 & -3 & 1 \\
0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{8.67}$$

The latter can be used to define a change to a new integral basis $\mathbf{H} \equiv \mathbb{S}\mathbf{I}$ which, by construction, obeys a system of DEQs in Jordan form,

$$d\mathbf{H} = \epsilon \; \mathbb{J} \,\mathrm{dlog}(z)\mathbf{H}\;. \tag{8.68}$$

In particular, the truly diagonal elements $\mathbf{H}_i$, say $i = 2, 3, 4, 6, 10, 11, 12, 13, 14$ (not belonging to any block-diagonal sector), obey a trivial first-order DEQ of the form,

$$\frac{d\mathbf{H}_i}{dz} = \epsilon \frac{\mathbb{J}_{ii}}{z}\mathbf{H}_i\,. \tag{8.69}$$

Therefore, their expression is of the type

$$\mathbf{H}_i = z^{\mathbb{J}_{ii}\epsilon}\,\mathbf{H}_{i,0}\;, \tag{8.70}$$

where $\mathbf{H}_{i,0}$ is a boundary constant which might still depend on ϵ. Among the possible choices of i, we look at the element $i = 11$,

$$\mathbf{H}_{11} = \mathbb{S}_{11,j}\mathbf{I}_j = z^{2\epsilon}\,\mathbf{H}_{11,0}\,, \tag{8.71}$$

from which we infer the behaviour around $z = 0$ of the following combination of canonical integrals,

$$\mathbf{H}_{11,0} = \lim_{z\to 0} z^{-2\epsilon}\left(-\frac{3}{2}\mathrm{I}_1 - 2\mathrm{I}_2 + \frac{2}{3}\mathrm{I}_3 - \frac{8}{3}\mathrm{I}_4 - \mathrm{I}_5 - 3\mathrm{I}_6 - 5\mathrm{I}_7 + -\frac{3}{2}\mathrm{I}_9 + \mathrm{I}_{10} - \mathrm{I}_{11} - 2\mathrm{I}_{12} - 3\mathrm{I}_{13} + \mathrm{I}_{14}\right)\,, \tag{8.72}$$

that involves the integral I_{14}. On the other hand, $\mathbf{H}_{11}$ can be computed by taking the limit $z \to 0$ on the r.h.s. of eq. (8.72) directly at the *integrand-level*, i.e. by evaluating the integrals $\mathbf{I}$ in the limit $m \to 0$. To this aim, we need to pull out the prefactor $m^{4\epsilon}$ coming from the integration measure defined in eq. (5.113), and consider the definition of the canonical integrals $\mathbf{I}$ in terms of the linear-ϵ basis $\mathbf{F}$ given in eq. (8.58),

$$\mathbf{H}_{11,0} = (-s)^{2\epsilon}\left(2s\mathrm{F}_3 + 3s\mathrm{F}_6 - s\mathrm{F}_{11} + 2s\mathrm{F}_{12} + s^2\mathrm{F}_{14}\right)\Big|_{m=0}. \tag{8.73}$$

In the latter equation, we took into account the vanishing of the massless tadpole in dimensional regularization and the symmetries arising from the massless limit of the $\mathbf{F}$ integrals. After applying the IBPs to the massless integrals, the contributions due to all subtopologies cancel and the contribution of the *massless non-planar vertex* $\mathrm{F}_{14}|_{m=0}$ [256] is the only one left,

$$\mathbf{H}_{11,0} = (-s)^{2+2\epsilon}\,\mathrm{F}_{14}|_{m=0}\,, \tag{8.74}$$

where

$$\mathrm{F}_{14}|_{m=0} = (-s)^{-2-2\epsilon}\,\mathcal{F}(\epsilon) \tag{8.75}$$

with

$$\mathcal{F}(\epsilon) \equiv 1 - 7\,\zeta_2\,\epsilon^2 - 27\,\zeta_3\,\epsilon^3 - \frac{57}{2}\,\zeta_4\,\epsilon^4 + \mathcal{O}(\epsilon^5)\,. \tag{8.76}$$

Therefore,

$$\mathbf{H}_{11,0} = \mathcal{F}(\epsilon)\,. \tag{8.77}$$

Finally the boundary constant of integral I_{14} can be determined by demanding the equality of eq. (8.72) and eq. (8.77).

All results have been numerically checked with the help of the computer codes GiNaC and SecDec.

Auxiliary vertex integral for eq. (8.61)

We conclude this section by showing how the boundary value of I_8 and I_{10} at $s = 0$ can be extracted from the solution of the system of DEQs for the auxiliary vertex integrals

$$\int \widetilde{\mathrm{d}^d k_1}\widetilde{\mathrm{d}^d k_2}\,\frac{D_5^{n_5}D_6^{n_6}D_7^{n_7}}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}}\,,\quad n_i \geq 0\,, \tag{8.78}$$

identified by the set of denominators

$$D_1 = k_1^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_1+p_1)^2 - m^2, \quad D_4 = (k_1+k_2+p_1+p_2)^2,$$
$$D_5 = (k_1+p_2)^2, \quad D_6 = (k_2+p_1)^2, \quad D_7 = (k_2+p_1+p_2)^2, \tag{8.79}$$

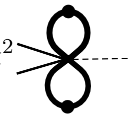
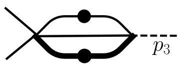
and by external momenta p_1, p_2 and p_3 satisfying

$$p_1^2 = p_2^2 = 0\,, \quad (p_1+p_2)^2 = p_3^2\,. \tag{8.80}$$

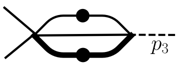
All integrals belonging to this family can be reduced to a set of 5 MIs, whose dependence on p_3^2 is parametrized in terms of the dimensionless variable

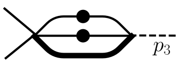
$$x = -\frac{p_3^2}{m^2}\,. \tag{8.81}$$

The basis of integrals

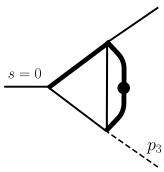
$\mathrm{I}_1 = \epsilon^2$  , $\qquad \mathrm{I}_2 = -\epsilon^2 p_3^2$ 

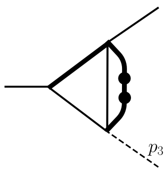
,

$\mathrm{I}_3 = \epsilon^2\, 2m^2$ 

$+\ \epsilon^2(m^2 - p_3^2)$ 

,

$\mathrm{I}_4 = -\,\epsilon^3\, p_3^2$ 

, $\quad \mathrm{I}_5 = -\epsilon^4\, p_3^2 m^2$ 

, (8.82)

fulfils a canonical system of DEQs,

$$d\mathbf{I} = \epsilon\, \mathrm{d}\mathbb{A}\, \mathbf{I}\,, \tag{8.83}$$

where

$$\mathrm{d}\mathbb{A} = \mathbb{M}_1\ \mathrm{dlog}\, x + \mathbb{M}_2\ \mathrm{dlog}(1+x) + \mathbb{M}_3\ \mathrm{dlog}(1-x)\,, \tag{8.84}$$

with

$$\mathbb{M}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & -2 & -2 \end{pmatrix}, \quad \mathbb{M}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \\ 0 & -4 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{M}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & -2 & -2 \\ \frac{1}{2} & -1 & \frac{1}{2} & 2 & 2 \end{pmatrix}. \tag{8.85}$$

In the Euclidean region $x > 0$ the general solution of the system of DEQs can be expressed in terms of HPLs. The boundary constants $\mathrm{I}_{4,5}$, which are the only MIs appearing for the first time in this computation, can be fixed by demanding their regularity at $x \to 0$. In this way, we obtain

$$\mathrm{I}_i(\epsilon, x) = \sum_{k=2}^{4} \mathrm{I}_i^{(k)}(x,y)\epsilon^k + \mathcal{O}(\epsilon^5)\,, \tag{8.86}$$

with

$$\mathrm{I}_4^{(2)}(x) = \ 0\,,$$

$$
\begin{aligned}
\mathrm{I}_4^{(3)}(x) &= -\zeta_2 H(1;x) + 2H(1,0,-1;x)\,,\\
\mathrm{I}_4^{(4)}(x) &= \zeta_3 H(1;x) - \zeta_2 H(0,1;x) + 2H(0,1,0,-1;x) - 8H(1,0,-1,-1;x)\,, \qquad (8.87)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{I}_5^{(2)}(x) &= \frac{1}{2}H(0,-1;x)\,,\\
\mathrm{I}_5^{(3)}(x) &= \zeta_2 H(1;x) - 2H(0,-1,-1;x) - \frac{1}{2}H(0,0,-1;x) - 2H(1,0,-1;x))\,,\\
\mathrm{I}_5^{(4)}(x) &= -\zeta_3 H(1;x) + \zeta_2 H(0,-1;x) 8H(0,-1,-1,-1;x) - 3H(0,-1,0,-1;x)\\
&\quad + 2H(0,0,-1,-1;x) + \frac{3}{2}H(0,0,0,-1;x) + 8H(1,0,-1,-1;x)\,. \qquad (8.88)
\end{aligned}
$$

The analytic continuation of these expressions to $x \to -1$ ($p_3^2 = m^2$) produces the smooth limits

$$
\begin{aligned}
\mathrm{I}_4(\epsilon,-1) &= -\left(\frac{5\zeta_3}{4} - 3\zeta_2\log(2)\right)\epsilon^3 - \left(8\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{33}{8}\zeta_4 + \frac{\log^4(2)}{3} - 2\zeta_2\log^2(2)\right)\epsilon^4\,,\\
\mathrm{I}_5(\epsilon,-1) &= -\frac{\zeta_2}{2}\epsilon^2 - \left(\frac{\zeta_3}{4} + 3\zeta_2\log(2)\right)\epsilon^3\\
&\quad - \left(-8\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{65}{4}\zeta_4 - \frac{\log^4(2)}{3} + 2\zeta_2\log^2(2)\right)\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)\,, \qquad (8.89)
\end{aligned}
$$

which have been used in eq. (8.62), where the different overall sign is due to the kinematic factors relating the canonical MIs I_i to the corresponding F_i, as defined by eq. (8.58) and eq. (8.82)).

8.5 Conclusions

In this chapter, we have considered the two-loop planar box-diagrams contributing to the two-loop virtual QED correction to the elastic muon-electron scattering. By working in the massless electron approximation, we employed the method of differential equations and the Magnus exponential series to identify a canonical set of master integrals. Boundary conditions were derived from the regularity requirements at pseudothresholds, or from the knowledge of the integrals at special kinematic points, which have been evaluated by means of auxiliary, simpler systems of differential equations. The resulting MIs are expressed, in their expansion around four space-time dimensions, in terms of generalized polylogarithms.

The scattering of energetic muons on atomic electrons has been recently proposed as an ideal framework to determine the leading hadronic contribution to the anomalous magnetic moment of the muon. The ambitious measurement of the differential cross section of the $\mu e \to \mu e$ process with an accuracy of 10ppm, aimed by the future experiment MUonE, requires, on the theoretical side, the knowledge of the QED corrections at NNLO. The present calculation represents the first step towards the analytic computation of the NNLO corrections to differential cross section for μe scattering. At the time being, we are working towards the evaluation of the missing non-planar integral topology. Once the analytic expressions of all relevant MIs is available, they will be employed in the computation of the two-loop virtual amplitude, which will be addressed through the adaptive integrand decomposition algorithm AIDA.

Chapter 9

Maximal Cuts and Feynman integrals beyond polylogarithms

In this chapter we discuss the solution of systems of differential equations for master integrals which remain coupled in the $\epsilon \to 0$ limit. In such cases, the master integrals obey irreducible higher-order differential equations and they are not expressible in terms of polylogarithmic functions. The standard solving strategy consists in first determining the homogeneous solutions at $\epsilon = 0$, and then applying Euler variation of constants in order to obtain the ϵ-expansion of the master integrals in terms of iterated integrals. However, no general technique for finding the homogeneous solutions of higher-order differential equations exists. We hereby show that the maximal-cuts of the master integrals, when computed along integration contours compatible with IBPs, provides, by definition, a full set of homogeneous solutions. We argue that such maximal-cuts constitute a natural generalization of the concept of leading singularity, for coupled system of equations. We discuss several two-loop examples, whose homogeneous solutions are expressed in terms of elliptic integrals and. Finally we apply these techniques to the three-loop massive banana graph in $d = 2 - 2\epsilon$, which satisfy an irreducible third-order differential equation. The content of this chapter is based on the publications [2, 4], in collaboration with L. Tancredi, and on original research in collaboration with P. Mastrolia and U. Schubert.

9.1 Coupled differential equations for Feynman integrals

In this last chapter, we turn our attention back to the general structure of the first-order DEQs satisfied by the MIs, which was discussed in chapter 5. For simplicity, we discuss systems of DEQs in one single variables but the following discussion equally applies to multi-scale problems. In particular, we reconsider the DEQs (5.23) for the sector of $n > 1$ MIs $\mathbf{F}(\epsilon, x)$,

$$\partial_x \mathbf{F}(\epsilon, x) = \mathbb{A}(\epsilon, x)\mathbf{F}(\epsilon, x) + \mathbf{S}(\epsilon, x)\,, \tag{9.1}$$

where the inhomogeneous term $\mathbf{S}(\epsilon, x)$ stems for the contributions of MIs from lower sectors, which we assume to be known. As usual, we suppose the MIs to be normalized in such a way that the matrix $\mathbb{A}(\epsilon, x)$ admits an expansion in positive powers of ϵ,

$$\mathbb{A}(\epsilon, x) = \mathbb{A}_0(x) + \sum_{k>0} \mathbb{A}_k(x)\epsilon^k\,. \tag{9.2}$$

According to the discussion of chapter (5), the solution of eq. (9.1) is streamlined if we are able to determine a $n \times n$ matrix $\mathbb{G}(x)$ that solves the DEQ in $\epsilon = 0$.

$$\partial_x \mathbb{G}(x) = \mathbb{A}_0 \, \mathbb{G}(x) \,. \tag{9.3}$$

In fact, $\mathbb{G}(x)$ can be used to define a new set of MIs $\mathbf{I}(\epsilon, x)$,

$$\mathbf{F}(\epsilon, x) = \mathbb{G}(x) \, \mathbf{I}(\epsilon, x) \,, \tag{9.4}$$

which satisfy a simplified system of DEQs,

$$\partial_x \mathbf{I}(\epsilon, x) = \hat{\mathbb{A}}(\epsilon, x) \mathbf{I}(\epsilon, x) + \hat{\mathbf{S}}(\epsilon, x) \,, \tag{9.5}$$

with

$$\hat{\mathbb{A}}(\epsilon, x) = \mathbb{G}^{-1}(x) \mathbb{A}(\epsilon, x) \mathbb{G}(x) \,, \quad \hat{\mathbf{S}}(\epsilon, x) = \mathbb{G}^{-1}(x) \mathbf{S}(\epsilon, x), \tag{9.6}$$

which allows to decouple, order-by -rder in ϵ, the DEQs for the coefficients of the Taylor expansion of the MIs around $\epsilon = 0$.

The solution of eq. (9.5) is particularly straightforward if $\mathbb{A}_k(x) = 0$, for $k > 1$,

$$\mathbb{A}(\epsilon, x) = \mathbb{A}_0(x) + \epsilon \mathbb{A}_1(x) \,. \tag{9.7}$$

In this case, if we assume the MIs and the inhomogeneous term to be Taylor expanded as

$$\begin{aligned} \mathbf{I}(\epsilon, x) &= \sum_{k=0}^{\infty} \mathbf{I}^{(k)}(x) \epsilon^k \,, \\ \mathbf{S}(\epsilon, x) &= \sum_{k=0}^{\infty} \mathbf{S}^{(k)}(x) \epsilon^k \,, \end{aligned} \tag{9.8}$$

we can integrate independently eq. (9.5) for all coefficients $\mathbf{I}^{(k)}(x)$,

$$\partial_x \mathbf{I}^{(k)}(x) = \hat{\mathbb{A}}_1(x) \mathbf{I}^{(k-1)}(x) + \hat{\mathbf{S}}^{(k)}(x) \,, \tag{9.9}$$

with

$$\hat{\mathbb{A}}_1(x) = \mathbb{G}^{-1}(x) \mathbb{A}_1(x) \mathbb{G}(x) \,, \quad \hat{\mathbf{S}}^{(k)}(x) = \mathbb{G}^{-1}(x) \mathbf{S}^{(k)}(x) \tag{9.10}$$

and obtain

$$\begin{aligned} \mathbf{I}^{(0)}(x) &= \mathbf{c}_0 + \int_{x_0}^{x} \mathrm{d}t \, \hat{\mathbf{S}}^{(0)}(t) \,, \\ \mathbf{I}^{(k)}(x) &= \mathbf{c}_k + \int_{x_0}^{x} \mathrm{d}t \, \hat{\mathbb{A}}_1(t) \mathbf{I}^{(k-1)}(t) + \int_{x_0}^{x} \mathrm{d}t \, \hat{\mathbf{S}}^{(k)}(t) \,. \end{aligned} \tag{9.11}$$

where x_0 is an arbitrary integration base-point and $\mathbf{c}_i$ are integration constants to be fixed by imposing suitable boundary conditions.

If $\mathbb{A}_0(x)$ is triangular, there is no actual need to separate the solution of the DEQs for $\mathbf{F}(\epsilon, x)$ from the rest of the system, and we can determine, both $\mathbf{F}(\epsilon, x)$ and their subtopologies at once, has we have seen in several cases throughout chapters 6-8. However, the picture changes dramatically if $\mathbb{A}_0(x)$ has no triangular form (and no change of basis which triangularizes it is found). In such cases, the MIs obey irreducible DEQs of degree n.

Let us consider, as an example, a sector with two MIs $\mathrm{F}_1(\epsilon, x)$ and $\mathrm{F}_2(\epsilon, x)$ fulfilling a non-triangular system of DEQs in the $\epsilon \to 0$ limit. If we denote by $\mathbf{H}(x)$ the homogeneous solutions of the DEQs obeyed by $\mathbf{F}(\epsilon, x)$ at $\epsilon = 0$,

$$\begin{cases} \partial_x\, H_1(x) = a_{11}(x) H_1(x) + a_{12}(x) H_2(x) \\ \partial_x\, H_2(x) = a_{21}(x) H_1(x) + a_{22}(x) H_2(x)\,. \end{cases} \tag{9.12}$$

we can differentiate both sides of the first of eqs. (9.12) and make use of

$$H_2(x) = \frac{1}{a_{12}} \left(\partial_x\, H_1(x) - a_{11} H_1(x) \right)\,, \tag{9.13}$$

in order to obtain a second-order DEQ for $H_1(x)$,

$$\begin{aligned} \partial_x^2\, H_1(x) = & \left(a_{11} + \frac{\partial_x a_{12} + a_{12} a_{22}}{a_{12}} \right) \partial_x\, H_1(x) + \\ & + \left(\partial_x a_{11} + a_{12} a_{21} - \frac{\partial_x a_{12} + a_{12} a_{22}}{a_{12}} \right) H_1(x)\,. \end{aligned} \tag{9.14}$$

Therefore, solving the 2×2 coupled system of first-order DEQs (9.12) is equivalent to solve a second-order homogeneous DEQ for $H_1(x)$. Once a solution for $H_1(x)$ is known, $H_2(x)$ can be obtained in terms of $H_1(x)$ and $\partial_x\, H_1(x)$, as prescribed by eq. (9.13). This equivalence generalizes to $n \times n$ systems, which can be cast into n-th order DEQ for the homogeneous solution one of the MIs, say $H_1(x)$

$$\partial_x^{(n)}\, H_1(x) = \sum_{j=0}^{(n-1)} b_i\, \partial_x^i H_1(x)\,, \tag{9.15}$$

and a set of relations expressing the others $(n-1)$ MIs in terms of $H_1(x)$ and its first $(n-1)$ derivatives,

$$H_i(x) = \sum_{j=0}^{n-1} c_{ij}\, \partial_x^{(j)}\, H_1(x)\,, \quad i = 2\,, \dots\,, n\,, \tag{9.16}$$

where $\partial_x^{(0)} H_1(x) \equiv H_1(x)$ and the coefficients b_i and b_{ij} are rational functions of the entries a_{ij} of $\mathbb{A}_0(x)$ and their derivatives.

Eq.(9.15) admits n *independent solutions* $H_1^{(i)}(x)$, $i = 1, \dots, n$. The corresponding n solutions for $H_j^{(i)}(x)$, $j > 1$, are fixed by eq. (9.16). Hence, the solution of the system (9.1) requires the determination of a non-sparse $n \times n$ matrix $\mathbb{G}(x)$, defined by

$$\mathbb{G}(x) = \begin{pmatrix} H_1^{(1)}(x) & \dots & H_1^{(n)}(x) \\ \dots & \dots & \dots \\ H_n^{(1)}(x) & \dots & H_n^{(n)}(x) \end{pmatrix}\,. \tag{9.17}$$

We stress once more that only n out of the $n \times n$ entries of $\mathbb{G}(x)$ are independent and actually need to be computed, since the knowledge of one row of $\mathbb{G}(x)$ allows to determine all others. As we can see from eq. (9.11), the kernels of the iterated integrals that define the Taylor expansion of the inhomogeneous solution around $\epsilon = 0$, depend on $\mathbb{G}(x)$, $\mathbb{A}_0(x)$, and on the inverse matrix $\mathbb{G}^{-1}(x)$,

$$\mathbb{G}^{-1}(x) = \frac{1}{W(x)}\text{adj}(\mathbb{G}(x))\,, \tag{9.18}$$

where $W(x) = \det(\mathbb{G})$ is the Wronskian determinant and $\text{adj}(\mathbb{G}(x))$ is the *adjugate* matrix. The entries of the adjugate matrix are defined through the co-factors of $\mathbb{G}(x)$ and, therefore, they can be written in terms of the homogeneous solutions $H_i^{(j)}$ as well. For instance, in a 2×2 case, we have

$$\mathbb{G}^{-1}(x) = \frac{1}{W(x)}\begin{pmatrix} H_2^{(2)} & -H_1^{(2)} \\ -H_2^{(1)} & H_1^{(1)} \end{pmatrix}. \tag{9.19}$$

In addition, if we recall that for any invertible matrix $\mathbb{B}(x)$

$$\partial_x(\det\mathbb{B}) = \text{tr}\left(\mathbb{B}^{-1}\partial_x\mathbb{B}\right)\det\mathbb{B}\,, \tag{9.20}$$

we can use eq. (9.3) in order to derive a DEQ for $W(x)$,

$$\partial_x W(x) = \text{tr}\left(\mathbb{G}^{-1}\partial_x\mathbb{G}\right)W(x) = \text{tr}\left(\mathbb{G}^{-1}\mathbb{A}_0(x)\mathbb{G}\right)W(x) = \text{tr}\left(\mathbb{A}_0(x)\right)W(x)\,, \tag{9.21}$$

where, in the last equality we have used the ciclic invariance of the trace.Such equation can be easily solved by quadrature and yields to

$$W(x) = W(x_0)e^{\int_{x_0}^{x}\text{d}t\,\text{tr}(\mathbb{A}_0(t))}\,, \tag{9.22}$$

which is known as *Abel theorem*. Therefore, regardless of the complexity of the entries of $\mathbb{G}(x)$, the Wronskian can be expressed in terms of simple functions, since it always fulfils a first order DEQ. In particular we observe that, if we perform a trivial rescaling of the MIs,

$$\mathbf{F}(x) = e^{\int_{x_0}^{x}\text{d}t\,\text{tr}(\mathbb{A}_0(t))}\tilde{\mathbf{F}}(x) \tag{9.23}$$

we obtain a homogeneous system of DEQs

$$\partial_x\tilde{\mathbf{H}}(x) = \left(\mathbb{A}_0(x) - e^{\int_{x_0}^{x}\text{d}t\,\text{tr}(\mathbb{A}_0(t))}\mathbb{1}\right)\tilde{\mathbf{H}}(x) \equiv \tilde{\mathbb{A}_0}(x)\tilde{\mathbf{H}}(x)\,, \tag{9.24}$$

where $\tilde{\mathbb{A}_0}(x)$ is a traceless matrix. Hence, according to eq. (9.21), we have

$$\partial_x W(x) = 0 \quad \to \quad W(x) = W_0\,, \tag{9.25}$$

i.e. it is always possible choose a basis of MIs whose homogeneous solutions have constant Wronskian (which, however, cannot be determined directly from Abel theorem). This simple observation, together with eq. (9.18), implies that the inhomogeneous solution is determined through a nested integration of (products of) the homogeneous solutions $H_i^{(j)}(x)$, convoluted with rational functions originating from the entries of $\mathbb{A}_0(x)$.

In summary, in the presence of a sector of n MIs that obey systems of coupled DEQs, we can determine their Taylor expansion around $\epsilon = 0$ in two steps:

- We first determine a complete set of solutions $\mathbb{G}(x)$ associated to a homogeneous system at $\epsilon = 0$, and we then perform a change of basis of the type (9.4) in order to bring the DEQs into an ϵ-factorized form;

- We recursively determine the Taylor expansion of the inhomogeneous solutions by iteratively integrating over kernel functions that involve the entries of $\mathbb{G}(x)$, as prescribed by eq. (9.11).

This strategy is not different from the one applied in chapter 5 to the full set of MIs: by choosing, within a non-trivial sector, a basis of MIs $\mathbf{F}(\epsilon, x)$ that obey ϵ-linear homogeneous DEQs, we can rotate away the $\mathcal{O}(\epsilon)$ term of homogeneous solution and define a new basis of MIs $\mathbf{I}(\epsilon, x)$ whose system of DEQs decouples order-by-order in the ϵ-expansion. Let us observe that, although the full ϵ-factorization of the system requires a starting matrix $\mathbb{A}(\epsilon, x)$ of the type (9.7), in case of a polynomial dependence of on ϵ the rotation of the MIs through the homogeneous solution matrix is anyway enough to decouple, at each order in ϵ, the coefficients of the Taylor expansion of the MIs.
The whole complexity of a coupled system in $\epsilon = 0$ dwells in the determination of the homogeneous solution matrix $\mathbb{G}(x)$. In fact, as we have already observed in section 5.5, if $\mathbb{A}_0(x)$ is non-triangular, series representation of the homogeneous solution, such the the Magnus exponential, cannot be used to define $\mathbb{G}(x)$, since they cannot be re-summed. One possible way (and, up to now, the only available one) to determine $H_i^{(j)}(x)$ is to attempt a solution of the n-th order homogeneous DEQ (9.15). However, unless eq. (9.15) can be cast into the standard form of some well-studied class of DEQs, this is indeed a difficult task, since no general strategy for the solution of higher-order DEQs is available. In the next section, we propose a systematic way to determine, independently on the size n of the system of DEQs, an integral representation of the complete set of homogeneous solutions.

9.2 Maximal-cuts and homogeneous solutions

In order to understand how to compute the solutions of the homogeneous system of DEQs of the type (9.3), let us go back to the definition of generalized cut of Feynman integrals (2.140), which was introduced in chapter 2. In particular, we consider *maximal-cuts* (2.141), i.e. the simultaneous cut of all denominators that characterize the integral. The maximal-cut of a m-denominator Feynman integral with all propagators raised to power one,

$$I(\epsilon, x; 1\,, \dots\,, 1\,; b_1\,, \dots\,, b_r) = \int \prod_{i=1}^{\ell} \widetilde{\mathrm{d}^d k_j}\, \frac{S_1^{b_1} \cdots S_r^{b_r}}{D_1 \cdots D_m}\,, \tag{9.26}$$

corresponds to the integral

$$\mathrm{MCut}[I(\epsilon, x; 1, 1\,, \dots\,, 1\,; b_1\,, \dots\,, b_r)] = \int \prod_{i=1}^{\ell} \widetilde{\mathrm{d}^d k_j}\, S_1^{b_1} \cdots S_r^{b_r} \delta(D_1) \cdots \delta(D_m)\,, \tag{9.27}$$

and we have seen in section 2.6 how to obtain consistent operative extensions of such definition which accommodate higher powers of the loop propagators, for instance by introducing the Baikov representation of the Feynman integral, as in eq. (2.147). It is an obvious consideration that the same m-denominator cut would vanish if applied to *any* of the subtopologies of $I(\epsilon, x; 1, 1\,, \dots\,, 1\,; b_1\,, \dots\,, b_r)$. In fact, the full set of δ-functions cannot be supported by integrals with fewer denominators. Despite its simplicity, this consideration has striking consequences. If we imagine to apply a maximal-cut to both sides of an IBP and, consequently, of a DEQ of the type (9.1), we would be left with

$$\partial_x \mathrm{MCut}[\mathbf{F}(\epsilon, x)] = \mathbb{A}(\epsilon, x)\, \mathrm{MCut}[\mathbf{F}(\epsilon, x)]\,, \tag{9.28}$$

since all the inhomogeneous terms related to subtopologies would drop,

$$\mathrm{MCut}[\mathbf{S}(\epsilon, x)] = 0\,. \tag{9.29}$$

Therefore, the maximal-cut of a Feynman integral is, *by construction*, a homogeneous solution of the DEQs satisfied by the uncut integral. The connection between unitarity cuts and DEQs is not new. In [102], for example, the second order DEQ satisfied by the two-loop massive sunrise was solved by inferring the homogeneous solution from the calculation of the imaginary part of the graph, which is indeed related to its maximal-cut. More in general, this connection has been largely exploited in the so-called reverse unitarity method [267], while a new way of solving IBPs using the information coming from the unitarity cuts has been proposed in [58]. Finally, in [268] it was observed that maximal-cuts constitute as well homogeneous solutions of the dimensional recurrence relations satisfied by Feynman integrals.

The crucial observation, which, to our believe, has not been fully exploited before, is that the computation of the maximal-cuts can be used as a systematic method to determine an explicit integral representation of the homogeneous solutions, independently on the complexity and of the order of the DEQs under consideration.

However, as argued in the previous section, the definition of the matrix $\mathbb{G}(x)$ requires the knowledge of n independent solutions and the maximal-cut defined in eq. (9.27) provides, apparently, only one of them. Nonetheless, we observe that eq. (9.27) is an incomplete definition of the maximal-cut operation, since it lacks a proper specification of the integration bounds. In fact, as we have discussed in chapter 4, at one-loop the simultaneous cut of all propagators corresponds to a *maximum-cut*, since the number of linear constraints imposed by the on-shell conditions always corresponds to the number of loop variables. Therefore, by replacing each denominator with a δ-function, the loop integral is completely localized.

As already observed in section 2.6, this is not true anymore in the general multi-loop case, where the number of loop variables is larger than the number of constraints imposed by the maximal-cut. This means that, after cutting all propagators, we still have to integrate the cut-integrand over the non-trivial region spanned by the unlocalized variables. Therefore, a more precise definition of the maximal-cut of a ℓ loop Feynman integral with m denominators is

$$\mathrm{MCut}[I(\epsilon, x; 1, 1\,, \dots\,, 1\,; b_1\,, \dots\,, b_r)] = \int_R \prod_{i=1}^{\ell} \widetilde{\mathrm{d}^d k_j}\, S_1^{b_1}\, \cdots\, S_r^{b_r} \delta(D_1)\, \cdots\, \delta(D_m)\,, \tag{9.30}$$

where R is the hypersurface in the space spanned by the loop momenta identified by the condition $D_1 = \dots = D_m = 0$. As we are about to argue, R can be generally decomposed in different patches R_i, $R = R_1 \cup R_2 \cup \dots \cup R_n$, such that the integration over each patch still provides a definition of the maximal-cut that obeys the correct homogeneous IBPs and ensures eq. (9.29) to be satisfied. Thus, the computation of the maximal-cut over *different regions* R_i can provide an integral representation of *different homogeneous solutions.*

In order to understand what are the admissible integration bounds, let us consider the general expression of a d-dimensional maximal-cut after all δ-functions have been evaluated,

$$\mathrm{MCut}[\mathrm{F}_i(d, x)] = C(d, x) \int_{\mathcal{C}_i} \mathrm{d}a_1\, \mathrm{d}a_2 \dots \mathrm{d}a_n P(x; a_1\,, a_2\,, \dots\,, a_n)^{\frac{d-\alpha}{2}}\,, \tag{9.31}$$

where $C(d,x)$ is some function of the space-time dimensions and of the external kinematics x, the integration variables a_i correspond to the ISPs, and α is some integer number, which depends on the topology of the integral, i.e. the number of loops and external legs. Finally, $\mathcal{C}_i$ is the integration contour in the (generally complex) space spanned by the a_i, which corresponds to the integration region R_i in the momentum-space representation of eq. (9.30). This general structure of the maximal-cut, which is common to all the integration methods adopted to perform the cuts, is particular transparent in the Baikov parametrization, where $P(x; a_1, a_2, \ldots, a_n)$ corresponds to the Gram-determinant of internal and external momenta, evaluated in the origin $z_i = 0$ of the hyperplane spanned by the denominators. In particular, in the following discussion, we assume eq. (9.31) to be a representation of the maximal-cut of $\mathrm{F}_i(d,x)$ in terms of a minimum number of integration variables. When using phase-space integration techniques, such a minimal representation is obtained by integrating away all the irrelevant angular directions. Conversely, in the Baikov representation the minimum number of integration variables can obtained through a loop-by-loop parametrization of the integration momenta, as the one derived in [60].

From eq. (9.31) we see that, in an arbitraryl number of dimensions (and, in particular, in the integer dimensions $d = 2, 4$ we are typically interested in) the integrand is a multivalued function with a root-type polydromy. In general the branch cuts of the integrand can have a rather complicated structure, since the position of the branching points $\bar{\mathbf{a}}^{(j)}(x)$,

$$\bar{\mathbf{a}}^{(j)}(x) = \{\bar{a}_1^{(j)}(x), \bar{a}_2^{(j)}(x), \ldots, \bar{a}_n^{(j)}(x)\}, \tag{9.32}$$

which are determined by the zeroes of $P(x; a_1, a_2, \ldots, a_n)$,

$$P(x; \bar{a}_1^{(j)}, \bar{a}_2^{(j)}, \ldots, \bar{a}_n^{(j)}) = 0, \tag{9.33}$$

migrates in the n-dimensional complex manifold described by the a_i, according to the value of the kinematic variable x, as explicitly indicated in eq. (9.32). It is exactly this branch-cut structure that determines what are the admissible integration contours $\mathcal{C}_i$. In fact, we observe that any contour which is delimited by the curves defined by (9.32) and which does not cross any branch cut of the integrand is a a valid integration path, since eq. (9.33) ensures the vanishing of any surface terms that might arise from a total derivative of the cut integrand. Hence, the maximal-cut evaluated along the contour $\mathcal{C}_i$ satisfies the homogeneous part of the IBPs and DEQs of the corresponding uncut integral.

Due to the potential symmetries of $P(x; a_1, a_2, \ldots, a_n)$ under reparametrization of the integration variables a_i (which in turn is inherited from the invariance of the integral under redefinition of the loop momenta) some of the $\mathcal{C}_i$ might be equivalent. However, we conjecture that, given a MIs $\mathrm{F}_i(x)$ satisfying an irreducible n-th order homogeneous DEQ, it is always possible to determine n *independent contours* $\mathcal{C}_i$ which correspond to a complete set of solutions of the homogeneous DEQs. In particular, in $\epsilon = 0$, where the homogeneous eq. (9.28) becomes ($\mathbf{F}(x) \equiv \mathbf{F}(0,x)$)

$$\partial_x \mathrm{MCut}[\mathbf{F}(x)] = \mathbb{A}_0(x)\, \mathrm{MCut}[\mathbf{F}(x)], \tag{9.34}$$

we can identify the j-th homogeneous solution associated to the MI $\mathrm{F}_i(x)$ with

$$H_i^{(j)}(x) = \mathrm{MCut}_{\mathcal{C}_j}[\mathrm{F}_i(x)], \quad j = 1, 2, \ldots n. \tag{9.35}$$

Thus, the homogeneous solution matrix (9.17) can beoperatively identified as

$$\mathbb{G}(x) = \begin{pmatrix} \text{MCut}_{\mathcal{C}_1}[\text{F}_1(x)] & \dots & \text{MCut}_{\mathcal{C}_n}[\text{F}_1(x)] \\ \dots & \dots & \dots \\ \text{MCut}_{\mathcal{C}_1}[\text{F}_n(x)] & \dots & \text{MCut}_{\mathcal{C}_n}[\text{F}_n(x)] \end{pmatrix} . \tag{9.36}$$

Let us stress once more that we always assume all $\mathbf{F}(\epsilon, x)$ to be well behaved at $\epsilon = 0$, so that the integration over the loop momenta and the limit $\epsilon \to 0$ commute and the cuts (9.35) correspond to the limit $\epsilon \to 0$ of eq. (9.31).

Analogous conjectures have been put forward in [62, 63] and, indeed, they are supported by the existence of a connection between the number of MIs of a given sector (which, in the case of a maximally coupled systems, corresponds to the degree of the of DEQs satisfied by the MIs) and the number of inequivalent contours in cut complex manifold identified by eq.(9.33) [269]. In [60, 62, 63] maximal-cuts of several one and two-loop integrals were computed by retaining the full dependence on d. It should be remarked that the DEQs of a sector of MIs are usually maximally coupled, even when $\mathbb{A}_0(x)$ is triangular. Therefore, the d-dimensional homogeneous matrices $\mathbb{G}(\epsilon, x)$ that have been derived, for instance, in [62] are always non-sparse, even in the cases where the MIs are expressible in terms of polylogarithms. Nevertheless it can be checked that, consistently with the triangular form of $\mathbb{A}_0(x)$, $\mathbb{G}(\epsilon, x)$ becomes a sparse matrix at $\epsilon = 0$ and its non-zero entries match (up to some trivial similarity transformation) the homogeneous solutions that can be obtained through the Magnus exponential. In this respect, we stress once more that the whole complexity of the analytic expression of a MI originates from the behaviour of its DEQs in the $\epsilon \to 0$ limit.

The determination of the number of independent contours and their explicit localization is, mathematically speaking, related to the study of the dimension of the so-called cohomology group associated to the variety identified by $P(x; \bar{\mathbf{a}}_1^{(j)}) = 0$. The formal mathematical treatment of this problem is beyond the scope of the present discussion but, in all considered cases, the independent contours could be always determined from considerations based on elementary complex analysis.

