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Gaffney–Friedrichs inequality for differential forms on Heisenberg groups

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Abstract. In this paper, we will prove several generalized versions, dependent on different boundary conditions, of the classical Gaffney–Friedrichs inequality for differential forms on Heisenberg groups. In the first part of the paper, we will consider horizontal differential forms and the horizontal differential. In the second part, we shall prove the counterpart of these results in the context of Rumin's complex.

Contents

1. Introduction

Let M^n be a smooth compact manifold of dimension n with boundary ∂M^n. If u is a differential form of degree h on M^n, $0 \leq h \leq n$, we set

$$u_{\mathrm{t}} := \nu \,\lrcorner\, (\nu \wedge u), \quad u_\nu := \nu \,\lrcorner\, u,$$

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where ν denotes the (Riemannian) outward unit normal vector along ∂M^n. Thus, one gets the orthogonal decomposition formula

$$u = u_{\mathrm{t}} + \nu \wedge u_\nu.$$

Denote now by $W^{1,2}(M^n, \bigwedge^h TM^n)$ the Sobolev space of differential forms on M^n of degree h. The classical Friedrichs–Gaffney inequality (see [26], [27], [36], and [42]) states that there exists a geometric constant $C > 0$ such that

$$\|u\|_{W^{1,2}(M^n,\bigwedge^h TM^n)} \le C\,\big(\|du\|_{L^2(M^n,\bigwedge^{h+1} TM^n)} + \|\delta u\|_{L^2(M^n,\bigwedge^{h-1} TM^n)} + \|u\|_{L^2(M^n,\bigwedge^h TM^n)}\big) \tag{1.1}$$

for every differential h-form $u \in W^{1,2}(M^n, \bigwedge^h TM^n)$ with vanishing either the tangential component u_{t} or the normal component u_ν on ∂M^n. Here d and δ denote, respectively, the differential and the codifferential of the de Rham complex in M^n.

Let $\mathcal{U}$ be a bounded open set with smooth boundary. If $\vec{F}\colon \mathcal{U} \to \mathbb{R}^n$ is a vector field, then (1.1) reduces to the inequality

$$\|\nabla \vec{F}\|_{L^2(\mathcal{U})^{n^2}} \le C\big(\|\mathrm{div}\, \vec{F}\|_{L^2(\mathcal{U})} + \|\mathrm{curl}\, \vec{F}\|_{L^2(\mathcal{U})^n} + \|\vec{F}\|_{L^2(\mathcal{U})^n}\big),$$

under suitable boundary conditions.

Roughly speaking, the conditions $u_{\mathrm{t}} = 0$ or $u_\nu = 0$ on ∂M^n imply the vanishing of some geometric quantities living on the boundary; see, [14], [42]. We remark that these conditions can be replaced by more complicated conditions, which can be written as linear combinations of the previous ones; for more details, we refer to Section 5.3.2 of [14].

Several generalizations of (1.1) can be found in the literature. We mention among others the Gaffney–Friedrichs inequality for Lipschitz domains proved in [34] and, above all, from our point of view, the recent papers by Tseng and Yau [44], [45] (see also [46]) for generalizations of the Gaffney–Friedrichs inequality (associated with symplectic Laplacians) in compact symplectic manifolds (thus of even dimension) with smooth boundaries of contact type.

The aim of the present paper is to prove a Gaffney–Friedrichs inequality for differential forms in Heisenberg groups.

By Darboux' theorem, Heisenberg groups can be seen as the prototype of contact manifolds (necessarily of odd dimension). Therefore our results are in some sense complementary to those in [44], [45].

Heisenberg groups will be presented in more detail in Section 2. Here we just recall that the Heisenberg group $\mathbb{H}^n$ is the $(2n + 1)$-dimensional Lie group with nilpotent, stratified Lie algebra $\mathfrak{h}$ of step 2 given by

$$\mathfrak{h} = \mathrm{span}\,\{X_1, \dots, X_n, Y_1, \dots, Y_n\} \oplus \mathrm{span}\,\{T\} := \mathfrak{h}_1 \oplus \mathfrak{h}_2,$$

where the only nontrivial commutation rules are $[X_j, Y_j] = T$ for any $j = 1, \dots, n$.

It is well known that $\mathbb{H}^n$ can be identified with $\mathbb{R}^{2n+1}$ through the (Lie group) exponential map. The stratification of the algebra induces a family of nonisotropic dilations in the group, again via the exponential map.

Since the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at the identity $e = 0 \in \mathbb{H}^n$, there is a natural left-invariant Riemannian metric in $\mathbb{H}^n$ making the basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$ orthonormal.

In addition, by left translation of $\mathfrak{h}_1$ one obtains a tangent subbundle of $T\mathbb{H}^n$ still denoted by $\mathfrak{h}_1$. We refer to $\mathfrak{h}_1$ as to the *horizontal layer*, and to

$$X_1, \dots, X_n, Y_1, \dots, Y_n,$$

as to the *horizontal derivatives* of $\mathbb{H}^n$. Moreover, we write

$$\nabla_H u := (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u)$$

whenever u is any smooth real function on $\mathbb{H}^n$.

If $0 \le h \le 2n$, the sections of $\bigwedge_h \mathfrak{h}_1$ are called *horizontal h-vectors,* while the sections of $\bigwedge^h \mathfrak{h}_1$ are called *horizontal h-covectors.*

Throughout this paper we shall denote by Ω_H^h, $0 \le h \le 2n$, the space of all horizontal h-forms, and by ϑ the 1-form on $\mathbb{H}^n$ such that $\ker \vartheta = \exp(\mathfrak{h}_1)$ and $\vartheta(T) = 1$.

It is to mention that the horizontal differential $d_H := d - \vartheta \wedge \mathcal{L}_T$ acts between horizontal differential forms in the sense that $d_H \colon \Omega_H^h \to \Omega_H^{h+1}$. Unfortunately, the diagram (Ω_H^*, d_H) defined by

$$0 \longrightarrow \Omega_H^0 \xrightarrow{d_H} \Omega_H^1 \xrightarrow{d_H} \Omega_H^1 \xrightarrow{d_H} \cdots$$

is not a differential complex, since d_H^2 does not vanish, in general, precisely because of the lack of commutativity in $\mathfrak{h}$. This difficulty is overcome by introducing the *Rumin complex* (E_0^*, d_c), which is a “natural” complex of differential forms, (chain) homotopic to the de Rham complex. We refer to [38] for the original definition, as well as to [4], [5], [6], [7]. Precise definitions of the complex (E_0^*, d_c) will be given in Section 8.1.

Thought the construction of Rumin’s forms may appear very technical, we will see in Section 8.1 that the complex (E_0^*, d_c) arises “naturally” in geometric measure theory starting from the notion of *intrinsic submanifolds* of $\mathbb{H}^n$ (see [22]) and, above all, of linear submanifolds in $\mathbb{H}^n$.

A further non-Euclidean feature arising typically from the geometry of $\mathbb{H}^n$ we have to deal with is the following. Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a smooth, bounded open set. We need to remark that in our paper we are dealing with different “boundary measures” on $\partial\mathcal{U}$. First, an intrinsic notion of perimeter measure $|\partial\mathcal{U}|_{\mathbb{H}^n}$ has been introduced in [28]; we refer the reader to [28], [18], [19], [20] for a detailed presentation. For simplicity, we shall denote the $\mathbb{H}^n$-perimeter measure by the symbol $d\sigma_H$. However, beside the $\mathbb{H}^n$-perimeter measure, we can actually consider both the $2n$-dimensional Euclidean Hausdorff measure $d\mathcal{H}^{2n}$ and the Riemannian measure $d\sigma$, defined in terms of the Riemannian structure in $\mathbb{H}^n$ induced by the fixed inner product in $\mathfrak{h}$. As a matter of fact, our results will fail to be completely analogous to the classical ones ultimately because $d\sigma$ and $d\sigma_H$ are not equivalent. This problem will be discussed later in this introduction.

We can now describe the content of this paper. Our aim is to prove Gaffney–Friedrichs-type inequalities for both (Ω_H^*, d_H) and (E_0^*, d_c) (remember we use the notation (Ω_H^*, d_H) even if $d_H^2 \neq 0$). If $\mathcal{U} \subsetneq \mathbb{H}^n$ is a smooth, bounded open set we are looking for estimates of the form

$$\|u\|_{W^{1,2}_{\mathbb{H}}(\mathcal{U},\bigwedge^h \mathfrak{h}_1)} \le C\left(\|d_H u\|_{L^2(\mathcal{U},\bigwedge^h \mathfrak{h}_1)} + \|d_H^* u\|_{L^2(\mathcal{U},\bigwedge^h \mathfrak{h}_1)} + \|u\|_{L^2(\mathcal{U},\bigwedge^h \mathfrak{h}_1)}\right),$$

under suitable boundary conditions.

Here $W^{1,2}_{\mathbb{H}}(\mathcal{U}, \bigwedge^h \mathfrak{h}_1)$ denotes the space of horizontal differential forms such that all their coefficients with respect to some coordinate frame belong to $W^{1,2}_{\mathbb{H}}(\mathcal{U})$ (that is, they belong to $L^2(\mathcal{U})$ together with all their horizontal derivatives).

Analogously, when dealing with forms of Rumin's complex, we are looking for estimates of the form

$$\|u\|_{W^{1,2}_{\mathbb{H}}(\mathcal{U},E_0^h)} \le C\left(\|d_c u\|_{L^2(\mathcal{U},E_0^{h+1})} + \|\delta_c u\|_{L^2(\mathcal{U},E_0^{h-1})} + \|u\|_{L^2(\mathcal{U},E_0^h)}\right), \tag{1.2}$$

under suitable boundary conditions. If $\Xi_0^h = \{\xi_i^h : 1 \le i \le \dim E_0^h\}$ is a smooth orthonormal basis of E_0^h, we denote by $W^{1,2}_{\mathbb{H}}(\mathcal{U}, E_0^h)$ the space of differential forms $u = \sum_j u_j \xi_j^h \in L^2(\mathcal{U}, E_0^h)$ such that

$$\|\nabla_H u\|^2 := \sum_{i,j} \left(|X_i u_j|^2 + |Y_i u_j|^2\right) \in L^1(\mathcal{U}),$$

endowed with its associated norm. In this case we confine ourselves to degrees $h \neq n,\ n+1$, in order to deal only with both the intrinsic differential d_c and codifferential δ_c of order 1. The remaining cases will be considered in a future paper. If $\mathcal{U} = \mathbb{H}^n$, inequality (1.2) is well known (see, e.g., [38]).

We can now state our main results, which correspond to the choice of different boundary conditions. Our approach is largely inspired by that of Csató, Dacorogna and Kneuss in [14]. In fact, several delicate algebraic manipulations we carry out in this paper are the counterpart in our setting of those presented in [14].

Denoting by n_H the *horizontal normal* to $\partial\mathcal{U}$, that is, the orthogonal projection onto $\bigwedge_1 \mathfrak{h}_1$ of the Riemannian outward unit normal n along $\partial\mathcal{U}$, we can define a horizontal unit normal vector to $\partial\mathcal{U}$ by setting $\nu_H := \mathrm{n}_H/\|\mathrm{n}_H\|$ at each point $p \in \partial\mathcal{U}$ where $\mathrm{n}_H(p) \neq 0$. These points are the so-called "non-characteristic points" of $\partial\mathcal{U}$, and we write $\mathrm{char}\,(\partial\mathcal{U})$ to indicate the set of all characteristic points of the boundary, i.e., the set of points $p \in \partial\mathcal{U}$ where $\mathrm{n}_H(p) = 0$. We recall that if $\partial\mathcal{U}$ is of class $\mathbf{C}^2$, then $\mathrm{char}\,(\partial\mathcal{U})$ is "small" (see, for more details, Remark 2.12 below). It is not surprising that the presence of the characteristic set $\mathrm{char}\,(\partial\mathcal{U})$ is at the origin of most of the "pathologies", at least from the Riemannian point view, we are facing in the context of Heisenberg groups. Unfortunately, in general $\mathrm{char}\,(\partial\mathcal{U}) \neq \emptyset$; for instance, the characteristic set is always non-empty when $\mathcal{U}$ is diffeomorphic to a ball. Outside $\mathrm{char}\,(\partial\mathcal{U})$ we set

$$u_{\mathrm{t}} := \nu_H \,\lrcorner\, (\nu_H \wedge u), \quad u_{\nu_H} := \nu_H \,\lrcorner\, u,$$

and we obtain the decomposition formula

$$u = u_{\mathrm{t}} + \nu_H \wedge u_{\nu_H}.$$

As a first thing, we need a counterpart of the condition "either $u_n = 0$ or $u_t = 0$" of the Riemannian case. When dealing with horizontal forms, it becomes "either $u_{\nu_H} = 0$ or $u_t = 0$", which will be called "condition (DN)" later on. This boundary condition represents the natural generalization to the horizontal geometry of $\partial\mathcal{U}$ of the classical Dirichlet–Neumann boundary conditions. On the other hand, when dealing with the Rumin complex, if J represents the linear operator known as *almost complex structure* of $\mathbb{H}^n$ (see Section 4), then it is possible to show that the condition "$(Ju)_{\mathrm{t}} = 0$" implies that "$u_{\nu_H} = 0$". Thus the condition "either $u_n = 0$ or $u_t = 0$" becomes "either $(Ju)_{\mathrm{t}} = 0$ or $u_t = 0$".

Nevertheless, it is worth observing that these conditions are not sufficient in order to prove our main results. In fact, we will need to introduce further boundary conditions, obtaining three different statements.

In Propositions 5.16 and 5.20 we introduce conditions $(J\nu_H)$ and $(\widetilde{J\nu_H})$. We define also the horizontal Dirichlet integral as

$$D_H(u) = \|d_H u\|^2_{L^2(\mathcal{U},\Omega_H^{h+1})} + \|d_H^* u\|^2_{L^2(\mathcal{U},\Omega_H^{h-1})}.$$

With these preliminaries in hand, our first formulation of the Gaffney–Friedrichs inequality for horizontal forms, which is stated in Theorem 6.1, reads basically as follows.

Theorem 1.1. *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain (i.e., bounded, connected open set) with boundary of class $\mathbf{C}^2$. If $\Omega_H^*(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ denotes the complexification of $\Omega_H^*(\overline{\mathcal{U}})$, let $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ be a horizontal h-form, with $1 \le h < n$, and assume that:*

(i) *u satisfies condition (DN) (see Proposition 5.11);*

(ii) *u satisfies either condition $(J\nu_H)$ (see Proposition 5.16) or condition $(\widetilde{J\nu_H})$ (see Proposition 5.20).*

Let $\mathcal{V}$ be an open neighborhood of $\mathrm{char}(\partial\mathcal{U})$ *(in the relative topology). Then, there exist geometric constants C_0, C_1 and C_2 (depending only on $\mathcal{U}$, $\mathcal{V}$, and on the integers h and n) such that*

$$D_H(u) + C_0 \int_{\partial\mathcal{U}\cap\mathcal{V}} \|u\|^2\, d\sigma \ge C_1 \int_{\mathcal{U}} \|\nabla_H u\|^2\, dV - C_2 \int_{\mathcal{U}} \|u\|^2\, dV. \tag{1.3}$$

Remark 1.2. If $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ is a horizontal h-form with $n+1 \le h \le 2n$, then (1.3) still holds, provided that $*_H u$ satisfies (i) and (ii); see Remark 8.13.

We stress that condition (ii) above can be dropped if u is "Kähler-symmetric" on the boundary $\partial\mathcal{U}$, i.e. if $u = \sum_{I,J} u_{I,J} \vartheta^I \wedge \vartheta^{\bar{J}}$ satisfies $u_{I,J} = \pm u_{J,I}$ on $\partial\mathcal{U}$ for all I, J; see Remark 5.18.

As a matter of fact, Theorem 1.1 is not completely satisfying because of the presence of the boundary integral on the left-hand side of (1.3).

Roughly speaking, we had to cut-off a small region around $\mathrm{char}(\partial\mathcal{U})$, and this requires two comments. First, trivially, Theorem 1.1 yields the precise counterpart of the Riemannian inequality when $\mathrm{char}(\partial\mathcal{U}) = \emptyset$ (this happens, for instance, when $\mathcal{U}$ is a thin torus; see, e.g., [13]). Second, and more importantly, we observe that the boundary integral on left-hand side of inequality (1.3) cannot be reabsorbed on the right-hand side, as we do classically using Ehrling's inequality. This is due to the presence in the boundary term of the Riemannian measure $d\sigma$.

To be more precise, we would like to stress the following points:

- Functions in $W^{1,2}_{\mathbb{H}}(\mathcal{U})$ admit L^2-continuous traces on the boundary $\partial\mathcal{U}$ with respect to both measures $d\sigma_H$ (see [15]) and $d\sigma$ on $\partial\mathcal{U}$ (see [2], [3]). However, in the first case, the trace map is compact under mild assumptions on $\partial\mathcal{U}$ (e.g., if $\partial\mathcal{U}$ is assumed sufficiently "flat" at characteristic points) whereas compactness fails to hold, in the second case, near characteristic points. Away from the characteristic set, the second result follows from the first one.
- Both sides of (1.3) turn out to be continuous with respect to the convergence in $W^{1,2}_{\mathbb{H}}(\mathcal{U}, \bigwedge^* \mathfrak{h}_1)$. The statement is trivial for the right-hand side, but is quite delicate for the boundary term on the left-hand side, since it relies on the trace theorems of [2], [3].
- Because of the lack of compactness of the trace operator from $W^{1,2}_{\mathbb{H}}(\mathcal{U})$ to $L^2(\partial\mathcal{U}, d\sigma)$, the L^2-norm of the trace of u in the left-hand side of (1.3) cannot be controlled with an arbitrary small constant $\delta > 0$ times the L^2-norm of $\nabla_H u$, and hence cannot be reabsorbed in the right-hand side.

Thus, in order to obtain a statement closer to the classical Gaffney–Friedrichs inequality, we have to make a geometric assumption on the characteristic set of the boundary $\partial\mathcal{U}$; see "condition (H)" in Definition 3.4 below. In rough words, condition (H) expresses the fact that characteristic points are isolated, and that $\partial\mathcal{U}$ is sufficiently flat at these points. In fact, this assumption is somehow related to the geometric conditions for trace theorems in [15], [2], [3] (see also [35]).

Subsequently, to avoid the presence of the boundary integral on the left-hand side of (1.3), in Proposition 5.16 we introduce "condition $(J\nu_H)$", a geometric condition used in a second formulation of the main inequality (see Theorem 6.3), which reads essentially as follows.

Theorem 1.3. *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain with boundary of class $\mathbf{C}^2$ satisfying condition (H) (see Definition 3.4). Let $u \in \Omega^h_H(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ be a horizontal h-form with $1 \le h < n$, and assume that:*

(i) *u satisfies condition (DN) (see Proposition 5.11);*

(ii) *u satisfies condition $(J\nu_H^*)$ (see Remark 5.18).*

Then, there exist geometric constants $\widetilde{C}_1$ and $\widetilde{C}_2$, only dependent on $\mathcal{U}$, and on the integers h and n, such that

$$D_H(u) \ge \widetilde{C}_1 \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \widetilde{C}_2 \int_{\mathcal{U}} \|u\|^2 \, dV. \tag{1.4}$$

Remark 1.4. If $u \in \Omega^h_H(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ is a horizontal h-form with $n+1 \le h \le 2n$, then (1.4) still holds provided that $*_H u$ satisfies (i) and (ii), where $*_H$ denotes the Hodge duality operator between horizontal forms.

As above, hypothesis (ii) can be dropped if u is "Kähler-symmetric" on the boundary.

In Section 7 we introduce two new conditions (see (7.2) and (7.3)). These are then used in Theorem 7.1, which is our final formulation of the main inequality.

Theorem 1.5. *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain with boundary of class $\mathbf{C}^2$ satisfying condition (H) (see Definition 3.4). Let $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ be a horizontal h-form with $1 \le h < n$, and assume that either*

(i) $u_{\nu_H} = 0$,

(ii) *u satisfies the condition (7.2),*

or

(j) $u_t = 0$,

(jj) *u satisfies the condition (7.3).*

Then, there exist geometric constants $\widetilde{C}_1$ and $\widetilde{C}_2$, only dependent on $\mathcal{U}$, and on the integers h and n, such that

$$D_H(u) \ge \widetilde{C}_1 \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \widetilde{C}_2 \int_{\mathcal{U}} \|u\|^2 \, dV. \tag{1.5}$$

Remark 1.6. If $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ is a horizontal h-form with $n+1 \le h \le 2n$, then (1.5) still holds, provided that $*_H u$ satisfies (i) and (ii) or (j) and (jj).

As above, conditions (ii) and (jj) can be dropped if u is "Kähler-symmetric" on the boundary.

Finally, for the counterpart of Theorems 1.1, 1.3, 1.5 in the case $h = n$ we refer the reader to Theorem 6.7 (see Section 6).

Theorems 1.1, 1.3, and 1.5 have a natural counterpart in the setting of Rumin's complex; see Theorems 8.21, 8.23, and 8.24.

The three different boundary conditions just discussed naturally arise as a consequence of an integration by parts that involves the (intrinsically 2nd order) differential operator T. When performing this computation, we carry out some elementary, but not trivial, algebraic manipulations that, in a sense, are modeled on the standard Kählerian structures of $\mathbb{H}^n$.

It is worth observing that the first and third conditions cannot be easily related one to another and that the second condition turns out to be stronger than the other two.

Let us give an overview of the organization of this paper.

In Section 2 we gather the basic notions concerning Heisenberg groups and differential forms. We also state some more or less known preliminary results.

Section 3 is devoted to prove some trace theorems in $\mathbb{H}^n$.

In Section 4 we collect some standard results of Kähler geometry in the context of Heisenberg groups.

Section 5 contains the technical core of the paper, with explicit estimates of the boundary terms that occur by integrating by parts the so-called horizontal Dirichlet integral D_H.

As a consequence of these estimates, in Sections 6 and 7, we state and prove our Gaffney–Friedrichs-type inequalities in (Ω_H^*, d_H).

Finally, in Section 8, after providing a basic introduction to Rumin's complex, we state our Gaffney–Friedrichs-type inequalities in (E_0^*, d_c).

Last but not least, the authors express their gratitude to the referee for his deep and thoughtful advices and his invaluable help.

