

# On an interior Calderón operator and a related Steklov eigenproblem for Maxwell's equations\*

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Dedicated to Professor Nicholas D. Alikakos  
on the occasion of his retirement

**Abstract:** We discuss a Steklov-type problem for Maxwell's equations which is related to an interior Calderón operator and an appropriate Dirichlet-to-Neumann type map. The corresponding Neumann-to-Dirichlet map turns out to be compact and this provides a Fourier basis of Steklov eigenfunctions for the associated energy spaces. With an approach similar to that developed by Auchmuty for the Laplace operator, we provide natural spectral representations for the appropriate trace spaces, for the Calderón operator itself and for the solutions of the corresponding boundary value problems subject to electric or magnetic boundary conditions on a cavity.

**Keywords:** Interior Calderón operator, Maxwell's equations, Steklov eigenfunctions, Trace spaces.

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## 1 Introduction

In a homogeneous isotropic medium filling a domain  $\Omega$  in  $\mathbb{R}^3$  the *time-harmonic* Maxwell's equations read

$$\operatorname{curl}E - i\omega\mu H = 0, \quad \operatorname{curl}H + i\omega\varepsilon E = 0, \quad (1.1)$$

where  $E, H$  are, respectively, the spatial parts of the electric and the magnetic field,  $\varepsilon$  and  $\mu$  are the electric permittivity and the magnetic permeability of the medium, and  $\omega > 0$  is the angular frequency (we have adopted the time convention  $e^{-i\omega t}$ ). In the considered case of homogeneous isotropic media,  $\varepsilon$  and  $\mu$  are constants, therefore  $E$  and  $H$  are automatically divergence-free.

Of *sine qua non* importance in electromagnetics is the following boundary value problem, involving the so-called “perfect conductor” condition on the boundary  $\Gamma$  of  $\Omega$ , i.e., the tangential trace of the electric field on  $\Gamma$  is given by a fixed vector  $m$ :

$$\begin{cases} \operatorname{curl}E - i\omega\mu H = 0, & \operatorname{curl}H + i\omega\varepsilon E = 0, & \text{in } \Omega, \\ \nu \times E = m, & & \text{on } \Gamma. \end{cases} \quad (1.2)$$

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The interior Calderón operator is defined as the mapping of the tangential component of the electric field to the tangential component of the magnetic field on  $\Gamma$ , i.e.,  $m \mapsto \nu \times H$ . This is the origin of another term used for this operator, namely the “electric to magnetic boundary component map”. Calderón operators are also known as capacity, or impedance, or admittance, or Poincaré-Steklov operators.

Operating by curl on (1.2) and setting  $\tilde{m} := -i\omega \varepsilon m$  we obtain

$$\begin{cases} \operatorname{curl} \operatorname{curl} H - \omega^2 \varepsilon \mu H = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl} H = \tilde{m}, & \text{on } \Gamma. \end{cases} \quad (1.3)$$

The corresponding interior Calderón operator for (1.3), maps  $\tilde{m}$  to  $\nu \times H$ .

Let us note that, in view of the vector identity  $\operatorname{curl} \operatorname{curl} w = \operatorname{grad} \operatorname{div} w - \Delta w$  and the fact that  $H$  is divergence-free, (1.3) can also be written as

$$\begin{cases} \Delta H + \omega^2 \varepsilon \mu H = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl} H = \tilde{m}, & \text{on } \Gamma. \end{cases}$$

Based on the sign of  $\varepsilon$  and of  $\mu$ , materials can be classified as

- (RH) When  $\varepsilon > 0$ ,  $\mu > 0$  the materials are called “right handed” or “double positive”, and exhibit forward propagating waves. These are the conventional materials of electromagnetics, e.g., dielectrics.
- (LH) When  $\varepsilon < 0$ ,  $\mu < 0$  the materials are called “left handed” or “double negative”, and they exhibit backward propagating waves. They are not found in Nature, but are physically realizable.
- (SRR) The materials with  $\varepsilon > 0$ ,  $\mu < 0$  are called of “split ring resonator” structure, and exhibit evanescent decaying waves and no transmission (typical examples are ferrites, microstructured magnets and split rings).
- (TW) The materials with  $\varepsilon < 0$ ,  $\mu > 0$  are called of “thin wire” structure, and they exhibit evanescent decaying waves and no transmission (typical examples are plasmas and fine wire structures).
- (LH), (SRR) and (TW) are included in the so-called “metamaterials”, [22].

Note that the results in the present work cover all the above cases (the coefficient  $\alpha$  of our approach, that corresponds to the physical constant  $\omega^2 \varepsilon \mu$  appearing in the time-harmonic Maxwell’s equations, can be of any sign).

Let us note that the study of the operators “curl curl” and “curl curl  $- \varrho^2 I$ ” is essential not only in the mathematical theory of classical electromagnetics, but in other related important applications areas, such as, e.g., the theory of superconductors, magnetohydrodynamics (MHD)(where the equations - consisting of an elegant and subtle coupling of the Navier-Stokes and Maxwell’s equations - govern the motion of electrically conducting viscous incompressible fluids in a magnetic field (see, e.g., [3]), and in particular in the ideal linear MHD equations that describe the stability properties of a “tokamak”, i.e., a device which uses a powerful magnetic field to confine a hot plasma in the shape of a torus), etc.

The classical Steklov eigenproblem for a bounded smooth domain  $\Omega$  in  $\mathbb{R}^n$  with boundary  $\Gamma$  reads

$$\begin{cases} \Delta v = 0, & \text{in } \Omega, \\ D_\nu v = \lambda v, & \text{on } \Gamma, \end{cases}$$

where the unknown  $v$  is a real or complex-valued function called Steklov eigenfunction and the unknown  $\lambda$  is a non-negative real number called Steklov eigenvalue. Here  $\nu$  denotes the unit outer normal to  $\Gamma$  and  $D_\nu v$  the normal derivative of  $v$ . The Steklov eigenvalues can be equivalently defined as the eigenvalues of the celebrated DtN (Dirichlet-to-Neumann) map defined from  $H^{1/2}(\Gamma)$  to  $H^{-1/2}(\Gamma)$  by  $g \mapsto D_\nu v$  where  $v$  is the solution to the Dirichlet problem

$$\begin{cases} \Delta v = 0, & \text{in } \Omega, \\ v = g, & \text{on } \Gamma. \end{cases}$$

It turns out that the non-zero eigenvalues are the reciprocals of the eigenvalues of the corresponding NtD (Neumann-to-Dirichlet) map which can be considered as a compact self-adjoint map from  $L^2(\Gamma)$  to itself. In particular, the eigenvalues have finite multiplicity and can be represented as a non-decreasing divergent sequence. We refer to [14, 21] for an introduction to Steklov-type problems, and to [17] for an interesting application of the problem.

Although the study of Steklov eigenvalues has a long history (see [20]) and Steklov boundary conditions have been considered for many classes of operators, in the literature there are not so many results concerning Steklov-type eigenvalues for Maxwell's equations; an interesting exception are those contained in the very recent papers [5, 7].

We note that in [5] (along the approach introduced in [4] for the Helmholtz equation) the use of Steklov eigenvalues for Maxwell's equations is suggested to detect changes in a scatterer using remote measurements of the scattered wave, i.e., as a novel "target signature" for nondestructive testing via inverse scattering. Because the Steklov eigenvalue problem for Maxwell's equations is not a standard eigenvalue problem for a compact operator, a modified Steklov problem is proposed, that restores compactness. In particular it is shown that it is possible to measure Steklov eigenvalues for a bounded inhomogeneous scatterer by solving a sequence of modified far field equations. To this end, the authors perturb the usual far field equation of the linear sampling method by using the far field pattern of an auxiliary impedance problem related to the modified Steklov problem. In order to measure the modified Steklov eigenvalues of a domain from far field measurements, the authors prove (i) the existence of modified Steklov eigenvalues, (ii) the well-posedness of the corresponding auxiliary exterior impedance problem, and (iii) provide theorems on the detection of modified Steklov eigenvalues from far field measurements. While our present paper was at the stage of review, two interesting manuscripts appeared in arXiv, continuing in a sense the work in [5]: in [15], both the original Steklov eigenvalue problem for Maxwell's equations and the aforementioned modified Steklov problem are studied, and their Fredholmness and approximation is analyzed. The original eigenvalue problem in the selfadjoint case is studied in [16], where it is established that, apart for a countable set of particular frequencies, the spectrum consists of three disjoint parts: the essential spectrum consisting of the point zero, an infinite sequence of positive eigenvalues which accumulate only at infinity, and an infinite sequence of negative eigenvalues which accumulate only at zero. See also the very recent work by Cogar [8], [10] and Cogar and Monk [9].

The aim of the present paper is to furnish a natural Steklov problem for a class of Maxwell's equations, which intrinsically exhibits a discrete spectrum, and use it in the spirit of [2] to provide spectral representations for the associated trace spaces and the solutions of the corresponding boundary value problems. Our starting point is the observation that, in the mathematical theory of electromagnetism, a natural counterpart of the NtD map above is the aforementioned celebrated Calderón operator. The Calderón operator and its variants have been and are studied extensively, see, e.g., [6]. Here we focus on a Calderón operator associated with the following boundary value problem: for a bounded, connected, open set  $\Omega$  in  $\mathbb{R}^3$  of class  $C^{1,1}$  and  $\alpha \in \mathbb{R}$ ,  $\theta > 0$  consider the interior<sup>1</sup> problem

$$\begin{cases} \operatorname{curl} \operatorname{curl} u - \alpha u - \theta \operatorname{grad} \operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl} u = f, & \text{on } \Gamma, \end{cases} \quad (1.4)$$

where  $u$  is the unknown vector field. We point out that the penalty term  $\theta \operatorname{grad} \operatorname{div} u$  is introduced in the equation in order to guarantee the coercivity of the quadratic form associated with the operator, as done e.g., in [11]. Note that the boundary operator  $\nu \times \operatorname{curl} u$  in (1.4) can be considered as the “electromagnetic version” of the boundary operator  $D_\nu v$  on  $\Gamma$  usually associated with the scalar Laplace operator. Indeed, the boundary conditions in (1.4) are the natural boundary conditions arising from the integration by parts formula

$$\int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} \varphi \, dx = \int_{\Omega} \operatorname{curl} \operatorname{curl} u \cdot \varphi \, dx - \int_{\Gamma} (\nu \times \operatorname{curl} u) \cdot \varphi \, d\sigma, \quad (1.5)$$

which is valid for sufficiently regular vector fields  $u, \varphi$ , while the Neumann boundary conditions for the scalar Laplace operator are the natural boundary conditions arising from the integration by parts formula

$$\int_{\Omega} \operatorname{grad} v \cdot \operatorname{grad} \psi \, dx = - \int_{\Omega} \Delta v \, \psi \, dx + \int_{\Gamma} D_\nu v \, \psi \, d\sigma,$$

which, again, is valid for sufficiently regular scalar functions  $v, \psi$ .