Example

Let us consider a paradigmatic example, which we will encounter in several two-loop computations. We suppose that the calculation of a maximal-cut in some finite number of space-time dimensions yields to a one-fold integral of the type

$$\text{MCut}\left(\text{F}(x)\right) = \int_{\mathcal{C}} \frac{da}{\sqrt{\pm R_4(a, x)}} , \tag{9.37}$$

where $R_4(a, x)$ is a fourth-degree polynomial in a with four distinct roots,

$$R_4(a, x) = (a - \bar{a}_1(x))(a - \bar{a}_2(x))(a - \bar{a}_3(x))(a - \bar{a}_4(x)) , \tag{9.38}$$

where $a_i(x)$ are real-valued function of x. The $a_i(x)$ correspond to the branching points of the integrand and their ordering on the real axis depends on the value of x. In the following we will assume, without loss of generality,

$$\bar{a}_1 < \bar{a}_2 < \bar{a}_3 < \bar{a}_4 . \tag{9.39}$$

These four branching points must be connected through branch cuts. Figure 9.2 shows (in red) one possible choice of the branch cuts, according to the sign we pick in the

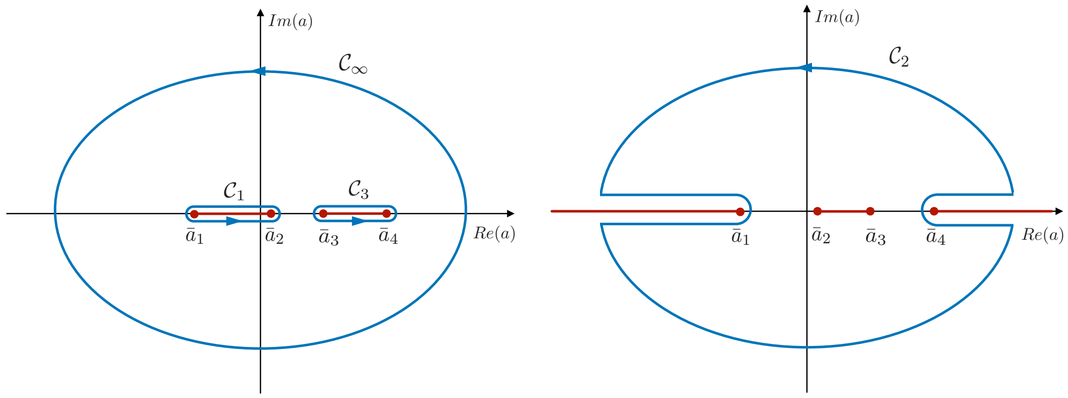


Figure 9.1: Left panel: The contours $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_\infty$. The branches of the integrand for the *positive sign* in the root in Eq. (9.96) are drawn in red. Right panel: The contour $\mathcal{C}_2$. The branches of the integrand for the *negative sign* in the root in Eq. (9.96) are drawn in red.

definition of the integrand (9.37). By following the above reasoning, we can identify three possible contours, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, which enclose a branch cut (i.e. that are limited by a couple of branching points), without crossing the other one. Therefore eq. (9.37), when it is evaluated along any of these contours, provides a solution of the homogeneous DEQ satisfied by $\mathrm{F}(x)$. However, it is easy to show that $\mathcal{C}_1$ and $\mathcal{C}_3$ are not independent. In fact, as it was already observed [102], we can introduce a fourth auxiliary contour $\mathcal{C}_\infty$, which, as shown in figure 9.2, encloses the point $a = \infty$. The integral along $\mathcal{C}_\infty$ is vanishing, since the integrand goes as $1/a^2$ when $a \to \infty$,

$$\oint_{\mathcal{C}_\infty} \frac{da}{\sqrt{R_4(a,x)}} = 0\,. \tag{9.40}$$

Moreover, the integrand is analytic everywhere but along the branch cuts. Therefore, we can to shrink $\mathcal{C}_\infty$ to encircle the two branch cuts and obtain from eq. (9.40)

$$\oint_{\mathcal{C}_\infty} \frac{da}{\sqrt{R_4(a,x)}} = \oint_{\mathcal{C}_1} \frac{da}{\sqrt{R_4(a,x)}} + \oint_{\mathcal{C}_3} \frac{db}{\sqrt{R_4(b,x)}} = 0\,, \tag{9.41}$$

which proves the equivalence between $\mathcal{C}_1$ and $\mathcal{C}_3$. In this way, we are left with two independent contours, say $\mathcal{C}_1$ and $\mathcal{C}_2$, that provide precisely two independent solutions of the DEQ satisfied by $\mathrm{F}(x)$. Now, by shrinking $\mathcal{C}_1$ and $\mathcal{C}_2$ around the corresponding branch cuts, we find an equivalent representation of the homogeneous solutions as one-dimensional real integrals,

$$\begin{aligned}
\mathrm{MCut}_{\mathcal{C}_1}[F(x)] &= \oint_{\mathcal{C}_1} \frac{da}{\sqrt{R_4(a,x)}} = 2\,i \int_{\bar{a}_1}^{\bar{a}_2} \frac{da}{\sqrt{-R_4(a,x)}} = -2\,i \int_{\bar{a}_3}^{\bar{a}_3} \frac{da}{\sqrt{-R_4(a,x)}}\,,\\
\mathrm{MCut}_{\mathcal{C}_2}[F(x)] &= \oint_{\mathcal{C}_2} \frac{da}{\sqrt{-R_4(a,x)}} = 2\,i \int_{\bar{a}_2}^{\bar{a}_3} \frac{da}{\sqrt{R_4(a,x)}} = 2\,i \left(\int_{-\infty}^{\bar{a}_1} + \int_{\bar{a}_4}^{\infty}\right) \frac{da}{\sqrt{R_4(a,x)}}\,,
\end{aligned} \tag{9.42}$$

where the sign of the roots on the r.h.s. in order to keep the integrals real-valued. Finally, we observe that the integrals defined in eqs. (9.42) can be expressed in terms *elliptic integrals* of the first kind,

$$\mathrm{K}(w) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-w\,z^2)}} \quad \text{with} \quad \Re(w) < 1\,, \tag{9.43}$$

In fact, by performing the change of variables

$$t^2 = \frac{(\bar{a}_4 - \bar{a}_2)(a - \bar{a}_1)}{(\bar{a}_2 - \bar{a}_1)(\bar{a}_4 - a)}\,,$$
$$t^2 = \frac{(\bar{a}_1 - \bar{a}_3)(a - \bar{a}_2)}{(\bar{a}_3 - \bar{a}_2)(\bar{a}_1 - a)}\,, \tag{9.44}$$

which map, respectively, the integration ranges $[\bar{a}_1, \bar{a}_2]$ and $[\bar{a}_2, \bar{a}_3]$ into $[0, 1]$, we obtain

$$\text{MCut}_{\mathcal{C}_1}[F(x)] = \frac{2}{\sqrt{(\bar{a}_3 - \bar{a}_1)(\bar{a}_4 - \bar{a}_2)}} \text{K}\,(w)\,, \tag{9.45}$$
$$\text{MCut}_{\mathcal{C}_2}[F(x)] = \frac{2}{\sqrt{(\bar{a}_3 - \bar{a}_1)(\bar{a}_4 - \bar{a}_2)}} \text{K}\,(1-w)\,, \tag{9.46}$$

where

$$w = \frac{(\bar{a}_2 - \bar{a}_1)(\bar{a}_4 - \bar{a}_3)}{(\bar{a}_3 - \bar{a}_1)(\bar{a}_4 - \bar{a}_2)}\,. \tag{9.47}$$

The fact that eqs. (9.45) are solutions of the same DEQ is consistent with the theory of complete elliptic integrals, which shows that $\text{K}(w)$ and $\text{K}(1-w)$ are two independent solutions of the same second-order DEQ,

$$\partial_w^2 \,\text{K}(w) + \left(\frac{1}{w} - \frac{1}{1-w}\right) \partial_w^2 \,\text{K}(w) - \frac{1}{4}\left(\frac{1}{w} + \frac{1}{1-w}\right) \text{K}(w) = 0\,. \tag{9.48}$$

□

As we will see in the next sections, all the known cases of two-loop integrals that do not evaluate to multiple polylogarithms (or, more generally, to Chen iterated integrals), are found to admit a representation of their maximal-cut of the type given in eq. (9.37) and, hence, to fulfil a second-order DEQ which fall in the class of eq. (9.48). This evidence might suggest that, at least at two-loop level, the theory of elliptic integrals could be sufficient to provide the homogeneous solutions to all coupled systems of DEQs.

Nonetheless, at higher-loop level one can encounter larger systems of coupled DEQs, the first example being the three-loop massive banana graph, which requires the solution of 3×3 system of coupled DEQs. In its generality, the proposed method can be used to determine a representation of the homogeneous solutions of a coupled system of DEQs, independently on the size of the system and, even more importantly, it provides an explicit integral representation of the homogeneous solution also when the properties of the corresponding higher-order DEQ are unknown.

9.2.1 Unit leading singularity

Before moving to the study of some explicit examples, we conclude this section with a simple observation on the connection between the leading singularity of a set of coupled MIs and the solution of their homogeneous system of DEQs. For simplicity, let us consider a 2×2 case, and let us invert the relation between the ϵ-linear basis $\mathbf{F}(x)$ and the ϵ-factorized basis $\mathbf{I}(x)$, given in eq. (9.4),

$$\begin{pmatrix} \text{I}_1(x) \\ \text{I}_2(x) \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} H_2^{(2)} & -H_1^{(2)} \\ -H_2^{(1)} & H_1^{(1)} \end{pmatrix} \begin{pmatrix} \text{F}_1(x) \\ \text{F}_2(x) \end{pmatrix}, \tag{9.49}$$

with $W(x) = H_1^{(1)} H_2^{(2)} - H_1^{(2)} H_2^{(1)}$. If we recall that, by definition, $\text{MCut}_{\mathcal{C}_j}[\text{F}_i] = H_i^{(j)}$, we can cut both sides of eq. (9.49) along the first contour and obtain

$$\begin{pmatrix} \text{MCut}_{\mathcal{C}_1}[\text{I}_1(x)] \\ \text{MCut}_{\mathcal{C}_1}[\text{I}_2(x)] \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} H_2^{(2)} & -H_1^{(2)} \\ -H_2^{(1)} & H_1^{(1)} \end{pmatrix} \begin{pmatrix} H_1^{(1)} \\ H_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{9.50}$$

Similarly, by cutting on the second contour, we have

$$\begin{pmatrix} \text{MCut}_{\mathcal{C}_2}[\text{I}_1(x)] \\ \text{MCut}_{\mathcal{C}_2}[\text{I}_2(x)] \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} H_2^{(2)} & -H_1^{(2)} \\ -H_2^{(1)} & H_1^{(1)} \end{pmatrix} \begin{pmatrix} H_1^{(2)} \\ H_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.51}$$

This result obviously extends to systems of arbitrary size: the maximal-cuts of the the MIs $\mathbf{I}(x)$ along any of the n contours $\mathcal{C}_i$ is either 1 or 0. More precisely, we have

$$\text{MCut}_{\mathcal{C}_i}[\text{I}_j(x)] = \delta_{ij}\,. \tag{9.52}$$

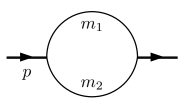
As we have seen in section 5.4, in [81] it was conjectured that MIs with unit leading singularity obey canonical system of DEQs. In this perspective, eq. (9.52) can be interpreted as a generalization of such conjecture to the case of coupled DEQs. In the presence of n coupled DEQs, we can associate to each MI n distinct leading singularities, corresponding to the maximal-cut of the MI evaluated along the n independent integration contours, that is to say to the n independent solutions of the corresponding homogeneous equation. Therefore, if we identify the maximal-cut matrix $\mathbb{G}(x)$ as the *leading-singularity matrix*, we immediately see from eq. (9.52) that a basis of MIs with leading singularity matrix equal to the identity matrix satisfies an ϵ-factorized homogeneous system of DEQs.

9.3 One-loop maximal-cuts

In this section, we compute the maximal-cut of a series of massive one-loop integrals and we show that they obey the homogeneous part of the DEQs satisfied by the corresponding MIs. At one loop, for any topology, we always find at most one MIs. This implies that one-loop integrals satisfy at most first-order DEQs. Consistently, the maximal-cut of a one-loop integral provides a unique homogeneous solution, since it localizes completely the loop momentum without leaving any additional integration to be performed. The maximal-cuts discussed in this section are first derived in arbitrary continuous dimension d and their value at integer dimensions $d = 2\,, 4$ (where they happen to be finite) is then derived from the $\epsilon \to 0$ limit of the resulting expressions. The results hereby derived will be used in the next section for the computation of maximal-cuts of two-loop integrals which contain the considered one-loop topologies as sub-loops.

One-loop massive bubble

As a first elementary example, we consider a one-loop bubble with arbitrary masses,

$$\text{F}(d; s, m_1^2, m_2^2) =$$ 

$$= \int \widetilde{\text{d}^d k} \frac{1}{(k^2 - m_1^2)((k-p)^2 - m_2^2)}\,, \tag{9.53}$$

where $p^2 = s$. This integral satisfies first-order DEQs in s, m_1^2 and m_2^2.

In arbitrary d the one in s reads

$$\partial_s \text{F}(\epsilon; s, m_1^2, m_2^2) = \text{S}(d, s, m_1^2, m_2^2)$$

$$+\frac{1}{2}\left(\frac{3-d}{(m_1-m_2)^2-s}+\frac{3-d}{(m_1+m_2)^2-s}+\frac{2-d}{s}\right)\partial_s \mathrm{F}(\epsilon; s, m_1^2, m_2^2)\ , \tag{9.54}$$

where the inhomogeneous term $\mathrm{S}(d, s, m_1^2, m_2^2)$ depend on the tadpoles with masses m_1 and m_2. The homogeneous part of eq. (9.54)

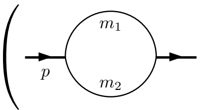
$$\partial_s H(d, s, m_1^2, m_2^2) = \frac{1}{2}\left(\frac{3-d}{(m_1-m_2)^2-s}+\frac{3-d}{(m_1+m_2)^2-s}+\frac{2-d}{s}\right) H(d, s, m_1^2, m_2^2) \tag{9.55}$$

can be easily integrated and yields to

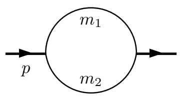
$$H(d, s, m_1^2, m_2^2) = c\,(-s)^{1-\frac{d}{2}}\left(m_1^4 - 2m_1^2\left(m_2^2+s\right)+\left(m_2^2-s\right)^2\right)^{\frac{d-3}{2}}\ , \tag{9.56}$$

where c is a multiplicative constant. This result can be recovered by computing the the d-dimensional maximal-cut of the one-loop bubble,

$$\mathrm{MCut}\left(\ \right) = \int \widetilde{\mathrm{d}^d k}\ \delta(k^2-m^2)\delta((k-p)^2-m^2)\,. \tag{9.57}$$



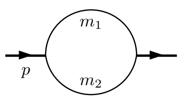
The integral can be evaluated either by moving to the rest frame of the external particle, $p^\alpha = (\sqrt{s}, \vec{0})$, or by introducing the Baikov representation of the bubble integral,



$$= (-s)^{1-\frac{d}{2}} \int \frac{\mathrm{d}z_1 \mathrm{d}z_2}{z_1 z_2}\left(-4s(m_1^2+z_1)+(m_1^2-m_2^2+s+z_1-z_2)^2\right)^{\frac{d-3}{2}}\ , \tag{9.58}$$

where we have omitted the irrelevant overall constant and labelled the denominators as $z_1 = k^2 - m_1^2$ and $z_2 = (k-p)^2 - m_2^2$. The maximal-cut of $F(d, s, m_1^2, m_2^2)$ is obtained from eq. (9.58) by replacing $1/z_i \to \delta(z_i)$,

$$\mathrm{MCut}\left(\ \right) = (-s)^{1-\frac{d}{2}}\left(m_1^4 - 2m_1^2\left(m_2^2+s\right)+\left(m_2^2-s\right)^2\right)^{\frac{d-3}{2}}\ , \tag{9.59}$$



and it correctly reproduces the homogeneous solution $H(d, s, m_1^2, m_2^2)$, given in eq. (9.56). A comment on eq. (9.59) is in order. In general, if a Feynman integral depends on more than one kinematic scale, the solution of the DEQ in one single invariant cannot capture the full dependence on all the remaining scales. For instance, in the present case, if we had solved only homogeneous DEQ in m_1^2, we would have not been able to determine the overall dependence on $(-s)^{(1-d/2)}$ of eq. (9.59). Nevertheless, the maximal-cut provides, by definition, the homogeneous solution of the DEQs in *all* kinematic invariants and, hence, it resolves its dependence on the whole set of scales.

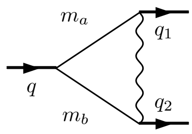
Finally, for later convenience, we extract the maximal-cut of an equal-mass bubble in $d = 2$. Since the massive bubble integral is well behaved in two-dimensions, we can read such maximal-cut directly form eq. (9.56), by setting $d = 2$ and $m_1 = m_2 = m$,

$$H(d=2, s, m^2, m^2) = \frac{1}{\sqrt{s(s-4m^2)}}\,. \tag{9.60}$$

One-loop massive triangle

As a second example, we consider a one-loop triangle in $d = 4 - 2\epsilon$ with two massive internal propagators,

$$\mathrm{F}_1(\epsilon; q^2, q_1^2, q_2^2, m_a^2, m_b^2) =$$



$$= \int \widetilde{\mathrm{d}^d k} \frac{1}{D_1 D_2 D_3}\,, \tag{9.61}$$

where the denominators are defined as

$$D_1 = k^2 - m_a^2\,, \quad D_2 = (k - q_1)^2\,, \quad D_3 = (k - q_1 - q_2)^2 - m_b^2\,, \tag{9.62}$$

and, in total generality, $q^2 \neq q_1^2 \neq q_2^2 \neq 0$. IBPs reduction allow us to obtain first-order DEQs for F_1 in any of the external invariants. For instance, if we differentiate w.r.t. q^2, we obtain (by neglecting all the inhomogeneous terms),

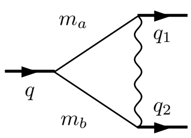
$$\begin{aligned}\partial_{q^2} H_1(\epsilon; q^2, q_1^2, q_2^2, m_a^2, m_b^2) = &\frac{(q_1^2 + q_2^2 - q^2)}{q_1^4 - 2q_1^2(q_2^2 + q^2) + (q_2^2 - q^2)^2} H_1(\epsilon; q^2, q_1^2, q_2^2, m_a^2, m_b^2) \\ &+ \epsilon\, C(q^2, q_1^2, q_2^2, m_a^2, m_b^2)\, H_1(\epsilon; q^2, q_1^2, q_2^2, m_a^2, m_b^2)\,,\end{aligned} \tag{9.63}$$

where $C(q^2, q_1^2, q_2^2, m_a^2, m_b^2)$ is a cumbersome rational function.

In the simpler case where $m_b = m_a = m$, the latter reads

$$\begin{aligned}C(q^2, q_1^2, &q_2^2, m^2) = C(q^2, q_1^2, q_2^2, m^2, m^2) \\ &= \frac{((m^2 + q_1^2)(q_1^2 - q_2^2) + q^2(m^2 - q_1^2))((m^2 + q_2^2)(q_1^2 - q_2^2) + q^2(q_2^2 - m^2))}{2\left(q_1^4 - 2q_1^2(q_2^2 + q^2) + (q_2^2 - q^2)^2\right)\left(q^2(m^2 - q_1^2)(m^2 - q_2^2) + m^2(q_1^2 - q_2^2)^2\right)}\,.\end{aligned} \tag{9.64}$$

The solution eq. (9.63) can determined by computing the d-dimensional maximal-cut of $\mathrm{F}_1(\epsilon; q^2, q_1^2, q_2^2, m_a^2, m_b^2)$. In this case, it is particularly convenient to adopt the Baikov parametrization and rewrite eq. (9.61) as



$$= G(q_1, q_2)^{\frac{3-d}{2}} \int \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 G(k, q_1, q_2)^{\frac{d-4}{2}} \frac{1}{z_1 z_2 z_3}\,, \tag{9.65}$$

where $z_i = D_i$ and $G(v_1, \ldots, v_n)$ is the Gram determinant of the vectors $v_1, \ldots, v_n$. In particular, for $G(q_1, q_2)$, we have (omitting normalization factors)

$$G(q_1, q_2) = -\frac{1}{4}\left((q^2 + q_1^2 - q_2^2)^2 - 4\, q_1^2\, q_2^2\right)\,. \tag{9.66}$$

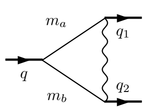
By using the definition of the denominators given in eq. (9.3), we can write the scalar products k^2, $k \cdot q_1$ and $k \cdot q_2$ in terms of the z_i variables and express $G(k, q_1, q_2)$ as a polynomial in the latter,

$$\begin{aligned}G(k, q_1, q_2) =&\frac{1}{4}(q_1^2\left(m_a^2 + q_1^2 + z_1 - z_2\right)\left(m_b^2 + q_1^2 - q^2 - z_2 + z_3\right) \\ &+ q_2^2\left(m_a^2 + q_1^2 + z_1 - z_2\right)\left(m_b^2 + q_1^2 - q^2 - z_2 + z_3\right) \\ &- q^2\left(m_a^2 + q_1^2 + z_1 - z_2\right)\left(m_b^2 + q_1^2 - q^2 - z_2 + z_3\right)\end{aligned}$$

$$
\begin{aligned}
&- q_2^2\left(m_a^2+q_1^2+z_1-z_2\right)^2+2q_1^2q_2^2\left(m_a^2+z_1\right)\\
&- q_1^4\left(m_a^2+z_1\right)-q_2^4\left(m_a^2+z_1\right)+2q_1^2q^2\left(m_a^2+z_1\right)\\
&+ 2q_2^2q^2\left(m_a^2+z_1\right)-q^4\left(m_a^2+z_1\right)\\
&- q_1^2\left(m_b^2+q_1^2-q^2-z_2+z_3\right)^2\,\big)\,. \qquad (9.67)
\end{aligned}
$$

The maximal-cut $\mathrm{F}_1(\epsilon;q^2,q_1^2,q_2^2,m_a^2,m_b^2)$ is obtained by simply replacing $1/z_i\to\ \delta(z_i)$ in eq. (9.65),

$$\mathrm{MCut}\left(\text{[diagram]}\right)=G(q_1,q_2)^{\frac{3-d}{2}}\int \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\,G(k,q_1,q_2)^{\frac{4-d}{2}}\delta(z_1)\delta(z_2)\delta(z_3)$$



$$
\begin{aligned}
&=\left((q^2+q_1^2-q_2^2)^2-4\;q_1^2\,q_2^2)\right)^{\frac{3-d}{2}}\times\big(m_a^2(m_b^2(q_1^2+q_2^2-q^2)\\
&+q_2^2(q_1^2-q_2^2+q^2))+q_1^2q^2(m_b^2-q_2^2)-m_b^2q_1^2(m_b^2+q_1^2-q_2^2)-m_a^4\,q_2^2\big)^{\frac{d-4}{2}}\,. \qquad (9.68)
\end{aligned}
$$

By direct computation, we can immediately verify that eq. (9.68) is exactly a solution of (9.69), as well as of the DEQs in the other invariants.

Although in this simple case we are able to determine a closed expression for the maximal-cut in arbitrary dimensions, it is interesting to see what happens in the $\epsilon\to 0$ limit. Quite remarkably, we observe that, in such limit, that entire dependence on the masses m_a and m_b disappears from the homogeneous DEQ,

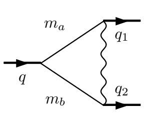
$$\partial_{q^2}\,H(0;q^2,q_1^2,q_2^2,m_a^2,m_b^2)=\frac{(q_1^2+q_2^2-q^2)}{q_1^4-2q_1^2(q_2^2+q^2)+(q_2^2-q^2)^2}H(0;q^2,q_1^2,q_2^2,m_a^2,m_b^2)\,, \qquad (9.69)$$

which can be readily solved by quadrature,

$$H_1(0;q^2,q_1^2,q_2^2,m_a^2,m_b^2)=\frac{c}{\sqrt{(q^2+q_1^2-q_2^2)^2-4\;q^2\,q_1^2}}\,, \qquad (9.70)$$

where c is an arbitrary integration constant. This result is, of course, in agreement with the four dimensional limit of eq.(9.68): the full dependence of the maximal-cut on the internal masses is retained in $G(k,q_1,q_2)$, which is raised to the power $(d-4)/2$. Therefore, we obtain

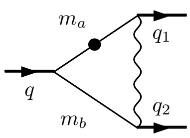
$$\mathrm{MCut}\left(\text{[diagram]}\right)\Bigg|_{\epsilon=0}=\frac{1}{\sqrt{(q^2+q_1^2-q_2^2)^2-4\;q_1^2\,q_2^2}}\,, \qquad (9.71)$$



which exactly corresponds to $H_1(0;q^2,q_1^2,q_2^2,m_a^2,m_b^2)$. This result has some important consequences.

Let us consider a similar three-point integral, but with one of the two massive propagators raised to power to, for example

$$\mathrm{F}_2(\epsilon;q^2,q_1^2,q_2^2,m_a^2,m_b^2)=\text{[diagram]}=\int\widetilde{\mathrm{d}^dk}\frac{1}{D_1^2D_2D_3} \qquad (9.72)$$



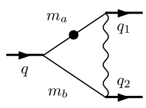
such that

$$\mathrm{F}_2(\epsilon;q^2,q_1^2,q_2^2,m_a^2,m_b^2)=\partial_{m_a^2}\mathrm{F}_1(\epsilon;q^2,q_1^2,q_2^2,m_a^2,m_b^2)\,.$$

By following the prescriptions given in section (2.6) for computing the maximal-cut in the presence of higher powers of denominators, we can identify

$$\mathrm{MCut}\left[\mathrm{F}_2(\epsilon; q^2, q_1^2, q_2^2, m_a^2, m_b^2)\right] = \partial_{m_a^2} \mathrm{MCut}\left[\mathrm{F}_1(\epsilon; q^2, q_1^2, q_2^2, m_a^2, m_b^2)\right] . \qquad (9.73)$$

Thus, if we specialize the above identity to $\epsilon = 0$ and make use of eq. (9.71), we obtain

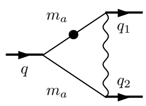
MCut (

) $\Big|_{\epsilon=0}$

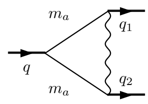
$$= \partial_{m_a^2} \frac{1}{\sqrt{(q^2 + q_1^2 - q_2^2)^2 - 4\, q^2\, q_1^2}} = 0\,. \qquad (9.74)$$

We should stress once more that this result is strictly valid in four dimensions. In fact, we see from eq. (9.65) that by taking a derivative w.r.t. m_a^2 we produce an overall factor $(d-4)$, which goes to zero as $\epsilon \to 0$.

A similar conclusion can be drawn inspecting the integration by parts identities. For simplicity, let us consider again the equal mass case, $m_b = m_a = m$. From IBPs we can show that

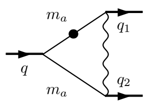


$$= \frac{(d-4)((m^2 + q_2^2)(q_2^2 - q_1^2) + q^2(m^2 - q_2^2))}{2q^2(m^2 - q_1^2)(m^2 - q_2^2) + 2m^2(q_1^2 - q_2^2)^2}$$



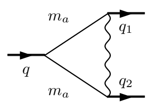
$$+ \text{subtopologies}\,. \qquad (9.75)$$

Clearly, when we apply the maximal-cut to the r.h.s. of eq. (9.75), all subtopologies drop and we are left with

MCut (

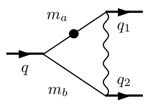
)

$$= \frac{\epsilon((m^2 + q_2^2)(q_2^2 - q_1^2) + q^2(m^2 - q_2^2))}{2q^2(m^2 - q_1^2)(m^2 - q_2^2) + 2m^2(q_1^2 - q_2^2)^2}$$

MCut (

) . (9.76)

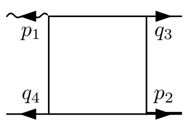
This identity, together with the $m_a = m_b$ limit of eq. (9.68), allows us to compute the d-dimensional maximal-cut of the triangle integral with a squared propagator. Consistently, since the r.h.s. of eq. (9.76) is proportional to ϵ, the four-dimensional limit we find again

MCut (

) $\Big|_{\epsilon=0}$ $= 0\,,.$ (9.77)

One-loop massive box

As a last one-loop example, we consider a massive box in $d = 4 - 2\epsilon$,

$\mathrm{F}(\epsilon; s, t, p_2^2, q_3^2, q_4^2, m^2) =$ 

$$= \int \widetilde{\mathrm{d}^d k} \frac{1}{D_1 D_2 D_3 D_4}\,, \qquad (9.78)$$

where the denominators are chosen to be

$$D_1 = k^2 - m^2\,, \quad D_2 = (k - p_1)^2 - m^2\,,$$

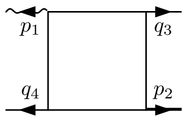
$$D_3 = (k - p_1 - q_3)^2 - m^2\,, \quad D_4 = (k - p_1 - q_3 - p_2)^2 - m^2\,, \tag{9.79}$$

and the kinematics is $p_1^2 = 0$, $p_2^2 \neq q_3^2 \neq q_4^2$. The Mandelstam invariants are given by

$$s = (p_1 + q_3)^2\,, \quad t = (q_3 + p_2)^2\,, \quad u = (p_1 + p_2)^2 = -s - t - p_2^2 - q_3^2 - q_4^2\,. \tag{9.80}$$

IBPs reduction allow us to obtain, in arbitrary d, first-order DEQs in any of the external invariants as well as in m^2, which we do not report explicitly. The corresponding homogeneous solution can determined by computing the d-dimensional maximal-cut of $\mathrm{F}(\epsilon; s, t, p_2^2, q_3^2, q_4^2, m^2)$.

In Baikov parametrization, eq. (9.78) is rewritten (up to an irrelevant overall constant) as



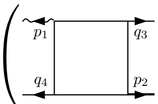
$$= G(p_1, q_3, p_2)^{\frac{4-d}{2}} \int \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}z_4 G(z_i)^{\frac{d-5}{2}} \frac{1}{z_1 z_2 z_3 z_4}\,, \tag{9.81}$$

where $z_i = D_i$ and

$$G(p_1, q_3, p_2) = \frac{1}{4}\left(p_2^2(q_3^2 - s)(q_4^2 - t) - (q_3^2 + q_4^2 - s - t)(q_3^2 q_4^2 - st)\right). \tag{9.82}$$

For brevity, we do not give the explicit expression of the polynomial $G(z_i)$, which can be easily obtained by rewriting the Gram determinant $G(k, q_1, q_2)$ in terms of the z_i variables. The maximal-cut $\mathrm{F}(\epsilon; s, t, p_2^2, q_3^2, q_4^2, m^2)$ is now obtained by simply replacing $1/z_i \to \delta(z_i)$,

$$\mathrm{MCut}\left(\right)$$



$$\begin{aligned} &= G(p_1, p_2, q_3)^{\frac{4-d}{2}} \int \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}z_4\, G(z_i)^{\frac{d-5}{2}} \delta(z_1)\delta(z_2)\delta(z_3)\delta(z_4) \\ &= \left(p_2^2(q_3^2 - s)(q_4^2 - t) - (q_3^2 + q_4^2 - s - t)(q_3^2 q_4^2 - st)\right)^{\frac{4-d}{2}} \times \\ &\quad \left(4m^2(p_2^2(q_3^2 - s)(q_4^2 - t) - (q_3^2 + q_4^2 - s - t)(q_3^2 q_4^2 - st)) + (q_3^2 q_4^2 - st)^2\right)^{\frac{d-5}{2}}. \end{aligned} \tag{9.83}$$

By direct computation, we have verified that eq. (9.83) solves the DEQs in the all kinematic invariants. Finally, since the box integral is well-behaved in four-dimensions, we can extract the value of the four-dimensional maximal-cut from the smooth $\epsilon \to 0$ limit of eq. (9.83). At $\epsilon = 0$ the contribution of $G(p_1, q_3, p_2)$ drops and we are left with

$$H(0; s, t, p_2^2, q_3^2, q_4^2, m^2) = \frac{1}{\sqrt{P(s, t, p_2^2, q_3^2, q_4^2)}}\,, \tag{9.84}$$

where we have defined

$$P(s, t, p_2^2, q_3^2, q_4^2) = 4m^2(p_2^2(q_3^2 - s)(q_4^2 - t) - (q_3^2 + q_4^2 - s - t)(q_3^2 q_4^2 - st)) + (q_3^2 q_4^2 - st)^2\,. \tag{9.85}$$

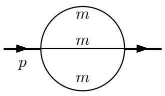
9.4 Two-loop maximal-cuts

In this section, we consider a few examples of two-loop integrals which fulfil irreducible second-order DEQs and we show how to obtain a complete set of homogeneous solutions by computing the maximal-cut of the MIs along different, independent contours. Once the homogeneous solutions are known, Euler variation of constants can be used in order to write down an integral representation of the full inhomogeneous solution.

The elliptic sunrise

We start by discussing the homogeneous solutions the two-loop sunrise with equal masses, which was the first discovered example of Feynman integral that does not evaluate to multiple polylogarithms [102].

The integral family of the two-loop sunrise is defined as ($a_i \geq 0$)



$$= I_{a_1\,a_2\,a_3\,a_4\,a_4} = \int \widetilde{\mathrm{d}^d k_1}\widetilde{\mathrm{d}^d k_2} \frac{(k_1\cdot p)^{a_4}(k_2\cdot p)^{a_5}}{(k_1^2-m^2)^{a_1}(k_2^2-m^2)^{a_2}((k_2-k_1+p)^2-m^2)^{a_3}}. \tag{9.86}$$

The three-denominator integrals belonging to the family (9.86) are reduced to two MIs $\mathbf{F}(\epsilon, x)$ via IBPs. We choose to work with the basis

$$\mathrm{F}_1(\epsilon,x) = (1+2\epsilon) I_{1\,1\,1\,0\,0}\,, \qquad \mathrm{F}_2(\epsilon,x) = m^2 I_{2\,1\,1\,0\,0}\,. \tag{9.87}$$

We study the DEQs in $d = 2-2\epsilon$, since the expansion of these MIs around $d = 2$ turns out to be simpler than the one around $d = 4$. The latter which can be nevertheless recovered by means of dimensional recurrence relations [270].

If we neglect the subtopology $I_{2\,0\,2\,0\,0}$, the two MIs satisfy an ϵ-linear coupled system of DEQs in the variable $x = s/m^2$ (we hereby indicate with $\mathbf{H}$ the vector of homogeneous solutions associated to F), *where*A_i(x)*are*2 ×2 matrices that do not depend on ϵ,

$$\mathbb{A}_0(x) = \begin{pmatrix} -\frac{1}{x} & -\frac{3}{x} \\ -\frac{1}{4(x-1)} + \frac{1}{3x} - \frac{1}{12(x-9)} & -\frac{1}{x-1} + \frac{1}{x} - \frac{1}{x-9} \end{pmatrix},$$

$$\mathbb{A}_1(x) = \begin{pmatrix} -\frac{2}{x} & -\frac{6}{x} \\ -\frac{3}{4(x-1)} + \frac{1}{x} - \frac{1}{4(x-9)} & -\frac{2}{x-1} + \frac{3}{x} - \frac{2}{x-9} \end{pmatrix}. \tag{9.88}$$

In $\epsilon = 0$ the system reduces to ($\mathbf{H}(0,x) \equiv \mathbf{H}(x)$)

$$\partial_x \mathbf{H}(x) = \mathbb{A}_0(x)\, \mathbf{H}(x)\,, \tag{9.89}$$

and, according to eq. (9.14), it can be rephrased as second-order homogeneous DEQ for one of the two MIs, say $H_1(x)$,

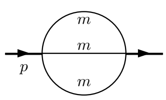
$$\left[\partial_x^2 + \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-9}\right)\partial_x + \left(-\frac{1}{3x} + \frac{1}{4(x-1)} + \frac{1}{12(x-9)}\right)\right] H_1(x) = 0\,, \tag{9.90}$$

and a linear relation which expresses $H_2(x)$ in terms of $H_1(x)$ and its derivative,

$$H_2(x) = -\frac{1}{3}(1 + x\partial_x) H_1(x)\,. \tag{9.91}$$

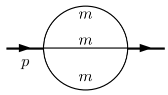
The two independent solutions of eq. (9.90) can be found from the direct computation of the maximal-cut of $\mathrm{F}_1(x)$,

MCut



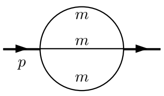
$$= \int \widetilde{\mathrm{d}^2 k_1}\delta(k_1^2 - m^2) \int \widetilde{\mathrm{d}^2 k_2}\delta(k_2^2 - m^2)\delta\left((k_2 - k_1 + p)^2 - m^2\right). \tag{9.92}$$

Has we have explicitly indicated in eq. (9.92), we can proceed by first using two δ-functions in order to localize the integral over k_2. This amounts to compute the maximal-cut of a one-loop bubble with external momentum $(p - k_1)$. Hence, according to eq. (9.60), we have



$$\mathrm{MCut}\left(\quad \right) = \int \widetilde{\mathrm{d}^2 k_1} \frac{\delta(k_1^2 - m^2)}{\sqrt{(k_1 - p)^2 \left((k_1 - p)^2 - 4m^2 \right)}} \,. \tag{9.93}$$

The residual integrand depends on k_1^2 and $k_1 \cdot p$. Therefore, we can adopt the Baikov parametrization of a one-loop two-point function for k_1. This nested computation of the cut of multi-loop Feynman integrals is equivalent to the loop-by-loop construction of Baikov parametrization, which has been introduced in [60]. In addition, since $\mathrm{F}_1(x)$ is well behaved in $d = 2$, we can obtain the two-dimensional result directly from the $d \to 2$ limit of the d-dimensional expression,



$$\mathrm{MCut}\left(\quad \right) = \lim_{d \to 2} (s)^{\frac{2-d}{2}} \int \mathrm{d}z_1 \mathrm{d}z_2 \frac{\delta(z_1)}{\sqrt{z_2 (z_2 - 4m^2)}} G(z_1, z_2)^{\frac{d-3}{2}} \,, \tag{9.94}$$

where we have introduced $z_1 = k_1^2 - m^2$ and $z_2 = (k_1 - p)^2$. In eq. (9.94), $G(z_1, z_2)$ is the polynomial in z_i originated from the Gram determinant $G(k_1, p)$,

$$G(z_1, z_2) = 4s(z_1 + m^2) - (m^2 + s + z_2 - z_2)^2 \,. \tag{9.95}$$

The integration over z_1 has now become trivial. By defining $a = z_2/m^2$, we obtain

$$\mathrm{MCut}\left(S(u)\right) = \oint \frac{da}{\sqrt{\pm a\,(a-4)\,(a - (\sqrt{u} - 1)^2)\,(a - (\sqrt{u} + 1)^2)}} = \oint_{\mathcal{C}} \frac{da}{\sqrt{\pm R_4(a, x)}} \,, \tag{9.96}$$

which matches the integral of eq. (9.37).

The position of the four branching-points of $R_4(a, x)$ depends on the values of x. If we suppose, for definiteness, $x > 9$, their ordering is

$$0 < 4 < (\sqrt{x} - 1)^2 < (\sqrt{x} + 1)^2 \,, \tag{9.97}$$

so that, according to eq. (9.45), the two independent solutions of the second-order DEQ (9.90) are

$$\begin{aligned} H_1^{(1)}(x) &= \int_0^{(\sqrt{x}-1)^2} \frac{da}{\sqrt{-R_4(a, x)}} \,, \\ H_1^{(2)}(x) &= \int_{(\sqrt{x}-1)^2}^{(\sqrt{x}+1)^2} \frac{da}{\sqrt{R_4(a, x)}} \,. \end{aligned} \tag{9.98}$$

With the change of variables (9.44), we can rewrite these solutions in terms of complete elliptic integrals of the first type,

$$\begin{aligned} H_1^{(1)}(x) =& \frac{1}{\sqrt{(\sqrt{x} + 3)(\sqrt{x} - 1)^3}} \mathrm{K}(\omega) \,, \\ H_1^{(2)}(x) =& \frac{1}{\sqrt{(\sqrt{x} + 3)(\sqrt{x} - 1)^3}} \mathrm{K}(1 - \omega) \,, \end{aligned} \tag{9.99}$$

with

$$\omega = \frac{16\sqrt{x}}{\sqrt{(\sqrt{x}+3)(\sqrt{x}-1)^3}}\,. \tag{9.100}$$

Finally, we can obtain the corresponding homogeneous solutions for $H_2(x)$ by applying the differential operator (9.91) to eq. (9.99),

$$\begin{aligned} H_2^{(1)}(x) =& \frac{(-x+2\sqrt{x}+3)\,\mathrm{K}(\omega)+(\sqrt{x}-1)^2\,\mathrm{E}(\omega)}{4\,(x-2\sqrt{x}-3)\,\sqrt{x^2-6x+8\sqrt{x}-3}}\,, \\ H_2^{(2)}(x) =& \frac{4\mathrm{K}(1-\omega)-(\sqrt{x}-1)^2\,\mathrm{E}(1-\omega)}{4\,(x-2\sqrt{x}-3)\,\sqrt{x^2-6x+8\sqrt{x}-3}}\,, \end{aligned} \tag{9.101}$$

where $\mathrm{E}(\omega)$ is the complete elliptic integral of the second kind

$$\mathrm{E}(w) = \int_0^1 \mathrm{d}z \frac{(1-w\,z^2)}{\sqrt{(1-z^2)}} \quad \text{with} \quad \Re(w) < 1\,. \tag{9.102}$$

In summary, the computation of the maximal-cut of $\mathrm{F}_1(x)$ along two independent contours, which are determined by the singularity structure of the integrand, allowed us to build the 2×2 matrix of homogeneous solutions

$$\mathbb{G}(x) = \begin{pmatrix} H_1^{(1)} & -H_1^{(2)} \\ H_2^{(1)} & H_2^{(2)} \end{pmatrix}\,, \tag{9.103}$$

whose independence can be checked *a posteriori* from the computation of the Wronskian determinant,

$$\begin{aligned} W(x) =& -\frac{1}{4(x-9)(x-1)}\left(\mathrm{E}(\omega)\mathrm{K}(1-\omega)+\mathrm{E}(1-\omega)\mathrm{K}(\omega)-\mathrm{K}(\omega)\mathrm{K}(1-\omega)\right) \\ =& -\frac{\pi}{8(x-9)(x-1))}\,,. \end{aligned} \tag{9.104}$$

In the second equality we have used the Legendre relation between elliptic integrals,

$$\mathrm{E}(\omega)\mathrm{K}(1-\omega)+\mathrm{E}(1-\omega)\mathrm{K}(\omega)-\mathrm{K}(\omega)\mathrm{K}(1-\omega) = \frac{\pi}{2}\,. \tag{9.105}$$

Despite the complexity of the individual entries of $\mathbb{G}(x)$, $W(x)$ has a remarkably simple form which is, indeed, consistent with Abel theorem. In fact, according to eq. (9.21), we have

$$\partial_x W(x) = \left(-\frac{1}{x-1}-\frac{1}{x-9}\right)W(x)\,, \tag{9.106}$$

which gives, in complete agreement with eq. (9.104),

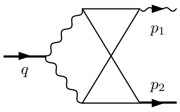
$$W(x) = c_0\frac{1}{(x-9)(x-1)}\,. \tag{9.107}$$

The analytic continuation of the homogeneous solution (9.103) to all kinematic regions and the derivation of the ϵ-expansion (9.11) of the inhomogeneous solution has been studied in detail in [271].

