ON 2-STEP, CORANK 2, NILPOTENT SUB-RIEMANNIAN METRICS*

DAVIDE BARILARI†, UGO BOSCAIN‡, AND JEAN-PAUL GAUTHIER§

Abstract. In this paper we study the nilpotent 2-step, corank 2 sub-Riemannian metrics that are nilpotent approximations of general sub-Riemannian metrics. We exhibit optimal syntheses for these problems. It turns out that in general the cut time is not equal to the first conjugate time but has a simple explicit expression. As a by-product of this study we get some smoothness properties of the spherical Hausdorff measure in the case of a generic 6-dimensional, 2-step, corank 2 sub-Riemannian metric.

Key words. sub-Riemannian geometry, optimal synthesis, spherical Hausdorff measure

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1. Introduction.

1.1. Sub-Riemannian manifolds. In this paper, by a sub-Riemannian manifold we mean a triple $(M, \Delta, \mathbf{g})$, where M is a connected orientable smooth manifold of dimension n, Δ is a smooth vector distribution of constant rank $m < n$, satisfying the Hörmander condition, and $\mathbf{g}$ is an Euclidean structure over Δ.

A sub-Riemannian manifold has the natural structure of a metric space, where the distance is the so-called Carnot–Caratheodory distance

$$d(q_0, q_1) = \inf \left\{ \int_0^T \sqrt{\mathbf{g}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \mid \gamma : [0, T] \to M \text{ is a Lipschitz curve,} \right.$$
$$\left. \gamma(0) = q_0, \gamma(T) = q_1, \ \ \dot{\gamma}(t) \in \Delta_{\gamma(t)} \text{ a.e. in } [0, T] \right\}.$$

As a consequence of the Hörmander condition d is actually a distance inducing the topology of the manifold. This is the Rashevsky–Chow theorem; see, for instance, [3] or [18].

Throughout this paper we assume that the structure is 2-step bracket generating, i.e.,

$$T_q M = \Delta_q + [\Delta, \Delta]_q \quad \text{for every } q \in M,$$

and we quote a 2-step sub-Riemannian metric by its rank and its dimension, i.e., with the pair (m, n). The quantity $k = n - m$ is called the *corank* of the structure.

It is known from Mitchell [20] that the Hausdorff dimension of M, as a metric space, is $\mathfrak{h} = 2n - m > n$. In this paper we focus on the case $(m, m + 2)$.

A sub-Riemannian manifold is left-invariant if $M = G$, a Lie group, and both Δ and $\mathbf{g}$ are left-invariant over G.

Locally, the pair $(\Delta, \mathbf{g})$ can be specified by the data of a set of m smooth vector fields spanning Δ, being an orthonormal frame for $\mathbf{g}$, i.e.,

$$(1) \qquad \Delta_q = \operatorname{span}\{X_1(q), \ldots, X_m(q)\}, \qquad \mathbf{g}_q(X_i(q), X_j(q)) = \delta_{ij}.$$

In this case, the set $\{X_1, \ldots, X_m\}$ is called a *local orthonormal frame* for the sub-Riemannian metric.

The sub-Riemannian metric can also be expressed locally in "control form" as follows. We consider the control system

$$(2) \qquad \dot{q} = \sum_{i=1}^{m} u_i X_i(q), \quad u_i \in \mathbb{R},$$

and the problem of finding the shortest curve that joins two fixed points q_0, $q_1 \in M$ is naturally formulated as the optimal control problem,

$$(3) \qquad \int_0^T \sqrt{\sum_{i=1}^{m} u_i^2(t)} \, dt \to \min, \qquad q(0) = q_0, \quad q(T) = q_1.$$

A *geodesic* for the sub-Riemannian metric is a curve, parametrized by constant velocity, such that every short enough piece of it is a local minimizer of the length. For the sub-Riemannian metrics given in control form all the geodesics can be computed with Pontryagin's Maximum Principle (PMP) [22]. In the 2-step bracket generating case, it is known that there is no strict abnormal minimizer, and all geodesics are projections on M of the trajectories of the Hamiltonian system associated with the Hamiltonian over T^*M

$$(4) \qquad H(\lambda, q) = \frac{1}{2} \sum_{i=1}^{m} \langle \lambda, X_i(q) \rangle^2, \qquad \lambda \in T_q^*M,$$

and corresponding to the level set $\{H = c\}$ for $c > 0$.

1.2. Nilpotent approximation. Consider a sub-Riemannian manifold $(M, \Delta, \mathbf{g})$ and fix a point $q \in M$. The Lie bracket induces a skew symmetric tensor bilinear mapping

$$(5) \qquad [\cdot, \cdot]_q : \Delta_q \times \Delta_q \to T_q M / \Delta_q.$$

Then, for every $Z^* \in (T_q M / \Delta_q)^*$, we have

$$Z^*([X, Y] + \Delta_q) = \langle A_{Z^*}(X), Y \rangle_{\mathbf{g}}$$

for some $\mathbf{g}$-skew symmetric endomorphism A_{Z^*} of Δ_q.

Remark 1 (notation). We denote by $\mathcal{L}_q$ the k-dimensional space of skew symmetric endomorphisms of Δ_q obtained by taking the union of all the A_{Z^*} at q. This notation is used in the appendix.

The space $L_q = \Delta_q \oplus T_q M / \Delta_q$ is endowed with the structure of a 2-step nilpotent Lie algebra by setting

$$[(V_1, W_1), (V_2, W_2)] = (0, [V_1, V_2] + \Delta_q).$$

The associated simply connected nilpotent Lie group is denoted by G_q and the exponential mapping Exp: $L_q \to G_q$ is one to one and onto. By translation, the metric $\mathbf{g}_q$ over Δ_q allows us to define a left-invariant sub-Riemannian metric over G_q.

DEFINITION 2. *The sub-Riemannian metric on G_q defined above is called the nilpotent approximation of $(M, \Delta, \mathbf{g})$ at q.*

Any k-dimensional vector subspace $\mathcal{V}_q$ of T_qM, transversal to Δ_q, allows us to identify L_q and G_q to $T_qM \simeq \Delta_q \oplus T_qM/\Delta_q$.

Fix $q_0 \in M$. We can chose coordinates x in Δ_{q_0} such that the metric $\mathbf{g}_{q_0}$ is the standard Euclidean metric, and for any linear coordinate system y in $\mathcal{V}_{q_0}$, there are skew symmetric matrices $L_1, \ldots, L_k \in so(m)$ such that the mapping (5) writes

$$[X, Y] + \Delta_{q_0} = \begin{pmatrix} X'L_1Y \\ \vdots \\ X'L_kY \end{pmatrix},$$

where X' denotes the transpose of the vector X. Then the nilpotent approximation written in control form is

(6)
$$\begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, m, \\ \dot{y}_1 = \dfrac{1}{2}x'L_1u, \\ \phantom{\dot{y}_1} \vdots \\ \dot{y}_k = \dfrac{1}{2}x'L_ku. \end{cases}$$

The construction of the nilpotent approximation given in Definition 2 makes sense for any sub-Riemannian metric, but it coincides with the standard one (see [6, 9]) in the 2-step bracket generating case only.

PROPOSITION 3. *The distribution is 2-step bracket generating if and only if the endomorphisms of Δ_q, L_i, $i = 1, \ldots, k$ (respectively, the matrices L_i when coordinates y in $\mathcal{V}_q$ are chosen), are independent.*

In the 2-step bracket generating case these linear coordinates y in T_qM/Δ_q may be chosen in such a way that the endomorphisms L_i, $i = 1, \ldots, k$, are orthonormal with respect to the Hilbert–Schmidt norm $\langle L_i, L_j \rangle = \frac{1}{m}\text{trace}(L_i'L_j)$. This choice defines a canonical Euclidean structure in T_qM/Δ_q and a corresponding volume in T_qM/Δ_q. Then using the Euclidean structure over Δ_q we get a canonical Euclidean structure over $\Delta_q \oplus T_qM/\Delta_q$. The choice of the vector subspace $\mathcal{V}_q$ induces an Euclidean structure on T_qM which depends on the choice of $\mathcal{V}_q$, but the associated volume on T_qM is independent on this choice.

DEFINITION 4. *This volume form on M is called the Popp measure.*

The Popp measure is a smooth volume form.

1.3. Statement of the results.

1.3.1. History. The main purpose of this paper is to build the optimal synthesis for $(m, m+2)$ nilpotent sub-Riemannian metrics, i.e., the set of all trajectories starting from the identity of the group and realizing the minimum of the distance, with a precise description of their cut time.

Optimal syntheses are in general very difficult to obtain. Usually the steps are the following:

- Apply first order necessary conditions for optimality (which in the case of sub-Riemannian manifolds are given by the PMP) to reduce the set of candidate optimal trajectories. This first step can be very difficult since one should find solutions of a Hamiltonian system, which is not integrable in general.

- Use higher order necessary conditions to reduce further the set of optimal trajectories. This step usually leads to the computation of the conjugate locus, i.e., the set of points up to which geodesics are locally optimal.
- Prove that no strict abnormal extremal is optimal (for instance, by using conditions such as the so-called Goh condition [6, 3]). If one fails to go beyond this step, then one can hardly get an optimal synthesis, since no general technique exists to treat abnormal minimizers.
- Among all solutions of the first order necessary conditions, find the optimal ones. One has to prove that for each point of a candidate optimal trajectory, there is no other trajectory among the selected ones reaching that point. The first point after which a first order trajectory loses global optimality is called a *cut point*. The union of all cut points is the *cut locus*.

