THE GHS AND OTHER INEQUALITIES FOR THE TWO-STAR MODEL

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Abstract. We consider the two-star model, a family of exponential random graphs indexed by two real parameters, h and α, that rule respectively the total number of edges and the mutual dependence between them. Borrowing tools from statistical mechanics, we study different classes of correlation inequalities for edges, that naturally emerge while taking the partial derivatives of the (finite size) free energy. In particular, under a mild hypothesis on the parameters, we derive first and second order correlation inequalities and then prove the so-called GHS inequality. As a consequence, the average edge density turns out to be an increasing and concave function of the parameter h, at any fixed size of the graph.

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1. Introduction

Correlation inequalities are an important tool in equilibrium statistical mechanics. They are used to estimate moments and correlations in ferromagnetic systems, allowing in turn to obtain analyticity properties of some physical observables (such as magnetization and susceptibility) and to prove/disprove the presence of a phase transition. Among these inequalities, we find the Griffiths, Hurst and Sherman (GHS) inequality, that rules the three-particle interactions and is mainly known for providing convexity properties of relevant functionals. As the Griffiths, Kelley and Sherman (GKS) inequality [12, 15], it was firstly proved for the classical Ising model, to show that the average magnetization is a concave function of the positive external field [14], and then extended to general classes of even ferromagnets that can be derived out of the Ising model [8, 13, 5, 19].

However, the aforementioned result is only one of the different implications entailed by the GHS inequality. For example, it has been used to characterize possible phase transitions, to prove monotonicity of correlation length, and to derive critical exponent inequalities for the Ising model on $\mathbb{Z}^d$; to obtain monotonicity of mass gap and to estimate coupling constants in φ^4 field theory; or also to show convexity-preserving properties of certain differential equations and diffusion processes. For further details we refer the reader to [9] and references therein.

In the present paper we consider a family of exponential random graphs known as two-star model [20]. Specifically, we consider a Gibbs probability measure on the set of all simple graphs on n vertices, whose Hamiltonian depends on the densities of edges and two-star graphs. Our goal is to study some correlation inequalities for such a model, with a particular focus on the GHS inequality.

In comparison with ferromagnetic systems, the major difference is that the Gibbs measure of our system, being supported on $\{0,1\}^{\binom{n}{2}}$, does not enjoy $\mathbb{Z}_2$-simmetry. As a consequence, although the positivity of the support of the measure allows to easily deduce positivity of the moments and derive the Fortuin, Kasteleyn and Ginibre (FKG) inequality [10], higher order correlations are non-trivial to analyze.

The manuscript is organized as follows. In Section 2 we introduce the two-star model and we define the corresponding free energy function. Moreover, we briefly recall some recent results about its asymptotic behavior, including the characterization of the phase diagram and some limit theorems for the edge density. Section 3 is devoted to correlation inequalities and it collects our main results. We first provide the formal definition of the aforementioned FKG, GKS and GHS inequalities in the context of a two-star model with generalized parameters (see Eq. (3.1)). In Subsection 3.1 we show that the FKG and GKS inequalities hold for this model, under a mild hypothesis on the coefficients, and then we derive some preliminary results used afterwards in the proof of the GHS inequality, that is the core of the present work (see Theorem 3.9). The statement of this result is given in Subsection 3.2 together with its proof. This is mainly based on ideas from [16], where an alternative and simplified strategy of the original proof has been devised. We conclude the paper bringing back the results to the classical two-star model, and making a few comments about some immediate consequences of the derived correlation inequalities.

2. Model and background

2.1. **Two-star model.** Let us consider the set $\mathcal{G}_n$ of all simple graphs on n labeled vertices that are identified with the elements of the set $[n] = \{1, 2, 3, \dots, n\}$. We define a probability distribution on $\mathcal{G}_n$ by means of the *homomorphism densities* of the subgraphs of the graph. Specifically, if $G \in \mathcal{G}_n$ and H is a given simple subgraph, we define

$$t(H, G) := \frac{|\hom(H, G)|}{|V(G)|^{|V(H)|}}, \tag{2.1}$$

i.e. the probability that a random mapping $V(H) \mapsto V(G)$ from the vertex set of H to the vertex set of G is edge-preserving.

Fix $j \in \mathbb{N}$. For $k = 1, \dots, j$, let H_k denote a k-star graph; an undirected graph with one *root* vertex and k other vertices connected with the root, and otherwise disconnected. The so-called j-star model is a family of exponential random graphs that, for any choice of the parameter $\boldsymbol{\beta} = (\beta_k)_{k=1,\dots,j}$, with $\beta_k \in \mathbb{R}$, is identified by the Gibbs probability density

$$\mu_{n;\boldsymbol{\beta}}(G) = \frac{\exp\left(H_{n,\boldsymbol{\beta}}(G)\right)}{Z_{n;\boldsymbol{\beta}}} \quad \text{for } G \in \mathcal{G}_n. \tag{2.2}$$

The function $H_{n,\boldsymbol{\beta}}$, called Hamiltonian, is given by

$$H_{n;\boldsymbol{\beta}}(G) = n^2 \sum_{k=1}^{j} \beta_k t(H_k, G) \tag{2.3}$$

and the normalizing factor

$$Z_{n;\boldsymbol{\beta}} = \sum_{G \in \mathcal{G}_n} \exp\left(H_{n;\boldsymbol{\beta}}(G)\right) \tag{2.4}$$

is the *partition function*.

In the present setting we focus on the *two-star model*, characterized by a Gibbs measure that depends only on the densities of edges and two-star graphs. Under this assumption, the measure can be conveniently expressed as follows.

Let $\mathcal{E}_n$ denote the edge set of the complete graph on n vertices, with elements labeled from 1 to $\binom{n}{2}$. If $i,\, j \in \mathcal{E}_n$ are neighboring edges, we write $i \sim j$ and we identify the unordered pair $\{i,j\}$ with the resulting two-star graph, that will be called *wedge* $\{i,j\}$ in short. Let $\mathcal{W}_n := \{\{i,j\} : i,j \in \mathcal{E}_n,\, i \sim j\}$ be the set of wedges of $\mathcal{E}_n$, and set $\mathcal{A}_n := \{0,1\}^{|\mathcal{E}_n|}$, $|\cdot|$ being the cardinality of a set.