A two-loop elliptic triangle

As a second example, we consider a two-loop non-planar triangle with two off-shell legs and four massive propagators, which enters the non-planar corrections to $H+$jet production mediated by a massive fermion loop. The integral family ($a_i \geq 0$)

$$I_{a_1\,a_2\,a_3\,a_4\,a_5\,a_6\,a_7} = $$



$$\int \widetilde{\mathrm{d}^d k_1}\,\widetilde{\mathrm{d}^d k_2}\;\frac{D_7^{a_7}}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6}}\,, \tag{9.108}$$

is dentified by the set of denominators

$$\begin{gathered} D_1=(k_1-p_1)^2, \quad D_2=(k_2-p_1)^2-m^2, \quad D_3=(k_1+p_2)^2, \\ D_4=(k_1-k_2+p_2)^2-m^2, \quad D_5=(k_1-k_2)^2-m^2, \quad D_6=k_2^2-m^2, \quad D_7=k_1^2\,, \end{gathered} \tag{9.109}$$

and by the external kinematics $p_1^2=0$, $p_2^2\neq 0$ and $q^2=(p_1+p_2)^2=s\neq 0$. The six-denominator integrals belonging to the family (9.143) can be reduced to two MIs via IBPs, which we choose to be

$$\mathrm{F}_1=(s-p_2^2)^2 I_{1\,1\,1\,1\,1\,1\,0}\,, \qquad \mathrm{F}_2=\frac{(s-p_2^2)^2(s+16m^2)}{2(1+2\epsilon)} I_{1\,2\,1\,1\,1\,1\,0}\,. \tag{9.110}$$

If we neglect all subtopologies, the latter satisfy two sets of ϵ-linear DEQs in the variables $x=-s/m^2$ and $y=-p_2^2/m^2$,

$$\begin{aligned} \partial_x \mathbf{F}(\epsilon,x,y) &= \left(\mathbb{A}_{0\,,x}(x,y)+\epsilon\,\mathbb{A}_{1\,,x}(x,y)\right)\mathbf{F}(\epsilon,x,y)\,, \\ \partial_y \mathbf{F}(\epsilon,x,y) &= \left(\mathbb{A}_{0\,,y}(x,y)+\epsilon\,\mathbb{A}_{1\,,y}(x,y)\right)\mathbf{F}(\epsilon,x,y)\,, \end{aligned} \tag{9.111}$$

where $\mathbb{A}_{0\,,i}(x,y)$ and $\mathbb{A}_{1\,,i}(x,y)$ are 2×2 matrices that do not depend on ϵ,

$$\mathbb{A}_{0\,,x}(x,y)=\begin{pmatrix} 0 & \frac{16x}{(x-16)(y-x)^2}+\frac{8}{(x-16)(y-x)} \\ \frac{(x-16)(x+y)}{2x(x^2-2xy-16x+y^2)} & -\frac{2(x-y-8)}{x^2-2xy-16x+y^2}-\frac{3}{y-x}+\frac{16}{(x-16)x} \end{pmatrix},$$

$$\mathbb{A}_{1\,,x}(x,y)=\begin{pmatrix} \frac{2}{y-x} & \frac{32x}{(x-16)(y-x)^2}+\frac{16}{(x-16)(y-x)} \\ \frac{2(x-16)(x+y)}{x(x^2-2xy-16x+y^2)} & -\frac{2(x-y-8)}{x^2-2xy-16x+y^2}-\frac{4}{y-x}-\frac{2}{x} \end{pmatrix},$$

$$\mathbb{A}_{0\,,y}(x,y)=\begin{pmatrix} 0 & -\frac{16x}{(x-16)(y-x)^2} \\ \frac{16-x}{x^2-2xy-16x+y^2} & \frac{2(x-y)}{x^2-2xy-16x+y^2}+\frac{3}{y-x} \end{pmatrix},$$

$$\mathbb{A}_{1\,,y}(x,y)=\begin{pmatrix} \frac{2}{x-y} & -\frac{32x}{(x-16)(x-y)^2} \\ -\frac{4(x-16)}{x^2-2x(y+8)+y^2} & -\frac{2x^2-4xy-64x+2y^2}{(x-y)(x^2-2x(y+8)+y^2)} \end{pmatrix}. \tag{9.112}$$

In $\epsilon=0$ the systems become

$$\partial_x \mathbf{F}(x,y) = \mathbb{A}_{0\,,x}(x,y)\mathbf{F}(x,y)\,,$$

$$\partial_y \mathbf{F}(x,y) = \mathbb{A}_{0,y}(x,y)\mathbf{F}(x,y)\,, \tag{9.113}$$

and, according to eq. (9.14), they can be rephrased as second-order homogeneous DEQs for one of the two MIs.

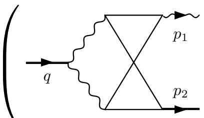
For $\mathrm{F}_1(x,y)$ such equations read

$$\partial_x^2\,\mathrm{F}_1(x,y) + \left(\frac{1}{y-x} - \frac{1}{x+y} + \frac{1}{x} + \frac{2(x-y-8)}{x^2-2xy-16x+y^2}\right)\partial_x\,\mathrm{F}_1(x,y) + \left(\frac{1}{x(y-x)} + \frac{1}{(y-x)^2} - \frac{y+4}{x\,(x^2-2xy-16x+y^2)}\right)\mathrm{F}_1(x,y) = 0\,, \tag{9.114}$$

and

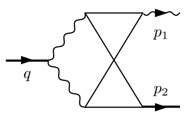
$$\partial_y^2\,\mathrm{F}_1(x,y) - \left(\frac{1}{y-x} + \frac{2(x-y)}{x^2-2xy-16x+y^2}\right)\partial_y\,\mathrm{F}_1(x,y) + \left(\frac{1}{(y-x)^2} - \frac{1}{x^2-2xy-16x+y^2}\right)\mathrm{F}_1(x,y) = 0\,. \tag{9.115}$$

The two independent solutions of this set of second-order DEQs can be found by direct computation of the maximal-cut of $\mathrm{F}_1(x,y)$. We start by computing the maximal-cut of one of the two sub-loops, namely k_2, which corresponds to a one-loop massive box of the type studied in section 9.3. In this way, using eq. (9.84), we obtain

MCut ()

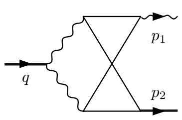
$$= (s-p_2^2)^2 \int \widetilde{\mathrm{d}^4 k_1}\,\delta\left((k_1-p_1)^2\right)\delta\left((k_1+p_2)^2\right) \times \frac{1}{\sqrt{P\left(k_1^2,(k_1-p_1+p_2)^2,p_2^2,0,0\right)}}\,, \tag{9.116}$$

with P being the polynomial defined in eq. (9.85) and where we have used that, when the two additional cuts are applied, $(k_1-p_1)^2 = (k_1+p_2)^2 = 0$. The two remaining δ-functions can be solved by introducing the d-dimensional Baikov parametrization of a one-loop triangle for k_1,

MCut ()

$$= (s-p_2^2)^2 \lim_{d\to4} G(p_1,p_2)^{\frac{3-d}{2}} \int \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \frac{\delta(z_1)\delta(z_2)}{\sqrt{P(z_i)}} G(z_i)^{\frac{d-4}{2}}\,, \tag{9.117}$$

where $z_1 = D_1$, $z_3 = D_5$ and the ISP z_3 is defined as $z_3 = (k^2+p_2^2-s)/2$. In eq. (9.117) $G(z_i)$ stems for the polynomial in the z_i variables which originates from the Gram determinant $G(k_1,p_1,p_2)$, whose explicit form is not reported for ease of writing. From eq. (9.117) we see immediately that, in the four-dimensional limit, there is no contribution from $G(z_i)$ and that the maximal-cut is simply obtained by evaluating $P(z_i)$ at $z_1 = z_2 = 0$,

MCut ()

$$= (s-p_2^2) \int_{\mathcal{C}} \mathrm{d}z_3 \frac{1}{\sqrt{P(z_i)}}\bigg|_{z_{1,2}=0}$$

$$
\begin{aligned}
=&(s-p_2^2)\int_{\mathcal{C}} \mathrm{d}a \, \frac{1}{\sqrt{(1-a^2)\left(16m^2 s+(1-a^2)\,(s-p_2^2)^2\right)}} \\
=&\frac{(s-p_2^2)}{\sqrt{p_2^4-2\,p_2^2\,s+s(s+16\,m^2)}}\int_{\mathcal{C}} \frac{\mathrm{d}a}{\sqrt{(1-a^2)\,(1-\omega\,a^2)}}\,.
\end{aligned} \tag{9.118}
$$

In the second equality we have performed the change of variables $z_3=(s-p_2^2)a$ and omitted, as usual, all multiplicative constants. In the last equality we have defined

$$
\omega=\frac{(s-p_2^2)^2}{p_2^4-2\,p_2^2\,s+s(s+16\,m^2)}\,. \tag{9.119}
$$

As for the case of the two-loop sunrise, the final integral is in the form (9.37) and, consequently, its evaluation along the two independent contours identified by the branching-points ± 1 and $\pm 1/\sqrt{\omega}$ yields to elliptic integrals of the first kind.

In particular, we immediately recognize, in the kinematic region $\omega<1$,

$$
\int_{-1}^{1} \frac{\mathrm{d}a}{\sqrt{(1-a^2)\,(1-\omega\,a^2)}}=2\int_{0}^{1} \frac{\mathrm{d}a}{\sqrt{(1-a^2)\,(1-\omega\,a^2)}}=\mathrm{K}(\omega)\,. \tag{9.120}
$$

Therefore, the two homogeneous solutions read

$$
\begin{aligned}
H_1^{(1)}(x,y)=&\frac{(s-p_2^2)}{\sqrt{p_2^4-2\,p_2^2\,s+s(s+16\,m^2)}}\mathrm{K}(\omega) \\
=&\frac{(x-y)}{\sqrt{x^2-2x(y+8)+y^2}}\mathrm{K}\left(\frac{(x-y)^2}{x^2-2(y+8)x+y^2}\right)\,, \\
H_1^{(2)}(x,y)=&\frac{(s-p_2^2)}{\sqrt{p_2^4-2\,p_2^2\,s+s(s+16\,m^2)}}\mathrm{K}(1-\omega) \\
=&\frac{(x-y)}{\sqrt{x^2-2x(y+8)+y^2}}\mathrm{K}\left(-\frac{16x}{x^2-2(y+8)x+y^2}\right)\,.
\end{aligned} \tag{9.121}
$$

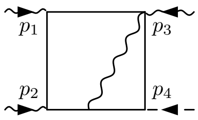
By direct computation, we have checked that $H_1^{(1)}(x,y)$ and $H_1^{(2)}(x,y)$ solve both second-order DEQs (9.114) and (9.115). Finally we observe that, in the limit $p_2^2\to 0$, we have

$$
\lim_{y\to 0} H_1^{(1)}(x,y)=\frac{x}{\sqrt{x(x-16)}}\mathrm{K}\left(\frac{x}{x-16}\right)\,, \tag{9.122}
$$

which provides the homogeneous solution for the differential equations of the corresponding non-planar two-loop triangle with only one off-shell leg. This integral enters the two-loop corrections to $t\bar{t}$ production in gluon-fusion, mediated by a massive-quark loop. In [104], the analytic continuation of the homogeneous solution to the physical region as been studied throughly and it has been used to derive a one-fold integral representation of the inhomogeneous solution.

A two-loop elliptic box

As a final example, we consider the planar six-denominator box computed in [103], in the framework of two-loop virtual corrections to $H+\mathrm{jet}$. The corresponding integral family ($a_i \geq 0$),



$$= I_{a_1\,a_2\,a_3\,a_4\,a_5\,a_6\,a_7\,a_8\,a_9} = \int \widetilde{\mathrm{d}^d k_1}\,\widetilde{\mathrm{d}^d k_2}\;\frac{(k_1\cdot p_3)^{a_7}(k_2\cdot p_1)^{a_8}(k_2\cdot p_2)^{a_9}}{D_1^{a_1}D_2^{a_2}D_3^{a_3}D_4^{a_4}D_5^{a_5}D_6^{a_6}}\,, \tag{9.123}$$

is identified by the denominators

$$D_1 = k_1^2 - m^2,\quad D_2 = (k_1-p_1)^2 - m^2,\quad D_3 = (k_1-p_1-p_2)^2 - m^2,$$
$$D_4 = (k_1-k_2+p_3)^2,\quad D_5 = k_2 - m^2,\quad D_6 = (k_2-p_1-p_2-p_3)^2 - m^2, \tag{9.124}$$

and by external momenta $p_1^2 = p_2^2 = p_3^2 = 0$, $p_4^2 = (p_1+p_2+p_3)^2 = m_h^2$. The Mandelstam invariants are defined as

$$s = (p_1+p_2)^2\,,\quad t = (p_1+p_3)^2\,,\quad u = (p_2+p_3)^2 = m_h^2 - s - t. \tag{9.125}$$

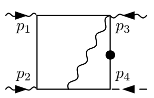
The IBPs reduction of the integral family of eq. (9.123) returns four six-denominator MIs, which we choose to be

$$\mathrm{F}_1 = I_{1\,1\,1\,1\,1\,1\,0\,0\,0}\,,\qquad \mathrm{F}_2 = I_{1\,2\,1\,1\,1\,1\,0\,0\,0}$$
$$\mathrm{F}_3 = I_{1\,1\,1\,1\,2\,1\,0\,0\,0}\,,\qquad \mathrm{F}_4 = I_{1\,1\,1\,1\,1\,1\,-1\,0\,0}. \tag{9.126}$$

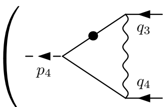
The master integrals $\mathbf{F}$ fulfil systems of first-order DEQs in the kinematic invariants. We have verified that, in $d=4$, the DEQs for F_4 are completely decoupled. Therefore, in the limit $\epsilon \to 0$, we can restrict our analysis to the homogeneous systems for the first three MIs, which read

$$\begin{cases} \partial_x \mathrm{F}_1(\vec{x}) = & a_{11}(\vec{x})\,\mathrm{F}_1(\vec{x}) + a_{12}(\vec{x})\,\mathrm{F}_2(\vec{x}) + a_{13}(\vec{x})\,\mathrm{F}_3(\vec{x}) \\ \partial_x \mathrm{F}_2(\vec{x}) = & a_{21}(\vec{x})\,\mathrm{F}_1(\vec{x}) + a_{22}(\vec{x})\,\mathrm{F}_2(\vec{x}) + a_{23}(\vec{x})\,\mathrm{F}_3(\vec{x}) \\ \partial_x \mathrm{F}_3(\vec{x}) = & a_{33}(\vec{x})\,\mathrm{F}_3(\vec{x})\,, \end{cases} \tag{9.127}$$

where $x \in \vec{x} = \{s, t, m_h^2, m^2\}$ and $a_{ij}(\vec{x})$ are rational functions of the Mandelstam invariants. We immediately see from eq. (9.127) that also the third MIs is decoupled, since none of its homogeneous DEQs contains terms proportional to either $\mathrm{F}_1(\vec{x})$ or $\mathrm{F}_2(\vec{x})$. In this respect, it is interesting to observe that $\mathrm{F}_3(\vec{x})$ has vanishing maximum-cut in four-dimensions. In fact, $\mathrm{F}_3(\vec{x})$ contains as sub-loop a one-loop triangle with one massive square denominator. In the previous section, we have seen that the maximal-cut of such triangle vanishes in $\epsilon = 0$. Hence, as a direct consequence of eq. (9.74), we trivially get

$$\mathrm{MCut}\left(\right.$$


$$\left.\right) = \int \widetilde{\mathrm{d}^4 k_1}\;\delta(k_1^2 - m^2)\,\delta((k_1-p_1)^2 - m^2)\,\delta((k_1-p_1-p_2)^2 - m^2)$$

$$\times\ \mathrm{MCut}\left(\right.$$


$$\left.\right) = 0. \tag{9.128}$$

A zero maximal-cut provides, indeed an obvious solution to the DEQs for $\mathrm{F}_3(\vec{x})$, since it is decoupled from $\mathrm{F}_1(\vec{x})$ and $\mathrm{F}_2(\vec{x})$. However, $\mathrm{MCut}\,(\mathrm{F}_3(\vec{x})) = 0$ does not prevent such DEQs to admit another non-trivial solutions which, although it is not captured by the maximal-cut, can be obtained independently by quadrature. This example reveals a close connection between the decoupling of the DEQ for a MI and the vanishing of its maximal-cut. In fact, precisely a vanishing maximal-cut could be seen as the hint of the decoupling of the DEQ, since only a decoupled equation would be automatically satisfied by a zero solution without imposing strong constraints on the maximal-cuts of the other MIs.

After decoupling $\mathrm{F}_3(\vec{x})$, we are left with two homogeneous first-order DEQs for the MIs F_1 and F_2,

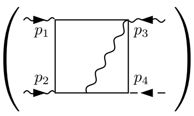
$$\begin{cases} \partial_x H_1(\vec{x}) = & a_{11}(\vec{x})\, H_1(\vec{x}) + a_{12}(\vec{x})\, H_2(\vec{x}) \\ \partial_x H_2(\vec{x}) = & a_{21}(\vec{x})\, H_1(\vec{x}) + a_{22}(\vec{x})\, H_2(\vec{x})\,. \end{cases} \tag{9.129}$$

These systems can be rephrased as a second-order DEQs for one of the two integrals. For instance, if we choose $\mathrm{F}_1(\vec{x})$ and we differentiate w.r.t. the internal mass, we have

$$\begin{aligned} &\partial^2_{m^2} H_1(\vec{x}) = \\ &\frac{s^2\left(48m^4 - 16m^2(t+m_h^2) + (t-m_h^2)^2\right) + 48m^4t^2 + 16m^2st(6m^2 - t + m_h^2)}{m^2\left((m_h^2-4m^2)^2s^2 + t^2(s-4m^2)^2 - 2(4m^2+m_h^2)st(s-4m^2)\right)}\partial_{m^2}H_1(\vec{x}) \\ &- \frac{2\left(s^2(-6m^2+t+m_h^2) + st(-12m^2+t-m_h^2) - 6m^2t^2\right)}{m^2\left((m_h^2-4m^2)^2s^2 + t^2(s-4m^2)^2 - 2(4m^2+m_h^2)st(s-4m^2)\right)} H_1(\vec{x})\,, \end{aligned} \tag{9.130}$$

Other three similar equations can be determined by taking derivatives w.r.t. s, t and m_h^2.

The two independent solutions of this set of second-order DEQs can be found by direct computation of the maximal-cut of $\mathrm{F}_1(x)$. As in the previous examples, we start by computing the maximal-cut of one of the two sub-loops, namely k_2, which corresponds to a one-loop triangle with three off-shell legs of the type studied in 9.3. In this way, using eq. (9.69), we obtain

MCut (

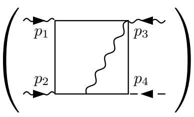
)

$$= \int \widetilde{\mathrm{d}^4 k_1}\, \delta(k_1^2 - m^2)\delta((k_1-p_1)^2 - m^2)\delta((k_1-p_1-p_2)^2 - m^2) \times \frac{1}{\sqrt{P_2(k^2, k\cdot p_3)}}\,, \tag{9.131}$$

where we defined

$$P_2(k^2, k\cdot p_3) = m^4 + ((k_1+p_3)^2 - m_h^2)^2 - 2m^2((k_1+p_3)^2 + m_h^2)\,. \tag{9.132}$$

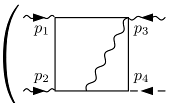
The three remaining δ-function can be easily solved by introducing the d-dimensional Baikov parametrization of a one-loop box for k_1,

MCut (

)

$$= \lim_{d\to 4} G(p_1,p_2,p_3)^{\frac{4-d}{2}} \int \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4 \frac{\delta(z_1)\delta(z_2)\delta(z_3)}{\sqrt{P_2(z_i)}} G(z_i)^{\frac{d-5}{2}}\,, \tag{9.133}$$

where $z_i = D_i$, $i = 1,2,3$ and the ISP z_4 is defined as $z_4 = s - k_1\cdot(p_1+p_2+p_3)$. In eq. (9.133) $G(z_i)$ stems for the polynomial in the z_i variables which originates from the Gram determinant $G(k_1,p_1,p_2,p_3)$, whose explicit form is not reported for ease of writing. From eq. (9.133) we see immediately that in four dimensions there is no contribution from the external Gram determinant $G(p_1, p_2, p_3)$, and the maximal-cut is simply obtained by evaluating the integrand in the origin of the hyperplane spanned by the denominators z_1, z_2 and z_3,

MCut ()

$$= \int_{\mathcal{C}} \mathrm{d}z_4 \frac{1}{\sqrt{P_2(z_i)}\sqrt{G_4(z_i)}}\bigg|_{z_{1,2,3}=0} = \frac{1}{s}\int_{\mathcal{C}} \mathrm{d}a \frac{1}{\sqrt{R_4(a,\vec{x})}}\,, \tag{9.134}$$

where $R_4(a, \vec{x})$ is a fouth-degree polynomial of the type (9.38), whose roots are given by

$$a_4 = \frac{1}{2}\left(s + t - 2\sqrt{\frac{-t\,u\,m^2}{s}}\right), \qquad a_2 = \frac{1}{2}\left(s + t + 2\sqrt{\frac{-t\,u\,m^2}{s}}\right),$$
$$a_3 = \frac{s + m_h^2 - 2\sqrt{m_h^2 m^2}}{2}\,, \qquad a_4 = \frac{s + m_h^2 + 2\sqrt{m_h^2 m^2}}{2}\,. \tag{9.135}$$

If we suppose to be in the kinematic region $a_4 < a_2 < a_3 < a_4$ (any other region can be reached by analytic continuation), the two homogeneous solutions for $\mathrm{F}_1(\vec{x})$ are, according to eq. (9.42),

$$H_1^{(1)}(\vec{x}) = \int_{a_4}^{a_2} \frac{dz_4}{\sqrt{-R_4(a,\vec{x})}}\,,$$
$$H_1^{(2)}(\vec{x}) = \int_{a_2}^{a_3} \frac{da}{\sqrt{R_4(a,\vec{x})}}\,. \tag{9.136}$$

With the usual change of variables (9.44), we can express the two solutions in terms of complete elliptic integrals of the first kind,

$$H_1^{(1)}(\vec{x}) = \frac{1}{\sqrt{Y}}\mathrm{K}(\omega)\,,$$
$$H_1^{(2)}(\vec{x}) = \frac{1}{\sqrt{Y}}\mathrm{K}(1-\omega) \tag{9.137}$$

with

$$\omega = \frac{16 m^2 \sqrt{-s\,t\,u\,m_h^2}}{Y}\,,$$
$$Y = s\left(m_h^2 - t\right)^2 - 4m^2\left(m_h^2(s-t) + t(s+t) - 2\sqrt{-s\,t\,u\,m_h^2}\right). \tag{9.138}$$

We observe that $H_1^{(1)}(\vec{x})$ has a smooth behaviour in the limit of vanishing internal mass $m^2 \to 0$,

$$H_1^{(1)}(\vec{x}) \underset{m^2\to 0}{=} \frac{1}{s(m_h^2 - t)}, \tag{9.139}$$

which reproduces the correct result for the maximal-cut of a six-denominator box with massless propagators. In addition, we have verified explicitly that eqs. (9.137) satisfy the second-order DEQ (9.130), as well as the ones in the Mandelstam invariants s, t and m_h^2. An alternative representation has been determined, in the physical region

$s\,, m_h^2 > 0$, $t\,, u < 0$ in [60].
In this respect, we would like stress that different (but equivalent) representations of the homogeneous solutions can be obtained by exploiting the symmetry properties of the DEQ (9.48) satisfied by $\mathrm{K}(\omega)$, for instance under the transformation $\omega \to 1/\omega$. In fact, if $\mathrm{K}(x)$ and $\mathrm{K}(1-x)$ are solutions of (9.48), then another couple of solutions is given by $1/\sqrt{x}\mathrm{K}(1/x)$ and $1/\sqrt{x}\mathrm{K}(1-1/x)$. However, since any second-order DEQ admits only two independent homogeneous solutions, we must have

$$\begin{aligned} \frac{1}{\sqrt{x}}\mathrm{K}\left(\frac{1}{x}\right) =& c_1\,\mathrm{K}(x) + c_2\,\mathrm{K}(1-x)\,, \\ \frac{1}{\sqrt{x}}\mathrm{K}\left(1-\frac{1}{x}\right) =& c_3\,\mathrm{K}(x) + c_3\,\mathrm{K}(1-x) \end{aligned} \tag{9.140}$$

for some (complex) constants c_i. Of course, since $\mathrm{K}(x)$ develops a branch cut when $x > 1$, one should assign a small imaginary part to x, which determines the sign of the imaginary parts of the coefficients c_i. For example, if we adopt the prescription $x \to x + i\,0^+$ and we assume $0 < x < 1$, we find

$$\begin{aligned} \frac{1}{\sqrt{x}}\mathrm{K}\left(\frac{1}{x}\right) &= \mathrm{K}(x) - i\,\mathrm{K}(1-x)\,, \\ \frac{1}{\sqrt{x}}\mathrm{K}\left(1-\frac{1}{x}\right) &= \mathrm{K}(1-x)\,. \end{aligned} \tag{9.141}$$

This means that we can equivalently take as homogeneous solutions of eq. (9.130) iether $H_1^{(1)}(\vec{x})$ and $H_1^{(2)}(\vec{x})$ defined above or the two new solutions

$$\begin{aligned} \widetilde{H}_1^{(1)}(\vec{x}) &= \frac{1}{\sqrt{s\,m^2}(-s\,t\,u\,m_h^2)^{1/4}}\mathrm{K}\left(\frac{1}{\omega}\right)\,, \\ \widetilde{H}_1^{(2)}(\vec{x}) &= \frac{1}{\sqrt{s\,m^2}(-s\,t\,u\,m_h^2)^{1/4}}\mathrm{K}\left(1-\frac{1}{\omega}\right)\,. \end{aligned} \tag{9.142}$$

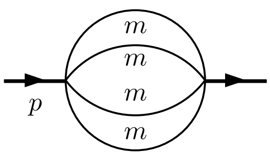
The arguments of the elliptic integrals appearing in eq. (9.142) match the ones found in [103], where the second order DEQs were solved by parametrizing the Mandelstam variables in terms of an additional dimensionless parameter and by matching the second-order DEQ w.r.t. such parameters with the elliptic DEQ (9.48).

9.5 The three-loop massive banana graph

In this final section, we apply the DEQs method to calculation of the three-loop massive banana graph in $d = 2-2\epsilon$ space-time dimensions. This problem constitutes a clean –yet valuable– test of the generality of the method proposed in section 9.2 for the computation of the homogeneous solutions as it involves the solution of a genuine third-order DEQ. For related related studies on this integral, we refer the reader to [105, 106].

9.5.1 System of differential equations

We consider the three-loop two-point integral family defined by ($a_i \geq 0$)



$= I_{a_1\,a_2\,a_3\,a_4\,a_5\,a_6\,a_7\,a_8\,a_9}$

$$= \int \widetilde{\mathrm{d}^d k_1}\, \widetilde{\mathrm{d}^d k_2} \widetilde{\mathrm{d}^d k_3}\; \frac{(k_1 \cdot p)^{a_6} (k_2 \cdot p)^{a_7} (k_3 \cdot p)^{a_8} (k_1 \cdot k_2)^{a_9}}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4}}\,, \tag{9.143}$$

where $p^2 = s \neq 0$ and the loop denominators are given by

$$\begin{aligned} D_1 = k_1^2 - m^2\,, \quad D_2 = k_2^2 - m^2\,, \\ D_3 = (k_1 - k_3)^2 - m^2\,, \quad D_4 = (k_2 - k_3 - p)^2 - m^2\,. \end{aligned} \tag{9.144}$$

Our integration measure is defined as

$$\int \widetilde{\mathrm{d}^d k} = \frac{(m^2)^{\frac{4-d}{2}}}{i\pi^{d/2}\Gamma\left(\frac{4-d}{2}\right)} \int \frac{\mathrm{d^d} k}{(2\pi)^d}\,, \tag{9.145}$$

in such a way that the one-loop tadpole integral reads

$$\int \frac{\widetilde{\mathrm{d}^d k}}{k^2 - m^2} = \frac{2m^2}{d-2}\,. \tag{9.146}$$

The three-denominator integrals belonging to the family (9.143) are usually referred to as three-loop banana graphs or three-loop sunrise graphs. Besides one single subtopology (which corresponds to the three-loop tadpole), IBPs reduction returns three independent MIs, which we choose as

$$\begin{aligned} \mathrm{F}_1(\epsilon, x) =& (1+2\epsilon)(1+3\epsilon)(m^2)^{-2} I_{1\,1\,1\,1\,0\,0\,0\,0\,0}\,, \\ \mathrm{F}_2(\epsilon, x) =& (1+2\epsilon)(m^2)^{-1} I_{2\,1\,1\,1\,0\,0\,0\,0\,0}\,, \\ \mathrm{F}_3(\epsilon, x) =& I_{2\,2\,1\,1\,0\,0\,0\,0\,0}\,, \end{aligned} \tag{9.147}$$

where the normalization factors have been defined for later convenience. For the three-loop tadpole we chose, instead, the master integral $\mathrm{F}_0 = I_{2\,2\,2\,0\,0\,0\,0\,0\,0}$, which in our normalization becomes simply

$$\mathrm{F}_0(\epsilon; m^2) = I_{2\,2\,2\,0\,0\,0\,0\,0\,0} = 1\,. \tag{9.148}$$

We study the DEQs satisfied by the MIs in the dimensionless variable

$$x = \frac{4m^2}{s}\,, \tag{9.149}$$

and, as for the case of the two-loop sunrise, we work in $d = 2 - 2\epsilon$, since the scalar massive bubbles are both IR finite and UV finite in two dimensions. The ϵ-linear system of DEQs in x satisfied by $\mathrm{F}_1(x)$, $\mathrm{F}_2(x)$ and $\mathrm{F}_3(x)$ is

$$\partial_x \mathbf{F}(\epsilon, x) = \left(\mathbb{A}_0(x) + \epsilon\, \mathbb{A}_1(x)\right) \mathbf{F}(\epsilon, x) + \mathbf{S}(x)\,, \tag{9.150}$$

where $\mathbb{A}_i(x)$ are 3×3 matrices that do not depend on ϵ,

$$\mathbb{A}_0(x) = \begin{pmatrix} \frac{1}{x} & \frac{4}{x} & 0 \\ -\frac{1}{4(x-1)} & \frac{1}{x} - \frac{2}{x-1} & \frac{3}{x} - \frac{3}{x-1} \\ \frac{1}{8(x-1)} - \frac{1}{8(4x-1)} & \frac{1}{x-1} - \frac{3}{2(4x-1)} & \frac{1}{x} - \frac{6}{4x-1} + \frac{3}{2(x-1)} \end{pmatrix}, \tag{9.151}$$

$$\mathbb{A}_1(x) = \begin{pmatrix} \frac{3}{x} & \frac{12}{x} & 0 \\ -\frac{1}{x-1} & \frac{2}{x} - \frac{6}{x-1} & \frac{6}{x} - \frac{6}{x-1} \\ \frac{1}{2(x-1)} - \frac{1}{2(4x-1)} & \frac{3}{x-1} - \frac{9}{2(4x-1)} & \frac{1}{x} - \frac{12}{4x-1} + \frac{3}{x-1} \end{pmatrix}, \tag{9.152}$$

(9.153)

and $\mathbf{S}(x)$i s the inhomogeneous term originated from the massive tadpole (9.148),

$$\mathbf{S}(x) = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2(4x-1)} \end{pmatrix} . \tag{9.154}$$

In $\epsilon = 0$ the system becomes (as usual, we denote by $\mathbf{H}(x)$ the vector of homogeneous solutions associated to the MIs $\mathbf{F}(\epsilon, x)$),

$$\partial_x \mathbf{H}(x) = \mathbb{A}_0(x)\mathbf{H}(x) , \tag{9.155}$$

and it can be rephrased in terms of a third-order DEQ for one of the MIs, say $H_1(x)$,

$$\left[\partial_x^3 + \frac{3(8x-5)}{2(x-1)(4x-1)}\partial_x^2 + \frac{4x^2-2x+1}{(x-1)x^2(4x-1)}\partial_x + \frac{1}{x^3(4x-1)}\right] H_1(x) = 0 . \tag{9.156}$$

Once the solutions of eq. (9.156) are known, $H_2(x)$ and $H_3(x)$ can be expressed in terms of $H_1(x)$ and its first two derivatives,

$$H_2(x) = \frac{1}{4}\left[x\partial_x - 1\right] H_1(x) , \tag{9.157}$$

$$H_3(x) = \frac{1}{12}\left[x^2(1-x)\partial_x^2 - (1+x)x\partial_x + 1\right] H_1(x). \tag{9.158}$$

Eq. (9.156) admits three independent solutions. Our goal is to build, starting from the maximal-cut of $\mathrm{F}_1(x)$, a 3×3 matrix of homogeneous solutions,

$$\mathbb{G}(x) = \begin{pmatrix} H_1^1(x) & H_1^{(2)}(x) & H_1^{(3)}(x) \\ H_2^{(1)}(x) & H_2^{(2)}(x) & H_2^{(3)}(x) \\ H_3^{(1)}(x) & H_3^{(2)}(x) & H_3^{(3)}(x) \end{pmatrix} , \tag{9.159}$$

which can then be used in order to bring the system (9.150) into an ϵ-factorized form. We observe that the singularity structure of eq. (9.156), or equivalently of the system (9.155) is rather simple, as it is characterized by only four regular singular points $x = 0$, $x = 1/4$, $x = 1$ and $x = \pm\infty$, which correspond, respectively, to $s = \pm\infty$, $s = 16m^2$, $s = 4m^2$ and $s = 0$. This structure closely resembles the one of the two-loop massive sunrise, given in eq. (9.90), whose solutions have been expressed in terms of complete elliptic integrals of the first and second kind. Therefore, it is particularly interesting to investigate the class of functions that appear in the solution of eq. (9.156) and how they generalize the ones required for the integration of the two-loop sunrise graph. We anticipate that, as we will see in the next section, the three-loop case can be solved in terms of *products of complete elliptic integrals of first and second kind.*

Before entering the detailed discussion of the computation of the maximal-cuts, let us recall that, in order to transform the system of DEQs (9.150) to the ϵ-factorized form,

$$\partial_x \mathbf{I}(\epsilon, x) = \epsilon\, \tilde{\mathbb{A}}_1(x)\mathbf{I}(\epsilon, x) + \tilde{\mathbf{S}}(x) , \tag{9.160}$$

with $\mathbf{F}(\epsilon, x) = \mathbb{G}(x)\mathbf{I}(\epsilon, x)$ and

$$\tilde{\mathbb{A}}_1(x) = \mathbb{G}^{-1}(x)\, \mathbb{A}_1(x)\, \mathbb{G}(x) , \quad \tilde{\mathbf{S}}(x) = \mathbb{G}^{-1}(x)\mathbf{S}(x) , \tag{9.161}$$

we need, besides the homogeneous matrix $\mathbb{G}(x)$, its inverse.$\mathbb{G}^{-1}(x)$ can be directly determined from the entries of $\mathbb{G}(x)$ as dictated by eq. (9.18),

$$\mathbb{G}^{-1}(x) = \frac{1}{W(x)} \begin{pmatrix} H_3^{(3)}H_2^{(2)} - H_2^{(3)}H_3^{(2)} & H_1^{(3)}H_3^{(2)} - H_3^{(3)}H_1^{(2)} & H_2^{(3)}H_1^{(2)} - H_1^{(3)}H_2^{(2)} \\ H_3^{(1)}H_2^{(3)} - H_2^{(1)}H_3^{(3)} & H_1^{(1)}H_3^{(3)} - H_3^{(1)}H_1^{(3)} & H_2^{(1)}H_1^{(3)} - H_1^{(1)}H_2^{(3)} \\ H_2^{(1)}H_3^{(2)} - H_3^{(1)}H_2^{(2)} & H_3^{(1)}H_1^{(2)} - H_1^{(1)}H_3^{(2)} & H_1^{(1)}H_2^{(2)} - H_2^{(1)}H_1^{(2)} \end{pmatrix}, \tag{9.162}$$

where $W(x)$ is the Wronskian determinant. As we have already discussed, Abel theorem allows us to obtain, up to an overall constant, the expression of $W(x)$ independently from the actual expression of $\mathbb{G}(x)$. In fact, according to eq. (9.21), $W(x)$ satisfies the first-order DEQ,

$$\partial_x W(x) = \frac{8x^2 - 17x + 6}{2x(x-1)(4x-1)} W(x)\,, \tag{9.163}$$

which immediately yields to

$$W(x) = \frac{c_0 x^3}{\sqrt{(1-4x)^3(1-x)}}\,. \tag{9.164}$$

9.5.2 The maximal-cut of the banana graph

In this section, we compute the maximal-cut of the scalar banana graph $F_1(x)$ in $d = 2$, which is defined by the $\epsilon \to 0$ limit of eq. (9.147). As for the two-loop examples discussed in section 9.4, we can evaluate the maximal-cut of the graph by cutting individual sub-loops first, in order to reuse the information on the maximal-cuts of integral topologies with a lower number of loops, and obtain a representation of the maximal-cut in terms of a minimal number of integration variables.

In particular, we observe that $\mathrm{F}_1(x)$ can be written as an integral over two one-loop bubbles,

$$\mathrm{F}_1(x) = \int \widetilde{\mathrm{d}^2 k_3} \int \widetilde{\mathrm{d}^2 k_1} \frac{1}{(k_1^2 - m^2)((k_1 - k_3)^2 - m^2)} \times \int \widetilde{\mathrm{d}^2 k_2} \frac{1}{(k_2^2 - m^2)((k_2 - (k_3 + p))^2 - m^2)}. \tag{9.165}$$

Therefore, we can localize the integrals over k_1 and k_2 independently, by using the expression (9.60) of the maximal-cut of a one-loop massive bubble in $d = 2$,

$$\mathrm{MCut}[\mathrm{F}_1(x)] = \int_{\mathcal{C}} \frac{\mathrm{d}^2 k_3}{\sqrt{k_3^2(k_3^2 - 4\,m^2)}\sqrt{(k_3+p)^2\,((k_3+p)^2 - 4m^2)}}\,. \tag{9.166}$$

The choice of the integration contour $\mathcal{C}$ will be specified later on.

The simultaneous cut of the four propagators left us with a two-fold integral over the loop momentum k_3, which can be conveniently parametrized by decomposing both the loop momentum and the external one in terms of two arbitrary massless momenta p_1^μ and p_2^μ,

$$p^\mu = p_1^\mu + p_2^\mu\,, \qquad k_3^\mu = a\, p_1^\mu + b\, p_2^\mu\,, \tag{9.167}$$

with $p_1^2 = p_2^2 = 0$. In this way, the scalar products appearing in eq. (9.166) are written as

$$k_3^2 = a\,b\,s \qquad (k_3 + p)^2 = (a+1)(b+1)\,s\,, \tag{9.168}$$

and the integration measure becomes

$$\int_{\mathcal{C}} \widetilde{\mathrm{d}^2 k_3} = \frac{s}{2} \int_{\mathcal{C}} \mathrm{d}a\, \mathrm{d}b\,, \tag{9.169}$$

where, in the r.h.s., $\mathcal{C}$ represents a still unspecified contour in the space spanned by the components a and b. In this parametrization, the maximal-cut becomes (up to a multiplicative constant)

$$\mathrm{MCut}[\mathrm{F}_1(x)] = x \int_{\mathcal{C}'} \frac{\mathrm{d}a\, \mathrm{d}b}{\sqrt{R_8(a,b,x)}}\,, \tag{9.170}$$

where $R_8(a,b,x)$ is the rank-eight polynomial in the variables a and b

$$R_8(a,b,x) = a\, b\, (a\, b - x)(a+1)(b+1)\left((a+1)(b+1) - x\right)\,. \tag{9.171}$$

We now need to understand what are the possible integration bounds that correspond to the independent solutions of eq. (9.156). According to the reasoning of section 9.2, *any* portion of *any* region (9.174) which is delimited by the zeroes of $R_8(a,b,x)$ (and the point at infinity) is a valuable solution to the homogeneous DEQ, since it ensure the vanishing of all boundary terms. The analysis is simpler if we consider one integral at a time.

