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OPTIMAL INSTALLATION OF SOLAR PANELS WITH PRICE IMPACT: A SOLVABLE SINGULAR STOCHASTIC CONTROL PROBLEM*

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Abstract. We consider a price-maker company which generates electricity and sells it in the spot market. The company can increase its level of installed power by irreversible installations of solar panels. The electricity price evolves as an Ornstein–Uhlenbeck process, whose drift is negatively impacted by the current level of the company's installed power. The company aims at maximizing the total expected profits from selling electricity in the market, net of the total expected proportional costs of installation. This problem is modeled as a *two-dimensional degenerate singular stochastic control problem*. We find that the optimal installation strategy is triggered by a curve which separates the *waiting region*, where it is not optimal to install additional panels, and the *installation region*, where it is. Such a curve is the unique strictly increasing solution of a first-order ordinary differential equation. Finally, we show numerically the dependence of the optimal installation strategy on the model's parameters.

Key words. singular stochastic control, irreversible investment, variational inequality, Ornstein–Uhlenbeck process, market impact

AMS subject classifications. 93E20, 49L20, 91B70, 60G99

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1. Introduction.

This paper proposes a model where a company can increase its current electricity production by irreversible investments in solar panels, while maximizing net profits. Irreversible investment problems have been widely studied in the context of real options and optimal capacity expansion. Related models in the economics literature can be found, for example, in the monograph [11]. Other relevant works in the mathematical literature are [9, 21, 24, 26], among many others.

We consider an infinitely lived profit maximizing company which is a large player in the market. The company can install solar panels in order to increase its production level of electricity up to a maximum level (given, for example, by physical constraints). The electricity generated will immediately be sold in the market, and while installing additional panels, the company incurs constant proportional costs. As it is assumed that the company is a large market player, its activities have an impact on the electricity price. In particular, we assume that the long-term electricity price level is negatively affected by the current level of installed power; that is, the electricity price will tend to move toward a lower price level if the electricity production is increased. Therefore, the company has to install solar panels carefully in order to avoid permanently low electricity prices which clearly decrease the marginal profits from selling electricity in the market.

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The mathematical formulation of the model leads to a *two-dimensional degenerate singular stochastic control problem* (see, for example, [17, 18] as early contributions) whose components are the electricity price and the current level of installed power which is purely controlled. In particular, we derive an explicit solution to a problem of singular stochastic control where the drift of one component of the state process (the electricity price) is linearly affected by the monotone process giving the cumulative amount of control (the level of installed power). In our model the electricity price evolves as an Ornstein–Uhlenbeck process, and dealing with such a process makes the problem more difficult in comparison to, for example, a geometric Brownian motion setting, due to the unhandy and nonexplicit expressions of the fundamental solutions of the second-order ordinary differential equation (ODE) involving the infinitesimal generator of the underlying Ornstein–Uhlenbeck process (see [7, Chapter II.10] for an introduction and Lemma A.1 for the analytical form of the fundamental solutions in our case). While a suitable transformation of the state variables leads to stochastic dynamics that have been studied (see, for example, [5, 13]), the corresponding value function is significantly different from those considered in the relevant literature, and another approach is required (see also Remark 2.4). Our ansatz involves the previously mentioned fundamental solutions to be computed on both components of the state variable, which is in contrast to the common case, where the corresponding fundamental solutions do solely depend on the underlying diffusion which represents, for instance, the economic indicator like the asset's price. It is worth noticing that our mathematical formulation shares similarities with the recent article [12] where an Ornstein–Uhlenbeck process is linearly controlled in the drift by a purely controlled variable. However, due to nonidentical sets of admissible controls, the methodology and results of [12] are different with respect to ours. We refer to Remark 2.1 for more details.

Price impact models have gained the interest of many researchers in recent years. Some of these works are also formulated as a *singular stochastic control* problem and study questions of optimal execution: [4] and [5] take into account a multiplicative and transient price impact, whereas [16] considers an exponential parametrization in a geometric Brownian motion setting allowing for a permanent price impact. Also, [1] presents an irreversible capital accumulation problem with permanent price impact, while [13] considers an extraction problem with Ornstein–Uhlenbeck dynamics and transient price impact. In all of the aforementioned papers dealing with *singular stochastic controls* [1, 4, 5, 13, 16], the agents' actions are modeled in such a way that they can lead to an immediate jump in the price. Instead, in our setting, the agent's singular control does not cause price jumps and rather lead to a long-term impact on the underlying price process by linearly affecting the mean-reversion level.