As a consequence of these difficulties, optimal syntheses in sub-Riemannian geometry have been obtained in few cases.

The most studied cases are those of left-invariant sub-Riemannian metrics. (See, for instance, [7] for a classification in the 3-dimensional (3D) case.) The first optimal synthesis was obtained for the Heisenberg group in [16, 17]. Then complete optimal syntheses were obtained for the 3D simple Lie groups $SU(2)$, $SO(3)$, $SL(2)$ with the metric induced by the Killing form in [11, 12]. Impressive work was done by Yuri Sachkov, who obtained the optimal synthesis for the group of motions of the plane $SE(2)$ (see [21, 25]).

In dimension larger than 3, only nilpotent groups have been attacked. The complete optimal synthesis was obtained in [5] in the contact nilpotent case. Some results were obtained by Sachkov for the Engel and Cartan groups [23, 24].

When a Lie group structure is not available there are also some results: the optimal synthesis was obtained for a neighborhood of the starting point in the 3D contact case in [4, 2, 15] and in the 4-dimensional (4D) quasi-contact case in [14]. The optimal synthesis was obtained in the important Martinet nilpotent case, where abnormal minimizers can be optimal (see [8]). They also solved the problem for certain perturbations of this case where strictly abnormal minimizers occur (see [10]).

To our knowledge, no other case has been solved.

It is interesting to notice that when the sub-Riemannian metric is invariant by certain continuous transformation (e.g., rotations), then most of the cut points are automatically conjugate. This happens, for instance, on the Heisenberg group, on $SU(2)$, and in the contact nilpotent case.

Remark 5 (notation). In the case of our nilpotent approximations, covectors in $T_q^* M$ can be identified with vectors in $T_q M$ via the Euclidean structure of $T_q M$ given by the choice of $\mathcal{V}_q$. In our coordinates (x, y), these covectors/vectors are typically denoted by (u_0, r).

For nilpotent $(m, m + 1)$ sub-Riemannian metrics that are nilpotent approximations of general sub-Riemannian metrics, the control systems can be written as

(C1)
$$\begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, m, \\ \dot{y} = \dfrac{1}{2} x' L u, & L \text{ skew symmetric}, \end{cases}$$

Denote by $\sigma(L)$ the set of all moduli of eigenvalues of the matrix L. The following fact is proved in [5].

THEOREM 6 (see [5]). *Arclength geodesics of system* (C1)*, starting from the origin, are parametrized by an initial covector* $\lambda_0 = (u_0, r) \in S^{m-1} \times \mathbb{R}$*, and they are*

optimal until time

$$t_{cut}(\lambda_0) = \frac{2\pi}{|r| \max \sigma(L)}$$

with the understanding $t_{cut}(\lambda_0) = +\infty$ *if* $r = 0$. *Moreover,* $t_{cut}(\lambda_0) = t_{conj}(\lambda_0)$.

The proof of this result is based on the fact that geodesics can be expressed in terms of usual trigonometric functions and, thanks to a certain monotonicity property, the cut locus can be explicitly computed and is exactly equal to the conjugate locus.

1.3.2. Optimal synthesis for the nilpotent $(m, m + 2)$ case. The main result of this paper is the optimal synthesis in the case of a nilpotent approximation in the $(m, m + 2)$ case. In this case the control system can be written in coordinates $q = (x_1, \ldots, x_k, y_1, y_2)$ as

(C2)
$$\begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, m, \\ \dot{y}_1 = \dfrac{1}{2} x' L_1 u, \\ \dot{y}_2 = \dfrac{1}{2} x' L_2 u, & L_1, L_2 \text{ skew symmetric.} \end{cases}$$

Set $r_1 = |r| \cos \theta$, $r_2 = |r| \sin \theta$, and $L_\theta = \cos(\theta) L_1 + \sin(\theta) L_2$.

THEOREM 7. *Arclength geodesics of system* (C2), *starting from the origin, are parametrized by an initial covector* $\lambda_0 = (u_0, r) \in S^{m-1} \times \mathbb{R}^2$, *and they are optimal until time*

$$t_{cut}(\lambda_0) = \frac{2\pi}{\max \sigma(r_1 L_1 + r_2 L_2)} = \frac{2\pi}{|r| \max \sigma(L_\theta)}$$

with the understanding $t_{cut}(\lambda_0) = +\infty$ *if* $r = 0$. *Moreover, in general,* $t_{cut}(\lambda_0) \neq t_{conj}(\lambda_0)$.

The last statement in Theorem 7 says that in the corank 2 case the cut and the conjugate time coincide only in some particular cases, which we explicitly describe in the $(4, 6)$ case (see Theorem 8).

The corank 2 case is more difficult than the corank 1 case precisely because the cut locus is not equal to the conjugate locus. (We are not able to compute the latter explicitly.)

Explicit expression of geodesics for this optimal synthesis is given in section 2.

1.3.3. The nilpotent $(4, 6)$ case. In the nilpotent $(4, 6)$ case our first result is the following.

THEOREM 8. *The following properties are equivalent:*

(P1) *The first conjugate locus is equal to the cut locus.*

(P2) *The linear coordinates y in $T_q M / \Delta_q$ can be chosen in such a way that the pair (L_1, L_2) of 4×4 skew symmetric matrices belongs to the set $(Q \cup \widehat{Q})^2$.*

Here Q (resp., $\widehat{Q}$) denotes the set of pure quaternions (resp., pure skew quaternions); see Appendix 4.1.

Our second result is a continuation of the paper [5] for corank 1, where the following result is proved for general sub-Riemannian metrics.

THEOREM 9 (see [5]). *In the $(m, m + 1)$ case the Radon–Nykodym derivative of the spherical Hausdorff measure with respect to the Popp measure is a C^3 function but is not C^5 in general.*

Here we show the following result.

THEOREM 10. *For a generic $(4,6)$ sub-Riemannian metric,[1] the Radon–Nykodym derivative of the spherical Hausdorff measure with respect to the Popp measure is $\mathcal{C}^1$.*

In a previous paper [5] it is shown that the Radon–Nikodym derivative of the spherical Hausdorff measure with respect to the Popp measure is inversely proportional (as a function of q) to the volume of the unit sub-Riemannian ball of the nilpotent approximation at q. Then Theorem 10 is a byproduct of the optimal synthesis given here.

Note that in the corank 1 case, the higher differentiability of the Radon–Nikodym derivative is due to the fact that the conjugate locus is equal to the cut locus, which is not the case here.

Due to the complexity of the computations even in this low dimensional case, it is not easy to determine the real degree of differentiability of Hausdorff measure. This is still an interesting open question.

1.4. Organization of the paper. Section 2 is devoted to the construction of the optimal synthesis for $(m, m + 2)$ nilpotent sub-Riemannian metrics and, as a consequence, to the proof of Theorem 7. In sections 2.1 and 2.2 we compute the exponential map. In section 2.3 we prove that geodesics are optimal up to t_{cut}. Finally, in section 2.4 we show that the cut time does not coincide, in general, with the first conjugate time. In section 3 we give the proofs of Theorems 8 and 10.

In the appendix we recall basic facts about quaternions, we prove a technical lemma, and, applying an Abraham transversality theorem, we prove that generically for the $(4,6)$ case a certain "bad set" is made of isolated points, which permits us to conclude about the differentiability of the Radon–Nikodym derivative (Theorem 10).

2. Exponential map and synthesis.

2.1. Hamiltonian equations in the (m, n) case. The purpose of this section is to compute the exponential map, i.e., the set of all geodesics, parametrized by length, starting form the origin of the control system (6), i.e., the system

$$(7) \qquad \begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, m, \\ \dot{y}_h = \dfrac{1}{2} x' L_h u, & h = 1, \ldots, k. \end{cases}$$

Let $L_h = (b_{ij}^h)$ for $h = 1, \ldots, k$. Then the control system can be written in the form $\dot{q} = \sum_{i=1}^m u_i X_i(q)$, where $q = (x, y)$ and

$$X_i = \partial_{x_i} + \frac{1}{2} \sum_{j,h} b_{ij}^h x_j \partial_{y_h}, \qquad i = 1, \ldots, m.$$

Setting $Y_h = \partial_{y_h}$ for $h = 1, \ldots, k$, the commutation relations are

$$(8) \qquad\qquad [X_i, X_j] = \sum_{h=1}^k b_{ij}^h Y_h, \qquad i, j = 1, \ldots, m,$$

$$(9) \qquad [X_i, Y_j] = [Y_j, Y_h] = 0, \qquad i = 1, \ldots, m, \qquad j, h = 1, \ldots, k.$$

Define the functions on T^*M that are linear on fibers,

$$(10) \qquad\qquad u_i(\lambda, q) = \langle \lambda, X_i(q) \rangle, \qquad i = 1, \ldots, m,$$

$$(11) \qquad\qquad r_h(\lambda, q) = \langle \lambda, Y_h(q) \rangle, \qquad h = 1, \ldots, k.$$

---

[1]Which means for an open and dense subset of all $(4,6)$ sub-Riemannian metrics, endowed with the Whitney topology.

