Rend. Lincei Mat. Appl. 30 (2019), 195–204
DOI 10.4171/RLM/842

Mathematical Physics — *Integrability of the spatial restricted three-body problem near collisions* (*an announcement*), by FRANCO CARDIN and MASSIMILIANO GUZZO, communicated on November 9, 2018.[1]

ABSTRACT. — We present the integration of the spatial circular restricted three-body problem in a neighbourhood of its collision singularities by extending an idea of Tullio Levi-Civita.

KEY WORDS: Three-body problems, collisions in celestial mechanics, regularization, Hamilton–Jacobi equations

MATHEMATICS SUBJECT CLASSIFICATION: 70F07, 70F16, 70H20

The circular restricted three-body problem is defined by the motion of a body P of infinitesimally small mass in the gravitation field of two massive bodies P_1 and P_2, the primary and secondary body respectively, which rotate uniformly around their common center of mass. In a rotating frame we consider the Hamiltonian:

$$h(x, y, z, p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2} + p_x y - p_y x - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{1}$$

where $r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$ and $r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$ denote the distances of P from P_1, P_2; notice that the units of mass, length and time have been chosen so that the masses of P_1 and P_2 are $1-\mu$ and μ ($\mu \leq 1/2$) respectively, their coordinates are $(x_1, 0, 0) = (-\mu, 0, 0)$, $(x_2, 0, 0) = (1-\mu, 0, 0)$, and their revolution period is 2π.

For $\mu > 0$, no smooth constants of motion independent of the Hamilton function h are known, and this represents the major obstruction to the lack of explicit uniform representations of solutions of the problem. There is a long history around the existence/non-existence of first integrals for the three-body problem as well as for general Hamiltonian systems. Theorems of non-existence of such constant of motions are due to Bruns [2] (whose result concerns algebraic first integrals) and Poincaré [13], revisited in [14, 1, 8]. Actually, whenever we discuss about the theorem of non-existence of Poincaré for the restricted three-body problem, we are speaking precisely on uniform first integrals analytic with respect to the mass parameter μ in domains which, when represented using the Delaunay

[1] The purpose of this paper is to announce and present results which are to appear (see reference [4] in the paper).

variables (L, G, l, g) (for the planar problem), have the form $D \times \mathbb{T}^2$ where $D \subseteq \mathbb{R}^2$ is any open subset of the actions L, G with $L > 0$, [1]. The theorem of Poincaré leaves the door open for the integration of the system in domains which are not invariant under translations of the angles (l, g). The interest in these kind of integrations depends on the specific domain. For example, when the domain is a neighbourhood of the collision set:

$$\mathscr{C}_j = \{(x, y, z, p_x, p_y, p_z) : (x, y, z) = (x_j, 0, 0)\}, \quad j = 1, 2,$$

even restricted to constant energy levels, the integration would allow to solve the (open) problem of close encounters, which we formulate as follows. Let σ be arbitrarily small; for any motion $(x(t), y(t), z(t))$ entering the ball $B_{(x_j,0,0)}(\sigma) \subseteq \mathbb{R}^3$ (centered at $(x_j, 0, 0)$ of radius σ) at time $t = t_1$ and leaving it at time t_2, express $(x(t_2), y(t_2), z(t_2), p_x(t_2), p_y(t_2), p_z(t_2))$ as an explicit function of $(x(t_1), y(t_1), z(t_1), p_x(t_1), p_y(t_1), p_z(t_1))$.

In a remarkable paper [11] T. Levi-Civita performed the integration of the *planar* circular restricted three-body problem in a neighbourhood of a collision set $\mathscr{C}_j$ through the introduction of a transformation which nowadays bears the name of Levi-Civita (LC hereafter) regularization; explicitly:

$$x = x_j + u_1^2 - u_2^2 \tag{2}$$
$$y = 2u_1u_2 \tag{3}$$
$$dt = r_j\, ds, \tag{4}$$