In our model the firm's installation strategy is represented by an increasing control, possibly non-absolutely continuous, and we take into account a running payoff function which depends linearly on the level of installed power and on the electricity price. In order to solve this problem, we follow the approach (quite classical in optimal stopping and singular control) of finding a classical solution to the associated Hamilton–Jacobi–Bellman (HJB) equation via the principle of smooth fit along the so-called free boundary, which separates the waiting and installation regions (see the beginning of section 4 for a more detailed description). This free boundary is also unknown and can be more or less hard to find, depending on the particular structure of the problem. In one-dimensional problems, it reduces to one or two points, characterized by algebraic equations; see, e.g., [14] and references therein. Instead, in two-dimensional problems like the one that we are studying, the free boundary turns out to be the graph of a function F, which is thus to be found with the help of functional equations which originate from $C^{2,1}$ regularity. While in some cases these functional equations are easy and give directly the free boundary F, possibly in terms of integrals to be computed numerically (see, e.g., [1, 2, 4, 13, 21] for examples), our functional equation turns out to be an ODE, whose solution cannot be computed analytically but that we prove to admit a unique global solution. (See [5] for an analogous result and also Remark 2.4 for highlights on the differences of methods used there versus ours.) Then, we characterize the geometry of the *waiting* and *installation* regions. We show that the optimal installation strategy is such that the company keeps the state process inside the *waiting* region. In particular, the state process is pushed toward the free boundary by installing a block of solar panels immediately if the initial electricity price is above the critical threshold (if the maximum level of installed power, that the company is able to reach, is not sufficiently high, the company will immediately install the maximum number of panels). Thereafter, the joint process will be reflected along the free boundary. The construction of the reflected diffusion relies on ideas in [10] that are based on the transformation of probability measures in the spirit of Girsanov. The uniqueness of the optimal diffusion process then follows by the global Lipschitz continuity of our free boundary. Our results are finally complemented by a numerical illustration of our problem, together with a discussion of the dependence on the model parameters. This comparative statics analysis shows interesting new behaviors. For instance, we find a nonmonotone dependence on the electricity price mean-reversion speed and a nontrivial dependence of the free boundary on the upper bound of the firm's production level.

The rest of the paper is organized as follows. In section 2 we introduce the setting and formulate the problem. In section 3 we provide preliminary results and a verification theorem. Then, in section 4 we derive an expression of the free boundary via an ODE, and an explicit solution is constructed. Finally, section 5 studies the dependence of the free boundary with respect to the model parameters.

2. Model and problem formulation.

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with a filtration $\mathbb{F}$ satisfying the usual conditions and carrying a standard one-dimensional $\mathbb{F}$-Brownian motion W.

We consider an infinitely lived company which installs solar panels and sells the electricity produced by those panels instantaneously in the spot market. The level of installed power can be increased at constant proportional cost $c \geq 0$ due to the installation costs of panels. It is assumed that the firm cannot reduce the number of solar panels, and thus the installation is irreversible. The current level of installed power is described by the process $(Y_t^{y,I})_{t\geq 0}$, which is given by

$$Y_t^{y,I} = y + I_t, \tag{2.1}$$

where the initial level of installed power is denoted by $y \geq 0$, and I_t is identified as the company's control variable: it is an $\mathbb{F}$-adapted nonnegative and increasing càdlàg process $I = (I_t)_{t\geq 0}$, where I_t represents the total power installed within the interval $[0, t]$. In the following, $(I_t)_{t\geq 0}$ is also referred to as the installation strategy. Moreover, we assume that the level of installed power cannot exceed a given $\bar{y} \in [y, \infty)$ since, for example, only a finite number of solar panels can be installed. The set of admissible installation strategies is therefore defined as

$$\mathcal{I}^{\bar{y}}(y) := \{I : \Omega \times [0,\infty) \mapsto [0,\infty) : (I_t)_{t\geq 0} \text{ is } \mathbb{F}\text{-adapted}, t \mapsto I_t \text{ is increasing, càdlàg, with } I_{0-} = 0 \leq I_t \leq \bar{y} - y \text{ a.s.}\}.$$

We write $\mathcal{I}^{\bar{y}}(y)$ in order to stress the dependence on both the initial level of installed power y and the maximum possible level $\bar{y}$.