The interior Calderón operator  $\mathcal{C}$  is here defined by

$$\mathcal{C}(f) = \nu \times u, \quad (1.6)$$

where  $u$  is the solution of (1.4), see Section 3.3 for more details. Thus, the Calderón operator establishes a correspondence between the electric and magnetic fields on  $\Gamma$  as follows:

$$\nu \times \operatorname{curl} u \longmapsto \nu \times u.$$

Now, the corresponding to the NtD-map in the case of our problem is

$$\nu \times \operatorname{curl} u \longmapsto u,$$

the eigenvalues of which are the reciprocals of the eigenvalues of the following Steklov-type problem for Maxwell's equations

$$\begin{cases} \operatorname{curl} \operatorname{curl} u - \alpha u - \theta \operatorname{grad} \operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \times \operatorname{curl} u = \lambda u, & \text{on } \Gamma. \end{cases} \quad (1.7)$$

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<sup>1</sup>The term “interior” is used to emphasize that the problem is cast in  $\Omega$  and not in the exterior of  $\Omega$ , as is done in the case of Calderón operators for scattering problems.

It is important to clarify that the energy space involved here is the classical space  $X_T(\Omega) = H(\text{curl}, \Omega) \cap H_0(\text{div}, \Omega)$  and that problem (1.7) has to be interpreted in the weak sense as follows<sup>2</sup>: find  $u \in X_T(\Omega)$  such that

$$\int_{\Omega} \text{curl} u \cdot \text{curl} \varphi \, dx - \alpha \int_{\Omega} u \cdot \varphi \, dx + \theta \int_{\Omega} \text{div} u \, \text{div} \varphi \, dx = -\lambda \int_{\Gamma} u \cdot \varphi \, d\sigma, \quad (1.8)$$

for all  $\varphi \in X_T(\Omega)$ . In particular, any vector field  $u \in X_T(\Omega)$  satisfies the condition  $u \cdot \nu = 0$  on  $\Gamma$ .

Note that the weak formulation (1.8) can be obtained from (1.7) by a standard procedure as follows: assume that  $u$  is a sufficiently regular solution of problem (1.7), say  $u \in (H^2(\Omega))^3$ , then multiplying both sides of the first equation in (1.7) by  $\varphi \in X_T(\Omega)$ , integrating by parts and using formula (1.5) and the boundary condition in (1.7), yield the validity of (1.8). By using the same calculations and the Fundamental Lemma of the Calculus of Variations, one can see that if  $u \in (H^2(\Omega))^3$  is a solution of (1.8) then it is also a solution of (1.7). We also note that using the weak formulations allows to avoid assuming extra smoothness assumptions on the boundary of  $\Omega$  required to guarantee the regularity of the solutions, see e.g., [28].

By the classical Gaffney inequality (Theorem 2.1), if  $\Omega$  is of class  $C^{1,1}$  the space  $X_T(\Omega)$  is continuously embedded into  $(H^1(\Omega))^3$ . Moreover, the trace operator is compact from  $H^1(\Omega)$  to  $L^2(\Gamma)$ . It follows that if the operator

$$u \mapsto \text{curl} \, \text{curl} u + \alpha u - \theta \, \text{grad} \, \text{div} u$$

is invertible, problem (1.7) has a discrete spectrum and provides a Fourier basis of eigenfunctions for the space  $TL^2(\Gamma)$  of square-summable vector fields tangent to  $\Gamma$ , see Section 3.2. For the sake of simplicity, these results are proved under the assumption that  $\alpha < A_1$  where  $A_1 > 0$  is the first eigenvalue of the associated Dirichlet problem, see (3.3) and (3.4). The analysis of the general case is discussed in Section 5, where we explain how to rule out the Dirichlet eigenfunctions associated with the Dirichlet eigenvalues smaller than  $\alpha$ .

We note that the boundary condition appearing in the Steklov problem discussed in [5, Equation (2.6)] is  $\nu \times \text{curl} u = \lambda u_T$  on  $\Gamma$ , where  $u_T$  is the tangential component of  $u$ ; accordingly the energy space used in [5] to treat that problem is  $\{u \in H(\text{curl}, \Omega) : u_T \in (L^2(\Gamma))^3\}$  and the corresponding eigenvectors turn out to be divergence free. Our boundary condition  $\nu \times \text{curl} u = \lambda u$  on  $\Gamma$ , is clearly stronger in the sense that it implies that our eigenfunctions are automatically tangential: this allows us to discard the part of the spectrum associated with possible non tangential eigenvectors which are responsible for the appearance of an accumulation point in the spectrum of the operator discussed in the counterexample in [5, p. 4383].

With an approach analogous to that in [2] concerning the Laplace operator, our Steklov problem allows us to achieve the following results:

- (1) we provide (Theorems 4.4, 4.20) a spectral representation for the solutions of problem (1.4),

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<sup>2</sup>Note that the minus sign in the right-hand side of (1.8) is due to the fact that the boundary condition in the problem is written in the customary form  $\nu \times \text{curl} u = \lambda u$ , and it is responsible for the appearance of negative spectra. The reader who is more familiar with classical eigenvalue problems for elliptic equations could write it in the form  $\text{curl} u \times \nu = \lambda u$ , which would change the sign of the eigenvalues.

- (2) we provide (see Theorem 4.4 and Remark 4.24) a spectral representation for the interior Calderón operator  $\mathcal{C}$ ,
- (3) we provide (see Theorem 4.12) a spectral representation for the trace space  $TH^{1/2}(\Gamma)$  and its dual  $TH^{-1/2}(\Gamma)$ ,
- (4) we provide (see Theorem 4.12) a spectral representation for the solutions to the following problem

$$\begin{cases} \operatorname{curl} \operatorname{curl} u - \alpha u - \theta \operatorname{grad} \operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \times u = f, & \text{on } \Gamma, \\ \nu \cdot u = 0, & \text{on } \Gamma. \end{cases} \quad (1.9)$$

Another approach to the representation of an exterior Calderón operator associated with a scattering problem for not necessarily spherical domains is proposed in [19]; there the appropriate series expansions are performed with respect to generalized harmonics (the set of eigenfunctions to the Laplace-Beltrami operator for the domain's boundary). Further, the norm in an appropriate trace space of the exterior Calderón operator is obtained in view of an eigenproblem for a suitable quadratic form. Let us note that in the case of a sphere the eigenfunctions of the Laplace-Beltrami operator are the spherical harmonics hence the classical Steklov eigenfunctions.

The present paper is organized as follows: Section 2 is devoted to preliminaries and notation; Section 3 is devoted to the study of problems (1.4) and (1.7); in particular, in Section 3.3 we introduce the Calderón operator and the associated NtD-type map. Section 4 is devoted to the above mentioned spectral representations. Finally, Section 5 includes an appendix devoted to the study of the case  $\alpha > A_1$ .

## 2 Preliminaries and notation

Let  $\Omega$  be a bounded open set in  $\mathbb{R}^3$  with sufficiently smooth boundary  $\Gamma := \partial\Omega$ . By  $L^2(\Omega)$ ,  $H^1(\Omega)$ ,  $H_0^1(\Omega)$ ,  $L^2(\Gamma)$ ,  $H^{1/2}(\Gamma)$ ,  $H^{-1/2}(\Gamma)$ , we denote the standard Lebesgue and Sobolev spaces. We will also employ the following spaces:

- $H(\operatorname{curl}, \Omega) = \{u \in (L^2(\Omega))^3 : \operatorname{curl} u \in (L^2(\Omega))^3\}$ ,  
with norm:  $\|u\|_{H(\operatorname{curl}, \Omega)} = \left( \|u\|_{(L^2(\Omega))^3}^2 + \|\operatorname{curl} u\|_{(L^2(\Omega))^3}^2 \right)^{1/2}$
- $H(\operatorname{div}, \Omega) = \{u \in (L^2(\Omega))^3 : \operatorname{div} u \in L^2(\Omega)\}$ ,  
with norm:  $\|u\|_{H(\operatorname{div}, \Omega)} = \left( \|u\|_{(L^2(\Omega))^3}^2 + \|\operatorname{div} u\|_{L^2(\Omega)}^2 \right)^{1/2}$
- $H_0(\operatorname{div}, \Omega) = \{u \in H(\operatorname{div}, \Omega) : \nu \cdot u = 0 \text{ on } \Gamma\}$
- $X_{\text{T}}(\Omega) = H(\operatorname{curl}, \Omega) \cap H_0(\operatorname{div}, \Omega)$ ,  
with norm:  $\|u\|_{H(\operatorname{curl}, \Omega) \cap H_0(\operatorname{div}, \Omega)} = \left( \|u\|_{(L^2(\Omega))^3}^2 + \|\operatorname{curl} u\|_{(L^2(\Omega))^3}^2 + \|\operatorname{div} u\|_{L^2(\Omega)}^2 \right)^{1/2}$
- $X_{\text{T}}(\operatorname{div} 0, \Omega) = \{u \in X_{\text{T}}(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega\}$
- $TL^2(\Gamma) = \{u \in (L^2(\Gamma))^3 : \nu \cdot u = 0 \text{ on } \Gamma\}$
- $TH^{1/2}(\Gamma) = \{u \in (H^{1/2}(\Gamma))^3 : \nu \cdot u = 0 \text{ on } \Gamma\}$

- $TH^{-1/2}(\Gamma) = (TH^{1/2}(\Gamma))'$

By  $\text{div}_\Gamma$  and  $\text{grad}_\Gamma$  we denote the usual tangential operators.

For details on these operators and spaces we refer to [1], [6], [12], [13], [18], [23], [25].

Throughout this paper, we consider bounded open sets of class  $C^{1,1}$  in which case  $X_T(\Omega)$  is continuously embedded in  $(H^1(\Omega))^3$ , and compactly embedded in  $(L^2(\Omega))^3$ . In particular, the following theorem holds (cf. e.g. [13, Theorem 3.8]).

**Theorem 2.1** *If  $\Omega$  is a bounded open set in  $\mathbb{R}^3$  of class  $C^{1,1}$  then  $X_T(\Omega)$  is continuously embedded in  $(H^1(\Omega))^3$ , in particular there exists  $c > 0$  such that*

$$\|u\|_{(H^1(\Omega))^3} \leq c (\|u\|_{L^2(\Omega)} + \|\text{curl}u\|_{L^2(\Omega)} + \|\text{div}u\|_{L^2(\Omega)}),$$

for all  $u \in X_T(\Omega)$ .

Thus, for bounded open sets of class  $C^{1,1}$  we have that<sup>3</sup>

$$X_T(\Omega) = \{u \in (H^1(\Omega))^3 : u \cdot \nu = 0\}.$$

For a smooth vector field  $w$  defined on  $\bar{\Omega}$ , its “*tangential trace*”  $\nu \times w$  on  $\Gamma$  is denoted by  $\gamma_T w$ , while its “*tangential components trace*”  $\nu \times (w \times \nu)$  on  $\Gamma$  is denoted by  $\pi_T w$ .

Since we are primarily interested in  $\pi_T$ , we note that  $\pi_T((H^1(\Omega))^3) = TH^{1/2}(\Gamma)$ ,  $\pi_T((H^{1/2}(\Gamma))^3) = TH^{1/2}(\Gamma)$  and  $(\pi_T((H^{1/2}(\Gamma))^3))' = TH^{-1/2}(\Gamma)$ .

Moreover, we have that  $\pi_T$  is a compact operator from  $(H^1(\Omega))^3$  to  $TL^2(\Gamma)$ . Note that we shall often use the same symbol for a function and its trace.