2. Preliminaries on horizontal forms

2.1. Heisenberg groups and horizontal forms

In this section we give a quick overview of Heisenberg groups and we fix our notation. For more details, the reader is referred to [11], [22], [29], [43]. Let $\mathbb{H}^n$ be the n-th Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates of the first kind. A point $p \in \mathbb{H}^n$ is written as a triple $p = (x, y, t)$, where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

If $p = (x, y, t)$, $p' = (x', y', t') \in \mathbb{H}^n$, then the Lie group operation is defined as

$$p \cdot p' := \Big(x + x', y + y', t + t + \frac{1}{2} \sum_{j=1}^{n} \big(x_j y_j' - y_j x_j'\big)\Big).$$

If p^{-1} denotes the inverse of $p \in \mathbb{H}^n$, then $p^{-1} = (-x, -y, -t)$. Moreover, if $q \in \mathbb{H}^n$ and $r > 0$, then left translations and intrinsic dilations are defined by setting

$$\tau_q p := q \cdot p, \quad \delta_r p := (rx, ry, r^2 t).$$

We endow $\mathbb{H}^n$ with the homogeneous norm

$$\varrho(p) := \big(\big(\|x\|^2_{\mathbb{R}^n} + \|y\|^2_{\mathbb{R}^n}\big)^2 + t^2\big)^{1/4},$$

which is (up to a constant) the so-called *Koranyi norm.* The associated gauge-distance is defined as $d_\varrho(p, q) := \varrho(p^{-1} \cdot q)$; see, e.g., [43]. The homogeneous dimension of $(\mathbb{H}^n, d_\varrho)$ (w.r.t. the dilations δ_r) is the integer $Q := 2n + 2$, which coincides with its Hausdorff dimension with respect to the metric d_ϱ. We notice that Q is strictly greater than the topological dimension of $\mathbb{H}^n$, which is $2n + 1$.

Let $\mathfrak{h}$ denote the Lie algebra of all left invariant vector fields of $\mathbb{H}^n$. We assume that the basis of $\mathfrak{h}$ is given by

$$X_i := \partial_{x_i} - \frac{y_i}{2}\partial_t, \quad Y_i := \partial_{y_i} + \frac{x_i}{2}\partial_t \quad \forall\, i = 1, \dots, n; \quad T := \partial_t.$$

The only non-trivial commutation relations are $[X_i, Y_i] = T$ for any $i = 1, \dots, n$. The subspace $\mathfrak{h}_1$ of $\mathfrak{h}$ generated by the vector fields $\{X_1, Y_1, \dots, X_n, Y_n\}$ is called *horizontal subspace.* Denoting by $\mathfrak{h}_2$ the linear span of T, we have

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2,$$

which simply means that the Lie algebra is stratified.

Throughout this paper, we endow $\mathfrak{h}$ with the inner product $\langle\cdot,\cdot\rangle$ that makes the basis $\{X_1, Y_1, \dots, X_n, Y_n, T\}$ orthonormal. We refer to $\langle\cdot,\cdot\rangle$ as the *Riemannian metric* in $\mathfrak{h}$ and we denote by $\|\cdot\|$ its associated norm. For later use, we set

$$W_{2i-1} := X_i, \quad W_{2i} := Y_i \quad \forall\, i = 1, \dots, n; \quad W_{2n+1} := T.$$

For any $f\colon \mathbb{H}^n \longrightarrow \mathbb{R}$ of class $\mathbf{C}^1$ we denote by $\nabla_H f$ the horizontal gradient of f (i.e., $\nabla_H f := \sum_{i=1}^{2n}(W_i f)W_i$) and by ∇f the Riemannian gradient of f (i.e., $\nabla f := \sum_{i=1}^{2n+1}(W_i f)W_i \equiv (\nabla_H f, Tf)$).

Furthermore, for any $\mathbf{C}^1$ horizontal vector field $\Phi = \sum_{i=1}^{2n} \varphi_i W_i$ we denote by $\operatorname{div}_H \Phi := \sum_{i=1}^{2n} W_i\varphi_i$ the horizontal divergence of Φ and by Δ_K the non-negative horizontal sub-Laplacian (i.e., the *Kohn Laplacian*) defined, for any function f of class $\mathbf{C}^2$, by setting

$$\Delta_K f := -\operatorname{div}_H\,(\nabla_H f) = -\sum_{i=1}^{2n} W_i^2 f.$$

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, which is dual to the standard basis $\{X_1, Y_1, \dots, X_n, Y_n, T\}$, is the family of covectors

$$\{dx_1, dy_1, \dots, dx_n, dy_n, \vartheta\},$$

where ϑ denotes the *contact form* of $\mathbb{H}^n$ given by $\vartheta := dt - \frac{1}{2}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i)$. The inner product on $\mathfrak{h}$ gives rise to an inner product on $\bigwedge^1 \mathfrak{h}$, denoted in the same way. In particular, $\langle\cdot,\cdot\rangle$ makes the basis $\{dx_1, dy_1, \dots, dx_n, dy_n, \vartheta\}$ an orthonormal basis. In accordance with our previous notation, we set

$$\psi_{2i-1} := dx_i, \quad \psi_{2i} := dy_i \quad \forall\, i = 1, \dots, n; \quad \psi_{2n+1} := \vartheta.$$

We clearly have $\psi_l(W_m) = \delta_l^m$ for every $l, m = 1, \dots, 2n+1$, where δ_l^m denotes the Kronecker delta function. The volume form of $\mathbb{H}^n$ is the left-invariant $(2n+1)$-form $dV := \psi_1 \wedge \cdots \wedge \psi_{2n+1}$.

Set $\bigwedge_0 \mathfrak{h} := \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and

$$\bigwedge\nolimits_k \mathfrak{h} := \operatorname{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\} =: \operatorname{span}\Psi_k,$$
$$\bigwedge\nolimits^k \mathfrak{h} := \operatorname{span}\{\psi_{i_1} \wedge \cdots \wedge \psi_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\} =: \operatorname{span}\Psi^k,$$

for any $k = 1, \dots, 2n+1$.

If the degree k of the form is fixed and $I = (i_1, \dots, i_k)$ is a multi-index, then we write

$$\psi^I := \psi_{i_1}^k \wedge \cdots \wedge \psi_{i_k}^k.$$

The action of a k-covector ψ on a k-vector v is denoted by $\langle \psi | v \rangle$.

We observe that the inner product $\langle\cdot,\cdot\rangle$ can be canonically extended to $\bigwedge_k \mathfrak{h}$ and $\bigwedge^k \mathfrak{h}$ in a way that Ψ_k and Ψ^k are both orthonormal bases.

The above definitions can be reformulated by replacing $\mathfrak{h}$ with the horizontal subspace $\mathfrak{h}_1$ and by setting

$$\bigwedge\nolimits_k \mathfrak{h}_1 := \operatorname{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\},$$
$$\bigwedge\nolimits^k \mathfrak{h}_1 := \operatorname{span}\{\psi_{i_1} \wedge \cdots \wedge \psi_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\},$$

for any $k = 1, \dots, 2n$. By definition, the *symplectic 2-form* of $\mathbb{H}^n$ is given by $d\vartheta = -\sum_{i=1}^n dx_i \wedge dy_i \in \bigwedge^2 \mathfrak{h}_1$.

If $1 \le k \le 2n+1$, the "Hodge star operator" and its dual operator (denoted in the same way), i.e.,

$$* : \bigwedge\nolimits_k \mathfrak{h} \leftrightarrow \bigwedge\nolimits_{2n+1-k} \mathfrak{h} \quad \text{and} \quad * : \bigwedge\nolimits^k \mathfrak{h} \leftrightarrow \bigwedge\nolimits^{2n+1-k} \mathfrak{h},$$

are the isomorphisms defined, for any $v, w \in \bigwedge_k \mathfrak{h}$ and $\varphi, \psi \in \bigwedge^k \mathfrak{h}$, by

$$v \wedge *w := \langle v, w \rangle\, W_1 \wedge \cdots \wedge W_{2n+1} \quad \text{and} \quad \varphi \wedge *\psi := \langle \varphi, \psi \rangle\, \psi_1 \wedge \cdots \wedge \psi_{2n+1}.$$

For any $v \in \bigwedge_k \mathfrak{h}$ we define $v^\sharp \in \bigwedge^k \mathfrak{h}$ via the identity $\langle v^\sharp | w \rangle = \langle v, w \rangle$ for any $w \in \bigwedge_k \mathfrak{h}$. The inverse operator on covectors is denoted as $\alpha \mapsto \alpha^\flat$.

It is well known that the Lie algebra $\mathfrak{h}$ can always be identified with the tangent space at the identity $e = 0 \in \mathbb{H}^n$, i.e., $\mathfrak{h} \cong T_e\mathbb{H}^n$. In particular, $\mathfrak{h}_1$ can be identified with a subspace of $T_e\mathbb{H}^n$, denoted by $\bigwedge_1 \mathfrak{h}_1$. Moreover, $\bigwedge_1 \mathfrak{h}_1$ defines by left translation a smooth subbundle of the tangent bundle $T\mathbb{H}^n$ which, with a slight abuse of notation, is still denoted by $\bigwedge_1 \mathfrak{h}_1$. By definition, the sections of $\bigwedge_1 \mathfrak{h}_1$ are called *horizontal vector fields.*

Analogously, if $0 \le h \le 2n+1$, then $\bigwedge^h \mathfrak{h}$ defines by left translation a vector bundle still denoted by $\bigwedge^h \mathfrak{h}$ and if $0 \le h \le 2n$, then $\bigwedge^h \mathfrak{h}_1$ defines (again by left translation) a vector bundle still denoted by $\bigwedge^h \mathfrak{h}_1$.

If $0 \le h \le 2n+1$, we denote by Ω^h the vector space of differential h-forms on $\mathbb{H}^n$ (i.e., the vector space of all smooth sections of $\bigwedge^h \mathfrak{h}$). Furthermore, if $0 \le h \le 2n$, we denote by Ω^h_H the vector space of horizontal differential h-forms on $\mathbb{H}^n$ (i.e., the vector space of all smooth sections of $\bigwedge^h \mathfrak{h}_1$).

Definition 2.1. Let $\alpha \in \Omega^h_H$. Throughout this paper, we shall set

$$d_H \alpha := d\alpha - \vartheta \wedge \mathcal{L}_T\, \alpha,$$

where the symbol $\mathcal{L}_T$ stands for "Lie derivative" along the vector field T.

Roughly speaking, the operator d_H represents the exterior differential along the horizontal distribution and is only defined for any h-form $\alpha \in \Omega^h$ such that $i_T(\alpha) = 0$, where the symbol i_T stands for "interior product" of α with T, which is defined by the formula

$$\langle i_T(\alpha) | v \rangle := \langle \alpha | T \wedge v \rangle$$

for all $v \in \bigwedge_{h-1} \mathfrak{h}$; see, for example, [33], p. 235.

We recall the following useful identity: If X, Y are vector fields, then

$$[\mathcal{L}_X, i_Y] = i_{[X,Y]};$$

see Corollary 6.4.12 in [1].

Moreover, we define the "horizontal Hodge star operator" and its dual operator (again denoted in the same way), i.e.,

$$*_H : \bigwedge\nolimits_k \mathfrak{h}_1 \to \bigwedge\nolimits_{2n-k} \mathfrak{h}_1 \quad \text{and} \quad *_H : \bigwedge\nolimits_k \mathfrak{h}_1 \leftrightarrow \bigwedge\nolimits_{2n-k} \mathfrak{h}_1,$$

as $v \wedge *_H w := \langle v, w \rangle W_1 \wedge \cdots \wedge W_{2n}$ and $\varphi_H \wedge *_H \psi := \langle \varphi, \psi \rangle \psi_1 \wedge \cdots \wedge \psi_{2n}$ for every $v, w \in \bigwedge_k \mathfrak{h}_1$ and every $\varphi, \psi \in \bigwedge^k \mathfrak{h}_1$.

We notice that, under our current assumptions, we have $\frac{(d\vartheta)^n}{n!} = \psi_1 \wedge \cdots \wedge \psi_{2n}$; see, e.g., [31], p. 44, Remark 1.2.22.

The next identities follow from [38], p. 292.

Lemma 2.2. *If $k \geq n$ and $\beta \in \bigwedge^k \mathfrak{h}_1$, with $n \leq k \leq 2n$, then*

$$*_H \beta = *(\vartheta \wedge \beta).$$

If $0 \leq k \leq n$ and $\alpha \in \bigwedge^k \mathfrak{h}_1$, then

$$*\alpha = (-1)^k \vartheta \wedge *_H \alpha.$$

For the sake of completeness, we recall some standard results concerning wedge product and interior multiplication; see Definition 2.11 and Propositions 2.14 and 2.16 in [14].

Definition 2.3. If $\alpha \in \Omega^k$ and $\mu \in \Omega^\ell$, with $1 \leq \ell < k \leq 2n+1$, we set

$$\mu \,\lrcorner\, \alpha := (-1)^{k-\ell} * (\mu \wedge (*\alpha)).$$

Lemma 2.4. *If $1 \leq k \leq 2n+1$, $\alpha \in \Omega^k$, $\beta \in \Omega^{k-1}$ and $\mu \in \Omega^1$, then*

$$\mu \,\lrcorner\, \alpha = i_{\mu^\flat} \alpha.$$

Moreover, we have

$$\langle \mu \,\lrcorner\, \alpha, \beta \rangle = \langle \alpha, \mu \wedge \beta \rangle.$$

By using Lemma 2.2 we obtain the following.

Lemma 2.5. *If $\alpha \in \Omega_H^k$ and $\mu \in \Omega_H^\ell$, with $1 \leq \ell < k \leq n$, then*

$$\mu \,\lrcorner\, \alpha = *_H(\mu \wedge *_H \alpha).$$

In addition, we need to recall a useful result.

Lemma 2.6. *If $\alpha \in \Omega_H^k$, $\beta \in \Omega_H^\ell$ and $\gamma \in \Omega_H^r$, with $0 \leq k + \ell \leq r \leq 2n$, then*

$$(\alpha \wedge \beta) \,\lrcorner\, \gamma = (-1)^{k+\ell} \alpha \,\lrcorner (\beta \,\lrcorner\, \gamma).$$

Moreover, if $k + \ell = r$, then

$$\langle \alpha \wedge \beta, \gamma \rangle = (-1)^{\ell(k+1)} \langle \beta, \alpha \,\lrcorner\, \gamma \rangle = (-1)^k \langle \alpha, \beta \,\lrcorner\, \gamma \rangle.$$

We also define the *horizontal codifferential* $\delta_H\colon \Omega_H^{h+1} \to \Omega_H^h$ by setting

$$\delta_H := - *_H d_H *_H .$$

Observe that

$$\int_{\mathbb{H}^n} \langle \delta_H \alpha, \beta\rangle \ dV = \int_{\mathbb{H}^n} \langle \alpha, d_H \beta\rangle \ dV$$

for all $\beta \in \Omega^{h-1}$ with compact support. Finally, let $\Delta_H\colon \Omega_H^h \to \Omega_H^h$ be the *horizontal sub-Laplacian operator* defined as

$$\Delta_H := d_H \delta_H + \delta_H d_H .$$

Definition 2.7 (The operators L and Λ)**.** From now on, we shall set

$$L\alpha := -d\vartheta \wedge \alpha, \quad \Lambda := L^*,$$

(i.e., L^* denotes the adjoint of L w.r.t. the inner product $\langle \cdot, \cdot \rangle$).

The following identity can be found in [38]; see also [44].

Lemma 2.8. *If* $\alpha \in \bigwedge^h \mathfrak{h}_1$*, then we have* $[\Lambda, L]\alpha = (n-h)\alpha$.

Note that

$$\langle i_Z \alpha, \beta\rangle = \left\langle \alpha, Z^{\#} \wedge \beta \right\rangle$$

for every $\alpha \in \bigwedge^{h+1} \mathfrak{h}_1$, $\beta \in \bigwedge^h \mathfrak{h}_1$ and $Z \in \bigwedge_1 \mathfrak{h}_1$. Hence, it follows that

$$\Lambda = \sum_{k=1}^n i_{Y_k} i_{X_k} .$$

2.2. Decomposition of forms on the boundary of a domain I

We begin with the definition of horizontal normal to the boundary of a domain (i.e., bounded, connected open set).

Definition 2.9. Let $E \subset \mathbb{H}^n$ be an open set with boundary ∂E of class $\mathbf{C}^1$. We denote by n_H the (non-unit) *horizontal normal* to ∂E defined as follows: n_H is the Riemannian orthogonal projection on $\bigwedge_1 \mathfrak{h}_1$ of the Riemannian outward unit normal n to ∂E. Thus we have $\mathrm{n} = \mathrm{n}_H + \mathrm{n}_T T$.

In particular, if (locally) $\partial E = \{f = 0\}$, where $f\colon \mathbb{H}^n \to \mathbb{R}$ is a $\mathbf{C}^1$ function with non-vanishing horizontal gradient, then $\mathrm{n}_H = \|\nabla f\|^{-1} \nabla_H f$, where ∇f is the Riemannian gradient of f and $\|\nabla f\|$ denotes its norm.

We define a horizontal unit normal vector to ∂E by setting $\nu_H := \mathrm{n}_H / \|\mathrm{n}_H\|$ at each point $p \in \partial E$ where $\mathrm{n}_H(p) \neq 0$. These points are the so-called *non-characteristic points* of ∂E and we usually write $\mathrm{char}\,(\partial E)$ to indicate the *characteristic set* of ∂E (i.e., the set of points $p \in \partial E$ where $\mathrm{n}_H(p) = 0$). We explicitly note that $\nu_H = \sum_{i=1}^{2n} (\nu_H)_i W_i$, where $(\nu_H)_i := \langle \nu_H, W_i \rangle$.

To avoid cumbersome notation, in the sequel we will still denote by n, n_H and ν_H, their dual 1-forms $\mathrm{n}^{\#}$, $\mathrm{n}_H^{\#}$ and $\nu_H^{\#}$.

Besides, we adapt to our framework a standard notation; see, e.g., [14] or [42]. More precisely, we shall set

$$\alpha_{\rm t} := \nu_H \lrcorner (\nu_H \wedge \alpha), \quad \alpha_{\nu_H} := \nu_H \lrcorner \alpha \quad \forall \alpha \in \Omega^h_H(\overline{\mathcal{U}}).$$

We thus obtain the useful decomposition formula

$$\alpha = \alpha_{\rm t} + \nu_H \wedge \alpha_{\nu_H} \quad \forall \alpha \in \Omega^h_H(\overline{\mathcal{U}}).$$

The next result will be needed later.

Lemma 2.10. *If $\alpha \in \Omega^h_H$, then*

$$\nu_H \wedge \alpha_{\nu_H} = 0 \quad \textit{if and only if} \quad \alpha_{\nu_H} = 0. \tag{2.1}$$

Proof. Suppose that $\nu_H \wedge \alpha_{\nu_H} = 0$. By Lemma 2.4 one has

$$0 = \langle \nu_H \wedge \alpha_{\nu_H}, \alpha\rangle = \langle \nu_H \wedge (\nu_H \lrcorner \alpha), \alpha\rangle = \langle \nu_H \lrcorner \alpha, \nu_H \lrcorner \alpha\rangle = |\alpha_{\nu_H}|^2.$$

The reverse implication is trivial. □

We conclude this section by recalling the horizontal Green's formulas valid in our setting; for similar statements, see Theorem 4.9 in [7].

Here and elsewhere, we make use of the standard notation $\mathcal{D} \equiv \mathbf{C}^\infty_0$.

Definition 2.11. Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain with boundary of class $\mathbf{C}^2$. For every $\alpha, \beta \in \Omega^h(\overline{\mathcal{U}}) := \mathcal{D}(\overline{\mathcal{U}}, \bigwedge^h \mathfrak{h})$, we set

$$\langle \alpha, \beta\rangle_{L^2(\mathcal{U})} := \int_{\mathcal{U}} \langle \alpha, \beta\rangle \, dV.$$

In addition, if $\alpha \in \Omega^{h-1}_H(\overline{\mathcal{U}}) := \mathcal{D}(\overline{\mathcal{U}}, \bigwedge^{h-1} \mathfrak{h}_1)$ and $\beta \in \Omega^h_H(\overline{\mathcal{U}}) := \mathcal{D}(\overline{\mathcal{U}}, \bigwedge^h \mathfrak{h}_1)$ are intrinsic forms, it follows that

$$\langle d_H\alpha, \beta\rangle_{L^2(\mathcal{U})} = \langle \alpha, \delta_H\beta\rangle_{L^2(\mathcal{U})} + \int_{\partial\mathcal{U}} \langle {\rm n} \wedge \alpha, \beta\rangle \, d\sigma. \tag{2.2}$$

These formulas also hold when $\alpha \in \mathbf{C}^1(\overline{\mathcal{U}}, \bigwedge^{h-1} \mathfrak{h}_1)$, $\beta \in \mathbf{C}^1(\overline{\mathcal{U}}, \bigwedge^h \mathfrak{h}_1)$.

Note that the outward unit normal ${\rm n}(p)$ at any point $p \in \partial\mathcal{U}$ is given by ${\rm n}(p) = {\rm n}_H(p) + {\rm n}_T(p)T$, where ${\rm n}_H(p)$ is the (orthogonal) projection of ${\rm n}(p)$ onto the horizontal subspace $\bigwedge_1 \mathfrak{h}_1$ at $p \in \partial\mathcal{U}$. Thus, after the natural identification ${\rm n} \cong {\rm n}^\#$, we get $\langle {\rm n} \wedge \alpha, \beta\rangle = \langle {\rm n}_H \wedge \alpha, \beta\rangle$, since both α and β are horizontal. Eventually, we obtain the formula

$$\int_{\partial\mathcal{U}} \langle {\rm n} \wedge \alpha, \beta\rangle \, d\sigma = \int_{\partial\mathcal{U}} \langle {\rm n}_H \wedge \alpha, \beta\rangle \, d\sigma = \int_{\partial\mathcal{U}} \langle \nu_H \wedge \alpha, \beta\rangle \, d\sigma_H,$$

where σ_H denotes the intrinsic perimeter measure in $\mathbb{H}^n$.

2.3. Perimeter measure in $\mathbb{H}^n$

We briefly recall the notion of intrinsic perimeter measure in Heisenberg groups and some related facts.