Remark 2.1. The absence of an upper bound $\bar{y}$ would lead to a different problem. In fact, the paper [12], in which the mathematical formulation is similar to ours, studies a problem where the controlled variable is nonmonotone (but still with bounded variation) and not bounded. The authors are then forced to use viscosity theory and arrive to formulate a system of equations which lacks the initial conditions and is not able to construct an analytical solution, as instead we do.

We assume that the current level of electricity production, which is proportional to $Y_t^{y,I}$, affects the electricity market price. In particular we assume that, when following an installation strategy $I \in \mathcal{I}^{\bar{y}}(y)$, the market electricity price is of Ornstein–Uhlenbeck type having its mean instantaneously reduced at time t by $\beta Y_t^{y,I}$, for some $\beta > 0$, and therefore the spot price $X^{x,y,I}$ evolves as[1]

$$dX_t^{x,y,I} = \kappa\left(\left(\mu - \beta Y_t^{y,I}\right) - X_t^{x,y,I}\right)dt + \sigma dW_t, \quad X_{0-}^{x,y,I} = x > 0. \tag{2.2}$$

Remark 2.2. It is common to represent electricity prices via a mean-reverting behavior and to include seasonal fluctuations and daily spikes; see, e.g., [6, 8, 22] and citations therein. Here, we do not represent the spikes and seasonal fluctuations. On the one hand, the installation time of solar panels usually takes several days or weeks, while they have a much longer lifespan (usually some decades). This makes the company indifferent to daily or weekly spikes and/or seasonality.[2] On the other hand, time homogeneous Ornstein–Uhlenbeck processes give tractable analytical tools (like known solutions of related ODEs). With the same arguments, we are also neglecting the stochastic and seasonal effects of solar production: in fact, since here we are interested in a long-term optimal behavior, we interpret the average electricity produced in a generic unit of time as proportional to the installed power.

Remark 2.3. The model in (2.2) also allows for negative prices, which can indeed be observed in some markets (e.g., in Germany; cf. [25]), which, however, do not seem to be persistent. One way to model this is to assume that the long-term mean of X remains positive, no matter what the level of $Y_t \in [0, \bar{y}]$, for all $t \geq 0$, is. This is achieved by imposing

$$\mu - \beta\bar{y} > 0. \tag{2.3}$$

For the solar energy application motivating our mathematical problem, it would be sensible to require the above inequality, but our mathematical derivation does not require it. Thus, all the results that follow are independent of whether the condition in (2.3) holds or not.

The company aims at maximizing the total expected profits from selling electricity in the market, net of the total expected costs of installation. That is, the company aims at determining

$$V(x,y) := \sup_{I \in \mathcal{I}^{\bar{y}}(y)} \mathcal{J}(x,y,I), \quad (x,y) \in \mathbb{R} \times [0,\bar{y}], \tag{2.4}$$

[1]There are several studies on the presence of price impact in electricity markets. For instance, [8] shows how to incorporate a market impact due to cross-border trading in electricity markets, and [27] models the price impact of wind electricity production on power prices.

[2]This can be justified if we interpret our fundamental price to be, for example, a weekly average price as in [2], where a calibration of this model to the Italian power market data is also carried out.

where for any $I \in \mathcal{I}^{\bar{y}}(y)$

$$\mathcal{J}(x,y,I) := \mathbb{E}\left[\int_0^\infty e^{-\rho t} X_t^{x,y,I}\left(\alpha Y_t^{y,I}\right) dt - c\int_0^\infty e^{-\rho t} dI_t\right], \quad \alpha > 0. \tag{2.5}$$

In (2.5), the parameter α is the proportional factor between the average electricity produced in a generic unit of time and the current level of installed power. Thus, the running gain $\alpha X_t^{x,y,I} Y_t^{y,I}$ can be viewed as a time-averaged revenue deriving from solar production.

For the sake of simplicity, we set $\alpha = 1$ in the following. In fact, the problem of finding an optimal control $I \in \mathcal{I}^{\bar{y}}(y)$ in (2.5) does not change for $\alpha > 0$ upon introducing a new cost factor $\tilde{c} = \frac{c}{\alpha}$.

