Journal of the History of Economic Thought,
Volume 40, Number 1, March 2018

ECONOMIC DYNAMICS AND THE CALCULUS OF VARIATIONS IN THE INTERWAR PERIOD

BY

MARIO POMINI

Analogies with rational mechanics played a pivotal role in the search for formal models in economics. In the period between the two world wars, a small group of mathematical economists tried to extend this view from statics to dynamics. The main result was the extensive application of calculus of variations to obtain a dynamic representation of economic variables. This approach began with the contributions put forward by Griffith C. Evans, a mathematician who, in the first phase of his scientific career, published widely in economics. Evans's research was further developed by his student Charles Roos. At the international level, this dynamic approach found its main followers in Italy, within the Paretian tradition. During the 1930s, Luigi Amoroso, the leading exponent of the Paretian School, made major contributions, along with his student Giulio La Volpe, that anticipated the concept of temporary equilibrium. The analysis of the application of the calculus of variations to economic dynamics in the interwar period raises a set of questions on the application of mathematics designed to study mechanics and physics to economics.

I. INTRODUCTION

The study of economic dynamics became a relevant topic in the field of mathematical economics in the period between the two world wars. Great progress was made in relation to the theory of business cycles, reflecting the fact that economic dynamics has become a substantially independent subject of study (Tinbergen 1934). At the international level, different competing research programs arose in the 1930s (Kyun 1988; Boumans 2005, pp. 21–74), and at the end of this period Paul Samuelson's

Department of Economics and Management, University of Padua, Italy. I would like to thank two anonymous referees for helpful comments on an earlier draft. Email: mario.pomini@unipd.it.

ISSN 1053-8372 print; ISSN 1469-9656 online/18/0100057-79
doi:10.1017/S1053837217000116

approach emerged as the dominant perspective, as expressed in the final part of his *Foundations of Economic Analysis* (1947). The major historical contribution toward understanding the complex situation of this period continues to be E. Roy Weintraub's book *Stabilizing Dynamics* (1991), in which he focuses his historical reconstruction on the literature associated with the approach pursued by Samuelson.

In this paper, my aim is to consider a different, more mathematically oriented approach to economic dynamics, which flourished in the same period but moved in a different direction. This approach began with the contributions put forward by Griffith Conrad Evans, a mathematician who, in the first phase of his scientific career, published widely in economics (1924, 1925, and 1930). His *Mathematical Introduction to Economics* (1930) can be viewed as one of the most original contributions to the field of mathematical economics in the interwar period. Evans's research was further developed in the United States by his student Charles Roos, who, in a remarkable series of papers, tried to develop the ideas of his teacher (1925, 1927, 1928, and 1934). At the international level, this dynamic approach found its main followers in Italy, within the Paretian tradition. During the 1930s, Luigi Amoroso (1938, 1940, and 1942), the main exponent of the Paretian School and a well-known scholar among the small community of mathematical economists, made major contributions, along with his student Giulio La Volpe (1936 and 1938).

This approach has two main distinguishing features. From the analytical point of view, the most important innovation proposed by Evans and the others is represented by the application of the calculus of variations. This analytical component derived directly from the analogy with rational mechanics. In a period in which the mathematical backgrounds of economists were modest, it is not surprising that the greatest achievements came from mathematicians or from economists with a sound background in natural sciences. From the economic point of view, these economists shared the view of dynamic economics as an equilibrium phenomenon, in the sense that the movements of the economic variables were viewed as a result of optimal choices over time. For this reason, we can define this approach as an attempt to extend the notion of equilibrium from the static framework to the dynamic one.

To complete the panorama that emerged in the interwar period, there are other limited cases. The calculus of variation was used initially by Jan Tinbergen in his doctoral thesis in 1929, prepared under the supervision of his teacher, the professor of theoretical physics Paul Ehrenfest. In this early work, Tinbergen tried to bring out the analogies between economics and physics. He was mainly interested in cyclic problems, referring to adaptation of the supply to seasonal cycles and business cycle (Boumans 1993). In the second part of the 1930s, another relevant economist who tried to extend dynamic analysis throughout the variational calculus was Gerard Tintner (1937, 1938). In particular, he introduced the calculus of variation to extend Roos's approach to the utility function (Duarte 2016).

In this paper, I consider the main results put forward by this analytical approach to dynamics based on calculus of variations and explore some reasons why these economists have been marginalized in the period that followed the Second World War. The sudden decline of this topic, in spite of the relevant analytical achievements, deserves consideration because two decades later, in the 1960s, functional calculus in the new format of optimal control became a milestone for economic dynamics (Wulwick 1995).

II. AT THE ORIGIN OF THE IDEA OF DYNAMIC EQUILIBRIUM: PARETO'S CONTRIBUTION

We can find the source of the idea of dynamics as an equilibrium process directly in the works of Vilfredo Pareto. He introduces the concept of equilibrium in the field of dynamics in paragraphs 586 and 928 of his *Corso* ([1896–97] 1941), in the chapter entitled "Principi generali dell'evoluzione sociale," which is devoted mainly to the study of the equilibrium of the society and the causes of its changes, in sociological terms. Pareto observes that society is never in a state of rest, but "is driven by a general movement which slowly changes it. Such movement is generally designated by the term 'evolution'. In mechanics, the d'Alembert's principle allows us to study the dynamic state of a system completely. In political economy, for the time being, we can only catch a glimpse of a similar principle" (Pareto [1896–97] 1941, p. 642).

What Pareto has in mind in these few pages is the direct extension of the mechanical analogy from the static to the dynamic case, even though he immediately realized that this would be a difficult task. Since the static is followed by the dynamic in rational mechanics, Pareto wonders whether this extension might not also be possible in the case of economics. This accounts for his recourse to the idea of dynamic equilibrium, a concept that is applicable to rational mechanics, in which the equilibrium is expressed in analytical terms by d'Alembert's principle. In mechanics, d'Alembert's principle adds the new force of inertia (defined as the negative of the product of mass times acceleration) to the acting forces to produce equilibrium. This means that the equations for the motion of any mechanical system may be derived in the form of equilibrium equations of force, exactly as in the static case (Amoroso 1921, p. 465). Indeed, using this method, dynamics is reduced to statics (Donzelli 1997). As an engineer, Pareto enquired as to what form this principle might take in economics with respect to consumption, and he sketched the following equation in a note:

$$(\phi_a - \frac{\delta x_a}{\delta r_a})\delta r_a + (\phi_b - \frac{\delta x_b}{\delta r_b})\delta r_b + ... = 0 \qquad (1)$$

where the term φ_i denotes the marginal utility of a single good, which is the active force in mechanical terms, while $\delta x_i / \delta r_i$ are hypothetical inertial forces relative to consumption, i.e., the cost of change in the level of consumption. The equation [1] is nothing other than a formal transposition of d'Alembert's principle in economics, where the path over time in consumption derives from the equilibrium between marginal utility (tastes) and resistance to change in the other differential component (obstacles), exactly as in the static case.

Having introduced the analogy, Pareto went no further; however, as an economist devoted to experimental observations, he regarded the equation [1] as being of little use (Boianovsky and Tarascio 1998). In fact, Pareto pointed out that the form of the relationship that expresses relations that indicate the hypothetical inertia ($\delta x_a / \delta r_a$) must be known. In addition, if we do not know how these differential equations are made, we will certainly not be able to contemplate obtaining the system's dynamic trajectory. Experience is essential in rational mechanics, where φ_i represents the impressed forces, while $\delta x_a / \delta r_a$ are the forces of inertia, $(\delta x_a / \delta r_a) = m(d^2 r_a / dt^2)$, but in economics this was still not possible. In Pareto's view, not every type of mathematical formalization was significant for economics, only the type that led to well-defined functional forms based on empirical observations. Pareto had no predilection for formal rigor unless it furnished a good understanding of real phenomena (McLure 2001, ch. 3).

