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THE MUMFORD-TATE GROUP OF 1-MOTIVES

by Cristiana BERTOLIN

Introduction.

The Mumford-Tate group of a 1-motive M defined over $\mathbb{C}$, $MT(M)$, is an algebraic $\mathbb{Q}$-group acting on the Hodge realization of M and endowed with an increasing filtration $W_\bullet$. In this paper we study the structure and the degeneracies of this group.

After recalling some definitions, in Section 1 we investigate the structure of $MT(M)$: the main result of this section is the structural lemma (1.4), where we prove that the unipotent radical of $MT(M)$, which is $W_{-1}(MT(M))$, is an extension by $W_{-2}(MT(M))$ of the unipotent radical of the Mumford-Tate group of a 1-motive without toric part, and that $W_{-1}(MT(M))$ injects into a "generalized" Heisenberg group. As a corollary, we can compute the dimension of $MT(M)$ (corollary on dimensions 1.5). We then explain how to reduce to the study of the Mumford-Tate group of a direct sum of 1-motives whose torus's character group and whose lattice are both of rank 1 (Theorem 1.7).

In Section 2, we classify the degeneracies of $MT(M)$, i.e., those phenomena which imply the decrease of the dimension of $MT(M)$. The 1-motive M is *deficient* if $W_{-2}(MT(M)) = 0$. This degeneracy was discovered by K. Ribet and O. Jacquinot (cf. [JR87]). The 1-motive M is *quasi-deficient* if $W_{-1}(MT(M))$ is abelian and it is *depressive* if the dimension

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of $W_{-2}(MT(M))$ is not maximal. The last two degeneracies are new. Then there are the "trivial degeneracies": *the trivial deficient* and *the trivial quasi-deficient*. We call them trivial because they are induced by trivial phenomena: for example, they appear when the 1-motive is without toric part, or without abelian part, or when its underlying extension is split, ... At the end of this section we give several examples.

Using the structural lemma, in Section 3 we give a geometrical interpretation of deficience and quasi-deficience and we show that these two degeneracies are trivial; more precisely a deficient 1-motive is trivially deficient and a quasi-deficient 1-motive is trivially quasi-deficient (Theorems 3.5 and 3.7).

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1. The structure of the Mumford-Tate group.

1.1. *A 1-motive M over* $\mathbb{C}$ consists of

(a) a finitely generated free $\mathbb{Z}$-module X,

(b) an extension G, defined over $\mathbb{C}$, of an abelian variety A by a torus T,

(c) a homomorphism $u : X \longrightarrow G(\mathbb{C})$.

The 1-motive $M = (X, A, T, G, u)$ can be view also as a complex of commutative group schemes concentrated in degree 0 and 1: $M = [X \xrightarrow{u} G]$. A *morphism of 1-motives* is a morphism of complexes of commutative group schemes. *An isogeny* between two 1-motives $M_1 = [X_1 \xrightarrow{u_1} G_1]$ and $M_2 = [X_2 \xrightarrow{u_2} G_2]$ is a morphism of 1-motives such that $f_X : X_1 \longrightarrow X_2$ is injective with finite cokernel, and $f_G : G_1 \longrightarrow G_2$ is surjective with finite kernel.

We can define an increasing filtration $W_\bullet$ on $M = [X \xrightarrow{u} G]$ in the following way: $W_0(M) = M, W_{-1}(M) = [0 \longrightarrow G], W_{-2}(M) = [0 \longrightarrow T]$. If we denote $\mathrm{Gr}_n^W = W_n/W_{n-1}$, we have $\mathrm{Gr}_0^W(M) = [X \longrightarrow 0], \mathrm{Gr}_{-1}^W(M) = [0 \longrightarrow A]$ and $\mathrm{Gr}_{-2}^W(M) = [0 \longrightarrow T]$.

The Cartier dual of M is the 1-motive $M^\vee = (X^\vee, A^\vee, T^\vee, G^\vee, u^\vee)$, where $X^\vee = \mathrm{Hom}(T, \mathbb{G}_m)$, $A^\vee$ is the dual abelian variety of A, $T^\vee$ is the torus with character group X, $G^\vee = \mathrm{Ext}^1(M/W_{-2}(M), \mathbb{G}_m)$ and $u^\vee$ comes from the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(M, \mathbb{G}_m) \longrightarrow X^\vee \xrightarrow{u^\vee} G^\vee \longrightarrow \mathrm{Ext}^1(M, \mathbb{G}_m) \longrightarrow \cdots$$

associated to the short exact sequence

$$0 \longrightarrow T \longrightarrow M \longrightarrow M/W_{-2}(M) \longrightarrow 0.$$

The notion of biextension allows for a more symmetric description of 1-motives: Consider the 7-uplet $(X, X^\vee, A, A^\vee, v, v^\vee, \psi)$ where X and $X^\vee$ are two finitely generated free $\mathbb{Z}$-modules, A and $A^\vee$ are two abelian varieties dual to each other, $v : X \longrightarrow A$ and $v^\vee : X^\vee \longrightarrow A^\vee$ are two homomorphisms, and ψ is a trivialization of the pull-back by $(v, v^\vee)$ of the Poincaré biextension $\mathcal{P}$ of $(A, A^\vee)$. From this 7-uplet $(A, A^\vee, X, X^\vee, v, v^\vee, \psi)$ it is easy to reconstruct the 1-motive $M = (X, A, T, G, u)$ and its Cartier dual $M^\vee = (X^\vee, A^\vee, T^\vee, G^\vee, u^\vee)$ (cf. [D75] (10.2.12)).

1.2. To each 1-motive $M = (X, A, T, G, u)$ defined over $\mathbb{C}$, one can attach a mixed Hodge structure: Let $T_\mathbb{Z}(M) = \mathrm{Lie}(G) \times_G X$ be the fibred product of $\mathrm{Lie}(G)$ and X over G. The $\mathbb{Q}$-vector space $T_\mathbb{Q}(M) = T_\mathbb{Z}(M) \otimes_\mathbb{Z} \mathbb{Q}$ is called *the Hodge realization* of M. The filtration $W_\bullet$ on M induces an increasing filtration $W_\bullet$ on $T_\mathbb{Z}(M)$:

$$\begin{aligned} W_0(T_\mathbb{Z}(M)) &= T_\mathbb{Z}(M), \\ W_{-1}(T_\mathbb{Z}(M)) &= \mathrm{H}_1(G, \mathbb{Z}), \\ W_{-2}(T_\mathbb{Z}(M)) &= \mathrm{H}_1(T, \mathbb{Z}). \end{aligned}$$

In particular we have that $\mathrm{Gr}_0^W(T_\mathbb{Z}(M)) \cong X, \mathrm{Gr}_{-1}^W(T_\mathbb{Z}(M)) \cong \mathrm{H}_1(A, \mathbb{Z})$ and $\mathrm{Gr}_{-2}^W(T_\mathbb{Z}(M)) \cong \mathrm{H}_1(T, \mathbb{Z})$. According to [D75] (10.1.3) on $T_\mathbb{Z}(M) \otimes_\mathbb{Z} \mathbb{C}$ we define a decreasing filtration $F^\bullet$ such that the triplet $(T_\mathbb{Z}(M), W_\bullet, F^\bullet)$ we obtain is a $\mathbb{Z}$-mixed Hodge structure without torsion, of level $\leqslant 1$, of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$, and with $\mathrm{Gr}_{-1}^W(T_\mathbb{Z}(M))$ polarisable.

Let $\mathcal{MHS}_\mathbb{Q}$ be the neutral tannakian category of $\mathbb{Q}$-mixed Hodge structures. Denote $T(M) = (T_\mathbb{Q}(M), W_\bullet, F^\bullet)$ the $\mathbb{Q}$-mixed Hodge structure attached to the 1-motive M and $\langle T(M) \rangle^\otimes$ the neutral tannakian subcategory of $\mathcal{MHS}_\mathbb{Q}$ generated by $T(M)$. This subcategory is endowed with the fiber functor ω which sends each object of $\langle T(M) \rangle^\otimes$ to its underlying vector space. By [DM82] 2.11 $\mathrm{Aut}^\otimes(\omega)$ is representable by a closed algebraic $\mathbb{Q}$-subgroup P of $\mathbf{GL}(T_{\mathbb{Q}}(M))$ and ω defines an equivalence of tensorial categories $\langle T(M)\rangle^{\otimes} \longrightarrow (\mathrm{Rep}_{\mathbb{Q}}(P),\otimes)$ where $\mathrm{Rep}_{\mathbb{Q}}(P)$ is the category of finite dimensional representations of P over $\mathbb{Q}$. We call P *the Mumford-Tate group of M* and often we denote it by $MT(M)$. By definition the notion of Mumford-Tate group is stable under isogeny and duality and so in this article we can work modulo isogenies. By [S72] Chapter 2 §2, P is endowed with an increasing filtration, $W_\bullet$, defined over $\mathbb{Q}$:

$W_i(P) = P \qquad \forall\, i \geqslant 0,$

$W_{-1}(P) = \{g \in P \mid (g-\mathrm{id})T_{\mathbb{Q}}(M) \subseteq \mathrm{H}_1(G,\mathbb{Q}), (g-\mathrm{id})\mathrm{H}_1(G,\mathbb{Q}) \subseteq \mathrm{H}_1(T,\mathbb{Q}), (g-\mathrm{id})\mathrm{H}_1(T,\mathbb{Q}) = 0\},$

$W_{-2}(P) = \{g \in P \mid (g-\mathrm{id})T_{\mathbb{Q}}(M) \subseteq \mathrm{H}_1(T,\mathbb{Q}), (g-\mathrm{id})\mathrm{H}_1(G,\mathbb{Q}) = 0\},$

$W_i(P) = 0 \qquad \forall\, i \leqslant -3.$

The inclusion $W_{-2}(P) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}\big(\mathrm{Gr}_0^W(T_{\mathbb{Q}}(M)), \mathrm{H}_1(T,\mathbb{Q})\big)$ implies that

$$\dim_{\mathbb{Q}} W_{-2}(P) \leqslant \mathrm{rank} X \cdot \mathrm{rank} X^\vee.$$

Since $W_{-1}(P)$ is a unipotent group, the derived group of $W_{-1}(P)$ is contained in $W_{-2}(P)$. Let $\mathrm{Gr}_0^W(P) = P/W_{-1}(P)$. According to [By83] §2.2, we know that $\mathrm{Gr}_0^W(P)$ acts trivially on $\mathrm{Gr}_0^W(T_{\mathbb{Q}}(M))$ and by homotheties on $\mathrm{Gr}_{-2}^W(T_{\mathbb{Q}}(M))$, and that the image of $\mathrm{Gr}_0^W(P)$ in $\mathbf{GL}(\mathrm{Gr}_{-1}^W(T_{\mathbb{Q}}(M)))$ is the Mumford-Tate group of A, $MT(A)$. In particular, $\mathrm{Gr}_0^W(P)$ is reductive and $W_{-1}(P)$ is the unipotent radical of P.

1.3. Let $M = (X, X^\vee, A, A^\vee, v, v^\vee, \psi)$ be a 1-motive defined over $\mathbb{C}$ and P its Mumford-Tate group. Let $\mathcal{H}_M = \mathbb{Q}(1)^{\mathrm{rk}X\cdot\mathrm{rk}X^\vee} \times \mathrm{H}_1(A,\mathbb{Q})^{\mathrm{rk}X} \times \mathrm{H}_1(A^\vee,\mathbb{Q})^{\mathrm{rk}X^\vee}$. Consider the following group law on $\mathcal{H}_M$: if $(\vec{s},\vec{x},\vec{x}^\vee)$, $(\vec{t},\vec{y},\vec{y}^\vee)$ are two elements of $\mathcal{H}_M$ we define

$$(1.3.1)\quad (\vec{s},\vec{x},\vec{x}^\vee)\circ(\vec{t},\vec{y},\vec{y}^\vee) = \Big(\vec{s}+\vec{t}+(\langle x_i, y_j^\vee\rangle_{\mathbb{Q}})_{\substack{i=1,\dots,\mathrm{rk}X\\ j=1,\dots,\mathrm{rk}X^\vee}}, \vec{x}+\vec{y}, \vec{x}^\vee+\vec{y}^\vee\Big)$$

where $\langle\, ,\rangle_{\mathbb{Q}} : \mathrm{H}_1(A,\mathbb{Q}) \times \mathrm{H}_1(A^\vee,\mathbb{Q}) \longrightarrow \mathbb{Q}(1)$ is the Weil pairing. We call $\mathcal{H}_M$ *the generalized Heisenberg group associaded to M.*

Let $M^{sc} = M \oplus M^\vee / W_{-2}(M \oplus M^\vee)$ and $P^{sc} = MT(M^{sc})$. Using the canonical isomorphisms $\mathrm{Hom}(X;\mathrm{H}_1(T,\mathbb{Q})) \cong \mathbb{Q}(1)^{\mathrm{rk}X\cdot\mathrm{rk}X^\vee}$, $\mathrm{Hom}(X;\mathrm{H}_1(A,\mathbb{Q})) \cong \mathrm{H}_1(A,\mathbb{Q})^{\mathrm{rk}X}$ and $\mathrm{Hom}(X^\vee;\mathrm{H}_1(A^\vee,\mathbb{Q})) \cong \mathrm{H}_1(A^\vee,\mathbb{Q})^{\mathrm{rk}X^\vee}$, from the definition of $W_\bullet$ we have the inclusions

$$(1.3.2)\qquad \begin{aligned} &\iota : W_{-2}(P) \longrightarrow \mathbb{Q}(1)^{\mathrm{rk}X\cdot\mathrm{rk}X^\vee},\\ &i : W_{-1}(P^{sc}) \longrightarrow \mathrm{H}_1(A,\mathbb{Q})^{\mathrm{rk}X} \times \mathrm{H}_1(A^\vee,\mathbb{Q})^{\mathrm{rk}X^\vee}. \end{aligned}$$

1.4. STRUCTURAL LEMMA.

(1) *The following sequence:*

$$0 \longrightarrow W_{-2}(P) \longrightarrow W_{-1}(P) \longrightarrow W_{-1}(P^{sc}) \longrightarrow 0 \tag{1.4.1}$$

is exact.

(2) *There exists an injective homomorphism of groups* $I : W_{-1}(P) \longrightarrow \mathcal{H}_M$ *such that the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & W_{-2}(P) \arrow[r] \arrow[d, hook, "\iota"] & W_{-1}(P) \arrow[r] \arrow[d, hook, "I"] & W_{-1}(P^{sc}) \arrow[r] \arrow[d, hook, "i"] & 0 \\
0 \arrow[r] & \mathbb{Q}(1)^{\mathrm{rk}X\cdot\mathrm{rk}X^\vee} \arrow[r] & \mathcal{H}_M \arrow[r] & \mathrm{H}_1(A,\mathbb{Q})^{\mathrm{rk}X} \times \mathrm{H}_1(A^\vee,\mathbb{Q})^{\mathrm{rk}X^\vee} \arrow[r] & 0
\end{tikzcd}$$

Proof.

(1) Since $\langle T(M^{sc})\rangle^{\otimes}$ is contained in $\langle T(M)\rangle^{\otimes}$, by 2.21 [DM82] we have the surjective homomorphism

$$\begin{aligned} \lambda : P &\longrightarrow P^{sc} \\ g &\longmapsto \lambda(g) = g_{|T_{\mathbb{Q}}(M^{sc})}. \end{aligned}$$

By definition of M^{sc}, we have that $g \in \mathrm{Ker}(\lambda)$ if and only if both the restrictions of g to $W_0/W_{-2}T_{\mathbb{Q}}(M)$ and to $W_{-1}T_{\mathbb{Q}}(M)$ are trivial. But this means that $W_{-2}(P) \cong \mathrm{Ker}(\lambda)$, and so we obtain the exact sequence

$$0 \longrightarrow W_{-2}(P) \longrightarrow P \longrightarrow P^{sc} \longrightarrow 0 \tag{1.4.2}$$

which implies (1.4.1) since P preserves the filtration $W_\bullet$.

(2) The existence of $I : W_{-1}(P) \longrightarrow \mathcal{H}_M$ is given by (1.4.1) and by the inclusions defined in (1.3.2). We only have to prove that I is an homomorphism of groups. In order to simplify notations, we identify $W_{-1}(P)$ with $I(W_{-1}(P)) \subseteq \mathcal{H}_M$. If $g_1 = (\vec{s}, \vec{x}, \vec{x}^\vee)$ and $g_2 = (\vec{t}, \vec{y}, \vec{y}^\vee)$ are any two elements of $W_{-1}(P)$ we have

$$g_1 \circ g_2 = (\vec{s} + \vec{t} + \Upsilon(\vec{x}, \vec{x}^\vee, \vec{y}, \vec{y}^\vee), \vec{x} + \vec{y}, \vec{x}^\vee + \vec{y}^\vee) \tag{1.4.3}$$

where Υ is a map from $\mathrm{H}_1(A,\mathbb{Q})^{\mathrm{rk}X} \times \mathrm{H}_1(A^\vee,\mathbb{Q})^{\mathrm{rk}X^\vee}$ to $\mathbb{Q}(1)$. In order to determine Υ, we have to understand how $g_1 \circ g_2$ acts on the Hodge realization $T_{\mathbb{Q}}(M)$ of M. Let $M_1 = M/W_{-2}M$, $M_2 = M^\vee/W_{-2}M^\vee$ and

$P_i = MT(M_i)$ for $i = 1,2$. According to 2.21 [DM82] we have the surjections $pr_i : W_{-1}(P^{sc}) \longrightarrow W_{-1}(P_i)$.