Remark 2.4. It is worth noticing that the controlled state process (2.2) can be transformed to similar dynamics studied in [5, 13] by considering the process $\tilde{X} := X^{x,y,I} + \beta Y^{y,I}$. Then, (2.2) leads to

$$d\tilde{X}_t^{x,y,I} = \kappa\left(\mu - \tilde{X}_t^{x,y,I}\right) dt + \sigma dW_t + \beta dY_t^{y,I}, \quad \tilde{X}_{0-}^{x,y,I} = \tilde{x} := x + \beta y > 0, \tag{2.6}$$

and the value function V for any $(x,y) \in \mathbb{R} \times [0,\bar{y}]$ is given by

$$V(x,y) := \sup_{I \in \mathcal{I}^{\bar{y}}(y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\tilde{X}_t^{x,y,I} Y_t^{y,I} - \beta\left(Y_t^{y,I}\right)^2\right) dt - c\int_0^\infty e^{-\rho t} dI_t\right]. \tag{2.7}$$

Here, the structure of the value function is highly different from that in [5, 13] (cf. (2.5) in [5] and (2.3) in [13], respectively). To see this, we apply Itô's formula to get

$$\begin{aligned} de^{-\rho t}\left(\tilde{X}_t^{x,y,I} Y_t^{y,I} - \beta(Y_t^{y,I})^2\right) &= e^{-\rho t}\left[\kappa Y_t^{y,I}\left(\mu - \tilde{X}_t^{x,y,I}\right) - \rho\left(\tilde{X}_t^{x,y,I} Y_t^{y,I} - \beta(Y_t^{y,I})^2\right)\right] dt \\ &\quad + e^{-\rho t}\sigma Y_t^{y,I} dW_t + e^{-\rho t}\left(\tilde{X}_t^{x,y,I} - \beta Y_t^{y,I}\right) dI_t. \end{aligned} \tag{2.8}$$

Then, using (2.8) in (2.7) gives

$$\begin{aligned} \rho V(x,y) = \sup_{I \in \mathcal{I}^{\bar{y}}(y)} \mathbb{E}\bigg[&\int_0^\infty e^{-\rho t}\kappa Y_t^{y,I}\left(\mu - \tilde{X}_t^{x,y,I}\right) dt \\ &+ \int_0^\infty e^{-\rho t}\left(\tilde{X}_t^{x,y,I} - \beta Y_t^{y,I} - c\rho\right) dI_t\bigg] + \left(\tilde{x}y - \beta y^2\right). \end{aligned}$$

Our setting is not covered by [5, 13], and neither in adverse. This is, apart from the presence of running gains $\int_0^\infty e^{-\rho t}\kappa Y_t^{y,I}(\mu - \tilde{X}_t^{x,y,I})dt$, due to the fact that the integrand of the second term inside the expectation depends also on Y. In [13], where the problem is solved by a combined use of a calculus of variation method developed by [5] and, as we do here, of a direct study on the HJB equation, the authors rely on the chain rule to derive the solution to the problem. This method fails here; see also Remark 4.1. The paper [5] considers, conversely to this setting and [13], a nonlinear integrand where the presence of the corresponding Y is not covered. The interesting calculus of variations method followed there leads to an ODE which characterizes the solution to the problem (analoguous to our results). However, that approach cannot be applied directly when working with linear integrands because it would require conditions which restrict the parameter space (see also Remark 4.10 in [13]).

3. A verification theorem.

The aim of this section is to provide a verification theorem which characterizes the solution to our problem.

The admissible noninstallation strategy is denoted by $I^0 \equiv 0$, and we indicate the electricity price process implied by I^0 by $(X_t^{x,y})_{t\geq 0}$, that is, $X_t^{x,y} \equiv X_t^{x,y,I^0}$. Also, we denote $X^x = X^{x,0} = X^{x,0,I^0}$.

LEMMA 3.1. *For all $x \in \mathbb{R}$, $y \in [0,\bar{y}]$, and $I \in \mathcal{I}^{\bar{y}}(y)$, it holds that*

$$|X_t^{x,y,I}| \leq |X_t^x| + \kappa\beta\bar{y}t. \tag{3.1}$$

Moreover, $\mathbb{E}[X_t^{x,y}] = m_t$ and $\mathrm{Var}[X_t^{x,y}] = \sigma_t^2$, where

$$m_t := xe^{-\kappa t} + (\mu - \beta y)(1 - e^{-\kappa t}) \quad \text{and} \quad \sigma_t := \sigma\sqrt{\frac{1-e^{-2\kappa t}}{2\kappa}}. \tag{3.2}$$

Finally,

$$\mathbb{E}[|X_t^{x,y}|] \leq K(1+|x|), \tag{3.3}$$

for a suitable constant $K > 0$ which does not depend on t and y.