Let us analyse first with the integral over a[1], by rewriting eq. (9.170) as

$$\mathrm{MCut}[\mathrm{F}_1(x)] = x \oint_{\mathcal{C}_b} \frac{\mathrm{d}b}{b(b+1)} \oint_{\mathcal{C}_a} \frac{da}{\sqrt{(a-a_1)\,(a-a_2)\,\,(a-a_3)\,(a-a_4)}}\,, \tag{9.172}$$

where we introduced the abbreviations

$$a_1 = -1\,, \qquad a_2 = \frac{x}{b+1} - 1\,, \qquad a_3 = 0\,, \qquad a_4 = \frac{x}{b}\,. \tag{9.173}$$

From eq. (9.172) we see that the integral in da is an integral over a square root of a quartic polynomial of the type studied in section 9.2, whose branching-points a_i, which are defined in eq. (9.173) vary as a function of b and x. Thus, for a fixed value of x each branching-point identifies different curves in the ba-plane, which we draw in different colours in figure 9.2. Moreover, if we suppose, without loss of generality, $1/2 < x < 1$ (i.e. $4m^2 < s < 8m^2$) we can divide the ba-plane into five different regions, depending on the value of the variable b,

$$\begin{aligned} &\mathrm{I}: b \in (-\infty, -1)\,, && \mathrm{II}: b \in (-1, -x)\,,\\ &\mathrm{III}: b \in (-x, x-1)\,, && \mathrm{IV}: b \in (x-1, 0)\,,\\ &\mathrm{V}: b \in (0, \infty)\,. \end{aligned} \tag{9.174}$$

which correspond to a particular ordering of the branching-points (9.173). Therefore, when moving within each of these regions, no branch cut is crossed, since the sign of argument of the square root in eq. (9.172) is fixed.

According to eq. (9.42), for each of the regions (9.174) $J = \mathrm{I}, \mathrm{II}, , \ldots, \mathrm{V}$, we can define two independent functions $f_i^J(x)$, $i = 1, 2$, by varying the integration range of a. Hence, we have a total of 10 putative homogeneous solutions,

$$f_1^{\mathrm{I}}(x) = x \int_{-\infty}^{-1} \mathrm{d}b \int_{x/(b+1)-1}^{-1} \frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}}\,,$$

[1] Notice that the starting integral is symmetric in a and b, so the following analysis can be equally repeated by inverting the role of the two variables.

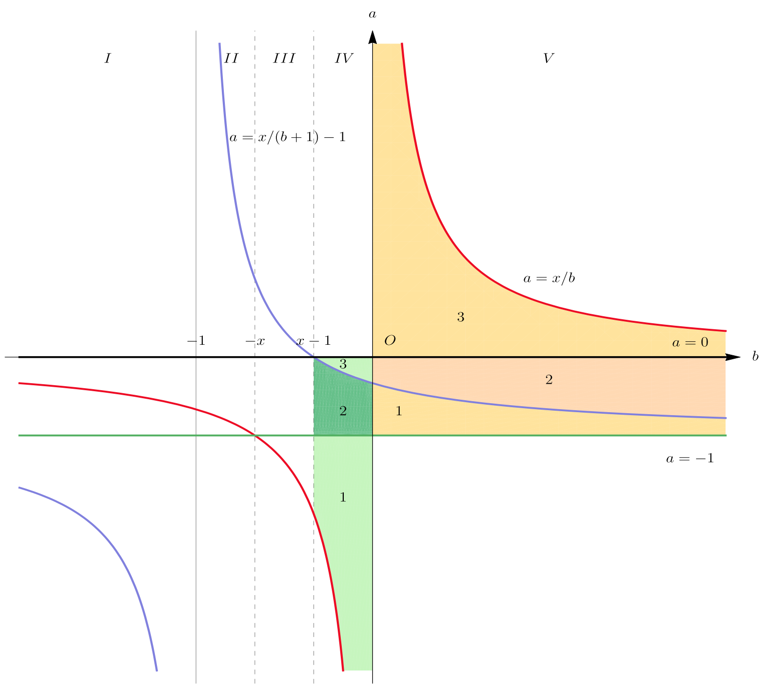


Figure 9.2: Solide lines represent the set of points where the argument of the square root in eq. (9.170) changes sign.

$$f_2^{\mathrm{I}}(x) = x \int_{-\infty}^{-1} \mathrm{d}b \int_{-1}^{x/b} \frac{\mathrm{d}a}{\sqrt{R_8(a,b,x)}} , \tag{9.175}$$

$$\begin{aligned} f_1^{\mathrm{II}}(x) =& x \int_{-1}^{-x} \mathrm{d}b \int_{-1}^{x/b} \mathrm{d}a \frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}} , \\ f_2^{\mathrm{II}}(x) =& x \int_{-1}^{-x} \mathrm{d}b \int_{x/b}^{0} \frac{\mathrm{d}a}{\sqrt{R_8(a,b,x)}} , \end{aligned} \tag{9.176}$$

$$\begin{aligned} f_1^{\mathrm{III}}(x) =& x \int_{-x}^{x-1} \mathrm{d}b \int_{x/b}^{-1} \frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}} , \\ f_2^{\mathrm{III}}(x) =& x \int_{-x}^{x-1} \mathrm{d}b \int_{-1}^{0} \frac{\mathrm{d}a}{\sqrt{R_8(a,b,x)}} , \end{aligned} \tag{9.177}$$

$$\begin{aligned} f_1^{\mathrm{IV}}(x) =& x \int_{x-1}^{0} \mathrm{d}b \int_{x/b}^{-1} \frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}} , \\ f_2^{\mathrm{IV}}(x) =& x \int_{x-1}^{0} \mathrm{d}b \int_{-1}^{x/(b+1)-1} \frac{\mathrm{d}a}{\sqrt{R_8(a,b,x)}} , \end{aligned} \tag{9.178}$$

$$\begin{aligned} f_1^{\mathrm{V}}(x) =& x \int_{0}^{\infty} \mathrm{d}b \int_{-1}^{x/(b+1)-1} \frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}} , \\ f_2^{\mathrm{V}}(x) =& x \int_{0}^{\infty} \mathrm{d}b \int_{x/(b+1)-1}^{0} \frac{\mathrm{d}a}{\sqrt{R_8(a,b,x)}} . \end{aligned} \tag{9.179}$$

The sign in the square-root has been chosen in such a way that every $f_i^J(x)$ is real. We should notice that many of these functions are related to each other by simple symmetry transformations.

First of all, the integrand in eq. (9.170) is symmetric under $\{a \leftrightarrow b\}$ and $\{a \to -a-1,\ b \to -b-1\}$ (and, hence, under a combination of these two symmetries). It is easy to see that under the transformation $\{a \to -a-1,\ b \to -b-1\}$ the regions I and V, and the regions II and IV are mapped into each other. Consistently, we find

$$f_1^{\mathrm{I}}(x) = f_1^{\mathrm{V}}(x)\,, \quad f_2^{\mathrm{I}}(x) = f_2^{\mathrm{V}}(x)\,, \quad f_1^{\mathrm{II}}(x) = f_1^{\mathrm{IV}}(x)\,, \quad f_2^{\mathrm{II}}(x) = f_2^{\mathrm{IV}}(x)\,. \tag{9.180}$$

The effect of the symmetry $\{a \leftrightarrow b\}$ is less immediate. Let us consider $f_1^{\mathrm{V}}(x)$ as an example. By using $R_8(a,b,x) = R_8(b,a,x)$, we can rename a and b, exchange the order of integration, and obtain

$$\begin{aligned} f_1^{\mathrm{V}}(x) &= x\int_0^{\infty} \mathrm{d}a \int_{-1}^{x/(a+1)-1} \frac{\mathrm{d}b}{\sqrt{-R(a,b,x)}} = x\int_{-1}^{x-1}\mathrm{d}b\int_0^{x/(b+1)-1}\frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}} \\ &= x\int_{-1}^{-x}\mathrm{d}b\int_0^{x/(b+1)-1}\frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}} + x\int_{-x}^{x-1}\mathrm{d}b\int_0^{x/(b+1)-1}\frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}} \\ &= f_1^{\mathrm{III}}(x) + f_1^{\mathrm{IV}}(x)\,. \end{aligned} \tag{9.181}$$

In the last step, we used eq. (9.180). With a similar calculation we can show that

$$f_2^{\mathrm{V}}(x) = \frac{1}{2} f_2^{\mathrm{III}}(x) + f_2^{\mathrm{IV}}(x)\,. \tag{9.182}$$

The six linear relations (9.180)-(9.181)(9.182) reduce down to four the number of independent functions, which we can choose to be

$$f_1^{\mathrm{V}}(x)\,, \quad f_2^{\mathrm{V}}(x)\,, \quad f_1^{\mathrm{IV}}(x)\,, \quad f_2^{\mathrm{IV}}(x)\,. \tag{9.183}$$

Although we have found four independent functions, they are not yet guaranteed to be actual solutions of the DEQ(9.156). In fact, a solution is obtained only when the maximal-cut is integrated over a region of the ba-plane bounded by branching-points. It is easy to see from figure 9.2 and from the definition of the functions (9.178, 9.179) that, while $f_1^{\mathrm{V}}(x)$ and $f_2^{\mathrm{V}}(x)$ indeed fulfil this requirement, $f_1^{\mathrm{IV}}(x)$ and $f_2^{\mathrm{IV}}(x)$ apparently do not. Nonetheless, we should remind that, for any given region in b, different choices of the integration boundaries in a can produce equivalent results, as summarized in eq. (9.42).

Let us look more closely at figure 9.2, where the integration region corresponding to the four functions are represented by the shaded areas 1 and 2. For both region IV and V, the integration over the areas 1 and 3 produce equivalent results. For instance, if we specialize eq. (9.2) to region IV,

$$\int_{x/b}^{-1}\frac{\mathrm{d}a}{\sqrt{-R(a,b,x)}} = \int_{x/(b+1)-1}^{0}\frac{\mathrm{d}a}{\sqrt{-R(a,b,x)}}\,,$$

we get

$$f_1^{\mathrm{IV}}(x) = x\int_{x-1}^{0}\mathrm{d}b\int_{x/b}^{-1}\frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}} = x\int_{x-1}^{0}\mathrm{d}b\int_{x/(b+1)-1}^{0}\frac{\mathrm{d}a}{\sqrt{-R_8(a,b,x)}}\,, \tag{9.184}$$

As it is clear from the figure, the second integral representation in (9.184) is now integrated in a region bounded by the branching-points. As a consequence, $f_1^{\mathrm{IV}}(x)$ is a a solution of the third-order DEQ. The same cannot be said of $f_2^{\rm IV}(x)$, which cannot be rewritten as an integral over a region bounded by branching-point only.

The complete set of three independent solutions is therefore given by $f_1^{\rm V}(x)$, $f_2^{\rm V}(x)$ and $f_1^{\rm IV}(x)$. Interestingly, if we want to build a viable solution of the homogeneous DEQ which involves the function $f_2^{\rm IV}(x)$, we should consider the combination

$$f(x) = f_2^{\rm II}(x) + f_2^{\rm III}(x) + f_2^{\rm IV}(x). \tag{9.185}$$

As this function is defined by integrating in a region delimited by the branching-points of $R_8(a,b,x)$, it must be a solution of the third-order DEQ. On the other hand, we have already determined three independent solutions. Hence, it must be possible to write $f(x)$ as a linear combination of $f_1^{\rm V}(x)$, $f_2^{\rm V}(x)$ and $f_1^{\rm IV}(x)$. Consistently, if we make use of eqs. (9.180), (9.181) and (9.182) we find

$$f(x) = f_2^{\rm III}(x) + 2\, f_2^{\rm IV}(x) = 2\, f_2^{\rm V}(x), \tag{9.186}$$

which proves that $f(x)$ is not independent from the chosen homogeneous solutions.

The homogeneous solutions

The above analysis allowed us to determine three independent homogeneous solutions in form of the two-fold integral representations

$$f_1^{\rm IV}(x)\,, \qquad f_1^{\rm V}(x)\,, \qquad f_2^{\rm V}(x)\,, \tag{9.187}$$

which are defined in eqs. (9.178) and (9.179). In the following, we show that these integrals can be evaluated in terms of known functions.

First of all we perform the integral in da which is type defined in eq. (9.37) ad yields, in all cases, to a complete elliptic integral of the first kind. By using the change of variables (9.44), we obtain

$$f_1^{\rm IV}(x) = 2x \int_{x-1}^{0} \frac{{\rm d}b}{\sqrt{b(b+1)}\sqrt{b(b+1)+x}} {\rm K}\left(1 - \frac{x^2}{b(b+1)+x}\right)\,, \tag{9.188}$$

$$f_1^{\rm V}(x) = 2x \int_{0}^{\infty} \frac{{\rm d}b}{\sqrt{b(b+1)}\sqrt{b(b+1)+x}} {\rm K}\left(\frac{x^2}{b(b+1)+x}\right)\,, \tag{9.189}$$

$$f_2^{\rm V}(x) = 2x \int_{0}^{\infty} \frac{{\rm d}b}{\sqrt{b(b+1)}\sqrt{b(b+1)+x}} {\rm K}\left(1 - \frac{x^2}{b(b+1)+x}\right)\,. \tag{9.190}$$

These one-fold integral representations are already more convenient for numerical evaluation than the original two-fold ones. Nonetheless, with some less trivial manipulation, we can evaluate analytically also the integrals over db. We will work out in detail the expressions for $f_1^{\rm V}(x)$ and $f_2^{\rm V}(x)$. A similar calculation can be performed also for the first function, $f_1^{\rm IV}(x)$ butwe will not need its explicit expression in order to build the matrix of homogeneous solutions, as it will soon become clear.

Let us consider eqs. (9.188) (9.189)) and perform the change of variables

$$b(b+1) = y^2\,, \qquad \int_0^\infty {\rm d}{\rm d}b = \int_0^\infty \frac{2y}{\sqrt{1+4y^2}} {\rm d}y\,, \tag{9.191}$$

so that the two integrals become

$$f_1^{\rm V}(x) = 2x \int_0^\infty \frac{{\rm d}y}{\sqrt{(y^2+x)\,(y^2+1/4)}} {\rm K}\left(\frac{x^2}{y^2+x}\right)\,, \tag{9.192}$$

$$f_2^{\rm V}(x) = 2x\int_0^\infty \frac{{\rm d}y}{\sqrt{(y^2+x)\,(y^2+1/4)}}{\rm K}\left(\frac{y^2+x(1-x)}{y^2+x}\right)\,. \tag{9.193}$$

Next, we introduce three parameters α, β and Γ, which are implicitly defined by

$$(\alpha+\beta)^2 = x\,, \quad (\alpha-\beta)^2 = x(1-x)\,, \quad \Gamma = \frac{1}{2}\,. \tag{9.194}$$

In general, for a given value of x, four different pairs of solutions exist to these equations, and we can choose any of these. For definiteness, we pick

$$\alpha = \frac{\sqrt{x}+\sqrt{x(1-x)}}{2}\,, \qquad \beta = \frac{\sqrt{x}-\sqrt{x(1-x)}}{2}\,, \tag{9.195}$$

where we are assuming $0 < x < 1$. For any given pair of solutions, $f_1^{\rm V}(x)$ and $f_2^{\rm V}(x)$ read

$$f_1^{\rm V}(x) = 2x\int_0^\infty \frac{{\rm d}y}{\sqrt{(y^2+(\alpha+\beta)^2)(y^2+\Gamma^2)}}{\rm K}\left(\frac{2\alpha\beta}{y^2+(\alpha+\beta)^2}\right)\,, \tag{9.196}$$

$$f_2^{\rm V}(x) = 2x\int_0^\infty \frac{{\rm d}y}{\sqrt{(y^2+(\alpha+\beta)^2)(y^2+\Gamma^2)}}{\rm K}\left(\frac{y^2+(\alpha-\beta)^2}{y^2+(\alpha+\beta)^2}\right)\,. \tag{9.197}$$

The integrals (9.196)-(9.197) are now in a standard form. In particular, the calculation of (9.197) is discussed in [272] (see eq. (33) therein). Suitable extensions of the methods described there allow us to compute also the integral (9.196). As a result, $f_1^{\rm V}(x)$ and $f_2^{\rm V}(x)$ are both expressed in terms of products of complete elliptic integrals of the first kind,

$$f_1^{\rm V}(x) = 2x{\rm K}(k_-^2){\rm K}(k_+^2) \tag{9.198}$$

$$f_2^{\rm V}(x) = 4x\left({\rm K}(k_-^2){\rm K}(1-k_+^2)+{\rm K}(k_+^2){\rm K}(1-k_-^2)\right)\,, \tag{9.199}$$

where we have defined

$$k_\pm = \frac{\sqrt{(\Gamma+\alpha)^2-\beta^2}\pm\sqrt{(\Gamma-\alpha)^2-\beta^2}}{2\Gamma} \qquad \text{with} \qquad k_- = \left(\frac{\alpha}{\Gamma}\right)\frac{1}{k_+} = \frac{2\alpha}{k_+}\,. \tag{9.200}$$

The explicit derivation of eqs. (9.198)-(9.198) is given in appendix D.

It is easy to prove by direct calculation that these functions solve the third-order DEQ (9.156) for every choice of α,β and, in particular, for the choice we made in eq. (9.195).

Although we have computed only two of the solutions, we can easily identify from eqs. (9.198)-(9.199) three independent solutions of the third-order DEQ (9.156) as the three functions

$$\begin{aligned} H_1^{(1)}(x) =& x\,{\rm K}\left(k_+^2\right){\rm K}\left(k_-^2\right)\,, \\ H_1^{(2)}(x) =& x\,{\rm K}\left(k_+^2\right){\rm K}\left(1-k_-^2\right)\,, \\ H_1^{(3)}(x) =& x\,{\rm K}\left(1-k_+^2\right){\rm K}\left(k_-^2\right)\,. \end{aligned} \tag{9.201}$$

These results allow us to fix completely the first row of the matrix $\mathbb{G}(x)$. The other two rows, which correspond to the homogeneous solutions of ${\rm F}_2(x)$ and ${\rm F}_3(x)$, can then be obtained from eqs. (9.157), (9.158). We do not report the results here for brevity, but we have checked that the Wronskian of these solutions agrees with the result of Abel theorem (9.164),

$$W(x) = -\frac{\pi^3 x^3}{512\sqrt{(1-4x)^3(1-x)}} \,. \tag{9.202}$$

By inspecting the three solutions, it is natural to wonder what happens if we consider one further products of elliptic integrals,

$$H_1^{(4)}(x) = x\,\mathrm{K}\left(1-k_+^2\right)\mathrm{K}\left(1-k_-^2\right) \,. \tag{9.203}$$

It is simple to prove by direct calculation that also eq. (9.203) solves the homogeneous DEQ (9.156). Of course, since a third-order DEQ admits only three independent solutions, this last solution cannot be linearly independent from the previous three. The linear dependence can be checked numerically, for instance through the PSLQ algorithm. Since the four functions develop imaginary parts for $x < 0$ or $x > 1/4$, the exact relation between the solutions depends on the value of the variable x and on the convention for their analytic continuation. With the choice (9.195) and by and taking $0 < x < \frac{1}{4}$ (where all solutions eqs. (9.201,9.203) are real valued), we find

$$H_1^{(4)}(x)|_{0<x<\frac{1}{4}} = \frac{1}{3}H_1^{(1)}(x)|_{0<x<\frac{1}{4}} \,. \tag{9.204}$$

One last comment is in order. We have computed explicitly two out of the three integrals (9.187) and we have nonetheless been able to extract a representation for *all* three independent solutions of the DEQ. Indeed, if $f_1^{\mathrm{IV}}(x)$ is also a solution of the the DEQ, and if the solutions chosen in eq. (9.201) are independent, it must be possible to write $f_1^{\mathrm{IV}}(x)$ as a linear combination of the latter. Consistently, if we consider the range $1/4 < x < 1$, where $f_1^{\mathrm{IV}}(x)$ is real-valued, and we adopt the prescription $x \to x + i\,0^+$, we find

$$f_1^{\mathrm{IV}}(x) = 4\left(H_1^{(1)}(x) + i\,H_1^{(2)}(x) - i\,H_1^{(3)}(x)\right) \,, \tag{9.205}$$

which, as expected, shows that also $f_1^{\mathrm{IV}}(x)$ is a solution of the third-order DEQ satisfied by the banana graph.

9.5.3 The third-order differential equation as a symmetric square

In the previous section we showed that the independent solutions of the third-order DEQ (9.156) can be found by integrating the maximal-cut of the three-loop massive banana graph along independent contours. In [105], an alternative representation of the solutions was found, in a different context, by observing that eq. (9.156) is a *symmetric square*.

A third-order differential operator $L_3(x)$,

$$L_3(x) = \partial_x^3 + c_2(x)\partial_x^2 + c_1(x)\partial_x + c_0(x) \,, \tag{9.206}$$

is said to be a *symmetric square* if three independent solutions $g_i(x)$ of the corresponding homogeneous equation can be written as

$$g_1(x) = (f_1(x))^2 \,, \qquad g_2(x) = f_1(x)\,f_2(x) \,, \qquad g_3(x) = (f_2(x))^2 \,, \tag{9.207}$$

where the two functions $f_1(x)$ and $f_2(x)$ are in turn solutions of the second-order DEQ defined by the differential operator

$$L_2(x) = \partial_x^2 + a_1(x)\partial_x + a_0(x) \,. \tag{9.208}$$

Testing whether a third-order differential operator is a symmetric square and building the corresponding second-order differential operator is very simple. By starting from the coefficients of $L_3(x)$ in eq. (9.206) we can build the following two combinations

$$\alpha_1(x) = \frac{1}{3}c_2(x)\,, \qquad \alpha_0(x) = \frac{c_1(x) - \alpha_1'(x) - 2\alpha_1^2(x)}{4}\,, \tag{9.209}$$

where $\alpha_1'(x) = \partial_x \alpha_1(x)$. Now, if the following relation is satisfied

$$4\,\alpha_0(x)\alpha_1(x) + 2\,\frac{d\alpha_0(x)}{dx} = c_0(x)\,, \tag{9.210}$$

then the differential operator $L_3(x)$ is the symmetric square of the second-order differential operator

$$L_2(x) = \frac{d^2}{d^2x} + \alpha_1(x)\frac{d}{dx} + \alpha_0(x)\,. \tag{9.211}$$

It is straightforward to check that the third-order DEQ (9.156) is a symmetric square. Given the coefficients

$$c_2(x) = \frac{3(8x-5)}{2(x-1)(4x-1)}\,, \qquad c_1(x) = \frac{4x^2-2x+1}{(x-1)x^2(4x-1)}\,, \qquad c_0(x) = \frac{1}{x^3(4x-1)}\,, \tag{9.212}$$

we find

$$\alpha_1 = \frac{8x-5}{2(x-1)(4x-1)}\,, \qquad \alpha_0 = -\frac{2x-1}{4(x-1)x^2(4x-1)}\,, \tag{9.213}$$

which indeed satisfy

$$4\,\alpha_0(x)\alpha_1(x) + 2\,\partial_x\alpha_0(x) = \frac{1}{x^3(4x-1)} = c_0(x)\,. \tag{9.214}$$

Therefore the DEQ obeyed by the three-loop banana graph can be written as symmetric square combinations of the two independent solutions of the second-order DEQ

$$\left[\frac{d^2}{d^2x} + \frac{8x-5}{2(x-1)(4x-1)}\frac{d}{dx} - \frac{2x-1}{4(x-1)x^2(4x-1)}\right] f(x) = 0\,, \tag{9.215}$$

Eq. (9.215) can be solved in terms of a class of special functions known as it Heun functions and it can be shown that such functions can be rewritten as a product of elliptic integrals

$$\begin{aligned}
\mathcal{H}_1^{(1)}(x) &= \mathrm{K}\left(\omega_+\right)\mathrm{K}\left(\omega_-\right)\,,\\
\mathcal{H}_1^{(2)}(x) &= \mathrm{K}\left(\omega_+\right)\mathrm{K}\left(1-\omega_-\right)\,,\\
\mathcal{H}_1^{(3)}(x) &= -\frac{1}{3}\mathrm{K}\left(1-\omega_+\right)\mathrm{K}\left(1-\omega_-\right)\,,
\end{aligned} \tag{9.216}$$

where we defined

$$\omega_\pm = \frac{1}{4x}\left(2x + (1-2x)\sqrt{\frac{x-1}{x}} \pm \sqrt{\frac{4x-1}{x}}\right)\,. \tag{9.217}$$

For a detailed derivation of these results, we refer the reader to [105] and references therein, where it is also shown that the solutions (9.216) are explicitly real for for $x >$ and that the elliptic integrals $\mathrm{K}(\omega_\pm)$ satisfy

$$\mathrm{K}(\omega_+) = \left(\sqrt{4-\frac{4}{x}} - \sqrt{1-\frac{1}{x}}\right)\mathrm{K}(\omega_-)\,, \quad x > 1\,. \tag{9.218}$$

Being solutions of the same DEQ, the representations (9.201) and (9.216) must be equivalent, although they look quite different. Proving by algebraic manipulations this equivalence is highly non-trivial, in particular since the exact relations among the two depends on the particular kinematic region. In fact, every time we cross a singular point of the DEQ (9.156), each of the three solutions (9.216) gets mapped to a linear combination of the three functions H_i

To give an example, we consider he region $x > 1$ where the functions (9.216) are real and we use the prescription $x \to x + i0^+$ to analytically continue the solutions found in (9.201). By making use of the PSLQ algorithm we can verify (with virtually arbitrary precision) that

$$\begin{aligned}
\mathcal{H}_1^{(1)}(x) &= 2\,i\,H_1^{(1)}(x) - H_1^{(2)}(x) + H_1^{(3)}(x)\,,\\
\mathcal{H}_1^{(2)}(x) &= -2\,i\,H_1^{(1)}(x) + 3H_1^{(2)}(x) - H_1^{(3)}(x)\,,\\
\mathcal{H}_1^{(3)}(x) &= -i\,H_1^{(2)}(x) + i\,H_1^{(3)}(x)\,,
\end{aligned} \tag{9.219}$$

which shows the equivalence between the two sets of solutions.

With this representation of the homogeneous solutions of $\mathrm{F}_1(x)$, we can determine the other two rows of the matrix $\mathbb{G}(x)$ by differentiating (9.201) as in eq. (9.158). Obviously, the Wronskian of this new matrix $\mathbb{G}(x)$ still satisfies Abel Theorem (9.164),

$$W(x) = \frac{\pi^3 x^3}{64\sqrt{(4x-1)^3(x-1)}}\,. \tag{9.220}$$

In the last two sections, we have obtained two equivalent representations of the matrix $\mathbb{G}(x)$ defined in eq. (9.159). In both cases, however, the homogeneous solutions are well-defined only on a limited interval of the x-axis, since they have branching-points in all the four singular points $x = 0$, $x = 1/4$, $x = 1$ and $x = \infty$ of the DEQ (9.156). In particular, it can be verified that the solutions built in this section are real for $x > 1$ but they develop an imaginary part whenever $x < 1$. In addition, they have discontinuities in all other singular points. On the other hand, the solutions found in (9.201) turn out to be real only for $0 < x < 1/4$. In order to properly analytically continue the results for every value of x, we need to build other sets of solutions, similar to (9.216) or (9.201), but real-valued in in the remaining regions $(-\infty, 0)$, $(1/4, 1)$, and then match the different real representations of $\mathbb{G}(x)$ across the singular points. Indeed, in every patch (a, b) real solutions can be built from simple linear combinations of (9.201) or (9.216). Many different combinations are possible and we refer to appendix C for the details of one possible choice. In order to distinguish these different sets of real-valued solutions we introduce the notation $H_{(k)}^{[i;\,(a,b)]}(x)$, which stems for the k-th homogeneous solution of i-th MI which is real for $a < x < b$. The corresponding matrix of solutions is then indicated as $\mathbb{G}^{(a,b)}(x)$. Finally, we normalize the homogeneous solutions for all values of x in such a way that the corresponding Wronskian $W^{(a,b)}(x)$ is equal to (9.220), up to a possible overall factor i, which appears when the argument of the square-root becomes negative.

9.5.4 The inhomogeneous solution

In this section, we make use of the procedure described in section 9.1 in order to compute the series expansion around $d = 2$ of the solution of the system (9.150). The following discussion holds for any of the kinematic regions $a < x < b$ located by the four singular points of the DEQs and, hence we keep giving as understood the subscript (a, b).

Both bases of MIs $\mathbf{F}_i(x)$ and $\mathbf{I}_i(x)$ are finite in $d = 2$ and they can be Taylor-expanded as

$$\mathrm{F}_i(\epsilon, x) = \sum_{k=0}^{\infty} \mathrm{F}_i^{(k)}(x)\epsilon^k \,, \qquad \mathrm{I}_i(\epsilon, x) = \sum_{k=0}^{\infty} \mathrm{I}_i^{(k)}(x)\epsilon^k \quad i = 1, 2, 3. \tag{9.5.1}$$

By substituting eq. (9.5.1) into (9.160), we obtain a particularly simple set of chained first-order DEQs for the coefficients $\mathrm{I}_i^{(k)}(x)$,

$$\partial_x \begin{pmatrix} \mathrm{I}_1^{(0)} \\ \mathrm{I}_2^{(0)} \\ \mathrm{I}_3^{(0)} \end{pmatrix} = \mathbb{G}^{-1}(x) \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2(1-4x)} \end{pmatrix} \tag{9.5.2}$$

and

$$\partial_x \begin{pmatrix} \mathrm{I}_1^{(n)}(x) \\ \mathrm{I}_2^{(n)}(x) \\ \mathrm{I}_3^{(n)}(x) \end{pmatrix} = \epsilon\, \mathbb{G}^{-1}(x)\mathbb{A}_1(x)G(x) \begin{pmatrix} \mathrm{I}_1^{(n-1)}(x) \\ \mathrm{I}_2^{(n-1)}(x) \\ \mathrm{I}_3^{(n-1)}(x) \end{pmatrix} , \quad n > 0. \tag{9.5.3}$$

From eq. (9.5.3) we see that the MIs $\mathbf{I}(x)$ have a manifest iterative structure (see eq. (9.11)), since each coefficient $\mathbf{I}^{(n)}(x)$ can be simply written as an integral of the lower order term $\mathbf{I}^{(k-1)}(x)$, convoluted with the integration kernel $\mathbb{G}^{-1}(x)\mathbb{A}_1(x)\mathbb{G}(x)$. Together with the integration of the lowest order (9.5.2), this kernel specifies the class of functions required at every order in ϵ.

Once $\mathbf{I}^{(n)}(x)$ have been determined, the corresponding term of the ϵ-expansion of the original MIs $\mathbf{F}(x)$ can be obtained by applying the rotation matrix $\mathbb{G}(x)$ back to the integrals $\mathbf{I}(x)$, according to the definition (9.4),

$$\begin{pmatrix} \mathrm{F}_1^{(n)}(x) \\ \mathrm{F}_2^{(n)}(x) \\ \mathrm{F}_3^{(n)}(x) \end{pmatrix} = \mathbb{G}(x) \begin{pmatrix} \mathrm{I}_1^{(n)}(x) \\ \mathrm{I}_2^{(n)}(x) \\ \mathrm{I}_3^{(n)}(x) \end{pmatrix} . \tag{9.5.4}$$

In the remaining part of this section, we will limit ourselves to the determination of the order zero terms $\mathbf{F}^{(0)}(x)$. Of course, this does not provide a complete solution to the problem, as the calculation of the leading term of the ϵ-expansion does not require to integrate over the kernel $\mathbb{G}^{-1}(x)\mathbb{A}_1(x)\mathbb{G}(x)$. Although the method described in this section can be used to provide a representation suitable for numerical evaluation also for the higher orders terms, it would be advisable to first study and classify the properties of the functions defined by repeated integrations over the kernel above, in order to have analytic control over the solution. This problem will be addressed in future work.

The DEQs (9.5.2) can be readily solved by quadrature,

$$\begin{aligned}
\mathrm{I}_1^{(0)}(x) &= c_1^{(0)} + \int_{x_0}^{x} \mathrm{d}t \frac{1}{1-4t} \mathcal{R}_1(t)\,, \\
\mathrm{I}_2^{(0)}(x) &= c_2^{(0)} + \int_{x_0}^{x} \mathrm{d}t \frac{1}{1-4t} \mathcal{R}_2(t)\,, \\
\mathrm{I}_3^{(0)}(x) &= c_3^{(0)} + \int_{x_0}^{x} \mathrm{d}t \frac{1}{1-4t} \mathcal{R}_3(t)\,,
\end{aligned} \tag{9.5.5}$$

where the integration base-point x_0 can be arbitrarily chosen and the integration constants $c_i^{(0)}$ have to be fixed by imposing suitable boundary conditions. The integrands $\mathcal{R}_i(x)$ are combinations of products of two homogeneous solutions, which originate from the entries of $\mathbb{G}^{-1}(x)$ (see eq. (9.162)),

$$\begin{aligned}
\mathcal{R}_1(x) =& \frac{1}{2W(x)} \left[H_2^{(3)}(x) H_1^{(2)}(x) - H_1^{(3)}(x) H_2^{(2)}(x) \right]\,, \\
\mathcal{R}_2(x) =& \frac{1}{2W(x)} \left[H_2^{(1)}(x) H_1^{(3)}(t) - H_1^{(1)}(x) H_2^{(3)}(x) \right]\,, \\
\mathcal{R}_3(x) =& \frac{1}{2W(x)} \left[H_1^{(1)}(x) H_2^{(2)}(t) - H_2^{(1)}(x) H_1^{(2)}(x) \right]\,.
\end{aligned} \tag{9.5.6}$$

Therefore eqs. (9.5.5) and (9.5.6) completely specify the inhomogeneous solution at order zero once the boundary constants $c_i^{(0)}$ have been fixed, for instance by imposing the regularity of the solutions at specific kinematic points.
We have already observed that the system (9.150), or equivalently the third-order DEQ (9.156), have regular singular points at $x = 1$ and $x = \pm\infty$, which correspond, respectively, to the pseudo-thresholds $s = 4m^2$ and $s = 0$ of the equal-mass banana graph. We can show that demanding the regularity of $\mathrm{F}_i^{(0)}(x)$ at such points is sufficient to fix the three integration constants $c_i^{(0)}$. In fact, by imposing regularity directly on the system of DEQs, we can determine three independent linear relations, which must be satisfied by the MIs at the pseudo-thresholds. In particular, regularity at $x \to 1^{\pm}$ requires

$$\lim_{x\to 1^{\pm}} \left(\mathrm{F}_3^{(0)}(x) + \frac{2}{3}\mathrm{F}_2^{(0)}(x) + \frac{1}{12}\mathrm{F}_1^{(0)}(x) \right) = 0\,, \tag{9.5.7}$$

whereas at $x \to \pm\infty$, we find

$$\lim_{x\to \pm\infty} \left(\mathrm{F}_2^{(0)}(x) + \frac{1}{4}\mathrm{F}_1^{(0)}(x) \right) = 0\,, \tag{9.5.8}$$

$$\lim_{x\to \pm\infty} \left(\mathrm{F}_3^{(0)}(x) - \frac{1}{16}\mathrm{F}_1(x)^{(0)} \right) = \frac{1}{8}\,. \tag{9.5.9}$$

It is worths observing that, since $x \to \pm\infty$ corresponds to $s \to 0^{\pm}$, the two conditions (9.5.8, 9.5.9) consistently reproduce the IBPs identities between the three-loop vacuum diagrams to which the MIs are reduced in the zero-momentum limit.

It is particularly convenient to fix explicitly the boundary constants by working in the region $1 < x < \infty$, since the end-points of this region corresponds exactly to the two pseudo-thresholds. If we specify eq. (9.5.5) to the interval $(1, \infty)$ and apply the rotation (9.5.6), we get

$$\mathrm{F}_1^{(0)}(x) = \quad H^{(1)}_{[1;\,(1,\infty)]}(x) \left[c_1^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t} \mathcal{R}_1^{(1,\infty)}(t) \right]$$

$$
\begin{aligned}
& + H^{(2)}_{[1;\,(1,\infty)]}(x)\left[c_2^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t}\mathcal{R}_2^{(1,\infty)}(t)\right] \\
& + H^{(3)}_{[1;\,(1,\infty)]}(x)\left[c_3^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t}\mathcal{R}_3^{(1,\infty)}(t)\right], \\
\mathrm{F}_2^{(0)}(x) = \; & H^{(1)}_{[2;\,(1,\infty)]}(x)\left[c_1^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t}\mathcal{R}_1^{(1,\infty)}(t)\right] \\
& + H^{(2)}_{[2;\,(1,\infty)]}(x)\left[c_2^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t}\mathcal{R}_2^{(1,\infty)}(t)\right] \\
& + H^{(3)}_{[2;\,(1,\infty)]}(x)\left[c_3^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t}\mathcal{R}_3^{(1,\infty)}(t)\right], \\
\mathrm{F}_3^{(0)}(x) = \; & H^{(1)}_{[3;\,(1,\infty)]}(x)\left[c_1^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t}\mathcal{R}_1^{(1,\infty)}(t)\right] \\
& + H^{(2)}_{[3;\,(1,\infty)]}(x)\left[c_2^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t}\mathcal{R}_2^{(1,\infty)}(t)\right] \\
& + H^{(3)}_{[3;\,(1,\infty)]}(x)\left[c_3^{(0)} + \int_1^x \mathrm{d}t \frac{1}{1-4t}\mathcal{R}_3^{(1,\infty)}(t)\right],
\end{aligned}
\tag{9.5.10}
$$

where we have chosen as integration base-point $x_0 = 1$ and re-introduced the superscript $(1,\infty)$ for all quantities that require analytic continuation. We remark that, when applied to eq. (9.5.10), the definition (9.5.6) of the function $\mathcal{R}_i^{(1,\infty)}(x)$ must be interpreted in terms of the homogeneous solutions $\mathbb{G}^{(1,\infty)}(x)$, which are defined in (C.1.4). Due to the choice of the integration base-point, in the $x \to 1^+$ limit all integrals appearing in the r.h.s. of (9.5.10) vanish and the MIs become

$$
\lim_{x\to 1^+} \mathrm{F}_i^{(0)}(x) = \lim_{x\to 1^+}\left(c_1^{(0)} H^{(1)}_{[i;\,(1,\infty)]}(x) + c_2^{(0)} H^{(2)}_{[i;\,(1,\infty)]}(x) + c_3^{(0)} H^{(3)}_{[i;\,(1,\infty)]}(x)\right). \tag{9.5.11}
$$

The limiting behaviours of the homogeneous solutions at the two pseudo-thresholds are discussed in appendix C and it is easy to verify that, when the expansions at $x \to 1^+$ (C.2.4) are plugged into eq. (9.5.11), the regularity constraint (9.5.7) is satisfied by demanding

$$
c_3^{(0)} = -3 c_1^{(0)}. \tag{9.5.12}
$$

In a similar way, we can impose regularity at $x \to +\infty$ by making use of the expansions (C.2.6). Remarkably, due to the presence of logarithmic divergences $\ln(1/x)$ in the expansion of the homogeneous solutions which must cancel in the expression of the MIs, eq. (9.5.8) allows us to fix at once $c_1^{(0)}$ and $c_2^{(0)}$,

$$
\begin{aligned}
c_1^{(0)} = & \frac{1}{3}\int_1^\infty dt \frac{1}{(1-4t)}\mathcal{R}_3^{(1,\infty)}(t) = \frac{1}{3}\int_0^1 \frac{dy}{y(y-4)}\mathcal{R}_3^{(1,\infty)}\left(\frac{1}{y}\right), \\
c_2^{(0)} = & -\int_1^\infty \frac{dt}{(1-4t)}\mathcal{R}_2^{(1,\infty)}(t) = -\int_0^1 \frac{dy}{y(y-4)}\mathcal{R}_2^{(1,\infty)}\left(\frac{1}{y}\right).
\end{aligned}
\tag{9.5.13}
$$

In the second equalities, we have simply performed the change of variable $t \to 1/y$ in order to map the integration range to $0 < y < 1$. As a consistency check, we have verified that these values of the integration constants are in agreement also with the regularity condition for the third master, given by eq. (9.5.9).