Having discarded the mechanical analogy, Pareto himself moved in a completely different direction in order to build equilibrium dynamics. This second and final attempt to offer an analytical basis to economic dynamics is contained in the brief appendix to his 1901 article, "Le Nuove Teorie Economiche," entitled "Le equazioni dell'equilibrio dinamico" (The equations of dynamic equilibrium). Here, Pareto adopts a more traditional strategy, in both interpretative and analytical terms. The analytical expedient used by Pareto consists in assuming that all the economic transactions are referred not to a single instant of time but to a generic time interval *dt*, of arbitrary amplitude. For the remaining part, the formal scheme of general equilibrium does not change even if it is no longer possible to maintain the division of the static equations into three main groups: exchange, production, and capitalization. In dynamic analysis, the phenomenon of consumption cannot be separated from that of production throughout the process of saving. Apart from this formal modification (the introduction of a new variable, *dt*), the system of general economic equilibrium with its different systems of equations remains similar to what has been found in statics, with only one difference. This is that the solutions will not be values, but functions that are exogenous and dependent upon time. Indeed, the only dynamic variable in the new Paretian model is saving, which in fact is represented as a first derivative. However, this solution also seems to lead to more problems than it actually solves. The note concludes with the following observation that Pareto made:

> We have thus obtained the equations of dynamic equilibrium. Now it would be easy to work out the ones which refer to at least the main oscillations. However, if we want more particulars from our equations, they will become more complicated, which they are already enormously. Therefore, for the moment, a totally different road needs to be pursued: instead of complicating the equations, we need to find a way to simplify them, even at the cost of sacrificing many particulars of the phenomenon.
>
> In the system of equations exposed, there is a system of simultaneous differential equations, and in general it will be impossible to integrate them except by approximation. And this is also the only method that can be pursued to resolve the equations of the system. (Pareto 1901, p. 2)

This second approach to the dynamics equilibria received little attention among the Paretians, who preferred to develop the approach that drew analogies with rational mechanics.[1] The first attempt to apply the indications put forward by Pareto was advanced by one of Maffeo Pantaleoni's young students, Amoroso, in a brief note published in the *Atti della Accademia Reale dei Lincei* (1912), "Contributo alla teoria matematica della dinamica economica." Amoroso graduated in mathematics, followed Pareto, and tried to derive the equations of economic motion by directly applying D'Alembert's principle. In his brief contribution, Amoroso proposed a new set of equations to dynamize the static scheme of general equilibrium:

[1]The only relevant exception was the case of Sensini, who considered this aspect in his *Corso di Economia Pura,* published in 1955.

$$m\ddot{x}_i - \Theta_i = \frac{\partial U}{\partial x_i} - \lambda p_i \qquad i = 1,\ldots..,n$$
$$\sum\nolimits_{i=1}^{n} p_s x_s = \Gamma(t) \tag{2}$$

In the system [2], the second equation represents the usual budget constraint with the difference that now the income is not constant, but dependent over time. The first set of equations is the direct transposition of D'Alembert's principle to the realm of economics. The economic agents are treated as bodies moving under the action of mechanical forces. On the right side, we find the first-order condition that characterizes the optimal choice of consumption. On the left side, Amoroso introduces D'Alembert's principle, according to which the system of forces in motion, $m\ddot{x}_i$, is compensated by the inertia force, Θ_i. With the system represented by the equation [2], the analogy between economics and analytical mechanics in the field of dynamics is complete, from the mathematical aspect. But the young mathematician also expresses some doubts about this procedure, observing that in order to build economic dynamics we need two additional elements. The first is to consider utility function and inertia function as formal instruments that measure the intensity of forces and not only as indices of force, as in the Paretian theory. The second is even more problematic because Amoroso sets out to derive the form of these functions directly from empirical observations. According to the young Amoroso, the analogies with analytical mechanics are essential in terms of mathematical perspective but problematic from the economic point of view. Economic reasoning requires an interpretative content that might serve for the explanation of real phenomena.

Amoroso returned to the dynamics in the final part of *Lezioni di economia matematica* (1921). This book includes the lectures given at the University of Bari, ending with a long paragraph (§ 66) dedicated to the relationships between statics and dynamics in economics. Amoroso's view reinforced Pareto's so clearly that the emerging discipline of mathematical economics had to be built on the basis of rational mechanics, but any attempt to introduce a formal model had to be abandoned. In the years that followed, Amoroso turned away from the field of economic dynamics and concentrated his interest on other topics, such as the diffusion of general equilibrium theory in the Italian context, or analysis of some crucial analytical aspect of the static theory, as in his 1928 paper about the problem of the existence of the equilibrium, which attracted Joseph Schumpeter's attention. Amoroso returned to this project to dynamize the general equilibrium only at the end of the 1930s.

In the 1920s, the view on dynamic equilibrium was divided between two strands of analysis: one that was statistically oriented; the other, mathematical. The first was developed by Henry Ludwell Moore, an American follower of Léon Walras, whom he met on a trip in Europe in the summer of 1903 and with whom he maintained a regular correspondence (Raybout 2013). Moore sought to dynamize the general equilibrium equations by means of statistics. His 1929 *Synthetic Economics* received great acclaim, but this line of inquiry found few proponents because it was regarded as problematic from the theoretical point of view (Mordecai 1930). By contrast, the mathematical strand underwent greater development. In the United States, it had been developed a few years earlier with the studies by Griffith Evans (1924 and 1930) and his student Charles Roos (1925, 1927, and 1934). Evans subsequently returned to his interest in pure mathematics, while Roos became primarily interested in statistical applications of demand theory. In Italy, dynamic equilibrium attracted interest at the Paretian School in the 1930s. Almost all of the Paretians made contributions in this field, following different approaches. These two lines of inquiry were closely interconnected, as we will see, and not only from the scientific point of view.

III. EVANS AND THE BIRTH OF ECONOMIC DYNAMICS

In spite of the important contributions made by Weintraub (1998 and 2002), there is probably much that remains to be considered with respect to Evans's role in the birth of economic dynamics. Evans was a pure mathematician who dealt with economics as a secondary research field during the first phase of his outstanding academic career. His articles on economics were published in journals of mathematics, rather than in those of economics. After the 1930s, Evans abandoned economic theory and returned to his studies in physics and mathematics.

His initial interest in theoretical physics, and especially in potential theory, led to a doctorate at the Rice Institute (Texas) with a PhD thesis on Vito Volterra's integral equations. Because completion of a doctoral program at that time required a period of study in Europe, Evans obtained a scholarship to continue his studies on applied mathematics in Rome, working under Volterra himself in the period from 1910 to 1912. He was subsequently offered a position at the Rice Institute and remained until 1932, when he was appointed to a chair at Berkeley and given the task of reorganizing the Department of Mathematics. Evans built his reputation among American mathematicians mainly as an expert in the field of functional calculus (Morrey 1983).

Evans and Volterra became friends and remained in contact with each other for many years, as is revealed by their regular correspondence. However, it is not clear whether Evans's decision to pay attention to mathematical economics was influenced by the great Italian mathematician. In his letters, Evans discusses only the problems of mathematical physics, in particular those relating to the integration of functional equations. There are two circumstances in which he refers to mathematical economics. In a letter[2] written in 1920, Evans informs Volterra that he is working also in the field of mathematical economics, preparing a communication for the annual meeting of the Mathematical Society. This first contribution would be published in 1922 in the *American Mathematical Monthly*. A second, more relevant, reference can be found in a letter of 1925 in which Evans tells Volterra about the project he is working on to write a text on mathematical economics that would be similar to Amoroso's book *Lezioni di economia matematica*. In the summer of the same year, he was invited to Chicago by the Department of Mathematics to give two courses: one on the integral functions and the other one on mathematical economics. Evans's book was ready, and it was published in 1930 under the title *Mathematical Introduction to Economics*, and in the preface he quotes the main mathematical texts of the Italian tradition, Pareto's

[2]The letters considered are included in the "Vito Volterra" Archive in the Accademia Nazionale dei Lincei, Rome.

Manuale in the French translation, Amaroso's *Lezioni*, and Alfonso De Pietri Tonelli's *Trattato di economia sperimentale*. We can speculate that the influence that Volterra had on Evans in the field of mathematical economics was mainly methodological. In economics, scientific models must be based directly on the underlying reality, a reality directly confirmed through experimentation and observation. This was a traditional attitude that was entirely different from that of the search for axiomatization that would prevail in economics in the postwar period (Weintraub 2002).