where (2), (3) are equivalent to the complex transformation: $\xi = \zeta^2$, $\xi = (x - x_j) + iy$, $\zeta = u_1 + iu_2$, while (4) is a parametrization of the physical time t into the proper time s. In a much less quoted part of the paper [11] Levi-Civita proved the existence of an integral of the Hamilton–Jacobi equation of the Hamiltonian representing the regularized planar circular restricted three-body problem, which we call the Levi-Civita Hamiltonian, in a neighbourhood of the collision singularity at P_j. In particular, he proved the existence of a second first integral, independent of h, defined in a neighbourhood of the the collision singularity at P_j and represented by a series analytic at $(u_1, u_2) = (0, 0)$. The coefficients of this series can be explicitly computed iteratively up to any arbitrary large order, so that the problem of planar close encounters can be solved explicitly[2].

The regularization of the spatial restricted three-body problem has been done by Kustaanheimo and Stiefel [9, 10] many decades after Levi-Civita, but the integrability of the regularized Hamiltonian, which we call the Kustaanheimo–Stiefel

[2] As a matter of fact, Levi-Civita constructed the solution of the Hamilton–Jacobi equation only for the collision singularity at P_1. Nevertheless, Levi-Civita's argument is valid also in a neighbourhood of the singularity at the secondary body P_2. Formally this extension is achieved by exchanging $1 - \mu$ with μ within the series. We notice that while the series at P_1 is analytic also in $\mu = 0$, the series at P_2 is not.

Hamiltonian, has never been addressed. Here, our purpose is precisely to extend to the fully spatial case the point of view followed by Levi-Civita, thus offering a complete integrability of the spatial problem near collisions.

Regularizations of spatial problems are dramatically more complicate than regularizations of the planar problem, see [12]. As for the Levi-Civita regularization, the Kustaanheimo–Stiefel regularization (KS hereafter) is defined by the introduction of a transformation on the space variables and by a time-reparametrization; but the KS space transformation is more complicate than the LC space transformation, since for an algebraic reason that we explain below (related to the extension of complex numbers to a space of quaternions) it is a map from a space of redundant variables u_1, u_2, u_3, u_4 to a space of Cartesian variables q_1, q_2, q_3. Following [9, 10], we introduce the projection map:

$$\begin{aligned} \pi : \mathbb{R}^4 &\to \mathbb{R}^3 \\ (u_1, u_2, u_3, u_4) &\mapsto \pi(u_1, u_2, u_3, u_4) = (q_1, q_2, q_3), \end{aligned} \tag{5}$$

where $(q_1, q_2, q_3, 0) = A(u)u$, and:

$$A(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \tag{6}$$

is a matrix which plays a central role in the KS regularization, it is a linear homogeneous function of $u_1, \ldots, u_4$ and satisfies $A(u)A^T(u) = |u|^2\mathscr{I}$. Matrices with such properties exist only for $n = 1, 2, 4, 8$ (see [7]), and the lack of this result for $n = 3$ is precisely the reason for the definition of the KS regularization in a 4-dimensional space. Then, for any motion in the KS variables we introduce the parametrization of time (4); notice that we have $r_j = |u|^2$. The space and time transformations (4), (5) have been used to represent the regularized equations of motions of the spatial circular restricted three-body problem in various forms (see [3] for a review of the subject). Below, we present it with the notations which we find useful to investigate the transformation of the KS Hamiltonian with respect to a suitable sub-group of SO(4).

The KS Hamiltonian

We first perform the phase-space translation

$$X = x - x_j, \quad Y = y, \quad Z = z, \quad P_x = p_x, \quad P_y = p_y - x_j, \quad P_z = p_z, \tag{7}$$

conjugating h to the Hamiltonian (to fix ideas we present all these computations for $j = 2$, so that the reference system defined above will be called planetocentric):

$$
\begin{aligned}
(8)\quad & H(X,Y,Z,P_x,P_y,P_z) = \\
&= \frac{P_x^2+P_y^2+P_z^2}{2} + P_xY - P_yX - \frac{\mu}{\sqrt{X^2+Y^2+Z^2}} \\
&- (1-\mu)\left(\frac{1}{\sqrt{(X+1)^2+Y^2+Z^2}} - 1 + X\right) - (1-\mu) - \frac{(1-\mu)^2}{2},
\end{aligned}
$$