Notice that there is a one-to-one correspondence between graphs $G \in \mathcal{G}_n$ and elements $x = (x_i)_{i \in \mathcal{E}_n} \in \mathcal{A}_n$ so that, if G corresponds to x, it holds that

$$t(H_1, G) = \frac{2}{n^2} \sum_{i \in \mathcal{E}_n} x_i \qquad t(H_2, G) = \frac{2}{n^3} \sum_{\{i,j\} \in \mathcal{W}_n} x_i x_j + \frac{2}{n^3} \sum_{i \in \mathcal{E}_n} x_i\,, \tag{2.5}$$

with H_1 an edge and H_2 a wedge.

Hence, we may look at the Hamiltonian of the two-star model as a function on $\mathcal{A}_n$ defined by

$$H_{n;\beta_1,\beta_2}(x) = \frac{2\beta_2}{n} \sum_{\{i,j\} \in \mathcal{W}_n} x_i x_j + 2\left(\beta_1 + \frac{\beta_2}{n}\right) \sum_{i \in \mathcal{E}_n} x_i\,. \tag{2.6}$$

Notice that this Hamiltonian is asymptotically equivalent to

$$H_{n;\alpha,h}(x) = \frac{\alpha}{n} \sum_{\{i,j\} \in \mathcal{W}_n} x_i x_j + h \sum_{i \in \mathcal{E}_n} x_i\,, \tag{2.7}$$

where, for convenience, we have set $h = 2\beta_1$ and $\alpha = 2\beta_2$. In the following, we will focus on the corresponding two-star model, having Gibbs density on $\mathcal{A}_n$ given by

$$\mu_{n;\alpha,h}(x) = \frac{\exp\left(H_{n;\alpha,h}(x)\right)}{Z_{n;\alpha,h}} \quad \text{with} \quad Z_{n;\alpha,h} = \sum_{x \in \mathcal{A}_n} \exp\left(H_{n;\alpha,h}(x)\right)\,. \tag{2.8}$$

Accordingly, we will denote the related measure and expectation by $\mathbb{P}_{n;\alpha,h}$ and $\mathbb{E}_{n;\alpha,h}$, respectively.

2.2. **Free energy.** The *free energy* is a key function in the context of statistical mechanics, as it encodes most of the asymptotic properties of the system. Specifically, the finite and infinite size free energies associated with (2.7) are

$$f_{n;\alpha,h} := \frac{1}{n^2} \ln Z_{n;\alpha,h} \quad \text{and} \quad f_{\alpha,h} := \lim_{n\to+\infty} f_{n;\alpha,h} \,. \tag{2.9}$$

To understand the important role of the free energy, we first observe that its partial derivatives w.r.t. α and h respectively give the average edge and wedge densities of the model. More precisely, if we denote by E_n the number of edges of the graph G, and by W_n the number of wedges of G, we get

$$\frac{\partial f_{n;\alpha,h}}{\partial h} = \frac{\mathbb{E}_{n;\alpha,h}\left(E_n\right)}{n^2} \quad \text{and} \quad \frac{\partial f_{n;\alpha,h}}{\partial \alpha} = \frac{\mathbb{E}_{n;\alpha,h}\left(W_n\right)}{n^3} \,. \tag{2.10}$$

The characterization of the infinite size free energy, together with its analytical properties, then provides a relevant tool to infer some structural properties of the graph.

As an application of Theorems 4.1 and 6.4 in [6], for any $(\alpha, h) \in \mathbb{R}^2$ we have that

$$f_{\alpha,h} = \sup_{0\le u\le 1} \left(\frac{\alpha u^2}{2} + \frac{hu}{2} - \frac{1}{2} I(u) \right) = \frac{\alpha (u^*)^2}{2} + \frac{hu^*}{2} - \frac{1}{2} I(u^*), \tag{2.11}$$

where $I(u) = u \ln u + (1-u)\ln(1-u)$ and $u^* = u^*(\alpha, h)$ is a maximizer that solves the fixed-point equation

$$\frac{e^{2\alpha u+h}}{1+e^{2\alpha u+h}} = u. \tag{2.12}$$

Depending on the parameters, Eq (2.12) can have more than one solution at which the supremum in (2.11) is attained. Having multiplicity of optimizers translates in the possibility of having limiting graphs with very different edge densities.

2.3. **Phase diagram.** We collect here the relevant features of the phase diagram of the two-star model, that can be obtained as a special case of some of the results in [21].

The infinite size free energy $f_{\alpha,h}$ is well-defined in $\mathbb{R}^2$. Moreover, it is analytic in the whole plane except for a continuous critical curve

$$\mathcal{M} := \{(\alpha, h) \in (\alpha_c, +\infty) \times (-\infty, h_c) : h = q(\alpha)\} \,,$$

starting at the critical point $(\alpha_c, h_c) = (2, -2)$ and contained in the cone $\alpha > 2$, $h < -2$. In particular, the system undergoes a first order phase transition across the curve and a second order phase transition at the critical point (see [21], Thms. 2.1 & 2.2). The scalar problem (2.11) admits one solution in the uniqueness the region $\mathcal{U} := \mathbb{R}^2 \setminus \mathcal{M}$; while, it has two solutions along the curve $\mathcal{M}$ (see [21], Prop. 3.2). A qualitative graphical representation of the phase diagram is provided in Fig. 2.1.

An analogous analysis has been performed in a sparse regime in [1], in the directed graph case in [2], and for a mean-field version of the model in [3].

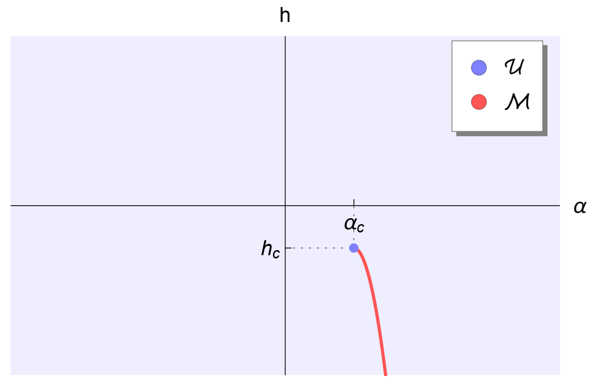

FIGURE 2.1. Phase space (α, h) for the two-star model (2.7). The blue region, that includes the critical point, is the uniqueness region $\mathcal{U}$ for the maximization problem (2.11); whereas, the red curve corresponds to the critical curve $\mathcal{M}$ along which (2.11) admits two solutions.