These functions can be treated as coordinates on the fibers of T^*M to solve the Hamiltonian system given by the PMP; see section 1.1. This Hamiltonian system is associated with the Hamiltonian

$$(12) \qquad H(\lambda, q) = \frac{1}{2}\sum_{i=1}^{m}\langle\lambda, X_i(q)\rangle^2 = \frac{1}{2}\sum_{i=1}^{m}u_i^2(\lambda, q), \qquad \lambda \in T_q^*M.$$

Remark 11. The geodesics parametrized by length correspond to the level set $\{H = 1/2\}$. Notice that for systems of type $\dot{q} = \sum_{i=1}^{m}u_iX_i(q)$ with fixed initial and final points, the problem of finding length-parametrized curves minimizing the length is equivalent to the problem of minimizing time with the constraint $\{\|u\| \le 1\}$.

For a function $a \in C^\infty(T^*M)$ we have that along the sub-Riemannian flow

$$(13) \qquad \dot{a} = \{a, H\} = \sum_{i=1}^{m}\{a, u_i\}u_i,$$

where $\{a, b\}$ denotes the Poisson bracket of two functions in T^*M. The following lemma gives a way of computing the covector $\lambda(t)$, a solution of the Hamiltonian system associated with (12) in the coordinates (u, r).

LEMMA 12. *If $u(t)$ and $r(t)$ are a solution of* (13) *corresponding to level set* $\{H = 1/2\}$*, then they satisfy*

$$\begin{cases} \dot{u}(t) = (r_1L_1 + \cdots + r_kL_k)u(t), & u(0) = u_0, & \|u_0\| = 1, \\ \dot{r}(t) = 0. \end{cases}$$

Proof. Recall that if $a_i(\lambda, q) = \langle\lambda, Z_i(q)\rangle$ for some vector fields Z_i, $i = 1, 2$, then

$$\{a_1, a_2\} = \langle\lambda, [Z_1, Z_2]\rangle.$$

Applying (13) for $a = r_h$ and using (9) we get

$$\dot{r}_h = \sum_{h=1}^{k}\{r_h, u_i\}u_i = 0 \quad \Rightarrow \quad r_h = \text{const}.$$

Similarly, using (8), one finds

$$\dot{u}_i = \sum_{i=1}^{n}\{u_i, u_j\}u_j = \sum_{i=1}^{n}b_{ij}^k r_k u_j. \qquad \square$$

Remark 13. In the following, geodesics are parametrized by the initial covector $\lambda(0) = (p(0), r(0)) = (u_0, r)$, since $r = \text{const}$ and $u_i = \langle\lambda, X_i\rangle$ and at the starting point we have $X_i(0) = \partial_{x_i}$.

2.2. Exponential map in the corank 2 case. From now on we focus on the case $(m, m + 2)$, i.e., when the corank k is equal to 2. We can write the equation of geodesics starting from the origin as follows:

$$(14) \qquad \begin{cases} x(t) = \displaystyle\int_0^t e^{s(r_1L_1 + r_2L_2)}u_0 ds, & x(0) = 0, \\[3mm] y_1(t) = \dfrac{1}{2}\displaystyle\int_0^t x(s)'L_1\, u(s)ds, & y_1(0) = 0, \\[3mm] y_2(t) = \dfrac{1}{2}\displaystyle\int_0^t x(s)'L_2\, u(s)ds, & y_2(0) = 0. \end{cases}$$

Remark 14 (notation). In the following we denote by $E_{L_1,L_2}^{u_0,r_1,r_2}(t)$ the geodesic, parametrized by the length, and starting from the origin, defined by (14), associated with L_1, L_2.

DEFINITION 15. *The matrices L_1, L_2 being fixed, the* exponential map *is the map* $\mathcal{E} : \mathbb{R}^+ \times \Lambda \to \mathbb{R}^n$ *defined by*

$$\mathcal{E}(t, u_0, r_1, r_2) = E_{L_1,L_2}^{u_0,r_1,r_2}(t), \qquad \Lambda = \{(u_0, r_1, r_2), u_0 \in S^{m-1}, r_i \in \mathbb{R}\}.$$

Remark 16. The optimal control problem

$$(15) \qquad \begin{cases} \dot{x}_i = u_i, & i = 1, \dots, m, \\ \dot{y}_1 = \dfrac{1}{2} x' L_1 u, \\ \dot{y}_2 = \dfrac{1}{2} x' L_2 u, \end{cases}$$

is invariant with respect to the following change of coordinates:
  (a) orthogonal changes of coordinates in the x space,
  (b) linear changes of coordinates in the y space.
Indeed, let M be a nonsingular orthogonal matrix $(M^{-1} = M')$ and define the new coordinates $\widetilde{x} = Mx$. Then

$$\dot{\widetilde{x}} = M\dot{x} = Mu =: \widetilde{u},$$

and

$$\dot{y}_i = x' L_i u = (Mx)' M L_i M'(Mu) = \widetilde{x}' M L_i M' \widetilde{u}.$$

Hence, in the new coordinates, L_i is changed for $\widetilde{L}_i := M L_i M'$.
  Also, it is easy to see that the change of coordinates

$$(16) \qquad \widetilde{y}_1 = \alpha_1 y_1 + \alpha_2 y_2, \qquad \widetilde{y}_2 = \beta_1 y_1 + \beta_2 y_2$$

corresponds to the change

$$\widetilde{L}_1 = \alpha_1 L_1 + \alpha_2 L_2, \qquad \widetilde{L}_2 = \beta_1 L_1 + \beta_2 L_2.$$

In other words, we can change L_1 and L_2 up to congruence and linear combinations.
  Using these arguments one immediately gets the next lemma.
  LEMMA 17. *Let $(r_1, r_2) =: (r \cos\theta, r \sin\theta)$ and $L_\theta := \cos\theta L_1 + \sin\theta L_2$, $\widetilde{L}_\theta := -\sin\theta L_1 + \cos\theta L_2$. Consider the rotation matrix $R_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ and the orthogonal matrix M such that $M L_\theta M'$ is block diagonal.*
  Denote $\Omega = \left(\begin{smallmatrix} M & 0 \\ 0 & R_\theta \end{smallmatrix}\right)$ and $\widetilde{u}_0 = Mu_0$. We have the equality

$$(17) \qquad \Omega E_{L_1,L_2}^{u_0,r_1,r_2}(t) = E_{L_\theta,\widetilde{L}_\theta}^{\widetilde{u}_0,r,0}(t).$$

Thanks to Lemma 17, one can always restrict to geodesics of the type

$$(18) \qquad \begin{cases} x(t) = \displaystyle\int_0^t e^{srL_\theta} u_0 ds, \\ y_1(t) = \dfrac{1}{2} \displaystyle\int_0^t x(s)' L_\theta\, u(s) ds, \\ y_2(t) = \dfrac{1}{2} \displaystyle\int_0^t x(s)' \widetilde{L}_\theta\, u(s) ds, \end{cases}$$

where L_θ is in the block diagonal form

$$
L_\theta = \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & a_\ell \\ & & & -a_\ell & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & a_\ell & \\ & & & -a_\ell & 0 & \\ & & & & & 0 \end{pmatrix},
$$

depending on the fact that m is even ($m = 2\ell$) or odd ($m = 2\ell + 1$), and where the geodesic is associated with the covector $(r_1, r_2) = (r, 0)$.

Remark 18. When we deal with a fixed sub-Riemannian metric we can assume also that the coordinates in the x space are chosen in such a way that $a_1 \geq a_i$ for every i. In this case

$$
\frac{2\pi}{r \max(\sigma(L_\theta))} = \frac{2\pi}{a_1 r}.
$$

2.3. Computation of the cut time. In this section we prove Theorem 7, i.e., we compute the last time at which a geodesic parametrized by length is optimal.

We first consider the case $r = 0$. In this case (14) can be easily integrated and gives the straight lines

$$
\begin{cases} x(t) = u_0 t, \\ y_i(t) = 0. \end{cases}
$$

This trajectory is optimal for any time (i.e., $t_{cut} = +\infty$) since the sub-Riemannian length of a geodesic coincides with the Euclidean length of its projection on the horizontal subspace $(x_1, \ldots, x_m)$, as follows from formula (3).

In what follows we use the notation $A = rL_\theta$, $\widetilde{A} = r\widetilde{L}_\theta$ and we focus on the case when A is even dimensional (i.e., $m = 2\ell$) and invertible. The case A noninvertible (in particular A odd dimensional) needs an obvious modification of the proof.

With this notation the system (18) is rewritten as

$$
(19) \quad \begin{cases} x(t) = A^{-1}(e^{tA} - I)u_0, \\[2mm] y_1(t) = \dfrac{1}{2} \displaystyle\int_0^t x(s)' A\, u(s)ds, \\[2mm] y_2(t) = \dfrac{1}{2} \displaystyle\int_0^t x(s)' \widetilde{A}\, u(s)ds. \end{cases}
$$

2.3.1. Maxwell points. Consider $\gamma(t) = E^{u_0, r, 0}(t)$, the geodesic associated with the problem (19) and with initial covector $(u_0, r, 0), r > 0$. Let us first show that there exists another geodesic reaching the point $\gamma(T^*)$ in time $T^* = 2\pi/(a_1 r)$. Using Arnold's terminology, points reached in the same time by more than one geodesic are called Maxwell points. At the end of this section we prove that γ cannot be optimal after T^*.