Evans's approach to economics came about in stages. His first article, "A Simple Theory of Competition," was published in 1922, and it contains a static analysis of the oligopolistic market. Evans takes for granted the existence of a well-defined quadratic cost function, $TC_i = Aq_i^2 + Bq_i + c$, and a well-defined market demand function, $q = ap + b$. He addresses the traditional problem regarding the determination of the price and quantity of equilibrium with the change in the number of firms. In this paper, Evans reveals his methodology, applying in economics the same framework used in classical mathematical physics. He starts with specific functional forms in order to obtain results that could be used for interpreting economic reality. He concludes the article with the following considerations:

> In a general system of economics we cannot for a proper discussion restrict ourselves to functions of a single variable. Mathematically this is not so essential a modification as it has been sometimes regarded. An extension of the problem which goes deeper is what is obtained when we remember that what a producer is interested in is not to make his momentary profit a maximum, but his total profit over a period of time of considerable extent, with reference to cost functions which are themselves changing as a whole with respect to time, such as in the instance discussed in the previous section. The mathematical discipline which enables us to find functions which make a maximum or a minimum quantities which depend upon them throughout periods of time is the calculus of functionals, or in special cases the calculus of variations. But the quantity which we want to make a maximum over a period of time need not be the total profit; it may be the total production, or whatever other quantity we wish to take as a desirable characteristic of the social system we discuss. The author regrets that at the present time he can refer only to his lecture courses for a further treatment of this point of view. Nevertheless it seems the most fruitful way that a really theoretical economics may be developed. (Evans 1922, pp. 380–381)

Evans used the same direct approach in his first contribution to dynamic economics as he did in his 1924 article, "The Dynamics of Monopoly," which was also published in the *American Mathematical Monthly*. In a period of strong price turbulence like that of the 1920s, Evans was the first, as noted by Tinbergen (1933), to consider a demand function that depended linearly on price variation ($\dot{p}$):

$$q = ap + b\dot{p} + c \tag{3}$$

In this new equation, the term $\dot{p}$ can be seen as introducing a speculative factor, where $a < 0$ and $b > 0$. The monopolist's problem, using a quadratic cost function, becomes that of maximizing the flow of profits throughout a given interval of time by choosing the optimal level of $p(t)$ as a function of time. In a note, Evans clarifies that he derived this approach to dynamics from Amoroso's 1921 book (Evans 1924, p. 78).

After the substitutions, the expression of profits depends at each instant of time on two magnitudes, the price level and its variation:

$$\int_{t_0}^{t_1} \pi_i(p, \dot{p})dt \tag{4}$$

Evans observes that the new equation [4] does not depend explicitly on time, and its maximization is a problem that extends from elementary mathematics and requires the more advanced calculus of variations. This approach was also too advanced for the mathematicians, as Evans observed: "An Editor of the Monthly had said that one should be obliged to present a certificate on character before being initiated into the mysteries of the Calculus of Variations to which study our present investigations belongs, since its fascination is so great that neophytes seek to introduce it into problems which would otherwise be perfectly simple" (Evans 1924, p. 79). The direct application of the Euler equation leads to a second-order differential equation that can be solved in this simple case, to arrive at the following general dynamic equation:

$$p(t) = \overline{p} + C_1 e^{mt} + C_2 e^{-mt} \tag{5}$$

where the two constants, C_1,C_2, are determined by means of the boundary conditions, $p_0(t_0)$, $p(t_1)$.

The Evans equation [5] can be considered the first explicit formulation of the idea of the dynamic equilibrium of one economic variable, in this case the price, and we shall see similar ones later. The equilibrium solution obtained in this way is dynamic in two respects. It is dynamic in descriptive terms, because the price is an exogenous function of time. The new result obtained is that the solution is not a single value as in the static theory, but a path in the given interval. The equilibrium solution is also dynamic in a normative respect, because the solution derives from an optimization process where $p(t)$ denotes the optimal trajectory of price. With Evans, the economic problem of defining the optimal path of prices becomes the mathematical problem of finding a function able to maximize the expression [4]. Evans's 1924 article contains his crucial contribution to the birth of dynamic economics in the modern sense. Subsequent minor studies (1925 and 1929) were collected in his 1930 book, *Mathematical Introduction to Economics*, but these did not bring any new results to the field of dynamic theory.

With his brief article of 1924, Evans paved the way for the mathematical theory of dynamic equilibrium. As Harold Hotelling (Hotelling 1931) notes in his review, an economic problem in the field of dynamics had for the first time been analyzed with the appropriate mathematical tools, which were highly advanced not only for the economists of the time but also for the mathematicians. The calculus of variations provided the instrument that was appropriate for the purpose of building economic dynamics. It was possible to go beyond the static state because the solution of equilibrium was itself a function of time. In the years that followed, Charles Roos, Evans's student at the Rice Institute, extended Evans's approach in this direction beyond the monopolistic case to make general equilibrium truly dynamic.

IV. ROOS AND THE DYNAMICS OF GENERAL EQUILIBRIUM

When he began his research, Roos was more of a mathematician than an economist, and subsequently he was attracted by the empirical research too. Roos also had greater influence than Evans within the community of economists (Dimand and Veloce 2007). Unlike Evans, he did not choose a university career, and his name is associated with being the first scientific director of the Cowles Commission and one of the founders of the Econometric Society (Louçã 2007). In 1937, Roos established the Econometric Institute in New York, of which he was director until his death in 1957. His writings on pure economic theory belong to the first part of his professional career.

Roos's first important contributions (1925 and 1927) constitute an extension in many directions of the line of inquiry pursued by Evans. These articles appeared mainly in mathematics journals, and this testifies to their dense analytical content. The first, "Mathematical Theory of Competition" (1925), appeared in the *American Journal of Mathematics*. In this article, Roos extends the analysis developed by Evans in the previous year to the case of several firms, using the same analytical scheme. Evans's dynamics of monopoly became Roos's dynamics of Cournot oligopoly. The main contribution that Roos made to economic dynamics is in extending Evans's perspective to more general cases. Roos's fundamental contribution is an article that was published in the *Journal of Political Economy*, entitled "A Dynamical Theory of Economics" (1927). In this article, he summarizes the main results of his mathematical research and tries to offer a way to dynamize the general economic equilibrium. The purpose of this article, as evidenced by its title, is ambitious, because Roos intended to construct a dynamic scheme for the general economic equilibrium and, therefore, to succeed in a project in which both Walras and Pareto had failed.

The article is divided into three parts, the first of which summarizes earlier mathematical results for the audience of economists. Roos considers a case in which the total demand function by the n producers could be written as

$$G = q_1 + q_2 + \dots q_n = G(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n, \dot{p}, p, t) \tag{6}$$

This equation contains many variables that relate in particular to the quantity produced, to the rate of change in prices, and to the production. The cost function will also be a function of the rate of production, the price, and the acceleration of production:

$$TC_k = \Theta(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n, \dot{p}, p, t) \tag{7}$$

Inasmuch as the price and rates of production have been chosen as functions of the time, the total net profit of each firm throughout the period considered is given by the following integral:

$$\pi_k = \int_{t_0}^{t_1} (pq_k - TC_k)dt = \int_{t_0}^{t_1} F_k(q_1, \dot{q}_1, \dots, q_n, \dot{q}_n, p, \dot{p}, t)dt \qquad (k = 1, \dots, n) \tag{8}$$

The maximization of the equation [8] is not a standard problem: given the times, t_0 and t_1, the problem is that of choosing the single price, p, and the rates of production, $(q_1, \dots, q_n)$, for every producer, satisfying the demand equation such that the profit, π_k, of each producer is a maximum when the rate of production, q_n, alone is allowed to vary with the price, as in the Cournot case.