the constant terms being kept for compliance with the values of the original Hamiltonian h; we will use also the compact notation $\xi = (X,Y,Z,P_x,P_y,P_z)$. The traditional KS regularization is obtained from the space transformation (5) with $(q_1,q_2,q_3) = (X,Y,Z)$, and can be expressed in the following Hamiltonian form:

$$
\begin{aligned}
(9)\quad \mathscr{K}(u,U;E) &= \frac{1}{8}|U - b_{(0,0,1)}(u)|^2 - \frac{1}{2}|u|^2|(0,0,1)\times\pi(u)|^2 - |u|^2E_\mu - \mu \\
&- (1-\mu)|u|^2\left(\frac{1}{|\pi(u)+(1,0,0)|} - 1 + \pi(u)\cdot(1,0,0)\right), \\
E_\mu &= E + (1-\mu) + \frac{(1-\mu)^2}{2},
\end{aligned}
$$

where $U = (U_1,U_2,U_3,U_4)$ denote the conjugate momenta to $u = (u_1,u_2,u_3,u_4)$, and the vector potential $b_\omega(u)$ (in (9) we have $\omega = (0,0,1)$), is defined by

$$
(10)\qquad b_\omega(u) = 2A^T(u)\Lambda_\omega A(u)u, \quad \Lambda_\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 & 0 \\ \omega_3 & 0 & -\omega_1 & 0 \\ -\omega_2 & \omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

The Hamiltonian $\mathscr{K}$ is a regularization of the spatial three-body problem at P_2. This means that if $(u(s),U(s))$ is a solution of the Hamilton equations of $\mathscr{K}(U,u;E)$ with initial conditions satisfying:

(i) $u(0) \neq 0$;
(ii) $l(u(0),U(0)) = 0$, where $l(u,\dot u) = u_4\dot u_1 - u_3\dot u_2 + u_2\dot u_3 - u_1\dot u_4$ is called the *bilinear form*;
(iii) $\mathscr{K}(u(0),U(0);E) = 0$,

and s in a small neighbourhood of $s=0$, then $\xi(t) = (X(t),Y(t),Z(t),P_x(t),P_y(t),P_z(t))$ such that:

$$
(X(t(s)),Y(t(s)),Z(t(s)),0) = A(u(s))u(s)
$$

$$(P_x(t(s)), P_y(t(s)), P_z(t(s)), 0) = \frac{1}{2|u(t(s))|^2} A(u(t(s)) U(t(s)))$$
$$t(s) = \int_0^s |u(\tau)|^2 \, d\tau,$$

is a solution of the Hamilton equations of (8), and $H(\xi(0)) = E$.

Statement of the main result

Our integration of the spatial circular restricted three-body problem is based on the definition of a complete integral $W(u, v; E, \mu)$ of the Hamilton–Jacobi equation of Hamiltonian $\mathscr{K}(u, U; E)$ which is defined for all the values of the parameters $v = (v_1, \ldots, v_4)$ in a neighbourhood of the sphere $|v| = 1$, and is analytic in a neighbourhood of $u = 0$. At this regard, we can prove (the proof will appear elsewhere [4]):

Theorem 1. *For fixed values of E_* and of $\mu_* > 0$, there exists a complete integral $W(u, v; E, \mu)$ of the Hamilton–Jacobi equation:*

$$\mathscr{K}\left(u, \frac{\partial W}{\partial u}(u, v; E, \mu); E\right) = \mu(|v|^2 - 1) \tag{11}$$

depending on the four parameters v and on E, μ, which is analytic for E, μ, v in the set:

$$\{|\mu - \mu_*| < a, \ |E - E_*| < b, \ ||v| - 1| < c\}$$

and u in the (complex) ball:

$$\{u \in \mathbb{C}^4 : |v| < d\}$$

with suitable constants $a, b, c, d > 0$ (depending only on E_*, μ_*). The coefficients of the Taylor expansions of W with respect to the variables u can be explicitly computed iteratively to any arbitrary order; in particular we have:

$$W = \sqrt{8\mu} \sum_{j=1}^{4} v_j u_j + \mathcal{O}_3(u). \tag{12}$$

Remarks. (I) The complete integral W of the Hamilton–Jacobi equation defines a canonical transformation through the system

$$U_\ell = \frac{\partial W}{\partial u_\ell}(u, v; E, \mu), \quad \ell = 1, \ldots, 4 \tag{13}$$

$$n_\ell = \frac{\partial W}{\partial v_\ell}(u, v; E, \mu), \quad \ell = 1, \ldots, 4. \tag{14}$$

conjugating $\mathscr{K}(u, U; E)$ to the Hamiltonian:

$$\hat{\mathscr{K}}(n, v) = \mu(|v|^2 - 1).$$

We can prove the following technical points: (i) the sub-system (13) has a global analytic inversion: $v = \hat{v}(u, U; E, \mu)$ defined for u, U so that u belongs to some complex ball around $u = 0$ and U belongs to the image of the map: $v \mapsto \frac{\partial W}{\partial u}(u, v; E, \mu)$ with v in a suitable neighbourhood of the unit sphere $|v| = 1$; (ii) the canonical transformation preserves the bilinear relation, i.e. we have $l(u, U) = 0$ if and only if $l(\hat{n}, \hat{v}) = 0$. Therefore, by denoting with $(n, v) = (\hat{n}(u, U; E, \mu), \hat{v}(u, U; E, \mu))$ the canonical transformation, the solutions $(u(s), U(s))$ of the Hamilton equations of $\mathscr{K}(u, U; E)$ are obtained from:

$$(n(0) + 2\mu v(0)s, v(0)) = (\hat{n}(u(s), U(s); E), \hat{v}(u(s), U(s); E)). \tag{15}$$

Formula (15) provides all the solutions of the spatial circular restricted three-body problem in a neighbourhood of the collision set $\mathscr{C}_2$.

(II) The proof of Theorem 1 is achieved through several steps: first, a geometric analysis of the KS Hamiltonian is needed to identify the parameters $v_1, \ldots, v_4$, providing the conserved momenta of Hamiltonian $\hat{\mathscr{K}}(n, v)$; second, an analytic part based on the Cauchy–Kowaleski theorem is used to provide analytic solutions to the Hamilton–Jacobi equation. The geometric analysis is the real heart of the proof and is completely original with respect to the work of Levi-Civita, since the geometric part required by the planar case is rather simpler. In fact, we need to represent in the space of the fictitious variables $(u_1, \ldots, u_4)$ the rotations of the euclidean space (q_1, q_2, q_3) with matrices which are in SO(4) and leave invariant the bilinear form. Moreover, to subgroup of SO(4) that we obtain this way must be parameterized by four parameters $v_1, \ldots, v_4$, constrained to the unit sphere, such that the inversion of the system of equations (13, 14) has no singularities (which arise if, for example, we parameterize the subgroup with three Euler angles). The analytic part is instead the argument that we extend from the integration of the planar problem, with an additional care for the global definition of the family of particular solutions found.

(III) An additional interesting question concerns the existence of Cartesian first integrals $F(X, Y, Z, P_x, P_y, P_z)$ independent of $H(X, Y, Z, P_x, P_y, P_z)$ defined in a neighbourhood of the collision set $\mathscr{C}_j$. The existence of Cartesian first integrals is not granted a priori from the existence of first integrals of the KS Hamiltonian; for example $|v|^2$ and $l(n, v)$ do not provide, with evidence, Cartesian first integrals. A deeper reason is that the map π has not a global smooth inversion defined in a neighbourhood of $q = (X, Y, Z) = 0$ (see [6], where a similar problem is addressed for the global definition of chaos indicators for the spatial three-body problem). We need at least two local inversions $\hat{u}_\pm(q)$ of π to cover a full neighbourhood of $q = 0$, and consequently two local inversions $(\hat{u}_\pm(q), \ \hat{U}_\pm(q, p))$, where $p = (P_x, P_y, P_z)$, satisfying $l(\hat{u}_\pm, \hat{U}_\pm) = 0$ covering a neighbourhood of the collision set $\mathscr{C}_j$. For any $\xi = (q, p)$ in the common domain of the local inversions, we prove that there exists an angle α (depending on q)