In Section 4 we shall give an equivalent definition of the spaces  $TH^{1/2}(\Gamma)$  and  $TH^{-1/2}(\Gamma)$  based on an intrinsic spectral representation.

### 3 Interior problems

Let  $\Omega$  be a bounded, connected open set in  $\mathbb{R}^3$  with boundary  $\Gamma := \partial\Omega \in C^{1,1}$ . Let  $\alpha \in \mathbb{R}$  and  $\theta > 0$  be fixed. In this section, we discuss the solvability of problems (1.4) and (1.7). We begin by proving that for any fixed  $\eta \geq 0$  sufficiently large, the problem

$$\begin{cases} \text{curl curl}u - \alpha u - \theta \text{grad div}u = 0, & \text{in } \Omega, \\ \nu \times \text{curl}u - \eta u = f, & \text{on } \Gamma \end{cases} \quad (3.1)$$

has a (unique) solution for every datum  $f \in TL^2(\Gamma)$ .

The weak formulation of problem (3.1) reads: find  $u \in X_T(\Omega)$  such that

$$\int_{\Omega} \text{curl}u \cdot \text{curl}\varphi \, dx - \alpha \int_{\Omega} u \cdot \varphi \, dx + \theta \int_{\Omega} \text{div}u \, \text{div}\varphi \, dx + \eta \int_{\Gamma} u \cdot \varphi \, d\sigma = - \int_{\Gamma} f \cdot \varphi \, d\sigma, \quad (3.2)$$

for all  $\varphi \in X_T(\Omega)$ . Note that the equivalence of the two formulations for smooth solutions can be proved in the same way explained in the Introduction for the case  $\eta = 0$ . (Recall that using the weak formulation enables us to avoid extra regularity assumptions on  $\Omega$ .)

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<sup>3</sup>Although this space could be denoted with other more specific symbols, such as  $TH^1(\Omega)$ , we prefer to keep the above notation

We note that when  $\alpha > 0$  the quadratic form associated with the left-hand side of equation (3.2) is not necessarily positive and this complicates the analysis of the problem. In order to avoid a number of technicalities which would render the exposition much heavier, we find it not only simpler, but also natural to assume that  $\alpha < A_1$ , where

$$A_1 := \inf_{\substack{u \in (H_0^1(\Omega))^3 \\ u \neq 0}} \frac{\int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx}{\int_{\Omega} |u|^2 dx}. \quad (3.3)$$

See Section 5 for a more general condition allowing arbitrarily large values of  $\alpha$ .

Note that  $A_1 > 0$  by Theorem 2.1, and that  $A_1$  is the first eigenvalue of the problem

$$\int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} \varphi dx + \theta \int_{\Omega} \operatorname{div} u \operatorname{div} \varphi dx = A \int_{\Omega} u \cdot \varphi dx, \forall \varphi \in (H_0^1(\Omega))^3, \quad (3.4)$$

in the unknowns  $u \in (H_0^1(\Omega))^3$  (the Dirichlet eigenfunction) and  $A \in \mathbb{R}$  (the Dirichlet eigenvalue).

**Theorem 3.5** *Let  $\Omega$  be a bounded, connected open set in  $\mathbb{R}^3$  with  $C^{1,1}$  boundary. Let  $\alpha < A_1$  and  $\theta > 0$ . Then there exists  $c_{\alpha, \theta} \geq 0$  such that for any  $\eta \geq c_{\alpha, \theta}$  the quadratic form defined by the left-hand side of (3.2) is coercive in  $X_{\Gamma}(\Omega)$  and problem (3.2) has a unique solution  $u \in X_{\Gamma}(\Omega)$  for all  $f \in TL^2(\Gamma)$ . Moreover, for  $\alpha \leq 0$  one can take  $c_{\alpha, \theta} = 0$ . Finally, if  $\alpha \leq 0$  and  $\eta = 0$  and if, in addition,  $f$  satisfies the condition  $\operatorname{div}_{\Gamma} f = 0$  on  $\Gamma$ , then  $\operatorname{div} u = 0$  in  $\Omega$ .*

**Proof.** We first consider the case  $\alpha \leq 0$ . By a straightforward application of the Riesz Theorem combined with Theorem 2.1, and by noting that the right-hand side of equation (3.2) defines linear continuous operators belonging to the dual space of  $X_{\Gamma}(\Omega)$ , it is clear that the problem is uniquely solvable in  $X_{\Gamma}(\Omega)$  for any  $\eta \geq 0$ .

We now consider the case  $\alpha > 0$ . Let  $\beta \in (\alpha, A_1)$ . First of all we note that there exists  $M > 0$  such that

$$\int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx + M \int_{\Gamma} |u|^2 d\sigma \geq \beta \int_{\Omega} |u|^2 dx, \quad (3.6)$$

for all  $u \in X_{\Gamma}(\Omega)$ . Indeed, assume for contradiction that for any  $n \in \mathbb{N}$  there exists  $u_n \in X_{\Gamma}(\Omega)$  such that

$$\int_{\Omega} |\operatorname{curl} u_n|^2 dx + \theta \int_{\Omega} |\operatorname{div} u_n|^2 dx + n \int_{\Gamma} |u_n|^2 d\sigma \leq \beta \int_{\Omega} |u_n|^2 dx, \quad (3.7)$$

and normalize  $u_n$  by setting  $\int_{\Omega} |u_n|^2 dx = 1$ . By (3.7) the sequence  $u_n$ ,  $n \in \mathbb{N}$ , is bounded in  $X_{\Gamma}(\Omega)$ ; hence - possibly passing to a subsequence - there exists  $u \in X_{\Gamma}(\Omega)$  such that  $u_n \rightarrow u$  weakly in  $X_{\Gamma}(\Omega)$ ,  $u_n \rightarrow u$  strongly in  $L^2(\Omega)$ . In particular,  $\int_{\Omega} |u|^2 dx = 1$ . Moreover, since the trace operator is compact, using condition (3.7) allows to conclude that the trace of  $u$  is zero, hence  $u \in (H_0^1(\Omega))^3$ . By passing to the limit in inequality (3.7) and using the weak lower semicontinuity of norms, we conclude that

$$\begin{aligned} & \int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx \\ & \leq \liminf_{n \rightarrow \infty} \left( \int_{\Omega} |\operatorname{curl} u_n|^2 dx + \theta \int_{\Omega} |\operatorname{div} u_n|^2 dx \right) \\ & \leq \lim_{n \rightarrow \infty} \beta \int_{\Omega} |u_n|^2 dx = \beta \end{aligned} \quad (3.8)$$



which implies that the infimum in (3.3) is not larger than  $\beta$ , hence it is strictly smaller than  $A_1$ , a contradiction.

Let  $\epsilon \in (0, 1 - \frac{\alpha}{\beta})$ . By using (3.6) we deduce immediately that

$$\begin{aligned}
& \int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx - \alpha \int_{\Omega} |u|^2 dx + (1 - \epsilon)M \int_{\Gamma} |u|^2 d\sigma \\
&= \epsilon \left( \int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx \right) + (1 - \epsilon) \left( \int_{\Omega} |\operatorname{curl} u|^2 dx \right. \\
&\quad \left. + \theta \int_{\Omega} |\operatorname{div} u|^2 dx + M \int_{\Gamma} |u|^2 d\sigma \right) - \alpha \int_{\Omega} |u|^2 dx \\
&\geq \epsilon \left( \int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx \right) + ((1 - \epsilon)\beta - \alpha) \int_{\Omega} |u|^2 dx, \quad (3.9)
\end{aligned}$$

for all  $u \in X_T(\Omega)$ , which implies the coercivity in  $X_T(\Omega)$  (hence in  $(H^1(\Omega))^3$ ) of the quadratic form in the left-hand side of (3.2). The conclusion about the existence of a unique solution then follows choosing  $c_{\alpha, \theta} = (1 - \epsilon)M$  and proceeding as in the case  $\alpha < 0$ .

Now, assume that  $\operatorname{div}_{\Gamma} f = 0$  on  $\Gamma$ ,  $\alpha \leq 0$  and  $\eta = 0$ . Consider a solution  $\Phi \in H^2(\Omega)$  to the following problem

$$\begin{cases} \Delta \Phi = \operatorname{div} u, & \text{in } \Omega, \\ D_{\nu} \Phi = 0, & \text{on } \Gamma. \end{cases} \quad (3.10)$$

Let us first observe that the above problem has a solution  $\Phi \in H^1(\Omega)$ , since  $u \cdot \nu = 0$  on  $\Gamma$  implies that  $\int_{\Omega} \operatorname{div} u dx = 0$ ; furthermore,  $\Phi \in H^2(\Omega)$  because  $u \in H^1(\Omega)$  and therefore  $\operatorname{div} u \in L^2(\Omega)$ . Clearly,  $\operatorname{grad} \Phi \in X_T(\Omega)$  hence we can use  $\overline{\operatorname{grad} \Phi}$  as a test function in (3.2) and get

$$\begin{aligned}
& -\alpha \int_{\Omega} |\operatorname{grad} \Phi|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx = \int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} \overline{\operatorname{grad} \Phi} dx \\
& -\alpha \int_{\Omega} u \cdot \overline{\operatorname{grad} \Phi} dx + \theta \int_{\Omega} \operatorname{div} u \overline{\Delta \Phi} dx = - \int_{\Gamma} f \cdot \overline{\operatorname{grad} \Phi} d\sigma = 0, \quad (3.11)
\end{aligned}$$

where the last equality holds because  $\operatorname{div}_{\Gamma} f = 0$  on  $\Gamma$ , in view of the fact that  $f$  is tangential and hence  $f \operatorname{grad} \Phi = f \operatorname{grad}_{\Gamma} \Phi$  on  $\Gamma$ . It follows from (3.11) and the assumption  $\alpha \leq 0$  that  $\operatorname{div} u = 0$  in  $\Omega$ .  $\square$

**Remark 3.12** *We note that if  $\alpha \leq 0$ ,  $\theta > 0$ ,  $\eta = 0$  and  $\operatorname{div}_{\Gamma} f = 0$  on  $\Gamma$ , then problem (3.2) can be formulated directly in the energy space  $X_T(\operatorname{div} 0, \Omega)$ . Therefore the weak formulation can be stated as: find  $u \in X_T(\operatorname{div} 0, \Omega)$  such that*

$$\int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} \varphi dx - \alpha \int_{\Omega} u \cdot \varphi dx = - \int_{\Gamma} f \cdot \varphi d\sigma, \quad (3.13)$$

for all  $\varphi \in X_T(\operatorname{div} 0, \Omega)$ .