Roos observes that the problem of the maximization equation [8] requires the calculus of variations in the form of a generalized Lagrange type. He presents the mathematical details of this procedure in an article that was published the following year (Roos 1928). In 1927, he limited himself to presenting the solution that had to satisfy the following equations:

$$\frac{\partial F_k}{\partial q_k}+\frac{\partial F_k}{\partial \dot{p}}\frac{\partial \dot{G}}{\partial q_k}+\frac{\partial G}{\partial q_k}W_k-\frac{d}{dt}\left(\frac{\partial F_k}{\partial \dot{q}_k}+\frac{\partial F_k}{\partial \dot{p}}\frac{\partial \dot{G}_p}{\partial \dot{q}_k}+\frac{\partial \dot{G}_P}{\partial \dot{q}_k}W_k\right)=0 \qquad k=1,\ldots,n \tag{9}$$

where W_k is in an integral expression. The equations [9] are functional equations for the determination of $q_1, q_2, \ldots, q_2$ and p in terms of the time. Roos concludes that the problem is completely determined because we have given an initial condition and the n functional equations plus the differential equation of demand to determine $n+1$ functions $q_1, q_2, \ldots, q_2$ and p. In the simplest case, the system [9] is reduced to a system of Volterra integral equations that can be solved if the demand function assumes the linear form proposed by Evans.

In the second part, Roos considers the case in which the demand depends upon the history of prices as well as on the present price. Under this assumption, the demand function assumes the form of the Volterra integral equation:

$$q(t)=ap(t)+b+\int_{-\infty}^{t}\phi(t-\tau)p(\tau)d\tau \tag{10}$$

The meaning of the equation [10] is as follows: the quantity demanded, $q(t)$, depends on the current price, $p(t)$, but also on all past prices, weighted according to a function that decreases in time, $\phi(t-\tau)$. In addition, in this more complicated case, the structure of the mathematical problem is the same, and the solutions are expressed by the equations [9] (Roos 1927).

The final part of this work clarifies Roos's vision of a dynamical theory of general equilibrium. He writes:

> We have already discussed the problem of competition and monopoly from a modified Cournot point of view and obtained conditions which must be satisfied in order to insure the maximum profit for the producer over an interval of time. Let us replace the static general equilibrium of Walras and Pareto by a dynamic one in an attempt to show the relationship existing between the problem of competition and the theory of economic equilibrium. In developing this new theory, we shall show that the theory of Pareto is incomplete in several respects and endeavor to complete this theory. (Roos 1927, p. 647)

The general scheme advanced by Roos is the Paretian and Walrasian one, in which there are now m commodities and n firms. The quantities consumed and supplied depend on all prices and on the rates of production, as in the static case. The novelty introduced by Roos is that to the usual group of static equations descriptive of a general equilibrium, he adds n equations of profit [8], one for each commodity. With this assumption, the problem of general equilibrium in the economic system is reduced to a case of the calculus of variations, and it is possible, albeit only in principle, to determine the rate of production, $q_n(t)$, and the path of prices, $p_n(t)$, depending on time as an exogenous variable. Using the concept of maximum profit, Roos was able to pass from the dynamics of monopoly to the dynamics of general equilibrium, realizing the Paretian project that was advanced in the 1901 article. Roos concludes: "The time variable has been introduced directly into the equation defining general equilibrium, and it is shown that a dynamic equilibrium exists. If we added the equation $t = t_0$, the equations defining dynamic equilibrium become equations defining static equilibrium at the time $t = t_0$" (Roos 1927, p. 655).

Obtaining an operational result from this purely mathematical construction would require defining the form of all the functions introduced. This is what Roos would do in his 1930 essay on the theory of the business cycle.

In the 1930s, Roos tried to add empirical relevance to his project to build economic dynamics. He had already, in the article that has been mentioned, posed the question of how to find some possible form for the dynamic demand equation that is coherent with the statistical data. This issue lies at the center of *Dynamic Economics*, the book that opened the monographs of the Cowles Commission published in 1934. The book was structured as follows. In the second chapter, Roos notes the relevance of the Volterra-type equation for the analysis of the variation of the economic magnitudes through the time. In the remaining chapters, we find the application of this theoretical structure to different demand markets such as the gasoline market or agricultural products. A long chapter is dedicated to the factor that influences residential building. Roos considers different time series ranging from 1900 to 1940 and tries to apply his theoretical formula based on many factors affecting the length of the time required to act. In the case of building, the crucial factor is represented by credit. The result of this analysis was in general a dynamic patter based on distributed lag.

Roos's writings in the second half of the 1920s can be considered to be the most ambitious attempt to approach the problem of dynamizing the general equilibrium theory. In the years that followed, the quest for a more realistic treatment of economic dynamics led him in another direction. Criticizing the practice of de-trending (Roos 1934, ch. 1), he attempted to provide a causal explanation of business-cycle phenomena using the Volterra integral equations as a theoretical tool.

V. DYNAMIC EQUILIBRIUM IN THE PARETIAN TRADITION: GENERAL ASPECTS

The theory of dynamic equilibrium was the main research topic in Italy during the 1930s, and the Paretian School made significant contributions to it in that decade. All the members, and particularly Amoroso, Giulio La Volpe, Eraldo Fossati, Felice Vinci, and Giuseppe Palomba, made important contributions, although they had followed different routes (Pomini and Tusset 2009). For my purposes here, I shall restrict the treatment to Amoroso and La Volpe, not only because they probably made the most significant contributions to the dynamics, but also because they used functional calculus drawn from the American mathematicians. Palomba used Lotka–Volterra equations (Gandolfo 2008), while Fossati proposed a dynamic theory based on uncertainty, using traditional calculus (Fossati 1937).

It is important to point out that the Italian approach to the building of dynamic equilibrium has features that are different from the American ones. We have seen that Evans and Roos always started from the cost and demand functions. La Volpe and Amoroso moved from a different perspective. They emphasized the economic reasoning behind the supply and demand. Considering this point of view, we can say that the Italian scholars differed from their American counterparts because they regarded themselves primarily as economists. In modern terms, we can see that Italian economists tried to obtain a microeconomic foundation of dynamic equilibrium. To achieve that, Amoroso and La Volpe extended the static utility function to include the time element essentially by following the Austrian tradition, as shown, for instance, by Paul Rosenstein-Rodan in his 1934 article, "The Role of Time in Economic Theory." Using this approach, the economic agent plans the action within a given interval of time, which may also coincide in the case of the consumer with its lifetime. In analytical terms, this requires introducing an intertemporal utility function in relation to the quantity consumed throughout the various periods into which the period considered can be divided. The rational consumer will seek to maximize the new expression as follows:

$$U(c,t)=\int_{t_0}^{t_1} U(c)\,dt \tag{11}$$

and an analogous function characterizes the behavior of the firm. The equation [11] states that in an intertemporal context, the rational agent considers the sum of the utilities that are achievable in the period that is being considered. To make this expression analytically treatable, both La Volpe and Amoroso hypothesize that utility is separable in time, an assumption that would recur in all the subsequent literature. Hence, while Evans and Roos used only the profit function, the Italian economists concentrated on a problem that is much more general and intuitive from the economic point of view: that of the agent's intertemporal choice. The next problem was to characterize the structure of the intertemporal utility function. Amoroso opted for an analogy with rational mechanics, while La Volpe followed the Austrian view of the crucial role of expectations.

A second aspect, which is connected to the first one, concerns the type of mathematics used to solve the optimum problem. The Italian economists made no reference to the Volterra integral equation, whereas it was the core of Roos's theory. The Volterra equation is quite a complex and sophisticated mathematical tool, but it is also unnecessary when the analytical frame of reference is changed. Amoroso (1938, 1940) and La Volpe ([1936] 1993) solved the problem of intertemporal optimization in a simpler and more natural way by directly applying the techniques of the calculus of variations. In this setting, a new analytical problem was the characterization of the optimal choice when the planning horizon becomes infinite, a problem that was first addressed by Frank Ramsey (1928). In approaching this problem, La Volpe first anticipated a new (necessary) transversality condition that offered a rational economic interpretation as well. Amoroso also followed the path indicated by Ramsey, introducing a special kind of bliss point.