As already said in the introduction, if $E \subset \mathbb{H}^n$ is a measurable set, an intrinsic notion of $\mathbb{H}$-perimeter measure $|\partial E|_{\mathbb{H}^n}$ has been introduced in [28]; we refer the reader to [28], [18], [19], [20] for a detailed presentation. Here, we just have to recall that, if E has locally finite $\mathbb{H}^n$-perimeter (i.e., E is a $\mathbb{H}$-Caccioppoli set), then $|\partial E|_{\mathbb{H}^n}$ is a Radon measure in $\mathbb{H}^n$, which is left-invariant and $(2n+1)$-homogeneous (with respect to the dilations δ_r).

By definition, the $2n$-dimensional *Riemannian measure* of ∂E, later denoted as σ, is obtained by wedging together the elements of an oriented orthonormal coframe for ∂E and, because of its role in integration, we adopt the notation $d\sigma$, when it appears under the integral sign.

Remark 2.12. If ∂E is of class $\mathbf{C}^2$, the characteristic set $\mathrm{char}\,(\partial E)$ turns out to be "small" since both its $\mathbb{H}^n$-perimeter measure and its $2n$-dimensional Euclidean Hausdorff measure vanish. For later purposes, we recall that the Riemannian measure σ is equivalent (in the measure theoretic sense) to the Euclidean measure $\mathcal{H}^{2n}$. Hence, under our assumptions, $\sigma(\mathrm{char}\,(\partial E)) = 0$. For further properties of $\mathrm{char}\,(\partial E)$, see, e.g., [17], [25], [16], [9], [10].

We also need the following representation formula; see [12].

Proposition 2.13. *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a bounded open set with boundary $\partial\mathcal{U}$ of class $\mathbf{C}^1$. Then $\mathcal{U}$ is a $\mathbb{H}^n$-Caccioppoli set. Furthermore, the $\mathbb{H}^n$-perimeter measure is absolutely continuous with respect to the Euclidean $2n$-dimensional Hausdorff measure $\mathcal{H}^{2n}$. More precisely, if $\mathcal{A} \subseteq \mathbb{H}^n$ is an open set, then*

$$|\partial\mathcal{U}|_{\mathbb{H}^n}(\mathcal{A}) = \int_{\partial\mathcal{U}\cap\mathcal{A}} \Big(\sum_{i=1}^{n} \big(\langle X_i, n\rangle^2_{\mathbb{R}^{2n+1}} + \langle Y_i, n\rangle^2_{\mathbb{R}^{2n+1}}\big)\Big)^{1/2} d\mathcal{H}^{2n} = \int_{\partial\mathcal{U}\cap\mathcal{A}} \|\mathrm{n}_H\|\, d\sigma,$$

where n is the Euclidean outward unit normal and $d\sigma$ denotes the $2n$-dimensional Riemannian measure along $\partial\mathcal{U}$.

Definition 2.14. Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a bounded open set with boundary $\partial\mathcal{U}$ of class $\mathbf{C}^1$. For the sake of simplicity, throughout the paper we write

$$\sigma_H := |\partial\mathcal{U}|_{\mathbb{H}^n}.$$

3. Boundary terms and the trace map

3.1. Trace theorems in $\mathbb{H}^n$

From now on we shall assume that $\mathcal{U}$ is a domain with boundary $\partial\mathcal{U}$ of class $\mathbf{C}^2$. Firstly we state a trace theorem away from characteristic points.

Theorem 3.1. *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a bounded open set with boundary of class $\mathbf{C}^2$. Let $\mathcal{V} \subset \partial\mathcal{U}$ be a neighborhood of $\mathrm{char}(\partial\mathcal{U})$. Then, there exists a geometric constant $C_{\mathcal{V},\mathcal{U}} > 0$ such that for any $0 < \delta < 1$ one has*

$$\int_{\partial\mathcal{U}\setminus\mathcal{V}} |u|^2\, d\sigma_H \le C_{\mathcal{V},\mathcal{U},\delta} \int_{\mathcal{U}} |u|^2\, dV + \delta \int_{\mathcal{U}} \|\nabla_H u\|^2\, dV, \tag{3.1}$$

for any $u \in \mathbf{C}^1(\overline{\mathcal{U}})$, where $C_{\mathcal{V},\mathcal{U},\delta} := C_{\mathcal{V},\mathcal{U}} + C^2_{\mathcal{V},\mathcal{U}}/\delta$.

Proof. Let $\varphi \in \mathbf{C}^1_0(\partial\mathcal{U})$ be such that $\varphi = 1$ on $\partial\mathcal{U} \setminus \mathcal{V}$ and $\varphi = 0$ on $\mathcal{V}' \subset\subset \mathcal{V}$. Now let $\widetilde{\nu_H}$ denote the extension of ν_H to $\partial\mathcal{U}$ defined as $\widetilde{\nu_H} := \varphi\nu_H$. This extension is a horizontal vector field of class $\mathbf{C}^1$ on $\partial\mathcal{U}$ coinciding with ν_H out of $\mathcal{V}$. With a slight abuse of notation, we denote by $\widetilde{\nu_H}$ any $\mathbf{C}^1$ horizontal extension of $\widetilde{\nu_H}$ to the closure of $\mathcal{U}$, i.e., $\widetilde{\nu_H} \in \mathbf{C}^1(\overline{\mathcal{U}})$. It follows that both $\|\widetilde{\nu_H}\|$ and $\mathrm{div}_H\,(\widetilde{\nu_H})$ are continuous functions on $\overline{\mathcal{U}}$ and hence they are both bounded by some positive constant $C_{\mathcal{V},\mathcal{U}}$, only dependent on $\mathcal{V}$ and $\mathcal{U}$. By the previous assumptions we get

$$\begin{aligned}
\int_{\partial\mathcal{U}\setminus\mathcal{V}} |u|^2\, d\sigma_H &= \int_{\partial\mathcal{U}\setminus\mathcal{V}} |u|^2 \langle \nu_H, \nu_H\rangle\, d\sigma_H \\
&= \int_{\partial\mathcal{U}\setminus\mathcal{V}} |u|^2 \langle (\nu_H - \widetilde{\nu_H} + \widetilde{\nu_H}), \nu_H\rangle\, d\sigma_H \\
&= \underbrace{\int_{\partial\mathcal{U}\setminus\mathcal{V}} |u|^2 \langle (\nu_H - \widetilde{\nu_H}), \nu_H\rangle\, d\sigma_H}_{=0} + \int_{\partial\mathcal{U}\setminus\mathcal{V}} |u|^2 \langle \widetilde{\nu_H}, \nu_H\rangle\, d\sigma_H \\
&\leq \int_{\partial\mathcal{U}} |u|^2 \langle \widetilde{\nu_H}, \nu_H\rangle\, d\sigma_H \qquad (\text{since } \langle \widetilde{\nu_H}, \nu_H\rangle = \varphi \text{ on } \mathcal{V}\cap\partial\mathcal{U}) \\
&= \int_{\partial\mathcal{U}} \left\langle \left(|u|^2\widetilde{\nu_H}\right), \nu_H\right\rangle d\sigma_H .
\end{aligned}$$

By the divergence theorem for $\mathbf{C}^2$ hypersurfaces and the very definition of the $\mathbb{H}^n$-perimeter measure σ_H, we can make the following calculations:

$$\begin{aligned}
\int_{\partial\mathcal{U}} \langle (|u|^2\widetilde{\nu_H}), \nu_H\rangle\, d\sigma_H &= \int_{\mathcal{U}} \mathrm{div}_H\,(|u|^2\widetilde{\nu_H})\, dV \\
&= \int_{\mathcal{U}} |u|^2\, \mathrm{div}_H\,(\widetilde{\nu_H})\, dV + \int_{\mathcal{U}} 2\,|u|\,\langle \nabla_H|u|, \widetilde{\nu_H}\rangle\, dV \\
&\leq C_{\mathcal{V},\mathcal{U}} \left(\int_{\mathcal{U}} |u|^2\, dV + \int_{\mathcal{U}} 2|u|\big\|\nabla_H|u|\big\|\, dV\right).
\end{aligned}$$

Finally, since

$$2C_{\mathcal{V},\mathcal{U}}\,|u|\,\big\|\nabla_H|u|\big\| \leq \frac{C^2_{\mathcal{V},\mathcal{U}}}{\delta}\,|u|^2 + \delta\big\|\nabla_H|u|\big\|^2 \leq \frac{C^2_{\mathcal{V},\mathcal{U}}}{\delta}\,|u|^2 + \delta\,\big\|\nabla_H u\big\|^2,$$

the claim easily follows. □

Notice that (3.1) contains the "error term" $\int_{\partial\mathcal{U}\cap\mathcal{V}} |u|^2\, d\sigma$, which depends on the choice of $\mathcal{V}$. This is a novelty with respect to the classical trace theorems. The error is actually related to the presence of characteristic points on $\partial\mathcal{U}$, as will be shown in Example 3.10 below.

Remark 3.2. In the Riemannian setting, a "global inequality" akin to (3.1) follows by Ehrling's theorem (see, e.g., [42], Lemma 1.5.3), provided that the trace operator $\mathcal{T}\colon W^{1,2}(\mathcal{U}) \to L^2(\partial\mathcal{U}, d\sigma)$ is compact. Later on, in Definition 3.4, we introduce a geometric assumption on $\partial\mathcal{U}$ that is called "condition (H)" implying that an Ehrling-type inequality still holds for the norm in $L^2(\partial\mathcal{U}, d\sigma_H)$ (see Theorem 3.9).

Thus, to get rid of the "error term" in Theorem 3.1, we need an assumption on the domain $\mathcal{U}$, ensuring that its characteristic set $\mathrm{char}(\partial\mathcal{U})$ contains only isolated points and a certain amount of "flatness" at the boundary, near $\mathrm{char}(\partial\mathcal{U})$.

Remark 3.3. Locally near any point $p_0 \in \mathrm{char}(\partial\mathcal{U})$, the boundary is a *t-graph* (i.e., Euclidean graph with respect to the hyperplane $t = 0$). Hence (locally around p_0) there is a $\mathbf{C}^2$ defining function $g\colon \mathbb{H}^n \to \mathbb{R}$, $g(x,y,t) = t - f(x,y)$, such that

$$\mathrm{N}_H := \nabla_H g = \nabla_H(t - f(x,y)) = \left(-\frac{y}{2} - \nabla_x f, \frac{x}{2} - \nabla_y f\right),$$

where we recall that $\mathrm{n}_H = \mathrm{N}_H/\|\nabla g\|$ and $\nu_H = \mathrm{N}_H/\|\mathrm{N}_H\|$. By compactness, there must exist a finite set $\{\mathcal{V}_i : i = 1, \ldots, N\}$ made of open subsets $\mathcal{V}_i \subset \partial\mathcal{U}$ such that $\mathrm{char}(\partial\mathcal{U}) \subset \bigcup_{i=1}^N \mathcal{V}_i$. Shrinking these sets, if necessary, we can assume that each $\mathcal{V}_i$ is a t-graph of class $\mathbf{C}^2$. Note that any characteristic point $p_0 \in \mathcal{V}_i \cap \mathrm{char}(\partial\mathcal{U})$ can be thought of as standing at $0 \in \mathbb{H}^n$. This second claim follows by left translating the set $\mathcal{V}_i$ by $-p_0$. Thus, if $f_i\colon \overline{\mathcal{V}_i} \subset \mathbb{R}^{2n} \to \mathbb{R}$ is a $\mathbf{C}^2$ function such that

$$\mathcal{V}_i = \{p = (x,y,t) \in \mathbb{H}^n : t = f_i(x,y) \quad \forall\,(x,y) \in \overline{\mathcal{V}}_i\},$$

we can always suppose $f_i(0,0) = 0$ and $\nabla_{\mathbb{R}^{2n}} f_i(0,0) = 0$. In this way, the point p_0 corresponds to $0 \in \mathbb{H}^n$ (here and elsewhere, $(0,0)$ denotes the null element in $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$).

We introduce a condition prescribing the behavior of $\partial\mathcal{U}$ near $\mathrm{char}(\partial\mathcal{U})$.

Definition 3.4 (Condition (H))**.** We say that the domain $\mathcal{U}$ satisfies condition (H) if there exists a finite family $\{\mathcal{V}_i : i = 1, \ldots, N\}$ of open subsets of $\partial\mathcal{U}$ such that $\mathrm{char}(\partial\mathcal{U}) \subset \bigcup_{i=1}^N \mathcal{V}_i$ and $\mathcal{V}_i \cap \partial\mathcal{U}$ is the t-graph of some function $f_i\colon \overline{\mathcal{V}}_i \subset \mathbb{R}^{2n} \to \mathbb{R}$ of class $\mathbf{C}^2$, i.e.,

$$\mathcal{V}_i \cap \partial\mathcal{U} = \{p = (x,y,t) \in \mathbb{H}^n : t = f_i(x,y) \quad \forall\,(x,y) \in \overline{\mathcal{V}}_i\}$$

and

$$\|\mathrm{Hess}_{\mathbb{R}^{2n}} f_i\| = O(\|\mathrm{N}_H^{(i)}\|)$$

for any $i = 1, \ldots, N$, where $\mathrm{N}_H^{(i)} := (-y/2 - \nabla_x f_i, x/2 - \nabla_y f_i)$.

Below we shall set $\|(x,y)\| := \sqrt{\|x\|_{\mathbb{R}^n}^2 + \|y\|_{\mathbb{R}^n}^2}$ for any $(x,y) \in \mathbb{R}^{2n}$.

Lemma 3.5. *If condition (H) holds, then $\mathrm{char}(\partial\mathcal{U})$ is discrete.*

Proof. Without loss of generality, by Remark 3.3, let $0 \in \mathbb{H}^n$ be a characteristic point of $\partial\mathcal{U} \cap \mathcal{V}_i$. In particular, we have $f_i(0,0) = 0$ and $\nabla_{\mathbb{R}^{2n}} f_i(0,0) = 0$. Hence $\|\mathrm{N}_H^{(i)}(x,y)\|\| \leq C\|(x,y)\|$ and

$$\|\mathrm{Hess}_{\mathbb{R}^{2n}} f_i\| = O(\|(x,y)\|) \quad \text{near } (0,0) \in \mathbb{R}^{2n}.$$

Again, by the mean value theorem,

$$\|\nabla_{\mathbb{R}^{2n}} f_i\| = O(\|(x,y)\|^2) \quad \text{near } (0,0) \in \mathbb{R}^{2n}.$$

Then, at each point $(x, y) \neq (0,0)$ we have

$$\|\mathrm{N}_H^{(i)}(x,y)\| = \left\|\frac{1}{2}(-y,x) - \nabla_{\mathbb{R}^{2n}} f_i\right\| \geq \frac{1}{2}\left|\|(x,y)\| - C\|(x,y)\|^2\right| > 0$$

near $(0,0) \in \mathbb{R}^{2n}$. This means that the characteristic point $0 \in \partial\mathcal{U} \cap \mathcal{V}$ is an isolated point of $\mathrm{char}(\partial\mathcal{U})$. □

In order to better illustrate the above condition (H), we consider a special case of domains in $\mathbb{H}^n$ satisfying it.

Remark 3.6. Suppose that, in a neighborhood of $0 \in \partial\mathcal{U}$, the boundary $\partial\mathcal{U}$ is the t-graph of the function $f(x,y) = \|(x,y)\|^{2\alpha}$ for some $\alpha \geq 3/2$. One checks that $\|\mathrm{N}_H\| = O(\|(x,y)\|)$ and that $\|\mathrm{Hess}_{\mathbb{R}^{2n}} f\| = O\left(\|(x,y)\|^{2(\alpha-1)}\right)$. Taken together, these facts show that condition (H) holds.

Next, we state a useful compactness criterion.

Theorem 3.7. *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a bounded open set of class $\mathbf{C}^2$. Let X be a Banach space and let $L\colon W^{1,2}_H(\mathcal{U}) \to X$ be a continuous linear map. Then L is compact if, and only if, the following property holds*:

$$\textit{For any } \delta > 0 \textit{ there exists } C(\delta) > 0 \textit{ such that}$$
$$\|Lu\|_X \leq \delta\|\nabla_H u\|_{L^2(\mathcal{U})} + C(\delta)\|u\|_{L^2(\mathcal{U})}.$$

Proof. The "only if" part is the well-known Ehrling's inequality (see, e.g., [42], Lemma 1.5.3). Thus we prove the "if" part by showing that L is completely continuous. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,2}_H(\mathcal{U})$ that weakly converges to 0. Then there exists $C > 0$ such that $\|u_n\|_{W^{1,2}_H(\mathcal{U})} \leq C$ for all $n \in \mathbb{N}$. Moreover, by Rellich's theorem (see, e.g., [28], Theorem 1.27) $u_n \to 0$ strongly in $L^2(\mathcal{U})$. Take now $\varepsilon > 0$ and set $\delta_\varepsilon := \varepsilon/(2C)$. In addition, choose $n_\varepsilon \in \mathbb{N}$ such that

$$\|u_n\|_{L^2(\mathcal{U})} < \frac{\varepsilon}{2C(\delta_\varepsilon)} \quad \text{for all } n > n_\varepsilon.$$

Then

$$\|Lu_n\|_X \leq \delta_\varepsilon \|\nabla_H u\|_{L^2(\mathcal{U})} + C(\delta_\varepsilon)\|u\|_{L^2(\mathcal{U})} < C\,\frac{\varepsilon}{2C} + C(\delta_\varepsilon)\,\frac{\varepsilon}{2C(\delta_\varepsilon)} = \varepsilon,$$

which shows that $Lu_n \to 0$ strongly in X, as wished. □

Lemma 3.8. *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a bounded open set with boundary of class $\mathbf{C}^2$ and suppose that condition (H) holds. Then there exists a $\mathbf{C}^1$ function $\widetilde{\mathrm{n}}_H\colon \bar{\mathcal{U}} \to \mathbb{R}^{2n}$ such that*:

(i) $\frac{\widetilde{\mathrm{n}}_H}{\|\widetilde{\mathrm{n}}_H\|} \equiv \nu_H$ *on* $\partial\mathcal{U} \setminus \mathrm{char}\,(\partial\mathcal{U})$;

(ii) $\mathrm{div}_H\, \widetilde{\mathrm{n}}_H = O(\|\widetilde{\mathrm{n}}_H\|)$;

(iii) $\langle \widetilde{\mathrm{n}}_H, \nabla_H \|\widetilde{\mathrm{n}}_H\|^2\rangle = O(\|\widetilde{\mathrm{n}}_H\|^3)$.

Proof. First of all, we notice that the problem can be localized near the boundary by introducing a cut-off function supported near $\partial\mathcal{U}$. Indeed, if $\widetilde{\mathrm{n}}_H$ satisfies (i)–(iii) in a neighborhood $\mathcal{M}$ of $\partial\mathcal{U}$, and if ψ is a cut-off function supported in $\mathcal{M}$ such that $\psi \equiv 1$ on a neighborhood of $\partial\mathcal{U}$, then $\psi\widetilde{\mathrm{n}}_H$ is a $\mathbf{C}^1$ function on $\overline{\mathcal{U}}$ that trivially satisfies (i). In addition

$$\mathrm{div}_H(\psi\widetilde{\mathrm{n}}_H) = \psi\mathrm{div}_H(\widetilde{\mathrm{n}}_H) + \langle\nabla_H\psi, \widetilde{\mathrm{n}}_H\rangle,$$

which is still $O(\|\widetilde{\mathrm{n}}_H\|)$. Analogously,

$$\langle\psi\widetilde{\mathrm{n}}_H, \nabla_H\|\psi\widetilde{\mathrm{n}}_H\|^2\rangle = \psi\|\widetilde{\mathrm{n}}_H\|^2\langle\widetilde{\mathrm{n}}_H, \nabla_H\psi\rangle + \psi^2\langle\widetilde{\mathrm{n}}_H, \nabla_H\|\widetilde{\mathrm{n}}_H\|^2\rangle = O(\|\widetilde{\mathrm{n}}_H\|^3)$$

when $\|\widetilde{\mathrm{n}}_H\| \to 0$. Therefore also (iii) holds.

Now we have to define $\widetilde{\mathrm{n}}_H$ away from the characteristic points and in each set $\mathcal{V}_i$ $(i = 1, \dots, N)$. Then the global extension $\widetilde{\mathrm{n}}_H$ is obtained by gluing up the local extensions by means of a partition of unity.

Clearly, away from characteristic points we can take $\widetilde{\mathrm{n}}_H := \nu_H$, since ν_H is a continuously differentiable function. Since $\widetilde{\mathrm{n}}_H$ never vanishes, (ii) and (iii) can be replaced by $\mathrm{div}_H\,\widetilde{\mathrm{n}}_H = O(1)$ and $\langle\widetilde{\mathrm{n}}_H, \nabla_H\|\widetilde{\mathrm{n}}_H\|^2\rangle = O(1)$, respectively.

We are left with the case of one of the $\mathcal{V}_i$'s. Let $i \in \{1, \dots, N\}$ be fixed and, for simplicity, we omit the index i in this proof. For any point in $\mathcal{V}$ we put $\widetilde{\mathrm{n}}_H(x, y, t) := \nabla_H(f(x, y) - t)$. Since $\widetilde{\mathrm{n}}_H(x, y, t) \neq 0$ for $(x, y, t) \neq 0$ (recall that $0 \in \mathbb{H}^n$ is an isolated characteristic point of $\partial\mathcal{U}$), it follows that at any point in $\mathcal{V} \setminus \mathrm{char}(\partial\mathcal{U})$ one has

$$\nu_H(x, y, t) = \frac{\widetilde{\mathrm{n}}_H(x, y, t)}{\|\widetilde{\mathrm{n}}_H(x, y, t)\|}.$$

This proves (i). Moreover, up to the sign, $\mathrm{div}_H\widetilde{\mathrm{n}}_H$ equals the trace of the Hessian of f_i, hence it is locally bounded and (ii) follows.