Set $u_0 = (u_1, u_2, u_3, \ldots, u_m)$ and consider the following variation of the horizontal covector:

$$
u_0^\omega = (\cos \omega\, u_1 + \sin \omega\, u_2, -\sin \omega\, u_1 + \cos \omega\, u_2, u_3, \ldots, u_m), \qquad \omega \in [0, 2\pi].
$$

Denote $\gamma^\omega(t) = (x^\omega, y_1^\omega, y_2^\omega) := E^{u_0^\omega, r, 0}(t)$ the geodesic associated with this variation.

Claim. There exists $\omega \neq 0$ such that $\gamma(T^*) = \gamma^\omega(T^*)$.

Proof of the claim. Denote by $M_A(t) := A^{-1}(e^{tA} - I)$ and notice that

$$
M_A(t) = A^{-1}(e^{tA} - I) = \begin{pmatrix} \frac{\sin a_1 rt}{a_1 r} & \frac{1-\cos a_1 rt}{a_1 r} & & & \\ \frac{-1+\cos a_1 rt}{a_1 r} & \frac{\sin a_1 rt}{a_1 r} & & & \\ & & \ddots & & \\ & & & \frac{\sin a_\ell rt}{a_\ell r} & \frac{1-\cos a_\ell rt}{a_\ell r} \\ & & & \frac{-1+\cos a_\ell rt}{a_\ell r} & \frac{\sin a_\ell rt}{a_\ell r} \end{pmatrix}.
$$

In other words we can write

$$
M_A(t) = \begin{pmatrix} D_1(t) & & \\ & \ddots & \\ & & D_\ell(t) \end{pmatrix},
$$

where
(20)

$$
D_i(t) = \begin{pmatrix} \frac{\sin a_i rt}{a_i r} & \frac{1-\cos a_i rt}{a_i r} \\ \frac{-1+\cos a_i rt}{a_i r} & \frac{\sin a_i rt}{a_i r} \end{pmatrix} = 2\frac{\sin(a_i rt/2)}{a_i r} \begin{pmatrix} \cos(a_i rt/2) & \sin(a_i rt/2) \\ -\sin(a_i rt/2) & \cos(a_i rt/2) \end{pmatrix}.
$$

We prove our claim by steps.

(i) From (20) it is easy to see that

$$
x^\omega(T^*) - x(T^*) = A^{-1}(e^{T^*A} - I)(u_0^\omega - u_0) = M_A(T^*)(u_0^\omega - u_0) = 0 \qquad \forall \omega \in [0, 2\pi]
$$

since $e^{T^*A} - I$ (and so $M_A(T^*)$) has its first 2×2 block equal to zero.

(ii) Now we show that $y_1^\omega(T^*) = y_1(T^*)$ for all $\omega \in [0, 2\pi]$. Indeed from (19), we get

$$
\begin{aligned}
y_1(t) &= -\frac{1}{2}u_0' \int_0^t (e^{-sA} - I)e^{sA}ds\, u_0 \\
&= \frac{1}{2}u_0' \int_0^t (e^{sA} - I)ds\, u_0 \\
&= \frac{1}{2}\langle M_A(t)u_0, u_0 \rangle + \frac{1}{2}t\|u_0\|^2,
\end{aligned}
$$

and

$$
y_1^\omega(T^*) - y_1(T^*) = \frac{1}{2}\left(\langle M_A(T^*)u_0^\omega, u_0^\omega \rangle - \langle M_A(T^*)u_0, u_0 \rangle\right) + \frac{T^*}{2}(\|u_0^\omega\|^2 - \|u_0\|^2).
$$

First notice that

$$
\|u_0^\omega\|^2 = \|u_0\|^2 \qquad \forall \omega \in [0, 2\pi].
$$

Moreover, setting $u_0^\omega = u_0 + v^\omega$ we get (we omit T^* in the argument of M_A)

$$
\langle M_A u_0^\omega, u_0^\omega \rangle - \langle M_A u_0, u_0 \rangle = \langle M_A v^\omega, v^\omega \rangle + \langle (M_A + M_A')v^\omega, u_0 \rangle = 0
$$

since the first 2×2 block of M_A that is zero at T^* and v^ω has a nonzero component only in the first two entries.

Remark 19. Note that (i) and (ii) are just the manifestation of the fact that forgetting about the second vertical component y_2, we are facing the corank 1 case, for which T^* is a cut time and there is a rotational symmetry that implies that it is also a conjugate time.

(iii) Now one can proceed in a similar way and compute

$$
\begin{aligned}
y_2(t) &= \frac{1}{2} \int_0^t x(s)' \widetilde{A} u(s) ds \\
&= \frac{1}{2} u_0' \int_0^t (A^{-1}(e^{sA} - I))' \widetilde{A} e^{sA} ds\, u_0 \\
&= -\frac{1}{2} u_0' \int_0^t (e^{-sA} - I) A^{-1} \widetilde{A} e^{sA} ds\, u_0 \\
&= \langle C(t) u_0, u_0 \rangle,
\end{aligned}
$$

where we set

$$
(21) \qquad C(t) = \frac{1}{2} \int_0^t (e^{-sA} - I) A^{-1} \widetilde{A} e^{sA} ds.
$$

Since all matrices appearing in (21) but $\widetilde{A}$ are 2×2 block diagonal, the first 2×2 diagonal block of

$$
K(s) := (e^{-sA} - I) A^{-1} \widetilde{A} e^{sA}
$$

is the product of the respective blocks. A direct computation shows that it is

$$
\frac{\alpha_0}{a_1} \begin{pmatrix} 1 - \cos(a_1 rs), & -\sin(a_1 rs) \\ \sin(a_1 rs), & 1 - \cos(a_1 rs) \end{pmatrix},
$$

where $\begin{pmatrix} 0 & \alpha_0 \\ -\alpha_0 & 0 \end{pmatrix}$ denotes the first 2×2 block of $\widetilde{A}$. Integrating from 0 to T^* one obtains for the first block of $C(T^*)$

$$
(22) \qquad \frac{\pi \alpha_0}{a^2 r^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

As before, we set $u_0^\omega = u_0 + v^\omega$ and we get (omiting T^* in the argument of C)

$$
(23) \qquad y_2^\omega(T^*) - y_2(T^*) = \langle C v^\omega, v^\omega \rangle + \langle (C + C') v^\omega, u_0 \rangle.
$$

Using (22) and

$$
\begin{aligned}
\|v^\omega\|^2 &= ((\cos \omega - 1) u_1 + \sin \omega\, u_2)^2 + (-\sin \omega\, u_1 + (\cos \omega - 1) u_2)^2 \\
&= 4(u_1^2 + u_2^2) \sin^2(\omega/2),
\end{aligned}
$$

one gets that (23) is linear with respect to the variables

$$
\cos \omega - 1 = -2 \sin^2(\omega/2), \qquad \sin \omega = 2 \cos(\omega/2) \sin(\omega/2).
$$

In other words, if we prescribe that the expression (23) is zero, we get

$$
(24) \qquad C_0 \sin(\omega/2)(C_1 \cos(\omega/2) + C_2 \sin(\omega/2)) = 0
$$

for some suitable constants C_0, C_1, C_2 that do not depend on ω. The claim is proved since (24) has always a nontrivial solution $\widetilde{\omega} \in [0, 2\pi]$. $\quad\blacksquare$

Let us now show that $\gamma(t)$ cannot be optimal after T^*. From the previous computation we have $\dot{\gamma}(T^*) \neq \dot{\gamma}^{\widetilde{\omega}}(T^*)$. By contradiction if γ is optimal after time T^*, then the concatenation of $\gamma^{\widetilde{\omega}}|_{[0,T^*]}$ and $\gamma|_{[T^*,T^*+\varepsilon]}$ (for some $\varepsilon > 0$) is optimal as well, which is impossible since all optimal trajectories are projections of the Hamiltonian system associated with (12) and they are smooth.

2.3.2. Optimality of geodesics. In this section we prove that $\gamma(t) = E^{u_0, r, 0}(t)$, $r > 0$, is optimal up to its first Maxwell time $T^* = 2\pi/(a_1 r)$. To this extent, consider the following auxiliary optimal control problem.

(P) Let $T < T^*$ and set $(\bar{x}, \bar{y}_1, \bar{y}_2) = \gamma(T)$. Find a length-parametrized trajectory of the system

(25)
$$\begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, m, \\ \dot{y}_1 = \dfrac{1}{2} x' L_1 u, \\ \dot{y}_2 = \dfrac{1}{2} x' L_2 u, \end{cases}$$

starting from the origin, and reaching the hyperplane $\{x = \bar{x}\}$ in time T, maximizing the y_1 coordinate.

Remark 20. Notice that $\bar{y}_1 > 0$ since $r > 0$ implies that the trajectory is not contained in the hyperplane $\{y_1 = 0\}$.

LEMMA 21. *The following assertions hold:* (i) *There exists a solution γ^* of the problem* (P). (ii) *γ^* is a length minimizer.* (iii) *$\gamma^*(t) = E^{\widetilde{u}_0, \widetilde{r}, 0}(t)$ for some $(\widetilde{u}_0, \widetilde{r})$.*

Proof. Let us prove (i). In problem (P), since we deal with length-parametrized trajectories, we can assume that the set of controls in (25) is $U = \{\|u\| \leq 1\}$. The existence of a solution to (P) can be obtained with standard arguments using the compactness and convexity of the set of admissible velocities (see [3, 13]).