A third element that distinguishes the Italian tradition is that, from the outset, it focused on the general equilibrium. The frame of reference was always that of a multiplicity of operators interacting with each other through markets. The treatment then followed the standard procedure of constructing the equations relative to demand and supply and then the equilibrium between the two. The only concern was that the system should be determined; that is, the number of equations should be equal to that of the unknowns. The fact that the Italian economists conducted the analysis in terms of general equilibrium derived from their theoretical project, which was to dynamize the system of Walras and, above all, of Pareto.

In spite of these differences, the similarities between American and Italian economists should also be emphasized. In the 1930s, Evans, Roos, and Amoroso were part of the small international group of mathematical economists (Fisher 1930). The articles that Evans and Roos wrote on dynamics were well known to the Italian mathematical economists, and from this point of view, the Italian economists were more open to international debate than many authors would admit. More importantly, they shared the same research program in building economic dynamics on the idea of dynamic equilibrium.

VI. THE TEMPORARY EQUILIBRIUM OF LA VOLPE

In order to obtain the dynamic equilibrium equations, it is necessary to offer some analytical structure to the intertemporal utility equation and then to apply the optimization techniques. Here, the intuition of the economist comes into play. La Volpe and Amoroso gave different interpretations for the influence of the time element. La Volpe formulated a very modern view, which was subsequently taken up in the 1960s by the theory of the consumption life cycle, while Amoroso proposed a view based on the inertia of behavior. Following a chronological order, I shall first consider the view that La Volpe sets out in his 1936 book, *Studies on the Theory of General Dynamic Economic Equilibrium*, which was his main theoretical contribution and probably the most outstanding achievement by the Paretian School (Nicola 2000).

In his introduction, La Volpe underlines that his objective is to reach a microeconomic foundation of the theory of dynamic general equilibrium. In his own words:

> I have thus attempted to construct a micro-dynamic theory of general equilibrium showing how market equilibrium is established in every instant as is shown by the behavior of economic subjects, consumers and firms on the basis of expectations and plans for the future and the way in which this equilibrium changes continually over time by means of changes in individual plans. This is the salient point of the theory. Whilst it is certainly true that it is difficult to predict the future, impossible to avoid errors in projecting present market tendencies into the future and easy to overestimate one's own perspective evaluations in one way or another, it is equally true that, in this way, economic activities are regulated at every moment. (La Volpe [1936] 1993, p. 7)

The reasoning in *Studies* follows the usual Paretian scheme: the first part considers the problem of dynamic consumption; the second analyzes production; and the third examines compatibility between the agents' choices, that is, the equality between the number of equations and the number of unknowns. La Volpe reveals the influence of the Austrian school for which dynamic analysis was a multi-period analysis influenced by the expectations of economic agents (particularly Rosenstein-Rodan 1934). He assumes that current consumer choices are based on all lifetime-expected consumptions, taking into account the intertemporal budget constraint. The distinctive feature of La Volpe's scheme is that it also depends on the expectations formed at time t_0 and is valid for subsequent periods, represented by τ, with $\tau > t_0$. For this reason, he speaks of a "future utility evaluation function." The total utility in a given interval of time (t_0, τ) for the agent, *i,* is expressed as follows:

$$\int_{t_0}^{\tau} U_i(C_0(t_0,\nu),...,C_{h+m}(t_0,\nu))d\nu \quad (12)$$

considering a generic basket of goods and services $(C_0,...,C_{h+m})$ and given the expectations established over time, t_0. At each instant, consumers must make a choice to determine their optimal consumption patterns, taking into account the intertemporal budget constraints. These constraints consist of expectations concerning the quantity of labor and other services that consumers may offer in the future, the amount of profit that they will furnish with respect to their financial choices, and the planned flow of consumption. Using La Volpe's notation, the budget constraint becomes the following expression (the time suffixes have been omitted in order to avoid encumbering the notation):

$$\sum_{j=0}^{h} p_j(H_{ji} - C_{ji}) + rF_i + R_i - \dot{F}_i - \sum_{j=h+1}^{h+m} p_j C_{ji} = 0 \quad (13)$$

where the first term in the equation [13] represents the result of services (which are between 0 and h in number) purchased or furnished (H_j) in the time interval, including labor. The second term, rF, indicates the interest on accumulated savings, while the third, R, is the proportion of income from shareholding. There are two further terms: the variation in savings by unit of time, $\dot{F}$; and the final summation, which indicates spending for consumption (consumption goods range from h_{h+1} to h_{h+m}). La Volpe hypothesizes that, at this stage of the analysis, both current and expected prices, as well as both the current and expected interest rates, are constant, so that the unknowns are reduced to the determination of current and future consumption and of planned financing.

Maximization of the equation [12] under the intertemporal budget constraint is a specific problem that is resolved through the calculus of variations. La Volpe was not a mathematician, and to obtain these results he enlisted the help of two talented young Neapolitan mathematicians, Giulio Andreoli and Gianfranco Cimmino, who both worked in the field of functional calculus and whom he thanks in his preface. La Volpe obtained the following equation, which describes the optimal consumption path for each good or service:

$$\frac{U_{C_{ji}}}{p_j} = Ae^{-rt} \quad (14)$$

where the parameter A is a constant of integration that must be determined. The equation [14] establishes the fundamental achievement of La Volpe's approach to the determination of the consumer's dynamic equilibrium. He speaks of two laws of consumer equilibrium. The first states that, where t is given, consumption should be distributed in such a way as to match the marginal weighted utility rates, exactly as in the static case. The original feature is in the second law, according to which the distribution of maximum satisfaction over time requires that the marginal weighted utility rate should decrease in line with interest rates, taking into account the constant $A_,$ which depends on the initial conditions. The equation [14] is in fact a system of equations, because it must be related to goods and services, h, and the number, m, of consumers who, along with budget constraint and the transversality condition, make the system determined. La Volpe observes the following:

> This system of equations represents the dynamic equilibrium and the historical movement of consumers. In fact, given t, it supplies the conditions that must exist in the plans in order that they, at any given moment, set themselves for the future, and, in function of t, describes the movement over time of individual consumer savings, through the constant succession of economic plans. (La Volpe [1936] 1993, p. 24)

La Volpe's dynamic consumer equilibrium is, then, according to the definition given by John Hicks in *Value and Capital* (1939), an anticipation of the concept of temporary equilibrium because it is determined on the basis of future prices that the consumer forecasts on the market. If these vary, then the entire optimal consumption profile will also change instantly. For this reason, Micho Morishima (1993) has pointed out that La Volpe anticipated, by a number of years, Hicks's important attempt to extend schemes of economic equilibrium to encompass time through the influence exerted by expectations.

VII. A NEW TRANSVERSALITY CONDITION

There is an aspect to La Volpe's account of dynamic equilibrium that should be emphasized in particular, both from the mathematical viewpoint and from that of economic interpretation, because it represents an undeniable advance in the application of functional calculus to economics. The Euler equation is normally a second-order differential equation containing two arbitrary constants. In terms of the problem with fixed initial and terminal points, as outlined by Evans and Roos, the two given boundary conditions provide sufficient information. La Volpe adopts a different way to consider the final condition. He assumes that this condition holds the following:

$$F_i(t,\tau) = 0 \quad \textit{if} \quad t = \tau \tag{15}$$

This equation states that in the final instant of the period being considered, savings must be nil, so the individual must have used all the financial resources at his disposal. The equation [15] will later become the transversality condition in the case in which the terminal state tends to infinity. As a Paretian, La Volpe was concerned about the scant realism of this hypothesis, and to remedy this limitation of the model, he also considered the case of bequests, the "need to bestow," as he put it (La Volpe [1936] 1993, pp. 44–48).