Finally, we prove (iii). For any $j, k = 1, \dots, n$, one has

$$\begin{aligned}
\frac{1}{2}\,\partial_{x_j}\|\widetilde{\mathrm{n}}_H\|^2 &= \sum_{k=1}^{n}\Big(\frac{y_k}{2} + \partial_{x_k}f\Big)\partial^2_{x_jx_k}f + \sum_{k=1}^{n}\Big(\frac{x_k}{2} - \partial_{y_k}f\Big)\Big(\frac{\delta_k^j}{2} - \partial^2_{x_jy_k}f\Big)\\
&= \sum_{k=1}^{n}\Big(\frac{y_k}{2} + \partial_{x_k}f\Big)\partial^2_{x_jx_k}f - \sum_{k=1}^{n}\Big(\frac{x_k}{2} - \partial_{y_k}f\Big)\partial^2_{x_jy_k}f - \frac{1}{2}\,\partial_{y_j}f + \frac{x_j}{4}\\
&= O(\|\widetilde{\mathrm{n}}_H\|^2) - \frac{1}{2}\,\partial_{y_j}f + \frac{x_j}{4}.
\end{aligned}$$

Here we used that the sum of the two first terms in the second line above is nothing but the inner product between $\nabla_H(f(x, y) - t) = \widetilde{\mathrm{n}}_H(x, y, t)$ and the j-th column of the Hessian matrix $\mathrm{Hess}_{\mathbb{R}^{2n}}f$, which can be estimated by using condition (H).

Analogously, it turns out that

$$\frac{1}{2}\,\partial_{y_j}\|\widetilde{\mathrm{n}}_H\|^2 = O(\|\widetilde{\mathrm{n}}_H\|^2) + \frac{1}{2}\,\partial_{x_j}f + \frac{y_j}{4}.$$

Therefore, we get

$$\begin{aligned}\frac{1}{2}\langle \widetilde{\mathrm{n}}_H, \nabla_H \|\widetilde{\mathrm{n}}_H\|^2\rangle &= \langle \widetilde{\mathrm{n}}_H, \overrightarrow{O}(\|\widetilde{\mathrm{n}}_H\|^2)\rangle - \sum_j \frac{x_j}{4}\Big(\frac{y_j}{2} + \partial_{x_j} f\Big) + \sum_j \frac{y_j}{4}\Big(\frac{x_j}{2} - \partial_{y_j} f\Big) \\ &\quad + \sum_j \frac{1}{2}\partial_{y_j} f\Big(\frac{y_j}{2} + \partial_{x_j} f\Big) + \sum_j \frac{1}{2}\partial_{x_j} f\Big(\frac{x_j}{2} - \partial_{y_j} f\Big) \\ &= \langle \widetilde{\mathrm{n}}_H, \overrightarrow{O}(\|\widetilde{\mathrm{n}}_H\|^2)\rangle = O(\|\widetilde{\mathrm{n}}_H\|^3),\end{aligned}$$

as wished. □

We conclude this subsection with the following Ehrling-type inequality.

Theorem 3.9. *Suppose that $\mathcal{U} \subsetneq \mathbb{H}^n$ is a bounded open set of class $\mathbf{C}^2$ satisfying condition (H). Then, for any $\delta > 0$ there exists $C(\delta) > 0$ such that*

$$\|\mathcal{T} u\|_{L^2(\partial\mathcal{U}, d\sigma_H)} \le \delta \|\nabla_H u\|_{L^2(\mathcal{U})} + C(\delta)\|u\|_{L^2(\mathcal{U})}$$

for any $u \in \mathbf{C}^1(\overline{\mathcal{U}})$. In particular, the map

$$\mathcal{T} : W^{1,2}_H(\mathcal{U}) \to L^2(\partial\mathcal{U}, d\sigma_H)$$

is compact.

Proof. Let $\nu_H^\tau := \widetilde{\mathrm{n}}_H/\sqrt{\tau^2 + \|\widetilde{\mathrm{n}}_H\|^2}$, where $\tau \in \mathbb{R}$. By Lemma 3.8 we have

$$\begin{aligned}\int_{\partial\mathcal{U}} |u|^2\, d\sigma_H &= \int_{\partial\mathcal{U}} |u|^2 \Big\langle \frac{\widetilde{\mathrm{n}}_H}{\|\widetilde{\mathrm{n}}_H\|}, \nu_H \Big\rangle\, d\sigma_H \\ &= \lim_{\tau\to 0} \int_{\partial\mathcal{U}} |u|^2\, \langle \nu_H^\tau, \nu_H\rangle\, d\sigma_H = \lim_{\tau\to 0}\int_{\mathcal{U}} \mathrm{div}_H(|u|^2 \nu_H^\tau)\, dV.\end{aligned}$$

On the other hand,

$$\int_{\mathcal{U}} \mathrm{div}_H(|u|^2\nu_H^\tau)\, dV = 2\int_{\mathcal{U}} \langle u\nabla_H u, \nu_H^\tau\rangle\, dV + \int_{\mathcal{U}} |u|^2 \mathrm{div}_H(\nu_H^\tau)\, dV =: I_1 + I_2.$$

By using (ii), (iii) and Lemma 3.8, we get that

$$I_2 = \int_{\mathcal{U}} |u|^2 \frac{\mathrm{div}_H \widetilde{\mathrm{n}}_H}{\sqrt{\tau^2 + \|\widetilde{\mathrm{n}}_H\|^2}}\, dV - \frac{1}{2}\int_{\mathcal{U}} |u|^2 \Big\langle \widetilde{\mathrm{n}}_H, \frac{\nabla_H \|\widetilde{\mathrm{n}}_H\|^2}{(\tau^2 + \|\widetilde{\mathrm{n}}_H\|^2)^{3/2}}\Big\rangle\, dV \le C\int_{\mathcal{U}} |u|^2\, dV.$$

Moreover,

$$I_1 \le 2C\int_{\mathcal{U}} |u\nabla_H u|\, dV \le \delta\int_{\mathcal{U}} |\nabla_H u|^2\, dV + \frac{C^2}{\delta}\int_{\mathcal{U}} |u|^2\, dV,$$

completing the proof of the first part of the theorem. The second part follows from Theorem 3.7. □

Example 3.10. We already pointed out that an Ehrling-type inequality for the norm in $L^2(\partial\mathcal{U}, d\sigma)$ is true for general $\mathbf{C}^2$ open sets $\mathcal{U}$ away from characteristic points, as we can see using Theorem 3.9 and keeping in mind that $d\sigma$ and $d\sigma_H$ are equivalent away from char$(\partial\mathcal{U})$. However, the example below shows that Ehrling's inequality (and hence compactness of the trace, which is still continuous by [2], [3]) fails to hold for the norm in $L^2(\partial\mathcal{U}, d\sigma)$, even for sets satisfying condition (H).

Let $\varrho(p)$ be the Korányi norm, let $B_0(1) := \{p = (x, y, t) \in \mathbb{H}^n : \varrho(p) \le 1\}$ and set $\mathcal{U} := \{t \ge 0\} \cap B_0(1)$. The hyperplane $\{t = 0\}$ has a unique isolated characteristic point at $0 \in \mathbb{H}^n$. In particular, let $S_0 := \partial\mathcal{U} \cap \{t = 0\}$, $u \in \mathcal{D}(B_0(1))$, and denote by $\mathcal{T}u$ the *trace of* u along the boundary.

Now, let us analyze the (possible) validity of the following statement:

$$\forall\,\varepsilon > 0\ \exists\, C_\varepsilon : \quad \underbrace{\int_{S_0} (\mathcal{T}u)^2\, d\sigma}_{=:\|\mathcal{T}u\|^2_{L^2(S_0)}} \le \varepsilon \underbrace{\int_{\mathcal{U}} \|\nabla_H u\|^2\, dV}_{=:\|\nabla_H u\|^2_{L^2(\mathcal{U})}} + C_\varepsilon \underbrace{\int_{\mathcal{U}} u^2\, dV}_{=:\|u\|^2_{L^2(\mathcal{U})}} . \tag{3.2}$$

By a homogeneity argument, we show that (3.2) cannot hold. To this aim, set

$$u_K := K^n u(Kx, Ky, K^2 t)$$

for some $K \in \mathbb{R}_+$, and suppose that $u \neq 0$ along S_0. It is elementary to check the following identities:

- $\|\mathcal{T}u_K\|^2_{L^2(S_0)} = \|\mathcal{T}u\|^2_{L^2(S_0)}$,
- $\|\nabla_H u_K\|^2_{L^2(\mathcal{U})} = \|\nabla_H u\|^2_{L^2(\mathcal{U})}$,
- $K^2\|u_K\|^2_{L^2(\mathcal{U})} = \|u\|^2_{L^2(\mathcal{U})}$.

By assuming the validity of (3.2), with u replaced by u_K, we get

$$\forall\,\varepsilon > 0\ \exists\, C_\varepsilon : \quad \|\mathcal{T}u_K\|^2_{L^2(S_0)} \le \varepsilon\|\nabla_H u_K\|^2_{L^2(\mathcal{U})} + C_\varepsilon\|u_K\|^2_{L^2(\mathcal{U})}.$$

Hence

$$\forall\,\varepsilon > 0\ \exists\, C_\varepsilon : \quad \|\mathcal{T}u\|^2_{L^2(S_0)} \le \varepsilon\|\nabla_H u\|^2_{L^2(\mathcal{U})} + \frac{C_\varepsilon}{K^2}\|u\|^2_{L^2(\mathcal{U})}.$$

By the arbitrariness of ε, $K \in \mathbb{R}_+$ (and since the L^2-norm of u can be assumed to be fixed) one readily obtains that the trace of u must be zero, which is a contradiction.

Remark 3.11. When there is no possibility of misunderstanding we shall write u instead of $\mathcal{T}u$.

4. Kähler geometry in Heisenberg groups

4.1. Basic notions of Kähler geometry in $\mathbb{H}^n$

We now introduce the Kählerian structures of $\mathbb{H}^n$, in order to make some explicit computations, and recall some lemmata from [38], which will be used in sequel.

Firstly, we note that the base manifold of the n-th Heisenberg group $\mathbb{H}^n$ can always be identified with $\mathbb{C}^n \times \mathbb{R}$, so that any point $p = (x, y, t) \in \mathbb{H}^n$ is seen as a couple (z, t), where $z = (z_1, \ldots, z_k, \ldots, z_n) \in \mathbb{C}^n$ and $z_k = x_k + iy_k$ for any $k = 1, \ldots, n$.

Let J be the unique endomorphism of $\mathfrak{h}_1$ ("almost complex structure") such that

$$J^2 = -\mathrm{Id}, \quad d\vartheta(Z_1, JZ_2) = -d\vartheta(JZ_1, Z_2)$$

for all horizontal vector fields $Z_1, Z_2 \in \bigwedge_1 \mathfrak{h}_1$ (in particular, one has $Y_i = JX_i$ and $X_i = -JY_i$ for any $i = 1, \ldots, n$).

It is not difficult to check that the inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{h}_1$ is precisely the Riemannian metric compatible with both the almost complex structure J and the symplectic form $d\vartheta$, since

$$d\vartheta(Z_1, Z_2) = \langle Z_1, JZ_2 \rangle .$$

One has $J^* = -J$, and hence $\langle JZ_1, JZ_2 \rangle = \langle Z_1, Z_2 \rangle$ for any $Z_1, Z_2 \in \bigwedge_1 \mathfrak{h}_1$.

It is a standard fact that an almost complex structure J induces a bigrading on $\bigwedge_1 \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C}$ (i.e., the complexified horizontal subspace); see [31], p. 27.

Thus, we have $\bigwedge_1 \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge_{1,0} \mathfrak{h}_1 \oplus \bigwedge_{0,1} \mathfrak{h}_1$. This bigrading naturally extends to the complex of horizontal differential forms; see [38]. In particular, we have $\Omega_H^h \otimes_{\mathbb{R}} \mathbb{C} = \sum_{p+q=h} \Omega_H^{p,q}$, where we recall that $\overline{\Omega_H^{p,q}} = \Omega_H^{q,p}$. The (real) inner product on $\bigwedge_1 \mathfrak{h}_1$ extends in the obvious way to a (complex valued) Hermitian inner product on the complexification $\bigwedge_1 \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C}$, still denoted as $\langle \cdot, \cdot \rangle$. Clearly, one has $\langle av, bw \rangle = a\bar{b}\langle v, w \rangle$ for every $v, w \in \bigwedge_1 \mathfrak{h}_1$ and every $a, b \in \mathbb{C}$. We now set

$$Z_k := \frac{X_k - iY_k}{\sqrt{2}}, \quad Z_{\bar{k}} := \frac{X_k + iY_k}{\sqrt{2}} \left(= \overline{Z_k}\right) \quad \forall\, k = 1, \ldots, n.$$

The family of horizontal vector fields $\{Z_1, Z_{\bar{1}}, \ldots, Z_n, Z_{\bar{n}}\}$ is an orthonormal basis of $\bigwedge_1 \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C}$ (w.r.t. the Hermitian inner product induced on the complexified horizontal subspace). For each k, let $\vartheta^k := Z_k^\#$, $\vartheta^{\bar{k}} := Z_{\bar{k}}^\#$. By duality, we get that $\{\vartheta^1, \vartheta^{\bar{1}}, \ldots, \vartheta^n, \vartheta^{\bar{n}}\}$ is an orthonormal basis of $\bigwedge^1 \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge^{1,0} \mathfrak{h}_1 \oplus \bigwedge^{0,1} \mathfrak{h}_1$. We notice explicitly that

$$\vartheta^k = Z_k^\# = \frac{dz_k}{\sqrt{2}} = \frac{dx_k + idy_k}{\sqrt{2}} \quad \text{and} \quad \vartheta^{\bar{k}} = Z_{\bar{k}}^\# = \frac{d\bar{z}_k}{\sqrt{2}} = \frac{dx_k - idy_k}{\sqrt{2}}.$$

It is easy to see that $JZ_k = iZ_k$ and $JZ_{\bar{k}} = -iZ_{\bar{k}}$. Denoting still by J the operator induced by J on differential forms, we have

$$J\alpha = i^{p-q}\alpha \quad \forall\, \alpha \in \Omega_H^{p,q} \otimes_{\mathbb{R}} \mathbb{C},$$

and if $\Pi^{p,q} \colon \Omega_H^h \otimes_{\mathbb{R}} \mathbb{C} \to \Omega_H^{p,q}$ denotes the natural projection, we get

$$J = \sum_{p+q=h} i^{p-q}\, \Pi^{p,q} \quad \text{on} \quad \Omega_H^h;$$

see [31], Definition 1.2.10.

Definition 4.1. Let $\partial_k := Z_k$ and $\partial_{\bar{k}} := Z_{\bar{k}}$ for any $k = 1, \ldots, n$.

If $u = \sum_I u_{I,J} \vartheta^I \wedge \vartheta^{\bar{J}}$, we will set:

- $\partial_k u := \sum_{I,J} (Z_k u_{I,J}) \vartheta^I \wedge \vartheta^{\bar{J}}$ and $\partial_{\bar{k}} u := \sum_{I,J} (Z_{\bar{k}} u_{I,J}) \vartheta^I \wedge \vartheta^{\bar{J}}$,

- $e_k u = \vartheta^k \wedge u$ and $e_{\overline{k}} u = \vartheta^{\overline{k}} \wedge u$,
- $i_k u := i_{Z_k} u$ and $i_{\overline{k}} := i_{Z_{\overline{k}}} u$,

for any $k = 1, \dots, n$. In Kähler coordinates, it turns out that

$$L = i \sum_{k=1}^{n} e_k e_{\overline{k}} \quad \text{and} \quad \Lambda = i \sum_{k=1}^{n} i_k i_{\overline{k}};$$

see Definition 2.7.

Just as in [38], p. 294, we can prove the following result.

Proposition 4.2. *Let $1 \le p, q \le n$. We have*

$$\sum_{k=1}^{n} e_k i_k = p \,\mathrm{Id} \quad \text{on } \Omega_H^{p,q} \quad \text{and} \quad \sum_{k=1}^{n} e_{\bar{k}} i_{\bar{k}} = q \,\mathrm{Id} \quad \text{on } \Omega_H^{p,q}.$$

As a consequence,

$$\sum_{k=1}^{n} e_k i_k = \sum_{p=1}^{n} p \,\Pi^{p,q} \quad \text{and} \quad \sum_{k=1}^{n} e_{\bar{k}} i_{\bar{k}} = \sum_{q=1}^{n} q \,\Pi^{p,q}.$$

Again, one has $\Lambda = L^*$ (w.r.t. the Hermitian inner product). We use the decomposition

$$d_H := d_H^{1,0} + d_H^{0,1}, \tag{4.1}$$

where $d_H^{1,0} \colon \Omega_H^{p,q} \to \Omega_H^{p+1,q}$ and $d_H^{0,1} \colon \Omega_H^{p,q} \to \Omega_H^{p,q+1}$. Moreover, for notational simplicity, we write $\partial := d_H^{1,0}$ and $\overline{\partial} := d_H^{0,1}$, so that (4.1) reads as

$$d_H = \partial + \overline{\partial}.$$

We stress that if $u \in \Omega^0$, then

$$\|\bar{\partial} u\|^2 + \|\partial u\|^2 = \|\nabla_H u\|^2. \tag{4.2}$$

Furthermore, we have

$$\partial_k := i_k \partial \quad \text{and} \quad \partial_{\overline{k}} := i_{\overline{k}} \overline{\partial} \quad \text{for every } k = 1, \dots, n.$$

In the sequel, we shall need the multi-index notation. More precisely, let I, J be multi-indices such that $p_I := |I|$ and $q_J := |J|$ (with $p_I, q_J \le n$), so that we can assume that $I = (i_1, \dots, i_{p_I})$ and $J = (j_1, \dots, j_{q_J})$. Set now $\vartheta^I := \vartheta^{i_1} \wedge \dots \wedge \vartheta^{i_{p_I}}$ and $\vartheta^{\bar{J}} := \vartheta^{\bar{j}_1} \wedge \dots \wedge \vartheta^{\bar{j}_{q_J}}$. We observe that if $h = p_I + q_J$, the elements $\vartheta^I \wedge \vartheta^{\bar{J}}$ form a basis of $\Omega_H^h \otimes_{\mathbb{R}} \mathbb{C}$. Hence, using Kähler coordinates, any $u \in \Omega_H^h \otimes_{\mathbb{R}} \mathbb{C}$ can be uniquely written as $u = \sum_I u_{I,J} \vartheta^I \wedge \vartheta^{\bar{J}}$, $|I| = p_I$, $|J| = q_J$, with $h = p_I + q_J$. Finally, we set

$$d_H^J := J^{-1} d_H J, \quad \delta_H^J := J^{-1} \delta_H J.$$

It is not difficult to see that the following identities hold:

$$d_H^J = J^{-1} d_H J = -i(\partial - \overline{\partial}), \quad \delta_H^J = J^{-1} \delta_H J = i(\partial^* - \overline{\partial}^*),$$

where ∂^* and $\overline{\partial}^*$ are the L^2-formal adjoints of ∂ and $\overline{\partial}$, respectively.

The calculation below can be found, for instance, in [38].

Lemma 4.3. *The following chain of identities holds*:

$$i n T = \sum_{k=1}^{n} \partial_k \partial_{\overline{k}} - \partial_{\overline{k}} \partial_k = -\Big(\sum_{k=1}^{n} \partial_{\overline{k}} i_k \partial \Big) - \Big(- \sum_{k=1}^{n} \partial_k i_{\overline{k}} \overline{\partial} \Big) = \partial^* \partial - \overline{\partial}^* \overline{\partial}.$$

We recall the *Kähler identities;* see, e.g., Proposition 3.1.12 in [31].

Lemma 4.4. *We have* $[\Lambda, \partial] = i \overline{\partial}^*$ *and* $[\Lambda, \overline{\partial}] = -i\, \partial^*$. *These identities in turn imply that* $[\partial^*, L] = -i\overline{\partial}$.

For the next proposition, see, for instance, either formula (8) in [38], or [47], pp. 41-43.

Proposition 4.5. *With the previous notation, the following identities hold*:

(i) $[\Lambda, d_H] = -\delta_H^J$;

(ii) $[\Lambda, d_H^J] = \delta_H$;

(iii) $[\Lambda, \delta_H^J] = 0$.

4.2. Kähler geometry of domains in $\mathbb{H}^n$

In Kähler coordinates, we have

$$\mathrm{n}_H \equiv \mathrm{n}_H^{\#} = \sum_{k=1}^{n} \big(\mathrm{n}_k \vartheta^k + \mathrm{n}_{\overline{k}} \vartheta^{\overline{k}} \big) = \mathrm{n}_H^{1,0} + \mathrm{n}_H^{0,1},$$

where $\mathrm{n}_H^{1,0} := \sum_{k=1}^n \mathrm{n}_k \vartheta_k$ and $\mathrm{n}_H^{0,1} := \sum_{k=1}^n \mathrm{n}_{\overline{k}} \vartheta_{\overline{k}}$. Accordingly, we set

$$\nu_H^{1,0} := \frac{\mathrm{n}_H^{1,0}}{\|\mathrm{n}_H\|} \quad \text{and} \quad \nu_H^{0,1} := \frac{\mathrm{n}_H^{0,1}}{\|\mathrm{n}_H\|}.$$

The operators ∂ and $\overline{\partial}$, and their adjoints ∂^* and $\overline{\partial}^*$, satisfy the integration by parts formulas below:

$$\begin{aligned} \int_{\mathcal{U}} \langle \partial\alpha, \beta \rangle \, dV &= \int_{\mathcal{U}} \langle \alpha, \partial^* \beta \rangle \, dV + \int_{\partial\mathcal{U}} \langle \mathrm{n}_H^{1,0} \wedge \alpha, \beta \rangle \, d\sigma \\ &= \int_{\mathcal{U}} \langle \alpha, \partial^* \beta \rangle \, dV + \int_{\partial\mathcal{U}} \langle \nu_H^{1,0} \wedge \alpha, \beta \rangle \, d\sigma_H \end{aligned} \tag{4.3}$$

for every $\alpha \in \Omega_H^{p-1,q}, \beta \in \Omega_H^{p,q}$, and

$$\begin{aligned} \int_{\mathcal{U}} \langle \overline{\partial}\alpha, \beta \rangle \, dV &= \int_{\mathcal{U}} \langle \alpha, \overline{\partial}^* \beta \rangle \, dV + \int_{\partial\mathcal{U}} \langle \mathrm{n}_H^{0,1} \wedge \alpha, \beta \rangle \, d\sigma \\ &= \int_{\mathcal{U}} \langle \alpha, \overline{\partial}^* \beta \rangle \, dV + \int_{\partial\mathcal{U}} \langle \nu_H^{0,1} \wedge \alpha, \beta \rangle \, d\sigma_H \end{aligned} \tag{4.4}$$

for every $\alpha \in \Omega_H^{p,q-1}, \beta \in \Omega_H^{p,q}$; see, e.g., [37], Chap. 3. More generally, all these formulas hold when α and β are horizontal differential forms of class $\mathbf{C}^1$ on $\overline{\mathcal{U}}$ (i.e., $\alpha \in \mathbf{C}^1(\overline{\mathcal{U}}, \bigwedge^{p-1,q} \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C})$ and $\beta \in \mathbf{C}^1(\overline{\mathcal{U}}, \bigwedge^{p,q} \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C})$).