To prove (ii) assume by contradiction that there exists a trajectory of (25) reaching the point $(\bar{x}, y_1^*, y_2^*) = \gamma^*(T)$ in time $T_0 < T$. By small time controllability there exists a trajectory of system (25) reaching in time T the point $(\bar{x}, \widehat{y}_1, \widehat{y}_2)$ with $\widehat{y}_1 > y_1^*$ contradicting the fact that γ^* maximizes the y_1 coordinate. The fact that γ^* is also a length minimizer follows from Remark 11.

To prove (iii) observe that γ^* satisfies the PMP (see again [3]) for the problem of minimizing $-y_1 = -\int_0^T \dot{y}_1 dt = -\int_0^T x' L_1 u\, dt$, i.e., with the Hamiltonian

(26)
$$\mathcal{H}_u = \sum_{i=1}^m \langle \lambda, u_i X_i \rangle + \nu x' L_1 u$$
$$= pu + r_1 x' L_1 u + r_2 x' L_2 u + \nu x' L_1 u,$$

where $\lambda = (p, r_1, r_2)$ are the dual variables to (x, y_1, y_2) in $T^* M$. In formula (26) ν is a nonnegative constant. The Hamiltonian equations give

$$\begin{cases} \dot{r}_1 = -\dfrac{\partial \mathcal{H}_u}{\partial y_1} = 0, \\ \dot{r}_2 = -\dfrac{\partial \mathcal{H}_u}{\partial y_2} = 0, \\ \dot{p}' = -\dfrac{\partial \mathcal{H}_u}{\partial x} = -(r_1 L_1 + r_2 L_2 + \nu L_1) u. \end{cases}$$

Since the final point is constrained on the set $\{x = \bar{x}\}$, the transversality conditions give $r_1 = 0, r_2 = 0$. Hence we have

$$\mathcal{H}_u = (p + \nu x' L_1)u,$$

(27)
$$\dot{p}' = -\nu L_1 u.$$

Notice that actually $\nu > 0$. Otherwise the trajectory is a straight line contained in the plane $\{y_1 = y_2 = 0\}$; see Remark 20. The maximality condition and the condition that the final time is fixed in such a way that trajectories are parameterized by length give

$$\mathcal{H}_{u(t)}(x(t), y_1(t), y_2(t), p(t), r_1, r_2) = \max_v \mathcal{H}_v(x(t), y_1(t), y_2(t), p(t), r_1, r_2) = 1,$$

(28)
$$u(t) = \frac{p'(t) - \nu L_1 x(t)}{\|p'(t) - \nu L_1 x(t)\|} = p'(t) - \nu L_1 x(t).$$

Notice that a geodesic for the problem (25) associated with the covector (u_0, r_1, r_2) corresponds to a control

(29)
$$u(t) = p' - r_1 L_1 x - r_2 L_2 x,$$

where

(30)
$$\dot{p}' = (-r_1 L_1 - r_2 L_2)(p' - r_1 L_1 x - r_2 L_2 x).$$

Comparing (27)–(28) with (29)–(30) it follows that γ^* is a geodesic for the problem (25) corresponding to an initial covector $(\tilde{u}_0, \nu, 0)$ for some $\tilde{u}_0$ with $\|\tilde{u}_0\| = 1$. Then (iii) is proved for $\tilde{r} = \nu$.   □

We have the following claim.

Claim. $\gamma^* = \gamma$, i.e., $\tilde{u}_0 = u_0$ and $\tilde{r} = r$.

Proof of the claim. It is enough to prove that the parameters u_0, r such that a geodesic $\gamma(t) = (x(t), y_1(t), y_2(t)) = E^{u_0, r, 0}$ satisfies $x(T) = \bar{x}$ with $T < T^*$ are unique.

From the computations in sections 2.3 we know that

$$x(t) = M_A(t)u_0, \qquad \text{where} \quad A = rL_1, \quad \text{and} \quad M_A(t) = A^{-1}(e^{tA} - I).$$

In particular, using the nonsingularity of A, the equality at $t = T$ gives

(31)
$$u_0 = M_A^{-1}(T)\bar{x}.$$

Computing the norm of vectors in equality (31), it follows that

(32) $$1 = \|u_0\|^2 = \sum_{i=1}^{\ell} \frac{\rho_i(\bar{x})^2}{T^2} \frac{a_i rT/2}{\sin(a_i rT/2)}, \qquad \text{where} \quad \rho_i(x) = (x_{2i-1}^2 + x_{2i}^2)^{1/2}.$$

Notice that the right-hand side of (32) is the sum of monotonic functions with respect to the variable rT on the segment $[0, 2\pi/a_1]$ ($T < T^*$ implies $rT \leq 2\pi/a_1$).

Moreover, since the curve is length-parametrized we have $\|x(T)\| \leq T$. As a consequence there exists a unique solution rT of (32) in the segment $[0, 2\pi/a_1]$. In particular r is uniquely determined and u_0 is uniquely recovered from (31).   □

Since $\gamma = \gamma^*$ and γ^* is length minimizer for every $T < T^*$, it follows that $t_{cut} = T^*$.

2.4. First conjugate time. In this section we prove that in the corank 2 case, the cut time is not equal to the first conjugate time, in general. This is deeply different from the corank 1 case, where the cut locus always coincides with the first conjugate locus.

It is enough to show that the cut time is not conjugate in the $(4, 6)$ case. Define the Jacobian of the exponential map

$$(33) \qquad J_{\mathcal{E}}(t, u_0, r_1, r_2) := \det\left(\frac{\partial \mathcal{E}}{\partial t}, \frac{\partial \mathcal{E}}{\partial u_0}, \frac{\partial \mathcal{E}}{\partial r_1}, \frac{\partial \mathcal{E}}{\partial r_2}\right).$$

Remark 22. Recall that the first conjugate time t_{conj} for the geodesic corresponding to the covector (u_0, r_1, r_2) is the first time $t > 0$ for which we have

$$(34) \qquad J_{\mathcal{E}}(t, u_0, r_1, r_2) = 0.$$

We have to prove that (34) is not satisfied when $t = t_{cut}$.

To compute $J_{\mathcal{E}}$ we use the following trick. Let $\mathfrak{L} = p dx + r_1 dy_1 + r_2 dy_2$ be the Liouville form.

LEMMA 23. *We have* $\mathfrak{L}(\frac{\partial \mathcal{E}}{\partial t}) = 1$, $\mathfrak{L}(\frac{\partial \mathcal{E}}{\partial u_0}) = \mathfrak{L}(\frac{\partial \mathcal{E}}{\partial r_i}) = 0$.

Proof. The first equality follows from the fact that the Hamiltonian is homogeneous of degree 2. Indeed set $\lambda = (p, r_1, r_2)$ and $q = (x, y_1, y_2)$, we have $\mathfrak{L} = \lambda dq$ and

$$\mathfrak{L}\left(\frac{\partial \mathcal{E}}{\partial t}\right) = \left\langle \lambda, \frac{\partial q}{\partial t} \right\rangle = \left\langle \lambda, \frac{\partial H}{\partial \lambda} \right\rangle = 2H = 1,$$

since length-parametrized trajectory belongs to the set $\{H = 1/2\}$. The second and third identities follow from the fact that the Liouville form is preserved by the Hamiltonian flow; hence the values of $\mathfrak{L}(\frac{\partial \mathcal{E}}{\partial u_0})$ and $\mathfrak{L}(\frac{\partial \mathcal{E}}{\partial r_i})$ are constant with respect to t. In particular at $t = 0$ they are annihilated by the Liouville form. $\qquad\square$

If we compute the exponential map in a neighborhood of a geodesic with $(r_1, r_2) = (r, 0)$ with $r \neq 0$ using the identity $r_1 dy_1 = \mathfrak{L} - p dx - r_2 dx_2$ and Lemma 23 we get

$$J_{\mathcal{E}} = dx \wedge dy_1 \wedge dy_2 \left(\frac{\partial \mathcal{E}}{\partial t}, \frac{\partial \mathcal{E}}{\partial u_0}, \frac{\partial \mathcal{E}}{\partial r_1}, \frac{\partial \mathcal{E}}{\partial r_2}\right)$$

$$(35) \qquad = \frac{1}{r_1} dx \wedge dy_2 \left(\frac{\partial \widetilde{\mathcal{E}}}{\partial u_0}, \frac{\partial \widetilde{\mathcal{E}}}{\partial r_1}, \frac{\partial \widetilde{\mathcal{E}}}{\partial r_2}\right),$$

where $\widetilde{\mathcal{E}}(t, u_0, r_1, r_2) = (x(t, u_0, r_1, r_2), y_2(t, u_0, r_1, r_2))$ denote the exponential map where y_1 is removed. More precisely, (35) is the function of (t, u_0, r_1, r_2) given by

$$J_{\mathcal{E}} = \frac{1}{r_1} \det\begin{pmatrix} \frac{\partial x}{\partial u_0} v_1 & \frac{\partial x}{\partial u_0} v_2 & \frac{\partial x}{\partial u_0} v_3 & \frac{\partial x}{\partial r_1} & \frac{\partial x}{\partial r_2} \\ \frac{\partial y_2}{\partial u_0} v_1 & \frac{\partial y_2}{\partial u_0} v_2 & \frac{\partial y_2}{\partial u_0} v_3 & \frac{\partial y_2}{\partial r_1} & \frac{\partial y_2}{\partial r_2} \end{pmatrix},$$

where v_1, v_2, v_3 are 3 independent tangent vectors to the 3-sphere $\{u_0 \in \mathbb{R}^4, \|u_0\| = 1\}$. We select

$$(36) \qquad v_1 = \begin{pmatrix} -u_2 \\ u_1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ -u_4 \\ u_3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -u_4 \\ 0 \\ 0 \\ u_1 \end{pmatrix}.$$