2.4. **Limiting distribution for the edge density.** We summarize some results on the asymptotic behavior of the edge density of the two-star model. By retracing the proofs in [4], we can obtain the following strong law of large numbers and standard central limit theorem:

$$\frac{2E_n}{n^2} \xrightarrow[n\to\infty]{\text{a.s.}} u^*(\alpha, h) \qquad \text{w.r.t. } \mathbb{P}_{n;\alpha,h}, \text{ for } (\alpha, h) \in \mathcal{U} \tag{2.13}$$

and

$$\sqrt{2}\,\frac{E_n - \mathbb{E}_{n;\alpha,h}(E_n)}{n} \xrightarrow[n\to\infty]{\text{d}} \mathcal{N}(0, v(\alpha, h)) \qquad \text{w.r.t. } \mathbb{P}_{n;\alpha,h}, \text{ for } (\alpha, h) \in \mathcal{U}\backslash\{(\alpha_c, h_c)\}, \tag{2.14}$$

where $\mathcal{N}(0, v(\alpha, h))$ is a centered Gaussian distribution with variance $v(\alpha, h) := \partial_h u^*(\alpha, h)$, being u^* the unique maximizer of (2.11). A further result can be also given in the multiplicity region; for all $(\alpha, h) \in \mathcal{M}$, it holds

$$\frac{2E_n}{n^2} \xrightarrow[n\to\infty]{\text{d}} \kappa\delta_{u_1^*(\alpha,h)} + (1-\kappa)\delta_{u_2^*(\alpha,h)} \qquad \text{w.r.t. } \mathbb{P}_{n;\alpha,h},$$

where u_1^*, u_2^* solve the maximization problem in (2.11) and $0 < \kappa < 1$ is a suitable (unknown) constant.

Similar limit theorems are obtained, with different techniques, in [18], where also results on the partial sum of the degrees can be found.

3. Correlation inequalities

In statistical mechanics the study of correlations between particles, so as the analysis of local functions, is often performed with the help of two important inequalities, both related to the sign of the derivatives of the free energy; the **GKS inequality** and the **GHS inequality** (see [11, 14, 15, 16] and references therein for further details). We aim at deriving the analogs of these two inequalities for our reference measure $\mu_{n;\alpha,h}$, given in (2.8).

To understand the connection between the GKS inequality and the sign of the derivatives of the free energy, we introduce a slightly more general setting.

Let $\boldsymbol{\alpha} = (\alpha_{ij})_{i,j\in\mathcal{E}_n}$ and $\boldsymbol{h} = (h_i)_{i\in\mathcal{E}_n}$ be two collections of real numbers (we write $\boldsymbol{\alpha} \geq 0$ (resp. $\boldsymbol{h} \geq 0$) as a shortcut for $\alpha_{ij} \geq 0$ (resp. $h_i \geq 0$) for all $i, j \in \mathcal{E}_n$).

For $x \in \mathcal{A}_n$, we define the Hamiltonian

$$H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x) = \frac{1}{n} \sum_{\{i,j\}\in\mathcal{W}_n} \alpha_{ij} x_i x_j + \sum_{i\in\mathcal{E}_n} h_i x_i \,. \tag{3.1}$$

In analogy with (2.8) and (2.9), we denote by $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ the Gibbs measure obtained from (3.1), by $\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ the corresponding expectation, and we set $f_{n;\boldsymbol{\alpha},\boldsymbol{h}} := \frac{1}{n^2} \ln Z_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ to be the finite size free energy. Observe that we recover the Hamiltonian (2.7) and the related Gibbs measure $\mu_{n;\alpha,h}$ by setting $\alpha_{ij} \equiv \alpha$, for all $i, j \in \mathcal{E}_n$, and $h_i \equiv h$, for all $i \in \mathcal{E}_n$.

Let $A \subseteq \mathcal{E}_n$ be a given subset of edges. The GKS inequality deals with expectations and covariances of random variables of the type $x_A := \prod_{i\in A} x_i$, with the convention that $x_\emptyset = 1$.

Definition 3.1 (GKS inequality). *The Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ on $\mathcal{A}_n$ satisfies the GKS inequality if, for all $A, B \subseteq \mathcal{E}_n$,*

$$\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_A x_B) \geq \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_A) \cdot \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_B) \,. \tag{3.2}$$

Remark 3.2. *Notice that, by choosing $A = \{i\}$ and $B = \{j\}$, with $i \neq j$, from the GKS inequality it follows that x_i and x_j are positively correlated under $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$.*

A useful link between the correlations of the system and the partial derivatives of the free energy w.r.t. the parameters h_i's is provided by the MacLaurin expansion of the log moment generating function of $x \in \mathcal{A}_n$.

The coefficients of this expansion are the so-called *Ursell functions*, that are formally defined, for $\ell \in \mathbb{N}$ and any choice of $i_1, \ldots, i_\ell \in \mathcal{E}_n$, by

$$u_\ell(i_1, \ldots, i_\ell) := n^2 \frac{\partial^\ell}{\partial h_{i_1} \ldots \partial h_{i_\ell}} f_{n;\boldsymbol{\alpha},\boldsymbol{h}} \,. \tag{3.3}$$

For instance, this yields

$$u_1(i) = \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i), \tag{3.4}$$

$$u_2(i,j) = \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j) - \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i)\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_j), \tag{3.5}$$

$$\begin{aligned} u_3(i,j,k) = {} & \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j x_k) - \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i)\, \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_j x_k) - \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_j)\, \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_k) \\ & - \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_k)\, \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j) + 2\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i)\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_j)\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_k). \end{aligned} \tag{3.6}$$

While the GKS inequality implies $u_2(i,j) \geq 0$, giving positive correlation between the random variables x_i and x_j, the GHS inequality concerns the sign of the Ursell function $u_3(i,j,k)$.