*In order to see that, under these assumptions, problems (3.2) and (3.13) are equivalent, we can argue as follows. Consider a fixed  $f \in TL^2(\Gamma)$  with  $\operatorname{div}_{\Gamma} f = 0$  on  $\Gamma$ . By Theorem 3.5 the solution  $u$  of problem (3.2) is also a solution of problem (3.13). To prove the reverse statement, consider a solution  $u$  of (3.13) and a test function*

$\varphi \in X_{\mathbb{T}}(\Omega)$ . Along the same lines of the proof of Theorem 3.5, consider a solution  $\Psi \in H^2(\Omega)$  to the following problem

$$\begin{cases} \Delta \Psi = \operatorname{div} \varphi, & \text{in } \Omega, \\ D_{\nu} \Psi = 0, & \text{on } \Gamma. \end{cases} \quad (3.14)$$

Then we can decompose  $\varphi$  as  $\varphi = \tilde{\varphi} + \operatorname{grad} \Psi$ , where  $\tilde{\varphi} := \varphi - \operatorname{grad} \Psi$ . Since  $\operatorname{div} \tilde{\varphi} = 0$  and also  $\operatorname{div} u = 0$ , we have that

$$\int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} \varphi \, dx - \alpha \int_{\Omega} u \cdot \varphi \, dx + \theta \int_{\Omega} \operatorname{div} u \operatorname{div} \varphi \, dx = -\alpha \int_{\Omega} u \cdot \tilde{\varphi} \, dx + \int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} \tilde{\varphi} \, dx. \quad (3.15)$$

Moreover, since  $\operatorname{div}_{\Gamma} f = 0$  on  $\Gamma$  we have

$$\int_{\Gamma} f \cdot \varphi \, d\sigma = \int_{\Gamma} f \cdot (\tilde{\varphi} + \operatorname{grad} \Psi) \, d\sigma = \int_{\Gamma} f \cdot \tilde{\varphi} \, d\sigma. \quad (3.16)$$

By (3.15) and (3.16) we have that

$$\int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} \varphi \, dx - \alpha \int_{\Omega} u \cdot \varphi \, dx + \theta \int_{\Omega} \operatorname{div} u \operatorname{div} \varphi \, dx = - \int_{\Gamma} f \cdot \varphi \, d\sigma, \quad (3.17)$$

hence  $u$  is a solution of problem (3.2).

### 3.1 Resolvent operators

For any  $\alpha < A_1, \theta > 0$  and  $\eta \geq c_{\alpha, \theta}$  as in Theorem 3.5, we consider the operator  $\mathcal{L}_{\alpha, \theta}^{\eta}$  from  $X_{\mathbb{T}}(\Omega)$  to its dual  $(X_{\mathbb{T}}(\Omega))'$  defined by the pairing

$$\langle \mathcal{L}_{\alpha, \theta}^{\eta}(u), \varphi \rangle = \int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} \varphi \, dx - \alpha \int_{\Omega} u \cdot \varphi \, dx + \theta \int_{\Omega} \operatorname{div} u \operatorname{div} \varphi \, dx + \eta \int_{\Gamma} u \cdot \varphi \, d\sigma,$$

for all  $u, \varphi \in X_{\mathbb{T}}(\Omega)$ .

Next, we consider the operator  $\mathcal{J}$  from  $TL^2(\Gamma)$  to  $(X_{\mathbb{T}}(\Omega))'$  defined by the pairing

$$\langle \mathcal{J}(f), \varphi \rangle = \int_{\Gamma} f \cdot \varphi \, d\sigma,$$

for all  $f \in TL^2(\Gamma), \varphi \in X_{\mathbb{T}}(\Omega)$ .

Recall that  $\pi_{\mathbb{T}}(u)$  is the tangential components trace of  $u$ , which coincides with the trace of  $u$  on  $\Gamma$  for any  $u \in X_{\mathbb{T}}(\Omega)$ . By Theorem 3.5,  $\mathcal{L}_{\alpha, \theta}^{\eta}$  is invertible, hence we can introduce the operator  $\mathcal{A}_{\eta}^{\Gamma}$  defined from  $TL^2(\Gamma)$  to itself, by

$$\mathcal{A}_{\eta}^{\Gamma} := -\pi_{\mathbb{T}} \circ \left( \mathcal{L}_{\alpha, \theta}^{\eta} \right)^{-1} \circ \mathcal{J}.$$

**Theorem 3.18** *The operator  $\mathcal{A}_{\eta}^{\Gamma}$  is compact and self-adjoint in  $TL^2(\Gamma)$ .*

**Proof.** The compactness of  $\mathcal{A}_{\eta}^{\Gamma}$  follows from the fact that the classical trace operator from  $(H^1(\Omega))^3$  to  $(L^2(\Gamma))^3$  is compact and coincides with the operator  $\pi_{\mathbb{T}}$  on  $X_{\mathbb{T}}(\Omega)$ .

Regarding self-adjointness, it suffices to note that for all  $f, g \in TL^2(\Gamma)$  we have

$$\begin{aligned}
\left\langle \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J}(f), g \right\rangle_{(L^2(\Gamma))^3} &= \left\langle \mathcal{J}(\bar{g}), \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J}(f) \right\rangle \\
&= \left\langle \mathcal{L}_{\alpha, \theta}^\eta \left( \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J}(\bar{g}) \right), \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J}(f) \right\rangle \\
&= \left\langle \mathcal{L}_{\alpha, \theta}^\eta \left( \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J}(f) \right), \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J}(\bar{g}) \right\rangle \\
&= \left\langle \mathcal{J}(f), \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J}(\bar{g}) \right\rangle = \left\langle f, \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J}(g) \right\rangle_{(L^2(\Gamma))^3}. \quad (3.19)
\end{aligned}$$

□

For our purposes, it is also convenient to consider the operator  $\mathcal{A}_\eta^\Omega$  from  $X_T(\Omega)$  to itself defined by

$$\mathcal{A}_\eta^\Omega(u) = - \left( \mathcal{L}_{\alpha, \theta}^\eta \right)^{-1} \circ \mathcal{J} \circ \pi_T. \quad (3.20)$$

It is evident that

$$\mathcal{A}_\eta^\Gamma \circ \pi_T = \pi_T \circ \mathcal{A}_\eta^\Omega. \quad (3.21)$$

For  $\eta \geq 0$ , it is convenient to define the following sesquilinear form

$$\langle u, v \rangle_{\alpha, \theta}^\eta := \int_\Omega \operatorname{curl} u \cdot \operatorname{curl} \bar{v} \, dx - \alpha \int_\Omega u \cdot \bar{v} \, dx + \theta \int_\Omega \operatorname{div} u \operatorname{div} \bar{v} \, dx + \eta \int_\Gamma u \cdot \bar{v} \, d\sigma, \quad (3.22)$$

for all  $u, v \in X_T(\Omega)$  and to observe that, if  $\eta \geq c_{\alpha, \theta}$ , it defines a scalar product in  $X_T(\Omega)$  by Theorem 3.5, see also Theorem 2.1. In view of this, problem (3.2) can be written as

$$\langle u, \varphi \rangle_{\alpha, \theta}^\eta = - \langle f, \varphi \rangle_{(L^2(\Gamma))^3},$$

for all  $\varphi \in X_T(\Omega)$ , where  $\langle \cdot, \cdot \rangle_{(L^2(\Gamma))^3}$  denotes the standard scalar product in  $(L^2(\Gamma))^3$  defined by  $\langle f, \varphi \rangle_{(L^2(\Gamma))^3} := \int_\Gamma u \cdot \bar{\varphi} \, d\sigma$ .

Note that  $\langle \cdot, \cdot \rangle_{\alpha, \theta}^0$  is the sesquilinear form appearing in the left-hand side of (1.8).

Then we have the following result, the proof of which is similar to the one of the previous theorem.

**Theorem 3.23** *The operator  $\mathcal{A}_\eta^\Omega$  is compact and self-adjoint with respect to (3.22).*

### 3.2 The eigenvalue problem

In this subsection we consider the eigenvalue problem (1.7). Recall that the weak formulation of (1.7) is given in (1.8). It turns out that this eigenvalue problem can be recast as an eigenvalue problem for the operator  $\mathcal{A}_\eta^\Gamma$ , or for the operator  $\mathcal{A}_\eta^\Omega$ . Since these operators are compact and self-adjoint, their spectra can be easily described. In particular, we have the following result.

**Theorem 3.24** *Let  $\alpha < A_1$  and  $\theta > 0$ . The spectrum of the operator  $\mathcal{A}_\eta^\Omega$  can be represented as  $\{0\} \cup \{\gamma_n : n \in \mathbb{N}\}$  where  $\gamma_n$ ,  $n \in \mathbb{N}$ , are negative eigenvalues of finite multiplicity,  $\gamma_n \rightarrow 0$  as  $n \rightarrow \infty$ , and 0 is an eigenvalue of infinite multiplicity with eigenspace given by  $(H_0^1(\Omega))^3$ . Moreover, the point spectrum of the operator  $\mathcal{A}_\eta^\Gamma$  is given by  $\{\gamma_n : n \in \mathbb{N}\}$ . Furthermore, if  $\mathcal{A}_\eta^\Omega u = \gamma_n u$  for some  $u \in X_T(\Omega)$ , then  $\mathcal{A}_\eta^\Gamma \pi_T u = \gamma_n \pi_T u$ .*

**Proof.** It is easy to see that the operators  $\mathcal{A}_\eta^\Omega, \mathcal{A}_\eta^\Gamma$  have the same non-zero eigenvalues and that zero is an eigenvalue of infinite multiplicity for the operator  $\mathcal{A}_\eta^\Omega$  with eigenspace equal to  $(H_0^1(\Omega))^3$ . The rest of the proof follows by the Hilbert-Schmidt Theorem applied to the compact and self-adjoint operator  $\mathcal{A}_\eta^\Gamma$  and by implementing (3.21).  $\square$

Next, by Theorem 3.24 and using the Min-Max Principle for the compact, self-adjoint operator  $\mathcal{A}_\eta^\Omega$ , we obtain the following result.

**Theorem 3.25** *Let  $\alpha < A_1$  and  $\theta > 0$ . The eigenvalues of problem (1.7) form a sequence  $\lambda_n$ ,  $n \in \mathbb{N}$ , in  $\mathbb{R}$ , given by  $\lambda_n = \gamma_n^{-1} + \eta$ , for all  $n \in \mathbb{N}$  and the eigenfunctions coincide with those of the operator  $\mathcal{A}_\eta^\Omega$  associated with  $\gamma_n$ . Moreover,  $\lambda_n \rightarrow -\infty$ , as  $n \rightarrow \infty$ , and can be represented as*

$$\lambda_n = - \min_{\substack{V \subset X_T(\Omega) \\ \dim V = n}} \max_{u \in V \setminus (H_0^1(\Omega))^3} \frac{\int_\Omega (|\operatorname{curl} u|^2 - \alpha |u|^2 + \theta |\operatorname{div} u|^2) dx}{\int_\Gamma |\pi_T u|^2 dx}, \quad (3.26)$$

where, as usual, each eigenvalue is repeated as many times as its multiplicity.