From the computation of section 2.3 one easily gets

$$\frac{\partial x}{\partial u_0} = M_A(t) = A^{-1}(e^{tA} - I),$$

$$\frac{\partial x}{\partial r_1} = -A^{-1}A(A^{-1}(e^{tA} - I) + t\,e^{tA})u_0$$

$$= -(M_A(t) + t\,e^{tA})u_0,$$

$$\frac{\partial x}{\partial r_2} = -A^{-1}\widetilde{A}(A^{-1}(e^{tA} - I) + t\,e^{tA})u_0$$

$$= -A^{-1}\widetilde{A}(M_A(t) + t\,e^{tA})u_0.$$

Moreover (see again section 2.3),

(37) $$y_2(t) = \langle C(t)u_0, u_0 \rangle,$$

where

(38) $$C(t) = -\frac{1}{2}\int_0^t (e^{-sA} - I)A^{-1}\widetilde{A}\,e^{sA}ds.$$

The function $y_2(t)$ from (37), being a quadratic form with respect to u_0, gives

$$\frac{\partial y_2}{\partial u_0}v_i = \langle (C(t) + C'(t))u_0, v_i \rangle.$$

Now we compute these derivatives at $t = t_{cut} = \frac{2\pi}{ar}$, where $a > b$ are the moduli of the eigenvalues of A.

It is easily seen that

$$B := M_A(t_{cut}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sin(2\pi b/a)}{br} & \frac{2\sin^2(\pi b/a)}{br} \\ 0 & 0 & -\frac{2\sin^2(\pi b/a)}{br} & \frac{\sin(2\pi b/a)}{br} \end{pmatrix},$$

from which it follows that

$$\begin{pmatrix} Bv_1 & Bv_2 & Bv_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u_3\frac{2\sin^2(\pi b/a)}{br} - u4\frac{\sin(2\pi b/a)}{br} & u_1\frac{2\sin^2(\pi b/a)}{br} \\ 0 & u_4\frac{2\sin^2(\pi b/a)}{br} + u_3\frac{\sin(2\pi b/a)}{br} & u_1\frac{\sin(2\pi b/a)}{br} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix},$$

where the last identity defines the matrix M.

The Jacobian determinant of the exponential map computed at $t = t_{cut}$ is then expressed as follows:

$$J_{\mathcal{E}}(t_{cut}) = \det \begin{pmatrix} \dfrac{\partial x}{\partial u_0}v_1 & \dfrac{\partial x}{\partial u_0}v_2 & \dfrac{\partial x}{\partial u_0}v_3 & \dfrac{\partial x}{\partial r_1} & \dfrac{\partial x}{\partial r_2} \\[2ex] \dfrac{\partial y_2}{\partial u_0}v_1 & \dfrac{\partial y_2}{\partial u_0}v_2 & \dfrac{\partial y_2}{\partial u_0}v_3 & \dfrac{\partial y_2}{\partial r_1} & \dfrac{\partial y_2}{\partial r_2} \end{pmatrix}$$

$$= \det \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ Bv_1 & Bv_2 & Bv_3 & \dfrac{\partial x}{\partial r_1} & \dfrac{\partial x}{\partial r_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial y_2}{\partial u_0}v_1 & \dfrac{\partial y_2}{\partial u_0}v_2 & \dfrac{\partial y_2}{\partial u_0}v_3 & \dfrac{\partial y_2}{\partial r_1} & \dfrac{\partial y_2}{\partial r_2} \end{pmatrix}$$

$$(39) \qquad = \det \begin{pmatrix} 0 & 0 & 0 & \frac{\partial x_1}{\partial r_1} & \frac{\partial x_1}{\partial r_2} \\[1ex] 0 & 0 & 0 & \frac{\partial x_2}{\partial r_1} & \frac{\partial x_2}{\partial r_2} \\[1ex] 0 & M_{11} & M_{12} & \frac{\partial x_3}{\partial r_1} & \frac{\partial x_3}{\partial r_2} \\[1ex] 0 & M_{21} & M_{22} & \frac{\partial x_4}{\partial r_1} & \frac{\partial x_4}{\partial r_2} \\[1ex] \langle Cu_0, v_1 \rangle & \langle Cu_0, v_2 \rangle & \langle Cu_0, v_3 \rangle & \frac{\partial y_2}{\partial r_1} & \frac{\partial y_2}{\partial r_2} \end{pmatrix},$$

where we use the notation

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

From (39) it follows

$$(40) \qquad\qquad J_{\mathcal{E}}(t_{cut}) = \langle Cu_0, v_1 \rangle \cdot \det M \cdot \det N,$$

where N is the matrix

$$N = \begin{pmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_1}{\partial r_2} \\[1ex] \frac{\partial x_2}{\partial r_1} & \frac{\partial x_2}{\partial r_2} \end{pmatrix}.$$

It is easy to see from the explicit expression of the geodesics that in the general case when $a \neq b$, the three factors in (40) do not vanish identically in u_0, since the matrix $\widetilde{A}$ is arbitrary. The proof of Theorem 7 is then completed.

Remark 24. Notice that M is the zero matrix when $a = b$. Hence, in the $(4, 6)$ case, $t_{cut} = t_{conj}$ for those θ such that L_θ has double eigenvalue. Moreover, in this case the rank of the Jacobian matrix drops by 2, since the first three columns are proportional.

3. The nilpotent $(4, 6)$ case. In this section we restrict to the $(4,6)$ case. By the previous discussion the geodesics of the sub-Riemannian metric can be written as

(41)
$$\begin{cases} x(t) = A^{-1}(e^{tA} - I)u_0, \\ y_1(t) = \dfrac{1}{2} \displaystyle\int_0^t x(s)' A\, u(s)ds, \qquad A = rL_\theta, \ \widetilde{A} = r\widetilde{L}_\theta, \\ y_2(t) = \dfrac{1}{2} \displaystyle\int_0^t x(s)' \widetilde{A}\, u(s)ds, \end{cases}$$

and we can assume the matrix L_θ to be diagonal,

(42)
$$L_\theta = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \qquad a \geq b,$$

while $\widetilde{L}_\theta$ is an arbitrary skew symmetric matrix,

$$\widetilde{L}_\theta = \begin{pmatrix} 0 & \alpha_0 & \alpha_1 & \alpha_2 \\ -\alpha_0 & 0 & \alpha_3 & \alpha_4 \\ -\alpha_1 & -\alpha_3 & 0 & \alpha_5 \\ -\alpha_2 & -\alpha_4 & -\alpha_5 & 0 \end{pmatrix}.$$

3.1. Proof of Theorem 8. Recall that the cut time t_{cut} coincides with t_{conj} if and only if t_{cut} is a time that satisfies the equation

(43)
$$J_{\mathcal{E}}(t, u, \theta)\big|_{t=t_{cut}} = 0.$$

(P2) $\Rightarrow$ (P1). We consider separately two cases:
(a) L_1, L_2 both belong to the same subspace, either Q or $\widehat{Q}$. Then it is not restrictive to assume that $L_1, L_2 \in Q$. In this case all linear combinations of L_1, L_2 belong to Q, i.e., $L_\theta = \cos(\theta)L_1 + \sin(\theta)L_2 \in Q$ for every θ. In particular L_θ has a double eigenvalue for every $\theta \in [0, 2\pi]$. From the computation of section 2 it is easily seen that $a = b$ implies $M = 0$; hence from (40) it follows that $t_{cut} = t_{conj}$.
(b) $L_1 \in Q$ and $L_2 \in \widehat{Q}$ (L_1 and L_2 play the same role). By (55) we have $[L_1, L_2] = 0$. Let us prove then that this property implies (P1).
Indeed every two commuting skew symmetric matrices can be block diagonalized simultaneously in the same basis. Hence we can assume that choosing an appropriate coordinate system y_1, y_2, both L_1, L_2 are diagonal. As a consequence L_θ and $\widetilde{L}_\theta$ are also diagonal. Moreover, from (38) it is easily seen that if both L_θ and $\widetilde{L}_\theta$ are diagonal, C is 2×2 block diagonal, with the first block equal to cI, for some constant c (see also (22)).
In particular it follows that $\langle Cu_0, v_1 \rangle = 0$ and again (40) implies $t_{cut} = t_{conj}$.
(P1) $\Rightarrow$ (P2). By assumption the identity

(44) $\qquad J_{\mathcal{E}}(t, u_0, \theta)\big|_{t=t_{cut}(\theta)} = \langle C(\theta)u_0, v_1 \rangle \cdot \det M(u_0, \theta) \cdot \det N(u_0, \theta) = 0$

holds for every u_0 (the horizontal part of the initial covector) and every θ. Since the exponential map is linear with respect to u_0 in the x-variable and quadratic with

respect to u_0 in the y_i-variables, it follows that (44) is an analytic expression of (u_0, θ). (It is polynomial with respect to u_0 and trigonometric in θ.) In particular one of the three factors in (44) must vanish identically.

Assume that $\det M(u_0, \theta) \equiv 0$. Then from the explicit expression it is computed that

$$\det M(u_0, \theta) = \frac{4u_1 u_3}{b(\theta)^2 r^2} \sin^2 \left(\pi \frac{b(\theta)}{a(\theta)} \right),$$

and since $a(\theta) \geq b(\theta)$ by assumption, $\det M \equiv 0$ implies $a(\theta) = b(\theta)$ for all θ.