Definition 3.3 (GHS inequality)**.** *The Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ on $\mathcal{A}_n$ satisfies the GHS inequality if, for all $i,j,k \in \mathcal{E}_n$, $u_3(i,j,k) \le 0$ or, equivalently, if*

$$\begin{aligned}(3.7)\quad & \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j x_k) - \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i)\,\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_j x_k) - \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_j)\,\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_k) \\ & \quad - \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_k)\,\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j) + 2\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_i)\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_j)\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x_k) \le 0.\end{aligned}$$

Observe that, in our case, $u_1(i) \ge 0$ trivially, due to the fact that $x_i \in \{0,1\}$. The rest of the section is devoted to proving $u_2(i,j) \ge 0$ and $u_3(i,j,k) \le 0$.

3.1. **The FKG and GKS inequalities.** We start with a preliminary result, the FKG inequality, that will help us in deriving the more advanced inequalities (3.2) and (3.7).

We first show that the measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ on $\mathcal{A}_n$ satisfies a proper lattice condition. Recall that $\mathcal{A}_n$ is partially ordered by

$$(3.8)\qquad x \le y \quad \text{if} \quad x_i \le y_i \quad \text{for all } i \in \mathcal{E}_n.$$

Moreover, given two configurations $x, y \in \mathcal{A}_n$, the (pointwise) maximum and minimum configurations are defined as

$$(x \vee y)(i) := \max\{x_i, y_i\} \quad \text{and} \quad (x \wedge y)(i) := \min\{x_i, y_i\},$$

for all $i \in \mathcal{E}_n$. The following property holds true.

Lemma 3.4. *If $\boldsymbol{\alpha} \ge 0$, then the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ fulfills the FKG lattice condition*

$$(3.9)\qquad \mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x \vee y)\,\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x \wedge y) \ge \mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x)\,\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}(y) \quad \textit{for } x, y \in \mathcal{A}_n.$$

Proof. For a configuration $z \in \mathcal{A}_n$, let $E_z := \{i \in \mathcal{E}_n : z_i = 1\}$, namely the set of edges present in z. We have $E_{x\vee y} = E_x \cup E_y$ and $E_{x\wedge y} = E_x \cap E_y$. Observe that

- the edges in the configuration $x \vee y$ are the edges the configurations x and y have in common, the edges present in configuration x only and those present in configuration y only;
- the edges in the configuration $x \wedge y$ are the edges the configurations x and y have in common;
- the wedges in the configuration $x \vee y$ are the wedges the configurations x and y have in common, the wedges present in configuration x (resp. configuration y) only and the wedges you may create by superimposing the edges of the two configurations;
- the wedges in the configuration $x \wedge y$ are the wedges the configurations x and y have in common.

Therefore, verifying that (3.9) is satisfied reduces to show the validity of the inequality

$$(3.10)\qquad \exp\left\{\frac{1}{n}\sum_{\{i,j\}\in E} \alpha_{ij} x_i x_j\right\} \ge 1,$$

where

$$E = \left\{\{i,j\} : \{i,j\} \subset E_{x\vee y} \text{ is a wedge and } \{i,j\} \left[\begin{array}{c} \not\subset E_x \\ \not\subset E_y \end{array}\right.\right\}.$$

The conclusion follows as $\boldsymbol{\alpha} \geq 0$ by assumption. ■

An immediate consequence of Lemma 3.4 is the positive correlation of increasing random variables. Specifically, if f and g are increasing functions on $\mathcal{A}_n$ (i.e., $f(x) \leq f(y)$ if $x \leq y$), then we obtain the **FKG inequality**

$$\mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(fg) \geq \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(f) \cdot \mathbb{E}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(g)\,. \tag{3.11}$$

Corollary 3.5. *If $\boldsymbol{\alpha} \geq 0$, then the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ satisfies the GKS inequality.*

Proof. Notice that for all $A \subseteq \mathcal{E}_n$, the function $x_A = \prod_{i\in A} x_i$ is increasing in $x \in \mathcal{A}_n$. Hence, by applying the FKG inequality (3.11) to the functions $f(x) = x_A$ and $g(x) = x_B$, we immediately derive (3.2). ■

Remark 3.6. *A straightforward adaptation of the arguments of Lemma 3.4 applies to the edge-triangle model with generalized parameters. It is defined by the Hamiltonian*

$$H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x) = \frac{1}{n} \sum_{\{i,j,k\}\in\mathcal{T}_n} \alpha_{ijk} x_i x_j x_k + \sum_{i\in\mathcal{E}_n} h_i x_i\,, \tag{3.12}$$

where $\mathcal{T}_n = \{\{i,j,k\} \subset \mathcal{E}_n : \{i,j,k\}$ is a triangle$\}$, $\boldsymbol{\alpha} = (\alpha_{ijk})_{i,j,k\in\mathcal{T}_n}$ and $\boldsymbol{h} = (h_i)_{i\in\mathcal{E}_n}$. This implies that whenever $\boldsymbol{\alpha} \geq 0$, the FKG and GKS inequalities w.r.t. the Gibbs measure associated to (3.12) *are verified. The statement obviously applies to the classical model with $\alpha_{ijk} \equiv \alpha$, for all $i,j,k \in \mathcal{T}_n$, and $h_i \equiv h$ for all $i \in \mathcal{E}_n$, under the hypothesis $\alpha \geq 0$.*

We now provide two useful consequences of the GKS inequality. To state properly the results we need to introduce a few more notation; we need a suitable "restriction" of the system on a subset.

For $A \subseteq \mathcal{E}_n$, set $\mathcal{W}_A := \{\{i,j\} : i,j \in A\,,\ i \sim j\}$ and define the Hamiltonian

$$H_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x) = \frac{1}{n} \sum_{\{i,j\}\in\mathcal{W}_A} \alpha_{ij} x_i x_j + \sum_{i\in A} h_i x_i\,. \tag{3.13}$$

Let $\mu_{A;\boldsymbol{\alpha},\boldsymbol{h}}$ be the associated Gibbs measure, with normalizing constant $Z_{A;\boldsymbol{\alpha},\boldsymbol{h}}$ (partition function), and let $\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}$ denote the corresponding expectation.

