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Heteroclinic connections and Dirichlet problems for a nonlocal functional of oscillation type

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Abstract

We consider an energy functional combining the square of the local oscillation of a one-dimensional function with a double-well potential. We establish the existence of minimal heteroclinic solutions connecting the two wells of the potential. This existence result cannot be accomplished by standard methods, due to the lack of compactness properties. In addition, we investigate the main properties of these heteroclinic connections. We show that these minimizers are monotone, and therefore they satisfy a suitable Euler–Lagrange equation. We also prove that, differently from the classical cases arising in ordinary differential equations, in this context the heteroclinic connections are not necessarily smooth, and not even continuous (in fact, they can be piecewise constant). Also, we show that heteroclinics are not necessarily unique up to a translation, which is also in contrast with the classical setting. Furthermore, we investigate the associated Dirichlet problem, studying existence, uniqueness and partial regularity properties, providing explicit solutions in terms of the external data and of the forcing source, and exhibiting an example of discontinuous solution.

Keywords Heteroclinic connections · Multiple scale problems · Regularity of minimizers

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1 Introduction

One of the most classical problems in ordinary differential equations consists in the study of second-order equations coming from mechanical systems having a Hamiltonian structure. For instance, one can consider the simple one-dimensional case in which the Hamiltonian has the form

$$H(p,q) = \frac{p}{} - \quad (q), \qquad p, q \qquad ,$$

giving rise to the system of equations

$$\begin{aligned} \dot{q} &= \ _{p}H = p, \\ \dot{p} &= - \ _{q}H = \quad '(q \end{aligned}$$

Noticing that $\ddot{q}$, this system of ordinary differential equations reduces to the single second-order equation

$$q = \quad '(q \tag{1.1}$$

Equation (1.1) has a variational structure, coming from the action functional

$$\frac{q^2(t)}{2} \qquad (q(t))\,\mathrm{d}t \tag{1.2}$$

namely minimizers (or, more generally, critical points) of this functional satisfy (1.1): in jargon, one says that (1.1) is the Euler–Lagrange equation associated with the functional in (1.2).

A typical and concrete example of this setting is given by the equation of the pendulum: in this case, setting the Lyapunov stable equilibrium of the pendulum at and the Lyapunov unstable[1] ones at = , and considering unit gravity for simplicity, one can take

$$W(\ ) = \frac{1 \quad \mathrm{co}\ (\quad)}{} \tag{1.3}$$

in (1.1) and obtain the equation

$$\ddot{\ } = -\sin(\tag{1.4}$$

An interesting analogue of (1.1) in partial differential equations arises in the study of phase coexistence models, and in particular in the analysis of the Allen–Cahn equation

$$\Delta \ + \ - \ ^3 \quad 0 \tag{1.5}$$

If one considers the one-dimensional case, with the choice

[1] We remark that the Lyapunov stable equilibrium of the pendulum is variationally unstable, namely the second derivative of the action functional is negatively defined. Vice versa, the Lyapunov unstable equilibria of the pendulum are variationally stable, since the second derivative of the action functional is positively defined. The terminology related to Lyapunov stability is perhaps more common in the dynamical systems community, while the one dealing with variational stability is often adopted in the calculus of variations. See [16] for a detailed classical study relating variational considerations and Lyapunov stability properties.

$$W(\quad) = \frac{(1 \qquad {}^2)^2}{4} \tag{1.6}$$

then (1.5) can be also reduced to (1.1).

A well-established topic for the dynamical systems as in (1.4) is the search for heteroclinic orbits, that are orbits which connect two (Lyapunov unstable) equilibria. These solutions have the special feature of separating the phase space into regions in which solutions exhibit different topological behaviors (e.g. oscillations versus librations), and, in higher dimensions, they provide the essential building block to chaos.

The analogue of such heteroclinic connections for the phase coexistence problems in (1.5) provides a transition layer connecting two (variationally stable) pure phases of the system. In higher dimensions, these solutions constitute the cornerstone to describe at a large scale the phase separation, as well as the phase parameter in dependence to the distance from the interface.

In this article, we explore a brand new line of investigation focused on a nonlocal analogue of (1.1), in which the second derivative is replaced by a finite difference. More concretely, we will consider a functional similar to that in (1.2), but in which the derivative is replaced by an oscillation term. We recall that other nonlocal analogues of (1.1) have been considered in the literature, mainly replacing the second derivative with a fractional second derivative, see [4, 6, 14, 19, 20]. Other lines of investigation took into account the case in which the second derivative is replaced by a quadratic interaction with an integrable kernel, see [1] and the references therein.

The interest in this problem combines perspectives in pure and in applied mathematics. Indeed, from the theoretical point of view, nonlocal functionals typically exhibit a number of *novel features* that are worth exploring and provide *several conceptual difficulties* that are completely new with respect to the classical cases. On the other hand, in terms of applications, nonlocal functionals can capture *original and interesting phenomena* that cannot be described by the classical models.

In our framework, in particular, we take into account a nonlocal interaction which is not scale invariant. This type of nonlocal structures is closely related to several geometric motions that have been recently studied both for their analytic interest in the calculus of variations and for their concrete applicability in situations in which detecting different scales allows the preservation of details and irregularities in the process of removing white noises (e.g. in the digitalization of fingerprints, in which one wants to improve the quality of the image without losing relevant features at small scales). We refer in particular to [5, 7–13, 17] for several recent contributions in the theoretical and applied analysis of nonlocal problems without scale invariance.

While the previous literature mostly focuses on geometric evolution equations, viscosity solutions, perimeter type problems and questions arising in the calculus of variations, in this paper we aim at investigating the existence and basic properties of heteroclinic minimizers for nonlocal problems with lack of scale invariance.

Since this topic of research is completely new, we will need to introduce the necessary methodology from scratch. In particular, one cannot rely on standard methods, since:

- the problems taken into account do not possess standard compactness properties,
- the functional to minimize cannot be easily differentiated,
- the solutions found need not to be (and in general are not) regular,
- the solutions found need not to be (and in general are not) unique.