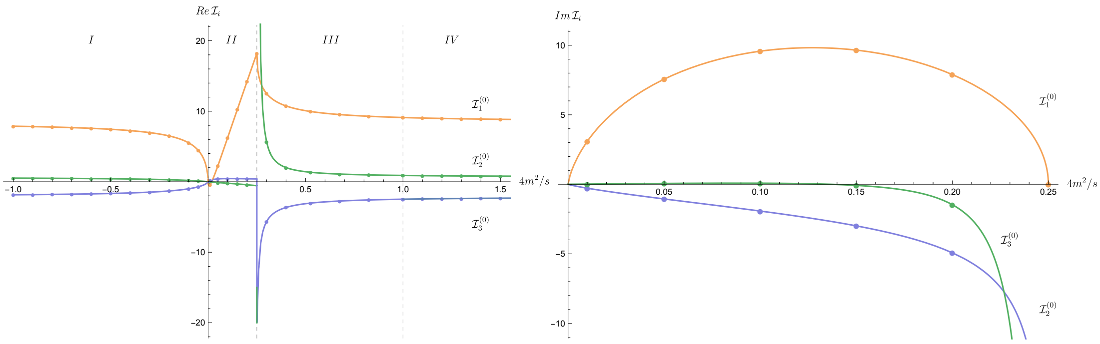


Figure 9.3: Real part (left panel) and imaginary part (right panel) of the finite term of the three MIs for the three-loop banana graph. The imaginary part is non-vanishing only in the range $0 < x < 1/4$, which corresponds to $s > 16m^2$. The numerical evaluation of the integral representation (9.5.10) (solid curves) is compared against the values obtained with SecDec (dots).

Although we were not able to determine an analytic expression of the boundary constants, their representation as definite integrals (9.5.13) allows a high-precision numerical evaluation,

$$c_1^{(0)} = -\,1.2064599496517629858762117245910770452963348722\ldots$$
$$c_2^{(0)} = +\,2.5819507507087486799289938331551672385057488393\ldots \qquad (9.5.14)$$

The representation (9.5.10) of the MIs, which is now fully determined, is valid for $1 < x < \infty$. The expression of $\mathbf{F}^{(0)}(x)$ in the other kinematic regions (and in particular in $0 < x < 1/4$, where the MIs develop an imaginary part) can be obtained by analytic continuation of eq. (9.5.10), along the line of the discussion of [271], as we illustrate in appendix C. As a summary of the results there presented, let us just stress that the determination of a set of explicitly real solutions $\mathbb{G}^{(a,b)}(x)$ in each region and the study of their leading behaviour in the proximity of the singular points can be used to define, for any value of x, a representation of the solutions which involves individually real-valued integrals and, therefore, allows a fast and accurate numerical evaluation. A plot of the numerical results obtained through our representation, compared against the computer code SecDec [67] is shown in figure 9.3.

9.6 Conclusions

In this chapter, we have addressed the problem of the solution of systems of differential equations which remain coupled in the $\epsilon \to 0$ limit. In this cases, the only available solving strategy consists in first determining a complete set of *independent homogeneous solutions* of the system of DEQs in $\epsilon = 0$ and then using Euler method of the variation of constants in order to reconstruct the complete inhomogeneous solution.

In practice, not dissimilarly from the case of triangular systems of DEQs, if we can start from a coupled system with a linear dependence on ϵ, the redefinition of the MIs through a rotation matrix built from the homogeneous solutions absorbs the $\mathrm{O}(\epsilon^0)$ part of the DEQs and brings the system into an ϵ-factorized form. The latter hugely facilitate the expression of the MIs as series expansion around $\epsilon = 0$ in terms of iterated integrals. Hence, when dealing with a system of n coupled first-order DEQs, which can always be rephrased in terms of a n-th order DEQ for one of the MIs, the main difficulty lies exactly in the determination of the complete set of n homogeneous solutions, since no general theory for higher-order DEQs is available.

Based on the simple observation that the maximal-cut of a Feynman integral fulfils, by construction, the homogeneous part of the corresponding DEQ, we have proposed to algorithmically determine a closed-form representation of the homogeneous solutions from the direct computation of the maximal-cuts of the coupled MIs [4].

At multi-loop level, the simultaneous cut of all denominators does not localize the integrand completely, so that the homogeneous solution is represented as a (multi-fold) integral over the region of the loop momentum space deformed by on-shell conditions of the loop denominators. Very interestingly, the integration domain can be split into different subregions, localized by the branch-cut structure of the integrand, which are still compatible with the homogeneous DEQs.

This observation suggests that it is possible to determine n-independent *integration domains*, which correspond to the n-independent solutions of the coupled system of DEQs [2, 62]. Hence, the computation of the maximal-cuts completely solves the problem of bringing the DEQs in ϵ-factorized form, independently from the size of the non-triangular blocks. This conjecture is supported by a growing number of examples of two-loop cases of 2×2 system of coupled DEQs, where the homogeneous solutions turn out to be evaluated in terms of *complete elliptic integrals*. In this chapter, we have successfully tested the proposed strategy on the three-loop massive banana graph, which constitute the simplest known example of MI obeying a genuine third-order DEQ [105, 106]. In this case, the computation of the maximal-cut has provided three independent solutions, which have been expressed in terms of *products of complete elliptic integrals*.

Having at disposal a closed integral representation of the homogeneous solutions is the first, fundamental step in the determination of the complete solution of the DEQs but, of course, is not the final one.

The solution of the homogeneous DEQs allows to write ϵ-expansion of the MIs in terms of iterated integrals over kernel functions which involve products of homogeneous solutions. Although a rather efficient numerical evaluation of such integrals is possible, a full analytic control of the solution is indeed advisable, both from a practical and a theoretical point of view.

To this aim, we need to understand and classify the properties of the new classes of functions which emerge from the nested integration over elliptic (and, presumably, even more complicated) kernels. Although very little is known about higher-order cases, a broad literature has been produced in the attempt of classifying the properties of the inhomogeneous solutions built from elliptic integrals, and many proposal for elliptic extensions of generalized polylogarithms have been put forward [102, 109–123].

These studies, which will be at the basis of the possibility to extend the concept of dlog-form to systems of coupled DEQs will deepen our understanding of the mathematical properties of Feynman integrals and they will definitely constitute one of the most fascinating subjects for future developments at the intersection between mathematics and particle physics.

Chapter 10

Conclusions

The stunning precision of the experimental measurements produced by the ongoing Run II of the Large Hadron Collider demands accurate theoretical predictions of the observed phenomena in order to test the fundamental mechanisms of the Standard Model and investigate possible indirect signals of new physics. Such accurate predictions require the evaluation of higher-order terms in the perturbative expansion of scattering amplitudes. The latter involve the computation of many loop integrals, which depend on a large number of external particles and mass-scales.

In this thesis, we have investigated new techniques for the analytical evaluation of multi-loop scattering amplitudes that proceeded by exploiting the wide range of consequences of the *unitarity* of the scattering matrix in Quantum Field Theory.

The basic concept of the so-called *generalized unitarity methods* consists in retrieving information about the mathematical properties of scattering amplitudes by analyzing their behaviour on *multiple cuts*, i.e. by enforcing the *on-shell* condition of their internal propagators.

Building upon this idea, we have employed generalized unitarity in the multi-loop extension of the *integrand decomposition method* and in the solution of *differential equations* for Feynman integrals.

In the first part of this thesis, we have presented the *adaptive integrand decomposition*, a new and simplified reduction algorithm that is suitable for both analytic and numerical application to general multi-loop amplitudes.

The integrand-level reduction of a multi-loop amplitude to a linear combination of scalar integrals entails the iterative multivariate polynomial division of integrand numerators modulo Gröbner basis of the ideal generated by the corresponding set of loop denominators.

With the present work, we have shown that this elaborate division procedure can be drastically simplified by replacing the expensive computation of Gröbner bases with the solution of systems of linear equations. This simplification is made possible by the introduction of a parametrization of Feynman integrands in terms of *longitudinal* and *transverse* variables. The first ones belong to the portion of space-time spanned by momenta of the external particles, while the the latter lie in the complementary, orthogonal space whose dimension acts as a regulator of loop divergencies.

The loop denominators are independent of the transverse coordinates and, as a consequence, the set of on-shell conditions that characterize a multiple-cut are linearized. In addition, the polynomial dependence of the numerator on the transverse components reveals the hyperspherical geometry of the transverse space, which allows their algorithmic integration through the expansion of the integrand in terms of *Gegenbauer poly-*

nomials. The systematic integration over the transverse variables eliminates spurious contributions step-by-step in the reduction procedure and thins down the intermediate expressions generated by the recursive structure of the algorithm.

These features depend exclusively on the external kinematics and are insensitive to the presence of internal masses and to the (non-)planar structure of the loop denominators. Therefore, they can be *adaptively* applied to all integrand topologies that contribute to an amplitude.

The outcome of integrand decomposition in the longitudinal and transverse space is the reduction of any multi-loop amplitude to a sum of irreducible integrands which are free from any spurious contribution and whose tensor numerators are expressed in terms of irreducible scalar products between loop momenta and external ones.

The overall simplified structure of this new reduction algorithm allowed, for the first time, the automation of the *analytic* integrand decomposition of one- and two-loop amplitudes, which we have implemented in the MATHEMATICA code AIDA (Adaptive Integrand Decomposition Algorithm). The code was used to test the feasibility of the full reduction chain on the calculation of the two-loop QED virtual corrections to the elastic scattering $\mu e \to \mu e$, which constituted the first non-trivial proof of concept of the applicability of the adaptive integrand decomposition method to multi-loop, multi-scale amplitudes.

In light of these results, we believe that integrand decomposition, in its adaptive formulation, is finally in the position to become an efficient and powerful tool for the computation of scattering amplitudes beyond one loop. In the perspective of an upcoming *NNLO revolution*, we are taking steps towards embedding the proposed algorithm into automated frameworks for next-to-next-to-leading order calculations.

In the second part of this thesis, we have explored the consequences of *maximal-unitarity* on the solution of the differential equations for Feynman integrals.

The differential equations method proved to be the best available tool for the analytic calculation of dimensionally regulated loop integrals. It allows to determine the expression of the master integrals as a series expansion around small values of the regulating parameter $\epsilon = (4-d)/2$ by solving their coupled first-order differential equations in the kinematic invariants.

In most of the known cases, it is possible to decouple such differential equations in the four-dimensional limit and express the coefficients of the ϵ-expansion of the master integrals in terms of *Chen iterated integrals*, i.e. repeated integrals over derivatives of logarithmic functions.

Such iterative structure of the solution is made manifest by the introduction of a *canonical basis* of master integrals, that is to say a basis of integrals which fulfils ϵ-factorized systems of differential equations in dlog-form.

In all respects, the determination of a canonical basis amounts the solution of the system of differential equations at $\epsilon = 0$. If the system of differential equations can be cast into an ϵ-linear form, such a solution can be used to define a similarity transformation of the integral basis which factorizes the ϵ-dependence of the system.

The solution of the differential equations at $\epsilon = 0$ can be efficiently computed through its Magnus exponential series which, for triangularizable systems, can be resummed into a closed analytic form. Besides defining the similarity transformation of the master integrals to a canonical form, the Magnus exponential determines their analytic expression in the whole phase space by acting as kinematical evolution operator on an arbitrary boundary value of the integrals.

In this thesis, we have applied the Magnus exponential method to determine two-loop three- and four-point integrals that depend on up to four kinematic scales. The considered integrals enter QCD corrections to the triple gauge couplings ZWW and γ^*WW, to the Higgs boson decay into an off-shell W-boson pair, $H \to WW$, and to the massive boson-pair production in the gluon-fusion channel, $gg \to HH$ and $gg \to WW$. In addition, we have computed all planar four-point integrals for the two-loop QED corrections to the elastic scattering $\mu e \to \mu e$. These integrals constitute an important part of the theoretical input needed for the determination of the leading hadronic contribution to the muon $g-2$, which is the goal of the recently proposed experiment MUonE to be realized at CERN. The same integrals are also useful for virtual corrections to $t\bar{t}$-production at colliders. Future studies will include the application of the techniques presented in this thesis to the evaluation of the last missing non-planar integral for μe scattering and to the computation, through the adaptive integrand decomposition algorithm, of the full virtual amplitude.

If the system of differential equations cannot be decoupled in four dimensions, the master integrals obey irreducible *higher order differential equations*. So far the lack of a general strategy for determining their homogeneous solutions has made their computation particularly challenging.

Based on the observation that maximal-cuts of Feynman integrals fulfil the homogeneous part of the full differential equations satisfied by the uncut integrals, we have proposed a general method to derive an integral representation of the homogeneous solutions by systematically computing the maximal-cut of the master integrals along different, independent contours. Remarkably, the number of such contours proved to correspond to the order of the differential equations satisfied by the master integrals and, therefore, to be sufficient for the determination of the complete set of homogeneous solutions. These solutions can then be used to define a change of basis which brings the coupled system of differential equations into an ϵ-factorized form. Consequently, the full inhomogeneous solution is determined by iterative integration over the homogeneous kernels.

In the last part of this work, we have applied this method to the determination of the homogeneous solutions of the master integrals for three- and four-point functions that enter two-loop corrections to $gg \to gg$ and $gg \to gH$ with the inclusion of finite *top*-quark mass effects. In all cases, the solutions of the corresponding homogeneous second-order differential equations turned out to be expressed in terms of *complete elliptic integrals*. In addition, as a proof of the generality of this technique, we have computed the leading term of the ϵ-expansion of the massive three-loop banana graph, which constitute the first remarkable example of Feynman integral satisfying an irreducible third-order differential equation.

As a whole, this thesis puts in a new light the role of unitarity in higher-order computations in perturbation theory. Generalized unitarity is revealed as the unifying *leitmotiv* of all the steps that lead to the prediction of scattering amplitudes beyond leading order. With this work we have shown that, far from being a pure computational tool, generalized unitarity methods provide strong insights into the connections of particle physics with the most advanced branches of mathematics, which include algebraic and differential geometry, and number theory. We have no doubt that further investigations of these connections will lead in the near feature to a deeper understanding of the structure of fundamental interactions.

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Appendix A

Longitudinal and transverse space for Feynman integrals

A.1 Spherical coordinates for multi-loop integrals

In this appendix, we derive the the $d = d_{\parallel} + d_{\perp}$ representation defined in eq. (2.43) for the multi-loop Feynman integral

$$I^{d\,(\ell,n)}_{a_1\ldots a_m}[\mathcal{N}] = \int \prod_{j=1}^{\ell} \frac{\mathrm{d}^d q_j}{\pi^{d/2}} \frac{\mathcal{N}(q_i)}{D_1^{a_1} D_2^{a_2} \cdots D_m^{a_m}}\,, \quad a_i \in \mathbb{N}\,. \tag{A.1.1}$$

We start by studying the properties of a set of auxiliary integrals of the type

$$I^{\kappa}_{\ell} = \int \prod_{i=1}^{\ell} d^m \boldsymbol{\lambda}_i I_{\ell}(\boldsymbol{\lambda}_i)\,, \tag{A.1.2}$$

where $\boldsymbol{\lambda}_i$ are vectors of an Euclidean space, which dimension m is first assumed to be an integer and then analytically continued to complex values. In particular, assuming the vectors $\boldsymbol{\lambda}_i$ to be decomposed w.r.t. an orthonormal basis $\{\mathbf{v}_i\}$,

$$\boldsymbol{\lambda}_i = \sum_{j=1}^{m} a_{ji} \mathbf{v}_j\,, \tag{A.1.3}$$

we are interested in the case where the integrand I^{κ}_{ℓ} depends explicitly, besides on the scalar products $\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j$, on a finite number $\kappa < m-1$ of components of each vector $\boldsymbol{\lambda}_i$, which, without loss of generality, we can choose to be a_{ji}, with $j \leq \kappa$. These auxiliary integrals will be later identified with the transverse integrals of the ℓ-loop Feynaman integrals (A.1.1). We will derive explicit formulae up to $\ell = 3$ and we will then deduce the general expression for an arbitrary number of loops.

- Let us start by considering the one-loop case, which involves the evaluation of integrals of the type

$$I^{\kappa}_1 = \int \mathrm{d}^m \boldsymbol{\lambda}_1\, \mathcal{I}_1(\boldsymbol{\lambda}_1)\,. \tag{A.1.4}$$

We assume $\boldsymbol{\lambda}_1$ to be decomposed with respect to an orthonormal basis $\{\mathbf{v}_i\}$ as

$$\boldsymbol{\lambda}_1 = \sum_{i=1}^{m} a_{i1} \mathbf{v}_i. \tag{A.1.5}$$

Regardless of the symmetries of the integrand, we can reparametrize I_1 in terms of spherical coordinates in m dimensions which, being $\{\mathbf{v}_i\}$ orthonormal, are defined by the well-known change of variables

$$\begin{cases} a_{11} &= \sqrt{\lambda_{11}} \cos\theta_{11}, \\ &\dots \\ a_{k1} &= \sqrt{\lambda_{11}} \cos\theta_{k1} \prod_{i=1}^{k-1} \sin\theta_{i1} \\ &\dots \\ a_{m1} &= \sqrt{\lambda_{11}} \prod_{i=1}^{m-1} \sin\theta_{i1}, \end{cases} \tag{A.1.6}$$

where $\sqrt{\lambda_{11}} \in [0, \infty)$ and all angles range over the interval $[0, \pi]$, except for $\theta_{(m-1)\,1} \in [0, 2\pi]$. Hence, by introducing the differential solid angle in M dimensions

$$d\Omega_{M-1} = (\sin\theta_1)^{M-3} \mathrm{d}\cos\theta_1 (\sin\theta_2)^{M-4} \mathrm{d}\cos\theta_2 \, \dots \, d\theta_{M-1}, \tag{A.1.7}$$

such that

$$\Omega_{M-1} = \int \mathrm{d}\Omega_{M-1} = \frac{2\pi^{\frac{M}{2}}}{\Gamma\left(\frac{M}{2}\right)}, \tag{A.1.8}$$

we can write (A.1.4) as

$$I_1^{\kappa} = \frac{1}{2} \int_0^{\infty} \mathrm{d}\lambda_{11} (\lambda_{11})^{\frac{m-2}{2}} \int \mathrm{d}\Omega_{m-1} \mathcal{I}_1(\lambda_{11}, \cos\theta_{i1}, \sin\theta_{i1}). \tag{A.1.9}$$

If the integrand is rotational invariant, i.e. it depends on $\lambda_{11} = \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_1$ only, we can integrate over all angular variables in such a way to obtain, by specifying eq. (A.1.8) for $M = m$,

$$I_1^{0} = \frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} \mathrm{d}\lambda_{11} (\lambda_{11})^{\frac{m-2}{2}} \mathcal{I}_1(\lambda_{11}). \tag{A.1.10}$$

However, in general cases, the integrand can show an explicit dependence on a subset of $\kappa < m-1$ components of $\boldsymbol{\lambda}_1$ which, with a suitable definition of the reference frame, can always be chosen to correspond to $\{a_{11}, \dots, a_{\kappa 1}\}$. In this way, according to eq. (A.1.6), the integrand solely depends on $\boldsymbol{\Lambda} = \{\lambda_{11}\}$ and $\boldsymbol{\Theta}_{\perp} = \{\theta_{11}, \dots, \theta_{\kappa 1}\}$, while all angles θ_{i1} with $i > \kappa$ can be still integrated out by using (A.1.8) with $M = m - \kappa$. Therefore, we have

$$I_1^{\kappa} = \Omega_{(m-\kappa-1)} \int \mathrm{d}\boldsymbol{\Lambda} \int \mathrm{d}^{\kappa}\boldsymbol{\Theta}_{\perp} \mathcal{I}_1(\boldsymbol{\Lambda}, \boldsymbol{\Theta}_{\perp}), \tag{A.1.11}$$

where we have defined

$$\int d\boldsymbol{\Lambda} = \int_0^{\infty} \mathrm{d}\lambda_{11} (\lambda_{11})^{\frac{m-2}{2}}, \tag{A.1.12}$$

and

$$\int \mathrm{d}^{\kappa}\boldsymbol{\Theta}_{\perp} = \prod_{i=1}^{\kappa} \int_{-1}^{1} \mathrm{d}\cos\theta_{i1} (\sin\theta_{i1})^{m-i-2}. \tag{A.1.13}$$

- In the two-loop case, we are interested in integrals of the type

$$I_2^\kappa = \int \mathrm{d}^m \boldsymbol{\lambda}_1 \mathrm{d}^m \boldsymbol{\lambda}_2 \mathcal{I}_2(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\,. \tag{A.1.14}$$

Also in this case, we assume the two vectors $\boldsymbol{\lambda}_i$ to be decomposed in terms of a common orthonormal basis $\{\mathbf{v}_i\}$,

$$\boldsymbol{\lambda}_1 = \sum_{i=1}^m a_{i1} \mathbf{v}_i, \tag{A.1.15a}$$

$$\boldsymbol{\lambda}_2 = \sum_{i=1}^m a_{i2} \mathbf{v}_i\,, \tag{A.1.15b}$$

and suppose that the integrand depends explicitly only on the first κ components a_{ij} of each vector, as well as on $\boldsymbol{\lambda}_1^2$, $\boldsymbol{\lambda}_2^2$ and $\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2$. We would like to map all integrals associated to this subset of components of each vectors into angular integrals. For I_1^κ, due to the choice of an orthonormal basis, this mapping was immediately achieved by parametrizing the integral in terms of spherical coordinates. In this case, there is an additional direction, corresponding to $\lambda_{12} = \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2$, we need to trace back after the change of coordinates is performed, since the integrand will generally depend on it. The simultaneous factorization of the integral over the relative orientation λ_{12} and over all relevant components of the two vectors can be obtained by expressing $\boldsymbol{\lambda}_2$ into a new orthonormal basis $\{\mathbf{e}_i\}$, which contains the vector $\mathbf{e}_1 \propto \boldsymbol{\lambda}_1$. From (A.1.15a) we see that, indeed, the set of vectors

$$\{\mathbf{v}'_i\} = \{\boldsymbol{\lambda}_1, \mathbf{v}_1, ..., \mathbf{v}_{m-1}\} \tag{A.1.16}$$

is a basis, although it is not an orthogonal one. Nevertheless, we can apply the Gram-Schimdt algorithm to pass from the arbitrary basis $\{\mathbf{v}'_i\}$ to an orthonormal one $\{\mathbf{e}_i\}$, given by

$$\begin{aligned} \mathbf{e}_1 &= \frac{\mathbf{u}_1}{|\mathbf{u}_1|}, \qquad \mathbf{u}_1 = \mathbf{v}'_1, \\ \mathbf{e}_k &= \frac{\mathbf{u}_k}{|\mathbf{u}_k|}, \qquad \mathbf{u}_k = \mathbf{v}'_k - \sum_{j=1}^{k-1} (\mathbf{v}'_k \cdot \mathbf{e}_j)\mathbf{e}_j, \qquad k \neq 1. \end{aligned} \tag{A.1.17}$$

By construction, the first vector of the new basis exactly corresponds to the direction of $\boldsymbol{\lambda}_1$. Applying the change of basis to (A.1.15b), we get

$$\boldsymbol{\lambda}_2 = \sum_{i=1}^m b_{i2} \mathbf{e}_i, \tag{A.1.18}$$

where the coefficients $\{b_{i2}\}$ are related to the components of both $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ with respect to $\{\mathbf{v}_i\}$ by

$$\begin{cases} b_{12} &= \frac{\lambda_{12}}{\sqrt{\lambda_{11}}} \\ b_{22} &= \frac{a_{12}\lambda_{11} - a_{11}\lambda_{12}}{\sqrt{\lambda_{11}}\sqrt{\lambda_{11} - a_{11}^2}} \\ & \dots \\ b_{k2} &= \frac{a_{k-1\,2}\left(\lambda_{11} - \sum_{i=1}^{k-2} a_{i1}^2\right) - a_{k-1\,1}\left(\lambda_{12} - \sum_{i=1}^{k-2} a_{i1} b_{i2}\right)}{\sqrt{\lambda_{11} - \sum_{i=1}^{k-2} a_{i1}^2}\sqrt{\lambda_{11} - \sum_{i=1}^{k-1} a_{i1}^2}} \\ & \dots \\ b_{m2} = & \frac{a_{m1} a_{m-1\,2} - a_{m-1\,1} a_{m\,2}}{\sqrt{a_{m1}^2 + a_{m-1\,1}^2}}. \end{cases} \tag{A.1.19}$$

Provided that both $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are now decomposed in two different -yet- orthonormal bases, we can introduce the change of variables

$$\begin{cases} a_{11} & = \sqrt{\lambda_{11}}\cos\theta_{11} \\ & \dots \\ a_{k1} & = \sqrt{\lambda_{11}}\cos\theta_{k1}\prod_{i=1}^{k-1}\sin\theta_{i1} \\ & \dots \\ a_{m1} & = \sqrt{\lambda_{11}}\prod_{i=1}^{m-1}\sin\theta_{i1}\,, \end{cases} \tag{A.1.20}$$

and

$$\begin{cases} b_{12} & = \sqrt{\lambda_{22}}\cos\theta_{12} \\ & \dots \\ b_{k2} & = \sqrt{\lambda_{22}}\cos\theta_{k2}\prod_{i=1}^{k-1}\sin\theta_{i2} \\ & \dots \\ b_{m2} & = \sqrt{\lambda_{22}}\prod_{i=1}^{m-1}\sin\theta_{i2}\,, \end{cases} \tag{A.1.21}$$

and express the integral I_2^κ into spherical coordinates as

$$I_2^\kappa = \frac{1}{4}\int_0^\infty \mathrm{d}\lambda_{11}\mathrm{d}\lambda_{22}(\lambda_{11})^{\frac{m-2}{2}}(\lambda_{22})^{\frac{m-2}{2}}\int \mathrm{d}\Omega_{(m-1)}\mathrm{d}\Omega_{(m-1)}\mathcal{I}_2(\lambda_{ij},\cos\theta_{ij},\sin\theta_{ij}). \tag{A.1.22}$$

By combining eq. (A.1.19) with the transformations (A.1.21)-(A.1.20), we immediately see that, as expected,

$$\lambda_{12} = \sqrt{\lambda_{11}\lambda_{22}}\cos\theta_{12}. \tag{A.1.23}$$

In addition, with some more algebra, we can express back the components of $\boldsymbol{\lambda}_2$ with respect to $\{\mathbf{v}_i\}$ in terms of the angular variables,

$$\begin{cases} a_{12} & = \sqrt{\lambda_{22}}(c_{\theta_{12}}c_{\theta_{11}} + c_{\theta_{22}}s_{\theta_{11}}s_{\theta_{12}}) \\ & \dots \\ a_{i2} & = \sqrt{\lambda_{22}}[c_{\theta_{12}}c_{\theta_{i1}}\prod_{j=1}^{i-1}s_{\theta_{j1}} + c_{\theta_{i+1\,2}}s_{\theta_{i1}}\prod_{j=1}^{i}s_{\theta_{j2}} \\ & \quad -c_{\theta_{i1}}\sum_{k=2}^{i}c_{\theta_{k2}}c_{\theta_{k-1\,1}}\prod_{j=1}^{k-1}s_{\theta_{j2}}(\delta_{ik}+(1-\delta_{ik})\prod_{l=1}^{i-k}s_{\theta_{k+l-1\,1}})]\,, \end{cases} \tag{A.1.24}$$

where, for ease of notation, we have denoted $s_{\theta_{ij}} = \sin\theta_{ij}$ and $c_{\theta_{ij}} = \cos\theta_{ij}$. In this way, the integral over each component $a_{i1} \notin \{a_{m-1\,1}, a_{m\,1}\}$ of $\boldsymbol{\lambda}_1$ is mapped into the integral over the angular variable θ_{i1} whereas each component $a_{i2} \notin \{a_{m-1\,2}, a_{m\,2}\}$ of $\boldsymbol{\lambda}_2$ can be expressed in terms of the angles θ_{j1} with $j \le i$ and θ_{j2} with $j \le i+1$.

Therefore, if we are dealing with an integrand depending on the first $\kappa < m-1$ components of both vectors, we can integrate out all angular variables θ_{i1}, $j > \kappa$ and θ_{i2}, $j > \kappa+1$. Specifically, if introduce the labelling

$$\begin{aligned} \boldsymbol{\Theta}_\Lambda =& \{\theta_{12}\}, \\ \boldsymbol{\Theta}_\perp =& \{\theta_{11},\,\dots,\,\theta_{\kappa 1},\theta_{22},\,\dots,\,\theta_{\kappa+1\,2}\}, \end{aligned} \tag{A.1.25}$$

we can rewrite I_2^κ as

$$I_2^\kappa = \Omega_{(m-\kappa-1)}\Omega_{(m-\kappa-2)}\int \mathrm{d}^3\boldsymbol{\Lambda}\int \mathrm{d}^{2\kappa}\boldsymbol{\Theta}_\perp\mathcal{I}_2(\boldsymbol{\Lambda},\boldsymbol{\Theta}_\perp), \tag{A.1.26}$$

where we have defined

$$\int d^3\mathbf{\Lambda} = \int_0^\infty \mathrm{d}\lambda_{11}\mathrm{d}\lambda_{22}(\lambda_{11})^{\frac{m-2}{2}}(\lambda_{22})^{\frac{m-2}{2}}\int \mathrm{d}\mathbf{\Theta}_\Lambda\,, \tag{A.1.27}$$

with

$$\int d\mathbf{\Theta}_\Lambda = \int_{-1}^1 \mathrm{d}\cos\theta_{12}(\sin\theta_{12})^{m-3}\,, \tag{A.1.28}$$

and

$$\int d^{2\kappa}\mathbf{\Theta}_\perp = \int_{-1}^1 \prod_{i=1}^{\kappa} \mathrm{d}\cos\theta_{i1}\mathrm{d}\cos\theta_{i+1\,2}(\sin\theta_{i1})^{m-i-2}(\sin\theta_{i+1\,2})^{m-i-3}\,. \tag{A.1.29}$$

- For the three-loop case, we study the auxiliary integral

$$I_3^\kappa = \int \mathrm{d}^m\boldsymbol{\lambda}_1\mathrm{d}^m\boldsymbol{\lambda}_2\mathrm{d}^m\boldsymbol{\lambda}_3\mathcal{I}_3(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)\,. \tag{A.1.30}$$

As usual, we assume the vectors $\boldsymbol{\lambda}_i$ to be initially decomposed in terms of the same orthonormal basis $\{\mathbf{v}_i\}$,

$$\boldsymbol{\lambda}_1 = \sum_{i=1}^m a_{i1}\mathbf{v}_i, \qquad \boldsymbol{\lambda}_2 = \sum_{i=1}^m a_{i2}\mathbf{v}_i, \qquad \boldsymbol{\lambda}_3 = \sum_{i=1}^m a_{i3}\mathbf{v}_i. \tag{A.1.31}$$

When moving to spherical coordinates, we want to keep trace of the three relative orientations

$$\lambda_{12} = \boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2, \quad \lambda_{23} = \boldsymbol{\lambda}_2\cdot\boldsymbol{\lambda}_3, \quad \lambda_{13} = \boldsymbol{\lambda}_3\cdot\boldsymbol{\lambda}_1, \tag{A.1.32}$$

together with the subset of κ components a_{ij}, $j\le\kappa$, of each vector. The proper transformation to spherical variables is reached in two steps:

1. First, we express the vectors $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$ in terms of the basis $\{\mathbf{e}_i\}$ defined by eq.(A.1.17), which contains the vector $\mathbf{e}_1 \propto \boldsymbol{\lambda}_1$,

$$\boldsymbol{\lambda}_2 = \sum_{i=1}^m b_{i2}\mathbf{e}_i, \tag{A.1.33a}$$

$$\boldsymbol{\lambda}_3 = \sum_{i=1}^m b_{i3}\mathbf{e}_i, \tag{A.1.33b}$$

where, similarly to (A.1.19), $\{b_{i2}\}$ and $\{b_{i3}\}$ are defined in terms of the components with respect to the basis $\{\mathbf{v}_i\}$ as

$$\begin{cases} b_{12} &= \frac{\lambda_{12}}{\sqrt{\lambda_{11}}} \\ b_{22} &= \frac{a_{12}\lambda_{11}-a_{11}\lambda_{12}}{\sqrt{\lambda_{11}}\sqrt{\lambda_{11}-a_{11}^2}} \\ & \dots \\ b_{k2} &= \frac{a_{k-1\,2}\left(\lambda_{11}-\sum_{i=1}^{k-2}a_{i1}^2\right)-a_{k-1\,1}\left(\lambda_{12}-\sum_{i=1}^{k-2}a_{i1}b_{i2}\right)}{\sqrt{\lambda_{11}-\sum_{i=1}^{k-2}a_{i1}^2}\sqrt{\lambda_{11}-\sum_{i=1}^{k-1}a_{i1}^2}} \\ & \dots \\ b_{m2} &= \frac{a_{m1}a_{m-1\,2}-a_{m-1\,1}a_{m2}}{\sqrt{a_{m1}^2+a_{m-1\,1}^2}}\,. \end{cases} \tag{A.1.34}$$

and

$$\begin{cases} b_{13} &= \frac{\lambda_{13}}{\sqrt{\lambda_{11}}} \\ b_{23} &= \frac{a_{13}\lambda_{11}-a_{11}\lambda_{13}}{\sqrt{\lambda_{11}}\sqrt{\lambda_{11}-a_{11}^2}} \\ & \dots \\ b_{k3} &= \frac{a_{k-1\,3}\left(\lambda_{11}-\sum_{i=1}^{k-2}a_{i1}^2\right)-a_{k-1\,1}\left(\lambda_{13}-\sum_{i=1}^{k-2}a_{i1}a_{i3}\right)}{\sqrt{\lambda_{11}-\sum_{i=1}^{k-2}a_{i1}^2}\sqrt{\lambda_{11}-\sum_{i=1}^{k-1}a_{i1}^2}} \\ & \dots \\ b_{m3} &= \frac{a_{m1}a_{m-1\,3}-a_{m-1\,1}a_{m3}}{\sqrt{a_{m1}^2+a_{m-1\,1}^2}}\,. \end{cases} \tag{A.1.35}$$

2. We build a new (non-orthogonal) basis

$$\mathbf{e}'_i = \{\boldsymbol{\lambda}_2, \mathbf{e}_1, \dots, \mathbf{e}_{m-1}\} \tag{A.1.36}$$

and we apply the Gram-Schmidt algorithm, in such a way to obtain an orthonormal basis $\{\mathbf{f}_i\}$,

$$\begin{aligned} \mathbf{f}_1 &= \frac{\mathbf{w}_1}{|\mathbf{w}_1|}, \qquad \mathbf{w}_1 = \mathbf{e}'_1, \\ \mathbf{f}_k &= \frac{\mathbf{w}_k}{|\mathbf{w}_k|}, \qquad \mathbf{w}_k = \mathbf{e}'_k - \sum_{j=1}^{k-1}(\mathbf{e}'_k\cdot\mathbf{f}_j)\mathbf{f}_j, \qquad k \neq 1, \end{aligned} \tag{A.1.37}$$

which first element is $\mathbf{f}_1 \propto \boldsymbol{\lambda}_2$. At this level, we can decompose $\boldsymbol{\lambda}_3$ as

$$\boldsymbol{\lambda}_3 = \sum_{i=1}^{m} c_{i3}\mathbf{f}_i, \tag{A.1.38}$$

with

$$\begin{cases} c_{13} &= \frac{\lambda_{23}}{\sqrt{\lambda_{22}}} \\ c_{23} &= \frac{b_{13}\lambda_{22}-b_{12}\lambda_{23}}{\sqrt{\lambda_{22}}\sqrt{\lambda_{22}-b_{12}^2}} \\ & \dots \\ c_{k3} &= \frac{b_{k-1\,3}\left(\lambda_{22}-\sum_{i=1}^{k-2}b_{i2}^2\right)-b_{k-1\,2}\left(\lambda_{23}-\sum_{i=1}^{k-2}b_{i2}b_{i3}\right)}{\sqrt{\lambda_{22}-\sum_{i=1}^{k-2}b_{i2}^2}\sqrt{\lambda_{22}-\sum_{i=1}^{k-1}b_{i2}^2}} \\ & \dots \\ c_{m3} &= \frac{b_{m2}b_{m-1\,3}-b_{m-1\,2}b_{m\,3}}{\sqrt{b_{m2}^2+b_{m-1\,2}^2}}\,. \end{cases} \tag{A.1.39}$$

Eqs.(A.1.31), (A.1.33a) and (A.1.38) give us a decomposition of the three vectors $\boldsymbol{\lambda}_i$ in terms of three different orthonormal basis. Hence, we can introduce spherical coordinates for $\boldsymbol{\lambda}_1$,

$$\begin{cases} a_{11} &= \sqrt{\lambda_{11}}\cos\theta_{11} \\ & \dots \\ a_{k1} &= \sqrt{\lambda_{11}}\cos\theta_{k1}\prod_{i=1}^{k-1}\sin\theta_{i1} \\ & \dots \\ a_{m1} &= \sqrt{\lambda_{11}}\prod_{i=1}^{m-1}\sin\theta_{i1}\,, \end{cases} \tag{A.1.40}$$

for $\boldsymbol{\lambda}_2$,

$$\begin{cases} b_{12} &= \sqrt{\lambda_{22}}\cos\theta_{12} \\ & \dots \\ b_{k2} &= \sqrt{\lambda_{22}}\cos\theta_{k2}\prod_{i=1}^{k-1}\sin\theta_{i2} \\ & \dots \\ b_{m2} &= \sqrt{\lambda_{22}}\prod_{i=1}^{m-1}\sin\theta_{i2} \end{cases} \tag{A.1.41}$$

and for $\boldsymbol{\lambda}_3$,

$$\begin{cases} c_{13} &= \sqrt{\lambda_{33}}\cos\theta_{23} \\ & \dots \\ c_{k3} &= \sqrt{\lambda_{33}}\cos\theta_{k3}\prod_{i=1}^{k-1}\sin\theta_{i3} \\ & \dots \\ c_{m3} &= \sqrt{\lambda_{33}}\prod_{i=1}^{m-1}\sin\theta_{i3}\,. \end{cases} \tag{A.1.42}$$

and rewrite I_3^κ as

$$\begin{aligned} I_3^\kappa = \frac{1}{8}\int_0^\infty \mathrm{d}\lambda_{11}(\lambda_{11})^{\frac{m-2}{2}}\int_0^\infty \mathrm{d}\lambda_{22}(\lambda_{22})^{\frac{m-2}{2}}\int_0^\infty \mathrm{d}\lambda_{33}(\lambda_{33})^{\frac{m-2}{2}}\times \\ \int d\Omega_{(m-1)}\int d\Omega_{(m-1)}\int d\Omega_{(m-1)}\mathcal{I}_3(\lambda_{ij},\cos\theta_{ij},\sin\theta_{ij}). \end{aligned} \tag{A.1.43}$$