A first consequence of the GKS inequality is a form of monotonicity, with respect to the volume, that can be established for the averages of x_Λ, with $\Lambda \subseteq \mathcal{E}_n$.

Lemma 3.7. *If the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ satisfies the GKS inequality then, for any $\Lambda \subseteq A \subseteq B \subseteq \mathcal{E}_n$,*

$$\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda) \leq \mathbb{E}_{B;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)\,. \tag{3.14}$$

Proof. Observe first that, for all $\Lambda \subseteq A \subseteq \mathcal{E}_n$, the function $\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)$ is non-decreasing in $\boldsymbol{\alpha}$. Indeed, by differentiating $\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}$ w.r.t α_{ij}, we get

$$\frac{\partial}{\partial\alpha_{ij}} \mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda) = \frac{1}{n} \left(\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda x_i x_j) - \mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda) \mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_i x_j) \right) \geq 0\,, \tag{3.15}$$

where the last inequality follows from the GKS inequality.

Now let $\mathcal{W}_{A,B} := \{\{i,j\} : i \in A\,, j \in B \setminus A\,, i \sim j\}$ and, for $s \in [0,1]$, consider the Hamiltonian

$$H_{B;\boldsymbol{\alpha}(s),\boldsymbol{h}}(x) := \frac{1}{n} \sum_{\{i,j\}\in\mathcal{W}_B\setminus\mathcal{W}_{A,B}} \alpha_{ij}x_ix_j + \frac{s}{n} \sum_{\{i,j\}\in\mathcal{W}_{A,B}} \alpha_{ij}x_ix_j + \sum_{i\in B} h_ix_i\,,$$

with corresponding Gibbs measure $\mu_{B;\boldsymbol{\alpha}(s),\boldsymbol{h}}$ and relative average $\mathbb{E}_{B;\boldsymbol{\alpha}(s),\boldsymbol{h}}$. Notice that, if $s = 1$, we obtain the system on the set B, so that $\mathbb{E}_{B;\boldsymbol{\alpha}(1),\boldsymbol{h}}(x_\Lambda) = \mathbb{E}_{B;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)$. Moreover, since $\mathcal{W}_B = \mathcal{W}_A \sqcup \mathcal{W}_{B\setminus A} \sqcup \mathcal{W}_{A,B}$, when $s = 0$, we get

$$H_{B;\boldsymbol{\alpha}(0),\boldsymbol{h}}(x) = H_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x) + H_{B\setminus A;\boldsymbol{\alpha},\boldsymbol{h}}(x)\,.$$

Then $\mu_{B;\boldsymbol{\alpha}(0),\boldsymbol{h}} = \mu_{A;\boldsymbol{\alpha},\boldsymbol{h}} \cdot \mu_{B\setminus A;\boldsymbol{\alpha},\boldsymbol{h}}$ and, as a consequence, being $\Lambda \subseteq A$, we have $\mathbb{E}_{B;\boldsymbol{\alpha}(0),\boldsymbol{h}}(x_\Lambda) = \mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)$. Finally, since $\boldsymbol{\alpha} \mapsto \mathbb{E}_{B;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)$ is a non-decreasing mapping and $\boldsymbol{\alpha}(0) < \boldsymbol{\alpha}(1)$, we conclude

$$\mathbb{E}_{A;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda) = \mathbb{E}_{B;\boldsymbol{\alpha}(0),\boldsymbol{h}}(x_\Lambda) \le \mathbb{E}_{B;\boldsymbol{\alpha}(1),\boldsymbol{h}}(x_\Lambda) = \mathbb{E}_{B;\boldsymbol{\alpha},\boldsymbol{h}}(x_\Lambda)\,,$$

as claimed. ■

A second consequence of the GKS inequality is a comparison between partition functions.

Lemma 3.8. *If the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ satisfies the GKS inequality then, for any $E, F \subseteq \mathcal{E}_n$,*

$$Z_{E;\boldsymbol{\alpha},\boldsymbol{h}} Z_{F;\boldsymbol{\alpha},\boldsymbol{h}} \le Z_{E\cup F;\boldsymbol{\alpha},\boldsymbol{h}} Z_{E\cap F;\boldsymbol{\alpha},\boldsymbol{h}}\,. \tag{3.16}$$

Proof. We follow some ideas developed in [16] to prove an analogous result for Ising spin systems. We set $K_1 := E \cap F$, $K_2 := E \setminus K_1$ and $K_3 := F \setminus K_1$, so that we can express the sets E, F, $E \cup F$ and $E \cap F$ as proper disjoint unions of the subsets K_i's. With this notation, the inequality (3.16) becomes equivalent to

$$L(\boldsymbol{\alpha}, \boldsymbol{h}) := \ln Z_{K_1\cup K_2\cup K_3;\boldsymbol{\alpha},\boldsymbol{h}} - \ln \frac{Z_{K_1\cup K_2;\boldsymbol{\alpha},\boldsymbol{h}} Z_{K_1\cup K_3;\boldsymbol{\alpha},\boldsymbol{h}}}{Z_{K_1;\boldsymbol{\alpha},\boldsymbol{h}}} \ge 0\,. \tag{3.17}$$