5. Boundary conditions and estimates of the boundary terms

5.1. Horizontal Dirichlet integral

Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain with smooth boundary of class $\mathbf{C}^2$. Below, we introduce the notion of horizontal Dirichlet integral.

Definition 5.1. Let u be a horizontal differential form, either in $\Omega_H^h(\overline{\mathcal{U}})$ or in $\Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $0 \le h \le 2n$. We define the *horizontal Dirichlet integral* as

$$D_H(u) := \int_{\mathcal{U}} (\langle d_H u, d_H u\rangle + \langle \delta_H u, \delta_H u\rangle)\, dV.$$

Furthermore, if $1 \le h < n$, we set

$$D_H^J(u) := D_H(u) - \frac{1}{n-h+1} D_H(Ju).$$

The main purpose of this section is to write the horizontal Dirichlet integral of u as the L^2-norm of $\nabla_H u$ up to an error term that will be estimated later in Sections 5.2 and 5.3.

Proposition 5.2 (see [38], Proposition 2)**.** *We have*

$$\Delta_H = \Delta_K - i \sum_{k=1}^{n} \left(e_k i_k - e_{\bar{k}} i_{\bar{k}}\right) \mathcal{L}_T.$$

In particular, if $u = \sum_{I,J} u_{I,J} \vartheta^I \wedge \vartheta^{\bar{J}} \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, *with* $|I| = p_I$, $|J| = q_J$, *and* $h = p_I + q_J$, *then*

$$\Delta_H u = \sum_{I,J} (\Delta_H u_{I,J})\, \vartheta^I \wedge \vartheta^{\bar{J}},$$

where

$$\Delta_H u_{I,J} = \Delta_K u_{I,J} - i(p_I - q_J) T u_{I,J}.$$

Proposition 5.3. *Let* $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, *with* $0 \le h \le 2n$. *Then*

$$D_H(u) = \int_{\mathcal{U}} \langle \Delta_H u, u\rangle\, dV + \int_{\partial\mathcal{U}} (\langle d_H u, \mathrm{n}_H \wedge u\rangle - \langle \delta_H u, \mathrm{n}_H \lrcorner\, u\rangle)\, d\sigma. \tag{5.1}$$

In addition, if $u = \sum_{I,J} u_{I,J} \vartheta^I \wedge \vartheta^{\bar{J}}$, *then*

$$\begin{aligned}\int_{\mathcal{U}} \langle \Delta_H u, u\rangle\, dV &= \int_{\mathcal{U}} \|\nabla_H u\|^2\, dV \\ &\quad - \sum_{I,J} \int_{\partial\mathcal{U}} \langle d_H u_{I,J}, \mathrm{n}_H\rangle \bar{u}_{I,J}\, d\sigma - i \sum_{I,J} (p_I - q_J) \int_{\mathcal{U}} \bar{u}_{I,J} T u_{I,J}\, dV.\end{aligned} \tag{5.2}$$

Proof. Assertion (5.1) is just an integration by parts. We have to prove (5.2). Keeping in mind that if $v \in \mathbf{C}^1(\overline{\mathcal{U}})$ is a (real or complex) 0-form we have

$$\int_{\mathcal{U}} \langle \Delta_K v, v\rangle\, dV = \int_{\mathcal{U}} \langle d_H v, d_H v\rangle\, dV - \int_{\partial\mathcal{U}} \bar{v} \langle d_H v, \mathrm{n}_H\rangle,$$

we compute

$$\int_{\mathcal{U}} \langle \Delta_H u, u\rangle \, dV = \sum_{I,J} \int_{\mathcal{U}} \bar{u}_{I,J} \Delta_H u_{I,J} \, dV$$

$$= \sum_{I,J} \int_{\mathcal{U}} \langle d_H u_{I,J} d_H u_{I,J}\rangle dV - \int_{\partial\mathcal{U}} \langle d_H u_{I,J}, \mathrm{n}_H\rangle \bar{u}_{I,J} d\sigma - i \sum_{I,J} (p_I - q_J) \int_{\mathcal{U}} \overline{u}_{I,J} T u_{I,J} dV$$

$$= \sum_{I,J} \int_{\mathcal{U}} \|\nabla_H u_{I,J}\|^2 \, dV - \int_{\partial\mathcal{U}} \langle d_H u_{I,J}, \mathrm{n}_H\rangle \bar{u}_{I,J} \, d\sigma - i \sum_{I,J} (p_I - q_J) \int_{\mathcal{U}} \bar{u}_{I,J} T u_{I,J} \, dV$$

$$= \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \sum_{I,J} \int_{\partial\mathcal{U}} \langle d_H u_{I,J}, \mathrm{n}_H\rangle \bar{u}_{I,J} \, d\sigma - i \sum_{I,J} (p_I - q_J) \int_{\mathcal{U}} \bar{u}_{I,J} T u_{I,J} \, dV.$$

□

Remark 5.4. Let us consider the following boundary integral:

$$\int_{\partial\mathcal{U}} f\big(u, \nabla_H u, \mathrm{n}_H\big) \, d\sigma,$$

where $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ and f is a (real-valued) continuous function that is linear in the third argument n_H. Let $\mathcal{V}_\epsilon \subset \partial\mathcal{U}$ be a family of open neighborhoods of $\mathrm{char}(\partial\mathcal{U})$ shrinking around $\mathrm{char}(\partial\mathcal{U})$ as long as $\epsilon \to 0$; in particular, we assume that $\mathcal{V}_{\epsilon_1} \subsetneq \mathcal{V}_{\epsilon_2}$ if $\epsilon_1 < \epsilon_2$ and that $\sigma(\mathcal{V}_\epsilon) \to 0$ as $\epsilon \to 0$ (by Remark 2.12 we already know that $\sigma(\mathrm{char}(\partial\mathcal{U})) = 0 = \sigma_H(\mathrm{char}(\partial\mathcal{U}))$.

By remembering that $d\sigma_H = \|\mathrm{n}_H\| d\sigma$ and that outside $\mathrm{char}(\partial\mathcal{U})$ we have set $\nu_H = \mathrm{n}_H / \|\mathrm{n}_H\|$, we get

$$\int_{\partial\mathcal{U}} f\big(u, \nabla_H u, \mathrm{n}_H\big) \, d\sigma = \lim_{\epsilon\to 0} \int_{\partial\mathcal{U}\setminus\mathcal{V}_\epsilon} f\big(u, \nabla_H u, \mathrm{n}_H\big) \, d\sigma$$

$$= \lim_{\epsilon\to 0} \int_{\partial\mathcal{U}\setminus\mathcal{V}_\epsilon} f\big(u, \nabla_H u, \nu_H\big) \, d\sigma_H =: \int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} f\big(u, \nabla_H u, \nu_H\big) \, d\sigma_H.$$

Combining Proposition 5.3 and Remark 5.4, we get the following corollary.

Corollary 5.5. *Let $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $0 \le h \le 2n$, and let us set*

$$\mathbf{A} := \int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \Big(- \sum_{I,J} \langle d_H u_{I,J}, \nu_H\rangle \bar{u}_{I,J} + \big(\langle d_H u, \nu_H \wedge u\rangle - \langle \delta_H u, \nu_H \,\lrcorner\, u\rangle\big)\Big) \, d\sigma_H,$$

and

$$\mathbf{B} := i \sum_{I,J} (p_I - q_J) \int_{\mathcal{U}} \bar{u}_{I,J} T u_{I,J} \, dV = i \Big\langle \mathcal{L}_T u, \sum_{k=1}^{n} (e_k i_k - e_{\bar{k}} i_{\bar{k}}) u \Big\rangle.$$

Then, we have

$$D_H(u) = \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV + \mathbf{A} - \mathbf{B} = \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV + \Re\mathrm{e}\, \mathbf{A} - \Re\mathrm{e}\, \mathbf{B}. \tag{5.3}$$

5.2. Estimate of the term A in (5.3)

The aim of this subsection is to show that we can write

$$\mathbf{A} = -\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \left(\langle d_H u_{\nu_H}, u_{\mathrm{t}}\rangle + \langle \delta_H u_{\mathrm{t}}, u_{\nu_H}\rangle\right) d\sigma_H + \text{“error term”},$$

and to provide sufficient conditions on the traces of u on the boundary $\partial\mathcal{U}$ to guarantee that

$$\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \left(\langle d_H u_{\nu_H}, u_{\mathrm{t}}\rangle + \langle \delta_H u_{\mathrm{t}}, u_{\nu_H}\rangle\right) d\sigma_H = 0;$$

see Proposition 5.11 below.

Definition 5.6 (The maps R_1, R_2)**.** Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain of class $\mathbf{C}^2$, let $\mu_H \in \mathbf{C}^1\left(\overline{\mathcal{U}}, \bigwedge^1 \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C}\right)$, with $0 \le h \le 2n$. We define the maps

$$R_1, R_2 : \mathbf{C}^0\Big(\overline{\mathcal{U}}, \bigwedge^h \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C}\Big) \longrightarrow \mathbf{C}^0\Big(\overline{\mathcal{U}}, \bigwedge^h \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C}\Big)$$

by setting

$$R_1(u) \equiv R_1^{\mu_H}(u) := \sum_{I,J} u_{I,J}\, d_H\big(\mu_H \lrcorner (\vartheta^I \wedge \vartheta^{\bar{J}})\big), \tag{5.4}$$

$$R_2(u) \equiv R_2^{\mu_H}(u) := \sum_{I,J} u_{I,J}\, \delta_H\big(\mu_H \wedge (\vartheta^I \wedge \vartheta^{\bar{J}})\big), \tag{5.5}$$

where $u = \sum_{I,J} u_{I,J} \vartheta^I \wedge \vartheta^{\bar{J}}$, $|I| = p_I$, $|J| = q_J$, and $h = p_I + q_J$. We also set $R_1^{\mu_H}(u) = 0$ if $h = 0$ and $R_2^{\mu_H}(u) = 0$ if $h = 2n$.

Note that the maps $R_1^{\mu_H}(u)$ and $R_2^{\mu_H}(u)$ are both linear in u and μ_H. The preceding definition is inspired by [14]; see Definition 5.1, p. 103. As a matter of fact, these maps turn out to be very useful because of well-known properties of the Lie derivative and, in particular, of Cartan's formula and of its dual version.

Remark 5.7. Let $u = \sum_{I,J} u_{I,J} \vartheta^I \wedge \vartheta^{\bar{J}}$. By using Cartan's formula and its dual version we get:

(i) $\mathcal{L}_{\mu_H}(u) = \mu_H \lrcorner\, d_H u + d_H(\mu_H \lrcorner\, u)$;

(ii) $\widetilde{\mathcal{L}_{\mu_H}}(u) := (-1)^{h(2n-h)} *_H \mathcal{L}_{\mu_H}(*_H u) = -\mu_H \wedge \delta_H u - \delta_H(\mu_H \wedge u)$.

In particular, one has $R_2^{\mu_H}(u) = (-1)^{h(2n-h)} *_H R_1^{\mu_H}(*_H u)$.

In addition, the following hold:

(iii) $\mathcal{L}_{\mu_H}(u) = \sum_{I,J} \langle d_H u_{I,J}, \mu_H\rangle \vartheta^I \wedge \vartheta^{\bar{J}} + R_1^{\mu_H}(u)$;

(iv) $\widetilde{\mathcal{L}_{\mu_H}}(u) = \sum_{I,J} \langle d_H u_{I,J}, \mu_H\rangle \vartheta^I \wedge \vartheta^{\bar{J}} + R_2^{\mu_H}(u)$.

Hence we obtain the identities:

$$\sum_{I,J}\langle d_H u_{I,J},\mu_H\rangle\vartheta^I\wedge\vartheta^{\bar J}=\mu_H\,\lrcorner\,d_H u+d_H(\mu_H\,\lrcorner\,u)-R_1^{\mu_H}(u)$$
$$=-\mu_H\wedge\delta_H u-\delta_H(\mu_H\wedge u)-R_2^{\mu_H}(u). \tag{5.6}$$

All these formulas can be checked by direct computations, as in the Euclidean case for which we refer the reader to Chap. 5 in [14].

If in Remark 5.7 we take $\mu_H=\nu_H$, we obtain the following result.

Lemma 5.8. *Let* $u=\sum_{I,J}u_{I,J}\vartheta^I\wedge\vartheta^{\bar J}$. *Then*

$$\sum_{I,J}\bar u_{I,J}\langle d_H u_{I,J},\nu_H\rangle+\langle R_1^{\nu_H}(u),u_{\mathrm t}\rangle+\langle R_2^{\nu_H}(u),\nu_H\wedge u_{\nu_H}\rangle$$
$$=\langle\nu_H\,\lrcorner\,d_H u,u_{\mathrm t}\rangle-\langle\delta_H u,u_{\nu_H}\rangle+\langle d_H u_{\nu_H},u_{\mathrm t}\rangle-\langle\delta_H\left(\nu_H\wedge u_{\mathrm t}\right),\nu_H\wedge u_{\nu_H}\rangle \tag{5.7}$$

at each point of $\partial\mathcal U\setminus\mathrm{char}(\partial\mathcal U)$.

Proof. Using Remark 5.7 yields

$$\langle\mathcal L_{\nu_H}u,u_{\mathrm t}\rangle=\langle\nu_H\,\lrcorner\,du,u_{\mathrm t}\rangle+\langle du_{\nu_H},u_{\mathrm t}\rangle=\langle\nu_H\,\lrcorner\,d_H u,u_{\mathrm t}\rangle+\langle d_H u_{\nu_H},u_{\mathrm t}\rangle\,. \tag{5.8}$$

Analogously, one has

$$\left\langle\widetilde{\mathcal L_{\nu_H}}u,\nu_H\wedge u_{\nu_H}\right\rangle=-\left(\langle\nu_H\wedge\delta_H u,\nu_H\wedge u_{\nu_H}\rangle+\langle\delta_H\left(\nu_H\wedge u\right),\nu_H\wedge u_{\nu_H}\rangle\right)$$
$$=-\left(\langle\delta_H u,u_{\nu_H}\rangle+\langle\delta_H\left(\nu_H\wedge u_{\mathrm t}\right),\nu_H\wedge u_{\nu_H}\rangle\right). \tag{5.9}$$

Adding together the left-hand sides of (5.8) and (5.9) and then using Remark 5.7 (see, in particular, formula (5.6)) yields

$$\langle\mathcal L_{\nu_H}u,u_{\mathrm t}\rangle+\left\langle\widetilde{\mathcal L_{\nu_H}}u,\nu_H\wedge u_{\nu_H}\right\rangle$$
$$=\sum_{I,J}\bar u_{I,J}\langle d_H u_{I,J},\nu_H\rangle+\langle R_1^{\nu_H}(u),u_{\mathrm t}\rangle+\langle R_2^{\nu_H}(u),\nu_H\wedge u_{\nu_H}\rangle.$$

Hence, by using (5.8) and (5.9), we deduce (5.7). □

We also need the following result (see [14], Lemma 5.5).

Lemma 5.9. *Let* u, μ_H, $R_1^{\mu_H}$ *and* $R_2^{\mu_H}$ *be as in Definition* 5.6. *Then*

$$R_1^{\mu_H}(\mu_H\wedge u)=\frac12 d_H\left(\|\mu_H\|^2\right)\wedge u+\mu_H\wedge R_1^{\mu_H}(u),$$
$$R_2^{\mu_H}(\mu_H\,\lrcorner\,u)=\frac12 d_H\left(\|\mu_H\|^2\right)\lrcorner\,u+\mu_H\,\lrcorner\,R_2^{\mu_H}(u).$$

The above formulas greatly simplify if we take $\|\mu_H\|=1$ and this can always be done, at least if both these quantities are restricted to the non-characteristic part of the boundary and we take $\mu_H=\nu_H$ (i.e., μ_H is the horizontal unit normal to $\partial\mathcal U\setminus\mathrm{char}(\partial\mathcal U)$).

Remark 5.10. For any $\alpha \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ the following holds on $\partial\mathcal{U} \setminus \mathrm{char}(\partial\mathcal{U})$:

- If $\nu_H \wedge \alpha = 0$, then $\nu_H \wedge d_H\alpha = 0$.
- If $\nu_H \,\lrcorner\, \alpha = 0$, then $\nu_H \,\lrcorner\, \delta_H\alpha = 0$.

These properties can be proved just as in the classical case, for which we refer to Theorem 3.23 in [14]. Thus, at each point of $\partial\mathcal{U} \setminus \mathrm{char}(\partial\mathcal{U})$, we deduce that:

- If $u_{\nu_H} = \nu_H \,\lrcorner\, u = 0$, then it follows that $\nu_H \wedge (\nu_H \,\lrcorner\, u) = 0$. Thus we get $\nu_H \wedge d_H(\nu_H \,\lrcorner\, u) = 0$ and $\langle d_H u_{\nu_H}, u_{\mathrm{t}}\rangle = 0$.
- If $\nu_H \wedge u = 0$, then $u_{\mathrm{t}} = \nu_H \,\lrcorner (\nu_H \wedge u) = 0$. Hence $\nu_H \,\lrcorner\, \delta_H(\nu_H \wedge u) = 0$ and $\langle \delta_H\left(\nu_H \wedge u_{\mathrm{t}}\right), \nu_H \wedge u_{\nu_H}\rangle = 0$.

We summarize the above discussion in the next proposition.

Proposition 5.11. *Let $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $1 \le h \le 2n$. Then*

$$\begin{aligned}\mathbf{A} = &-\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \left(\langle d_H u_{\nu_H}, u_{\mathrm{t}}\rangle + \langle \delta_H u_{\mathrm{t}}, u_{\nu_H}\rangle\right) d\sigma_H \\ &-\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \big\langle \widetilde{R}(u), u\big\rangle\, d\sigma_H,\end{aligned} \tag{5.10}$$

where

$$\big\langle \widetilde{R}(u), u\big\rangle := \big\langle R_1^{\nu_H}(u), u_{\mathrm{t}}\big\rangle + \big\langle R_2^{\nu_H}(u), \nu_H \wedge u_{\nu_H}\big\rangle.$$

Remark 5.12. The first boundary integral in (5.10) vanishes if

$$\text{either } u_{\mathrm{t}} = 0 \;\text{ or }\; u_{\nu_H} = 0 \text{ on } \;\partial\mathcal{U} \setminus \mathrm{char}(\partial\mathcal{U}) \quad (\text{condition (DN)})$$

and, in this case, we get

$$\Re\mathrm{e}\,\mathbf{A} = -\Re\mathrm{e} \int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \big\langle \widetilde{R}(u), u\big\rangle\, d\sigma_H. \tag{5.11}$$

Remark 5.13. Obviously, when $u_{\mathrm{t}} = 0$, it follows that $\langle \widetilde{R}(u), u\rangle = \langle R_2^{\nu_H}(u), u\rangle$. Finally, if $u_{\nu_H} = 0$, then $\langle \widetilde{R}(u), u\rangle = \langle R_1^{\nu_H}(u), u\rangle$.

Proof of Proposition 5.11. Let us start from the identity in Corollary 5.5. For what concerns the term $\mathbf{A}$, by using (5.7) and Remark 5.4, we get

$$\begin{aligned}\mathbf{A} &= -\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \Big(\sum_{I,J} \bar{u}_{I,J}\langle d_H u_{I,J}, \nu_H\rangle - \langle \nu_H \,\lrcorner\, d_H u, u_{\mathrm{t}}\rangle + \langle \delta_H u, u_{\nu_H}\rangle\Big)\, d\sigma_H \\ &= -\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \left(\langle d_H u_{\nu_H}, u_{\mathrm{t}}\rangle - \langle \delta_H\left(\nu_H \wedge u_{\mathrm{t}}\right), \nu_H \wedge u_{\nu_H}\rangle\right)\, d\sigma_H \\ &\quad + \int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \left(\langle R_1(u), u_{\mathrm{t}}\rangle + \langle R_2(u), \nu_H \wedge u_{\nu_H}\rangle\right)\, d\sigma_H.\end{aligned}$$

Then (5.10) follows since $\delta_H\left(\nu_H \wedge u_{\mathrm{t}}\right) = -\widetilde{\mathcal{L}_{\nu_H}}(u_{\mathrm{t}}) - \nu_H \wedge \delta_H u_{\mathrm{t}}$ and

$$\big\langle \widetilde{\mathcal{L}_{\nu_H}}(u_{\mathrm{t}}), \nu_H \wedge u_{\nu_H}\big\rangle = 0.$$

Thus, Remark 5.10 yields (5.11), and the remaining claims easily follow. □

Remark 5.14. If we look at identity (5.3) we see that $\Re e\mathbf{A}$ does not depend on the coordinates. In fact, by its very definition, $\Re e\mathbf{B}$ is independent of the coordinates and, in addition, a straightforward computation shows that the same assertion holds for the quantity $D_H(u)$ and for the L^2-norm of $\nabla_H u$. Now, if condition (DN) holds, then both quantities $R_1^{\nu_H}$ and $R_2^{\nu_H}$ are independent of the coordinates. In particular, their expressions in Kähler coordinates (5.4) and (5.5) can be replaced, when convenient, by their counterpart in a different system of coordinates.