such that:

$$u_+ = \mathscr{S}_\alpha^0 u_-, \quad U_+ = \mathscr{S}_\alpha^0 U_-, \quad v_+ = \mathscr{S}_\alpha^0 v_-, \quad n_+ = \mathscr{S}_\alpha^0 n_- \hat{v},$$

where $u_\pm := \hat{u}_\pm(q)$, $U_\pm := \hat{U}_\pm(q,q)$, $v_\pm := \hat{v}(u_\pm, U_\pm)$, $n_\pm := \hat{n}(u_\pm, U_\pm)$, and $\mathscr{S}_\alpha^0$ is the $SO(4)$ matrix:

$$\mathscr{S}_\alpha^0 = \begin{pmatrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ \sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix}. \tag{16}$$

Therefore, the first integrals $\hat{v}_\pm(\hat{u}(q), \hat{U}(q,p); E)$ depend in principle on the local inversion map, and we do not grant their extension to global smooth functions in any neighbourhood of the collision set. But, if we consider the dynamics in the n, v variables, we notice that the functions:

$$N_x = v_1 n_4 - v_4 n_1, \quad N_y = \frac{1}{2}(v_1 n_3 - n_1 v_3 + n_2 v_4 - n_4 v_2),$$

$$N_z = \frac{1}{2}(v_1 n_2 - n_1 v_2 + n_4 v_3 - n_3 v_4)$$

are first integrals and are invariant by application of the map $(n, v) \mapsto (\mathscr{S}_\alpha^0 n, \mathscr{S}_\alpha^0 v)$. Their local representatives: $\mathscr{N}_x^\pm(\xi)$, $\mathscr{N}_y^\pm(\xi)$, $\mathscr{N}_z^\pm(\xi)$ satisfy, in their common domain:

$$\mathscr{N}_x^+ = \mathscr{N}_x^-, \quad \mathscr{N}_y^+ = \mathscr{N}_y^-, \quad \mathscr{N}_z^+ = \mathscr{N}_z^-,$$

and therefore are the local representatives of functions $\mathscr{N}_x$, $\mathscr{N}_y$, $\mathscr{N}_z$ globally defined and smooth in a neighbourhood of the collision set. We consider the set of three first integrals:

$$(H,\ \mathscr{N}^2 := \mathscr{N}_x^2 + \mathscr{N}_y^2 + \mathscr{N}_z^2,\ \mathscr{N}_z).$$

We notice that, since $\mathscr{N}^2$, $\mathscr{N}_z$ are first integrals, we have:

$$\{H, \mathscr{N}^2\} = 0, \quad \{H, \mathscr{N}_z\} = 0.$$

The Poisson bracket $\{H, \mathscr{N}_z\} = 0$ is sufficient to grant the complete integrability of the planar circular restricted three-body problem in a neighbourhood of its collision singularities. It remains to understand if even the spatial case is completely integrable. At this regard, we notice that in the space of the variables n, v, we have:

$$\{N^2, N_z\} = l(n,v)a(n,v), \tag{17}$$

so that the two integrals are in involution on the level set $l(n, v) = 0$. The atypical Poisson bracket in (17) seems a rule for the KS regularization. For example, the elementary Poisson brackets of $q = \hat{q}(u)$; $p = \hat{p}(u, U)$ defined from $\hat{q}(u) = \pi(u)$, $(\hat{p}_1, \hat{p}_2, \hat{p}_3, 0) = \frac{1}{2|u|^2} A(u)U$, satisfy:

$$(18) \quad \{\hat{q}_i, \hat{p}_j\} = \delta_{ij}, \quad \{\hat{q}_i, \hat{q}_j\} = 0, \quad \{\hat{p}_i, \hat{p}_j\} = l(u, U)\phi_{ij}(u, U), \quad i, j = 1, 2, 3.$$

From (17) and (18) we get:

$$\{\mathcal{N}^2, \mathcal{N}_z\} = 0.$$

The existence of a complete set of Cartesian first integrals defined in a neighbourhood of the singularity offers a classification of the close encounters which uses the Cartesian variables.