By construction, the relative orientations of between the vectors $\boldsymbol{\lambda}_i$ are mapped into

$$\begin{aligned} \lambda_{12} &= \sqrt{\lambda_{11}\lambda_{22}}\cos\theta_{12}, \\ \lambda_{23} &= \sqrt{\lambda_{22}\lambda_{33}}\cos\theta_{13}, \\ \lambda_{31} &= \sqrt{\lambda_{11}\lambda_{33}}\left(\cos\theta_{12}\cos\theta_{13}+\sin\theta_{12}\sin\theta_{13}\cos\theta_{23}\right), \end{aligned} \tag{A.1.44}$$

and, by inverting (A.1.35) and (A.1.39), we can obtain the expressions of a_{i2} and a_{i3} as polynomials in $s_{\theta_{ij}}$ and $c_{\theta_{ij}}$. In particular, one can verify that each integral over $a_{i1} \notin \{a_{m-1\,1}, a_{m\,1}\}$ is mapped into the integral over the angular variable θ_{i1} and, as for $I_2^{(k)}$, each component $a_{i2} \notin \{a_{m-1\,2}, a_{m\,2}\}$ can be expressed in terms of the angles θ_{j1} with $j \leq i$ and θ_{j2} with $j \leq i+1$. Finally, each $a_{i3} \notin \{a_{m-1\,3}, a_{m\,3}\}$ turns out to be function of the angles θ_{j1} with $j \leq i$, θ_{j2} with $j \leq i+1$ and θ_{j3} with $j \leq i+2$. Therefore, if we are dealing with an integrand depending on the first $\kappa < m-1$ components of all $\boldsymbol{\lambda}_i$, we can integrate out all angular variables θ_{i1}, $j > \kappa$, θ_{i2}, $j > \kappa+1$ and θ_{i3}, $j > \kappa+2$. In particular, if we introduce

$$\begin{aligned} \boldsymbol{\Theta}_\Lambda =& \{\theta_{12}\,,\theta_{13}\,,\theta_{23}\}, \\ \boldsymbol{\Theta}_\perp =& \{\theta_{11}\,,\dots\,,\theta_{\kappa 1}\,,\theta_{22},\,\dots,\,\theta_{\kappa+1\,2}\,,\theta_{33},\,\dots,\,\theta_{\kappa+2\,3}\}, \end{aligned} \tag{A.1.45}$$

we can write

$$I_3^\kappa = \prod_{i=1}^{3}\Omega_{(m-\kappa-i)}\int \mathrm{d}^6\boldsymbol{\Lambda}\int \mathrm{d}^{3\kappa}\boldsymbol{\Theta}_\perp\mathcal{I}_3(\boldsymbol{\Lambda},\boldsymbol{\Theta}_\perp)\,, \tag{A.1.46}$$

where we have defined

$$\int \mathrm{d}^6\boldsymbol{\Lambda} = \int_0^\infty \mathrm{d}\lambda_{11}\mathrm{d}\lambda_{22}\mathrm{d}\lambda_{33}(\lambda_{11})^{\frac{n-2}{2}}(\lambda_{22})^{\frac{n-2}{2}}(\lambda_{33})^{\frac{m-2}{2}}\int d^3\boldsymbol{\Theta}_\Lambda\,, \tag{A.1.47}$$

with

$$\int \mathrm{d}^3\mathbf{\Theta}_\Lambda = \int_{-1}^{1} \mathrm{d}c_{\theta_{12}}\mathrm{d}c_{\theta_{13}}\mathrm{d}c_{\theta_{23}}(s_{\theta_{12}})^{n-3}(s_{\theta_{12}})^{m-3}(s_{theta_{23}})^{n-4}\,, \tag{A.1.48}$$

and

$$\int \mathrm{d}^{3\kappa}\mathbf{\Theta}_\perp = \int_{-1}^{1}\prod_{i=1}^{\kappa} \mathrm{d}c_{\theta_{i1}}\mathrm{d}c_{\theta_{i+1\,2}}\mathrm{d}c_{\theta_{i+2\,3}}(s_{\theta_{i1}})^{m-i-2}(s_{\theta_{i+1\,2}})^{m-i-3}(s_{\theta_{i+2\,3}})^{m-i-4}. \tag{A.1.49}$$

- The strategy adopted in the previous cases ca be generalized to ℓ-loop integrals of the type

$$I_\ell^\kappa = \int \prod_{i=1}^{\ell} d^m\boldsymbol{\lambda}_i \mathcal{I}_\ell(\boldsymbol{\lambda}_j), \tag{A.1.50}$$

in order to define a change of variable which maps a subset of κ components of each $\boldsymbol{\lambda}_i$, as well as their $\ell(\ell-1)/2$ relative orientations λ_{ij}, into angular variables. Starting from the decomposition of all vectors in terms of a single orthonormal basis, one can define, by recursively applying the Gram-Schimdt algorithm, $(\ell-1)$ auxiliary orthonormal basis carrying information both on $\kappa \leq m-1$ components of the original basis and on the relative orientations λ_{ij}. After all vectors have been decomposed into the proper orthonormal basis, we introduce m-dimensional polar coordinates and, we express the components of all $\boldsymbol{\lambda}_i$ with respect to $\{\mathbf{v}_i\}$ in terms of the angular variables. The final transformation has the form

$$\begin{cases} \lambda_{ij} \to & P\left[\lambda_{ll}, \sin[\mathbf{\Theta}_\Lambda], \cos[\mathbf{\Theta}_\Lambda]\right], \qquad i \neq j \\ a_{ji} \to & P\left[\lambda_{ll}, \sin[\mathbf{\Theta}_{\perp,\Lambda}], \cos[\mathbf{\Theta}_{\perp,\Lambda}]\right], \quad j \leq \kappa, \end{cases} \tag{A.1.51}$$

where P indicates a general polynomial and $\mathbf{\Theta}_\Lambda$ and $\mathbf{\Theta}_\perp$ label the sets of angles

$$\begin{aligned} \mathbf{\Theta}_\Lambda =& \{\theta_{ij}\}, \qquad 1 \leq i < j \leq \ell, \\ \mathbf{\Theta}_\perp =& \{\theta_{ij}\}, \qquad j \leq i \leq \ell+\kappa-1,\ \ 1 \leq j \leq \ell. \end{aligned} \tag{A.1.52}$$

Therefore, if the integrand I_ℓ^κ solely depends on κ components of each $\boldsymbol{\lambda}_i$, all angles θ_{ij}, $i \geq j+\kappa$ can be integrated out, yielding to

$$I_\ell^\kappa = \prod_{i=1}^{\ell} \Omega_{(m-\kappa-i)} \int \mathrm{d}^{\frac{\ell(\ell-1)}{2}}\mathbf{\Lambda} \int \mathrm{d}^{\ell\kappa}\mathbf{\Theta}_\perp \mathcal{I}_\ell(\mathbf{\Lambda}, \mathbf{\Theta}_\perp), \tag{A.1.53}$$

where we have defined

$$\int \mathrm{d}^{\frac{\ell(\ell+1)}{2}}\mathbf{\Lambda} = \int_0^\infty \prod_{i=1}^{\ell} d\lambda_{ii}(\lambda_{ii})^{\frac{m-2}{2}} \int \mathrm{d}^{\frac{\ell(\ell-1)}{2}}\mathbf{\Theta}_\Lambda\,, \tag{A.1.54}$$

with

$$\int \mathrm{d}^{\frac{\ell(\ell-1)}{2}}\mathbf{\Theta}_\Lambda = \int_{-1}^{1} \prod_{1\leq i<j\leq\ell}^{\ell} \mathrm{d}\cos\theta_{ij}(\sin\theta_{ij})^{m-2-i}\,, \tag{A.1.55}$$

and

$$\int \mathrm{d}^{\ell\kappa}\mathbf{\Theta}_\perp = \int_{-1}^{1} \prod_{i=1}^{\kappa}\prod_{j=1}^{\ell} \mathrm{d}\cos\theta_{i+j-1\,j}(\sin\theta_{i+j-1\,j})^{m-i-j-1}. \tag{A.1.56}$$

Having derived eq. (A.1.53), we can finally turn our attention to the arbitrary ℓ loop integral defined in eq. (A.1.1) and, by introducing the $q_i^\alpha = q_{\parallel\, i}^\alpha + \lambda_i^\alpha$ parametrization of the loop momenta discussed in sec. 2.3, we can rewrite it as

$$I_n^{d\,(\ell)}[\mathcal{N}] = \int \prod_{1=1}^{\ell} \frac{\mathrm{d}^{n-1} q_{\parallel\, i}}{\pi^{d/2}} \int \mathrm{d}^{d-n-1}\lambda_i \frac{\mathcal{N}(q_{\parallel\, i}^\alpha, \lambda_i^\alpha)}{\prod_j D_j(q_{\parallel\, i}^\alpha, \lambda_{ij})}, \tag{A.1.57}$$

where we have explicitly indicated that the denominators depend on the $d_\parallel$-dimensional momenta $q_{\parallel\, i}$ and on the scalar products λ_{ij} between the transverse vectors living in $d_\perp$ dimensions. Moreover, as observed in sec. 2.3, the numerator does not depend explicitly on components of λ_i^α lying in the -2ϵ portion of the transverse space, but rather on the scalar products λ_{ij} and on the four-dimensional components x_{ij}, $i > d_\parallel$,

$$\mathcal{N}(q_{\parallel\, i}^\alpha, \lambda_i^\alpha) \equiv \mathcal{N}(q_{\parallel\, i}^\alpha, \lambda_{ij}, x_{d_\parallel+1\, i}, \,\ldots,\, x_{4i}). \tag{A.1.58}$$

It is now clear that the integral over the transverse vectors λ^α corresponds to a $d_\perp$-dimensional integral of the type I_ℓ^κ with $\kappa = 4 - d_\parallel$. Therefore, by substituting (A.1.53) in (A.1.57), we obtain

$$\begin{aligned} I_n^{d\,(\ell)}[\mathcal{N}] =& \Omega_d^{(\ell)} \int \prod_{i=1}^{\ell} \mathrm{d}^{n-1} q_{\parallel\, i} \int \mathrm{d}^{\frac{\ell(\ell+1)}{2}} \mathbf{\Lambda} \int \mathrm{d}^{(4-d_\parallel)\ell} \mathbf{\Theta}_\perp \frac{\mathcal{N}(q_{i\,\parallel}, \mathbf{\Lambda}, \mathbf{\Theta}_\perp)}{\prod_j D_j(q_{\parallel\, i}, \mathbf{\Lambda})}, \\ \Omega_d^{(\ell)} =& \prod_{i=1}^{\ell} \frac{\Omega_{(d-4-i)}}{2\pi^{\frac{d}{2}}}, \end{aligned} \tag{A.1.59}$$

which is exactly the $d = d_\parallel + d_\perp$ parametrization introduced in eq. (2.43).

A.2 One-loop transverse integrals

In this appendix, we give explicit formulae for the $d = d_\parallel + d_\perp$ parametrization of one-loop integrals introduced in eq. (2.43) and, for every $d_\parallel = 0, \ldots 3$, we provide a catalogue of transverse space integrals.

Four-point integrals ($\ell = 1$, $d_\parallel = 3$) **:**

$$I_4^{d\,(1)}[\mathcal{N}\,] = \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\mathcal{N}(q)}{D_1 D_2 D_3 D_4}, \tag{A.2.1}$$

- Loop momentum decomposition, $q^\alpha = q_\parallel^\alpha + \lambda^\alpha$:

$$q_\parallel^\alpha = \sum_{i=1}^{3} x_i e_i^\alpha, \quad \lambda^\alpha = x_4 e_4^\alpha + \mu^\alpha, \tag{A.2.2}$$

- Transverse variable:

$$x_4 = \sqrt{\lambda^2} \cos\theta_1, \tag{A.2.3}$$

- $d = d_\parallel + d_\perp$ parametrization:

$$\begin{aligned} I_4^{d\,(1)}[\mathcal{N}\,] =& \frac{1}{\pi^2 \Gamma(\frac{d-4}{2})} \int \mathrm{d}^3 q_\parallel \int_0^\infty \mathrm{d}\lambda^2 (\lambda^2)^{\frac{d-5}{2}} \int_{-1}^{1} \mathrm{d}\cos\theta_1 (\sin\theta_1)^{d-6} \times \\ & \frac{\mathcal{N}(q_\parallel, \lambda^2, \cos\theta_1)}{D_1 D_2 D_3 D_4}, \end{aligned} \tag{A.2.4}$$

- Transverse tensor integrals ($n_4 \in \mathbb{N}$):

$$\begin{aligned}
I_4^{d\,(1)}[\,x_4^{2n_4+1}\,] =& 0\,,\\
I_4^{d\,(1)}[\,x_4^{2n_4}\,] =& \frac{(2n_4-1)!!}{\prod_{i=1}^{n_4}(d-5+2i)} I_4^{d\,(1)}[\,\lambda^{2n_4}\,]\\
=& \frac{(2n_4-1)!!}{2^{n4}} I_4^{d+2n_4\,(1)}[\,1\,]\,.
\end{aligned} \tag{A.2.5}$$

In the last equality, we have identified additional powers of λ^2 in the numerator with higher dimensional integrals.

Three-point integrals ($\ell = 1,\ d_\parallel = 2$) **:**

$$I_3^{d\,(1)}[\mathcal{N}\,] = \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\mathcal{N}(q)}{D_1 D_2 D_3}, \tag{A.2.6}$$

- Loop momentum decomposition, $q^\alpha = q_\parallel^\alpha + \lambda^\alpha$:

$$q_\parallel^\alpha = \sum_{i=1}^{2} x_i e_i^\alpha, \quad \lambda^\alpha = \sum_{i=3}^{4} x_i e_i^\alpha + \mu^\alpha, \tag{A.2.7}$$

- Transverse variables:

$$\begin{cases} x_3 = \sqrt{\lambda^2}\cos\theta_1 \\ x_4 = \sqrt{\lambda^2}\sin\theta_1\cos\theta_2, \end{cases} \tag{A.2.8}$$

- $d = d_\parallel + d_\perp$ parametrization:

$$\begin{aligned}
I_3^{d\,(1)}[\mathcal{N}\,] =& \frac{1}{\pi^2\Gamma(\frac{d-4}{2})} \int \mathrm{d}^2 q_\parallel \int_0^\infty \mathrm{d}\lambda^2 (\lambda^2)^{\frac{d-4}{2}} \int_{-1}^{1} \mathrm{d}\!\cos\theta_1 \mathrm{d}\!\cos\theta_2 \times\\
& (\sin\theta_1)^{d-5}(\sin\theta_2)^{d-6} \frac{\mathcal{N}(q_\parallel, \lambda^2, \{\cos\theta_1, \sin\theta_1, \cos\theta_2\})}{D_1 D_2 D_3},
\end{aligned} \tag{A.2.9}$$

- Transverse tensor integrals ($n_i \in \mathbb{N}$):

$$\begin{aligned}
I_3^{d\,(1)}[\,x_3^{m_3} x_4^{m_4}\,] =& 0 \quad \text{if} \quad m_3 \vee m_4 \quad \text{odd},\\
I_3^{d\,(1)}[\,x_3^{2n_3} x_4^{2n_4}\,] =& \frac{\prod_{i=3}^{4}(2n_i-1)!!}{\prod_{i=1}^{n_3+n_4}(d-4+2i)} I_3^{d\,(1)}[\,\lambda^{2(n_3+n_4)}\,]\\
=& \prod_{i=3}^{4} \frac{(2n_i-1)!!}{2^{n_i}} I_3^{d+2(n_3+n_4)\,(1)}[\,1\,].
\end{aligned} \tag{A.2.10}$$

In the last equality, we have identified additional powers of λ^2 in the numerator with higher dimensional integrals.

Two-point integrals with $p^2 \neq 0$ ($\ell = 1,\ d_\parallel = 1$) **:**

$$I_2^{d\,(1)}[\mathcal{N}\,] = \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\mathcal{N}(q)}{D_1 D_2}, \tag{A.2.11}$$

- Loop momentum decomposition, $q^\alpha = q_\parallel^\alpha + \lambda^\alpha$:

$$q_\parallel^\alpha = x_1 e_1^\alpha, \quad \lambda^\alpha = \sum_{i=2}^{4} x_i e_i^\alpha + \mu^\alpha, \tag{A.2.12}$$

- Transverse variables:

$$\begin{cases} x_2 = \sqrt{\lambda^2}\cos\theta_1 \\ x_3 = \sqrt{\lambda^2}\sin\theta_1\cos\theta_2, \\ x_4 = \sqrt{\lambda^2}\sin\theta_1\sin\theta_2\cos\theta_3, \end{cases} \tag{A.2.13}$$

- $d = d_\parallel + d_\perp$ parametrization:

$$\begin{aligned} I_2^{d\,(1)}[\mathcal{N}\,] =& \frac{1}{\pi^2\Gamma(\frac{d-4}{2})}\int \mathrm{d}q_\parallel \int_0^\infty \mathrm{d}\lambda^2 (\lambda^2)^{\frac{d-3}{2}} \int_{-1}^{1} \mathrm{d}\cos\theta_1 \mathrm{d}\cos\theta_2 \mathrm{d}\cos\theta_3 \times \\ & (\sin\theta_1)^{d-4}(\sin\theta_2)^{d-5}(\sin\theta_3)^{d-6}\times \\ & \frac{\mathcal{N}(q_\parallel,\lambda_{11},\cos\theta_1,\sin\theta_1,\cos\theta_2,\sin\theta_2,\cos\theta_3)}{D_0 D_1}, \end{aligned} \tag{A.2.14}$$

- Transverse tensor integrals ($n_i \in \mathbb{N}$):

$$\begin{aligned} I_2^{d\,(1)}[\,x_2^{m_2}x_3^{m_3}x_4^{m_4}\,] =& 0 \quad \text{if} \quad m_2 \vee m_3 \vee m_4 \quad \text{odd}, \\ I_2^{d\,(1)}[\,x_2^{2n_2}x_3^{2n_3}x_4^{2n_4}\,] =& \frac{\prod_{i=2}^{4}(2n_i-1)!!}{\prod_{i=1}^{n_2+n_3+n_4}(d-3+2i)} I_2^{d\,(1)}[\,\lambda^{2(n_2+n_3+n_4)}\,] \\ =& \prod_{i=2}^{4}\frac{(2n_i-1)!!}{2^{n_i}} I_2^{d+2(n_2+n_3+n_4)\,(1)}[\,1\,]\,. \end{aligned} \tag{A.2.15}$$

In the last equality, we have identified additional powers of λ^2 in the numerator with higher dimensional integrals.

Two-point integrals with $p^2 = 0$ ($\ell = 1$, $d_\parallel = 2$) :

$$I_2^{d\,(1)}[\mathcal{N}\,]|_{p^2=0} = \int \frac{\mathrm{d}^d q}{\pi^{d/2}}\frac{\mathcal{N}(q)}{D_0 D_1}, \tag{A.2.16}$$

- Loop momentum decomposition, $q^\alpha = q_\parallel^\alpha + \lambda^\alpha$:

$$q_\parallel^\alpha = \sum_{i=1}^{2} x_i e_i^\alpha, \quad \lambda^\alpha = \sum_{i=3}^{4} x_i e_i^\alpha + \mu^\alpha, \tag{A.2.17}$$

- Transverse variables:

$$\begin{cases} x_3 = \sqrt{\lambda^2}\cos\theta_1 \\ x_4 = \sqrt{\lambda^2}\sin\theta_1\cos\theta_2, \end{cases} \tag{A.2.18}$$

- $d = d_\parallel + d_\perp$ parametrization:

$$\begin{aligned} I_2^{d\,(1)}[\mathcal{N}\,]|_{p^2=0} =& \frac{1}{\pi^2\Gamma(\frac{d-4}{2})}\int \mathrm{d}^2 q_\parallel \int_0^\infty \mathrm{d}\lambda^2 (\lambda^2)^{\frac{d-4}{2}} \int_{-1}^{1} \mathrm{d}\cos\theta_1 (\sin\theta_1)^{d-5}\times \\ & \int_{-1}^{1} \mathrm{d}\cos\theta_2 (\sin\theta_2)^{d-6} \frac{\mathcal{N}(q_\parallel,\lambda^2,\{\cos\theta_1,\sin\theta_1,\cos\theta_2\})}{D_1 D_2}, \end{aligned} \tag{A.2.19}$$

- Transverse tensor integrals ($n_i \in \mathbb{N}$):

$$
\begin{aligned}
I_2^{d\,(1)}[\, x_3^{m_3} x_4^{m_4}\,]|_{p^2=0} =& 0 \quad \text{if} \quad m_3 \vee m_4 \quad \text{odd}, \\
I_2^{d\,(1)}[\, x_3^{2n_3} x_4^{2n_4}\,]|_{p^2=0} =& \frac{(2n_3-1)!!\,(2n_4-1)!!}{\prod_{i=1}^{n_3+n_4}(d-4+2i)} I_2^{d\,(1)}[\, \lambda^{2(n_3+n_4)}\,]|_{p^2=0} \\
=& \prod_{i=3}^{4} \frac{(2n_i-1)!!}{2^{n_i}} I_3^{d+2(n_3+n_4)\,(1)}[\,1\,]|_{p^2=0}.
\end{aligned}
\tag{A.2.20}
$$

In the last equality, we have identified additional powers of λ^2 in the numerator with higher dimensional integrals.

One-point integrals ($\ell = 1$, $d_{\|} = 0$) **:**

$$
I_1^{d\,(1)}[\mathcal{N}\,] = \int \frac{\mathrm{d}^d q}{\pi^{d/2}} \frac{\mathcal{N}(q)}{D_1}, \tag{A.2.21}
$$

- Loop momentum decomposition, $q^\alpha = \lambda^\alpha$:

$$
q^\alpha \equiv \lambda^\alpha = \sum_{i=1}^{4} x_i^\alpha e_i^\alpha + \mu^\alpha, \tag{A.2.22}
$$

- Transverse variables:

$$
\begin{cases}
x_1 = \sqrt{\lambda^2}\cos\theta_1, \\
x_2 = \sqrt{\lambda^2}\sin\theta_1\cos\theta_2, \\
x_3 = \sqrt{\lambda^2}\sin\theta_1\sin\theta_2\cos\theta_3 \\
x_4 = \sqrt{\lambda^2}\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4,
\end{cases}
\tag{A.2.23}
$$

- $d = d_{\|} + d_{\perp}$ parametrization:

$$
\begin{aligned}
I_1^{d\,(1)}[\mathcal{N}\,] =& \frac{1}{\pi^2\Gamma(\frac{d-4}{2})} \int_0^\infty \mathrm{d}\lambda^2 (\lambda^2)^{\frac{d-2}{2}} \int_{-1}^{1} \mathrm{d}\cos\theta_1 \mathrm{d}\cos\theta_2 d\cos\theta_3 \mathrm{d}\cos\theta_4 \times \\
& (\sin\theta^2)^{d-3}(\sin\theta_2)^{d-4}(\sin\theta_3)^{d-5}(\sin\theta_4)^{d-6}\times \\
& \frac{\mathcal{N}(\lambda^2,\cos\theta_1,\sin\theta_1,\cos\theta_2,\sin\theta_2,\cos\theta_3,\sin\theta_3,\cos\theta_4)}{D_1},
\end{aligned}
\tag{A.2.24}
$$

- Transverse tensor integrals ($n_i \in \mathbb{N}$):

$$
\begin{aligned}
I_1^{d\,(1)}[\, x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}\,] =& 0 \quad \text{if} \quad m_1 \vee m_2 \vee m_3 \vee m_4 \quad \text{odd}, \\
I_1^{d\,(1)}[\, x_1^{2n_1} x_2^{2n_2} x_3^{2n_3} x_4^{2n_4}\,] =& \frac{\prod_{i=1}^{4}(2n_i-1)!!}{\prod_{i=1}^{n_1+n_2+n_3+n_4}(d-3+2i)} I_2^{d\,(1)}[\, \lambda^{2(n_1+n_2+n_3+n_4)}\,] \\
=& \prod_{i=1}^{3} \frac{(2n_i-1)!!}{2^{n_i}} I_2^{d+2(n_1+n_2+n_3+n_4)\,(1)}[\,1\,].
\end{aligned}
\tag{A.2.25}
$$

In the last equality, we have identified additional powers of λ^2 in the numerator with higher dimensional integrals.

A.3 Two-loop transverse integrals

In this appendix, we give explicit formulae for the $d = d_\parallel + d_\perp$ parametrization of two-loop integrals introduced in eq. (2.43) and, for every $d_\parallel = 0, \ldots 3$, we provide a catalogue of transverse space integrals.

For ease of notation, we will denote $s_{\theta_{ij}} = \sin\theta_{ij}$ and $c_{\theta_{ij}} = \cos\theta_{ij}$. In all cases, the relative orientation of the transverse vectors is defined as

$$\lambda_{12} = \sqrt{\lambda_{11}\lambda_{22}}\cos\theta_{12}. \tag{A.3.1}$$

Four-point integrals $(\ell = 2,\, d_\parallel = 3)$ **:**

$$I_4^{d\,(2)}[\mathcal{N}] = \int \frac{\mathrm{d}^d q_1 \mathrm{d}^d q_2}{\pi^d}\frac{\mathcal{N}(q_1, q_2)}{D_1 \ldots D_n}, \tag{A.3.2}$$

- Loop momenta decomposition, $q^\alpha = q^\alpha_{\parallel\, i} + \lambda^\alpha_i$:

$$q^\alpha_{\parallel\, i} = \sum_{j=1}^{3} x_{ji} e^\alpha_j, \quad \lambda^\alpha_i = x_{4i} e^\alpha_4 + \mu^\alpha_i, \tag{A.3.3}$$

- Transverse variables:

$$\begin{cases} x_{41} = & \sqrt{\lambda_{11}}\, c_{\theta_{11}} \\ x_{42} = & \sqrt{\lambda_{22}}\,(c_{\theta_{11}} c_{\theta_{12}} + s_{\theta_{11}} s_{\theta_{12}} c_{\theta_{22}}), \end{cases} \tag{A.3.4}$$

- $d = d_\parallel + d_\perp$ parametrization:

$$\begin{aligned} I_4^{d\,(2)}[\mathcal{N}] =& \frac{2^{d-6}}{\pi^5\Gamma(d-5)} \int \mathrm{d}^3 q_{\parallel\, 1} \mathrm{d}^3 q_{\parallel\, 2} \int_0^\infty \mathrm{d}\lambda_{11}\mathrm{d}\lambda_{22} (\lambda_{11})^{\frac{d-5}{2}} (\lambda_{22})^{\frac{d-5}{2}} \times \\ & \int_{-1}^{1} \mathrm{d}c_{\theta_{12}} \mathrm{d}c_{\theta_{22}} \mathrm{d}c_{\theta_{11}}\, (s_{\theta_{12}})^{d-6}\, (s_{\theta_{11}})^{d-6} (s_{\theta_{22}})^{d-7} \frac{\mathcal{N}(q_1, q_2)}{D_1 \ldots D_n}, \end{aligned} \tag{A.3.5}$$

- Transverse tensor integrals (unless otherwise stated, we assume $i \neq j$):

$$\begin{aligned} I_4^{d\,(2)}[\, x_{4i} x_{4j}\,] =& \frac{1}{(d-3)} I_4^{d\,(2)}[\,\lambda_{ij}] \quad \forall i, j, \\ I_4^{d\,(2)}[\, x_{4i}^4\,] =& \frac{3}{(d-3)(d-1)} I_4^{d\,(2)}[\,\lambda_{ii}^2] \quad \forall i, j, \\ I_4^{d\,(2)}[\, x_{4i}^3 x_{4j}\,] =& \frac{3}{(d-3)(d-1)} I_4^{d\,(2)}[\,\lambda_{12}\lambda_{ii}], \\ I_4^{d\,(2)}[\, x_{41}^2 x_{42}^2\,] =& \frac{3}{(d-3)(d-1)} I_4^{d\,(2)}[\,2\lambda_{12}^2 + \lambda_{11}\lambda_{22}], \\ I_4^{d\,(2)}[\, x_{4i}^6\,] =& \frac{15}{(d-3)(d-1)(d+1)} I_4^{d\,(2)}[\,\lambda_{ii}^3], \\ I_4^{d\,(2)}[\, x_{4i}^5 x_{4j}\,] =& \frac{1}{(d-3)(d-1)(d+1)} I_4^{d\,(2)}[\,\lambda_{12}\lambda_{ii}^2], \\ I_4^{d\,(2)}[\, x_{4i}^4 x_{4j}^2\,] =& \frac{3}{(d-3)(d-1)(d+1)} I_4^{d\,(2)}[\lambda_{ii}(4\lambda_{12}^2 + \lambda_{11}\lambda_{22})], \\ I_4^{d\,(2)}[\, x_{42}^3 x_{41}^3\,] =& \frac{3}{(d-3)(d-1)(d+1)} I_4^{d\,(2)}[\lambda_{12}(2\lambda_{12}^2 + 3\lambda_{11}\lambda_{22})]. \end{aligned} \tag{A.3.6}$$

Moreover, in general, we have

$$I_4^{d\,(2)}[\, x_{41}^{\alpha_4} x_{42}^{\beta_4}\,] = 0, \quad \text{if} \quad \alpha_4 + \beta_4 = 2n + 1. \tag{A.3.7}$$

Three-point integrals ($\ell = 2,\ d_{\|} = 2$) **:**

$$I_3^{d\,(2)}[\mathcal{N}] = \int \frac{d^d q_1 d^d q_2}{\pi^d} \frac{\mathcal{N}(q_1,q_2)}{D_1 \dots D_n}, \tag{A.3.8}$$

- Loop momenta decomposition, $q^\alpha = q^\alpha_{\| i} + \lambda^\alpha_i$:

$$q^\alpha_{\| i} = \sum_{j=1}^{2} x_{ji} e^\alpha_j, \quad \lambda^\alpha_i = \sum_{j=3}^{4} x_{ji} e^\alpha_i + \mu^\alpha_i, \tag{A.3.9}$$

- Transverse variables:

$$\begin{cases} x_{31} = & \sqrt{\lambda_{11}} c_{\theta_{11}} \\ x_{41} = & \sqrt{\lambda_{11}} s_{\theta_{11}} c_{\theta_{21}} \\ x_{32} = & \sqrt{\lambda_{22}} (c_{\theta_{12}} c_{\theta_{11}} + s_{\theta_{12}} c_{\theta_{22}} s_{\theta_{11}}) \\ x_{42} = & \sqrt{\lambda_{22}} [c_{\theta_{12}} c_{\theta_{21}} s_{\theta_{11}} + s_{\theta_{12}} (c_{\theta_{32}} s_{\theta_{21}} s_{\theta_{22}} - c_{\theta_{11}} c_{\theta_{21}} c_{\theta_{22}})], \end{cases} \tag{A.3.10}$$

- $d = d_{\|} + d_{\perp}$ parametrization:

$$\begin{aligned} I_3^d[\mathcal{N}] =& \frac{2^{d-6}}{\pi^5 \Gamma(d-5)} \int \mathrm{d}^2 q_{\| 1} \mathrm{d}^2 q_{\| 2} \int_0^\infty \mathrm{d}\lambda_{11} \mathrm{d}\lambda_{22} (\lambda_{11})^{\frac{d-4}{2}} (\lambda_{22})^{\frac{d-4}{2}} \times \\ & \int_{-1}^1 \mathrm{d}c_{\theta_{12}} \mathrm{d}c_{\theta_{11}} \mathrm{d}c_{\theta_{21}} \mathrm{d}c_{\theta_{22}} \mathrm{d}c_{\theta_{32}}\, (s_{\theta_{12}})^{d-5}\, (s_{\theta_{11}})^{d-5} \times \\ & (s_{\theta_{21}})^{d-6} (s_{\theta_{22}})^{d-6} (s_{\theta_{32}})^{d-7} \frac{\mathcal{N}(q_1,q_2)}{D_1 \dots D_n}, \end{aligned} \tag{A.3.11}$$

- Transverse tensor integrals (unless otherwise stated, we assume $i \neq j$):

$$\begin{aligned} I_3^{d\,(2)}[\, x_{3i} x_{3j}\,] =& I_3^{d\,(2)}[\, x_{4i} x_{4j}\,] = \frac{1}{(d-2)} I_3^{d\,(2)}[\, \lambda_{ij}] \quad \forall i,j, \\ I_3^{d\,(2)}[\, x_{3i}^4\,] =& I_3^{d\,(2)}[\, x_{4i}^4\,] = \frac{3}{(d-2)d} I_3^{d\,(2)}[\, \lambda_{ii}^2], \\ I_3^{d\,(2)}[\, x_{3i}^3 x_{3j}\,] =& I_3^{d\,(2)}[\, x_{4i}^3 x_{4j}\,] = \frac{3}{(d-2)d} I_3^{d\,(2)}[\, \lambda_{ii}\lambda_{ij}], \\ I_3^{d\,(2)}[\, x_{31}^2 x_{32}^2\,] =& I_3^{d\,(2)}[\, x_{41}^2 x_{42}^2\,] = \frac{1}{(d-2)d} I_3^{d\,(2)}[\, 2\lambda_{12}^2 + \lambda_{11}\lambda_{22}], \\ I_3^{d\,(2)}[\, x_{3i}^2 x_{4j}^2\,] =& \frac{1}{(d-3)(d-2)d} I_3^{d\,(2)}[\, -2\lambda_{12}^2 + (d-1)\lambda_{11}\lambda_{22}], \\ I_3^{d\,(2)}[\, x_{3i}^2 x_{4i} x_{4j}\,] =& I_3^{d\,(2)}[\, x_{4i}^2 x_{3i} x_{3j}\,] = \frac{1}{(d-2)d} I_3^{d\,(2)}[\, \lambda_{12}\lambda_{ii}], \\ I_3^{d\,(2)}[\, x_{31} x_{41} x_{32} x_{32}\,] =& \frac{1}{(d-3)(d-2)d} I_3^{d\,(2)}[\, (d-2)\lambda_{12}^2 - \lambda_{11}\lambda_{22}]. \end{aligned} \tag{A.3.12}$$

Moreover, in general, we have

$$I_3^{d\,(2)}[\, x_{31}^{\alpha_3} x_{41}^{\alpha_4} x_{32}^{\beta_3} x_{42}^{\beta_4}\,] = 0, \quad \text{if} \quad \alpha_i + \beta_i = 2n+1. \tag{A.3.13}$$

Two-point integrals with $p^2 \neq 0$ ($\ell = 2$, $d_\parallel = 1$) :

$$I_2^{d\,(2)}[\mathcal{N}] = \int \frac{\mathrm{d}^d q_1 \mathrm{d}^d q_2}{\pi^d} \frac{\mathcal{N}(q_1, q_2)}{D_1 \dots D_n}, \tag{A.3.14}$$

- Loop momenta decomposition, $q^\alpha = q^\alpha_{\parallel\, i} + \lambda^\alpha_i$:

$$q^\alpha_{\parallel\, i} = x_{1i} e_1^\alpha, \quad \lambda_i^\alpha = \sum_{j=2}^{4} x_{ji} e_i^\alpha + \mu_i^\alpha, \tag{A.3.15}$$

- Transverse variables:

$$\begin{cases} x_{21} = & \sqrt{\lambda_{11}} c_{\theta_{11}}, \\ x_{31} = & \sqrt{\lambda_{11}} s_{\theta_{11}} c_{\theta_{21}}, \\ x_{41} = & \sqrt{\lambda_{11}} s_{\theta_{11}} s_{\theta_{21}} c_{\theta_{31}}, \\ x_{22} = & \sqrt{\lambda_{22}} (c_{\theta_{12}} c_{\theta_{11}} + s_{\theta_{12}} c_{\theta_{22}} s_{\theta_{11}}) \\ x_{32} = & \sqrt{\lambda_{22}} \left[c_{\theta_{12}} c_{\theta_{21}} s_{\theta_{11}} + s_{\theta_{12}} (c_{\theta_{32}} s_{\theta_{21}} s_{\theta_{22}} - c_{\theta_{11}} c_{\theta_{21}} c_{\theta_{22}}) \right] \\ x_{42} = & \sqrt{\lambda_{22}} [c_{\theta_{12}} c_{\theta_{31}} s_{\theta_{11}} s_{\theta_{21}} + s_{\theta_{12}} (c_{\theta_{42}} s_{\theta_{31}} s_{\theta_{22}} s_{\theta_{32}} \\ & \qquad - c_{\theta_{11}} c_{\theta_{31}} c_{\theta_{22}} s_{\theta_{21}} - c_{\theta_{21}} c_{\theta_{31}} c_{\theta_{32}} s_{\theta_{22}})], \end{cases} \tag{A.3.16}$$

- $d = d_\parallel + d_\perp$ parametrization:

$$\begin{aligned} I_2^d[\mathcal{N}] = & \frac{2^{d-6}}{\pi^5 \Gamma(d-5)} \int \mathrm{d}q_{\parallel\,1} \mathrm{d}q_{\parallel\,2} \int_0^\infty \mathrm{d}\lambda_{11} \mathrm{d}\lambda_{22} (\lambda_{11})^{\frac{d-3}{2}} (\lambda_{22})^{\frac{d-3}{2}} \times \\ & \int_{-1}^{1} \mathrm{d}c_{\theta_{12}} \mathrm{d}c_{\theta_{11}} \mathrm{d}c_{\theta_{21}} \mathrm{d}c_{\theta_{31}} \mathrm{d}c_{\theta_{22}} \mathrm{d}c_{\theta_{32}} \mathrm{d}c_{\theta_{42}} \times \\ & (s_{\theta_{12}})^{d-4} (s_{\theta_{11}})^{d-4} (s_{\theta_{21}})^{d-5} (s_{\theta_{31}})^{d-6} (s_{\theta_{22}})^{d-5} \times \\ & (s_{\theta_{32}})^{d-6} (s_{\theta_{42}})^{d-7} \frac{\mathcal{N}(q_1, q_2)}{D_1 \dots D_n}, \end{aligned} \tag{A.3.17}$$

- Transverse tensor integrals (unless otherwise stated, we assume $i \neq j$):

$$\begin{aligned} I_2^{d\,(2)}[\, x_{2i} x_{2j}\,] = & I_2^{d\,(2)}[\, x_{3i} x_{3j}\,] = I_2^{d\,(2)}[\, x_{4i} x_{4j}\,] = \frac{1}{(d-1)} I_2^{d\,(2)}[\, \lambda_{ij}], \quad \forall i, j, \\ I_2^{d\,(2)}[\, x_{2i}^4\,] = & I_2^{d\,(2)}[\, x_{3i}^4\,] = I_2^{d\,(2)}[\, x_{3i}^4\,] = \frac{3}{(d-1)(d+1)} I_2^{d\,(2)}[\, \lambda_{ii}^2], \\ I_2^{d\,(2)}[\, x_{2i}^3 x_{2j}\,] = & I_2^{d\,(2)}[\, x_{3i}^3 x_{3j}\,] = I_2^{d\,(2)}[\, x_{4i}^3 x_{4j}\,] = \frac{3}{(d-1)(d+1)} I_3^{d\,(2)}[\, \lambda_{ii} \lambda_{12}], \\ I_2^{d\,(2)}[\, x_{2i}^2 x_{2i}^2\,] = & I_2^{d\,(2)}[\, x_{3i}^2 x_{3i}^2\,] = I_2^{d\,(2)}[\, x_{4i}^2 x_{4i}^2\,] \\ = & \frac{1}{(d-1)(d+1)} I_2^{d\,(2)}[\, 2\lambda_{12}^2 + \lambda_{11} \lambda_{22}], \\ I_2^{d\,(2)}[\, x_{2i}^2 x_{3i}^2\,] = & I_2^{d\,(2)}[\, x_{2i}^2 x_{4i}^2\,] = I_2^{d\,(2)}[\, x_{3i}^2 x_{4i}^2\,] = \frac{1}{(d-1)(d+1)} I_2^{d\,(2)}[\, \lambda_{ii}^2], \\ I_2^{d\,(2)}[\, x_{2i}^2 x_{3j}^2\,] = & I_2^{d\,(2)}[\, x_{2i}^2 x_{4j}^2\,] = I_2^{d\,(2)}[\, x_{3i}^2 x_{4j}^2\,] \\ = & \frac{1}{(d-2)(d-1)(d+1)} I_2^{d\,(2)}[\, -2\lambda_{12}^2 + d \lambda_{11} \lambda_{22}], \end{aligned}$$

$$
\begin{aligned}
I_2^{d\,(2)}[\,x_{2i}^2 x_{3i} x_{3j}\,] =& I_2^{d\,(2)}[\,x_{2i}^2 x_{4i} x_{4j}\,] = \frac{1}{(d-1)(d+1)} I_2^{d\,(2)}[\,\lambda_{12}\lambda_{ii}],\\
I_2^{d\,(2)}[\,x_{3i}^2 x_{2i} x_{2j}\,] =& I_2^{d\,(2)}[\,x_{3i}^2 x_{4i} x_{4j}\,] = \frac{1}{(d-1)(d+1)} I_2^{d\,(2)}[\,\lambda_{12}\lambda_{ii}],\\
I_2^{d\,(2)}[\,x_{4i}^2 x_{2i} x_{2j}\,] =& I_2^{d\,(2)}[\,x_{4i}^2 x_{3i} x_{3j}\,] = \frac{1}{(d-1)(d+1)} I_2^{d\,(2)}[\,\lambda_{12}\lambda_{ii}],\\
I_2^{d\,(2)}[\,x_{21} x_{31} x_{22} x_{32}\,] =& \frac{1}{(d-2)(d-1)(d+1)} I_2^{d\,(2)}[\,(d-1)\lambda_{12}^2 - \lambda_{11}\lambda_{22}],\\
I_2^{d\,(2)}[\,x_{21} x_{41} x_{22} x_{42}\,] =& \frac{1}{(d-2)(d-1)(d+1)} I_2^{d\,(2)}[\,(d-1)\lambda_{12}^2 - \lambda_{11}\lambda_{22}],\\
I_2^{d\,(2)}[\,x_{31} x_{41} x_{32} x_{42}\,] =& \frac{1}{(d-2)(d-1)(d+1)} I_2^{d\,(2)}[\,(d-1)\lambda_{12}^2 - \lambda_{11}\lambda_{22}]. \qquad \text{(A.3.18)}
\end{aligned}
$$