$$\begin{tikzcd}[column sep=small]
 & & W_{-1}(P^{sc}) \arrow[dll, "pr_1"'] \arrow[d, hook, "i"] \arrow[drr, "pr_2"] & & \\
W_{-1}(P_1) \arrow[dd, hook] & & \mathrm{H}_1(A,\mathbb{Q})^{\mathrm{rk}X} \times \mathrm{H}_1(A^\vee,\mathbb{Q})^{\mathrm{rk}X^\vee} \arrow[ddl] \arrow[ddr] & & W_{-1}(P_2) \arrow[dd, hook] \\
 & & & & \\
\mathrm{H}_1(A,\mathbb{Q})^{\mathrm{rk}X} & & & & \mathrm{H}_1(A^\vee,\mathbb{Q})^{\mathrm{rk}X^\vee}
\end{tikzcd}$$

Let $\pi : W_{-1}(P) \longrightarrow W_{-1}(P^{sc})$ be the surjection constructed in (1). By definition of $W_{-1}(P_1)$ we have

$$\big(pr_1(\pi\, g_1) - \mathrm{id}\big)\, W_0/W_{-2}(T_{\mathbb{Q}}(M)) \subseteq \mathrm{H}_1(A,\mathbb{Q}),$$
$$\big(pr_1(\pi\, g_1) - \mathrm{id}\big)\, \mathrm{H}_1(A,\mathbb{Q}) = 0.$$

Hence modulo the canonical isomorphism $\mathrm{Hom}(X;\mathrm{H}_1(A,\mathbb{Q})) \cong \mathrm{H}_1(A,\mathbb{Q})^{\mathrm{rk}X}$ which allows us to identify $pr_1(\pi\, g_1) - \mathrm{id}$ with $\vec{x}$, we obtain that

$$\vec{x} : \mathrm{Gr}_0^W(T_{\mathbb{Q}}(M)) \longrightarrow \mathrm{H}_1(A,\mathbb{Q}). \tag{1.4.4}$$

Since $M_2^\vee = W_{-1}(M)$, $pr_2(\pi\, g_2)$ acts on a contravariant way on $W_{-1}T_{\mathbb{Q}}(M)$, and therefore we have

$$\big(pr_2(\pi\, g_2)' - \mathrm{id}\big)\, \mathrm{H}_1(G,\mathbb{Q}) \subseteq \mathrm{H}_1(T,\mathbb{Q}),$$
$$\big(pr_2(\pi\, g_2)' - \mathrm{id}\big)\, \mathrm{H}_1(T,\mathbb{Q}) = 0,$$

where the symbol $'$ denote the contravariant action. Consequently, modulo the canonical isomorphism $\mathrm{Hom}(X^\vee;\mathrm{H}_1(A^\vee,\mathbb{Q})) \cong \mathrm{H}_1(A^\vee,\mathbb{Q})^{\mathrm{rk}X^\vee}$ which allows us to identify $pr_2(\pi\, g_2)' - \mathrm{id}$ with $-\vec{y}^\vee$, we have that

$$-\vec{y}^\vee : \mathrm{H}_1(A,\mathbb{Q}) \longrightarrow \mathrm{H}_1(T,\mathbb{Q}). \tag{1.4.5}$$

Again modulo the canonical isomorphism $\mathrm{Hom}(X;\mathrm{H}_1(T,\mathbb{Q})) \cong \mathbb{Q}(1)^{\mathrm{rk}X\cdot\mathrm{rk}X^\vee}$, from (1.4.4) and (1.4.5) we finally get

$$(-\langle x_i, y_j^\vee\rangle_{\mathbb{Q}})_{i,j} : \mathrm{Gr}_0^W(T_{\mathbb{Q}}(M)) \xrightarrow{\vec{x}} \mathrm{H}_1(A,\mathbb{Q}) \xrightarrow{-\vec{y}^\vee} \mathrm{H}_1(T,\mathbb{Q}).$$

Hence $\Upsilon(\vec{x},\vec{x}^\vee,\vec{y},\vec{y}^\vee) = (-\langle x_i, y_j^\vee\rangle_{\mathbb{Q}})_{i,j}$ and using (1.4.3) we can conclude.

Remarks. — (1) If $\mathrm{rk}X = \mathrm{rk}X^\vee = 1$ then $\mathcal{H}$ is a Heisenberg group.

(2) In [D01] P. Deligne proposes to the author another approach to the study of the structure of the Mumford-Tate group of M: he suggests one investigate the W_{-1} of the Lie algebra of the motivic Galois group of M interpreted as a 1-motive.

1.5. COROLLARY ON DIMENSIONS.

(1) *If $A \neq 0$, let $\mathcal{A}$ be the connected component of the identity in the Zariski closure of $v \times v^\vee(X \times X^\vee)$ and $F = \mathrm{End}\,\mathcal{A} \otimes \mathbb{Q}$. Then*

$$\dim_{\mathbb{Q}} P = \dim_{\mathbb{Q}} MT(A) + \dim_{\mathbb{Q}} W_{-2}(P) + \dim_{\mathbb{Q}} \mathrm{Hom}_F\big(v \times v^\vee(X \times X^\vee); \mathrm{H}_1(\mathcal{A}, \mathbb{Q})\big).$$

(2) *If $A = 0$, let $\mathcal{T}$ be the connected component of the identity in the Zariski closure of $u(X)$ and $F' = \mathrm{End}\,\mathcal{T} \otimes \mathbb{Q}$. Then*

$$\dim_{\mathbb{Q}} P = 1 + \dim_{\mathbb{Q}} \mathrm{Hom}_{F'}\big(u(X); \mathrm{H}_1(\mathcal{T}, \mathbb{Q})\big).$$

Proof. — (1) From (1.4.1), we have that

$$\dim_{\mathbb{Q}} P = \dim_{\mathbb{Q}} \mathrm{Gr}_0^W(P) + \dim_{\mathbb{Q}} W_{-2}(P) + \dim_{\mathbb{Q}} W_{-1}(P^{sc}),$$

where $\mathrm{Gr}_0^W(P)$ is $MT(A)$. Since $W_{-1}(T_{\mathbb{Q}}(M^{sc})) = \mathrm{Gr}_{-1}^W(T_{\mathbb{Q}}(M^{sc}))$, we remark that $W_{-1}(P^{sc}) = \ker[P \longrightarrow MT(W_{-1}M^{sc})]$. Hence applying [A92] prop. 1 to M^{sc}, we find that

$$\dim_{\mathbb{Q}} W_{-1}(P^{sc}) = \dim_{\mathbb{Q}} \mathrm{Hom}_F\big(v \times v^\vee(X \times X^\vee); \mathrm{H}_1(\mathcal{A}, \mathbb{Q})\big).$$

(2) If $A = 0$, we observe that $\mathrm{Gr}_0^W(P) = MT(T)$, and therefore $\dim_{\mathbb{Q}} P = 1 + \dim_{\mathbb{Q}} W_{-1}(P)$. Since $W_{-1}(P) = \ker[P \longrightarrow MT(W_{-1}M)]$, applying [A92] Prop. 1 to M, we obtain

$$\dim_{\mathbb{Q}} W_{-1}(P) = \dim_{\mathbb{Q}} \mathrm{Hom}_{F'}\big(u(X); \mathrm{H}_1(\mathcal{T}, \mathbb{Q})\big).$$

1.6. Let $M = (X, X^\vee, A, A^\vee, v, v^\vee, \psi)$ be a 1-motive defined over $\mathbb{C}$. Denote $r = \mathrm{rank}\,X$ and $s = \mathrm{rank}\,X^\vee$. Moreover, let $\{x_i\}$ (resp. $\{x_j^\vee\}$) be a basis of X (resp. of $X^\vee$). We can also consider M as a complex $M = [X \longrightarrow G]$ where G is an extension of A by $T = \mathrm{Hom}(X^\vee, \mathbb{G}_m)$. Denote by $(x_j^\vee)_*(G)$ the pushout of G by $x_j^\vee : T \longrightarrow \mathbb{G}_m$, which is the extension of A by $\mathbb{G}_m$

parametrized by the point $v^\vee(x_j^\vee)$, and by $(x_j^\vee)_*(u(x_i))$ the point on the fibre of $(x_j^\vee)_*(G)$ above $v(x_i)$ corresponding to $\psi(x_i, x_j^\vee)$. Consider the 1-motive

$$M_{ij} = [x_i\mathbb{Z} \xrightarrow{u_{ij}} G_j],$$

where $G_j = (x_j^\vee)_*(G)$ and $u_{ij}(x_i) = (x_j^\vee)_*(u(x_i))$. We can also write M_{ij} in the form $(x_i\mathbb{Z}, x_j^\vee\mathbb{Z}, A, A^\vee, v_{ij}, v_{ij}^\vee, \psi_{ij})$ where $v_{ij} : x_i\mathbb{Z} \longrightarrow A$ and $v_{ij}^\vee : x_j^\vee\mathbb{Z} \longrightarrow A^\vee$ are the homomorphisms defined by $v_{ij}(x_i) = v(x_i)$ and $v_{ij}^\vee(x_j^\vee) = v^\vee(x_j)$ respectively, and ψ_{ij} is the restriction of ψ to $x_i\mathbb{Z} \times x_j^\vee\mathbb{Z}$.