On the proof of Theorem 1

Consider the reference frame defined by

$$(19) \qquad (x - x_j, y, z) = \lambda \mathcal{R}(q_1, q_2, q_3),$$

for any arbitrary matrix $\mathcal{R} \in SO(3)$ and $\lambda > 0$, and introduce the KS space and time regularization (5), (4). We obtain the regularization represented by the Hamiltonian:

$$(20) \quad \mathcal{K}_{\lambda\mathcal{R}}(u, U) = \frac{1}{8\lambda^2}|U - \lambda^2 b_\omega(u)|^2 - \frac{1}{2}\lambda^2|u|^2|\omega \times \pi(u)|^2 - \mu\lambda^{-1} - |u|^2 E_\mu$$
$$- (1 - \mu)|u|^2\Big(\frac{1}{|\lambda\pi(u) + e|} - 1 + \lambda\pi(u) \cdot e\Big).$$

where $\omega = \mathcal{R}^T(0, 0, 1)$, $e = \mathcal{R}^T(1, 0, 0)$.

The Hamiltonians $\mathcal{K}_{\lambda\mathcal{R}}(u, U; E)$ are conjugate to $\mathcal{K}(u, U; E)$ by a peculiar set of linear transformations of $\mathbb{R}^4$ parameterized by four parameters v, which are conjugate (through the map π) to the transformations (19). Precisely, for any $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4 \backslash 0$ we define the matrices:

$$\mathcal{S}_v = \begin{pmatrix} v_1 & -v_2 & -v_3 & -v_4 \\ v_2 & v_1 & -v_4 & v_3 \\ v_3 & v_4 & v_1 & -v_2 \\ v_4 & -v_3 & v_2 & v_1 \end{pmatrix}.$$

We have: $\mathcal{S}_v\mathcal{S}_v^T = |v|^2\mathcal{I}$, $l(\mathcal{S}_v u, \mathcal{S}_v U) = |v|^2 l(u, U)$ for all $(u, U) \in T^*\mathbb{R}^4$ and, for any $u \in \mathbb{R}^4$, we have:

$$(21) \qquad \pi(\mathcal{S}_v u) = \mathcal{R}_v \pi(u)$$

where:

$$\mathscr{R}_v = \begin{pmatrix} v_1^2 - v_2^2 - v_3^2 + v_4^2 & -2(v_1v_2 + v_3v_4) & -2(v_1v_3 - v_2v_4) \\ 2(v_1v_2 - v_3v_4) & v_1^2 - v_2^2 + v_3^2 - v_4^2 & -2(v_2v_3 + v_1v_4) \\ 2(v_1v_3 + v_2v_4) & -2(v_2v_3 - v_1v_4) & v_1^2 + v_2^2 - v_3^2 - v_4^2 \end{pmatrix} \tag{22}$$

satisfies: $\mathscr{R}_v\mathscr{R}_v^T = |v|^4\mathscr{I}$, and depends on the v as in the Euler–Rodrigues formula. Therefore, the map:

$$\Pi : \mathscr{S} = \bigcup_{v\in\mathbb{R}^4\setminus 0} S_v \to SO(3)$$
$$S_v \mapsto \Pi(S_v) = \frac{1}{|v|^2}\mathscr{R}_v,$$

is surjective. For any matrix $S_v \in \mathscr{S}$, we have the identity:

$$\mathscr{K}(S_v u, S_v^{-T}U) = |v|^2\mathscr{K}_{|v|^2\Pi(S_v)}(u, U). \tag{23}$$

which is crucial to relate the particular solutions $\tilde{W}(u; E, \mu, \kappa, v_1, \dots, v_4)$ of the Hamilton–Jacobi equation:

$$\mathscr{K}_{|v|^2\Pi(S_v)}\left(u, \frac{\partial\tilde{W}}{\partial u}\right) = \frac{\kappa}{|v|^2}, \tag{24}$$

to the solutions W of the Hamilton–Jacobi equation (11) through the formula:

$$W(u; E, \mu, v) = \tilde{W}(|v|^{-2}S_v^T u; E, \mu, \kappa_v, v), \tag{25}$$

with $\kappa_v = \mu(|v|^2 - 1)$. For fixed values E_* and $\mu > 0$, the Cauchy–Kowaleski theorem grants the existence of solutions of equation (24) satisfying:

$$\tilde{W}(0, u_2, u_3, u_4; E, \mu, \kappa, v_1, \dots, v_4) = 0$$

which are analytic for all the values of the parameters $(E, \kappa, v_1, \dots, v_4)$ in a suitable neighbourhood of $(E, \kappa) = (E_*, 0)$ and of the unit sphere $|v| = 1$, in the same common neighbourhood of $u = 0$. Finally, the function defined by (25) is the complete integral of the Hamilton–Jacobi equation (11).

ACKNOWLEDGMENTS. This reasearch has been supported by ERC project 677793 Stable and Chaotic Motions in the Planetary Problem.

REFERENCES

[1] V. I. ARNOLD - V. V. KOZLOV - A. I. NEISHTADT, *Mathematical aspects of classical and celestial mechanics, volume* 3 *of Encyclopaedia of Mathematical Sciences*, Springer Berlin, third edition, 2006. [Dynamical systems. III], Translated from the Russian original by E. Khukhro.

[2] H. Bruns, *Uber die Integrale des Vielkörper-problems*, Acta Mathematica, volume 11 (1887).

[3] A. Celletti, *The Levi-Civita, KS and Radial-Inversion Regularizing Transformations,* in *Singularities in Gravitational Systems*, Lecture Notes in Physics 590, D. Benest and Cl. Froeschlé (eds), Springer, pp. 25–48, 2002.

[4] F. Cardin - M. Guzzo, *Integrability of the spatial circular restricted three-body problem in a neighbourhood of the collision singularities*, Preprint 2018.

[5] R. Courant - D. Hilbert, *Methods of Mathematical Physics*, Vol. II, Wiley-VCH, 1962.

[6] M. Guzzo - E. Lega, *Geometric chaos indicators and computations of the spherical hypertube manifolds of the spatial circular restricted three-body problem*, Physica D, vol. 373, 35–58, 2018.

[7] A. Hurwitz, Math. Werke II, 565, 1933.

[8] E. Julliard Tosel, *Bruns theorem, the proof and some generalizations*, Celestial Mechanics and Dynamical Astronomy, vol. 76, 241–281, 2000.

[9] P. Kustaanheimo, *Spinor regularisation of the Kepler motion*, Annales Universitatis Turkuensis A 73, 1–7. Also Publications of the Astronomical Observatory Helsinki 102, 1964.

[10] P. Kustaanheimo - E. L. Stiefel, *Perturbation theory of Kepler motion based on spinor regularization*, Journal für die Reine und Angewandte Mathematik 218, 204–219, 1965.

[11] T. Levi-Civita, *Sur la régularisation qualitative du probléme restreint des trois corps*, Acta Math., vol. 30, 305–327, 1906.

[12] J. Moser, *Regularization of Kepler's problem and the averaging method of a manifold*, Comm. on Pure and Appl. Math., vol. 23, 609–636, 1970.

[13] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, vol. 1–3. Gauthier-Villars, Paris, 1892, 1893, 1899.

[14] C. L. Siegel - J. K. Moser, *Lectures on Celestial Mechanics*, reprint of the 1971 edition, Springer Berlin, Heidelberg, New York, 1971.

Received 26 September 2018,
and in revised form 19 October 2018.

Franco Cardin
Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
Via Trieste 63
35121 Padova, Italy
cardin@math.unipd.it

Massimiliano Guzzo
Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
Via Trieste 63
35121 Padova, Italy
guzzo@math.unipd.it