Notice that, if there is no interaction between the edges in K_1 and those in K_3, then

$$Z_{K_1\cup K_2\cup K_3;\boldsymbol{\alpha},\boldsymbol{h}} = Z_{K_1\cup K_2;\boldsymbol{\alpha},\boldsymbol{h}} Z_{K_3;\boldsymbol{\alpha},\boldsymbol{h}} \quad \text{and} \quad Z_{K_1\cup K_3;\boldsymbol{\alpha},\boldsymbol{h}} = Z_{K_1;\boldsymbol{\alpha},\boldsymbol{h}} Z_{K_3;\boldsymbol{\alpha},\boldsymbol{h}},$$

that yields $L(\boldsymbol{\alpha}, \boldsymbol{h}) = 0$. To conclude, it suffices to show that the function $L(\boldsymbol{\alpha}, \boldsymbol{h})$ is non-decreasing with respect to the interaction parameter $\boldsymbol{\alpha}$. To this purpose, we consider the change in $L(\boldsymbol{\alpha}, \boldsymbol{h})$, when an interaction of strength α_{ij}, between the edges $i \in K_1$ and $j \in K_3$, is added to the system. By differentiating w.r.t. α_{ij} we get

$$\frac{\partial}{\partial\alpha_{ij}} L(\boldsymbol{\alpha}, \boldsymbol{h}) = \frac{1}{n} \left(\mathbb{E}_{K_1\cup K_2\cup K_3;\boldsymbol{\alpha},\boldsymbol{h}}(x_ix_j) - \mathbb{E}_{K_1\cup K_3;\boldsymbol{\alpha},\boldsymbol{h}}(x_ix_j)\right) \ge 0\,, \tag{3.18}$$

where the last inequality follows from Lemma 3.7. All together this implies that $L(\boldsymbol{\alpha}, \boldsymbol{h}) \ge 0$. ■

3.2. **The GHS inequality.** We are now ready to derive our main result: the GHS inequality for the model associated with the Hamiltonian (3.1).

Theorem 3.9. *If $\boldsymbol{\alpha} \geq 0$, then the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$ satisfies the GHS inequality. In particular, for any choice of $i,j,k \in \mathcal{E}_n$, we have*

$$\frac{\partial^3}{\partial h_i \partial h_j \partial h_k} f_{n;\boldsymbol{\alpha},\boldsymbol{h}} \leq 0 \,. \tag{3.19}$$

The strategy of the proof is based on the trick of introducing a duplicate set of variables. Let $y \in \mathcal{A}_n$ be an independent copy of $x \in \mathcal{A}_n$, with the same energy as in (3.1), and let $\mathbb{E}$ denote the expectation with respect to the joint measure

$$\mu(x,y) := \frac{\exp\left\{H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x) + H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(y)\right\}}{Z^2_{n;\boldsymbol{\alpha},\boldsymbol{h}}}. \tag{3.20}$$

For any $i \in \mathcal{E}_n$, define the variables $z_i = x_i - y_i$ and $v_i = \frac{1}{2}(x_i + y_i)$. Notice that z_i takes value on $\{-1, 0, +1\}$, while v_i takes value on $\left\{0, \frac{1}{2}, 1\right\}$, and that the following equivalence of events holds for all $i \in \mathcal{E}_n$:

$$\{z_i \in \{-1,+1\}\} = \left\{v_i = \frac{1}{2}\right\} \quad \text{and} \quad \{v_i \in \{0,1\}\} = \{z_i = 0\} \,. \tag{3.21}$$

With standard notation, we set $z := (z_i)_{i \in \mathcal{E}_n}$ and $v := (v_i)_{i \in \mathcal{E}_n}$. Moreover, for any given $A \subseteq \mathcal{E}_n$, we define the functions $z_A := \prod_{i \in A} z_i$ and $v_A := \prod_{i \in A} v_i$.

Proposition 3.10. *Let $\boldsymbol{\alpha} \geq 0$. Then, for any $C, D \subseteq \mathcal{E}_n$, it holds true*

$$\mathbb{E}(z_C z_D) \geq \mathbb{E}(z_C)\mathbb{E}(z_D) \,, \tag{3.22}$$

$$\mathbb{E}(z_C v_D) \leq \mathbb{E}(z_C)\mathbb{E}(v_D) \,. \tag{3.23}$$

Remark 3.11. *It is easy to check that the Ursell function $u_3(i,j,k)$, given explicitly in* (3.6)*, can be written as a function of the random variables z_i's and v_i's as*

$$u_3(i,j,k) = \mathbb{E}(z_i z_j v_k) - \mathbb{E}(z_i z_j)\mathbb{E}(v_k) \,. \tag{3.24}$$

The statement of Theorem 3.9 is then a consequence of the inequality (3.23)*. Similarly, it can be shown that Eq.* (3.22) *implies the GKS inequality for the Gibbs measure $\mu_{n;\boldsymbol{\alpha},\boldsymbol{h}}$.*

Proof of Proposition 3.10. We first consider two general functions $\Phi(z)$ and $\Psi(v)$, with $z = (z_i)_{i \in \mathcal{E}_n}$ and $v = (v_i)_{i \in \mathcal{E}_n}$, and we try to express the average $\mathbb{E}(\Phi(z)\Psi(v))$ in a convenient form. Later we will focus on the functions $\Phi(z) = z_C$ and $\Psi(v) = v_D$.

Observe that, due to the identity $x_i x_j + y_i y_j = \frac{1}{2} z_i z_j + 2 v_i v_j$, the exponent of the joint measure (3.20) can be phrased in terms of the variables z and v. It yields

$$H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(x) + H_{n;\boldsymbol{\alpha},\boldsymbol{h}}(y) = \widehat{H^1}_{n;\boldsymbol{\alpha}}(z) + \widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v),$$

where

$$
(3.25)\qquad
\begin{aligned}
\widehat{H^1}_{n;\boldsymbol{\alpha}}(z) &= \frac{1}{2n}\sum_{\{i,j\}\in\mathcal{W}_n}\alpha_{ij}z_iz_j\,,\\
\widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v) &= \frac{2}{n}\sum_{\{i,j\}\in\mathcal{W}_n}\alpha_{ij}v_iv_j + 2\sum_{i\in\mathcal{E}_n}h_iv_i\,.
\end{aligned}
$$

Moreover, by exploiting the constraints (3.21), we can partition the state space of the couple (z,v) in a disjoint union, over subsets $A\subseteq\mathcal{E}_n$, of the sets

(3.26)

$$
\mathcal{S}_A := \left\{(z,v) : z_i = 0, v_i\in\{0,1\}\ \forall i\in A \text{ and } v_i = \frac{1}{2}, z_i\in\{-1,1\}\ \forall i\in A^c\right\}.
$$