From this it easily follows that L_θ has double eigenvalue for all θ, i.e., if we write

$$L_\theta = q(\theta) + \widehat{q}(\theta),$$

it follows that one of $\|q(\theta)\|$ and $\|\widehat{q}(\theta)\|$ is identically zero. (It is not a restriction to assume $\|\widehat{q}(\theta)\| \equiv 0$.) Hence $L_\theta \in Q$ for all θ, which implies in particular that $L_1, L_2 \in Q$.

It is not restrictive now to assume that $a(\bar{\theta}) \neq b(\bar{\theta})$ for some $\bar{\theta} \in [0, 2\pi]$. We show that the identities

(a) $\det N(u_0, \bar{\theta}) = 0$,
(b) $\langle C(\bar{\theta})u_0, v_1 \rangle = 0$

both imply that there exists a choice of the coordinates such that $L_1 \in Q$, $L_2 \in \widehat{Q}$. We give details only for case (b); the other case is similar. Considering (b) as an equation in the variables $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (the nondiagonal entries of the matrix $\widetilde{L}_\theta$), it is easy to see that the identity (b) can be written as the equation

$$F(\alpha_i, u_i) = 0,$$

where F is a quadratic form in the u_i whose coefficients depend linearly on α_i. Since this equation should be satisfied for all $u_0 = (u_1, \ldots, u_4)$, choosing values

$$u_0 \in \{(1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1)\},$$

one gets the set of four linear equations

(45)
$$\begin{cases} (a\alpha_4 + b\alpha_1) \cos \eta + (a\alpha_3 - b\alpha_2) \sin \eta = 0, \\ (a\alpha_3 - b\alpha_2) \cos \eta - (a\alpha_4 + b\alpha_1) \sin \eta = 0, \\ (a\alpha_2 - b\alpha_3) \cos \eta + (a\alpha_1 + b\alpha_4) \sin \eta = 0, \\ (a\alpha_1 + b\alpha_4) \cos \eta - (a\alpha_2 - b\alpha_3) \sin \eta = 0, \end{cases}$$

where we set $\eta = \pi b/a$, and for simplicity of notation we denote $a = a(\theta_0), b = b(\theta_0)$.

It is easy to show, using the fact that $a \neq b$, that this system has the unique solution

$$\alpha_1 = \cdots = \alpha_4 = 0,$$

which means that L_1 and L_2 are both diagonal. Due to this fact they can be written, as pure quaternions (see Appendix 4.1), as a linear combination of $i, \widehat{i}$,

(46)
$$L_1 = \alpha i + \widehat{\alpha}\widehat{i}, \qquad L_2 = \beta i + \widehat{\beta}\widehat{i}.$$

Performing the change of variables

$$\begin{pmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{pmatrix} = \begin{pmatrix} \alpha & \widehat{\alpha} \\ \beta & \widehat{\beta} \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

we find a system of coordinates such that $L_1 = i$ and $L_2 = \widehat{i}$, i.e., that satisfies (P2).

3.2. Proof of Theorem 10. In the paper [5] it is proved that if the sub-Riemannian manifold is regular[2] with Hausdorff dimension $\mathfrak{h}$, the Radon–Nikodym derivative of the spherical Hausdorff measure $\mathcal{S}^{\mathfrak{h}}$ with respect to the Popp measure μ, denoted $f_{\mathcal{S}\mu}$, is given by the volume of the unit ball in the nilpotent approximation, namely,

$$(47) \qquad f_{\mathcal{S}\mu}(q) = \frac{2^{\mathfrak{h}}}{\widehat{\mu}_q(\widehat{B}_q)}, \qquad q \in M,$$

where $\widehat{\mu}_q$ is the Popp measure defined on the nilpotent approximation G_q of the structure at the point q. Note that $\widehat{\mu}_q$ is the left-invariant measure on G_q that coincides with the Popp measure of the original sub-Riemannian metric at the point q.

Remark 25. Notice that in our $(m, m+2)$ case the structure is automatically regular since, by assumption, the distribution has constant rank and with one bracket we get all the tangent space.

Remark 26. In [5] it is proved that $f_{\mathcal{S}\mu}$ is a continuous function, which is bounded and bounded away from zero, in restriction to compact sets.

Remark 27. For the analysis of the regularity of (47) it is convenient to parametrize the nilpotent unit ball via the exponential map, as a function defined on the whole fiber in the cotangent space. In other words, we do not restrict to the set $\{\|u_0\| = 1\}$ and define for every $\lambda_0 = (u_0, r) \in \mathbb{R}^6$

$$\mathcal{E}(\lambda_0) = \pi(e^{\vec{H}}(\lambda_0)),$$

where H is the Hamiltonian defined in (12) and $e^{t\vec{H}}$ denotes the flow in T^*M of the Hamiltonian vector field associated with H. Using the homogeneity property $H(c\lambda) = c^2 H(\lambda)$ for all $c > 0$, we have that

$$e^{\vec{H}}(s\lambda) = e^{s\vec{H}}(\lambda) \qquad \forall\, s > 0.$$

In other words, we can recover the geodesic on the manifold with initial covector λ_0 as the image of the ray $\{t\lambda_0, t \in [0, 1]\} \subset T^*_{q_0}M$ that joins the origin to λ_0,

$$\mathcal{E}(t\lambda_0) = \pi(e^{\vec{H}}(t\lambda_0)) = \pi(e^{t\vec{H}}(\lambda_0)) = \gamma(t).$$

Due to the previous analysis and thanks to Remark 27, we can express the volume of the unit ball of the nilpotent approximation at a point p as follows:

$$(48) \qquad V(p) = \int_0^{2\pi} \int_0^{A(\theta,p)} \int_B J_{\mathcal{E}}(u, \theta, r, p)\, du\, dr\, d\theta,$$

where $J_{\mathcal{E}}$ is the Jacobian of the exponential map starting from p, expressed in the new variables, $B = \{u_0 = (u_1, \ldots, u_4), \|u_0\| \le 1\}$ is the 4D unit ball, and

$$A(\theta, p) = \frac{2\pi}{\max \sigma(L_\theta(p))}.$$

The problem of the regularity of the function (47) is then reduced to the regularity of the function

$$p \mapsto V(p),$$

---

[2]A sub-Riemannian manifold $(M, \Delta, \mathbf{g})$ is said to be *regular* if defining the distributions $\Delta^1 := \Delta$, $\Delta^{i+1} := \Delta^i + [\Delta^i, \Delta]$, the dimension of Δ^i_q does not depend on the point for all $i \geq 1$.

where p is a (6-dimensional) parameter. Since the family of sub-Riemannian metrics is smooth with respect to p, the exponential map smoothly depends on the parameter p. As a consequence the integrand in (48), being the Jacobian of the exponential map, is a smooth function of its variables.

In addition, the function $p \mapsto A(\theta, p)$ is Lipschitz, being the inverse of the "maximum moduli of eigenvalues" function, which is Lipschitz (see [19]). In particular $A(\theta, p)$ admits bounded first derivative almost everywhere with respect to (θ, p).

DEFINITION 28. *Define the following sets:*
- *Σ is the set of p such that there exists θ for which $L_\theta(p)$ has a double eigenvalue.*
- *Σ_0 is the set of p such that there exists a finite number of θ for which $L_\theta(p)$ has a double eigenvalue.*
- *Σ_∞ is the set of p such that for all θ, $L_\theta(p)$ has a double eigenvalue.*

Thanks to Lemma 32, for a generic sub-Riemannian metric, the set of points $p \in \Sigma_\infty$ is a union of isolated points. Moreover, due to the expression (54) of the eigenvalues in terms of the quaternions given in Appendix 4.1, the fact that L_θ has a double eigenvalue for all θ is written as $\|q(\theta)\| = 0$ for all θ (or the same for $\widehat{q}$). This condition is equivalent to the equation $\|q(\theta)\|^2 = 0$, which is analytic in θ. In particular this equation, if it is not identically satisfied, has a finite number of solutions in $[0, 2\pi]$.

Remark 29. Notice that the expression (54) for the eigenvalues provides a crucial obstruction for the generalization of the result to $m > 4$. From this it follows that for a generic sub-Riemannian metric, the set of critical points Σ is the disjoint union $\Sigma = \Sigma_0 \cup \Sigma_\infty$. Moreover, the set of points where $p \mapsto A(\cdot, p)$ is not smooth is contained in Σ.

Let us write the volume function, depending on the parameter p, as follows:

$$(49) \qquad V(p) = \int_{\theta=0}^{2\pi} \int_{r=0}^{A(\theta,p)} f(\theta, r, p) dr d\theta,$$

where we denote by

$$f(\theta, r, p) = \int_B J_{\mathcal{E}}(u, \theta, r, p) du.$$

Recall that f is smooth as a function of all its variables, while $A(\theta, p)$ is Lipschitz with respect to the parameters (θ, p). In particular it has bounded derivatives.