The equation [15] is not only a technical condition for pinning down the optimal trajectory. La Volpe is aware of its relevant economic content, in that it allows the horizon to be extended from the single period to the entire life cycle of the individual. By considering the equation [13], integrating it, and taking account of the transversality condition, he obtains the following expression:

$$\int_{t_0}^{\tau} \sum_{j=0}^{h+m} p_j C_{ji} e^{-rt} = F(t_0) + \int_{t_0}^{\tau} (\sum_{j=0}^{h} p_j H_{ji} + R_i) e^{-rt} \qquad (16)$$

where, to simplify the notation, the interest rate is constant. In the equation [16], the left-hand term represents the current value of consumption, calculated at time t, while the right-hand term is the current value of the individual's total wealth. La Volpe's conclusion is that the current value of expenditure on all consumption expected at a given instant by individuals for their lifetimes must be equal to the wealth possessed at that instant. In his own words:

> It is evident that if an individual saves for a certain amount of time, he will need to spend in future, and if the individual spends more than he owns, he will need to save.
>
> All this does not exclude the eventuality of individuals who choose to accumulate assets for a longer or shorter period of time in order to spend the profits but this works in any case on the assumption that sooner or later even assets will be consumed.… It is thus by deferring the consumption of present resources or anticipating the future consumption of resources that maximum well-being tends to be achieved. But individuals never reach this goal because they are continually forced to change direction. (La Volpe [1936] 1993, p. 29)

In this passage, La Volpe has anticipated and grasped the deeper meaning of the necessary condition for economic consistency that must be respected in the process of intertemporal optimization. This condition was later taken up in Franco Modigliani's life-cycle theory, as well as in the literature in the 1960s on optimal growth, but no mention has ever been made of La Volpe's pioneering work in first identifying and using it.

Moving from consumption to production, La Volpe's analysis of firm dynamic equilibrium was close to that of the consumer and is rather encouraged because investment is in itself a dynamic element. Furthermore, every investment project is finalized with a certain delay, and this explains why marginal costs must become greater as the productive period grows longer. Applying the calculus of variations once again, La Volpe determines the dynamic functions of the supply of goods (La Volpe [1936] 1993, p. 56). Taking the dynamic functions of demand by consumers and the dynamic functions of supply by firms, market equilibrium requires that excess of demand on the markets is nil at all times. The equation system contains $(1+h+m+1)$ unknowns: the wage, the prices of h productive services, the prices of m goods, and the interest rates with a corresponding equal number of functional relations. For every t value, the solutions give the equilibrium at that moment in time and, in functions of t, the description of the historic evolution of an economy starting from a given initial point.

After the Second World War, La Volpe intensified his academic activities, but was unable to build on the achievements of his 1936 essay. He published a considerable amount of work on a great variety of themes ranging from banking theory and comparative costs to theory of the market forms (Pomini 2009). Furthermore, as head of study service, he directed a number of empirical research studies, the results of which are published in *Ricerche Economiche*, a review that he transformed from a straightforward statistical bulletin to a front-ranking economic theory review. La Volpe returned frequently to the theme that was dearest to him—the economic dynamics of general economic equilibrium—and put forward a new methodology that he called "variational dynamics" (La Volpe 1967 and 1977). These were mostly complicated input-output models with delayed variables and were designed to fill the gap between abstract theories and the needs of empirical research. As has been observed (Di Matteo 1998), La Volpe's reserved nature, a characteristic, for that matter, of many in the Paretian School, led him to a certain isolation, which meant that he never achieved the academic recognition that he undoubtedly deserved. This acknowledgment arrived only much later when the 1936 essay was finally published in English (1993) and the groundbreaking nature of his theoretical perspective emerged.

VIII. FROM THE BUSINESS CYCLE TOWARD THE DYNAMICS OF GENERAL EQUILIBRIUM

The 1930s were a time of intense research in which Amoroso was mainly interested in building economic dynamics. In this period, he pursued two different research paths. At the beginning of the decade, he proposed an interesting theory of the business cycle, thus intervening in the intense theoretical debate that the Great Depression had generated. The turning point was represented in the article of 1929, "Le equazioni differenziali della dinamica economica," in which Amoroso indicates that he was favorably impressed by the concept of Roos's differential equation of demand. The article begins in the following way:

> Several years ago, mathematical economics seemed at a dead end. After Walras's and Pareto's general arrangement, which collected all the ideas on the theory of equilibrium in a single synthesis, it seemed that the last word had been given, at least pro tempore, and that the new method had produced, for the moment, all the fruits of which it was capable. This did not mean that new mathematical problems were not at the horizon; but rather that their solution seemed to be beyond the strength of human intellect.… The idea was a transformation from the static theory to a dynamic theory, but the way towards this solution was not visible. The interference of the strictly economic phenomena with the political and social ones, on the other hand, in the dynamic field, seemed so strict that a strictly economic analysis seemed void of meaning. Economic dynamics seemed to get lost in the wider sea of Sociology and Politics. (Amoroso 1929, p. 68)

Amoroso's first works on economic dynamics focus on the theory of the economic cycle. Toward the end of the 1920s and during the first years of the 1930s, Amoroso tried to elaborate on a mathematical model of the economic cycle, exploiting the introduction of the differential equations of demand. These first contributions to dynamic analysis are summarized in the essay dated 1932, entitled "Contributo alla teoria matematica della dinamica economica." This essay covers a very wide range of topics, ranging from the dynamization of the offer curve, to a non-monetary theory of the economic cycle, up to the actual study of demographic dynamics. Breaking away from the main opinion of the Italian economists, who looked at the monetary disturbance, Amoroso proposed a real theory of the economic cycle, to which other followers of

Pareto also turned. Another stage in the construction of Amoroso's dynamic theory is represented by the article dated 1933, entitled "La dinamica dell'impresa." This is a short technical contribution that examines the problem of the maximum of a monopolist when there are adjustment costs. The article, a note of a few pages, is important because for the first time Amoroso uses the calculus of variations.

It is during the second half of the decade that Amoroso returns to the project of his earlier years, that of dynamizing the equations of general economic equilibrium. The first important contribution is an article dated 1938, entitled "La teoria matematica del programma economico," followed shortly afterwards by an article published in *Econometrica* (1940), "The Transformation of Value in the Productive Process." His view of economic dynamics will find full expression in the text, dated 1942, "Meccanica Economica," which gathers the lectures held at Alto Istituto di Matematica in Rome.

In Amoroso's approach, the novelty in the consumption theory is represented by the explicit introduction of a principle that is analogous to that of inertia in mechanics, according to the project manifested in 1921. In general, as observed by Amoroso, the quantity of a specific good, c_i, that is consumed in a certain temporal interval is itself a function of time and represents the velocity of the flow in consumption. In turn, its derivative with respect to time, $\dot{c}_i$, can be interpreted as the acceleration, assessed algebraically in value and sign, of this flow of consumption. For Amoroso, if we want to give a realistic description of the consumer's behavior, it is necessary to take into consideration the role of habits and the psychological resistances that can accelerate or slacken the variation of the expenditure for consumption, the term $\dot{c}_i$. In formal terms, the utility function becomes as follows:

$$U = U(c_i, \dot{c}_i) \tag{17}$$

Amoroso gives the equation [17] the name "Lagrange's ophelimity" to distinguish it from that of Pareto. In terms of construction, Lagrange's ophelimity can be considered as an extension of Pareto's theory because if the resistances to change are null, $\dot{c}_i = 0$, the two functions of utility coincide, and there is the return to the static case.

The rational consumer will try to determine the maximum value of the trajectory of consumption represented by the equation [17] in a fixed temporal interval. From a mathematical viewpoint, the problem is clearly determined, since it means finding the optimal trajectory of the following functional:

$$\int_{t_0}^{t_1} U(c_i, \dot{c}_i)\,dt \tag{18}$$

We are in the presence of a typical problem of calculus of variations, whose necessary conditions, which are also sufficient for the type of equations that Amoroso has in mind, are represented by Euler's equation:

$$\frac{\partial U}{\partial c_i} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{c}_i}\right) \tag{19}$$

Amoroso calls this expression "Lagrange's marginal utility." The first term of the equation [19] represents the marginal utility in a traditional sense, while the second, which captures the dynamic element, constitutes the loss of utility originated by the presence of habits (inertia of the system), and in the final one, the costs are associated with changes. Should the consumer have to divide the income among more goods on the basis of the income, this condition of equilibrium implies that the total utility becomes maximum when the value of the Lagrange marginal utilities that are considered are the same as in the static case.