Hence, we can write

$$
(3.27)\qquad \mathbb{E}(\Phi(z)\Psi(v)) = \sum_{A\subseteq\mathcal{E}_n}\sum_{(z,v)\in\mathcal{S}_A}\Phi(z)\Psi(v)\frac{\exp\left\{\widehat{H^1}_{n;\boldsymbol{\alpha}}(z)+\widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v)\right\}}{Z^2_{n;\boldsymbol{\alpha},\boldsymbol{h}}}\,.
$$

It is easy to see that if $(z,v)\in\mathcal{S}_A$, and with the same notation introduced in (3.13), we obtain

$$
(3.28)\qquad \widehat{H^1}_{n;\boldsymbol{\alpha}}(z) = \frac{1}{2n}\sum_{\{i,j\}\in\mathcal{W}_{A^c}}\alpha_{ij}z_iz_j\,,\quad \text{with } z_i\in\{-1,1\},\ \forall i\in A^c
$$

and

$$
(3.29)\qquad
\begin{aligned}
\widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v) = {}& \frac{2}{n}\sum_{\{i,j\}\in\mathcal{W}_A}\alpha_{ij}v_iv_j + \sum_{i\in A}\left(2h_i + \frac{1}{n}\sum_{j\in A^c: j\sim i}\alpha_{ij}\right)v_i\\
&+\frac{1}{2n}\sum_{\{i,j\}\in\mathcal{W}_{A^c}}\alpha_{ij} + \sum_{i\in A^c}h_i\,,\qquad \text{with } v_i\in\{0,1\},\ \forall i\in A\,.
\end{aligned}
$$

In particular, on the set $\mathcal{S}_A$,

- the Hamiltonian $\widehat{H^1}_{n;\boldsymbol{\alpha}}(z)$ corresponds to the Hamiltonian of an Ising spin system on the set A^c, with inverse temperature $\boldsymbol{\beta} := \boldsymbol{\alpha}/2n$, magnetic field $\boldsymbol{h}=\mathbf{0}$, and associated Gibbs measure
$$
\mu^{\mathrm{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z) := \frac{e^{H^{\mathrm{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z)}}{Z^{\mathrm{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}}\,.
$$
- the Hamiltonian $\widehat{H^2}_{n;\boldsymbol{\alpha},\boldsymbol{h}}(v)$ corresponds to the two-star Hamiltonian on A given in (3.13), but with parameters $\boldsymbol{\alpha'} := 2\boldsymbol{\alpha}$ and $\boldsymbol{h'} := (h'_i)_{i\in\mathcal{E}_n}$, where $h'_i := 2h_i + \frac{1}{n}\sum_{j\in A^c:j\sim i}\alpha_{ij}$. Indeed, the two Hamiltonians differ only for the constant term $\frac{1}{2n}\sum_{\{i,j\}\in\mathcal{W}_{A^c}}\alpha_{ij} + \sum_{i\in A^c}h_i$ that, being irrelevant in the Gibbs measure, will be neglected. As before, we write $\mu_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}$ for the Gibbs measure related to the Hamiltonian (3.29).

Going back to Eq. (3.27), in view of the previous considerations, it turns out that

$$\mathbb{E}(\Phi(z)\Psi(v)) = \sum_{A\subseteq\mathcal{E}_n} P(A) f^{\Phi}(A) g^{\Psi}(A), \tag{3.30}$$

where, with self-explanatory notation, we set

$$f^{\Phi}(A) := \mathbb{E}^{\text{Is}}_{A^c,\boldsymbol{\beta},\mathbf{0}}(\Phi(z)|_{z_i=0,\forall i\in A}), \qquad g^{\Psi}(A) := \mathbb{E}_{A,\boldsymbol{\alpha'},\boldsymbol{h'}}(\Psi(v)|_{v_i=\frac{1}{2},\forall i\in A^c}) \tag{3.31}$$

and

$$P(A) := \frac{Z^{\text{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}} \cdot Z_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}}{Z^2_{n;\boldsymbol{\alpha},\boldsymbol{h}}}. \tag{3.32}$$

Notice that P is a probability on $\mathcal{E}_n$ by construction. Specializing the identity (3.30) to the functions $\Phi(z) = z_C$ and $\Psi(v) = v_D$, with $C, D \subset \mathcal{E}_n$, we get

$$\mathbb{E}(z_C v_D) = \sum_{A\subseteq\mathcal{E}_n} P(A)\mathbb{E}^{\text{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z_C|_{z_i=0,\forall i\in A})\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_i=\frac{1}{2},\forall i\in A^c}). \tag{3.33}$$

The proof of the two inequalities (3.23) and (3.22) is an immediate application of the FKG inequality relatively to P. Indeed the required hypotheses are fulfilled:

- If $\boldsymbol{\beta} \geq 0$ (equiv. $\boldsymbol{\alpha} \geq 0$), the GKS inequality for ferromagnetic Ising systems [11] guarantees that the function $\mathbb{E}^{\text{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z_C|_{z_i=0,\forall i\in A})$ is non-increasing in A, for any choice of $C \subseteq \mathcal{E}_n$.
- If $\boldsymbol{\alpha'} \geq 0$ (equiv $\boldsymbol{\alpha} \geq 0$), the function $\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_i=\frac{1}{2},\forall i\in A^c})$ is non-decreasing in A, for any choice of $D \subseteq \mathcal{E}_n$. This is a consequence of Lemma 3.7. Indeed, let $A \subseteq B$ and observe that

$$\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_{A^c}=\frac{1}{2}}) = \frac{1}{2^{|D\cap A^c|}}\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_{D\cap A})$$
$$\mathbb{E}_{B;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_{B^c}=\frac{1}{2}}) = \frac{1}{2^{|D\cap B^c|}}\mathbb{E}_{B;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_{D\cap B}).$$

  Since, by hypothesis, $D\cap A \subseteq D\cap B$ and $|D\cap B^c| \leq |D\cap A^c|$, Lemma 3.7 allows to conclude that

$$\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_i=\frac{1}{2},\forall i\in A^c}) \leq \mathbb{E}_{B;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_i=\frac{1}{2},\forall i\in B^c}). \tag{3.34}$$