Moreover, in general we have

$$
I_2^{d\,(2)}[\,x_{21}^{\alpha_2} x_{31}^{\alpha_3} x_{41}^{\alpha_4} x_{22}^{\beta_2} x_{32}^{\beta_3} x_{42}^{\beta_4}\,] = 0, \quad \text{if} \quad \alpha_i + \beta_i = 2n+1. \qquad \text{(A.3.19)}
$$

,

Two-point integrals with $p^2 = 0$ ($\ell = 2$, $d_\parallel = 2$) :

$$
I_2^{d\,(2)}[\mathcal{N}\,]|_{p^2=0} = \int \frac{\mathrm{d}^d q_1 \mathrm{d}^d q_2}{\pi^d} \frac{\mathcal{N}(q_1, q_2)}{D_1 \ldots D_n}, \qquad \text{(A.3.20)}
$$

- Loop momenta decomposition, $q_i^\alpha = q_{\parallel\,i}^\alpha + \lambda_i^\alpha$:

$$
q_{\parallel\,i}^\alpha = \sum_{j=1}^{2} x_{ji} e_j^\alpha, \quad \lambda_i^\alpha = \sum_{j=3}^{4} x_{ji} e_i^\alpha + \mu_i^\alpha, \qquad \text{(A.3.21)}
$$

- Transverse variables:

$$
\begin{cases}
x_{31} = & \sqrt{\lambda_{11}} c_{\theta_{11}} \\
x_{41} = & \sqrt{\lambda_{11}} s_{\theta_{11}} c_{\theta_{21}} \\
x_{32} = & \sqrt{\lambda_{22}} (c_{\theta_{12}} c_{\theta_{11}} + s_{\theta_{12}} c_{\theta_{22}} s_{\theta_{11}}) \\
x_{42} = & \sqrt{\lambda_{22}} [c_{\theta_{12}} c_{\theta_{21}} s_{\theta_{11}} + s_{\theta_{12}} (c_{\theta_{32}} s_{\theta_{21}} s_{\theta_{22}} - c_{\theta_{11}} c_{\theta_{21}} c_{\theta_{22}})],
\end{cases} \qquad \text{(A.3.22)}
$$

- $d = d_\parallel + d_\perp$ parametrization:

$$
\begin{aligned}
I_3^d[\mathcal{N}\,] =& \frac{2^{d-6}}{\pi^5 \Gamma(d-5)} \int \mathrm{d}^2 q_{\parallel\,1} d^2 q_{\parallel\,2} \int_0^\infty \mathrm{d}\lambda_{11} \mathrm{d}\lambda_{22} (\lambda_{11})^{\frac{d-4}{2}} (\lambda_{22})^{\frac{d-4}{2}} \times \\
& \int_{-1}^{1} \mathrm{d}c_{\theta_{12}} dc_{\theta_{11}} dc_{\theta_{21}} dc_{\theta_{22}} dc_{\theta_{32}} \, (s_{\theta_{12}})^{d-5} \, (s_{\theta_{11}})^{d-5} \times \\
& (s_{\theta_{21}})^{d-6} (s_{\theta_{22}})^{d-6} (s_{\theta_{32}})^{d-7} \frac{\mathcal{N}(q_1, q_2)}{D_1 \ldots D_n}, \qquad \text{(A.3.23)}
\end{aligned}
$$

- Transverse tensor integrals (unless specified we assume $i \neq j$):

$$
I_2^{d\,(2)}[\,x_{3i} x_{3j}\,]|_{p^2=0} = I_2^{d\,(2)}[\,x_{4i} x_{4j}\,]|_{p^2=0} = \frac{1}{(d-2)} I_2^{d\,(2)}[\,\lambda_{ij}]|_{p^2=0} \quad \forall i, j,
$$

$$\begin{aligned}
I_2^{d\,(2)}[\,x_{3i}^4\,]|_{p^2=0} =& I_2^{d\,(2)}[\,x_{4i}^4\,]|_{p^2=0} = \frac{3}{(d-2)d} I_2^{d\,(2)}[\,\lambda_{ii}^2]|_{p^2=0},\\
I_2^{d\,(2)}[\,x_{3i}^3 x_{3j}\,]|_{p^2=0} =& I_2^{d\,(2)}[\,x_{4i}^3 x_{4j}\,]|_{p^2=0} = \frac{3}{(d-2)d} I_2^{d\,(2)}[\,\lambda_{ii}\lambda_{ij}]|_{p^2=0},\\
I_2^{d\,(2)}[\,x_{31}^2 x_{32}^2\,]|_{p^2=0} =& I_2^{d\,(2)}[\,x_{41}^2 x_{42}^2\,]|_{p^2=0} = \frac{1}{(d-2)d} I_2^{d\,(2)}[\,2\lambda_{12}^2 + \lambda_{11}\lambda_{22}]|_{p^2=0},\\
I_2^{d\,(2)}[\,x_{3i}^2 x_{4j}^2\,]|_{p^2=0} =& \frac{1}{(d-3)(d-2)d} I_2^{d\,(2)}[\,-2\lambda_{12}^2 + (d-1)\lambda_{11}\lambda_{22}]|_{p^2=0},\\
I_2^{d\,(2)}[\,x_{3i}^2 x_{4i} x_{4j}\,]|_{p^2=0} =& I_2^{d\,(2)}[\,x_{4i}^2 x_{3i} x_{3j}\,] = \frac{1}{(d-2)d} I_2^{d\,(2)}[\,\lambda_{12}\lambda_{ii}]|_{p^2=0},\\
I_2^{d\,(2)}[\,x_{31} x_{41} x_{32} x_{32}\,]|_{p^2=0} =& \frac{1}{(d-3)(d-2)d} I_2^{d\,(2)}[\,(d-2)\lambda_{12}^2 - \lambda_{11}\lambda_{22}]|_{p^2=0}.
\end{aligned} \tag{A.3.24}$$

Moreover, in general, we have

$$I_2^{d\,(2)}[\,x_{31}^{\alpha_3} x_{41}^{\alpha_4} x_{32}^{\beta_3} x_{42}^{\beta_4}\,]|_{p^2=0} = 0, \quad \text{if} \quad \alpha_i + \beta_i = 2n+1. \tag{A.3.25}$$

One-point integrals ($\ell = 2$, $d_\parallel = 0$) **:**

$$I_1^{d\,(2)}[\mathcal{N}\,] = \int \frac{d^d q_1 d^d q_2}{\pi^d} \frac{\mathcal{N}(q_1,q_2)}{D_1 \ldots D_n}, \tag{A.3.26}$$

- Loop momenta decomposition, $q_i^\alpha = \lambda_i^\alpha$:

$$\lambda_i^\alpha = \sum_{j=1}^{4} x_{ji} e_i^\alpha + \mu_i^\alpha, \tag{A.3.27}$$

- Transverse variables:

$$\begin{cases}
x_{11} = & \sqrt{\lambda_{11}} c_{\theta_{11}}\\
x_{21} = & \sqrt{\lambda_{11}} s_{\theta_{11}} c_{\theta_{21}}\\
x_{31} = & \sqrt{\lambda_{11}} s_{\theta_{11}} s_{\theta_{21}} c_{\theta_{31}}\\
x_{41} = & \sqrt{\lambda_{11}} s_{\theta_{11}} s_{\theta_{21}} s_{\theta_{31}} c_{\theta_{41}}\\
x_{12} = & \sqrt{\lambda_{22}} (c_{\theta_{12}} c_{\theta_{11}} + s_{\theta_{12}} c_{\theta_{22}} s_{\theta_{11}})\\
x_{22} = & \sqrt{\lambda_{22}} [c_{\theta_{12}} c_{\theta_{21}} s_{\theta_{11}} + s_{\theta_{12}} (c_{\theta_{32}} s_{\theta_{21}} s_{\theta_{22}} - c_{\theta_{11}} c_{\theta_{21}} c_{\theta_{22}})]\\
x_{32} = & \sqrt{\lambda_{22}} [c_{\theta_{12}} c_{\theta_{31}} s_{\theta_{11}} s_{\theta_{21}} + s_{\theta_{12}} (c_{\theta_{42}} s_{\theta_{31}} s_{\theta_{22}} s_{\theta_{32}}\\
& - c_{\theta_{11}} c_{\theta_{31}} c_{\theta_{22}} s_{\theta_{21}} - c_{\theta_{21}} c_{\theta_{31}} c_{\theta_{32}} s_{\theta_{22}})]\\
x_{42} = & \sqrt{\lambda_{22}} [c_{\theta_{12}} c_{\theta_{41}} s_{\theta_{11}} s_{\theta_{21}} s_{\theta_{31}} + s_{\theta_{12}} (c_{\theta_{52}} s_{\theta_{41}} s_{\theta_{22}} s_{\theta_{32}} s_{\theta_{42}}\\
& - c_{\theta_{11}} c_{\theta_{41}} c_{\theta_{22}} s_{\theta_{21}} s_{\theta_{31}} - c_{\theta_{21}} c_{\theta_{41}} c_{\theta_{32}} s_{\theta_{22}} s_{\theta_{31}}\\
& - c_{\theta_{31}} c_{\theta_{41}} c_{\theta_{42}} s_{\theta_{22}} s_{\theta_{32}})],
\end{cases}$$

- $d = d_\parallel + d_\perp$ parametrization:

$$\begin{aligned}
I_1^d[\mathcal{N}\,] =& \frac{2^{d-6}}{\pi^5 \Gamma(d-5)} \int_0^\infty \mathrm{d}\lambda_{11} \mathrm{d}\lambda_{22} (\lambda_{11})^{\frac{d-2}{2}} (\lambda_{11})^{\frac{d-2}{2}} \times\\
& \int_{-1}^{1} \mathrm{d}c_{\theta_{12}} \mathrm{d}c_{\theta_{11}} \mathrm{d}c_{\theta_{21}} \mathrm{d}c_{\theta_{31}} \mathrm{d}c_{\theta_{41}} \mathrm{d}c_{\theta_{22}} \mathrm{d}c_{\theta_{32}} \mathrm{d}c_{\theta_{52}} \times\\
& (s_{\theta_{12}})^{d-3} (s_{\theta_{11}})^{d-3} (s_{\theta_{21}})^{d-3} (s_{\theta_{31}})^{d-5} (s_{\theta_{41}})^{d-6} \times\\
& (s_{\theta_{22}})^{d-4} (s_{\theta_{32}})^{d-5} \mathrm{d}c_{\theta_{42}} (s_{\theta_{42}})^{d-6} (s_{\theta_{52}})^{d-7} \frac{\mathcal{N}(q_1,q_2)}{D_1 \ldots D_n},
\end{aligned} \tag{A.3.28}$$

- Transverse tensor integrals:

$$I_1^{d\,(2)}[\,x_{1i}x_{1j}\,] = I_1^{d\,(2)}[\,x_{2i}x_{2j}\,] = I_1^{d\,(2)}[\,x_{3i}x_{3j}\,] = I_1^{d\,(2)}[\,x_{4i}x_{4j}\,] = \frac{1}{d} I_1^{d\,(2)}[\,\lambda_{ij}], \quad \forall i,j. \tag{A.3.29}$$

Moreover, in general, we have

$$I_1^{d\,(2)}[\,x_{11}^{\alpha_1}x_{21}^{\alpha_2}x_{31}^{\alpha_3}x_{41}^{\alpha_4}x_{12}^{\beta_1}x_{22}^{\beta_2}x_{32}^{\beta_3}x_{42}^{\beta_4}\,] = 0, \quad \text{if} \quad \alpha_i + \beta_i = 2n+1. \tag{A.3.30}$$

A.4 Gegenbauer polynomials

In this appendix, we recall the most relevant properties of Gegenbauer polynomials. *Gegenbauer polynomials* $C_n^{(\alpha)}(x)$ are orthogonal polynomials over the interval $[-1,1]$ with respect to the weight function

$$\omega_\alpha(x) = (1-x^2)^{\alpha-\frac{1}{2}} \tag{A.4.1}$$

and they can be defined through the generating function

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x) t^n. \tag{A.4.2}$$

These polynomials obey the orthogonality relation

$$\int_{-1}^{1} dx\, \omega_\alpha(x) C_n^{(\alpha)}(x) C_m^{(\alpha)}(x) = \delta_{mn} \frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!\,(n+\alpha)\Gamma^2(\alpha)}. \tag{A.4.3}$$

The explicit expression of the first Gegenbauer polynomials is given by

$$\begin{aligned} C_0^{(\alpha)}(x) &= 1, \\ C_1^{(\alpha)}(x) &= 2\alpha x, \\ C_2^{(\alpha)}(x) &= -\alpha + 2\alpha(1+\alpha)x^2, \\ &\cdots \end{aligned} \tag{A.4.4}$$

and it can inverted in order to express arbitrary powers of the variable x in terms of products of Gegenbauer polynomials,

$$\begin{aligned} x &= \frac{1}{2\alpha} C_0^{(\alpha)}(x) C_1^{(\alpha)}(x), \\ x^2 &= \frac{1}{4\alpha^2}[C_1^{(\alpha)}(x)]^2, \\ x^3 &= \frac{1}{4\alpha^2(1+\alpha)} C_1^{(\alpha)}(x)[\alpha C_0^{(\alpha)}(x) + C_2^{(\alpha)}(x)], \\ x^4 &= \frac{1}{4\alpha^2(1+\alpha)^2}[\alpha C_0^{(\alpha)}(x) + C_2^{(\alpha)}(x)]^2, \\ &\cdots \end{aligned} \tag{A.4.5}$$

These identities can be used in order to evaluate the integral of any polynomial in x, convoluted with the weight function $\omega_\alpha(x)$, by means of the orthogonality relation (A.4.3).

A.5 Four-dimensional bases

In this appendix, we provide the explicit definitions of the four-dimensional basis $\mathcal{E} = \{e_i^\alpha\}$ used throughout chapters 2-4 to decompose the four-dimensional part of the loop momenta $q_{[4]\,i}$,

$$q^\alpha_{[4]\,i} = p^\alpha_{0\,i} + x_{1i}e_1^\alpha + x_{2i}e_2^\alpha + x_{3i}e_3^\alpha + x_{4i}e_4^\alpha. \tag{A.5.1}$$

In the following, for any pair of massless vectors q_1^α and q_2^α, we denote by $\varepsilon^\alpha_{q_1,q_2}$ the spinor product

$$\varepsilon^\alpha_{q_1\,q_2} = \frac{1}{2}\langle q_1|\gamma^\alpha|q_2]. \tag{A.5.2}$$

$d = 4 - 2\epsilon$ bases

In the $d = 4 - 2\epsilon$ parametrization of Feynman integrals, we choose, independently from the number of external legs, a basis of massless vectors $\mathcal{E}$ defined in terms of two adjacent external momenta p_1 and p_2 by

$$e_1^\alpha = \frac{1}{1-r_1r_2}(p_1^\alpha - r_1p_2^\alpha), \quad e_2^\alpha = \frac{1}{1-r_1r_2}(p_2^\alpha - r_2p_1^\alpha), \quad e_3^\alpha = \varepsilon^\alpha_{e_1\,e_2}, \quad e_4^\alpha = \varepsilon^\alpha_{e_2,e_1}, \tag{A.5.3}$$

where

$$r_i = \frac{p_i^2}{\gamma} \qquad \text{with} \qquad \gamma = (p_1\cdot p_2)\left(1 + \sqrt{1 - \frac{p_1^2p_2^2}{(p_1\cdot p_2)^2}}\right). \tag{A.5.4}$$

In the case of two-point integrals, p_1 corresponds to the external momentum and p_2 is an arbitrary massless vector. In the case of one-point integrals, both p_1 and p_2 are chosen to be arbitrary massless vectors.

$d = d_\parallel + d_\perp$ bases

In the $d = d_\parallel + d_\perp$ parametrization of Feynman integrals with $n \le 4$ external legs, the four-dimensional basis $\mathcal{E}$ is chosen in such a way to satisfy the requirements

$$e_i \cdot p_j = 0, \qquad i > n-1, \quad \forall j = 1,\dots n-1, \tag{A.5.5a}$$
$$e_i \cdot e_j = \delta_{ij}, \quad i,j > n-1, \tag{A.5.5b}$$

where $\{p_1, p_2, \dots, p_{n-1}\}$ is the set of independent external momenta.

- **Four-point integrals**
  In case of four-point integrals $\mathcal{E}$ is defined as

$$\begin{aligned}
e_1^\alpha &= \frac{1}{1-r_1r_2}(p_1^\alpha - r_1p_2^\alpha),\\
e_2^\alpha &= \frac{1}{1-r_1r_2}(p_2^\alpha - r_2p_1^\alpha),\\
e_3^\alpha &= \frac{1}{i\sqrt{\beta}}\left[(\varepsilon_{e_2\,e_1}\cdot p_3)\,\varepsilon^\alpha_{e_1\,e_2} + (\varepsilon_{e_1\,e_2}\cdot p_3)\,\varepsilon^\alpha_{e_2\,e_1}\right],\\
e_4^\alpha &= \frac{1}{\sqrt{\beta}}\left[(\varepsilon_{e_2\,e_1}\cdot p_3)\,\varepsilon^\alpha_{e_1\,e_2} - (\varepsilon_{e_1\,e_2}\cdot p_3)\,\varepsilon^\alpha_{e_2\,e_1}\right].
\end{aligned} \tag{A.5.6}$$

  with $r_{1,2}$ given by (A.5.4) and $\beta = 2e_1\cdot e_2\,(\varepsilon_{e_1\,e_2}\cdot p_3)\,(\varepsilon_{e_1\,e_2}\cdot p_3)$.

- **Three-point integrals**
  For three-point integrals $\mathcal{E}$ is defined as

$$e_1^\alpha = \frac{1}{1-r_1 r_2}(p_1^\alpha - r_1 p_2^\alpha), \qquad e_2^\alpha = \frac{1}{1-r_1 r_2}(p_2^\alpha - r_2 p_1^\alpha),$$
$$e_3^\alpha = \frac{1}{i\sqrt{2e_1\cdot e_2}}\left(\varepsilon^\alpha_{e_1\, e_2} + \varepsilon^\alpha_{e_2\, e_1}\right), \qquad e_4^\alpha = \frac{1}{\sqrt{2e_1\cdot e_2}}\left(\varepsilon^\alpha_{e_1\, e_2} - \varepsilon^\alpha_{e_2, e_1}\right), \qquad \text{(A.5.7)}$$

  with $r_{1,2}$ given by (A.5.4).

- **Two-point integrals with $p^2 \neq 0$**
  For a two-point integral with massive external momentum p, we introduce two massless vectors q_1 and q_2 satisfying

$$p^\alpha = q_1^\alpha + \frac{p^2}{2q_1\cdot q_2} q_2^\alpha \qquad \text{(A.5.8)}$$

  and we define the massive auxiliary momentum q

$$q^\alpha = q_1^\alpha - \frac{p^2}{2q_1\cdot q_2} q_2^\alpha. \qquad \text{(A.5.9)}$$

  The basis $\mathcal{E}$ is therewith defined as

$$e_1^\alpha = \frac{1}{\sqrt{p^2}} p^\alpha, \qquad e_2^\alpha = \frac{1}{i\sqrt{p^2}} q^\alpha,$$
$$e_3^\alpha = \frac{1}{i\sqrt{2q_1\cdot q_2}}(\varepsilon^\alpha_{q_1\, q_2} + \varepsilon^\alpha_{q_2\, q_1}), \qquad e_4^\alpha = \frac{1}{\sqrt{2q_1\cdot q_2}}(\varepsilon^\alpha_{q_1\, q_2} - \varepsilon^\alpha_{q_2\, q_1}). \qquad \text{(A.5.10)}$$

- **Two-point integrals with $p^2 = 0$**
  In the case of two-point integrals with massless external momentum p, we introduce a massless auxiliary vector q_1 and we define the basis $\mathcal{E}$ as

$$e_1^\alpha = p^\alpha, \qquad e_2^\alpha = q_1^\alpha,$$
$$e_3^\alpha = \frac{1}{i\sqrt{2p\cdot q_1}}(\varepsilon^\alpha_{p\, q_1} + \varepsilon^\alpha_{q_1\, p}), \qquad e_4^\alpha = \frac{1}{\sqrt{2p\cdot q_1}}(\varepsilon^\alpha_{p\, q_1} - \varepsilon^\alpha_{q_1\, p}). \qquad \text{(A.5.11)}$$

- **One-point integrals**
  For one-point integrals, we introduce two arbitrary independent massless vectors q_1 and q_2 and we build a completely orthonormal basis $\mathcal{E}$,

$$e_1^\alpha = \frac{1}{\sqrt{2q_1\cdot q_2}}(q_1^\alpha + q_2^\alpha), \qquad e_2^\alpha = \frac{1}{i\sqrt{2q_1\cdot q_2}}(q_1^\alpha - q_2^\alpha),$$
$$e_3^\alpha = \frac{1}{i\sqrt{2q_1\cdot q_2}}(\varepsilon^\alpha_{q_1\, q_2} + \varepsilon^\alpha_{q_2\, q_1}), \qquad e_4^\alpha = \frac{1}{\sqrt{2q_1\cdot q_2}}(\varepsilon^\alpha_{q_1\, q_2} - \varepsilon^\alpha_{q_2\, q_1}). \qquad \text{(A.5.12)}$$

Appendix B

dlog-forms

In this appendix we collect the coefficient matrices of the dlog-forms for the master integrals discussed in chapters 6-8.

B.1 dlog-forms for X^0WW

We list below the coefficients matrix of the dlog-form for the master integrals of the first and second integral families, respectively defined by eqs. (6.7,6.25).

First integral family

For the first integral family, given in eq. (6.7) we have

$$\begin{aligned} d\mathbb{A} = {} & \mathbb{M}_1 \text{ dlog}(u) + \mathbb{M}_2 \text{ dlog}(z) + \mathbb{M}_3 \text{ dlog}(1-z) \\ & + \mathbb{M}_4 \text{ dlog}(\bar{z}) + \mathbb{M}_5 \text{ dlog}(1-\bar{z}) + \mathbb{M}_6 \text{ dlog}(z-\bar{z}) \\ & + \mathbb{M}_7 \text{ dlog}(1+u\,z\,\bar{z}) + \mathbb{M}_8 \text{ dlog}\left(1-u\,z(1-\bar{z})\right) \\ & + \mathbb{M}_9 \text{ dlog}\left(1-u\,\bar{z}(1-z)\right) + \mathbb{M}_{10} \text{ dlog}\left(1+u(1-z)(1-\bar{z})\right) , \end{aligned} \tag{B.1.1}$$

with

$$\mathbb{M}_1 = \left(\begin{array}{ccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-2&\frac{1}{2}&-\frac{3}{2}&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\frac{3}{2}&-2&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&\frac{1}{2}&0&0&0&0&0&\frac{1}{2}&\frac{1}{2}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&-2&2&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&-2&1&2&0\\
0&0&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right), \tag{B.1.2}$$

$$
\mathbb{M}_2 = \left(\begin{array}{ccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\frac{1}{2}&-\frac{1}{2}&0&0&0&0&0&0&0&0&2&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-\frac{1}{2}&0&-1&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-2&\frac{1}{2}&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-\frac{1}{2}&0&-\frac{1}{2}&-\frac{1}{2}&0&\frac{1}{2}&-\frac{1}{2}&0&0&-1&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&-1&\frac{1}{2}&0&-\frac{1}{2}\\
0&0&0&-1&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&-1&-\frac{1}{2}&\frac{3}{2}&0&0&0&-2&0&0&0&0&-2&-1&0&0&0&1&0&0&0&0&-2&1&0&-1\\
0&2&0&-2&2&1&-3&0&0&0&0&0&0&0&0&4&2&0&0&0&-2&0&0&0&0&4&-2&0&2
\end{array}\right), \qquad \text{(B.1.3)}
$$

$$
\mathbb{M}_3 = \left(\begin{array}{ccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&-2&-1&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\frac{3}{2}&0&\frac{3}{2}&0&0&0&0&0&0&0&0&0&0&0&3&3&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&-\frac{1}{2}&0&\frac{1}{2}&0&0&-1&0&0&0&0&0&2&1&1&1&1&0&0&0&0&0&0&0\\
0&\frac{1}{2}&0&\frac{1}{2}&\frac{1}{2}&0&-\frac{1}{2}&\frac{1}{2}&0&0&1&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&-1&0&\frac{1}{2}&\frac{1}{2}\\
0&0&0&1&0&0&-\frac{3}{2}&1&\frac{1}{2}&0&2&0&0&0&0&1&0&0&0&0&1&1&0&0&0&-2&0&1&1\\
0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-2&0&2&0&0&-3&2&1&0&0&0&0&0&0&2&0&0&0&0&2&2&0&0&0&-4&0&2&2
\end{array}\right), \qquad \text{(B.1.4)}
$$

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\mathbb{M}_4 = \left(\begin{array}{ccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-\frac{1}{2}&\frac{1}{2}&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\frac{3}{2}&-\frac{3}{2}&0&0&0&0&0&0&0&0&3&3&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-2&\frac{1}{2}&0&1&-\frac{1}{2}&0&0&0&0&0&0&0&0&-2&-1&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-\frac{1}{2}&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-\frac{1}{2}&0&-2&\frac{1}{2}&0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{1}{2}&0&\frac{1}{2}&\frac{1}{2}&0&-\frac{1}{2}&\frac{1}{2}&0&0&1&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&-1&-\frac{1}{2}&1&\frac{1}{2}\\
0&0&0&1&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&\frac{1}{2}&-\frac{3}{2}&0&0&0&2&0&0&0&0&1&1&0&0&0&1&0&0&0&0&-2&-1&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right), \qquad \text{(B.1.5)}
$$

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\mathbb{M}_5 = \left(\begin{array}{ccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&2&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&-1&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&1&0&0&0&0&0&0&0&-1&-2&-1&0&0&0&0&0&0&0\\
0&-\frac{1}{2}&0&-\frac{1}{2}&-\frac{1}{2}&0&\frac{1}{2}&-\frac{1}{2}&0&0&-1&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&-1&1&-\frac{1}{2}&-\frac{1}{2}\\
0&0&0&-1&0&0&\frac{3}{2}&-1&-\frac{1}{2}&0&-2&0&0&0&0&1&0&0&0&0&-2&-1&0&0&0&-2&2&-1&-1\\
0&-1&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right), \qquad \text{(B.1.6)}
$$

$$
\mathbb{M}_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0
\end{pmatrix}, \tag{B.1.7}
$$

$$
\mathbb{M}_7 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -\frac{1}{2} & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}, \tag{B.1.8}
$$

$$
\mathbb{M}_8 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -3 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\
1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & -3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\
-1 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1
\end{pmatrix}, \tag{B.1.9}
$$

$$
\mathbb{M}_9 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -3 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\
-1 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\
-1 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & -1
\end{pmatrix}, \tag{B.1.10}
$$

$$\mathbb{M}_{10} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
-1&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&-2&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&-4&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&-1&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&-1&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&-2&0&0&0&0&0&0&0&0&0&0&-4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&\frac{3}{2}&-1&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&-3&-2&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&1&0&0&0&-2&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&2&0&2&2&1&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0
\end{pmatrix}. \tag{B.1.11}$$

Second Integral family

For the second integral family, given in eq. (6.25) we have

$$\begin{aligned}
\mathrm{d}\mathbb{A} = {} & \mathbb{M}_1 \,\mathrm{dlog}(v) + \mathbb{M}_2 \,\mathrm{dlog}(1-v) + \mathbb{M}_3 \,\mathrm{dlog}(1+v) + \mathbb{M}_4 \,\mathrm{dlog}(z) \\
& + \mathbb{M}_5 \,\mathrm{dlog}(1-z) + \mathbb{M}_6 \,\mathrm{dlog}(\bar{z}) + \mathbb{M}_7 \,\mathrm{dlog}(1-\bar{z}) \\
& + \mathbb{M}_8 \,\mathrm{dlog}(z-\bar{z}) + \mathbb{M}_9 \,\mathrm{dlog}\left(z+v(1-z)\right) + \mathbb{M}_{10} \,\mathrm{dlog}\left(1-z(1-v)\right) \\
& + \mathbb{M}_{11} \,\mathrm{dlog}\left(\bar{z}+v(1-\bar{z})\right) + \mathbb{M}_{12} \,\mathrm{dlog}\left(1-\bar{z}(1-v)\right) \\
& + \mathbb{M}_{13} \,\mathrm{dlog}\left(v+z\bar{z}(1-v)^2\right) + \mathbb{M}_{14} \,\mathrm{dlog}\left(v+(1-z-\bar{z}+z\bar{z})(1-v)^2\right) \\
& + \mathbb{M}_{15} \,\mathrm{dlog}\left(v+z(1-v)^2\right) + \mathbb{M}_{16} \,\mathrm{dlog}\left(v+(1-z)(1-v)^2\right) \\
& + \mathbb{M}_{17} \,\mathrm{dlog}\left(v+\bar{z}(1-v)^2\right) + \mathbb{M}_{18} \,\mathrm{dlog}\left(v+(1-\bar{z})(1-v)^2\right) \,,
\end{aligned} \tag{B.1.12}$$

with

$$\mathbb{M}_1 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
-1&0&0&0&0&0&5&-6&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&2&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&2&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&1&1&-1&-2&4&0&0&0&0&0&0&0&0&-1&1&2&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0 \\
1&1&0&0&0&0&-2&4&-1&1&0&0&0&0&0&0&0&0&0&0&0&1&-1&2&0&0&0&0&0&0&0 \\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0 \\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&-\frac{1}{2}&0&0&0&0&0&-\frac{1}{2}&-\frac{1}{2}&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&-2&0&0&0&2&2&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&1&0&0&0&2&1&2&0&0 \\
1&2&4&0&0&-2&-2&1&2&0&0&4&0&0&0&0&0&0&0&0&2&0&-2&-2&0&0&4&2&2&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2
\end{pmatrix}, \tag{B.1.13}$$

$$
\mathbb{M}_2 = \left(\begin{array}{ccccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&8&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&8&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&-4&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&-4&1&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&0&0&0&0&-4&0&4&0&0\\
0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&0&0&0&0&-4&0&2&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&2&0&0&0&0&0&0&0&0&4
\end{array}\right), \tag{B.1.14}
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$$
\mathbb{M}_3 = \left(\begin{array}{ccccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-6&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-\frac{1}{2}&1&0&0&-1&\frac{1}{2}&0&0&0&0&0&0&0&0&0&1&-2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-2&0&0&0&0&0&0&0&0&-4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right), \tag{B.1.15}
$$

$$
\mathbb{M}_4 = \left(\begin{array}{ccccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-\frac{1}{2}&0&0&0&-\frac{1}{2}&1&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&1&1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
-\frac{1}{2}&0&0&-1&-\frac{1}{2}&1&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&1&1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-\frac{1}{2}&0&0&0&0&0&0&-\frac{3}{2}&1&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&-2&-1&-1&0&0&0&0&0&0&0\\
\frac{1}{2}&0&0&0&0&0&0&\frac{3}{2}&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0\\
\frac{1}{2}&0&0&0&0&0&0&\frac{3}{2}&-2&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\frac{1}{2}&0&0&0&0&0&0&\frac{3}{2}&-2&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&2&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&\frac{1}{2}&-1&0&2&-1&\frac{1}{2}&0&0&-1&0&0&0&-1&-1&1&0&0&2&1&1&0&0&0&0&0&0&0
\end{array}\right), \tag{B.1.16}
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$$\mathbb{M}_5 = \left(\begin{array}{ccccccccccccccccccccccccccccccc}
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\end{array}\right), \tag{B.1.17}$$

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\end{array}\right), \tag{B.1.18}$$

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\end{array}\right), \tag{B.1.19}$$

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\mathbb{M}_8 = \begin{pmatrix}
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\end{pmatrix}, \tag{B.1.24}$$

$$\mathbb{M}_{13} = \begin{pmatrix}
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\end{pmatrix}, \tag{B.1.25}$$

$$\mathbb{M}_{14} = \begin{pmatrix}
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\end{pmatrix}, \tag{B.1.26}$$

$$\mathbb{M}_{15} = \begin{pmatrix}
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\end{pmatrix}, \tag{B.1.27}$$

$$\mathbb{M}_{16} = \begin{pmatrix}
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\end{pmatrix}, \tag{B.1.28}$$

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\end{pmatrix}, \tag{B.1.29}$$

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\end{pmatrix}. \tag{B.1.30}$$

B.2 dlog-forms for massive boson-pair production

The dlog-form for the master integrals belonging to the integral family defined by eq. (7.2) is given by

$$\begin{aligned}
\mathrm{d}\mathbb{A} = {} & \mathbb{M}_1 \,\mathrm{dlog}(x) + \mathbb{M}_2 \,\mathrm{dlog}(4+x) + \mathbb{M}_3 \,\mathrm{dlog}(\sqrt{x} + \sqrt{4+x}) + \mathbb{M}_4 \,\mathrm{dlog}(y) \\
& + \mathbb{M}_5 \,\mathrm{dlog}(1+y) + \mathbb{M}_6 \,\mathrm{dlog}(4+y) + \mathbb{M}_7 \,\mathrm{dlog}(\sqrt{y} + \sqrt{4+y}) \\
& + \mathbb{M}_8 \,\mathrm{dlog}(x-4y) + \mathbb{M}_9 \,\mathrm{dlog}(x - y(4+y)) + \mathbb{M}_{10} \,\mathrm{dlog}(x^2 - y - xy(3+y)), \\
& + \mathbb{M}_{11} \,\mathrm{dlog}(\sqrt{x} + \sqrt{x-4y}) + \mathbb{M}_{12} \,\mathrm{dlog}(\sqrt{4+x} + \sqrt{x-4y}), \\
& + \mathbb{M}_{13} \,\mathrm{dlog}\left(\frac{2x - y(y+3) - (y+1)\lambda_y}{2x - y(y+3) + (y+1)\lambda_y}\right) + \mathbb{M}_{14} \,\mathrm{dlog}\left(\frac{\sqrt{x}(y+2) - \sqrt{x+4}\lambda_y}{\sqrt{x}(y+2) + \sqrt{x+4}\lambda_y}\right) \\
& + \mathbb{M}_{15} \,\mathrm{dlog}\left(\frac{-\sqrt{xy} + \sqrt{(y+4)(x-4y)}}{\sqrt{xy} + \sqrt{(y+4)(x-4y)}}\right) + \mathbb{M}_{16} \,\mathrm{dlog}\left(-\frac{\sqrt{x}\,(\lambda_y - 1 - y) + \sqrt{x-4y}}{\sqrt{x}\,(\lambda_y - 1 - y) - \sqrt{x-4y}}\right) \\
& + \mathbb{M}_{17} \,\mathrm{dlog}\left(\frac{\sqrt{x}\,(\lambda_y + 1 + y) - \sqrt{x-4y}}{\sqrt{x}\,(\lambda_y + 1 + y) + \sqrt{x-4y}}\right) + \\
& \mathbb{M}_{18} \,\mathrm{dlog}\left(\frac{\left(\sqrt{x-4y} + y\right)\left(y\left(\sqrt{x-4y} + \sqrt{x+4} + 4\right) + 2\left(\sqrt{x+4} + 2\right)\right)}{\left(\sqrt{x-4y} - y\right)\left(y\left(-\sqrt{x-4y} + \sqrt{x+4} + 4\right) + 2\left(\sqrt{x+4} + 2\right)\right)}\right),
\end{aligned} \tag{B.2.1}$$

with $\lambda_y = \sqrt{y}\sqrt{4+y}$ and

$$\mathbb{M}_1 = \begin{pmatrix}
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\end{pmatrix}, \tag{B.2.2}$$

$$\mathbb{M}_2 = \begin{pmatrix}
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\end{pmatrix}, \qquad (B.2.12)
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\end{array}\right), \tag{B.2.18}$$

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\end{array}\right). \tag{B.2.19}$$

B.3 dlog-forms for μe scattering

We list below the coefficients matrix of the dlog-forms

$$\mathrm{d}\mathbb{A} = \mathbb{M}_1 \ \mathrm{dlog}(x) + \mathbb{M}_2 \ \mathrm{dlog}(1+x) + \mathbb{M}_3 \ \mathrm{dlog}(1-x)$$

$$
\begin{aligned}
&+ \mathbb{M}_4 \,\mathrm{dlog}(y) + \mathbb{M}_5 \,\mathrm{dlog}(1+y) + \mathbb{M}_6 \,\mathrm{dlog}(1-y) \\
&+ \mathbb{M}_7 \,\mathrm{dlog}(x+y) + \mathbb{M}_8 \,\mathrm{dlog}\left(1 + x\, y\right) \\
&+ \mathbb{M}_9 \,\mathrm{dlog}\left(1 - y(1-x-y)\right) \,,
\end{aligned} \tag{B.3.1}
$$

for the master integrals in the first and second integral family, respectively defined in eqs. (8.24,8.50).

First integral family

For the first integral family, given in eq. (8.24), we have ($\mathbb{M}_3$ is vanishing for this integral family):

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\mathbb{M}_1 = \begin{pmatrix}
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-\frac{1}{4} & -1 & 0 & -6 & -\frac{19}{6} & \frac{17}{12} & -\frac{21}{8} & -1 & 0 & 0 & 2 & -\frac{1}{4} & 0 & 0 & 0 & 0 & -1 & -1 & 1 & \frac{1}{2} & 12 & 6 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 \\
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\end{array}\right), \qquad \text{(B.3.8)}
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\end{array}\right). \qquad \text{(B.3.9)}
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Second integral family

For the second integral family, given in eq. (8.50), we find:

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\mathbb{M}_1 = \left(\begin{array}{cccccccccccccccccccccccccccccccccccccccccc}
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\end{pmatrix}. \quad \text{(B.3.18)}
$$

Appendix C

Analytic continuation of the three-loop banana graph

In this appendix, we describe the analytic continuation of the MIs derived in sec. 9.5.4 to arbitrary values of $x = 4m^2/s$. We start from the analytic continuation of the homogeneous solutions by first defining, for each kinematic region $a < x < b$, a set of real-valued homogeneous solutions $\mathbb{G}^{(a,b)}(x)$ and then by matching their limiting behaviours in order to link them across the singularities of the DEQs. Finally, we apply these results to eq. (9.5.10) and obtain the analytic continuation of the inhomogeneous solution.

C.1 Homogeneous solutions

In sections 9.5.2-9.5.3 we have obtained, through different approaches, two different representation of the homogeneous solutions, corresponding to eqs. (9.201) and (9.216). Although the two representations have been shown to be completely equivalent, we decide to work with the latter, since it leads to more compact expressions. Therefore, we consider homogeneous solutions written in terms of products of complete elliptic integrals with arguments

$$\omega_{\pm} = \frac{1}{4x}\left(2x + (1-2x)\sqrt{\frac{x-1}{x}} \pm \sqrt{\frac{4x-1}{x}}\right) . \tag{C.1.1}$$

The solutions (9.216) are explicitly real in the region $(1, \infty)$. In order to define a set of real solutions in the other three regions, $(-\infty, 0)$, $(0, 1/4)$ and $(1/4, 1)$, we can make use of well-known identities between complete elliptic integrals, such as

$$\mathrm{K}\left(\frac{1}{z}\right) = \sqrt{z}\left(\mathrm{K}\left(z\right) - i\mathrm{K}\left(1-z\right)\right) , \quad \text{with} \quad z \to z + i0^+ , \tag{C.1.2}$$

which establish linear relations between elliptic integrals with different reality domains. Therefore, by extending the set of building-blocks of the homogeneous solutions to the elliptic integrals

$$\mathrm{K}\left(\omega_{\pm}\right) , \quad \mathrm{K}\left(1-\omega_{\pm}\right) , \quad \mathrm{K}\left(\frac{1}{\omega_{\pm}}\right) , \quad \mathrm{K}\left(\frac{1}{1-\omega_{\pm}}\right) , \quad \mathrm{K}\left(1-\frac{1}{\omega_{\pm}}\right) , \tag{C.1.3}$$

one can easily obtain, for each region (a, b), a matrix of homogeneous solutions $G^{(a,b)}(x)$ with real entries. In the following we list, for each region, one possible choice of real solutions for the first master integral, which correspond to the first row of $\mathbb{G}^{(a,b)}(x)$. As we have already observed, the other two rows can be obtained by applying the differential operators (9.157, 9.158) to the first one.