**Proof.** It is enough to observe that  $u \in X_T(\Omega)$  and  $\lambda \in \mathbb{R}$  satisfy equation (1.8) if and only if

$$\langle u, \varphi \rangle_{\alpha, \theta}^\eta = -(\lambda - \eta) \langle u, \varphi \rangle_{(L^2(\Gamma))^3}$$

for all  $\varphi \in X_T(\Omega)$ , and this holds if and only if  $\lambda - \eta < 0$  and  $\mathcal{A}_\eta^\Omega u = \gamma u$ , where  $\gamma = (\lambda - \eta)^{-1}$ . By Theorem 3.24 we deduce the existence of the sequence of eigenvalues  $\lambda_n$ ,  $n \in \mathbb{N}$ , by the sequence  $\gamma_n$ ,  $n \in \mathbb{N}$ . Since any eigenvalue  $\lambda_n$  is bounded above by  $\eta$  and  $\gamma_n \rightarrow 0$ , we deduce that  $\lambda_n \rightarrow -\infty$  as  $n \rightarrow \infty$ . Formula (3.26) follows by the Min-Max Principle applied to the operator  $-\mathcal{A}_\eta^\Omega$  in which case one eventually obtains the min-max characterization for  $-(\lambda_n - \eta)$  which yields (3.26).  $\square$

We note that the eigenvalue problem for the operator  $\mathcal{A}_\eta^\Omega$ , can be written in the form

$$\int_\Omega \operatorname{curl} u \cdot \operatorname{curl} \varphi dx - \alpha \int_\Omega u \cdot \varphi dx + \theta \int_\Omega \operatorname{div} u \operatorname{div} \varphi dx + \eta \int_\Gamma u \cdot \varphi d\sigma = -\lambda \int_\Gamma u \cdot \varphi d\sigma,$$

for all  $\varphi \in X_T(\Omega)$ , in the unknowns  $u \in X_T(\Omega)$  (the eigenvector) and  $\lambda$  (the eigenvalue). It follows by the previous results that the space  $X_T(\Omega)$  can be decomposed as an orthogonal sum with respect to the scalar product (3.22), namely

$$X_T(\Omega) = \operatorname{Ker} \mathcal{A}_\eta^\Omega \oplus (\operatorname{Ker} \mathcal{A}_\eta^\Omega)^\perp = (H_0^1(\Omega))^3 \oplus (\operatorname{Ker} \mathcal{A}_\eta^\Omega)^\perp.$$

We note that  $u \in (\operatorname{Ker} \mathcal{A}_\eta^\Omega)^\perp$  if and only if

$$\int_\Omega \operatorname{curl} u \cdot \operatorname{curl} \varphi dx - \alpha \int_\Omega u \cdot \varphi dx + \theta \int_\Omega \operatorname{div} u \operatorname{div} \varphi dx = 0, \quad (3.27)$$

for all  $\varphi \in (H_0^1(\Omega))^3$  or, equivalently, for all  $\varphi \in (C_c^\infty(\Omega))^3$ .

Thus,  $u \in (\operatorname{Ker} \mathcal{A}_\eta^\Omega)^\perp$  if and only if  $u$  is a weak solution in  $(H^1(\Omega))^3$  of the problem

$$\begin{cases} \operatorname{curl} \operatorname{curl} u - \alpha u - \theta \operatorname{grad} \operatorname{div} u = 0, & \text{in } \Omega, \\ \nu \cdot u = 0, & \text{on } \Gamma. \end{cases} \quad (3.28)$$

By setting<sup>4</sup>

$$\mathcal{H}(\Omega) := (\text{Ker}\mathcal{A}_\eta^\Omega)^\perp,$$

we can write

$$X_T(\Omega) = (H_0^1(\Omega))^3 \oplus \mathcal{H}(\Omega). \quad (3.29)$$

By these observations, we deduce the validity of the following

**Corollary 3.30** *The operator  $\mathcal{A}_\eta^\Gamma$  has no kernel, that is  $\text{Ker}\mathcal{A}_\eta^\Gamma = \{0\}$ .*

**Proof.** Assume that  $f \in \text{Ker}\mathcal{A}_\eta^\Gamma$ , that is  $-\pi_T \circ (\mathcal{L}_{\alpha,\theta}^\eta)^{-1} \circ \mathcal{J}(f) = 0$ . We set

$$u = (\mathcal{L}_{\alpha,\theta}^\eta)^{-1} \circ \mathcal{J}(f)$$

and we observe that, in particular,  $u$  satisfies equation (3.27) for all  $\varphi \in (H_0^1(\Omega))^3$  (note that in this step of the proof we have used only test functions vanishing at the boundary). Thus  $u \in \mathcal{H}(\Omega)$ . Since by assumption  $\pi_T u = 0$ , we have that  $u \in (H_0^1(\Omega))^3$ . Thus, by (3.29) we have  $u = 0$ . Going back to the equation satisfied by  $u$ , we obtain that  $\int_\Gamma f \cdot \varphi \, d\sigma = 0$  for all  $\varphi \in X_T(\Omega)$ , and this implies that  $f = 0$  since  $f$  is a tangential field (note that in this second step, we have used all test functions  $\varphi \in X_T(\Omega)$  in the weak formulation of the equation).  $\square$

### 3.3 Interior Calderón operator

In this section we introduce a Calderón operator associated with the interior problem (3.2). In order to identify the appropriate condition under which our Calderón operator is well-defined we need the following result.

In the sequel we denote by  $\Sigma = \{\lambda_n : n \in \mathbb{N}\}$  the set of Steklov eigenvalues of problem (1.8).

**Theorem 3.31** *Let  $\alpha < A_1$  and  $\theta > 0$ . Then the problem*

$$\begin{cases} \text{curl curl} u - \alpha u - \theta \text{grad div} u = 0, & \text{in } \Omega, \\ \nu \times \text{curl} u = \lambda u + f, & \text{on } \Gamma \end{cases} \quad (3.32)$$

*is uniquely solvable in  $X_T(\Omega)$  for all  $f \in TL^2(\Gamma)$  if and only if  $\lambda \notin \Sigma$ .*

**Proof.** Note that problem (3.32) can be written in the weak form as

$$\mathcal{L}_{\alpha,\theta}^\eta(u) = (\eta - \lambda)\mathcal{J}(u) - \mathcal{J}(f). \quad (3.33)$$

Inverting the operator  $\mathcal{L}_{\alpha,\theta}^\eta$ , problem (3.33) turns out to be equivalent to

$$u = (\lambda - \eta)\mathcal{A}_\eta^\Omega(u) - (\mathcal{L}_{\alpha,\theta}^\eta)^{-1} \mathcal{J}(f). \quad (3.34)$$

Since  $\mathcal{A}_\eta^\Omega$  is a compact operator, it follows by the Fredholm Alternative that problem (3.34) is uniquely solvable if and only if 1 is not an eigenvalue of the operator  $(\lambda - \eta)\mathcal{A}_\eta^\Omega$ , and this exactly means that  $\lambda \notin \Sigma$ .  $\square$

Then we can give the following definition.

---

<sup>4</sup>These functions are the analogues of the harmonic functions considered in [2].

**Definition 3.35** Assume that  $\alpha < A_1$  and  $\theta > 0$  are such that  $0 \notin \Sigma$ . The interior Calderón operator  $\mathcal{C}$  is the operator defined from  $TL^2(\Gamma)$  to itself, mapping any  $f \in TL^2(\Gamma)$  to  $\mathcal{C}(f) := \nu \times u$  on  $\Gamma$ , where  $u \in X_T(\Omega)$  is the solution of (1.4) (i.e., of (3.2) with  $\eta = 0$ ).

We conclude this section by discussing the condition  $0 \notin \Sigma$ . To do so, we consider two auxiliary eigenvalue problems. The first is the classical eigenvalue problem for the Neumann Laplacian

$$\begin{cases} -\Delta\phi = \lambda\phi, & \text{in } \Omega, \\ D_\nu\phi = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.36)$$

for  $\phi \in H^1(\Omega)$ , which is well-known to admit a divergent sequence  $\lambda_n^N$ ,  $n \in \mathbb{N}$ , of non-negative eigenvalues of finite multiplicity, with  $\lambda_1^N = 0$ . The second is the eigenvalue problem for the curl curl operator with “magnetic” boundary conditions

$$\begin{cases} \text{curl curl}u = \lambda u, & \text{in } \Omega, \\ \nu \times \text{curl}u = 0, & \text{on } \Gamma, \end{cases} \quad (3.37)$$

for  $u \in X_T(\text{div } 0, \Omega)$ , which also admits a divergent sequence  $\lambda_n^M$ ,  $n \in \mathbb{N}$ , of non-negative eigenvalues of finite multiplicity, with  $\lambda_1^M = 0$ , see e.g., [29] for the relation between problem (3.37) and the eigenvalue problem for Maxwell’s system.

Consider now problem (1.7) with  $\lambda = 0$ , namely

$$\begin{cases} \text{curl curl}u - \alpha u - \theta \text{grad div}u = 0, & \text{in } \Omega, \\ \nu \times \text{curl}u = 0, & \text{on } \Gamma. \end{cases} \quad (3.38)$$

Then we can prove the following theorem where we do not put any *a priori* restrictions on  $\alpha$  and  $\theta$ . For the values of  $\alpha$  and  $\theta$  for which  $\Sigma$  is well-defined, the following statement gives a necessary and sufficient condition for the validity of the hypothesis  $0 \notin \Sigma$ . Note that the following theorem could be considered as the “magnetic” version of [11, Theorem 1.1].

**Theorem 3.39** Problem (3.38) has a non-trivial solution  $u \in X_T(\Omega)$  if and only if  $\alpha \in \{\theta\lambda_n^N : n \in \mathbb{N}\} \cup \{\lambda_n^M : n \in \mathbb{N}\}$ .

**Proof.** Since the case  $\theta = 0$  is trivial, we assume that  $\theta \neq 0$ .

Clearly, for divergence free fields, (3.38) reads

$$\begin{cases} \text{curl curl}u = \alpha u, & \text{in } \Omega, \\ \nu \times \text{curl}u = 0, & \text{on } \Gamma, \end{cases} \quad (3.40)$$

and if  $\alpha = \lambda_j^M$  for some  $j \in \mathbb{N}$ , then the eigenfunctions corresponding to this  $\lambda_j^M$  solve problem (3.38).

Assume now that  $\lambda = \frac{\alpha}{\theta}$  is an eigenvalue of problem (3.36). Thus there exists a non-trivial solution  $\phi \in H^1(\Omega)$  to (3.36). In particular,  $\phi \in H^2(\Omega)$  by standard regularity theory. Then, it is readily seen that the function  $u = \text{grad } \phi$  is a non-trivial solution to (3.38) if  $\phi$  is not constant.

Thus, we have proved that if  $\alpha \in \{\theta\lambda_n^N : n \in \mathbb{N}\} \cup \{\lambda_n^M : n \in \mathbb{N}\}$  then there exists a non-trivial solution  $u \in X_T(\Omega)$  to (3.38).