Remark 5.15. We point out that from Lemma 5.9 it follows that

$$\langle R_2(u), \nu_H \wedge u_{\nu_H}\rangle = \langle R_2(u_{\nu_H}), u_{\nu_H}\rangle \quad \text{and} \quad \langle R_1(u), u_{\mathrm{t}}\rangle = \langle R_1(u_{\mathrm{t}}), u_{\mathrm{t}}\rangle.$$

5.3. Estimate of the term B in (5.3)

The aim of this subsection is to prove that

$$\begin{aligned}\Re e\,\mathbf{B} = {} & \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}}(\|\bar\partial u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\,dV \\ & - \frac{1}{n}\Im m\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})}\Big\langle \mathcal{L}_{J\nu_H}u, \sum_{k=1}^n (e_k i_k - e_{\bar k}\, i_{\bar k})u\Big\rangle\, d\sigma_H + \text{“error term”},\end{aligned}$$

where the “error term” depends only on the trace of u on the boundary (not on its derivatives) and will be estimated below under different assumptions. At the same time, we provide sufficient conditions on the traces of u on $\partial\mathcal{U}$ that guarantee that

$$\Im m\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})}\Big\langle \mathcal{L}_{J\nu_H}u, \sum_{k=1}^n (e_k i_k - e_{\bar k}\, i_{\bar k})u\Big\rangle = 0.$$

Proposition 5.16. *Let $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $1 \le h \le 2n$. Then*

$$\begin{aligned}\mathbf{B} = {} & \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}}(\|\bar\partial u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\,dV \\ & + \frac{i}{n}\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})}\Big\langle \mathcal{L}_{J\nu_H}u, \sum_{k=1}^n (e_k\, i_k - e_{\bar k}\, i_{\bar k})u\Big\rangle\, d\sigma_H \\ & - \frac{i}{n}\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})}\Big\langle R_1^{J\nu_H}u, \sum_{k=1}^n (e_k\, i_k - e_{\bar k}\, i_{\bar k})u\Big\rangle\, d\sigma_H.\end{aligned} \tag{5.12}$$

Proof. Let $v \in \mathbf{C}^1(\overline{\mathcal{U}})$ be a (complex-valued) 0-form and recall that

$$i n T = \sum_{k=1}^n (\partial_k\,\partial_{\bar k} - \partial_{\bar k}\,\partial_k).$$

By (4.3) and (4.4), we have

$$
\begin{aligned}
-i\int_{\mathcal{U}} \bar{v}\, Tv\, dV &= \frac{1}{n}\int_{\mathcal{U}} \bar{v}\sum_{k=1}^{n}(\partial_{\bar{k}}\partial_k - \partial_k\partial_{\bar{k}})\, v\, dV \\
&= \frac{1}{n}\int_{\mathcal{U}} \bar{v}\left(-\partial^*\partial + \bar{\partial}^*\bar{\partial}\right) v\, dV \\
&= \frac{1}{n}\int_{\mathcal{U}} (\|\bar{\partial}v\|^2 - \|\partial v\|^2)\, dV - \frac{1}{n}\int_{\partial\mathcal{U}} \left(\bar{v}\langle\bar{\partial}v, \mathrm{n}_H^{0,1}\rangle - \bar{v}\langle\partial v, \mathrm{n}_H^{1,0}\rangle\right) d\sigma \\
&= \frac{1}{n}\int_{\mathcal{U}} (\|\bar{\partial}v\|^2 - \|\partial v\|^2)\, dV + \frac{1}{n}\int_{\partial\mathcal{U}} \bar{v}\left\langle d_H v, \left(\mathrm{n}_H^{1,0} - \mathrm{n}_H^{0,1}\right)\right\rangle d\sigma \\
&= \frac{1}{n}\int_{\mathcal{U}} (\|\bar{\partial}v\|^2 - \|\partial v\|^2)\, dV + \frac{i}{n}\int_{\partial\mathcal{U}} \bar{v}\,\langle d_H v, J\mathrm{n}_H\rangle\, d\sigma,
\end{aligned}
$$

where we have used the identity $J\mathrm{n}_H = i(\mathrm{n}_H^{1,0} - \mathrm{n}_H^{0,1})$.

From these computations, by arguing as in Remark 5.4 and by applying (iii) of Remark 5.7, we get that the term $\mathbf{B}$ can be rewritten as follows:

$$
\begin{aligned}
\mathbf{B} &= -i\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}} \bar{u}_{I,J} T u_{I,J}\, dV \\
&= \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}} (\|\bar{\partial}u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\, dV \\
&\quad + \frac{i}{n}\sum_{I,J}(p_I - q_J)\lim_{\epsilon\to 0}\int_{\partial\mathcal{U}\setminus\mathcal{V}_\epsilon} \bar{u}_{I,J}\,\langle d_H u_{I,J}, J\nu_H\rangle\, d\sigma_H \\
&= \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}} (\|\bar{\partial}u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\, dV \\
&\quad + \frac{i}{n}\sum_{I,J}(p_I - q_J)\int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \bar{u}_{I,J}\,\langle d_H u_{I,J}, J\nu_H\rangle\, d\sigma_H \\
&= \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}} (\|\bar{\partial}u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\, dV \\
&\quad + \frac{i}{n}\int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \left\langle \mathcal{L}_{J\nu_H} u, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})u\right\rangle d\sigma_H \\
&\quad - \frac{i}{n}\int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \left\langle R_1^{J\nu_H} u, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})u\right\rangle d\sigma_H .
\end{aligned}
$$

This achieves the proof. □

The following assertion is a straightforward consequence of the identity (5.12).

Corollary 5.17. *Suppose that the following "condition $(J\nu_H)$" holds:*

$$
\Im\mathrm{m}\left\langle \mathcal{L}_{J\nu_H} u, \sum_{k=1}^{n} e_k i_k u\right\rangle = \Im\mathrm{m}\left\langle \mathcal{L}_{J\nu_H} u, \sum_{k=1}^{n} e_{\bar{k}}\, i_{\bar{k}} u\right\rangle \quad (\text{condition } (J\nu_H)).
$$

Then we have

$$\begin{aligned}\Re\mathrm{e}\,\mathbf{B} &= \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}}(\|\bar\partial u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\,dV \\ &\quad + \Im\mathrm{m}\,\frac{1}{n}\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})}\Big\langle R_1^{J\nu_H}u, \sum_{k=1}^{n}(e_k i_k - e_{\bar k}\, i_{\bar k})u\Big\rangle\, d\sigma_H. \end{aligned} \tag{5.13}$$

Remark 5.18. From Proposition 4.2 it follows that

$$\sum_{k=1}^{n}(e_k i_k - e_{\bar k}\, i_{\bar k}) = \sum_{p,q}(p-q)\,\Pi^{p,q}.$$

Therefore, we see that condition $(J\nu_H)$ is a compatibility condition on the bidegree components of the trace of u in the (tangent) direction $J\nu_H$.

In addition, we stress that condition $(J\nu_H)$ is written in a "geometric" form on $\partial\mathcal{U}$ and it could be replaced by the following condition $(J\nu_H^*)$, which is written "in coordinates":

$$\sum_{I,J}(p_I - q_J)\,\Im\mathrm{m}\,(\bar u_{I,J}\,\langle d_H\, u_{I,J}, J\nu_H\rangle) = 0 \tag{5.14}$$

at every point of $\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})$. The last condition is perhaps less "elegant" but has the advantage of not introducing an error term.

Typically, identity (5.14) holds, if the form $u = \sum_{I,J} u_{I,J}\vartheta^I\wedge\vartheta^{\bar J}$ is "Kähler-symmetric" on $\partial\mathcal{U}$, i.e., if

$$u_{I,J} = \pm u_{J,I} \quad \text{for all } I,\ J \text{ with } |I|+|J| = h.$$

We also observe that if u is Kähler-symmetric on all of $\overline{\mathcal{U}}$, then $\mathbf{B} = 0$, and the main inequality still holds under condition (DN).

Let us analyze the meaning of condition $(J\nu_H)$ in the case of horizontal 1-forms.

Example 5.19 (1-forms)**.** Let $u = \sum_{i=1}^n \big(u_i\vartheta^i + u_{\bar i}\vartheta^{\bar i}\big)$ be a 1-form, where we assume that $u_i := f_i + ig_i$ for any $i = 1,\dots,n$. Also recall that if u is real, then $u_{\bar i} = \bar u_i$ for any $i = 1,\dots,n$. Note that $J\nu_H = i(\nu_H^{1,0} - \nu_H^{0,1})$ and that, in this case, we have $p_i = 1$, $q_i = 0$ and $p_{\bar i} = 0$, $q_{\bar i} = 1$, $i = 1,\dots,n$. With these preliminaries, we may reformulate condition $(J\nu_H)$ as follows:

$$\sum_{i=1}^{n}\langle(f_i\nabla_H g_i - g_i\nabla_H f_i)\,, J\nu_H\rangle = \sum_{\bar i=1}^{n}\langle(f_{\bar i}\nabla_H g_{\bar i} - g_{\bar i}\nabla_H f_{\bar i})\,, J\nu_H\rangle\,. \tag{5.15}$$

The proof of (5.15) is an elementary exercise. In addition, we observe that if u is real, then (5.15) becomes

$$\sum_{i=1}^{n}\langle(f_i\nabla_H g_i - g_i\nabla_H f_i)\,, J\nu_H\rangle = 0$$

or, equivalently, $\sum_{i=1}^n (f_i\,\mathcal{L}_{J\nu_H}\, g_i - g_i\,\mathcal{L}_{J\nu_H}\, f_i) = 0$.

By using (iv) in Remark 5.7 we obtain the following dual result.

Proposition 5.20. *Let* $u \in \Omega^h_H(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, *with* $1 \leq h \leq 2n$. *Then*

$$\begin{aligned}\mathbf{B} &= \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}}(\|\bar\partial u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\,dV\\ &\quad + \frac{i}{n}\int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})}\Big\langle \widetilde{\mathcal{L}_{J\nu_H}} u, \sum_{k=1}^{n}(e_k i_k - e_{\bar k}\, i_{\bar k})u\Big\rangle\, d\sigma_H\\ &\quad - \frac{i}{n}\int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})}\Big\langle R_2^{J\nu_H} u, \sum_{k=1}^{n}(e_k i_k - e_{\bar k}\, i_{\bar k})u\Big\rangle\, d\sigma_H.\end{aligned}$$

In addition, suppose the following "condition $(\widetilde{J\nu_H})$*" holds:*

$$\Im\mathrm{m}\Big\langle \widetilde{\mathcal{L}_{J\nu_H}} u, \sum_{k=1}^{n} e_k i_k u\Big\rangle = \Im\mathrm{m}\Big\langle \widetilde{\mathcal{L}_{J\nu_H}} u, \sum_{k=1}^{n} e_{\bar k}\, i_{\bar k} u\Big\rangle \quad (\text{condition } (\widetilde{J\nu_H})).$$

Then

$$\begin{aligned}\Re\mathrm{e}\,\mathbf{B} &= \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}}\left(\|\bar\partial u_{I,J}\|^2 - \|\partial u_{I,J}\|^2\right)\,dV\\ &\quad - \frac{i}{n}\int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})}\Big\langle R_2^{J\nu_H} u, \sum_{k=1}^{n}(e_k i_k - e_{\bar k}\, i_{\bar k})u\Big\rangle\, d\sigma_H.\end{aligned}$$

Remark 5.21. Just as in Remark 5.18, we observe that condition $(\widetilde{J\nu_H})$ is written in a "geometric" form and that it could be replaced by condition $(J\nu_H^*)$. Again, this alternative condition has the advantage of not introducing an error term.

6. Gaffney–Friedrichs-type inequalities for horizontal forms

The first version of our main result reads as follows.

Theorem 6.1 (Gaffney–Friedrichs inequality)**.** *Let* $\mathcal{U} \subsetneq \mathbb{H}^n$ *be a domain with smooth boundary of class* $\mathbf{C}^2$. *Let* $u \in \Omega^h_H(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ *be a horizontal h-form with* $1 \leq h < n$, *and assume that*:

(i) *u satisfies condition (DN) (see Proposition 5.11)*;

(ii) *u satisfies either condition* $(J\nu_H)$ *(see Proposition 5.16) or condition* $(\widetilde{J\nu_H})$ *(see Proposition 5.20)*.

Let $\{\mathcal{V}_\epsilon\}_{\epsilon>0}$ *be a family of open neighborhoods of* $\mathrm{char}(\partial\mathcal{U})$ *(in the relative topology) shrinking around* $\mathrm{char}(\partial\mathcal{U})$ *when* $\epsilon \to 0$. *In addition, assume that* $\sigma(\mathcal{V}_\epsilon) \to 0$ *as* $\epsilon \to 0$. *Then, there exist geometric constants* C_0, C_1 *and* C_2 *such that*

$$D_H(u) + C_0\int_{\partial\mathcal{U}\cap\mathcal{V}_\epsilon}\|u\|^2\,d\sigma \geq C_1\int_{\mathcal{U}}\|\nabla_H u\|^2\,dV - C_2\int_{\mathcal{U}}\|u\|^2\,dV. \tag{6.1}$$

The constants C_0, C_1, C_2 *depend only on* $\mathcal{U}$, ϵ *and on the integers h and n.*

Furthermore, if $u \in \Omega^h_H(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$*is a horizontal* h*-form with* $n+1 \le h \le 2n$*, then* (6.1) *still holds provided that* $*_H u$ *satisfies* (i) *and* (ii)*.*

Remark 6.2. The constant C_2 may blow up as ϵ tends to 0^+. Indeed, let us define two constants:

- $C_{1,\epsilon} := 2n \max_{i,j=1,\dots,2n} \sup_{\partial\mathcal{U}\cap\mathcal{V}_\epsilon} |W_j(\mathrm{n}_H)_i - (\nu_H)_i W_j\|\mathrm{n}_H\||$,
- $C_{2,\epsilon} := \sup_{\partial\mathcal{U}\setminus\mathcal{V}_\epsilon} \|\mathcal{J}\mathrm{ac}_H \nu_H\|$,

where $\mathcal{J}\mathrm{ac}_H\nu_H = [W_j(\nu_H)_i]_{i,j=1,\dots,2n}$ denotes the horizontal Jacobian matrix of the unit horizontal normal ν_H. Since n_H is of class $\mathbf{C}^1$, the constant $C_{1,\epsilon}$ turns out to be globally bounded along $\partial\mathcal{U}$. On the other hand, we have $C_{2,\epsilon} = O(1/\|\mathrm{n}_H\|)$, and hence $C_{2,\epsilon}$ may diverge when $\epsilon \to 0^+$ (since $\|\mathrm{n}_H\| \to 0^+$ as $\epsilon \to 0^+$). Below, we shall prove the result with the constants

$$C_0 := C_{\dim} \cdot C_{1,\epsilon}, \quad C_1 := \frac{1}{n} - C_{\dim} \cdot C_{2,\epsilon} \cdot \delta, \quad C_2 := C_{\dim} \cdot C_{2,\epsilon} \cdot C_{\mathcal{V}_\epsilon,\mathcal{U},\delta},$$

where

$$0 < \delta < \min\left\{1, \frac{1}{n\, C_{\dim} C_{2,\epsilon}}\right\},$$

the constant $C_{\mathcal{V}_\epsilon,\mathcal{U},\delta}$ was defined in Theorem 3.1, and $C_{\dim}$ is a fixed dimensional constant that only depends on n.

Proof. Combining (5.3), (5.11) and (5.13) we obtain

$$\begin{aligned} D_H(u) = {} & \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}} (\|\bar\partial u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\, dV \\ & - \Re\mathrm{e} \int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \langle R_i^{\nu_H} u, u\rangle \, d\sigma_H \\ & - \Im\mathrm{m} \frac{1}{n} \int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \Big\langle R_j^{J\nu_H} u, \sum_{k=1}^n (e_k i_k - e_{\bar k}\, i_{\bar k}) u\Big\rangle \, d\sigma_H, \end{aligned}$$

where $i, j = 1, 2$. On the other hand, keeping in mind (4.2) and the fact that

$$|p_I - q_J| \le h < n,$$

we get

$$\begin{aligned} \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \frac{1}{n}\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}} (\|\bar\partial u_{I,J}\|^2 - \|\partial u_{I,J}\|^2)\, dV & \\ \ge \frac{n-h}{n} \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV, & \end{aligned}$$

so that

$$\begin{aligned} D_H(u) \ge {} & \frac{1}{n}\int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \Re\mathrm{e}\int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \langle R_i^{\nu_H} u, u\rangle \, d\sigma_H \\ & - \Im\mathrm{m} \frac{1}{n} \int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \Big\langle R_j^{J\nu_H} u, \sum_{k=1}^n (e_k i_k - e_{\bar k}\, i_{\bar k}) u\Big\rangle \, d\sigma_H. \end{aligned} \tag{6.2}$$

By arguing as in [14], Chapter 5, Section 2, it is not difficult to check that $R_1^{\nu_H}(u)$ and $R_2^{\nu_H}(u)$ satisfy the estimates

$$\|R_i^{\nu_H}(u)\| \leq C_{\rm dim}^i \,\|\mathcal{J}\mathrm{ac}_H \nu_H\| \|u\| \quad (i = 1, 2),$$

where $C_{\rm dim}^i := C^i(h, n)$ is a positive constant, dependent only on the integers h and n. Analogously, we have

$$\|R_i^{J\nu_H}(u)\| \leq C_{\rm dim}^i \,\|\mathcal{J}\mathrm{ac}_H \nu_H\| \|u\| \quad (i = 1, 2).$$

Moreover, a straightforward computation shows that $\|\mathcal{J}\mathrm{ac}_H \nu_H\|$ is of class $\mathbf{C}^1$ out of $\mathrm{char}(\partial\mathcal{U})$ and that $\|\mathcal{J}\mathrm{ac}_H \nu_H\| = O(1/\|\mathrm{n}_H\|)$ near $\mathrm{char}(\partial\mathcal{U})$.

Hence, keeping in mind Theorem 3.1, we make the following computations:

$$\begin{aligned}
D_H(u) &\geq \frac{1}{n}\int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - C_{\rm dim} \int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \|\mathcal{J}\mathrm{ac}_H \nu_H\| \|u\|^2 \, d\sigma_H \\
&\geq \frac{1}{n}\int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - C_{\rm dim} \int_{\partial\mathcal{U}\cap\mathcal{V}_\epsilon} \|\mathcal{J}\mathrm{ac}_H \nu_H\| \|u\|^2 \, d\sigma_H \\
&\quad - C_{\rm dim} \int_{\partial\mathcal{U}\setminus\mathcal{V}_\epsilon} \|\mathcal{J}\mathrm{ac}_H \nu_H\| \|u\|^2 \, d\sigma_H \\
&\geq \frac{1}{n}\int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - C_0 \int_{\partial\mathcal{U}\cap\mathcal{V}_\epsilon} \|u\|^2 \, d\sigma - C_{\rm dim} C_{2,\epsilon} \int_{\partial\mathcal{U}\setminus\mathcal{V}_\epsilon} \|u\|^2 \, d\sigma_H \\
&\geq \frac{1}{n}\int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - C_0 \int_{\partial\mathcal{U}\cap\mathcal{V}_\epsilon} \|u\|^2 \, d\sigma \\
&\quad - C_{\rm dim} C_{2,\epsilon} \Big(C_{\mathcal{V}_\epsilon,\mathcal{U},\delta} \int_{\mathcal{U}} \|u\|^2 \, dV + \delta \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV \Big),
\end{aligned}$$

and the assertion (6.1) follows, where the constant $C_{\mathcal{V}_\epsilon,\mathcal{U},\delta}$ was introduced in Theorem 3.1. □

Theorem 6.3 (Gaffney–Friedrichs inequality (2nd version))**.** *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain with boundary of class $\mathbf{C}^2$ satisfying condition (H) (see Definition 3.4). Let $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ be a horizontal h-form with $1 \leq h < n$, and assume that:*

(i) *u satisfies condition (DN) (see Proposition 5.11);*

(ii) *u satisfies condition $(J\nu_H^*)$ (see Remark 5.18).*

Then, there exist geometric constants $\widetilde{C}_1$ and $\widetilde{C}_2$, only dependent on $\mathcal{U}$ and on the integers h and n, such that

$$D_H(u) \geq \widetilde{C}_1 \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \widetilde{C}_2 \int_{\mathcal{U}} \|u\|^2 \, dV. \tag{6.3}$$

Furthermore, if $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ is a horizontal h-form with $n+1 \leq h \leq 2n$, then (6.3) still holds provided that $*_H u$ satisfies (i) *and* (ii)*.*

For the case $h = n$ we refer the reader to Theorem 6.7 below.

We start from the estimate (6.2) in the proof of Theorem 6.1, by proving a more effective estimate of the remaining terms. By Remarks 5.18 and 5.21, we are reduced to

$$D_H(u) \geq \frac{1}{n}\int_{\mathcal{U}} \|\nabla_H u\|^2\, dV - \Re\mathrm{e} \int_{\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})} \langle R_i^{\nu_H} u, u\rangle\, d\sigma_H. \tag{6.4}$$

To this end, let us study the quantities $R_i^{\nu_H}(u,u)$, $i=1,2$.