1.7. Theorem. — *The 1-motives M and $\oplus_{i=1}^r \oplus_{j=1}^s M_{ij}$ generate the same neutral tannakian subcategory of $\mathcal{MHS}_\mathbb{Q}$. In particular, $MT(M) = MT(\oplus_{i=1}^r \oplus_{j=1}^s M_{ij})$.*

Proof. — Via the isomorphism $\mathrm{Ext}^1(A, \mathbb{G}_m^s) \cong \Pi_{j=1}^s \mathrm{Ext}^1(A, \mathbb{G}_m)$, the extension G corresponds to the product of extensions $G_1 \times \ldots \times G_s$ and so

$$\oplus_{i=1}^r \oplus_{j=1}^s \Big(M/[\Pi_{\substack{1\leqslant k\leqslant r\\ k\neq i}} x_k\mathbb{Z} \longrightarrow \Pi_{\substack{1\leqslant l\leqslant s\\ l\neq j}} G_l]\Big) = \oplus_{i=1}^r \oplus_{j=1}^s M_{ij},$$

which implies that $\langle T(\oplus_{i=1}^r \oplus_{j=1}^s M_{ij})\rangle^\otimes \subset \langle T(M)\rangle^\otimes$.

To conclude, it is enough to show that $\langle T(M)\rangle^\otimes \subset \langle T(\oplus_{i=1}^r \oplus_{j=1}^s M_{ij})\rangle^\otimes$. Let $u_i = \Pi_j u_{ij}$ and $v_i = \Pi_j v_{ij}$. We remark that for each $i = 1, \ldots, r$ the homomorphisms $v_{ij}^\vee$ represent the same extension G_j and so we can let $v_j^\vee = v_{ij}^\vee$. In order to simplify computations, we identify X with $\mathbb{Z}^r$ and $X^\vee$ with $\mathbb{Z}^s$. If $\mathrm{d}_\mathbb{Z} : \mathbb{Z} \longrightarrow \mathbb{Z}^s$ is the diagonal homomorphism, for each $i = 1, \ldots, r$ we have

$$\big(\oplus_{j=1}^s M_{ij}\big)/[\mathbb{Z}^s/\mathrm{d}_\mathbb{Z}(\mathbb{Z}) \longrightarrow 0] = [\mathrm{d}_\mathbb{Z}(\mathbb{Z}) \xrightarrow{\Pi_j u_{ij}} \Pi_{j=1}^s G_j] \cong [\mathbb{Z} \xrightarrow{u_i} \Pi_{j=1}^s G_j].$$

Hence taking the sum for $i = 1, \ldots, r$, we obtain

$$\oplus_{i=1}^r \Big(\big(\oplus_{j=1}^s M_{ij}\big)/[\mathbb{Z}^s/\mathrm{d}_\mathbb{Z}(\mathbb{Z}) \longrightarrow 0]\Big) = [\mathbb{Z}^r \xrightarrow{\Pi_i u_i} (\Pi_{j=1}^s G_j)^r], \tag{1.7.1}$$

where $[\mathbb{Z}^r \xrightarrow{\Pi_i u_i} (\Pi_{j=1}^s G_j)^r]$ is the 1-motive

$$\begin{array}{ccccccccc} & & & & \mathbb{Z}^r & & & & \\ & & & & \Big\downarrow{\scriptstyle \Pi_i u_i} & \searrow{\scriptstyle \Pi_{i,j} v_{ij}} & & & \\ 0 & \longrightarrow & \mathbb{G}_m^{rs} & \longrightarrow & (\Pi_{j=1}^s G_j)^r & \longrightarrow & A^{sr} & \longrightarrow & 0 \end{array}$$

with the extension $(\Pi_{j=1}^s G_j)^r$ defined by the homomorphism $\Pi_{i,j} v_{ij}^\vee = (\Pi_j v_j^\vee)^r : (\mathbb{Z}^s)^r \longrightarrow (A^{\vee s})^r$. The Cartier dual of $[\mathbb{Z}^r \stackrel{\Pi_i u_i}{\longrightarrow} (\Pi_{j=1}^s G_j)^r]$ is the 1-motive

$$\begin{tikzcd}
 & & \mathbb{Z}^{rs} \arrow[d, "\Pi_i u_i^\vee"'] \arrow[dr, "\Pi_{i,j} v_{ij}^\vee"] & & \\
0 \arrow[r] & \mathbb{G}_m^r \arrow[r] & ((\Pi_{j=1}^s G_j)^r)^\vee \arrow[r] & (A^\vee)^{sr} \arrow[r] & 0.
\end{tikzcd}$$

If $\mathrm{d}_{\mathbb{Z}^s} : \mathbb{Z}^s \longrightarrow (\mathbb{Z}^s)^r$ is the diagonal homomorphism of $\mathbb{Z}^s$ in $(\mathbb{Z}^s)^r$, we observe that

(1.7.2)

$$\begin{aligned}
\Big([\mathbb{Z}^r \stackrel{\Pi_i u_i}{\longrightarrow} (\Pi_{j=1}^s G_j)^r]^\vee / [\mathbb{Z}^{rs}/\mathrm{d}_{\mathbb{Z}^s}(\mathbb{Z}^s) \longrightarrow 0]\Big)^\vee &\cong [\mathbb{Z}^s \stackrel{\Pi_i u_i}{\longrightarrow} ((\Pi_{j=1}^s G_j)^r)^\vee]^\vee \\
&= [\mathbb{Z}^r \stackrel{\Pi_i u_i}{\longrightarrow} (\widetilde{\Pi_{j=1}^s G_j})^r],
\end{aligned}$$

where $[\mathbb{Z}^r \stackrel{\Pi_i u_i}{\longrightarrow} (\widetilde{\Pi_{j=1}^s G_j})^r]$ is the 1-motive

$$\begin{tikzcd}
 & & \mathbb{Z}^{r} \arrow[d, "\Pi_i u_i"'] \arrow[dr, "\Pi_{i,j} v_{ij}"] & & \\
0 \arrow[r] & \mathbb{G}_m^s \arrow[r] & (\widetilde{\Pi_{j=1}^s G_j})^r \arrow[r] & A^{sr} \arrow[r] & 0
\end{tikzcd}$$

with the extension $(\widetilde{\Pi_{j=1}^s G_j})^r$ defined by the homomorphism $(\Pi_j v_j^\vee)^r$: $\mathbb{Z}^s \longrightarrow (A^{\vee s})^r \quad (1, \dots, 1) \longmapsto (\Pi_j v_j^\vee(1), \dots, \Pi_j v_j^\vee(1))$. Now the homomorphism

$$\begin{aligned}
+ : (A^s)^r &\longrightarrow A^s \\
(\vec{X}_1, \dots, \vec{X}_r) &\longmapsto \vec{X}_1 + \dots + \vec{X}_r
\end{aligned}$$

defines a surjection between the 1-motives $[\mathbb{Z}^r \stackrel{\Pi_{ij} v_{ij}}{\longrightarrow} (A^s)^r]$ and $[\mathbb{Z}^r \stackrel{\Pi_i v_i}{\longrightarrow} A^s]$, which lifts to a surjection from M to $[\mathbb{Z}^r \stackrel{\Pi_i u_i}{\longrightarrow} (\widetilde{\Pi_{j=1}^s G_j})^r]$. Hence M is a quotient of $[\mathbb{Z}^r \stackrel{\Pi_i u_i}{\longrightarrow} (\widetilde{\Pi_{j=1}^s G_j})^r]$, and so by (1.7.1) and (1.7.2), we can write M as a quotient of

$$\left\{\left(\oplus_{i=1}^r \Big(\big(\oplus_{j=1}^s M_{ij}\big) / [\mathbb{Z}^s/\mathrm{d}_{\mathbb{Z}}(\mathbb{Z}) \longrightarrow 0]\Big)\right)^\vee \Big/ [\mathbb{Z}^{rs}/\mathrm{d}_{\mathbb{Z}^s}(\mathbb{Z}^s) \longrightarrow 0]\right\}^\vee .$$

1.8. Let $M = [\mathbb{Z} \xrightarrow{u} G]$ be a 1-motive over $\mathbb{C}$, where G is the extension of A by $\mathbb{G}_m$ parametrized by $v^\vee(1) = Q \in A^\vee(\mathbb{C})$, and $u(1)$ is the point $S \in G(\mathbb{C})$ which lifts the point $v(1) = R \in A(\mathbb{C})$. Its Cartier dual is the 1-motive $M^\vee = [\mathbb{Z} \xrightarrow{u^\vee} G^\vee]$, where $G^\vee$ is defined by the point $R \in A(\mathbb{C})$ and $u^\vee(1) = S^\vee \in G^\vee(\mathbb{C})$ lifts the point $Q \in A^\vee(\mathbb{C})$.