Basically, for Amoroso, the theory of the dynamization of consumption arises through the extension of the marginal principle, introducing a relation that expresses the psychological resistances to change. There is full analogy with rational mechanics, expressed not only at an interpretative level but even more so on the analytical level, since, as in the equation [19], the consumer's dynamic behavior is traced back to a second-order differential equation, exactly as in the case of the motion equations of analytical mechanics. Some problems can arise from a mathematical perspective because the system is composed of a large number of differential equations whose solutions are not always possible, especially in some special cases. However, from a theoretical viewpoint, nothing hinders the ability to obtain the usual functions of demand, as Amoroso himself demonstrates with the particular specifications concerning the form of the intertemporal function of utility (Amoroso 1942, pp. 143–145).

The mathematical nature of the problem concerning the firm is analogous to that of the consumer, once the production is made dependent not only on the quantity of the factors used but also on their rate of variation ("Lezione XIV. La dinamica dell'impianto industriale"). As in the case of the dynamic treatment of consumption, Euler's equation leads to the formulation of a system of second-order differential equations that are analogous to those representing the fluctuating output of an industrial plant. These equations, in turn, determine the unknown functions, the quantities of factors used, x_n, considering $2n$ to be arbitrary constants that can be determined by imposing the starting and final configurations, or assigning starting configurations and initial velocities. In the first case, which Amoroso considers to be the most interesting one, it is possible to find a theory of a planned economy, which he will indicate as one of the highest theoretical achievements of corporative economic theory.

In spite of the publication of the article "The Transformation Value in the Productive Process" in *Econometrica* (1940), in the period following the Second World War, the approach proposed by Amoroso, based on the physical analogies, did not attract interest in the community of mathematical economists. In addition, in the Italian context, his dynamic system was considered to be more of a clever refinement of Pareto's system than a tool that could offer new possibilities for the development of dynamic analysis. For this reason, Amoroso was no longer mentioned, except by his student Palomba (1959), who probably did so more out of a sense of gratitude toward his teacher than because of deep conviction.

IX. FROM GENERAL DYNAMIC EQUILIBRIUM TO BALANCED ECONOMIC GROWTH

After the Second World War, the promising approach to economic dynamics based on variational calculus was not further developed. Only in the middle of the 1960s, with the application of the new techniques of optimal control, did this kind of mathematics return to the center of the scene in the field of dynamics. The new course was opened by David Cass with an article published in 1965. To understand this gap in the evolution of the formal dynamics, we can consider different elements.

First, we have seen that the main aim of the authors considered was to dynamize the theory of general economic equilibrium. This required rewriting Walras's and Pareto's equations in dynamic terms through the use of advanced mathematical techniques. But this project clashed with tendencies that emerged after the Second World War in general equilibrium theory, which moved in a different direction, that of giving it axiomatic foundations (Ingrao and Israel 1987). In addition, mathematics had now become profoundly different because it had switched from traditional calculus to set theory and convex analysis. This kind of inquiry was essentially static and relegated dynamic analysis to a wholly marginal role.

Second, the general economic equilibrium theory was probably not the tool best suited to constructing a dynamic theory. If the static theory had already proved to be a fragile mathematical construct, this feature was even more evident in the case of dynamics, where everything depended on variations in the magnitudes over time. For instance, La Volpe was fully aware of the interpretative limitations of his dynamic theory of the equilibrium. It is therefore not surprising that greater fortune was enjoyed by other research programs, such as Keynesian analysis, which were based on aggregate magnitudes that could also be measured. In the postwar period, the shift from statics to dynamics largely coincided with the birth of modern macroeconomics (Graziani 1991).

From the epistemological point of view, we noted that the economists considered emphasized the analogies between economics and rational mechanics. The idea of dynamic equilibrium can be considered as the extension into economics of the idea that the variables change through time, as in the analytic mechanics. The calculus of variations was the ideal tool for this kind of research. The result was a kind of dynamics that was totally exogenous: in the state of equilibrium, the quantities were moving in relation to time. Intelligible as these physical analogies may be, and useful in that they provide the necessary mathematical shapes, the scheme derived from these analogies still lacked sound economic meaning. In order to build economic dynamics, it was not necessary to consider the entire temporal path of the economic phenomena, but only the present state and its direction.

This approach would be introduced by Ragnar Frisch (1936) and developed by Samuelson in his influential book *Foundations of Economic Analysis*. With Samuelson, there was a profound shift in the path of economic dynamics. Within the new perspective, the equilibrium was conceived of as a case limit of the dynamic behavior of the system. How the system responds to changes (comparative statics) was considered more important than any particular position of equilibrium. Following this new perspective, Samuelson linked the notion of equilibrium to a different set of mathematical conditions from those related to Evans, Roos, or Amoroso. The new research program that Samuelson proposed rapidly obscured the previous one. It was more compatible with economic reasoning and did not require the sophisticated mathematical tools that functional calculus did. In this new framework, the concept of economic dynamic equilibrium would be restricted to that of balanced economic growth, a situation in which the economic variables grow at an exogenous constant rate (Boumans 2009).

This transformation paved the way for the application of the new techniques of optimal control in the 1960s with the aim to obtain a microfoundation of the consumption behavior.

X. CONCLUSIONS

Analogies with rational mechanics played a pivotal role in the search for formal models in economics (Grattan-Guinness 2010). This approach, which was pursued by Pareto and his followers, did have great success in the case of static equilibrium. In the period between the two world wars, a small group of economists who were mathematically oriented tried to extend this case to economic dynamics. The main result was the extensive application of calculus of variations to obtain a mathematical model of the change in the economic variables. In the 1920s, Evans and Roos paved the way and extended the Cournot model, which directly considered the differential element in terms of prices and quantities. On the Italian side, Amoroso and La Volpe moved toward a context that was more congenial to economic reasoning, using intertemporal relations and introducing some behavioral relations, such as expectations and habits. The general result was an exogenous dynamics based on long-term relations, in which the laws of motion depend explicitly on time. This view of the economic process was challenged by Samuelson, who, following Frisch, took a different path, which was less demanding with respect to analytical instruments and closer to economic reasoning.

REFERENCES

Allen, Roy George. 1937. *Mathematical Analysis for Economists*. London: Macmillan.

Amoroso, Luigi. 1912. "Contributo alla teoria matematica della dinamica economica." *Atti della Accademia dei Lincei* 21: 341–346.

———. 1921. *Lezioni di economia matematica*. Bologna: Zanichelli.

———. 1928. "Discussione del sistema di equazioni che definiscono l'equilibrio del consumatore." *Annali di Economia* 4: 30–41.

———. 1929. "Le equazioni differenziali nella dinamica economica." *Giornale degli Economisti e Rivista di Statistica* 28: 67–79.

———. 1932. "Contributo alla teoria matematica della dinamica economica." In *Nuova Collana degli Economisti V*. Torino: UTET, pp. 419–440.

———. 1933. "La dinamica dell'impresa." *Rivista italiana di Statistica, Economia e Finanza* 4: 442–451.

———. 1938. "La teoria matematica del programma economico." In Luigi Amoroso, ed., *Cournot nella economia e nella filosofia*. Padova: Cedam, pp. 74–93.

———. 1940. "The Transformation Value in the Productive Process." *Econometrica* 8: 1–11.

———. 1942. *Meccanica Economica*. Città di Castello: Macrì.

Arfken, George, and Hans Weber. 2001. *Mathematical Methods for Physicists*. New York: Harcourt Academic Press.

Boianovsky, Mauro, and Vincent Tarascio. 1998. "Mechanical Inertia and Economic Dynamics: Pareto on Business Cycles." *Journal of the History of Economic Thought* 20: 5–21.

Boumans, Marcel. 2005. *How Economists Model the Word into Numbers*. Abingdon: Routledge.

———. 2009. "Dynamizing Stability." *History of Political Economy* 41 (Annual suppl.): 127–146.

Cass, David. 1965. “Optimal Growth in an Aggregative Model of Capital Accumulation.” *Review of Economic Studies* 32: 233–240.

De Pietri Tonelli, Alfonso. 1921. *Trattato di Economia Razionale e Sperimentale*. Rovigo: Cedam.