- $-\infty < x < 0$:

$$\begin{aligned} H^{(1)}_{[1;\,(-\infty,0)]}(x) =& \mathrm{K}\left(\omega_+\right)\mathrm{K}\left(\omega_-\right)\,, \\ H^{(2)}_{[1;\,(-\infty,0)]}(x) =& \frac{1}{2}\left[\mathrm{K}\left(\omega_+\right)\mathrm{K}\left(1-\omega_-\right)+\mathrm{K}\left(1-\omega_+\right)\mathrm{K}\left(\omega_-\right)\right]\,, \\ H^{(3)}_{[1;\,(-\infty,0)]}(x) =& \frac{1}{\sqrt{1-\omega_-}\sqrt{1-\omega_+}}\mathrm{K}\left(\frac{1}{1-\omega_+}\right)\mathrm{K}\left(\frac{1}{1-\omega_-}\right)\,. \end{aligned} \tag{C.1.4}$$

with Wronskian

$$W^{(-\infty,0)}(x) = \frac{\pi^3 x^3}{64\sqrt{(1-4x)^3(1-x)}}\,. \tag{C.1.5}$$

- $0 < x < 1/4$:

$$\begin{aligned} H^{(1)}_{[1;\,(0,1/4)]}(x) =& \frac{1}{2}\left[\mathrm{K}\left(\omega_+\right)\mathrm{K}\left(1-\omega_-\right)+\mathrm{K}\left(1-\omega_+\right)\mathrm{K}\left(\omega_-\right)\right]\,, \\ H^{(2)}_{[1;\,(0,1/4)]}(x) =& -\frac{1}{2}\left[\mathrm{K}\left(\omega_+\right)\mathrm{K}\left(\omega_-\right)+\mathrm{K}\left(1-\omega_+\right)\mathrm{K}\left(1-\omega_-\right)\right]\,, \\ H^{(3)}_{[1;\,(0,1/4)]}(x) =& \mathrm{K}\left(\omega_-\right)\left[\mathrm{K}\left(\omega_+\right)+\frac{1}{\sqrt{\omega_+}}\mathrm{K}\left(\frac{1}{\omega_+}\right)\right]\,, \end{aligned} \tag{C.1.6}$$

with Wronskian

$$W^{(0,1/4)}(x) = \frac{\pi^3 x^3}{64\sqrt{(1-4x)^3(1-x)}}\,. \tag{C.1.7}$$

- $1/4 < x < 1$:

$$\begin{aligned} H^{(1)}_{[1;\,(1/4,1)]}(x) =& \frac{1}{2}\left[\mathrm{K}\left(\omega_+\right)\mathrm{K}\left(\omega_-\right)+\mathrm{K}\left(1-\omega_+\right)\mathrm{K}\left(1-\omega_-\right)\right]\,, \\ H^{(2)}_{[1;\,(1/4,1)]}(x) =& \frac{1}{\sqrt{\omega_+}}\mathrm{K}\left(\omega_-\right)\mathrm{K}\left(1-\frac{1}{\omega_+}\right)\,, \\ H^{(3)}_{[1;\,(1/4,1)]}(x) =& -\frac{1}{\sqrt{1-\omega_+}\sqrt{\omega_-}}\mathrm{K}\left(\frac{1}{1-\omega_+}\right)\mathrm{K}\left(\frac{1}{\omega_-}\right)\,, \end{aligned} \tag{C.1.8}$$

with Wronskian

$$W^{(1/4,1)}(x) = \frac{\pi^3 x^3}{64\sqrt{(4x-1)^3(1-x)}}\,. \tag{C.1.9}$$

- $1 < x < \infty$:

$$\begin{aligned} H^{(1)}_{[1;\,(1,\infty)]}(x) =& \mathrm{K}\left(\omega_+\right)\mathrm{K}\left(\omega_-\right)\,, \\ H^{(2)}_{[1;\,(1,\infty)]}(x) =& \mathrm{K}\left(\omega_+\right)\mathrm{K}\left(1-\omega_-\right)\,, \\ H^{(3)}_{[1;\,(1,\infty)]}(x) =& -\frac{1}{3}\mathrm{K}\left(1-\omega_+\right)\mathrm{K}\left(1-\omega_-\right)\,, \end{aligned} \tag{C.1.10}$$

with Wronskian

$$W^{(1,\infty)}(x) = \frac{\pi^3 x^3}{64\sqrt{(4x-1)^3(x-1)}}\,. \tag{C.1.11}$$

C.2 Analytic continuation of the homogeneous solution

Once real homogeneous solutions $\mathbb{G}^{(a,b)}(x)$ have been found in each region (a,b), we must match them across the four singular points $x = 0, 1/4, 1$ and $x = \pm\infty$ in order to analytically continue the homogeneous solutions the whole real axis $-\infty < x < \infty$. Given the matrices $^{(a,b)}(x)$ and $\mathbb{G}^{(b,c)}(x)$ of real solutions defined in the adjacent intervals (a,b) and (b,c), the analytic continuation amounts to define a matching matrix $\mathbb{M}^{(b)}$,

$$\mathbb{G}^{(b,c)}(x) = \mathbb{G}^{(a,b)}(x)\mathbb{M}^{(b)}\,, \tag{C.2.1}$$

which allows to continue $\mathbb{G}^{(a,b)}(x)$ to the region $b < x < c$. The matrix $M^{(b)}$ can be obtained by assigning a small imaginary part to $x \to x - i0^+$ (the sign of which is inherited from the Feynman prescription $s \to s + i0^+$) and by equating the $x \to b^+$ limit of the two sides of (C.2.1).
This procedure leads to the four matching matrices

$$\begin{aligned}
\mathbb{M}^{(0)} &= \begin{pmatrix} 0 & 1 & -1 \\ 2 & -3i & 3i \\ -i & -1/2 & 0 \end{pmatrix}, \quad &\mathbb{M}^{(1/4)} &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & -2i & 0 \\ 0 & -i & 0 \end{pmatrix}, \\
\mathbb{M}^{(1)} &= \begin{pmatrix} 1 & 0 & -1/3 \\ 2i & 3 & 2/3i \\ i/2 & 0 & i/6 \end{pmatrix}, \quad &\mathbb{M}^{(\infty)} &= \begin{pmatrix} 1 & -i & -3 \\ 0 & -1/3 & 2i \\ 0 & 0 & -3 \end{pmatrix},
\end{aligned} \tag{C.2.2}$$

which, consistently with eq. (C.2.1), satisfy

$$\mathbb{M}^{(0)}\mathbb{M}^{(1/4)}\mathbb{M}^{(1)}\mathbb{M}^{(\infty)} = 1\,. \tag{C.2.3}$$

The limits of the homogeneous solutions (C.1.4 -C.1.10) close to the singular points, which have been used to obtain (C.2.2), can be easily calculated with the help of computer algebra system such as MATHEMATICA and, therefore, we will not write them down explicitly. As an example, we will just list below the leading behaviour of the homogeneous solutions (C.1.10) at the end-points of the region $(1,\infty)$, which have been also used in sec. 9.5.4 in order to fix the boundary constants of the inhomogeneous solution.

The limit of $\mathbb{G}^{(1,\infty)}(x)$ for $x \to 1^+$ is

$$\begin{aligned}
\lim_{x\to 1^+} H^{(1)}_{[1;(1,\infty)]}(x) =& \mathrm{K}(r_+)\,\mathrm{K}(r_-) + \mathcal{O}\left(\sqrt{x-1}\right)\,, \\
\lim_{x\to 1^+} H^{(2)}_{[1;(1,\infty)]}(x) =& \mathrm{K}(r_+)\,\mathrm{K}(r_+) + \mathcal{O}\left(\sqrt{x-1}\right)\,, \\
\lim_{x\to 1^+} H^{(3)}_{[1;(1,\infty)]}(x) =& -\frac{1}{3}\mathrm{K}(r_+)\,\mathrm{K}(r_-) + \mathcal{O}\left(\sqrt{x-1}\right)\,, \\
\lim_{x\to 1^+} H^{(1)}_{[2;(1,\infty)]}(x) =& \frac{1}{26\sqrt{x-1}}\bigg(\mathrm{E}(r_-)\left(6\mathrm{E}(r_+) + \left(\sqrt{3}-9\right)\mathrm{K}(r_+)\right) \\
& - \mathrm{K}(r_-)\left(\left(9+\sqrt{3}\right)\mathrm{E}(r_+) - 6\mathrm{K}(r_+)\right)\bigg) + \mathcal{O}\left(\sqrt{x-1}\right)\,, \\
\lim_{x\to 1^+} H^{(2)}_{[2;(1,\infty)]}(x) =& \frac{1}{6}\left(\left(\sqrt{3}-3\right)\mathrm{K}(r_+)^2 - 2\left(\sqrt{3}-3\right)\mathrm{K}(r_+)\,\mathrm{E}(r_+) - 6\mathrm{E}(r_+)^2\right) \\
& + \mathcal{O}\left(\sqrt{x-1}\right) \\
\lim_{x\to 1^+} H^{(3)}_{[2;(1,\infty)]}(x) =& \frac{1}{12\sqrt{x-1}}\left(\mathrm{K}(r_+)\left(\mathrm{K}(r_-) - \mathrm{E}(r_-)\right) - \mathrm{K}(r_-)\,\mathrm{E}(r_+)\right) +
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{18}\left(\left(3+\sqrt{3}\right)\mathrm{K}\left(r_-\right)\mathrm{E}\left(r_+\right)-\mathrm{E}\left(r_-\right)\left(\left(\sqrt{3}-3\right)\mathrm{K}\left(r_+\right)+6\mathrm{E}\left(r_+\right)\right)\right)\\
&+\mathcal{O}\left(\sqrt{x-1}\right)\,,\\
\lim_{x\to1^+}H^{(1)}_{[3;\,(1,\infty)]}(x)=&\frac{1}{8\sqrt{x-1}}\left(\mathrm{K}\left(r_-\right)E\left(r_+\right)+\mathrm{K}\left(r_+\right)\left(\mathrm{E}\left(r_-\right)-\mathrm{K}\left(r_-\right)\right)\right)+\\
&136\left(\mathrm{K}\left(r_-\right)\left(4\left(3+\sqrt{3}\right)E\left(r_+\right)-3\mathrm{K}\left(r_+\right)\right)\right.\\
&\left.-4\mathrm{E}\left(r_-\right)\left(\left(\sqrt{3}-3\right)\mathrm{K}\left(r_+\right)+6\mathrm{E}\left(r_+\right)\right)\right)+\mathcal{O}\left(\sqrt{x-1}\right)\,,\\
\lim_{x\to1^+}H^{(2)}_{[3;\,(1,\infty)]}(x)=&\frac{1}{36}\left(\left(9-4\sqrt{3}\right)\mathrm{K}\left(r_+\right)^2+8\left(\sqrt{3}-3\right)\mathrm{K}\left(r_+\right)\mathrm{K}\left(r_+\right)+24\mathrm{E}\left(r_+\right)^2\right)\\
&+\mathcal{O}\left(\sqrt{x-1}\right)\,,\\
\lim_{x\to1^+}H^{(3)}_{[3;\,(1,\infty)]}(x)=&\frac{1}{24\sqrt{x-1}}\left(\mathrm{K}\left(r_-\right)E\left(r_+\right)+\mathrm{K}\left(r_+\right)\left(\mathrm{E}\left(r_-\right)-\mathrm{K}\left(r_-\right)\right)\right)+\\
&\frac{1}{108}(4\mathrm{E}\left(r_-\right)\left(6\mathrm{E}\left(r_+\right)+\left(\sqrt{3}-3\right)\mathrm{K}\left(r_+\right)\right)\\
&+\mathrm{K}\left(r_-\right)\left(3\mathrm{K}\left(r_+\right)-4\left(3+\sqrt{3}\right)\mathrm{E}\left(r_+\right)\right))+\mathcal{O}\left(\sqrt{x-1}\right)\,,
\end{aligned}
\tag{C.2.4}
$$

where we have defined

$$
r_\pm\equiv\lim_{x\to1^+}\omega_\pm=\frac{2\pm\sqrt{3}}{4}. \tag{C.2.5}
$$

The leading behaviour of $\mathbb{G}^{(1,\infty)}(x)$ for $x\to+\infty$ is, instead,

$$
\begin{aligned}
\lim_{x\to+\infty}H^{(1)}_{[1;\,(1,\infty)]}(x)=&\frac{\pi^2}{4}+\mathcal{O}\left(1/x\right)\,,\\
\lim_{x\to+\infty}H^{(2)}_{[1;\,(1,\infty)]}(x)=&\frac{3}{4}\pi\left(4\ln2-\ln\left(1/x\right)\right)+\mathcal{O}\left(1/x\right)\,,\\
\lim_{x\to+\infty}H^{(3)}_{[1;\,(1,\infty)]}(x)=&\frac{1}{2}\left(\ln\left(1/x\right)-4\ln2\right)^2+\mathcal{O}\left(1/x\right)\,,\\
\lim_{x\to+\infty}H^{(1)}_{[2;\,(1,\infty)]}(x)=&-\frac{\pi^2}{16}+\mathcal{O}\left(1/x\right)\,,\\
\lim_{x\to+\infty}H^{(2)}_{[2;\,(1,\infty)]}(x)=&\frac{3\pi}{16}\left(1-4\ln2+\ln\left(1/x\right)\right)+\mathcal{O}\left(1/x\right)\,,\\
\lim_{x\to+\infty}H^{(3)}_{[2;\,(1,\infty)]}(x)=&\frac{1}{16}(4\ln2-\ln\left(1/x\right)-2)(4\ln\left(1/x\right)-\ln\left(1/x\right))+\mathcal{O}\left(1/x\right)\,,\\
\lim_{x\to+\infty}H^{(1)}_{[3;\,(1,\infty)]}(x)=&-\frac{\pi^2}{64}+\mathcal{O}\left(1/x\right)\,,\\
\lim_{x\to+\infty}H^{(2)}_{[3;\,(1,\infty)]}(x)=&\frac{3\pi}{128}\left(-2\ln\left(1/x\right)-3+8\ln2\right)+\mathcal{O}\left(1/x\right)\,,\\
\lim_{x\to+\infty}H^{(2)}_{[3;\,(1,\infty)]}(x)=&\frac{x}{24}-\frac{1}{64}(4\ln2-\ln\left(1/x\right)-3)(4\ln2-\ln2)\ln\left(1/x\right)+\mathcal{O}\left(1/x\right)\,.
\end{aligned}
\tag{C.2.6}
$$

C.3 Analytic continuation of the inhomogeneous solution

We are finally in the position to analytically continue the inhomogeneous solutions to arbitrary values of x. As it is explicitly shown by eq. (9.5.10), the inhomogeneous solution is defined, for $x > 1$, in terms of integrals of the functions $\mathcal{R}_i^{(1,\infty)}(x)$ which, in turn, depend on the homogeneous solutions $\mathbb{G}^{(1,\infty)}(x)$ and on their Wronskian $W^{(1,\infty)}(x)$. Therefore, in order to extend the integral representation (9.5.10) to the other kinematic regions, it is sufficient to analytically continue the elements of $\mathbb{G}^{(1,\infty)}(x)$ appearing in the definition (9.5.6) of $\mathcal{R}_i^{(1,\infty)}(x)$, by making use of the matching matrices $\mathbb{M}^{(b)}$, as prescribed by eq. (C.2.1). In this way, all imaginary parts (whenever they are present) are made explicit and, as a result, we obtain a representation of the solution which involves, for any x, the evaluation real integrals only.

We start by considering the analytic continuation to $1/4 < x < 1$. The matching matrix $\mathbb{M}^{(1)}$, which has been defined in eq. (C.2.2), can be used in order to express the homogeneous solutions $\mathbb{G}^{(1,\infty)}$ in terms of the set of real solutions defined in $(1/4, 1)$,

$$\mathbb{G}^{(1,\infty)}(x) = \mathbb{G}^{(1/4,1)}(x)\mathbb{M}^{(1)}. \tag{C.3.1}$$

In addition, it is easy to see that, with the Feynman prescription $x \to x - i\epsilon$, the Wronskian is analytically continued for $1/4 < x < 1$ as

$$W^{(1,\infty)} = \frac{\pi^3 x^3}{64\sqrt{(4x-1)^3(x-1)}} = \frac{i\pi^3 x^3}{64\sqrt{(1-4x)^3(1-x)}} = iW^{(1/4,1)}. \tag{C.3.2}$$

By acting with eqs. (C.3.1) and (C.3.2) on the inhomogeneous solution (C.2.2), we can write the MIs in region $(1/4, 1)$ in terms of individually real-valued integrals as

$$\begin{aligned}
\mathrm{F}_1^{(0)}(x) = {} & H^{(1)}_{[1;\,(1/4,1)]}(x)\left(c_1^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_1^{(1/4,1)}(t)\right) \\
& + H^{(2)}_{[1;\,(1/4,1)]}(x)\left(c_2^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_2^{(1/4,1)}(t)\right) \\
& + H^{(3)}_{[1;\,(1/4,1)]}(x)\left(c_3^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_3^{(1/4,1)}(t)\right), \\
\mathrm{F}_2^{(0)}(x) = {} & H^{(1)}_{[2;\,(1/4,1)]}(x)\left(c_1^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_1^{(1/4,1)}(t)\right) \\
& + H^{(2)}_{[2;\,(1/4,1)]}(x)\left(c_2^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_2^{(1/4,1)}(t)\right) \\
& + H^{(3)}_{[2;\,(1/4,1)]}\left(c_3^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_3^{(1/4,1)}(t)\right), \\
\mathrm{F}_3^{(0)}(x) = {} & H^{(1)}_{[3;\,(1/4,1)]}(x)\left(c_1^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_1^{(1/4,1)}(t)\right) \\
& + H^{(2)}_{[3;\,(1/4,1)]}(x)\left(c_2^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_2^{(1/4,1)}(t)\right) \\
& + H^{(3)}_{[3;\,(1/4,1)]}(x)\left(c_3^{(0)} + \int_x^1 \frac{\mathrm{d}t}{1-4t}\mathcal{R}_3^{(1/4,1)}(t)\right),
\end{aligned} \tag{C.3.3}$$

where $\mathcal{R}_i^{(1/4,1)}(x)$ are combinations of homogeneous solutions defined by

$$\begin{aligned}
\mathcal{R}_1^{(1/4,1)}(x) = {} & \frac{i}{4W^{(1/4,1)}(x)}\Big[2H^{(1)}_{[1;\,(1/4,1)]}(x)H^{(2)}_{[2;\,(1/4,1)]}(x) \\
& - 2H^{(1)}_{[2;\,(1/4,1)]}(x)H^{(2)}_{[1;\,(1/4,1)]}(x) +
\end{aligned}$$

$$
\begin{aligned}
&i\left(H^{(3)}_{[2;\,(1/4,1)]}(x)H^{(2)}_{[1;\,(1/4,1)]}(x)-H^{(3)}_{[1;\,(1/4,1)]}(x)H^{(2)}_{[2;\,(1/4,1)]}(x)\right)\Bigg],\\
\mathcal{R}_2^{(1/4,1)}(x)=&-\frac{1}{6W^{(1/4,1)}(x)}\Bigg[H^{(1)}_{[2;\,(1/4,1)]}(x)\left(H^{(3)}_{[1;\,(1/4,1)]}(x)+4H^{(2)}_{[1;\,(1/4,1)]}(x)\right)-\\
&H^{(1)}_{[1;\,(1/4,1)]}(x)\left(H^{(3)}_{[2;\,(1/4,1)]}(x)+4H^{(2)}_{[2;\,(1/4,1)]}(x)\right)\Bigg],\\
\mathcal{R}_3^{(1/4,1)}(x)=&\frac{3i}{4W^{(1/4,1)}(x)}\Bigg[2H^{(1)}_{[1;\,(1/4,1)]}(x)H^{(2)}_{[2;\,(1/4,1)]}(x)\\
&-2H^{(1)}_{[2;\,(1/4,1)]}(x)H^{(2)}_{[1;\,(1/4,1)]}(x)+\\
&i\left(H^{(3)}_{[1;\,(1/4,1)]}(x)H^{(2)}_{[2;\,(1/4,1)]}(x)-H^{(3)}_{[2;\,(1/4,1)]}(x)H^{(2)}_{[1;\,(1/4,1)]}(x)\right)\Bigg].
\end{aligned}
\tag{C.3.4}
$$

We can now continue the solution to $0<x<1/4$, where the master integrals develop an imaginary part. The region $(0,1/4)$ must be linked to $(1,\infty)$ by passing through the region $(1/4,1)$. This means that, according to the definition (C.2.1), the homogeneous solutions $\mathbb{G}^{(1,\infty)}(x)$ are continued in terms of the real-valued solutions defined for $0<x<1/4$ as

$$
\mathbb{G}^{(1,\infty)}(x)=\mathbb{G}^{(0,1/4)}(x)\mathbb{M}^{(1/4)}\mathbb{M}^{(1)}\,, \tag{C.3.5}
$$

where $\mathbb{M}^{(1/4)}$ and $\mathbb{M}^{(1)}$ are the matching matrices given in eq. (C.2.2). In this case, the Wronskian is trivially continued,

$$
W^{(1,\infty)}=\frac{\pi^3x^3}{64\sqrt{(4x-1)^3(x-1)}}=\frac{\pi^3x^3}{64\sqrt{(1-4x)^3(1-x)}}=W^{(0,1/4)}\,, \tag{C.3.6}
$$

and, by making use of eqs. (C.3.5) and (C.3.6), we can write the master integrals in region $(0,1/4)$ as

$$
\begin{aligned}
\mathrm{F}_1^{(0)}(x)=&\quad H^{(1)}_{[1;\,(0,1/4)]}(x)\left(b_1^{(0)}+\int_{1/4}^{x}\frac{\mathrm{d}t}{1-4t}\mathcal{R}_1^{(0,1/4)}(t)\right)\\
&+H^{(2)}_{[1;\,(0,1/4)]}(x)\left(b_2^{(0)}+\int_{1/4}^{x}\frac{\mathrm{d}t}{1-4t}\mathcal{R}_2^{(0,1/4)}(t)\right)+\\
&H^{(3)}_{[1;\,(0,1/4)]}(x)\left(b_3^{(0)}+\int_{1/4}^{x}\frac{\mathrm{d}t}{1-4t}\mathcal{R}_3^{(0,1/4)}(t)\right),\\
\mathrm{F}_2^{(0)}(x)=&\quad H^{(1)}_{[2;\,(0,1/4)]}(x)\left(b_1^{(0)}+\int_{1/4}^{x}\frac{\mathrm{d}t}{1-4t}\mathcal{R}_1^{(0,1/4)}(t)\right)\\
&+H^{(2)}_{[2;\,(0,1/4)]}(x)\left(b_2^{(0)}+\int_{1/4}^{x}\frac{\mathrm{d}t}{1-4t}\mathcal{R}_2^{(0,1/4)}(t)\right)\\
&+H^{(3)}_{[3;\,(0,1/4)]}(x)\left(b_3^{(0)}+\int_{1/4}^{x}\frac{\mathrm{d}t}{1-4t}\mathcal{R}_3^{(0,1/4)}(t)\right),\\
\mathrm{F}_3^{(0)}(x)=&\quad H^{(1)}_{[3;\,(0,1/4)]}(x)\left(b_1^{(0)}+\int_{1/4}^{x}\frac{\mathrm{d}t}{1-4t}\mathcal{R}_1^{(0,1/4)}(t)\right)\\
&+H^{(2)}_{[3;\,(0,1/4)]}(x)\left(b_2^{(0)}+\int_{1/4}^{x}d\frac{\mathrm{d}t}{1-4t}\mathcal{R}_2^{(0,1/4)}(t)\right)
\end{aligned}
$$

$$+ H^{(3)}_{[3;\,(0,1/4)]}(x)\left(b^{(0)}_3 + \int_{1/4}^{x} \frac{\mathrm{d}t}{1-4t}\mathcal{R}^{(0,1/4)}_3(t)\right), \tag{C.3.7}$$

where the integration constants $b^{(0)}_i$ are defined by

$$b^{(0)}_i = c^{(0)}_i + \int_{1/4}^{1} \frac{\mathrm{d}t}{1-4t}\mathcal{R}^{(1/4,1)}_i(t)\,, \tag{C.3.8}$$

and $\mathcal{R}^{(0,1/4)}_i(x)$ are the combinations of homogeneous solutions

$$\begin{aligned}
\mathcal{R}^{(0,1/4)}_1(x) =& \frac{1}{4W^{(0,1/4)}(x)}\Bigg[H^{(1)}_{[1;\,(0,1/4)]}(x)(H^{(3)}_{[2;\,(0,1/4)]}(x) + 4H^{(2)}_{[2;\,(0,1/4)]}(x)) \\
&- H^{(1)}_{[2;\,(0,1/4)]}(x)\left(H^{(3)}_{[1;\,(0,1/4)]}(x) + 4H^{(2)}_{[1;\,(0,1/4)]}(x)\right) \\
&- 2i\left(H^{(3)}_{[2;\,(0,1/4)]}(x)H^{(2)}_{[1;\,(0,1/4)]}(x) - H^{(3)}_{[1;\,(0,1/4)]}(x)H^{(2)}_{[2;\,(0,1/4)]}(x)\right)\Bigg], \\
\mathcal{R}^{(0,1/4)}_2(x) =& \frac{1}{6W^{(0,1/4)}(x)}\Bigg[4H^{(3)}_{[2;\,(0,1/4)]}(x)H^{(2)}_{[1;\,(0,1/4)]}(x) - 4H^{(3)}_{[1;\,(0,1/4)]}(x)H^{(2)}_{[2;\,(0,1/4)]}(x) \\
&+ 3i\left(H^{(1)}_{[2;\,(0,1/4)]}(x)H^{(2)}_{[1;\,(0,1/4)]}(x) - H^{(1)}_{[1;\,(0,1/4)]}(x)H^{(2)}_{[2;\,(0,1/4)]}(x)\right)\Bigg], \\
\mathcal{R}^{(0,1/4)}_3(x) =& \frac{3}{4W^{(0,1/4)}(x)}\Bigg[H^{(3)}_{[1;\,(0,1/4)]}(x)H^{(1)}_{[2;\,(0,1/4)]}(x) - H^{(3)}_{[2;\,(0,1/4)]}H^{(1)}_{[1;\,(0,1/4)]}(x) \\
&+ 2i\left(H^{(3)}_{[2;\,(0,1/4)]}(x)H^{(2)}_{[1;\,(0,1/4)]}(x) - H^{(3)}_{[1;\,(0,1/4)]}(x)H^{(2)}_{[2;\,(0,1/4)]}(x)\right)\Bigg].
\end{aligned} \tag{C.3.9}$$

Finally, the expression of the master integrals in the Euclidean region $x < 0$ can be obtained by matching the homogeneous solutions at infinity, according to eq. (C.2.1),

$$\mathbb{G}^{(1,\infty)}(x) = \mathbb{G}^{(-\infty,0)}(x)\left(\mathbb{M}^{(\infty)}\right)^{-1}, \tag{C.3.10}$$

with the matching matrix $\mathbb{M}^{(\infty)}$ defined in eq.(C.2.2). The Wronskian can be directly continued to negative values of x,

$$W^{(1,\infty)} = \frac{\pi^3 x^3}{64\sqrt{(4x-1)^3(x-1)}} = \frac{\pi^3 x^3}{64\sqrt{(1-4x)^3(1-x)}} = W^{(-\infty,0)}\,, \tag{C.3.11}$$

and by acting again with (C.3.10) and (C.3.11) on eq. (9.5.10), we obtain the expression of the master integrals in region $(-\infty, 0)$,

$$\begin{aligned}
\mathrm{F}^{(0)}_1(x) =\;& H^{(1)}_{[1;\,(-\infty 0)]}(x)\left(d^{(0)}_1 + \int_0^{-1/x} \frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}^{(-\infty,0)}_1(1/y)\right) \\
&+ H^{(2)}_{[1;\,(-\infty 0)]}(x)\int_0^{-1/x} \frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}^{(-\infty,0)}_2(1/y) \\
&+ H^{(3)}_{[1;\,(-\infty 0)]}(x)\int_0^{-1/x} \frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}^{(-\infty,0)}_3(1/y)\,, \\
\mathrm{F}^{(0)}_2(x) =\;& H^{(1)}_{[2;\,(-\infty 0)]}(x)\left(d^{(0)}_1 + \int_0^{-1/x} \frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}^{(-\infty,0)}_1(1/y)\right) +
\end{aligned}$$

$$\begin{aligned}
& H^{(2)}_{[2;(-\infty 0)]}(x)\int_0^{-1/x}\frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}_2^{(-\infty,0)}\,(1/y) \\
& \quad + H^{(2)}_{[2;(-\infty 0)]}(x)\int_0^{-1/x}\frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}_3^{(-\infty,0)}\,(1/y)\ , \\
\mathrm{F}_3^{(0)}(x) = & \quad H^{(2)}_{[3;(-\infty 0)]}(x)\left(d_1^{(0)}+\int_0^{-1/x}\frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}_1^{(-\infty,0)}\,(1/y)\right) \\
& \quad + H^{(2)}_{[3;(-\infty 0)]}(x)\int_0^{-1/x}\frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}_2^{(-\infty,0)}\,(1/y) \\
& \quad + H^{(3)}_{[3;(-\infty 0)]}(x)\int_0^{-1/x}\frac{\mathrm{d}y}{y(1+4y)}\mathcal{R}_3^{(-\infty,0)}\,(1/y)\ , \qquad \text{(C.3.12)}
\end{aligned}$$

where $d_1^{(0)}$ is the integration constant

$$d_1^{(0)} = c_1^{(0)} + \int_0^1 \frac{\mathrm{d}y}{y(y-4)}\mathcal{R}_1^{(1,\infty)}\,(1/y)\ , \qquad \text{(C.3.13)}$$

and $\mathcal{R}_i^{(-\infty,0)}(x)$ are the combinations of homogenous solutions

$$\begin{aligned}
\mathcal{R}_1^{(-\infty,0)}(x) = & \frac{1}{2W^{(-\infty,0)}(x)}\Bigg[H^{(2)}_{[1;(-\infty,0)]}(x)\left(3H^{(1)}_{[2;(-\infty,0)]}(x)+H^{(3)}_{[2;(-\infty,0)]}(x)\right) \\
& - H^{(2)}_{[2;(-\infty,0)]}(x)\left(3H^{(1)}_{[1;(-\infty,0)]}(x)+H^{(3)}_{[1;(-\infty,0)]}(x)\right) \\
& + i\left(H^{(1)}_{[1;(-\infty,0)]}(x)H^{[2;(-\infty,0)]}_{(3)}(x) - H^{[2;(-\infty,0)]}_{(1)}(x)H^{[1;(-\infty,0)]}_{(3)}(x)\right)\Bigg]\ , \\
\mathcal{R}_2^{(-\infty,0)}(x) = & \frac{1}{6W^{(-\infty,0)}(x)}\Bigg[\left(H^{(1)}_{[1;(-\infty,0)]}H^{(3)}_{[2;(-\infty,0)]}(x) - H^{(1)}_{[2;(-\infty,0)]}(x)H^{(3)}_{[1;(-\infty,0)]}(x)\right) \\
& + 6i\left(H^{(1)}_{[1;(-\infty,0)]}(x)H^{(2)}_{[2;(-\infty,0)]}(x) - H^{(1)}_{[12\,(-\infty,0)]}(x)H^{(2)}_{[1;(-\infty,0)]}(x)\right)\Bigg]\ , \\
\mathcal{R}_3^{(1/4,1)}(x) = & \frac{1}{6W^{(-\infty,0)}(x)}\left[H^{(1)}_{[2;(-\infty,0)]}(x)H^{(2)}_{[1;(-\infty,0)]}(x) - H^{(1)}_{[1;(-\infty,0)]}(x)H^{(2)}_{[2;(-\infty,0)]}\right]. \\
& \qquad \text{(C.3.14)}
\end{aligned}$$

We stress here that similar results in the region $(-\infty,0)$ could be obtained by matching the solutions in $x=0$. This point, nevertheless, is more delicate, due to the divergence in the Wronskian, $1/W(x) \sim 1/x^3$ as $x \to 0$, and, hence we preferred to continue passing through $x = \pm\infty$.

Appendix D

Bessel moments and elliptic integrals

In this appendix, we give a brief derivation of eqs. (9.198)-(9.199). We first reproduce the proof of eq. (9.199), which was first presented in [272] (see eq.(33) therein), and then we use similar arguments to derive eq. (9.198).
The evaluation of eq. (9.197) requires the study of the integral

$$\pi \int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t^2+(a+b)^2)(t^2+c^2)}} \mathrm{K}\left(\frac{t^2+(a-b)^2}{t^2+(a+b)^2}\right) \tag{D.0.1}$$

The analytic expression of (D.0.1) can be obtained by first studying the following auxiliary integral

$$I_1(\omega) = \frac{2}{\pi}\int_0^\infty \mathrm{d}t\mathrm{d}z_1\mathrm{d}z_2\, K_0(az_1)K_0(bz_1)K_0\left(cz_2\right)\cos\left(tz_1\right)\cos\left((\omega+t)z_2\right), \tag{D.0.2}$$

where $K_0(x)$ is the modified Bessel function of the second kind,

$$K_0(x) = \int_0^\infty \mathrm{d}t \frac{\cos\left(xt\right)}{\sqrt{t^2+1}}, \tag{D.0.3}$$

which satisfies the identity

$$\int_0^\infty \mathrm{d}t K_0(at)\cos(\omega t) = \frac{2}{\pi}\frac{1}{\sqrt{a^2+\omega^2}}. \tag{D.0.4}$$

Eq. (D.0.4) allows to trivially perform the integration over z_2,

$$I_1(\omega) = \int_0^\infty \mathrm{d}t \frac{1}{\sqrt{(\omega+t)^2+c^2}}\int_0^\infty \mathrm{d}z_1\, K_0(az_1)K_0(bz_1)\cos\left(tz_1\right). \tag{D.0.5}$$

The integral over z_1 is now in standard form (see for instance eq.2.16.36.2 of [273]) and it can be evaluated in terms of an elliptic integral of the first kind,

$$I_1(\omega) = \pi\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t^2+(a+b)^2}\sqrt{(\omega+t)^2+c^2}} \mathrm{K}\left(\frac{t^2+(a-b)^2}{t^2+(a+b)^2}\right), \tag{D.0.6}$$

from which we immediately see that eq. (D.0.4) corresponds to the value of the auxiliary integral (D.0.2) at $\omega=0$. In order to evaluate $I_1(0)$, we go back to eq. (D.0.2) and we start by performing the t integration, for which we can use of the distribution identity

$$\int_0^\infty \mathrm{d}t\cos\left(tz_1\right)\cos\left((\omega+t)z_2\right) = \frac{\pi}{2}\cos(\omega z_1)\,\left(\delta(z_1-z_2)+\cos(\omega z_1)\delta(z_1+z_2)\right). \tag{D.0.7}$$

The term proportional to $\delta(z_1+z_2)$ in the r.h.s. of eq. (D.0.7) has no support in the region where $I_1(\omega)$ is defined, therefore we have

$$I_1(\omega) = \int_0^\infty \mathrm{d}z_1\, K_0(az_1)K_0(bz_1)K_0\left(cz_2\right)\cos\left(\omega z_1\right), \tag{D.0.8}$$

which, if we set $\omega=0$, reduces to

$$I_1(0) = \int_0^\infty \mathrm{d}z_1\, K_0(az_1)K_0(bz_1)K_0\left(cz_2\right). \tag{D.0.9}$$

This last integral is connected to the master formula (see eq(3.3) of [274])

$$\int_0^\infty \mathrm{d}t I_\mu(at)K_0(bt)K_0(ct) = \frac{1}{4c}W_\mu(k_+)W_\mu(k_-), \tag{D.0.10}$$

where $I_\mu(z)$ is the modified Bessel function of the first kind, the function $W_\mu(k)$ is related to associated Legendre polynomial $P^{\mu/2}_{-1/2}$,

$$W_\mu(k) = \frac{\sqrt{\pi}\Gamma\left(\frac{1+\mu}{2}\right)}{(1-k^2)^{1/4}}P^{\mu/2}_{-1/2}\left(\frac{2k^2}{2\sqrt{1-k^2}}\right), \tag{D.0.11}$$

and the arguments $k_\pm$ are defined by

$$k_\pm = \frac{\sqrt{(c+a)^2-b^2}\pm\sqrt{(c-a)^2-b^2}}{2c}. \tag{D.0.12}$$

The expansion of eq.(D.0.10) around $\mu=0$ allow us to express a set of integrals of three Bessel functions as a product of two complete elliptic integrals. In fact, by making use of

$$\begin{aligned} I_\mu(x) =& I_0(x) - \mu K_0(x) + \mathcal{O}(\mu^2),\\ W_\mu(k) =& 2\mathrm{K}(k^2) - \mu\pi\mathrm{K}(1-k^2) + \mathcal{O}(\mu^2), \end{aligned} \tag{D.0.13}$$

one can easily check that that eq. (D.0.10) implies

$$\int_0^\infty \mathrm{d}t I_0(at)K_0(bt)K_0(ct) = \frac{1}{c}\mathrm{K}(k_-^2)\mathrm{K}(k_+^2), \tag{D.0.14}$$

$$\int_0^\infty \mathrm{d}t K_0(at)K_0(bt)K_0(ct) = \frac{\pi}{2c}\left(\mathrm{K}(k_-^2)\mathrm{K}(1-k_+^2)+\mathrm{K}(k_+^2)\mathrm{K}(1-k_-^2)\right). \tag{D.0.15}$$

Thanks to eq. (D.0.15), we can finally evaluate $I_1(0)$,

$$I_1(0) = \frac{\pi}{2c}\left(\mathrm{K}(k_-^2)\mathrm{K}(1-k_+^2)+\mathrm{K}(k_+^2)\mathrm{K}(1-k_-^2)\right), \tag{D.0.16}$$

which proves eq. (9.199).

The proof of eq. (9.199), which requires the evaluation of the integral

$$\int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t^2+(a+b)^2)(t^2+c^2)}}\mathrm{K}\left(\frac{2ab}{t^2+(a+b)^2}\right), \tag{D.0.17}$$

proceeds along the same lines. We start from the auxiliary integral

$$I_2(\omega) = \frac{2}{\pi}\int_0^\infty \mathrm{d}t\mathrm{d}z_1\mathrm{d}z_2\, I_0(az_1)K_0(bz_1)K_0\left(cz_2\right)\cos\left(tz_1\right)\cos\left((\omega+t)z_2\right), \tag{D.0.18}$$

which, by using in order (D.0.4), becomes

$$\mathcal{I}_2(\omega) = \int_0^\infty \mathrm{d}t \frac{1}{\sqrt{(\omega+t)^2+c^2}} \int_0^\infty \mathrm{d}z_1\, I_0(az_1)K_0(bz_1)\cos(tz_1). \tag{D.0.19}$$

As in the previous case, the integral over z_1 can be evaluated in terms of an elliptic integral of the first kind (see for instance eq.2.16.36.2 of [273]),

$$I_2(\omega) = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{t^2+(a+b)^2}\sqrt{(\omega+t)^2+c^2}} \mathrm{K}\left(\frac{t^2+(a-b)^2}{t^2+(a+b)^2}\right), \tag{D.0.20}$$

from which we see that eq. (D.0.17) corresponds to $I_2(0)$. Therefore, in order to determine the value of the auxiliary integral at zero, we first make use of (D.0.7) in eq. (D.0.18) in order to integrate over t

$$I_2(\omega) = \int_0^\infty \mathrm{d}z_1\, I_0(az_1)K_0(bz_1)K_0(cz_2)\cos(\omega z_1). \tag{D.0.21}$$

and then, after setting $\omega = 0$,

$$I_2(0) = \int_0^\infty \mathrm{d}z_1\, I_0(az_1)K_0(bz_1)K_0(cz_2)\ , \tag{D.0.22}$$

we make use eq. (D.0.14) and obtain

$$I_2(0) = \frac{1}{c}\mathrm{K}(k_-^2)\mathrm{K}(k_+^2)\,, \tag{D.0.23}$$

which proves eq. (9.198).

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