The diagonal homomorphism $\mathrm{d} : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ of $\mathbb{Z}$ and the multiplication law $\mu : \mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m$ of $\mathbb{G}_m$ induce the morphisms of 1-motives $\mathrm{d} : [\mathbb{Z} \longrightarrow 0] \longrightarrow [\mathbb{Z} \times \mathbb{Z} \longrightarrow 0]$ and $\mu : [0 \longrightarrow \mathbb{G}_m \times \mathbb{G}_m] \longrightarrow [0 \longrightarrow \mathbb{G}_m]$. In [Be98] Thm 1, Bertrand constructs the 1-motive

$$\mathbf{M} := \mu_* \mathrm{d}^* (M \oplus M^\vee) = [\mathbb{Z} \xrightarrow{\mathbf{U}} \mathbf{G}]$$

where $\mathbf{G}$ is the extension of $\mathbb{A} = A \times A^\vee$ by $\mathbb{G}_m$ parametrized by the point $(Q, R) \in \mathbb{A}^\vee(\mathbb{C})$ and $\mathbf{U}(1)$ is the point $\mathbf{S} \in \mathbf{G}(\mathbb{C})$ which lifts the point $(R, Q) \in \mathbb{A}(\mathbb{C})$.

1.9. Lemma. — $MT(M) = MT(\mathbf{M})$.

Proof. — Let $P = MT(M)$, $\mathbf{P} = MT(\mathbf{M})$, $\mathbf{V} := (v \times v^\vee) : \mathbb{Z} \longrightarrow \mathbb{A}$, $\mathbf{V}^\vee := (v^\vee \times v) : \mathbb{Z} \longrightarrow \mathbb{A}^\vee$, $M^{sc} = [\mathbb{Z} \times \mathbb{Z} \xrightarrow{v \times v^\vee} A \times A^\vee]$ and $\mathbf{M}^{sc} = [\mathbb{Z} \times \mathbb{Z} \xrightarrow{\mathbf{V} \times \mathbf{V}^\vee} \mathbb{A} \times \mathbb{A}^\vee]$. According to (1.4.2), we have that $P/W_{-2}(P) = MT(M^{sc})$ and $\mathbf{P}/W_{-2}(\mathbf{P}) = MT(\mathbf{M}^{sc})$. Since

$$\mathbf{M}^{sc}/[0 \longrightarrow \mathbb{A}^\vee] = M^{sc},$$

we observe that $\langle T(M^{sc})\rangle^\otimes \subseteq \langle T(\mathbf{M}^{sc})\rangle^\otimes$. If $M_1 = [\mathbb{Z} \xrightarrow{v} A]$ and $M_2 = [\mathbb{Z} \xrightarrow{v^\vee} A^\vee]$, we remark that

$$M_1 \oplus M_2 \oplus M_2 \oplus M_1 / [\mathbb{Z}^4/\mathrm{d} \times \mathrm{d}(\mathbb{Z} \times \mathbb{Z}) \longrightarrow 0] \cong \mathbf{M}^{sc},$$

which implies that $\langle T(\mathbf{M}^{sc})\rangle^\otimes \subseteq \langle T(M^{sc})\rangle^\otimes$. Hence we have $\langle T(\mathbf{M}^{sc})\rangle^\otimes = \langle T(M^{sc})\rangle^\otimes$ and so we obtain that $P/W_{-2}(P) = \mathbf{P}/W_{-2}(\mathbf{P})$.

We now have to verify that $W_{-2}(P) = W_{-2}(\mathbf{P})$. Let $P^\vee = MT(M^\vee)$. By definition we have the inclusions

$$\begin{aligned} \lambda : & W_{-2}(P) \longrightarrow \mathrm{H}_1(\mathbb{G}_m, \mathbb{Q}), \\ \lambda^\vee : & W_{-2}(P^\vee) \longrightarrow \mathrm{H}_1(\mathbb{G}_m, \mathbb{Q}), \\ \Lambda : & W_{-2}(\mathbf{P}) \longrightarrow \mathrm{H}_1(\mathbb{G}_m, \mathbb{Q}). \end{aligned}$$

Since the notion of Mumford-Tate group is stable under duality we have $\lambda = \lambda^\vee$ and by construction of $\mathbf{M}$ we remark that $\Lambda = \lambda + \lambda^\vee = 2\lambda$.

Therefore we can conclude that

$$W_{-2}(P)=0 \iff \lambda=0 \iff \Lambda=0 \iff W_{-2}(\mathbf{P})=0$$
$$\dim_{\mathbb{Q}} W_{-2}(P)=1 \iff \lambda \text{ iso} \iff \Lambda \text{ iso} \iff \dim_{\mathbb{Q}} W_{-2}(\mathbf{P})=1,$$

which implies that $W_{-2}(P)=W_{-2}(\mathbf{P})$.

2. The degeneracies of the Mumford-Tate group.

2.1. Let M be a 1-motive defined over $\mathbb{C}$ and P its Mumford-Tate group. Consider the following subgroups of P:

$$\begin{aligned} U^{(1)}(P) &= \ker\big[P \longrightarrow MT(\mathrm{Gr}_{-1}^W M)\big],\\ U^{(2)}(P) &= \ker\big[P \longrightarrow MT(\mathrm{Gr}_{-2}^W M)\big],\\ U^{(3)}(P) &= \ker\big[P \longrightarrow MT(W_0/W_{-2}M)\big],\\ U^{(4)}(P) &= \ker\big[P \longrightarrow MT(W_{-1}M)\big]. \end{aligned}$$

Clearly $U^{(4)}(P)\subseteq U^{(1)}(P)\cap U^{(2)}(P)$ and $U^{(3)}(P)\subseteq U^{(1)}(P)$. If $\mathrm{Gr}_{-1}^W M\neq 0$, then $\langle T(W_{-2}M)\rangle^{\otimes}$ is a subcategory of $\langle T(\mathrm{Gr}_{-1}^W M)\rangle^{\otimes}$ and so the natural homomorphism $P\longrightarrow MT(\mathrm{Gr}_{-2}^W M)$ factorises via $P\longrightarrow MT(\mathrm{Gr}_{-1}^W M)$, which implies that $U^{(1)}(P)\subseteq U^{(2)}(P)$. Finally, if $P^\vee$ is the Mumford-Tate group of $M^\vee$, then $U^{(3)}(P)=U^{(4)}(P^\vee)$ and $U^{(4)}(P)=U^{(3)}(P^\vee)$.

2.2. Lemma

(i) $W_{-1}(P)=U^{(1)}(P)\cap U^{(2)}(P)$,

(ii) $W_{-2}(P)=U^{(3)}(P)\cap U^{(4)}(P)$.

Proof. — By [A92] Lemma 2 (c), P respects the filtration $W_\bullet$ of $T_{\mathbb{Q}}(M)$. Moreover each element of P acts trivially on $\mathrm{Gr}_0^W(T_{\mathbb{Q}}(M))$ and so if $g\in P$ we have $(g-\mathrm{id})T_{\mathbb{Q}}(M)\subseteq \mathrm{H}_1(G,\mathbb{Q})$. We remark also that

$$\begin{aligned} g\in U^{(1)}(P) &\iff g_{|\mathrm{Gr}_{-1}^W(T_{\mathbb{Q}}(M))}=\mathrm{id} \iff (g-\mathrm{id})\mathrm{H}_1(G,\mathbb{Q})\subseteq \mathrm{H}_1(T,\mathbb{Q})\\ g\in U^{(2)}(P) &\iff g_{|\mathrm{Gr}_{-2}^W(T_{\mathbb{Q}}(M))}=\mathrm{id} \iff (g-\mathrm{id})\mathrm{H}_1(T,\mathbb{Q})=0\\ g\in U^{(3)}(P) &\iff g_{|W_0/W_{-2}(T_{\mathbb{Q}}(M))}=\mathrm{id} \iff (g-\mathrm{id})T_{\mathbb{Q}}(M)\subseteq \mathrm{H}_1(T,\mathbb{Q})\\ g\in U^{(4)}(P) &\iff g_{|W_{-1}(T_{\mathbb{Q}}(M))}=\mathrm{id} \iff (g-\mathrm{id})\mathrm{H}_1(G,\mathbb{Q})=0. \end{aligned}$$

The result now follows from the definitions of $W_{-1}(P)$ and of $W_{-2}(P)$.

2.3. The Mumford-Tate group P of a 1-motive M can present some degeneracies which correspond to decreases of the dimension of P. We classify these degeneracies in the following way:

• M is *deficient* if $W_{-2}(P) = 0$. Among the deficient 1-motives, there are those which are trivially deficient: M is *trivially deficient* if $W_{-1}(P) = U^{(4)}(P)$ and $W_{-2}(P) = 0$ (or dually if $W_{-1}(P) = U^{(3)}(P) \cap U^{(2)}(P)$ and $W_{-2}(P) = 0$), or if $W_{-1}(P) = 0$.