We now prove the converse statement. Assume that  $u \in X_T(\Omega)$  is a non-trivial solution to (3.38) and assume that  $\alpha \notin \{\theta\lambda_n^N : n \in \mathbb{N}\}$ . Looking at the weak formulation

of problem (3.38), namely problem (1.8) with  $\lambda = 0$ , and setting  $\varphi = \text{grad}\psi$  with  $\psi \in H^2(\Omega)$  and  $D_\nu\psi = 0$ , we get

$$0 = -\alpha \int_{\Omega} u \cdot \text{grad}\psi \, dx + \theta \int_{\Omega} \text{div}u \, \Delta\psi \, dx = \theta \int_{\Omega} \text{div}u \left( \frac{\alpha}{\theta}\psi + \Delta\psi \right) dx. \quad (3.41)$$

Since  $\frac{\alpha}{\theta}$  is assumed to be in the resolvent of the Neumann Laplacian, the map  $\frac{\alpha}{\theta}I + \Delta$  is a bijection between  $\{\psi \in H^2(\Omega) : D_\nu\psi = 0\}$  (the domain of the Neumann Laplacian for  $\Omega$  of class  $C^{1,1}$ ) and  $L^2(\Omega)$ . Thus, by the arbitrary choice of  $\psi$  in (3.41), we deduce that  $\text{div}u = 0$ , hence  $u$  is a solution of (3.40). In particular,  $\alpha$  belongs to  $\{\lambda_n^M : n \in \mathbb{N}\}$ .  $\square$

## 4 Spectral representations

Let  $\Omega$  be a bounded connected open set in  $\mathbb{R}^3$  with boundary  $\Gamma := \partial\Omega \in C^{1,1}$ . Throughout this section we again assume that  $\alpha < A_1$  and  $\theta > 0$ . Recall that the space  $X_{\text{T}}(\Omega)$  can be considered as a Hilbert space with respect to the scalar product defined by (3.22). Moreover, the space  $\mathcal{H}(\Omega)$  of solutions to problem (3.28) is a closed subspace and it admits a Hilbert basis of Steklov eigenfunctions  $u_n^\Omega$ ,  $n \in \mathbb{N}$ , which satisfy the equation

$$\int_{\Omega} \text{curl}u_n^\Omega \cdot \text{curl}\varphi \, dx - \alpha \int_{\Omega} u_n^\Omega \cdot \varphi \, dx + \theta \int_{\Omega} \text{div}u_n^\Omega \, \text{div}\varphi \, dx = -\lambda_n \int_{\Gamma} u_n^\Omega \cdot \varphi \, d\sigma, \quad (4.1)$$

for all  $\varphi \in X_{\text{T}}(\Omega)$ . Note that equation (4.1) can be equivalently written as

$$\langle u_n^\Omega, \varphi \rangle_{\alpha, \theta}^\eta = -(\lambda_n - \eta) \langle u_n^\Omega, \varphi \rangle_{(L^2(\Gamma))^3}, \quad (4.2)$$

for all  $\varphi \in X_{\text{T}}(\Omega)$ .

In the sequel the eigenfunctions  $u_n^\Omega$ ,  $n \in \mathbb{N}$ , will be normalized with respect to the scalar product (3.22), namely we shall assume that

$$\langle u_n^\Omega, u_m^\Omega \rangle_{\alpha, \theta}^\eta = \delta_{nm},$$

where  $\delta_{nm}$  is the Kronecker symbol.

Taking into account Theorem 3.24 and the fact that the traces of the eigenfunctions  $u_n^\Omega$  provide also a basis of the space  $TL^2(\Gamma)$ , we set

$$u_n^\Gamma := \sqrt{|\lambda_n - \eta|} \pi_{\text{T}} u_n^\Omega \quad (4.3)$$

and, in view of (4.2), we observe that  $u_n^\Gamma$ ,  $n \in \mathbb{N}$ , is an orthonormal basis of  $TL^2(\Gamma)$ .

We proceed by proving the following theorem which provides a spectral representation for the solutions of problem (1.4) (i.e., of (3.2) with  $\eta = 0$ ) for data  $f \in TL^2(\Gamma)$  and a corresponding spectral representation for the associated Calderón operator.

Recall that, by Theorem 3.31, if  $0 \notin \Sigma$  then problem (1.4) is uniquely solvable. In Theorem 4.20 we shall prove the same result for data  $f \in TH^{-1/2}(\Gamma)$  and this will allow to extend the Calderón operator and define it as an operator from  $TH^{-1/2}(\Gamma)$  to  $TH^{1/2}(\Gamma)$ , as one would expect.

**Theorem 4.4** Assume that  $0 \notin \Sigma$ . Let  $f \in TL^2(\Gamma)$  be represented as

$$f = \sum_{n=1}^{\infty} c_n u_n^\Gamma,$$

where  $(c_n)_{n \in \mathbb{N}} \in \ell^2$ . Then the solution  $u \in X_T(\Omega)$  of problem (1.4) (i.e., of (3.2) with  $\eta = 0$ ), is given by

$$u = \sum_{n=1}^{\infty} \left( \frac{\sqrt{|\lambda_n - \eta|}}{\lambda_n} c_n \right) u_n^\Omega. \quad (4.5)$$

Moreover, the corresponding interior Calderón operator can be represented as

$$\mathcal{C}(f) = \nu \times \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} u_n^\Gamma. \quad (4.6)$$

**Proof.** Since  $(c_n)_{n \in \mathbb{N}} \in \ell^2$  and  $\lambda_n \rightarrow -\infty$ , it is obvious that the series (4.5) converges in  $X_T(\Omega)$ . By the continuity of the form  $\langle \cdot, \cdot \rangle_{\alpha, \theta}^0$  and of the trace operator, it follows that it is obvious that

$$\begin{aligned} \langle u, \varphi \rangle_{\alpha, \theta}^0 &= \sum_{n=1}^{\infty} \left( \frac{\sqrt{|\lambda_n - \eta|}}{\lambda_n} c_n \right) \langle u_n^\Omega, \varphi \rangle_{\alpha, \theta}^0 \\ &= - \sum_{n=1}^{\infty} c_n \sqrt{|\lambda_n - \eta|} \langle u_n^\Omega, \varphi \rangle_{(L^2(\Gamma))^3} \\ &= - \sum_{n=1}^{\infty} c_n \langle u_n^\Gamma, \varphi \rangle_{(L^2(\Gamma))^3} = - \langle f, \varphi \rangle_{(L^2(\Gamma))^3}, \end{aligned} \quad (4.7)$$

for all  $\varphi \in X_T(\Omega)$ . This means that  $u$  is a solution of problem (3.2) with  $\eta = 0$ . Formula (4.6) follows immediately from (4.5).  $\square$

In the spirit of [2], for  $s > 0$  we define the space  $T\mathcal{H}^s(\Gamma)$  by

$$T\mathcal{H}^s(\Gamma) := \left\{ f = \sum_{n=1}^{\infty} c_n u_n^\Gamma : \|f\|_{s, \Gamma} := \left( \sum_{n=1}^{\infty} |c_n|^2 |\lambda_n - \eta|^{2s} \right)^{1/2} < \infty \right\}, \quad (4.8)$$

endowed with the norm  $\|\cdot\|_{s, \Gamma}$ , and we define the space  $T\mathcal{H}^{-s}(\Gamma)$  as the dual of  $T\mathcal{H}^s(\Gamma)$ . Theorem 4.12(i) shows that for  $s = 1/2$  and  $s = -1/2$  this definition is equivalent to any of the classical definitions of the trace space: in other words  $T\mathcal{H}^{1/2}(\Gamma) = TH^{1/2}(\Gamma)$  and  $T\mathcal{H}^{-1/2}(\Gamma) = TH^{-1/2}(\Gamma)$ , see Section 2.

It is easy to see that the space  $T\mathcal{H}^{-s}(\Gamma)$  can be identified with a space of sequences, namely

$$\left\{ F = (c_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|F\|_{-s, \Gamma} := \left( \sum_{n=1}^{\infty} |c_n|^2 |\lambda_n - \eta|^{-2s} \right)^{1/2} < \infty \right\}, \quad (4.9)$$

with the understanding that the action of an element  $F = (c_n)_{n \in \mathbb{N}} \in T\mathcal{H}^{-s}(\Gamma)$  on  $f = \sum_{n=1}^{\infty} d_n u_n^\Gamma \in T\mathcal{H}^s(\Gamma)$  is given by the pairing

$$\langle F, f \rangle = \sum_{n=1}^{\infty} c_n d_n, \quad (4.10)$$



which means that  $\langle F, u_n^\Gamma \rangle = c_n$  for all  $n \in \mathbb{N}$ .

We note that any function  $F \in TL^2(\Gamma)$  defines an element of  $T\mathcal{H}^{-s}(\Gamma)$  by means of the formula

$$\langle F, f \rangle = \int_{\Gamma} F \cdot f \, d\sigma, \quad \forall f \in T\mathcal{H}^s(\Gamma).$$

In this case, in order to recover formula (4.10),  $F$  should be represented with respect to the basis  $\bar{u}_n^\Gamma := \overline{u_n^\Gamma}$ ,  $n \in \mathbb{N}$ . Indeed, if  $F = \sum_{n=1}^{\infty} c_n \bar{u}_n^\Gamma$ , then  $\langle F, f \rangle = \int_{\Gamma} F \cdot f \, d\sigma = \sum_{n,m=1}^{\infty} c_n d_m \langle u_m^\Gamma, u_n^\Gamma \rangle_{(L^2(\Gamma))^3} = \sum_{n=1}^{\infty} c_n d_n$ . Thus, we may think of  $\bar{u}_n^\Gamma$ ,  $n \in \mathbb{N}$ , as a dual basis and represent formally any element  $F \in T\mathcal{H}^{-s}(\Gamma)$  as  $F = \sum_{n=1}^{\infty} c_n \bar{u}_n^\Gamma$ . Note that since the coefficients of our operator are real, it follows that if  $u$  is an eigenfunction then also  $\bar{u}$  is an eigenfunction (see also the weak formulation (1.8)); hence one could choose a basis of real eigenfunctions in which case one should not worry about passing from  $u_n^\Gamma$  to its complex conjugate.)

As we have already mentioned in the introduction, the following theorem allows in particular to characterize the space  $T\mathcal{H}^{1/2}(\Gamma)$  as the trace space of  $X_{\mathbb{T}}(\Omega)$ . It also provides spectral representations for the solutions of the problem

$$\begin{cases} \operatorname{curl} \operatorname{curl} u - \alpha u - \theta \operatorname{grad} \operatorname{div} u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Gamma, \end{cases} \quad (4.11)$$

and of problem (1.9). We understand the solutions to problem (4.11) as functions  $u \in \mathcal{H}(\Omega)$  satisfying the condition  $\pi_{\mathbb{T}} u = u = f$ . Note that by (3.29) if  $f = 0$ , the unique solution is  $u = 0$  since  $u$  would have to belong also to the space  $(H_0^1(\Omega))^3$ . Thus, the solution to problem (4.11) for all admissible data  $f$  as below will be unique.