- If $\boldsymbol{\alpha} \geq 0$, the probability P, defined in (3.32) and acting on subsets of $\mathcal{E}_n$, satisfies the FKG lattice condition, namely

$$P(E)P(F) \leq P(E\cup F)P(E\cap F), \qquad \forall E, F \subseteq \mathcal{E}_n. \tag{3.35}$$

  According to the definition of P, the inequality (3.35) follows if the two inequalities

$$Z^{\text{Is}}_{E;\boldsymbol{\beta},\mathbf{0}} Z^{\text{Is}}_{F;\boldsymbol{\beta},\mathbf{0}} \leq Z^{\text{Is}}_{E\cup F;\boldsymbol{\beta},\mathbf{0}} Z^{\text{Is}}_{E\cap F;\boldsymbol{\beta},\mathbf{0}}$$

  and

$$Z_{E;\boldsymbol{\alpha'},\boldsymbol{h'}} Z_{F;\boldsymbol{\alpha'},\boldsymbol{h'}} \leq Z_{E\cup F;\boldsymbol{\alpha'},\boldsymbol{h'}} Z_{E\cap F;\boldsymbol{\alpha'},\boldsymbol{h'}}$$

  are simultaneously satisfied. The first inequality holds true as a consequence of the GKS inequality for Ising spin systems with $\boldsymbol{\beta} \geq 0$ and magnetic field $\boldsymbol{h} \geq 0$ (see [16], Lemma on p. 90). The second inequality is instead verified thanks to Lemma 3.8.

Thus, as P obeys the FKG lattice condition, and the functions $\mathbb{E}^{\text{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z_C|_{z_i=0,\forall i\in A})$ and $\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_i=\frac{1}{2},\forall i\in A^c})$ are respectively non-increasing and non-decreasing in A, from Eq. (3.33) we get

$$\begin{aligned}\mathbb{E}(z_C v_D) &\leq \sum_{A\subseteq\mathcal{E}_n} P(A)\mathbb{E}^{\text{Is}}_{A^c;\boldsymbol{\beta},\mathbf{0}}(z_C|_{z_i=0,\forall i\in A}) \sum_{A\subseteq\mathcal{E}_n} P(A)\mathbb{E}_{A;\boldsymbol{\alpha'},\boldsymbol{h'}}(v_D|_{v_i=\frac{1}{2},\forall i\in A^c}) \\ &= \mathbb{E}(z_C)\mathbb{E}(v_D)\,,\end{aligned} \tag{3.36}$$

providing inequality (3.23). The inequality (3.22) can be obtained in the same way by setting $\phi(z) = z_C z_D$, so that $g^{\psi}(A) = 1$, and by observing that $f^{\phi}(A)$ is non-decreasing in A, hence giving the reverse inequality. ■

Proof of Theorem 3.9. The statement follows readily from Remark 3.11 and Proposition 3.10. ■

3.3. **The GHS inequality for the two-star model.** Let $\alpha, h \in \mathbb{R}$. Recall that the two-star model is obtained, as a particular case, by setting $\alpha_{ij} \equiv \alpha$, for all $i,j \in \mathcal{E}_n$, and $h_i \equiv h$, for all $i \in \mathcal{E}_n$, in the Hamiltonian (3.1). This means that, whenever $\alpha \geq 0$, the GHS inequality holds true for the Gibbs measure $\mu_{n;\alpha,h}$, given in (2.2).

Observe that, by differentiating the free energy $f_{n;\alpha,h}$ w.r.t. h, we get the following identities in terms of the Ursell functions (3.3):

$$n^2\frac{\partial}{\partial h}f_{n;\alpha,h} = \sum_{i\in\mathcal{E}_n} u_1(i)\,,\quad n^2\frac{\partial^2}{\partial h^2}f_{n;\alpha,h} = \sum_{i,j\in\mathcal{E}_n} u_2(i,j)\,,$$
$$n^2\frac{\partial^3}{\partial h^3}f_{n;\alpha,h} = \sum_{i,j,k\in\mathcal{E}_n} u_3(i,j,k)\,,$$

and so on. Thus, not only the sign of each Ursell function provides a specific correlation inequality between the random variables x_i's, but also it gives a definite sign to a derivative of the free energy.

A direct computation easily shows that, being a variance, the second order partial derivative of $f_{n;\alpha,h}$ w.r.t. h is always non-negative. Thus, proving that $u_2(i,j) \geq 0$ (GKS inequality) is useful to know the covariance between x_i and x_j, but it is somehow irrelevant to the purpose of showing that the free energy is a convex function of h. On the contrary, the GHS inequality ($u_3(i,j,k) \leq 0$) is of particular importance, as it implies that the average edge density (2.10) is a concave function of the parameter h at any fixed size of the graph.

Explicitly, setting $m_n(\alpha,h) := \frac{\mathbb{E}_{n;\alpha,h}(E_n)}{n^2}$ and assuming that $\alpha \geq 0$, from the GKS and GHS inequalities we readily get

$$\frac{\partial m_n(\alpha,h)}{\partial h} = \frac{\partial^2 f_{n;\alpha,h}}{\partial h^2} \geq 0\,,\qquad \frac{\partial^2 m_n(\alpha,h)}{\partial h^2} = \frac{\partial^3 f_{n;\alpha,h}}{\partial h^3} \leq 0\,. \tag{3.37}$$

Understanding the limiting behavior of the above derivatives has then a twofold purpose. On the one hand, it allows to infer properties regarding the edge density and its limiting behavior; for example, the existence of $\lim_{n\to+\infty}\frac{\partial m_n(\alpha,h)}{\partial h}$ is fundamental for proving the standard central limit theorem in (2.14) (see [4]). On the other hand, it is crucial for detecting the occurence of phase transitions, that are generally associated with the emergence of singularities in the infinite size free energy. In particular, one can exploit convergence results on the derivatives of convex functions to guarantee that the limits and the derivatives w.r.t. the external field commute (see [7, Lemma V.7.5]), and then obtain proper regularity conditions. Notice that this procedure can be seen as an alternative approach to the investigation of the hypotheses that allow for the application of the Lee-Yang theorem [17]. However, in this respect, the convexity property (3.37) provides a more specific information that may enter in the characterization of further features of the model.

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