• M is *quasi-deficient* if $W_{-1}(P)$ is abelian. Since the derived group of $W_{-1}(P)$ is contained in $W_{-2}(P)$, *deficience implies quasi-deficience.* Among the quasi-deficient 1-motives, there are those which are trivially quasi-deficient: M is *trivially quasi-deficient* if $W_{-1}(P) = U^{(4)}(P)$ (or dually if $W_{-1}(P) = U^{(3)}(P) \cap U^{(2)}(P)$), or if $W_{-1}(P) = W_{-2}(P)$.

• M is *depressive* if $0 < \dim_{\mathbb{Q}} W_{-2}(P) < \mathrm{rank} X \cdot \mathrm{rank} X^{\vee}$.

2.4. *Examples*

(1) The 1-motives $M = (X, X^{\vee}, A, A^{\vee}, v, v^{\vee}, \psi)$ such that the image of v consists of torsion points are examples of trivially quasi-deficient 1-motives.

(2) The 1-motives without toric part or the 1-motives $M = [X \xrightarrow{u} G]$ with the image of u consisting of torsion points are trivially deficient 1-motives.

(3) In [JR87] Jacquinot and Ribet construct a deficient 1-motive using an abelian variety A with complex multiplication, a point in $A^{\vee}$ and a homomorphism $f : A^{\vee} \longrightarrow A$.

(4) In order to construct a quasi-deficient 1-motive it is enough to take a deficient 1-motive à la Jacquinot-Ribet and to "perturb" its trivialization ψ by an element of $\mathbb{G}_m$ which is not a root of unity.

(5) As example of quasi-deficient depressive 1-motives we can take the 1-motive $M = [\mathbb{Z} \xrightarrow{u} \mathbb{G}_m^2]$ with $u(1) = (q, q^2)$ and q not a root of unity. If we want a depressive but not quasi-deficient 1-motive, it is enough to take the 1-motive $M \oplus M$, with M any 1-motive which is not quasi-deficient. Unfortunately, for the moment we have only trivial examples of depression, i.e., examples where the depression can be explain in terms of deficience or isogeny.

3. Geometrical interpretation of deficience and of quasi-deficience.

3.1. We first introduce some notations due in great part to D. Bertrand (cf. [Be94], [Be98]): Let $M = (X, X^\vee, A, A^\vee, v, v^\vee, \psi)$ be a 1-motive over $\mathbb{C}$ and $\mathcal{P}$ the Poincaré biextension of $(A, A^\vee)$. An abelian subvariety B of $A \times A^\vee$ is said to be *isotropic* if the restriction of $\mathcal{P}$ to B is trivial or of order 2. For each pair $(x, x^\vee)$ in $X \times X^\vee$ consider the following conditions:

(i) *there exists an isotropic abelian subvariety $B_{(x,x^\vee)}$ of $A \times A^\vee$, which contains a sufficiently large multiple of $(v(x), v^\vee(x^\vee))$. We denote ξ the canonical splitting of the square of $\mathcal{P}_{|B_{(x,x^\vee)}}$;*

(ii) *if β is the restriction of $(v, v^\vee)$ to the group Z generated by a sufficiently large multiple of $(x, x^\vee)$ in $X \times X^\vee$, the trivializations $\psi_{|Z}$ and $\beta^*\xi$ of $\beta^*\mathcal{P}$ coincide.*

We say that M is *quasi-isotropic* (resp. *isotropic*) if (i) (resp. (i) and (ii)) is (resp. are) satisfied for each pair $(x, x^\vee)$ in $X \times X^\vee$. If we consider the 1-motive $M = (\mathbb{Z}, \mathbb{Z}, \mathcal{E}, \mathcal{E}^\vee, v, v^\vee, \psi)$ where $\mathcal{E}$ is an elliptic curve, $v(1) = P \in \mathcal{E}(\mathbb{C})$ and $v^\vee(1) = Q \in \mathcal{E}^\vee(\mathbb{C})$, the conditions (i) and (ii) become

(i′) $(P, Q) \in (\cup_n \mathcal{E}_n) \times \mathcal{E}^\vee \bigcup \mathcal{E} \times (\cup_n \mathcal{E}_n^\vee) \bigcup \big\{(R, S) \in \mathcal{E} \times \mathcal{E}^\vee / (\cup_n \mathcal{E}_n) \times (\cup_n \mathcal{E}_n^\vee) | (\mathcal{P}_{|B_{(P,Q)}})^2$ is trivial$\big\}$, where $\mathcal{E}_n$ (resp. $\mathcal{E}_n^\vee$) is the group of n-torsion points of $\mathcal{E}$ (resp. $\mathcal{E}^\vee$).

(ii′) $u(1)$ corresponds to one of the following points $\big\{(\alpha, (P, Q)) \in (\mathcal{P})_{(P,Q)} | \ \alpha \in \mathbb{G}_m$ is a root of unity$\big\}$.

Remark. — The condition (i) is equivalent to the existence of an antisymmetric homomorphism $g_{(x,x^\vee)} : A \longrightarrow A^\vee$ and of an integer $N_{(x,x^\vee)}$ such that $g_{(x,x^\vee)}(P) = N_{(x,x^\vee)} Q$ for each (P, Q) in $B_{(x,x^\vee)}$. The antisymmetric homomorphisms appear also in the construction of deficient 1-motives by Jacquinot and Ribet (cf. [JR87]). Also using antisymmetric homomorphisms, Breen interprets the Jacquinot and Ribet's deficient 1-motives in terms of alternate biextensions (cf. [Br87]).

3.2. Lemma. — *Let $M = (\mathbb{Z}, \mathbb{Z}, A, A^\vee, v, v^\vee, \psi)$ be a quasi-isotropic 1-motive over $\mathbb{C}$. Then M has the same Mumford-Tate group as the 1-motive $\mathcal{M} = [\mathbb{Z} \xrightarrow{u} B \times G_b]$ where B is an isotropic abelian subvariety of $A \times A^\vee$ containing a sufficiently large multiple b of $(v(1), v^\vee(1))$, $B \times G_b$ is the extension of $\mathbb{B} = B \times B^\vee$ by $\mathbb{G}_m$ parametrized by the point $(0, b)$ of $\mathbb{B}^\vee$,* *and $\mathcal{U}(1)$ is the point (b,α) of $B \times \mathbb{G}_m$ which lifts the point $(b,0)$ of $\mathbb{B}$. In particular if M is isotropic, then $\mathcal{U}(1) = (b,\alpha)$ with α a root of unity.*

Proof. — By hypothesis, there exists an isotropic abelian subvariety B of $A \times A^\vee$ containing a sufficiently large multiple b of $(v(1), v^\vee(1))$. According to [Be94] Lemma 1, there exists an isogeny $S : \mathbb{B} \longrightarrow \mathbb{A}$ whose restriction to B is the natural inclusion of B in $\mathbb{A}$ and such that $S^*\mathcal{P}$ is a multiple of the Poincaré biextension of $(B, B^\vee), \mathcal{P}_{(B,B^\vee)}$. Let $\mathbf{M}$ be the 1-motive associated to M in 1.8. Using the isogeny $S : \mathbb{B} \longrightarrow \mathbb{A}$, we see that $\mathbf{M}$ is isogeneous to the following 1-motive $\mathcal{M}$:

$$\begin{array}{ccccccccc} & & & & \mathbb{Z} & & & & \\ & & & & \mathcal{U}\downarrow & & & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & B \times G_b & \longrightarrow & \mathbb{B} & \longrightarrow & 0 \\ & & & & \mathcal{U}(1) & \longmapsto & (b,0). & & \end{array}$$

where $B \times G_b$ is the extension of $\mathbb{B}$ by $\mathbb{G}_m$ parametrized by the point $(0,b) \in \mathbb{B}^\vee$. Since M is quasi-isotropic, $\mathcal{P}_{(B,B^\vee)}$ admits a canonical splitting ξ, and therefore we obtain $\mathcal{U}(1) = (b,\alpha)$ with $\alpha \in \mathbb{G}_m$. If M is isotropic, we observe that $\mathcal{U}(1) = (b,\alpha)$ with α a root of unity. Finally, since $\mathbf{M}$ is isogeneous to $\mathcal{M}$, we have $MT(\mathbf{M}) = MT(\mathcal{M})$ and so by 1.9 we obtain $MT(M) = MT(\mathcal{M})$.