Remark 6.4. Let $\mu_H \in \mathbf{C}^1\big(\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})), \bigwedge^1 \mathfrak{h}_1 \otimes_{\mathbb{R}} \mathbb{C}\big)$ such that $\|\mu_H\| = 1$. Let $\varphi\colon \partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U}) \to \mathbb{R}$ be a smooth function. We want to estimate $R_i^{\varphi\mu_H}(u)$, $i=1,2$, for a horizontal h-form u. As pointed out in Remark 5.14, these estimates do not depend on the reference frame used. Therefore we write u with respect to the basis Ψ^h. We have

$$\begin{aligned} R_1^{\varphi\mu_H}(u) &= \varphi\Big(\sum_I u_I d_H\left(\mu_H \lrcorner\, \psi^I\right)\Big) + \sum_I u_I d_H\varphi \wedge \left(\mu_H \lrcorner\, \psi^I\right), \\ &= \varphi\, R_1^{\mu_H}(u) + d_H\varphi \wedge (\mu_H \lrcorner\, u)\,, \end{aligned}$$

and

$$\begin{aligned} R_2^{\varphi\mu_H}(u) &= \varphi\left(\sum_I u_I \delta_H\left(\mu_H \wedge \psi^I\right)\right) - \sum_I u_I\left(d_H\varphi \lrcorner \left(\mu_H \wedge \psi^I\right)\right) \\ &= \varphi\, R_2^{\mu_H}(u) - d_H\varphi \lrcorner\, (\mu_H \wedge u)\,. \end{aligned}$$

By condition (H), near the characteristic set, the boundary of $\mathcal{U}$ is a t-graph (i.e., $\partial\mathcal{U}$ is a Euclidean graph w.r.t. the hyperplane $t=0$) and so there exists a $\mathbf{C}^2$ defining function $g\colon \mathbb{H}^n \to \mathbb{R}$ of the form $g(x,y,t) = t - f(x,y)$. Hence

$$\mathrm{N}_H = \nabla_H g = \nabla_H(t - f(x,y)) = \Big(-\frac{y}{2} - \nabla_x f, \frac{x}{2} - \nabla_y f\Big).$$

Accordingly, we assume that $\nu_H = \mathrm{N}_H/\|\mathrm{N}_H\|$, where $\mathrm{N}_H := \nabla_H g$. Thus we get

$$R_1^{\nu_H}(u) = \frac{R_1^{\mathrm{N}_H}(u)}{\|\mathrm{N}_H\|} + d_H\Big(\frac{1}{\|\mathrm{N}_H\|}\Big) \wedge (\mathrm{N}_H \lrcorner\, u)\,, \tag{6.5}$$

where the second term vanishes on $\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})$ when $u_{\nu_H} = \nu_H \lrcorner\, u = 0$.
Similarly we get

$$R_2^{\nu_H}(u) = \frac{R_2^{\mathrm{N}_H}(u)}{\|\mathrm{N}_H\|} - d_H\Big(\frac{1}{\|\mathrm{N}_H\|}\Big) \lrcorner\, (\mathrm{N}_H \wedge u)\,, \tag{6.6}$$

and the second term vanishes on $\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})$ when $u_{\mathrm{t}} = \nu_H \lrcorner (\nu_H \wedge u) = 0$.

As we shall see below, formulas (6.5) and (6.6) are very important for our purposes. In particular, under the hypothesis $u_{\nu_H} = 0$ on $\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})$, we find that $R_1^{\nu_H}(u) = R_1^{\mathrm{N}_H}(u)/\|\mathrm{N}_H\|$. Furthermore, if $u_{\mathrm{t}} = 0$ on $\partial\mathcal{U}\setminus \mathrm{char}(\partial\mathcal{U})$, then the quantity $R_2^{\nu_H}(u)$ can be obtained by duality (via the horizontal Hodge star operator) from the computation of $R_1^{\nu_H}(u)$. Thus, let u be a horizontal h-form and let us compute

$$\begin{aligned}
R_1^{\nu_H}(u,u) &:= \langle R_1^{\nu_H}(u), u\rangle = \sum_{I,J} u_I \overline{u}_J \, \frac{\left\langle d_H\left(\mathrm{N}_H \,\lrcorner\, \psi^I\right), \psi^J\right\rangle}{\|\mathrm{N}_H\|} \\
&= \sum_{I,J}\sum_r u_I \overline{u}_J \, \frac{\left\langle d_H\left((\mathrm{N}_H)_r\, \psi_r \,\lrcorner\, \psi^I\right), \psi^J\right\rangle}{\|\mathrm{N}_H\|} \\
&= \sum_{I,J}\sum_{r,k} W_k\left((\mathrm{N}_H)_r\right) u_I \overline{u}_J \, \frac{\left\langle \psi_k \wedge \left(\psi_r \,\lrcorner\, \psi^I\right), \psi^J\right\rangle}{\|\mathrm{N}_H\|} \\
&= \frac{1}{\|\mathrm{N}_H\|}\Big\langle \sum_{r,k} W_k\left((\mathrm{N}_H)_r\right) \psi_k \wedge \left(\psi_r \,\lrcorner\, u\right), u\Big\rangle \\
&= \frac{1}{\|\mathrm{N}_H\|}\Big\langle \sum_{r,k} W_k\left((\mathrm{N}_H)_r\right) \psi_r \,\lrcorner\, u, \psi_k \,\lrcorner\, u\Big\rangle.
\end{aligned}$$

By condition (H) we have

$$\mathcal{J}\mathrm{ac}_H(\mathrm{N}_H) = \frac{1}{2} J - \mathrm{Hess}_{\mathbb{R}^{2n}} f.$$

Thus, using the skew-symmetry of the linear operator J, we get

$$\Re\mathrm{e}\, R_1^{\nu_H}(u,u) = O\Big(\frac{\|\mathrm{Hess}_{\mathbb{R}^{2n}} f\|}{\|\mathrm{N}_H\|}\Big)\|u\|^2, \tag{6.7}$$

and applying condition (H) yields $O(\|\mathrm{Hess}_{\mathbb{R}^{2n}} f\|/\|\mathrm{N}_H\|) = O(1)$.

Remark 6.5. More generally, let v be such that $J\nu_H \,\lrcorner\, v = 0$. Now, arguing as above we obtain

$$R_1^{J\nu_H}(v) = \frac{R_1^{J\mathrm{N}_H}(v)}{\|\mathrm{N}_H\|}.$$

Thus, as above we get

$$\mathcal{J}\mathrm{ac}_H(J\mathrm{N}_H) = -\frac{1}{2}\,\mathrm{Id} - J\,\mathrm{Hess}_{\mathbb{R}^{2n}} f,$$

and therefore

$$\Im\mathrm{m}\, R_1^{J\nu_H}(v,v) = O\Big(\frac{\|\mathrm{Hess}_{\mathbb{R}^{2n}} f\|}{\|\mathrm{N}_H\|}\Big)\|v\|^2 = O(1)\|v\|^2.$$

Eventually, we resume the above discussion in the following.

Lemma 6.6. *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain (bounded and open) with boundary of class $\mathbf{C}^2$ satisfying condition (H). Let u, v be horizontal h-forms defined on $\overline{\mathcal{U}}$, with $1 \le h \le n$. Then, we have the following*:

(i) *If $u_{\nu_H} = 0$ on $\partial\mathcal{U}$, then $\Re\mathrm{e}\, R_1^{\nu_H}(u,u) = O(\|u\|^2)$.*

(ii) *If $u_{\mathrm{t}} = 0$ on $\partial\mathcal{U}$, then $\Re\mathrm{e}\, R_2^{\nu_H}(u,u) = O(\|u\|^2)$.*

In addition, we have:

(iii) *If* $(Jv)_{\nu_H} = 0$ *on* $\partial\mathcal{U}$, *then* $\Im\mathrm{m}\, R_1^{J\nu_H}(v,v) = O(\|v\|^2)$.

(iv) *If* $(Jv)_{\mathrm{t}} = 0$ *on* $\partial\mathcal{U}$, *then* $\Im\mathrm{m}\, R_2^{J\nu_H}(v,v) = O(\|v\|^2)$.

In particular, it follows from definitions that $\Re\mathrm{e}\,\langle R(u), u\rangle = O(\|u\|^2)$.

Proof. The proof of (i) follows by using (6.7). Then (ii) follows from (i) by duality (using the horizontal Hodge star operator); see Remark 5.7. The last claim it is an immediate consequence of (i), (ii) and of the very definition of $R(u)$. Keeping in mind Remark 6.5, the assertions (iii) and (iv) follow in the same way. □

Proof of Theorem 6.3. From (6.4) we know that

$$D_H(u) \geq \frac{1}{n}\int_{\mathcal{U}} \|\nabla_H u\|^2\, dV - \Re\mathrm{e}\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \langle R_i^{\nu_H} u, u\rangle\, d\sigma_H.$$

By applying Lemma 6.6 and Theorem 3.9, it follows that

$$D_H(u) \geq \frac{1}{n}\int_{\mathcal{U}} \|\nabla_H u\|^2 dV - C\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \|u\|^2 d\sigma_H \geq \frac{1}{2n}\int_{\mathcal{U}} \|\nabla_H u\|^2 dV - C_n\|u\|_{L^2(\mathcal{U})}.$$

The proof easily follows. □

Theorem 6.7. *Suppose the assumptions of either Theorem* 6.1 *or Theorem* 6.3 *are satisfied, where the condition* $1 \leq h < n$ *is replaced by*

$$h = n \quad \textit{and} \quad \Pi^{n,0}u = 0 = \Pi^{0,n}u.$$

Then, the conclusions of Theorems 6.1 *and* 6.3 *hold.*

Moreover, if $u \in \Omega_H^{n,0} \cup \Omega_H^{0,n}$, *then estimates like* (6.1) *or* (6.3) *fail to hold.*

Proof. The first assertion follows by noticing that, during the proof of Theorems 6.1 and 6.3, the assumption $h < n$ has been used only in deriving inequality (6.2), where we used that if $u \in \Omega_H^{p,q}$, then $|p-q| < n$. But, trivially, the same conclusion holds if $h = n$ and $\Pi^{n,0}u = 0 = \Pi^{0,n}u$.

As for the second assertion, we take, for instance, $u = f\,\vartheta^{(1,2,\dots,n)}$, with $f \in \mathcal{D}(\mathcal{U})$. In such a case the estimates (6.1) and (6.3) coincide and represent nothing but a maximal subelliptic estimate for the operator $\Delta_K \pm inT$. But then $\Delta_K \pm inT$ would be hypoelliptic (see, e.g., [8], Theorem 4.1), contradicting the fact that the values $\pm n$ are "forbidden values" for the Kohn Laplacian in $\mathbb{H}^n$ (see, e.g., [43], Chap. XIII, section 2.3). □

7. Further Gaffney–Friedrichs inequalities for horizontal differential forms

As pointed out in Remark 5.18, the condition $(J\nu_H^*)$ of Theorem 6.3 is written "in coordinates". Therefore, we may replace it by a slightly different "geometric" condition.

To this end, we first observe that, in the proof of Theorem 6.3, condition $(J\nu_H^*)$ can be replaced by the following weaker one:

$$\left|\Im\mathrm{m}\Big\langle \partial_{J\nu_H} u, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})u\Big\rangle_{L^2(\partial\mathcal{U}, d\sigma_H)}\right| \leq C\,\|u\|^2_{L^2(\partial\mathcal{U}, d\sigma_H)}. \tag{7.1}$$

Let us still suppose that both conditions (DN) and (H) hold. If $u_{\nu_H} = 0$, we can argue as follows. By applying Remarks 5.7 and 5.18, we compute

$$\begin{aligned}
\Big\langle \partial_{J\nu_H} u\,, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})u\Big\rangle &= \Big\langle \partial_{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})Ju\Big\rangle \\
&= \Big\langle \big(\mathcal{L}_{J\nu_H} Ju - R_1^{J\nu_H} Ju\big), \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})Ju\Big\rangle \\
&= \Big\langle \mathcal{L}_{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})Ju\Big\rangle - \Big\langle R_1^{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})Ju\Big\rangle \\
&= \Big\langle \mathcal{L}_{J\nu_H} Ju, J\sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})u\Big\rangle - \Big\langle R_1^{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})Ju\Big\rangle \\
&= \Big\langle J^{-1}\mathcal{L}_{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})u\Big\rangle - \Big\langle R_1^{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})Ju\Big\rangle.
\end{aligned}$$

Now suppose that the following geometric condition holds:

$$\Im\mathrm{m}\Big\langle J^{-1}\mathcal{L}_{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})u\Big\rangle = 0. \tag{7.2}$$

Under this assumption, let us show that (7.1) holds. We have

$$\begin{aligned}
\Im\mathrm{m}\Big\langle R_1^{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})Ju\Big\rangle &= \Im\mathrm{m}\Big\langle R_1^{J\nu_H} v, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})v\Big\rangle \\
&= \Im\mathrm{m}\|N_H\|^{-1}\Big\langle R_1^{JN_H} v, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})v\Big\rangle \\
&= \Im\mathrm{m}\|N_H\|^{-1}\Big\langle \Big(-\frac{h}{2}\mathrm{Id} - J(\mathrm{Hess}_{\mathbb{R}^{2n}} f)\Big)v, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})v\Big\rangle \\
&= -\Im\mathrm{m}\frac{h}{2}\|N_H\|^{-1}\Big\langle v, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})v\Big\rangle \\
&\quad - \Im\mathrm{m}\|N_H\|^{-1}\Big\langle J(\mathrm{Hess}_{\mathbb{R}^{2n}} f)v, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})v\Big\rangle \\
&= -\Im\mathrm{m}\|N_H\|^{-1}\Big\langle J(\mathrm{Hess}_{\mathbb{R}^{2n}} f)v, \sum_{k=1}^{n}(e_k i_k - e_{\bar{k}}\, i_{\bar{k}})v\Big\rangle,
\end{aligned}$$

since

$$\Big\langle v, \sum_{k=1}^{n}(e_k i_k - e_{\bar k} i_{\bar k})v\Big\rangle = \sum_{k=1}^{n}\left(\|i_k v\|^2 - \|i_{\bar k} v\|^2\right)$$

is a real number. Thus keeping in mind that $\|v\| = \|Ju\| = \|u\|$ yields (7.1).

Analogously, if in condition (DN) one has $u_t = 0$, then we can argue in a similar way by assuming that:

$$\Im\mathrm{m}\Big\langle J^{-1}\widetilde{\mathcal{L}}_{J\nu_H} Ju, \sum_{k=1}^{n}(e_k i_k - e_{\bar k} i_{\bar k})u\Big\rangle = 0. \tag{7.3}$$

We summarize the previous arguments in the following.

Theorem 7.1 (Gaffney–Friedrichs inequality (3rd version))**.** *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain with boundary of class $\mathbf{C}^2$ satisfying condition (H) (see Definition 3.4). Let $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ be a horizontal h-form with $1 \le h < n$, and assume that either*

(i) $u_{\nu_H} = 0$,

(ii) *u satisfies the condition* (7.2),

or

(j) $u_t = 0$,

(jj) *u satisfies the condition* (7.3).

Then, there exist geometric constants $\widetilde{C}_1$ and $\widetilde{C}_2$, only dependent on $\mathcal{U}$ and on the integers h and n, such that

$$D_H(u) \ge \widetilde{C}_1 \int_{\mathcal{U}} \|\nabla_H u\|^2\, dV - \widetilde{C}_2 \int_{\mathcal{U}} \|u\|^2\, dV. \tag{7.4}$$

Furthermore, if $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ is a horizontal h-form with $n+1 \le h \le 2n$, then (7.4) *still holds provided that $*_H u$ satisfies either* (i) *and* (ii), *or* (j) *and* (jj). *Finally, the conclusions of Theorem 6.7 still hold in this case if $h = n$.*

8. Rumin's complex in Heisenberg groups

8.1. Rumin's complex

In this section we briefly sketch the main ideas in Rumin's construction of the intrinsic complex of differential forms in Heisenberg groups; see [38]. For a more general approach we refer the reader, for instance, to [40], [41], and [4].

First, we would like to discuss how Rumin's complex appears naturally in the geometric measure theory of Heisenberg groups. The starting point is the question "what is counterpart of a linear manifold in Heisenberg groups". As shown in [21], [23], this role is played by the homogeneous subgroups of $\mathbb{H}^n$, that is, in exponential coordinates, by the homogeneous subalgebras of $\mathfrak{h}$. It is well known that, in Euclidean spaces, linear submanifolds are the annihilators of homogeneous simple covectors, which are invariant under translations. Thus, is it natural to look for left-invariant homogeneous differential forms whose annihilator is a subalgebra of $\mathfrak{h}$.

By the Frobenius theorem, the annihilator of a left invariant differential form ω is a Lie subalgebra of $\mathfrak{h}$ if and only if $d\omega = 0$. On the other hand, when acting on left-invariant forms, the exterior differential d is nothing but its "algebraic" part, which in the sequel will be denoted as d_0; see Definition 8.2 below.

A natural choice for a class of intrinsic differential forms in $\mathbb{H}^n$ would be to take $\ker d_0$ as the ambient space. Nevertheless, this choice is not totally satisfying, since it fails to take into account a crucial algebraic property of linear manifolds in Euclidean spaces, which resides in the fact that they are complemented. Indeed, also complementary subspaces of a fixed subspace V can be viewed as annihilators of differential forms in the following sense:

If V is the annihilator of a *simple form* ω, then a complementary subspace W is the annihilator of the Hodge-dual form $*\omega$, where the Hodge duality must be taken with respect to an inner product making V and W orthogonal. Thus in order to obtain a satisfying notion of intrinsic h-covector in $\mathfrak{h}$, we have to choose once for all an inner product in $\mathfrak{h}$ and take

$$E_0^h = \ker d_0 \cap \ker(d_0 *).$$

Recall that $\mathfrak{h}$ is endowed with the inner product that makes the basis

$$\{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$$

orthonormal.

The family of vector spaces $(E_0^h)_{0 \le h \le n}$ can be equipped with an "exterior differential operator"

$$d_c : E_0^h \to E_0^{h+1}$$

making (E_0^*, d_c) a complex, which is chain homotopic to the de Rham complex. The definition of d_c is rather technical and will be given by Theorem 8.6 below. Essentially, d_c is defined as

$$d_c := \Pi_{E_0} d \Pi_E,$$

where Π_E is the projection onto a second complex (E^*, d), again homotopic to the de Rham complex, which is meant to take into account the lack of commutativity of $\mathfrak{h}$, and where Π_{E_0} is the orthogonal projection on E_0^* that minimizes the number of compatibility conditions for a differential form to be exact. We stress that d_c is an operator of order 1 in the horizontal derivatives, when acts on E_0^h with $h \neq n$, but of order 2 on E_0^n.

Definition 8.1. If $\alpha \in \bigwedge^1 \mathfrak{h}_1$, $\alpha \neq 0$, then we say that α has *weight* 1, and write $w(\alpha) = 1$. If $\alpha = \vartheta$, then we say that α has *weight* 2, and write $w(\alpha) = 2$. More generally, we say that $\alpha \in \bigwedge^h \mathfrak{h}$ has *pure weight* k when α is a linear combination of covectors $\psi_{i_1} \wedge \cdots \wedge \psi_{i_h}$ such that $w(\psi_{i_1}) + \cdots + w(\psi_{i_h}) = k$.

Note that, if $\alpha, \beta \in \bigwedge^h \mathfrak{h}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$. Moreover, we have (see, e.g., formula (13) in [4]):

$$\bigwedge^h \mathfrak{h} = \bigwedge^{h,h} \mathfrak{h} \oplus \bigwedge^{h,h+1} \mathfrak{h},$$

where $\bigwedge^{h,p} \mathfrak{h}$ denotes the linear span of $\Psi^{h,p} := \{\alpha \in \Psi^h : w(\alpha) = p\}$.

The ordinary exterior differential operator d splits into the sum of its weighted components. More precisely, we give the following definition.

Definition 8.2. Let $\alpha = \sum_{\psi^I \in \Psi^{h,p}} \alpha_i \psi^I$ be a smooth (simple) h-form of pure weight p. Then we shall write

$$d\alpha = d_0\,\alpha + d_1\,\alpha + d_2\,\alpha,$$

where $d_0\alpha$ has pure weight p, $d_1\alpha$ has pure weight $p+1$, and $d_2\alpha$ has pure weight $p+2$.

When acting on left-invariant forms, one has $d = d_0$, since d preserves the weight. Notice also that $d_1 = d_H$.

Using Cartan's identity (see, for example, [30], formula (9) p. 21) and the left-invariance of the forms $\psi^I \in \Psi^{h,p}$, it follows that

$$d_0\alpha = \sum_{\psi^I \in \Psi^{h,p}} \alpha_i\, d\psi^I.$$

Analogously, we have

$$d_1\alpha = \sum_{\psi^I \in \Psi^{h,p}} W_j(\alpha_i)\psi_j \wedge \psi^I, \quad d_2\alpha = \sum_{\psi^I \in \Psi^{h,p}} T(\alpha_i)\vartheta \wedge \psi^I.$$

We stress that d_0 is an *algebraic operator*, and therefore can be identified with an operator *acting on covectors*.

The following important notion due to Rumin can be found in [39], [40].

Definition 8.3. For any $0 \le h \le 2n+1$ we set $E_0^h := \operatorname{Ker} d_0 \cap \mathcal{R}(d_0)^\perp$, where $\mathcal{R}(d_0)$ denotes the range of d_0. The elements of E_0^h are called *intrinsic h-forms on $\mathbb{H}^n$.*

It is not difficult to see that $*E_0^h = E_0^{2n+1-h}$. Observe that, since this notion is invariant under left translations, the space E_0^h can be seen as the space of sections of a fiber subbundle of $\bigwedge^h \mathfrak{h}$, generated by left translation and still denoted as E_0^h. Since d_0 is algebraic, there is no ambiguity if we denote by E_0^* both the space of covectors and the spaces of the sections of the associated linear bundle. We also note that E_0^h inherits from $\bigwedge^h \mathfrak{h}$ the inner product $\langle \cdot, \cdot \rangle$ on the fibers.

Theorem 8.4 (See [39])**.** *With the notation of Definition 2.7, we have:*

- $E_0^1 = \bigwedge^1 \mathfrak{h}_1$.
- *If* $2 \le h \le n$, *then* $E_0^h = \bigwedge^h \mathfrak{h}_1 \cap \ker \Lambda$.
- *If* $n < h \le 2n+1$, *then* $E_0^h = \vartheta \wedge \ker L$.

We remark that an h-form in E_0^h has either weight h, if $1 \le h \le n$, or weight $h+1$, if $n < h \le 2n+1$. Let $\Xi_0^h = \{\xi_i^h : 1 \le i \le N_h\}$ be an orthonormal basis of E_0^h, where $N_h := \dim E_0^h$. Notice that we can always assume that $\xi_i^1 = \psi_i$ for any $i = 1, \dots, 2n$.

We have to define an "inverse" of the algebraic operator d_0 and this can be done as follows (see, e.g., Lemma 2.11 in [4]).

Lemma 8.5. *For any* $\beta \in \bigwedge^{h+1} \mathfrak{h}$ *there exists a unique* $\alpha \in \bigwedge^{h} \mathfrak{h} \cap (\ker d_0)^{\perp}$ *such that* $d_0\alpha - \beta \in (\mathcal{R}(d_0))^{\perp}$*. In the sequel, with a slight abuse of notation, we shall set* $d_0^{-1}\beta := \alpha$*.*

By construction, the operator d_0^{-1} is weight-preserving.

In the next theorem we summarize the main features of the intrinsic exterior differential d_c. For more details, we refer the reader to [39]; see also [40] and [4].