**Theorem 4.12** *The following statements hold*

(i) *The image of the trace operator  $\pi_{\mathbb{T}}$  is given by*

$$\pi_{\mathbb{T}}(X_{\mathbb{T}}(\Omega)) = T\mathcal{H}^{1/2}(\Gamma), \quad (4.13)$$

*hence  $T\mathcal{H}^{1/2}(\Gamma)$  coincides with the usual Sobolev space  $TH^{1/2}(\Gamma)$ .*

(ii) *Let  $f \in T\mathcal{H}^{1/2}(\Gamma)$  be represented as*

$$f = \sum_{n=1}^{\infty} c_n u_n^\Gamma, \quad (4.14)$$

*where  $(c_n \sqrt{|\lambda_n - \eta|})_{n \in \mathbb{N}} \in \ell^2$ . Then the solution  $u \in X_{\mathbb{T}}(\Omega)$  of problem (4.11) is given by*

$$u = \sum_{n=1}^{\infty} c_n \sqrt{|\lambda_n - \eta|} u_n^\Omega. \quad (4.15)$$

(iii) *Let  $f \in T\mathcal{H}^{1/2}(\Gamma)$  and let  $f \times \nu$  be represented as*

$$f \times \nu = \sum_{n=1}^{\infty} c_{n,\nu} u_n^\Gamma, \quad (4.16)$$

*where  $(c_{n,\nu} \sqrt{|\lambda_n - \eta|})_{n \in \mathbb{N}} \in \ell^2$ . Then the solution  $u \in X_{\mathbb{T}}(\Omega)$  of problem (1.9) is given by*

$$u = \sum_{n=1}^{\infty} c_{n,\nu} \sqrt{|\lambda_n - \eta|} u_n^\Omega. \quad (4.17)$$

**Proof.** Let  $u \in X_T(\Omega)$ . By (3.29) we have that  $u = u_0 + \tilde{u}$ , where  $u_0 \in (H_0^1(\Omega))^3$  and  $\tilde{u} \in \mathcal{H}(\Omega)$ . In particular,  $\tilde{u}$  can be written as  $\tilde{u} = \sum_{n=1}^{\infty} c_n u_n^\Omega$  with  $(c_n)_{n \in \mathbb{N}} \in \ell^2$ . Thus

$$\pi_T u = \pi_T \tilde{u} = \sum_{n=1}^{\infty} c_n \pi_T u_n^\Omega = \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{|\lambda_n - \eta|}} u_n^\Gamma,$$

which clearly implies that  $\pi_T u \in T\mathcal{H}^{1/2}(\Gamma)$ . Thus,  $\pi_T(X_T(\Omega)) \subset T\mathcal{H}^{1/2}(\Gamma)$ .

In order to prove the reverse inclusion, we consider a function  $f \in T\mathcal{H}^{1/2}(\Gamma)$  represented as in (4.14) and we observe that the series in (4.15) is convergent in  $X_T(\Omega)$ . Thus, if  $u$  is the function defined by (4.15), we have that

$$\pi_T u = \sum_{n=1}^{\infty} c_n \sqrt{|\lambda_n - \eta|} \pi_T u_n^\Omega = \sum_{n=1}^{\infty} c_n u_n^\Gamma = f, \quad (4.18)$$

which shows that  $f \in \pi_T(X_T(\Omega))$ . Thus,  $T\mathcal{H}^{1/2}(\Gamma) \subset \pi_T(X_T(\Omega))$ .

The equality (4.18) gives also the proof of statement (ii), because, by definition, the function  $u$  in (4.15) belongs to  $\mathcal{H}(\Omega)$  hence it is a solution of (3.28). Moreover, since  $u \cdot \nu = 0$  on  $\Gamma$ , we have that  $\pi_T u$  coincides with the trace of  $u$ , hence (with a slight abuse of notation) we have that  $\pi_T u = u$  on  $\Gamma$ . Thus, by (4.18), the function  $u$  satisfies also the condition  $u = f$  on  $\Gamma$  required by problem (4.11), hence  $u$  is a solution of (4.11).

The proof of statement (iii) follows directly by statement (ii) because the latter implies that the function  $u$  defined by (4.17) is a solution of problem (4.11) with  $f$  replaced by  $f \times \nu$ . Thus  $u = f \times \nu$  on  $\Gamma$ . It follows that  $\nu \times u = \nu \times (f \times \nu) = f$  as required in problem (1.9), hence the function  $u$  defined by (4.17) is a solution of problem (1.9).  $\square$

Theorem 4.12 allows to consider equation (1.4) also with a datum  $f$  replaced by an element  $F \in T\mathcal{H}^{-1/2}(\Gamma)$ , in which case the formulation would read as follows: find  $u \in X_T(\Omega)$  such that

$$\langle u, \varphi \rangle_{\alpha, \theta}^0 = -\langle F, \bar{\varphi} \rangle, \quad (4.19)$$

for all  $\varphi \in X_T(\Omega)$ . Indeed, the trace of  $\varphi$  on  $\Gamma$  belongs to  $T\mathcal{H}^{1/2}(\Gamma)$  hence the right-hand side of equality (4.19) is well-defined. Note that the following theorem is stated with the use of an orthonormal basis of eigenfunctions which are not necessarily real. However, as we mentioned before, it is always possible to select an orthonormal basis of real eigenfunctions. In accordance to the notation  $\bar{u}_n^\Gamma = \overline{u_n^\Gamma}$  introduced before, we set  $\bar{u}_n^\Omega := \overline{u_n^\Omega}$ .

**Theorem 4.20** *Assume that  $0 \notin \Sigma$ . Let  $F \in T\mathcal{H}^{-1/2}(\Gamma)$  be represented as*

$$F = \sum_{n=1}^{\infty} c_n \bar{u}_n^\Gamma, \quad (4.21)$$

where  $(c_n |\lambda_n - \eta|^{-1/2})_{n \in \mathbb{N}} \in \ell^2$ . Then the solution  $u \in X_T(\Omega)$  of problem (4.19) is given by

$$u = \sum_{n=1}^{\infty} \left( \frac{\sqrt{|\lambda_n - \eta|}}{\lambda_n} c_n \right) \bar{u}_n^\Omega. \quad (4.22)$$

**Proof.** Note that since  $(c_n|\lambda_n - \eta|^{-1/2})_{n \in \mathbb{N}} \in \ell^2$ , the series (4.22) converges in  $X_T(\Omega)$ . Let  $\bar{\varphi} = \sum_{n=1}^{\infty} d_n u_n^\Gamma \in T\mathcal{H}^{1/2}(\Gamma)$  with  $(d_n \sqrt{|\lambda_n - \eta|})_{n \in \mathbb{N}} \in \ell^2$ . By the continuity of the bilinear form  $\langle \cdot, \cdot \rangle_{\alpha, \theta}^0$  we have that

$$\begin{aligned} \langle u, \varphi \rangle_{\alpha, \theta}^0 &= \sum_{n=1}^{\infty} \left( \frac{\sqrt{|\lambda_n - \eta|}}{\lambda_n} c_n \right) \langle \bar{u}_n^\Omega, \varphi \rangle_{\alpha, \theta}^0 = - \sum_{n=1}^{\infty} c_n \sqrt{|\lambda_n - \eta|} \langle \bar{u}_n^\Omega, \varphi \rangle_{(L^2(\Gamma))^3} \\ &= - \sum_{n=1}^{\infty} c_n \int_{\Gamma} \bar{u}_n^\Gamma \sum_{m=1}^{\infty} d_m u_m^\Gamma d\sigma = - \sum_{n=1}^{\infty} c_n d_n = - \langle F, \bar{\varphi} \rangle, \end{aligned} \quad (4.23)$$

which means that  $u$  is a solution of problem (4.19).  $\square$

**Remark 4.24** *By formula (4.22), it follows that the interior Calderón operator defined in Definition 3.35 can be extended from  $TL^2(\Gamma)$  to  $T\mathcal{H}^{-1/2}(\Gamma)$  by setting*

$$\mathcal{C}(F) = \nu \times \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n} \bar{u}_n^\Gamma, \quad (4.25)$$

for all  $F \in T\mathcal{H}^{-1/2}(\Gamma)$  represented as in (4.21), with  $\mathcal{C}(F)$  being an element of  $T\mathcal{H}^{1/2}(\Gamma)$ .

## 5 Appendix: the case $\alpha > A_1$

Our approach allows to treat also the case  $a > A_1$ . First of all, one should note that by standard spectral theory, problem (3.4) has a divergent sequence of positive eigenvalues  $A_n$ ,  $n \in \mathbb{N}$  with finite multiplicity. Assume that  $\alpha \in \mathbb{R}$  is such that  $A_n < \alpha < A_{n+1}$  for some  $n \in \mathbb{N}$ .

Let  $V_n$  be the subspace of  $(H_0^1(\Omega))^3$  generated by all eigenfunctions associated with all eigenvalues  $A_k$  with  $k \leq n$ , and let

$$V_n^\perp = \{v \in X_T(\Omega) : \langle v, u \rangle_{\alpha, \theta}^0 = 0, \forall u \in V_n\}.$$

Clearly,  $V_n^\perp$  is a closed subspace of the  $X_T(\Omega)$ . Then we have the following result.

**Theorem 5.1** *Let  $\Omega$  be a bounded, connected open set in  $\mathbb{R}^3$  with  $C^{1,1}$  boundary. Assume that  $A_n < \alpha < A_{n+1}$  for some  $n \in \mathbb{N}$ , and let  $\theta > 0$ . Then*

$$X_T(\Omega) = V_n \oplus V_n^\perp, \quad (5.2)$$

and there exists  $c_{\alpha, \theta} \geq 0$  such that for any  $\eta \geq c_{\alpha, \theta}$  the quadratic form defined by the left-hand side of (3.2) is coercive in  $V_n^\perp$ .

**Proof.** We note that  $\langle \cdot, \cdot \rangle_{\alpha, \theta}^0$  is not necessarily a scalar product, hence the proof of (5.2) requires some justification. Given  $v \in X_T(\Omega)$ , the weak problem

$$\langle u, \varphi \rangle_{\alpha, \theta}^0 = \langle v, \varphi \rangle_{\alpha, \theta}^0, \quad \forall \varphi \in V_n,$$

has a solution  $u \in V_n$ . This can be proved by finding a critical point of the functional  $u \mapsto \frac{1}{2} \langle u, u \rangle_{\alpha, \theta}^0 - \langle v, u \rangle_{\alpha, \theta}^0$  in the finite dimensional space  $V_n$ . To do so, note that for all  $u \in V_n$  we have

$$\int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx \leq A_n \int_{\Omega} |u|^2 dx \quad (5.3)$$

hence  $\langle u, u \rangle_{\alpha, \theta}^0 \leq -\varrho \int_{\Omega} |u|^2 dx$ , where  $\varrho = \alpha - A_n > 0$ . Thus for  $u \in V_n$ ,  $u \neq 0$  we have

$$\begin{aligned} & \frac{1}{2} \langle u, u \rangle_{\alpha, \theta}^0 - \langle v, u \rangle_{\alpha, \theta}^0 \\ & \leq -\frac{\varrho}{2} \int_{\Omega} |u|^2 dx - \langle v, u \rangle_{\alpha, \theta}^0 = -\frac{\varrho}{2} \|u\|_{L^2(\Omega)}^2 - \|u\|_{L^2(\Omega)} \left\langle v, \frac{u}{\|u\|_{L^2(\Omega)}} \right\rangle_{\alpha, \theta}^0 \\ & \leq -\frac{\varrho}{2} \|u\|_{L^2(\Omega)}^2 - \|u\|_{L^2(\Omega)} \min_{\substack{\|u\|_{L^2(\Omega)}=1 \\ u \in V_n}} \langle v, u \rangle_{\alpha, \theta}^0, \end{aligned} \quad (5.4)$$

which implies that  $\frac{1}{2} \langle u, u \rangle_{\alpha, \theta}^0 - \langle v, u \rangle_{\alpha, \theta}^0 \rightarrow -\infty$  as  $\|u\|_{L^2(\Omega)} \rightarrow \infty$ . Thus the map  $u \mapsto \frac{1}{2} \langle u, u \rangle_{\alpha, \theta}^0 - \langle v, u \rangle_{\alpha, \theta}^0$  has actually a maximum in  $V_n$ , hence a critical point, as required.