3.3. Since A is an abelian variety defined over $\mathbb{C}$, we can view it as a quotient V/Λ of a $\mathbb{C}$-vector space V by a lattice Λ. By definition $A^\vee = V^\vee/\Lambda^\vee$, where $V^\vee = \mathrm{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C})$ is the $\mathbb{C}$-vector space of $\mathbb{C}$-antilinear forms, and $\Lambda^\vee = \{l \in V^\vee \mid \mathrm{Im}\, l(\Lambda) \subseteq \mathbb{Z}\}$. According to [LB92] §4, the duality between A and $A^\vee$ can be expressed in terms of the non degenerate $\mathbb{R}$-bilinear form

$$\begin{aligned} \langle\, , \rangle : V \times V^\vee &\longrightarrow \mathbb{R}, \\ (v,l) &\longmapsto \langle v,l\rangle = \mathrm{Im}\, l(v). \end{aligned}$$

By [LB92] §5, the Poincaré biextension $\mathcal{P}$ of $(A, A^\vee)$ is defined by the Hermitian form

$$\begin{aligned} H : (V \times V^\vee) \times (V \times V^\vee) &\longrightarrow \mathbb{C}, \\ (v_1,l_1),(v_2,l_2) &\longmapsto l_1(v_2) + \overline{l_2(v_1)} \end{aligned}$$

and by the semi-character

$$\begin{aligned} \chi : \Lambda \times \Lambda^\vee &\longrightarrow \mathbb{C}_1 = \{z \in \mathbb{C} \mid |z| = 1\} \\ (\lambda, l) &\longmapsto \mathrm{e}^{2i\pi\langle\lambda,l\rangle}. \end{aligned} \tag{3.3.1}$$

The first Chern class of $\mathcal{P}$, $c_1(\mathcal{P})$, can be identified with the Hermitian form H or with the real valued alternate form

$$E = \mathrm{Im}H : (V \times V^{\vee}) \times (V \times V^{\vee}) \longrightarrow \mathbb{R} \qquad (3.3.2)$$
$$(v_1, l_1), (v_2, l_2) \longmapsto \langle v_2, l_1 \rangle - \langle v_1, l_2 \rangle.$$

By [D75] (10.2.3), in Hodge realization the Poincaré biextension $\mathcal{P}$ defines a pairing $\langle\, ,\rangle_{\mathbb{Z}} : \mathrm{H}_1(A, \mathbb{Z}) \times \mathrm{H}_1(A^{\vee}, \mathbb{Z}) \longrightarrow \mathbb{Z}(1)$ which coincides with the pairing $\langle\, ,\rangle : V \times V^{\vee} \longrightarrow \mathbb{R}$ once we extend the scalars to $\mathbb{R}$ and we identify $\mathbb{Z}(1)$ with $\mathbb{Z}$ by sending $2i\pi$ to 1. From now on, we identify these two pairings.

3.4. LEMMA. — *Let $M = (X, X^{\vee}, A, A^{\vee}, v, v^{\vee}, \psi)$ be a 1-motive over $\mathbb{C}$. The following conditions are equivalent:*

(i) *$W_{-1}(MT(M))$ is abelian,*

(ii) *$\big((\langle x_i, y_j^{\vee}\rangle_{\mathbb{Q}} - \langle y_i, x_j^{\vee}\rangle_{\mathbb{Q}})_{\substack{i=1,\dots,\mathrm{rk}X \\ j=1,\dots,\mathrm{rk}X^{\vee}}} = \vec{0}$ for all $(\vec{x}, \vec{x}^{\vee}), (\vec{y}, \vec{y}^{\vee}) \in \mathrm{H}_1(B, \mathbb{Q})$, where B is the connected component of the identity in the Zariski closure of $v(X) \times v^{\vee}(X^{\vee})$.*

Proof. — Let $P = MT(M)$. According to the structural lemma, we have the following inclusions:

$$\begin{aligned} W_{-1}(P) &\subseteq W_{-2}(P) \times \mathrm{H}_1(B, \mathbb{Q}) \\ &\subseteq \mathbb{Q}(1)^{\mathrm{rk}X \cdot \mathrm{rk}X^{\vee}} \times \mathrm{H}_1(A, \mathbb{Q})^{\mathrm{rk}X} \times \mathrm{H}_1(A^{\vee}, \mathbb{Q})^{\mathrm{rk}X^{\vee}} \end{aligned}$$

and we know that the commutator of two elements $g_1 = (\vec{s}, \vec{x}, \vec{x}^{\vee})$ and $g_2 = (\vec{t}, \vec{y}, \vec{y}^{\vee})$ of $W_{-1}(P)$ is

$$[g_1, g_2] = \big((-\langle x_i, y_j^{\vee}\rangle_{\mathbb{Q}} + \langle y_i, x_j^{\vee}\rangle_{\mathbb{Q}})_{\substack{i=1,\ \ ,\mathrm{rk}X \\ j=1,\ \ ,\mathrm{rk}X^{\vee}}}, 0, 0\big)$$

where $\langle\, ,\rangle_{\mathbb{Q}}$ is the pairing obtained from $\langle\, ,\rangle_{\mathbb{Z}}$ by extension of scalars. The result now follows from the fact that $W_{-1}(P)$ is abelian if and only if $[g_1, g_2] = 0$ for each $g_1, g_2 \in W_{-1}(P)$.

3.5. THEOREM. — *Let $M = (X, X^{\vee}, A, A^{\vee}, v, v^{\vee}, \psi)$ be a 1-motive over $\mathbb{C}$. The following conditions are equivalent:*

(i) *M is quasi-isotropic,*

(ii) *M is quasi-deficient,*

(iii) *M is trivially quasi-deficient.*

Proof.

(i) $\Longrightarrow$ (ii): By hypothesis there exists an isotropic abelian subvariety B of $A \times A^\vee$. But this implies that $c_1(\mathcal{P}_{|B}) = 0$ and therefore by (3.3.2) and 3.4 we can conclude that $W_{-1}(P)$ is abelian.

(ii) $\Longrightarrow$ (i): From 3.4 and (3.3.2), we have $E_{|V_B \times V_B} = c_1(\mathcal{P}_{|B}) = 0$. Moreover, if we consider a special case of 3.4 (it is enough to take the pairs $(\vec{x},0)$ and $(0,\vec{x}^\vee)$ of $\mathrm{H}_1(B,\mathbb{Q})$), we find that $(\langle x_i, x_j^\vee\rangle_{\mathbb{Q}})_{i,j} = \vec{0}$ for each $(\vec{x},\vec{x}^\vee)$ in $\mathrm{H}_1(B,\mathbb{Q})$, which implies by (3.3.1) that $\chi_{|\Lambda_B} = 1$. Since $H_{|V_B \times V_B} = 0$ and $\chi_{|\Lambda_B} = 1$, we have that the restriction of the Poincaré biextension to B is trivial.

(i) $\Longrightarrow$ (iii): We have to prove that $W_{-1}\big(MT(M)\big) = U^{(4)}\big(MT(M)\big)$. Since by definition $U^{(4)}\big(MT(M)\big) \subseteq W_{-1}\big(MT(M)\big)$, it is enough to show that $W_{-1}\big(MT(M)\big) \subseteq U^{(4)}\big(MT(M)\big)$. Using the same notations as in 1.6, by Theorem 1.7 we have that $MT(M) = MT(\oplus_{i=1}^{\mathrm{rank} X} \oplus_{j=1}^{\mathrm{rank} X^\vee} M_{ij})$ where $M_{ij} = [x_i\mathbb{Z} \xrightarrow{u_{ij}} G_j]$. There exists the injection

$$MT(M) \longrightarrow \oplus_{i=1}^{\mathrm{rk} X} \oplus_{j=1}^{\mathrm{rk} X^\vee} MT(M_{ij}) \tag{3.5.1}$$

which respects the filtration $W_\bullet$. According to 3.2 we know that for each i,j there is a 1-motive $\mathcal{M}_{ij}$ such that $MT(M_{ij}) = MT(\mathcal{M}_{ij})$ and

$$\mathcal{M}_{ij} = [\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}} B_{ij} \times G_{b_{ij}}]$$

where B_{ij} is an isotropic abelian subvariety of $A \times A^\vee$ containing a sufficiently large multiple b_{ij} of $(v_{ij}(x_i), v_{ij}^\vee(x_j^\vee))$, $B_{ij} \times G_{b_{ij}}$ is the extension of $\mathbb{B}_{ij} = B_{ij} \times B_{ij}^\vee$ by $\mathbb{G}_m$ parametrized by $(0, b_{ij}) \in \mathbb{B}_{ij}^\vee$, and $\mathcal{U}_{ij}(1)$ is the point $(b_{ij}, \alpha_{ij}) \in B_{ij} \times \mathbb{G}_m$ which lifts the point $(b_{ij}, 0) \in \mathbb{B}_{ij}$. Consider the two 1-motives

$$\mathcal{M}_{ij}^1 := (\mathcal{M}_{ij}/[0 \longrightarrow B_{ij}]) = [\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}^1} G_{b_{ij}}],$$
$$\mathcal{M}_{ij}^2 := (\mathcal{M}_{ij}/[0 \longrightarrow G_{b_{ij}}]) = [\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}^2} B_{ij}],$$

where $\mathcal{U}_{ij}^1(1) = \alpha_{ij}$ and $\mathcal{U}_{ij}^2(1) = b_{ij}$. If $\mathrm{d} : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ is the diagonal homomorphism of $\mathbb{Z}$, we have $\big(\mathcal{M}_{ij}^1 \oplus \mathcal{M}_{ij}^2\big)/[\mathbb{Z} \times \mathbb{Z}/\mathrm{d}(\mathbb{Z}) \longrightarrow 0] \cong \mathcal{M}_{ij}$ and therefore $MT(\mathcal{M}_{ij}^1 \oplus \mathcal{M}_{ij}^2) = MT(\mathcal{M}_{ij})$. But since $\mathcal{M}_{ij}^{1\vee} = [\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}^{1\vee}} B_{ij} \times \mathbb{G}_m]$ with $\mathcal{U}_{ij}^{1\vee}(1) = (b_{ij}, \alpha_{ij})$, we remark that $MT(M_{ij}) = MT(\mathcal{M}_{ij}^{1\vee})$. Using (3.5.1) we obtain the injection

$$MT(M) \xrightarrow{\phi} \oplus_{i,j} MT\big([\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}^{1\vee}} B_{ij} \times \mathbb{G}_m]\big).$$