Theorem 8.6. *The de Rham complex* (Ω^*, d) *splits into the direct sum of two sub-complexes* (E^*, d) *and* (F^*, d)*, where we have set*

$$E := \ker d_0^{-1} \cap \ker \left(d_0^{-1}d\right), \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(dd_0^{-1}).$$

Furthermore, the following assertions hold:

(i) *Let* Π_E *be the (non-orthogonal) projection on* E *along* F*. For any* $\alpha \in E_0^h$ *one has either* $\Pi_E\alpha = \alpha - d_0^{-1}d_1\alpha$*, if* $1 \le h \le n$*, or* $\Pi_E\alpha = \alpha$*, if* $h > n$*.*

(ii) Π_E *is a chain map, i.e.,* $d\Pi_E = \Pi_E d$*.*

(iii) *Let* Π_{E_0} *denote the orthogonal projection from* $\bigwedge^* \mathfrak{h}$ *onto* E_0^**. Then, we have* $\Pi_{E_0} = \mathrm{Id} - d_0^{-1}d_0 - d_0d_0^{-1}$ *and* $\Pi_{E_0^{\perp}} = d_0^{-1}d_0 - d_0d_0^{-1}$*.*

(iv) *We have* $\Pi_{E_0} = \Pi_{E_0}\Pi_E\Pi_{E_0}$ *and* $\Pi_E = \Pi_E\Pi_{E_0}\Pi_E$*.*

Let $0 \le h \le 2n$ *and set* $d_c := \Pi_{E_0} d\Pi_E : E_0^h \to E_0^{h+1}$*. Then, we have:*

(v) $d_c^2 = 0$*.*

(vi) *The differential complex* (E_0^*, d_c) *is exact.*

(vii) *If* $h \neq n$*, then* $d_c \colon E_0^h \to E_0^{h+1}$ *is a homogeneous differential operator in the horizontal derivatives of order* 1*. Moreover,* $d_c \colon E_0^n \to E_0^{n+1}$ *is a homogeneous differential operator of order* 2*.*

Notice that for any smooth function $f \in E_0^0$ we have

$$d_c f = (\nabla_H f)^{\#} = \sum_{1=1}^{n} (X_i f dx_i + Y_i f dy_i)\,.$$

We can also define a codifferential δ_c, by taking the formal adjoint of d_c in $L^2(\mathbb{H}^n, E_0^*)$. More precisely, we set $\delta_c := d_c^*$.

Proposition 8.7. *On* E_0^h *we have* $\delta_c = (-1)^h * d_c *$*.*

For a proof, see, e.g., [24], Proposition 3.15.

Explicit calculations and further examples concerning Rumin's complex in Heisenberg groups can be found in [5].

Definition 8.8 (Sub-Laplacians on forms; see [38])**.** We define the operator $\Delta_{c,h}$ on E_0^h by setting

$$\Delta_{c,h} := \begin{cases} d_c\delta_c + \delta_c d_c & \text{if } h \neq n,\, n+1, \\ (d_c\delta_c)^2 + \delta_c d_c & \text{if } h = n, \\ d_c\delta_c + (\delta_c d_c)^2 & \text{if } h = n+1. \end{cases}$$

Notice that $\Delta_{c,0} = \Delta_K$ is the usual sub-Laplacian on $\mathbb{H}^n$.

Proposition 8.9 (see Proposition 4 in [38]). *Let* $1 \le h < n$, *and* $\alpha \in E_0^h$. *Then*

(i) $\delta_c \alpha = \delta_H \alpha$;

(ii) $d_c \alpha = d_H u - \dfrac{1}{n-h+1} L\Lambda(d_H \alpha)$.

The next lemma follows from the Kähler identities in Proposition 4.5.

Lemma 8.10. *For any* $u \in E_0^h$, *with* $0 \le h \le n$, *we have* $\Lambda(d_H u) = -\delta_H^J u$. *Furthermore* $\Lambda^2(d_H u) = 0$.

Proof. Keeping in mind that $\Lambda u = 0$, and using (i) of Proposition 4.5, yields

$$\Lambda(d_H u) = d_H \Lambda u - \delta_H^J u = -\delta_H^J u.$$

Moreover, by applying (iii) of Proposition 4.5 we obtain

$$\Lambda^2(d_H u) = \Lambda \delta_H^J u = \delta_H^J \Lambda u = 0.$$

□

Lemma 8.11. *Let* $u \in E_0^h$, *with* $0 \le h < n$. *Then*

$$d_c u = d_H u + \frac{1}{n-h+1} L\delta_H^J u. \tag{8.1}$$

Moreover, the following identity holds:

$$\|d_c u\|^2 + \frac{1}{n-h+1}\|\delta_H^J u\|^2 = \|d_H u\|^2.$$

Proof. By Proposition 8.9 and Lemma 8.10 we get

$$d_c u = d_H u - \frac{1}{n-h+1} L\Lambda(d_H u) = d_H u + \frac{1}{n-h+1} L\delta_H^J u.$$

In order to prove the second assertion, we note that by definition $d_c u$ is orthogonal to the range of L. Now since $L\delta_H^J u = -L\Lambda(d_H u)$ we get

$$\begin{aligned}
\|L\delta_H^J u\|^2 &= \|L\Lambda(d_H u)\|^2 = \langle L\Lambda(d_H u), L\Lambda(d_H u)\rangle = \langle \Lambda(d_H u), \Lambda L\Lambda(d_H u)\rangle \\
&= \left\langle \Lambda(d_H u), L\Lambda^2(d_H u)\right\rangle + (n-h+1)\langle \Lambda(d_H u), \Lambda(d_H u)\rangle \quad \text{(by Lemma 2.8)} \\
&= (n-h+1)\langle \Lambda(d_H u), \Lambda(d_H u)\rangle \quad \text{(by Lemma 8.10)} \\
&= (n-h+1)\|\delta_H^J u\|^2,
\end{aligned}$$

and the thesis follows. □

8.2. Decomposition of forms on the boundary of a domain II

This section is the counterpart of Section 2.2 and, roughly speaking, the idea here is to replace horizontal forms with intrinsic forms in E_0^*.

Recall that $\mathcal{U} \subsetneq \mathbb{H}^n$ is a domain with boundary of class $\mathbf{C}^2$.

With the notation of Section 2.2, if $\alpha \in E_0^h$, if $n < h \le 2n+1$, we have

$$\alpha = \vartheta \wedge \alpha_H \quad \text{with} \quad \alpha_H \in \Omega_H^{h-1}.$$

Now, writing $\alpha_H = (\alpha_H)_{\rm t} + \nu_H \wedge (\alpha_H)_{\nu_H}$, where we have set

$$(\alpha_H)_{\rm t} := \nu_H \,\lrcorner\, (\nu_H \wedge \alpha_H) \quad \text{and} \quad (\alpha_H)_{\nu_H} := \nu_H \,\lrcorner\, \alpha_H,$$

we obtain the decomposition formula

$$\alpha = \vartheta \wedge (\alpha_H)_{\mathrm{t}} + \vartheta \wedge \nu_H \wedge (\alpha_H)_{\nu_H}.$$

Thus if $\alpha \in E_0^h$, with $n < h \leq 2n+1$, we can set

$$\alpha_{\mathrm{t}} := \vartheta \wedge (\alpha_H)_{\mathrm{t}} \quad \text{and} \quad \alpha_{\nu_H} := -\vartheta \wedge (\alpha_H)_{\nu_H},$$

and again we obtain the identity

$$\alpha = \alpha_{\mathrm{t}} + \nu_H \wedge \alpha_{\nu_H}.$$

Clearly, it turns out that $\alpha_{\mathrm{t}} \perp \nu_H \wedge \alpha_{\nu_H}$.

The above definition is motivated by the following lemma.

Lemma 8.12. *If $\alpha \in E_0^h$, with $n < h \leq 2n+1$, then*

$$*\alpha_{\mathrm{t}} = \nu_H \wedge (*\alpha)_{\nu_H} \quad \textit{and} \quad *(\nu_H \wedge \alpha_{\nu_H}) = (*\alpha)_{\mathrm{t}}.$$

Proof. By Lemma 2.2 we have

$$*\alpha_{\mathrm{t}} = *_H(\alpha_H)_{\mathrm{t}} = \nu_H \wedge (*_H\alpha_H)_{\nu_H} = \nu_H \wedge (*\alpha)_{\nu_H}.$$

On the other hand,

$$\begin{aligned}(\nu_H \wedge \alpha_{\nu_H}) &= -*(\nu_H \wedge \vartheta \wedge (\alpha_H)_{\nu_H}) = *(\vartheta \wedge \nu_H \wedge (\alpha_H)_{\nu_H}) \\ &= *_H(\nu_H \wedge (\alpha_H)_{\nu_H}) = (*_H\alpha_H)_{\mathrm{t}} = (*\alpha)_{\mathrm{t}}.\end{aligned}$$

□

In particular, if $\alpha \in \Omega_H^h$, $1 \leq h \leq 2n$, we can always write

$$\alpha_{\mathrm{t}} := \nu_H \,\lrcorner\, (\nu_H \wedge \alpha), \quad \alpha_{\nu_H} := \nu_H \,\lrcorner\, \alpha,$$

and, as above, we have the decomposition formula

$$\alpha = \alpha_{\mathrm{t}} + \nu_H \wedge \alpha_{\nu_H} \quad \forall\, \alpha \in E_0^h.$$

Remark 8.13. We stress that combining (2.1) and Lemma 8.12, we obtain a very useful result: If $1 \leq h \leq 2n+1$, and $\alpha \in E_0^h$, then

$$\alpha_{\mathrm{t}} = 0 \text{ if and only if } (*\alpha)_{\nu_H} = 0 \quad \text{and} \quad \alpha_{\nu_H} = 0 \text{ if and only if } (*\alpha)_{\mathrm{t}} = 0.$$

Definition 8.14. From now on, we denote by $E_0^*(\overline{\mathcal{U}})$ the space of smooth sections of E_0^* over $\overline{\mathcal{U}}$. With a slight abuse of notation, we also denote by $E_0^*(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ the corresponding space of complex forms $\Gamma(\overline{\mathcal{U}}, E_0^* \otimes_{\mathbb{R}} \mathbb{C})$.

We conclude this section by formulating a Green-type identity for the Rumin differential d_c (compare with formula (2.2)).

Theorem 8.15 (Green identity in (E_0^*, d_c))**.** *Suppose that $\mathcal{U} \subsetneq \mathbb{H}^n$ is a domain with boundary of class $\mathbf{C}^2$. If $\alpha \in E_0^{h-1}(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, and $\beta \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $h \neq n$, $n+1$, then*

$$\begin{aligned}\langle d_c\alpha, \beta\rangle_{L^2(\mathcal{U})} &= \langle \alpha, \delta_c\beta\rangle_{L^2(\mathcal{U})} + \int_{\partial\mathcal{U}} \langle \mathrm{n} \wedge \alpha, \beta\rangle \; d\sigma \\ &= \langle \alpha, \delta_c\beta\rangle_{L^2(\mathcal{U})} + \int_{\partial\mathcal{U}} \langle \nu_H \wedge \alpha, \beta\rangle \; d\sigma_H.\end{aligned}$$

8.3. Gaffney–Friedrichs-type inequalities: Technical preliminaries

Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain with smooth boundary of class $\mathbf{C}^2$. Below, we generalize to E_0^* a classical definition which can be found in [36]; see, e.g., Definition 7.2.6 in p. 291 (also compare with Definition 5.1).

Definition 8.16. Let $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ be an intrinsic differential h-form, with $0 \le h \le 2n+1$. We define the *CC-Dirichlet integral* by setting

$$D_c(u) := \int_{\mathcal{U}} \left(\langle d_c u, d_c u\rangle + \langle \delta_c u, \delta_c u\rangle\right) dV.$$

It is clear from the definition that this quantity is a non-negative real number. Moreover, we remind the reader that $D_c(u) = D_c(*u)$.

Finally, it is worth observing that our main results for the complex (E_0^*, d_c) (see, more precisely, Theorems 8.21, 8.23, and 8.24) only concern the case $h \neq n, n+1$.

Proposition 8.17. *Let $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $1 \le h < n$. Then*

$$\begin{aligned} D_c(u) &= D_H(u) - \frac{1}{n-h+1}\int_{\mathcal{U}} \langle \delta_H^J u, \delta_H^J u\rangle\, dV \\ &\ge D_H(u) - \frac{1}{n-h+1} D_H(Ju) = D_H^J(u). \end{aligned}$$

Proof. By applying together Definition 8.16, identity (8.1) in Lemma 8.11, and Proposition 8.9, we get

$$\begin{aligned} \langle d_c u, d_c u\rangle + \langle \delta_c u, \delta_c u\rangle &= \langle d_H u, d_H u\rangle - \tfrac{1}{n-h+1}\langle \delta_H^J u, \delta_H^J u\rangle + \langle \delta_H u, \delta_H u\rangle \\ &= \langle d_H u, d_H u\rangle + \langle \delta_H u, \delta_H u\rangle - \tfrac{1}{n-h+1}\left\langle \delta_H^J u, \delta_H^J u\right\rangle \\ &= \langle d_H u, d_H u\rangle + \langle \delta_H u, \delta_H u\rangle - \tfrac{1}{n-h+1}\left\langle J^{-1}\delta_H J u, J^{-1}\delta_H J u\right\rangle \\ &= \langle d_H u, d_H u\rangle + \langle \delta_H u, \delta_H u\rangle - \tfrac{1}{n-h+1}\left\langle \delta_H J u, \delta_H J u\right\rangle, \end{aligned}$$

where we have used that $J^2 = -\mathrm{Id}$. Now since

$$\int_{\mathcal{U}} \langle \delta_H J u, \delta_H J u\rangle\, dV \le D_H(Ju),$$

the proof follows. $\square$

Lemma 8.18. *Let $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ be an intrinsic h-form, with $1 \le h \le n$. Then, at every point of $\partial\mathcal{U} \setminus \mathrm{char}(\partial\mathcal{U})$, the following implications hold:*

(i) $u_{\mathrm{t}} = 0 \Rightarrow (Ju)_{\nu_H} = 0$;

(ii) $(Ju)_{\mathrm{t}} = 0 \Rightarrow u_{\nu_H} = 0$.

Proof. We just prove (i), since the proof of (ii) is similar. Let $g\colon \mathbb{H}^n \to \mathbb{R}$ be a *defining function* for $\mathcal{U}$ of class $\mathbf{C}^2$. We are assuming that:

- $\mathcal{U} = \{x \in \mathbb{H}^n : g(x) < 0\}$;
- $g(x) = 0$ if and only if $x \in \partial\mathcal{U}$;
- $\nabla g \neq 0$ for all $x \in \partial\mathcal{U}$;

see, e.g., Chap. 2 in [32]. Now observe that $d_H g$ is parallel to ν_H, and that the hypothesis $u_{\rm t} = 0$ is equivalent to $d_H(gu) = 0$ on $\partial\mathcal{U}$. Indeed, if $u_{\rm t} = 0$, then $u = \nu_H \wedge (\nu_H \,\lrcorner\, u)$. On the other hand,

$$d_H(gu) = d_H g \wedge u = d_H g \wedge \nu_H \wedge (\nu_H \,\lrcorner\, u) = 0.$$

Moreover, if $d_H(gu) = 0$, then $d_H g \wedge u = 0$, and so $\nu_H \wedge u = 0$. This implies $u_{\rm t} = \nu_H \,\lrcorner (\nu_H \wedge u) = 0$.

On the other hand, by Lemma 8.10, if $u_{\rm t} = 0$, then

$$0 = -\Lambda d_H(gu) = \delta_H^J(gu) = J^{-1}\delta_H J(gu),$$

which implies

$$\delta_H(gJu) = \delta_H J(gu) = 0.$$

From this we get $\delta_H(gJu) = 0$ on $\partial\mathcal{U}$, and since $\delta_H(gJu) = -(d_H g \,\lrcorner\, Ju)$, the proof of (i) follows. □

By applying Proposition 5.11 to Ju, and by keeping into account that the first two integrals in (5.10) remain unchanged if we replace u with Ju, we find the following identity.

Proposition 8.19. *Let $u \in \Omega_H^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, and assume that either $(Ju)_{\rm t} = 0$ or $(Ju)_{\nu_H} = 0$ on $\partial\mathcal{U} \setminus \mathrm{char}(\partial\mathcal{U})$. Then*

$$D_H(Ju) = \int_{\mathcal{U}} \|\nabla_H u\|^2\, dV - i\sum_{I,J}(p_I - q_J)\int_{\mathcal{U}} \bar{u}_{I,J} T u_{I,J}\, dV - \int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \langle \widetilde{R}_J(u), u\rangle\, d\sigma_H,$$

where we have set $\widetilde{R}_J(u) := J^{-1}\widetilde{R}(Ju)$.

Combining now Propositions 8.17, 5.11, 8.19 together with Lemma 8.18, and formula (5.3) in Corollary 5.5, we obtain the next proposition.

Proposition 8.20. *Let $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $1 \le h < n$, and suppose that either $u_{\rm t} = 0$ or $(Ju)_{\rm t} = 0$ on $\partial\mathcal{U} \setminus \mathrm{char}(\partial\mathcal{U})$. Then*

$$\begin{aligned}
D_H^J(u) &= D_H(u) - \frac{1}{n-h+1} D_H(Ju)\\
&= \frac{n-h}{n-h+1}\int_{\mathcal{U}} \Big(\|\nabla_H u\|^2 - i\sum_{I,J}(p_I - q_J)\bar{u}_{I,J} T u_{I,J}\Big)\, dV - \int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \langle R(u), u\rangle\, d\sigma_H\\
&= \frac{n-h}{n-h+1} D_H(u) - \frac{n-h}{n-h+1}\Re\mathrm{e}\,\mathbf{A} - \int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \langle R(u), u\rangle\, d\sigma_H\\
&= \frac{n-h}{n-h+1} D_H(u) - \frac{1}{n-h+1}\Re\mathrm{e}\int_{\partial\mathcal{U}\setminus\mathrm{char}(\partial\mathcal{U})} \langle (\widetilde{R} - \widetilde{R}_J)u, u\rangle\, d\sigma_H,
\end{aligned}$$

where $R(u) := \widetilde{R}(u) - \frac{1}{n-h+1}\widetilde{R}_J(u)$.

8.4. Gaffney–Friedrichs inequalities for Rumin's complex: Main results

At this point, by using the estimates of the "error terms" proved in the preceding sections, Theorem 6.1 can be stated in (E_0^*, d_c) as follows.

Theorem 8.21 (Gaffney–Friedrichs inequality in (E_0^*) (1st version))**.** *Let $\mathcal{U} \subsetneq \mathbb{H}^n$ be a domain with boundary of class $\mathbf{C}^2$. Let $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $1 \leq h < n$, and assume that:*

- (i) *u satisfies either $u_t = 0$ or $Ju_t = 0$;*
- (ii) *u satisfies either condition $(J\nu_H)$ (see Proposition 5.16) or condition $(\widetilde{J\nu_H})$ (see Proposition 5.20).*

Let $\{\mathcal{V}_\epsilon\}_{\epsilon>0}$ be a family of open neighborhoods of $\mathrm{char}(\partial\mathcal{U})$ *(in the relative topology) shrinking around* $\mathrm{char}(\partial\mathcal{U})$ *when $\epsilon \to 0$. In addition, assume that $\sigma(\mathcal{V}_\epsilon) \to 0$ as $\epsilon \to 0$. Then, there exist geometric constants C_0, C_1, and C_2 such that*

$$D_c(u) + C_0 \int_{\partial\mathcal{U}\cap\mathcal{V}_\epsilon} \|u\|^2 \, d\sigma \geq C_1 \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - C_2 \int_{\mathcal{U}} \|u\|^2 \, dV. \tag{8.2}$$

The constants C_0, C_1, C_2 depend only on $\mathcal{U}$, ϵ, and on the integers h and n. Finally, if $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ with $n+1 < h \leq 2n$, then (8.2) still holds, provided that $*u$ satisfies (i) *and* (ii).

Remark 8.22. Just as in Remark 6.2, the constant C_2 may blow up as ϵ tends to 0^+.

We conclude by stating two alternative versions of the main inequality, for the Rumin's complex.

Theorem 8.23 (Gaffney–Friedrichs inequality in (E_0^*, d_c) (2nd version))**.** *Suppose that $\mathcal{U} \subsetneq \mathbb{H}^n$ is a domain with boundary of class $\mathbf{C}^2$ satisfying condition (H) (see Definition 3.4). Let $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$, with $1 \leq h < n$, and assume that:*

- (i) *either $u_t = 0$ or $Ju_t = 0$;*
- (ii) *u satisfies condition $(J\nu_H^*)$ (see Remark 5.18).*

Then, there exist geometric constants $\widetilde{C}_1$ and $\widetilde{C}_2$, depending only on $\mathcal{U}$, and on the integers h and n, such that

$$D_c(u) \geq \widetilde{C}_1 \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \widetilde{C}_2 \int_{\mathcal{U}} \|u\|^2 \, dV. \tag{8.3}$$

Furthermore, if $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ with $n+1 < h \leq 2n$, then (8.3) still holds provided that $*u$ satisfies (i) *and* (ii).

Theorem 8.24 (Gaffney–Friedrichs inequality in (E_0^*, d_c) (3rd version))**.** *Suppose that $\mathcal{U} \subsetneq \mathbb{H}^n$ is a domain with boundary of class $\mathbf{C}^2$ satisfying condition (H) (see Definition 3.4). Let $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ with $1 \leq h < n$, and assume that either*

- (i) *$Ju_t = 0$,*
- (ii) *u satisfies the condition (7.2),*

or

(j) $u_t = 0$,

(jj) *u satisfies the condition* (7.3).

Then, there exist geometric constants $\widetilde{C}_1$ *and* $\widetilde{C}_2$*, depending only on* $\mathcal{U}$*, and on the integers h and n, such that*

$$D_c(u) \geq \widetilde{C}_1 \int_{\mathcal{U}} \|\nabla_H u\|^2 \, dV - \widetilde{C}_2 \int_{\mathcal{U}} \|u\|^2 \, dV. \tag{8.4}$$

Furthermore, if $u \in E_0^h(\overline{\mathcal{U}}) \otimes_{\mathbb{R}} \mathbb{C}$ *with* $n+1 < h \leq 2n$*, then* (8.4) *still holds provided that* $*u$ *satisfies either* (i) *and* (ii)*, or* (j) *and* (jj)*.*

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