Then  $v = u + (v - u)$  with  $v - u \in V_n^\perp$ , hence  $X_T(\Omega) = V_n + V_n^\perp$ .

On the other hand, the relation  $V_n \cap V_n^\perp = \{0\}$  can be proved as follows. Assume by contradiction that there exists  $u \in V_n$  such that  $\langle u, \varphi \rangle_{\alpha}^0 = 0$  for all  $\varphi \in V_n$ , then  $\int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx = \alpha \int_{\Omega} |u|^2 dx$  which opposes (5.3). Thus, (5.2) holds.

We claim that

$$\tilde{A}_{n+1} := \inf_{\substack{v \in V_n^\perp \cap (H_0^1(\Omega))^3 \\ v \neq 0}} \frac{\int_{\Omega} |\operatorname{curl} v|^2 dx + \theta \int_{\Omega} |\operatorname{div} v|^2 dx}{\int_{\Omega} |v|^2 dx} > \alpha. \quad (5.5)$$

Indeed, assume for contradiction that  $\tilde{A}_{n+1} \leq \alpha$ . By standard arguments, we can find a minimizer  $v \in V_n^\perp \cap H_0^1(\Omega)$  such that  $\int_{\Omega} |\operatorname{curl} v|^2 dx + \theta \int_{\Omega} |\operatorname{div} v|^2 dx = \tilde{A}_{n+1} \int_{\Omega} |v|^2 dx$  and we can assume that  $\int_{\Omega} |v|^2 dx = 1$ . Consider now the  $(n+1)$ -dimensional space  $W_{n+1} = V_n + \langle v \rangle$ . Given an element  $w = u + \xi v \in W_n$ , with  $\xi \in \mathbb{C}$ , we easily see that

$$\begin{aligned} & \int_{\Omega} |\operatorname{curl} w|^2 dx + \theta \int_{\Omega} |\operatorname{div} w|^2 dx = \int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx \\ & \quad + |\xi|^2 \left( \int_{\Omega} |\operatorname{curl} v|^2 dx + \theta \int_{\Omega} |\operatorname{div} v|^2 dx \right) \\ & \quad + 2\operatorname{Re} \bar{\xi} \left( \int_{\Omega} \operatorname{curl} u \operatorname{curl} \bar{v} dx + \theta \int_{\Omega} \operatorname{div} u \operatorname{div} \bar{v} dx \right) \\ & = \int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx + |\xi|^2 \left( \int_{\Omega} |\operatorname{curl} v|^2 dx + \theta \int_{\Omega} |\operatorname{div} v|^2 dx \right) \\ & \quad + 2\alpha \operatorname{Re} \bar{\xi} \int_{\Omega} u \bar{v} dx \leq A_n \int_{\Omega} |u|^2 dx + |\xi|^2 \tilde{A}_{n+1} + 2\alpha \operatorname{Re} \bar{\xi} \int_{\Omega} u \bar{v} dx \\ & \leq \alpha \left( \int_{\Omega} |u|^2 dx + |\xi|^2 + 2\operatorname{Re} \bar{\xi} \int_{\Omega} u \bar{v} dx \right) = \alpha \int_{\Omega} |w|^2 dx \end{aligned} \quad (5.6)$$

This implies that

$$\sup_{\substack{w \in W_{n+1} \\ w \neq 0}} \frac{\int_{\Omega} |\operatorname{curl} w|^2 dx + \theta \int_{\Omega} |\operatorname{div} w|^2 dx}{\int_{\Omega} |w|^2 dx} \leq \alpha$$

and by the Min-Max Principle

$$A_{n+1} = \inf_{\substack{W \subset (H_0^1(\Omega))^3 \\ \dim W \leq n+1}} \sup_{\substack{w \in W \\ w \neq 0}} \frac{\int_{\Omega} |\operatorname{curl} w|^2 dx + \theta \int_{\Omega} |\operatorname{div} w|^2 dx}{\int_{\Omega} |w|^2 dx} \leq \alpha$$

which is clearly a contradiction since  $\alpha < A_{n+1}$ . The claim is proved.

Inequality (5.5) allows to prove that for a fixed  $\beta \in (\alpha, \tilde{A}_{n+1})$ , there exists  $M > 0$  such that

$$\int_{\Omega} |\operatorname{curl} u|^2 dx + \theta \int_{\Omega} |\operatorname{div} u|^2 dx + M \int_{\Gamma} |u|^2 d\sigma \geq \beta \int_{\Omega} |u|^2 dx, \quad (5.7)$$

for all  $u \in V_n^\perp$ : the argument is the same one used in the proof of Theorem 3.5, where the role of the space  $X_T(\Omega)$  is now played by  $V_n^\perp$ . Again, by the same argument used in the proof of Theorem 3.5, one can easily deduce by (5.7) the coercivity of the quadratic form in the statement.  $\square$

Now we observe that, whenever  $u \in V_n^\perp$ , one can equivalently consider, in the weak formulation of problem (3.2), only test functions  $\varphi \in V_n^\perp$ ; indeed, adding to  $\varphi$  a function  $\tilde{\varphi} \in V_n$  leaves invariant both sides of the equation. Therefore, all the analysis carried out for the various problems discussed in this paper, can still be performed without any essential modifications; it suffices to replace the energy space  $X_T(\Omega)$  by the smaller energy space  $V_n^\perp$ , thus excluding the Dirichlet eigenfunctions given by  $V_n$ .

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### References

- [1] Assous, F., Ciarlet, P., Labrunie, S., *Mathematical Foundations of Computational Electromagnetism*, Applied Mathematical Sciences - Vol. 198, Springer International Publishing AG, part of Springer Nature, Cham, Switzerland, 2018.
- [2] Auchmuty, G. Spectral characterization of the trace spaces  $H^s(\partial\Omega)$ , *SIAM J. Math. Anal.* 38 (2006), 894-905.
- [3] Boulmezaoud, T.-Z., Maday, Y., Amari, T., On the linear force-free fields in bounded and unbounded three-dimensional domains, *ESAIM Math. Model. Numer. Anal.* 33 (1999), 359-393.
- [4] Cakoni, F., Colton, D., Meng, S., Monk, P., Stekloff eigenvalues in inverse scattering, *SIAM J. Appl. Math.* 76 (2016), 1737-1763.

- [5] Camanõ, J., Lackner, C., Monk, P., Electromagnetic Stekloff eigenvalues in inverse scattering, *SIAM J. Math. Anal.* 49 (2017), 4376-4401.
- [6] Cessenat, M., *Mathematical Methods in Electromagnetics, Linear Theory and Applications*, Series on Advances in Mathematics for Applied Sciences - Vol. 41, World Scientific Publishing, Singapore, 1996.
- [7] Cogar, S., Colton, D., Monk, P., Eigenvalue problems in inverse electromagnetic scattering theory, in “Maxwell’s Equations: Analysis and Numerics” (Langer, U., Pauly, D., Repin S., eds.), Radon Series on Computational and Applied Mathematics vol. 24, De Gruyter, Berlin, 2019, 145-169.
- [8] Cogar, S., *Analysis of a trace class Stekloff eigenvalue problem arising in inverse scattering*, *SIAM J. Appl. Math.* 80 (2020), 881-905.
- [9] Cogar, S., Monk, P., *Existence and stability of electromagnetic Stekloff eigenvalues with a trace class modification*, arXiv:2005.14277v1 [math.AP] 28 May 2020.
- [10] Cogar, S., *Existence and stability of electromagnetic Stekloff eigenvalues with a trace class modification*, arXiv:2006.16428v1 [math.AP] 29 Jun 2020.
- [11] Costabel, M., Dauge, M., Maxwell and Lamé eigenvalues on polyhedra, *Math. Methods Appl. Sci.* 22 (1999), 243-258.
- [12] Dautray, R., Lions, J.-L., *Mathematical Analysis and Numerical Methods for Science and Technology: Vol. 3, Spectral Theory and Applications*, Springer-Verlag, Berlin, 1990.
- [13] Girault, V., Raviart P.-A., *Finite Element Approximation of the Navier-Stokes Equations*, Lecture Notes in Mathematics - No. 749, Springer, Berlin, 1981.
- [14] Girouard, A., Polterovich, I., Spectral geometry of the Steklov problem, *J. Spectr. Theory* 7 (2017), 321-359.
- [15] Halla, M., Electromagnetic Stekloff eigenvalues: approximation analysis, arXiv:1909.00689v2 (6 Sep 2019).
- [16] Halla, M., Electromagnetic Stekloff eigenvalues: existence and behavior in the selfadjoint case, arXiv:1909.01983 (4 Sep 2019).
- [17] Heyden S., Ortiz M., Functional optimality of the sulcus pattern of the human brain, *Mathematical Medicine and Biology: A Journal of the IMA* (2018) doi: 10.1093/imammb/dqy007 (15 pages).
- [18] Kirsch, A., Hettlich, F., *The Mathematical Theory of Time-Harmonic Maxwell’s Equations, Expansion-, Integral-, and Variational Methods*, Applied Mathematical Sciences - Vol. 190, Springer International Publishing, Cham, Switzerland, 2015.
- [19] Kristensson, G., Stratis, I. G., Wellander, N., Yannacopoulos, A. N., The exterior Calderón operator for non-spherical objects, *SN Partial Differ. Equ. Appl.* 1 (2020), (32 pages)

- [20] Kuznetsov, N., Kulczycki, T., Kwaśnicki, M., Nazarov, A., Poborchi, S., Polterovich, I., Siudeja, B., The legacy of Vladimir Andreevich Steklov, *Notices Amer. Math. Soc.* 61 (2014), 9-22.
- [21] Lamberti, P.D., Provenzano, L., Viewing the Steklov eigenvalues of the Laplace operator as critical Neumann eigenvalues, *Current Trends in Analysis and its Applications*, 171-178, Trends Math., Birkhäuser/Springer, Cham, Switzerland, 2015.
- [22] Li, J., A literature survey of mathematical study of metamaterials, *Int. J. Numer. Anal. Model.* 13 (2016), 230-243.
- [23] Monk, P., *Finite Element Methods for Maxwell's Equations*, Clarendon Press, Oxford, 2003.
- [24] Nédélec, J.-C., *Acoustic and Electromagnetic Equations, Integral Representations for Harmonic Problems*, Applied Mathematical Sciences - Vol. 144, Springer-Verlag, New York, 2001.
- [25] Roach, G. F., Stratis, I. G., Yannacopoulos, A. N., *Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2012.
- [26] ter Elst, A. F. M., Ouhabaz, E. M., Convergence of the Dirichlet-to-Neumann operator on varying domains, *Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics*, 147-154, Oper. Theory Adv. Appl., 250, Birkhäuser/Springer, Cham, Switzerland, 2015.
- [27] Weber, C., A local compactness theorem for Maxwell's equations, *Math. Methods Appl. Sci.* 2 (1980), 12-25.
- [28] Weber, C. Regularity theorems for Maxwell's equations. *Math. Methods Appl. Sci.* 3 (1981), no. 4, 523-536.
- [29] Zhang, Z., Comparison results for eigenvalues of curlcurl operator and Stokes operator, *Z. Angew. Math. Phys.* 69:104 (2018).

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