Let g be an element of $W_{-1}\big(MT(M)\big)$. Since

$$W_{-1}\big(\oplus_{i,j} MT([\mathbb{Z} \xrightarrow{\mathcal{U}^{1\vee}_{ij}} B_{ij} \times \mathbb{G}_m])\big) = U^{(4)}\big(\oplus_{i,j} MT([\mathbb{Z} \xrightarrow{\mathcal{U}^{1\vee}_{ij}} B_{ij} \times \mathbb{G}_m])\big),$$

we have that $\phi(g)_{|\oplus_{i,j} W_{-1}(T_{\mathbb{Q}}([\mathbb{Z} \longrightarrow B_{ij} \times \mathbb{G}_m]))} = 0$. But this implies that $g_{|W_{-1}(T_{\mathbb{Q}}(M))} = 0$, and so we can conclude that $g \in U^{(4)}\big(MT(M)\big)$.

3.6. LEMMA. — *Let $M = (\mathbb{Z}, \mathbb{Z}, A, A^\vee, v, v^\vee, \psi)$ be a 1-motive defined over $\mathbb{C}$. The following conditions are equivalent:*

(i) *M is isotropic,*

(ii) *M is deficient.*

Proof.

(i) $\Longrightarrow$ (ii) : This is done in [Be98] Thm 1, but unfortunatly the converse was proved only for 1-motives defined over a number field.

(ii) $\Longrightarrow$ (i) : We will prove that if M is not isotropic then it is not deficient. If M is not quasi-isotropic, by Theorem 3.5 it is not quasi-deficient, and so in particular it is not deficient. If M is quasi-isotropic, then according to 3.2 M has the same Mumford-Tate group as the 1-motive $\mathcal{M} = [\mathbb{Z} \xrightarrow{\mathcal{U}} B \times G_b]$, where B is an isotropic abelian subvariety of $A \times A^\vee$ containing a sufficiently large multiple b of $(v(1), v^\vee(1))$, $B \times G_b$ is the extension of $\mathbb{B} = B \times B^\vee$ by $\mathbb{G}_m$ parametrized by $(0, b) \in \mathbb{B}^\vee$ and $\mathcal{U}(1)$ is the point $(b, \alpha) \in B \times \mathbb{G}_m$ which lifts the point $(b, 0) \in \mathbb{B}$. Consider the two 1-motives

$$\begin{aligned} \mathcal{M}_1 &:= \mathcal{M}/[0 \longrightarrow B] = [\mathbb{Z} \xrightarrow{u_1} G_b], \\ \mathcal{M}_2 &:= \mathcal{M}/[0 \longrightarrow G_b] = [\mathbb{Z} \xrightarrow{u_2} B], \end{aligned}$$

where $u_1(1) = \alpha$ and $u_2(1) = b$. If $\mathrm{d} : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ is the diagonal homomorphism of $\mathbb{Z}$, we have

$$\big(\mathcal{M}_1 \oplus \mathcal{M}_2\big)/[\mathbb{Z} \times \mathbb{Z}/\mathrm{d}(\mathbb{Z}) \longrightarrow 0] = [\mathrm{d}(\mathbb{Z}) \xrightarrow{u_1 \times u_2} B \times G_b] \cong \mathcal{M}$$

and so $MT(\mathcal{M}_1 \oplus \mathcal{M}_2) = MT(\mathcal{M})$. But $\mathcal{M}_1^\vee = [\mathbb{Z} \xrightarrow{(u_1)^\vee} \mathbb{G}_m \times B]$ with $(u_1)^\vee(1) = (\alpha, b)$, and therefore

$$MT(M) = MT(\mathcal{M}_1^\vee). \tag{3.6.1}$$

Since M is not isotropic, α is not a root of unity and so by [B01] 5.4 we have $W_{-2}\big(MT(\mathcal{M}_1^\vee)\big) = W_{-1}\big(MT(\mathcal{M}_1^\vee/[0 \longrightarrow B])\big) \cong \mathrm{H}_1(\mathbb{G}_m, \mathbb{Q}) \neq 0$, which implies by (3.6.1) that M is not deficient.

3.7. Theorem. — *Let $M = (X, X^\vee, A, A^\vee, v, v^\vee, \psi)$ be a 1-motive defined over $\mathbb{C}$. The following conditions are equivalent:*

(i) *M is isotropic,*

(ii) *M is deficient,*

(iii) *M is trivially deficient.*

Proof. — Using the same notations as in 1.6, by Theorem 1.7 we have $MT(M) = MT(\oplus_{i=1}^{\mathrm{rank}X} \oplus_{j=1}^{\mathrm{rank}X^\vee} M_{ij})$ where $M_{ij} = [x_i\mathbb{Z} \xrightarrow{u_{ij}} G_j]$. There exists the injection

$$MT(M) \longrightarrow \oplus_{i=1}^{\mathrm{rk}X} \oplus_{j=1}^{\mathrm{rk}X^\vee} MT(M_{ij}), \tag{3.7.1}$$

which respects the filtration $W_\bullet$.

(i) $\Longleftrightarrow$ (ii): By (3.7.1) if all the M_{ij} are deficient then so is M. Moreover, according to 3.6 the 1-motive M_{ij} is deficient if and only if it is isotropic. Hence we can conclude that: M is deficient if and only if M_{ij} is deficient for each i, j, if and only if M_{ij} is isotropic for each i, j, if and only if M is isotropic.

(i) $\Longrightarrow$ (iii): According to 3.2 for each i, j there exists a 1-motive $\mathcal{M}_{ij}$ such that $MT(M_{ij}) = MT(\mathcal{M}_{ij})$ and

$$\mathcal{M}_{ij} = [\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}} B_{ij}] \oplus [0 \xrightarrow{0} G_{b_{ij}}]$$

where B_{ij} is an isotropic abelian subvariety of $A \times A^\vee$ containing a sufficiently large multiple b_{ij} of $(v_{ij}(x_i), v^\vee_{ij}(x^\vee_j))$, $G_{b_{ij}}$ is the extension of $B^\vee_{ij}$ by $\mathbb{G}_m$ parametrized by $b_{ij} \in B_{ij}$ and $\mathcal{U}_{ij}(1) = b_{ij}$. Since $[0 \xrightarrow{0} G_{b_{ij}}]^\vee = [\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}} B_{ij}]$ we have

$$MT(M_{ij}) = MT\big([\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}} B_{ij}]\big)$$

and so using (3.7.1) we obtain the injection

$$MT(M) \xrightarrow{\phi} \oplus_{i,j} MT\big([\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}} B_{ij}]\big).$$

By hypothesis we already know that $W_{-2}(MT(M)) = 0$. In order to conclude it is enough to prove that $W_{-1}\big(MT(M)\big) \subseteq U^{(4)}\big(MT(M)\big)$. Let g be an element of $W_{-1}\big(MT(M)\big)$. Since

$$W_{-1}\big(\oplus_{i,j} MT([\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}} B_{ij}])\big) = U^{(4)}\big(\oplus_{i,j} MT([\mathbb{Z} \xrightarrow{\mathcal{U}_{ij}} B_{ij}])\big),$$

we have $\phi(g)_{|\oplus_{i,j} W_{-1}(T_{\mathbb{Q}}([\mathbb{Z} \longrightarrow B_{ij}]))} = 0$ and therefore $g_{|W_{-1}(T_{\mathbb{Q}}(M))} = 0$. So we can conclude that $W_{-1}\big(MT(M)\big) \subseteq U^{(4)}\big(MT(M)\big)$.

Remark. — The direct proof that (ii) implies (i) answers a question of Y. André (cf. [Be98] Thm 1 Remark (iv)).

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