We want to prove that V is C^1 at any point p_0. To this extent, let us write

$$V(p) = \int_0^{2\pi} \int_0^{A(\theta,p_0)} f(\theta, r, p) dr d\theta + \int_0^{2\pi} \int_{A(\theta,p_0)}^{A(\theta,p)} f(\theta, r, p) dr d\theta.$$

The function

$$p \mapsto \int_0^{2\pi} \int_0^{A(\theta,p_0)} f(\theta, r, p) dr d\theta$$

is always smooth since it is the integral of a smooth function (with respect to p) on a fixed domain. Denote now

$$(50) \qquad W(p) := \int_{\theta=0}^{2\pi} \int_{A(\theta,p_0)}^{A(\theta,p)} f(\theta, r, p) dr d\theta.$$

We are left to prove that W is C^1 around p_0. Notice that by definition, $W(p_0) = 0$.

Assume that $p_0 \notin \Sigma$. Then, since both functions A and f in (50) are smooth, W is C^1 at p_0 and the derivative at a point p (in a neighborhood of p_0) is computed as follows:

$$(51) \quad \frac{\partial W}{\partial p_i}(p) = \int_0^{2\pi} \int_{A(\theta,p_0)}^{A(\theta,p)} \frac{\partial f}{\partial p_i}(\theta, r, p)drd\theta + \int_0^{2\pi} \frac{\partial A}{\partial p_i}(\theta, p)f(\theta, A(\theta, p), p)drd\theta.$$

Assume now that $p_0 \in \Sigma_0$. The first term in (51) is continuous. Moreover, since at p_0 there is only a finite number of θ such that L_θ has double eigenvalue we have

$$(52) \quad \frac{\partial A}{\partial p_i}(\theta, p) \underset{p \to p_0}{\longrightarrow} \frac{\partial A}{\partial p_i}(\theta, p_0) \qquad \text{a.e. } \theta \in [0, 2\pi].$$

Since $\frac{\partial A}{\partial p_i}$ is bounded and f is smooth, by Lebesgue's dominated convergence we have that the second term is also continuous.

Finally, consider the case when $p_0 \in \Sigma_\infty$. Since p_0 is an isolated point, the partial derivatives are defined and continuous in $N_{p_0} \setminus \{p_0\}$, where N_{p_0} is a neighborhood of p_0. We claim that

$$(53) \quad \frac{\partial W}{\partial p_i}(p) \longrightarrow 0 \qquad \text{when} \quad p \to p_0 \in \Sigma_\infty.$$

Indeed, by definition of Σ_∞, the cut time $A(\theta, p_0)$ coincides with the conjugate time, i.e., it satisfies the identity

$$J_\mathcal{E}(u, \theta, A(\theta, p_0), p_0) = 0 \qquad \forall \theta \in [0, 2\pi].$$

From this it follows that $f(\theta, A(\theta, p), p) \to 0$ for all θ, which easily implies (53). From the fact that W is continuous in N_{p_0}, C^1 in $N_{p_0} \setminus \{p_0\}$ and has partial derivatives tending to zero for $p \to p_0$, it follows that W is C^1 in N_{p_0}.

4. Appendix.

4.1. Quaternions. The Lie algebra $so(4)$ of 4×4 skew symmetric matrices is the direct sum

$$so(4) = Q \oplus \widehat{Q},$$

where Q is the space of pure quaternions and $\widehat{Q}$ is the set of pure skew quaternions. The space Q (resp., $\widehat{Q}$) is generated by the three matrices i, j, k (resp., $\widehat{i}, \widehat{j}, \widehat{k}$)

$$i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\widehat{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \widehat{j} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \widehat{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

If we endow $so(4)$ with the Hilbert–Schmidt scalar product

$$\langle L_1, L_2 \rangle = \frac{1}{4} \text{trace}(L_1' L_2),$$

then $i, j, k, \widehat{i}, \widehat{j}, \widehat{k}$ is an orthonormal basis.

The eigenvalues ω_1, ω_2 of $A = q + \widehat{q}$ satisfy

$$(54) \qquad -(\omega_{1,2})^2 = (\|q\| \pm \|\widehat{q}\|)^2.$$

As a consequence an element $A \in so(4)$ has a double eigenvalue if and only if $A \in Q \cup \widehat{Q}$.

Also pure quaternions and pure skew quaternions commute:

$$(55) \qquad [q, \widehat{q}] = 0, \quad q \in Q, \quad \widehat{q} \in \widehat{Q}.$$

4.2. Transversality lemma. Let S be the set of (m, n) smooth sub-Riemannian metrics over M, equipped with the Whitney topology. Due to the $\mathcal{C}^\infty$ structure, we have the existence of smooth bump functions and the results in this section are essentially local. Then we can assume that S is the set of m-tuples $F = (X_1, \dots X_m)$ of smooth independent vector fields on some open subset M of $\mathbb{R}^n$, satisfying

$$T_q M = \Delta_q + [\Delta, \Delta]_q \quad \text{for every } q \in M.$$

The vector fields $X_1, \dots, X_m$ form an orthonormal basis for the sub-Riemannian metric **g** they specify.

Let B be the bundle over M whose fiber at $q \in M$ is the variety of k-dimensional vector space of **g**-skew symmetric endomorphisms of Δ_q.

Let us consider the mapping

$$\begin{aligned} \rho : S \times M &\rightarrow B, \\ (F, q) &\rightarrow \mathcal{L}_q^F, \end{aligned}$$

where $\mathcal{L}_q^F$ has been defined in Remark 1. It is clear that ρ is $\mathcal{C}^\infty$.

Let us fix a point $F_0 \in S$, a point $q_0 \in M$, and coordinates (x, y) in M such that $q_0 = (0, 0)$ and the nilpotent approximation of F_0 reads in control form

$$(56) \qquad \begin{cases} \dot{x}_i = u_i, & i = 1, \dots, m, \\ \dot{y}_1 = \dfrac{1}{2} x' L_1 u, \\ \quad \vdots \\ \dot{y}_k = \dfrac{1}{2} x' L_k u. \end{cases}$$

In the coordinates y, the space $\mathcal{L}_q^{F_0}$ is the vector subspace spanned by the matrices $L_1, \dots L_k$. We have a natural gradation in formal power series of (x, y) induced by setting that the x_i have weight 1 and the y_i have weight 2. This induces a formal gradation on formal vector fields on M_{q_0} in which $\frac{\partial}{\partial x_i}$ have weight -1 and $\frac{\partial}{\partial y_i}$ have weight -2. The vector fields of the nilpotent approximation (56) have weight -1.

In control form the sub-Riemannian metric F_0 itself reads

$$(57) \qquad \begin{pmatrix} \dot{x} \\ \dot{y}_1 \\ \vdots \\ \dot{y}_k \end{pmatrix} = \begin{pmatrix} u \\ \frac{1}{2} x' L_1 u \\ \vdots \\ \frac{1}{2} x' L_k u \end{pmatrix} + H,$$

where H is a term of order > -1 as a formal u-dependent vector field. Then we take a smooth bump function $b(x, y)$ which is compactly supported in M and which is 1

in a neighborhood of $q_0 = (0,0)$. We consider the affine space $\mathcal{A}$ of variations of F_0 of the form

$$
(58) \qquad
\begin{pmatrix} \dot{x} \\ \dot{y}_1 \\ \vdots \\ \dot{y}_k \end{pmatrix}
=
\begin{pmatrix} u \\ \frac{1}{2}x'L_1u \\ \vdots \\ \frac{1}{2}x'L_ku \end{pmatrix}
+
\begin{pmatrix} 0 \\ \frac{1}{2}x'\delta L_1u \\ \vdots \\ \frac{1}{2}x'\delta L_ku \end{pmatrix}
b(x,y) + H.
$$

This defines new sub-Riemannian metrics $F_0 + \delta F$. Since S is open in the set of all rank m smooth sub-Riemannian metrics over M, then, for a small perturbation δF, we have $F_0 + \delta F \in S$.

To show that ρ_{q_0}, defined by $\rho_{q_0}(F) := \rho(q_0, F)$, is a submersion at q_0 on the fiber B_{q_0}, it is enough to observe that

$$
\begin{aligned}
\widehat{\rho}_{q_0} : \mathcal{A} &\rightarrow so(4)^k, \\
\delta F &\mapsto (L_1 + \delta L_1, \ldots, L_k + \delta L_k)
\end{aligned}
$$

is an affine submersion. Then we have proved the following lemma.

LEMMA 30. *The map ρ is a submersion.*

Now let us restrict to the (4,6) case.

DEFINITION 31. *We say that a point $q_0 \in M$ is critical for a sub-Riemannian metric F if all elements of the subspace $\mathcal{L}_{q_0}^F$ (from Remark 1) have a double eigenvalue.*

By formula (54), this means that whatever the coordinates y, the matrices $L_1(q_0)$, $L_2(q_0)$ both belong either to Q or to $\widehat{Q}$.

The dimension d_1 of the fiber of the bundle B is the dimension of the Grassmannian $G(2,6)$ of 2-subspaces of $\mathbb{R}^6$, i.e., $d_1 = 8$.

The dimension of the set of pairs L_1, L_2 that both belong to Q (resp., $\widehat{Q}$) is the dimension d_2 of the Grassmannian $G(2,3)$, i.e., $d_2 = 2$.

Let us define now the partially algebraic "wrong set" $W \subset B$ as follows: the fiber W_{q_0} is the set of 2-subspaces of the **g**-skew symmetric endomorphisms of Δ_{q_0}, whose elements have a double eigenvalue. The codimension of W in B is $d_1 - d_2 = 6$.

The next lemma follows from Lemma 30 and a noncompact version of Abraham's parametric transversality theorems [1].

LEMMA 32 (the (4,6) case). *The set of sub-Riemannian metrics that have only isolated critical points is open and dense in S.*

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