Dimand, Robert. 1988. “Early Mathematical Theories of Business Cycles.” In D. E. Moggridge, ed., *Keynes, Macroeconomics and Method*. London: Edward Elgar, pp. 155–161.

Dimand, Robert, and William Veloce. 2007. “Charles F. Roos, Harold T. Davis and the Quantitative Approach to Business Cycle Analysis at the Cowles Commission in the 1930s and Early 1940s.” *The European Journal of the History of Economic Thought* 13: 519–541.

Di Matteo, Massimo. 1993. “Foreword Part II.” In Giulio La Volpe, *Studies on the Theory of General Dynamic Economic Equilibrium*. Edited by Michio Morishima and Massimo Di Matteo. Basingstoke: Macmillan.

———. 1998. “Giulio La Volpe.” *Rivista Italiana degli Economisti* 3: 157–160.

Donzelli, Franco. 1997. “Pareto’s Mechanical Dream.” *History of Economic Ideas* 5: 127–178.

Duarte, Pedro G. 2016. “A Path through the Wilderness: Time Discounting in Growth Models.” *History of Political Economy* 48 (2): 161–181.

Evans, Griffith C. 1922. “A Simple Theory of Competition.” *American Mathematical Monthly* 29: 371–380.

———. 1924. “The Dynamics of Monopoly.” *American Mathematical Monthly* 31: 91–117.

———. 1925. “The Mathematical Theory of Economics.” *American Mathematical Monthly* 32: 104–110.

———. 1929. “Cournot on Mathematical Economics.” *Bulletin of American Mathematical Society* 35: 269–271.

———. 1930. *Mathematical Introduction to Economics*. New York: McGraw-Hill.

Fisher, Irving. 1930. “Mathematics in the Social Sciences.” *The Scientific Monthly* 30: 547–557.

Fossati, Eraldo. [1937] 1953. “Ricerca sulle relazioni tra il tempo e l’utilità.” In Eraldo Fossati, *Frammenti di teoria dinamica*. Bologna: Cappelli Editore, pp. 35–52.

Frisch, Ragnar. 1933. “Propagation Problems and Impulses Problems in Dynamics Economics.” In *Economic Essays in Honour of Gustav Cassel*. London: Allen and Unwin, pp. 171–205.

———. 1936. “On the Notion of Equilibrium and Disequilibrium.” *The Review of Economic Studies* 3: 100–105.

Gandolfo, Giancarlo. 2008. “Giuseppe Palomba and the Lotka–Volterra Equations.” *Rendiconti Lincei*: 347–357.

Grattan-Guinness, Ivor. 2010. “How Influential Was Mechanics in the Development of Neoclassical Economics? A Small Example of Large Questions.” *Journal of the History of Economic Thought* 32: 531–580.

Graziani, Augusto. 1991. “The Italian Economic Journals and Some Major Turning-Points in Economic Theory.” *Economic Notes* 21: 121–133.

Guerraggio, Angelo, and Giovanni Paoloni. 2011. *Vito Volterra*. Basel: Birkhäuser.

Hicks, John. 1939. *Value and Capital*. Oxford: Oxford University Press.

Hotelling, Harold. 1931. “Review of *Mathematical Introduction to Economics*.” *Journal of Political Economy* 39: 107–109.

Ingrao, Bruna, and Giorgio Israel. 1990. *The Invisible Hand*. Cambridge: MIT Press.

Kyun, Kim. 1988. *Equilibrium Business Cycle Theory in Historical Perspective.* Cambridge: Cambridge University Press.

Louçã, Francisco. 2007. *The Years of High Econometrics*. Abingdon: Routledge.

La Volpe, Giulio. [1936] 1993. *Studies on the Theory of General Dynamic Economic Equilibrium*. Houndmills: Macmillan. First published as *Studi sulla teoria dell’equilibrio economico dinamico generale*. Napoli: Jovene.

———. 1938. *Ricerche di dinamica economica corporativa: impostazioni e problemi*. Padova: Cedam.

———. 1967. “L’analisi variazionale come fondamento ei modelli econometrici.” *Ricerche Economiche* 21: 231–252.

———. 1977. "La dinamica eso-endogena e i problemi diretti e inversi nella ricerca economica." *Rivista di Politica Economica* 67: 111–152.

McLure, Michael. 2001. *Pareto, Economy and Society: The Mechanical Analogy*. London: Routledge.

Mirowski, Philip. 1989. *More Heat than Light: Economics as a Social Physics, Physics as Nature Economics*. Cambridge: Cambridge University Press.

Moore, Henry L. 1929. *Synthetic Economics*. New York: Macmillan.

Mordecai, Ezekiel. 1930. "Moore's Synthetic Economics." *The Quarterly Journal of Economics* 44 (4): 663–679.

Morrey, Carl B. 1983. "Griffith Conrad Evans." *Biographical Memoirs. National Academy of Science* 54: 127–155.

Nicola, Pier Carlo. 2000. *Mainstream Mathematical Economics in the 20th Century*. Berlin: Springer.

Palomba, Giuseppe. 1939. *Introduzione allo studio della dinamica economica*. Napoli: Jovene.

———. 1959. *Fisica economica*. Napoli: Jovene.

Pareto, Vilfredo. [1896–97] 1941. *Corso di Economia Politica*. Torino: Boringhieri.

———. 1901. "Le nuove teorie economiche con annessa appendice. Le equazioni dell'equilibrio dinamico." *Giornale degli Economisti* 23: 235–259.

Pomini, Mario. 2009. "Equilibrio dinamico ed aspettative in Giulio La Volpe." *Il Pensiero Economico Italiano* XVII: 331–349.

Pomini, Mario, and Gianfranco Tusset. 2009. "Habits and Expectations: Dynamic General Equilibrium in the Italian Paretian School." *History of Political Economy* 41: 311–343.

Ramsey, Frank. 1928. "A Mathematical Theory of Saving." *Economic Journal* 38: 543–559.

Raybout, Alain. 2013. "H. L. Moore and the Dynamic Complement of Pure Economics." *Oeconomica* 3: 221–240.

Roos, Charles. 1925. "A Mathematical Theory of Competition." *American Journal of Mathematics* 47: 163–165.

———. 1927. "A Dynamical Theory of Economics." *The Journal of Political Economy* 35: 632–656.

———. 1928. "Generalized Lagrange Problems in the Calculus of Variations." *American Mathematical Society* 30: 360–384.

———. 1930. "A Mathematical Theory of Price and Production Fluctuations and Economic Crisis." *Journal of Political Economy* 38: 501–522.

———. 1934. *Dynamic Economics*. Bloomington: The Principia Press.

Rosenstein-Rodan, Paul N. 1934. "The Role of Time in Economic Theory." *Economica* 1: 77–97.

Samuelson, Paul A. 1941. "The Stability of Equilibrium: Comparative Statics and Dynamics." *Econometrica* 9: 97–120.

———. 1947. *Foundations of Economic Analysis*. Cambridge: Harvard University Press.

Samuelson, Paul, and Robert Solow. 1956. "A Complete Capital Model Involving Heterogeneous Capital Goods." *The Quarterly Journal of Economics* 70: 537–562.

Sensini, Guido. 1955. *Corso di Economia Pura*. Roma: Maglione.

Tinbergen, Jan. 1933. "L'utilisation des équations fonctionelles et des nombres complexes dans les recherches économiques." *Econometrica* 1: 36–51.

———. 1934. "Annual Survey of Significant Developments in General Economic Theory." *Econometrica* 2: 13–36.

Weintraub, Roy. 1991. *Stabilizing Dynamics*. Cambridge: Cambridge University Press.

———. 1998. "From Rigor to Axiomatics: The Marginalization of Griffith C. Evans." *History of Political Economy* 30: 227–259.

———. 2002. *How Economics Became a Mathematical Science*. Durham: Duke University Press.

Wulwick, Nancy. 1995. "The Hamiltonian Formalism and Optimal Growth Theory." In Ingrid H. Rima, ed., *Measurement, Quantification and Economic Analysis*. New York: Routledge, pp. 157–175.