Then, the expected profits of the firm following the noninstallation strategy is described by the function $R : \mathbb{R} \times [0,\bar{y}] \mapsto \mathbb{R}$ such that

$$R(x,y) := \mathcal{J}(x,y,I^0) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} X_t^{x,y} y dt\right] = \frac{xy}{\rho+\kappa} + \frac{\mu\kappa y}{\rho(\rho+\kappa)} - \frac{\kappa\beta y^2}{\rho(\rho+\kappa)}. \tag{3.4}$$

Here the integral converges thanks to (3.3) and is computed thanks to (3.2). The following preliminary result provides a growth condition and a monotonicity property of the value function V, and its connection to the function R. The proof of the proposition can be found in Appendix B.

PROPOSITION 3.2. *There exists a constant $K > 0$ such that for all $(x,y) \in \mathbb{R} \times [0,\bar{y}]$ one has*

$$|V(x,y)| \leq K\big(1+|x|\big). \tag{3.5}$$

Moreover, $V(x,\bar{y}) = R(x,\bar{y})$, and V is increasing in x.

In a next step we derive the HJB, a particular partial differential equation which characterizes the solution to our problem.

For given and fixed $y \geq 0$, let $\mathcal{L}^y$ be the infinitesimal generator of the diffusion $X^{x,y}$ given by the second-order differential operator

$$\mathcal{L}^y u(x,y) := \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}u(x,y) + \kappa((\mu-\beta y) - x)\frac{\partial}{\partial x}u(x,y), \tag{3.6}$$

where $u(\cdot,y) \in C^2(\mathbb{R})$.

The HJB equation, for singular control problems as this one (following, for example, [15, Chapter VIII]), should identify with an appropriate solution w to the HJB equation

$$\max\{\mathcal{L}^y w(x,y) - \rho w(x,y) + xy, w_y(x,y) - c\} = 0, \quad (x,y) \in \mathbb{R} \times [0,\bar{y}), \tag{3.7}$$

with boundary condition

$$w(x, \bar{y}) = R(x, \bar{y}). \tag{3.8}$$

With reference to (3.7), we introduce the *waiting region*

$$\mathbb{W} := \{(x, y) \in \mathbb{R} \times [0, \bar{y}) : \mathcal{L}^y w(x, y) - \rho w(x, y) + xy = 0,\, w_y(x, y) - c < 0\}, \tag{3.9}$$

where we expect it not to be optimal to install additional solar panels, and the *installation region*

$$\mathbb{I} := \{(x, y) \in \mathbb{R} \times [0, \bar{y}) : \mathcal{L}^y w(x, y) - \rho w(x, y) + xy \leq 0,\, w_y(x, y) - c = 0\}, \tag{3.10}$$

where we expect it to be.

We move on by proving a verification theorem. It shows that an appropriate solution to the HJB equation (3.7) identifies with the value function, if an admissible installation strategy exists which keeps the state process (X, Y) inside the waiting region $\overline{\mathbb{W}}$ (here we have denoted by $\overline{\mathbb{W}}$ the closure of $\mathbb{W}$) with minimal effort, i.e., the level of installed power is increased only at the time when (X, Y) enters the installation region $\mathbb{I}$. These results are standard in singular control, but for the sake of completeness we provide a proof in Appendix B

Theorem 3.3 (verification theorem). *Suppose there exists a function $w : \mathbb{R} \times [0, \bar{y}] \mapsto \mathbb{R}$ such that $w \in C^{2,1}(\mathbb{R} \times [0, \bar{y}])$ solves the HJB equation* (3.7) *with boundary condition* (3.8) *and satisfies the growth condition*

$$|w(x, y)| \leq K\big(1 + |x|\big) \tag{3.11}$$

for a constant $K > 0$. Then $w \geq v$ on $\mathbb{R} \times [0, \bar{y}]$.

Moreover, suppose that for all initial values $(x, y) \in \mathbb{R} \times [0, \bar{y})$, there exists a process $I^\star \in \mathcal{I}^{\bar{y}}(y)$ such that

$$\left(X_t^{x,y,I^\star}, Y_t^{y,I^\star}\right) \in \overline{\mathbb{W}} \quad \textit{for all } t \geq 0,\ \mathbb{P}\textit{-a.s.}, \tag{3.12}$$

$$I_t^\star = \int_{0-}^t \mathbb{1}_{\left\{\left(X_s^{x,y,I^\star}, Y_s^{y,I^\star}\right) \in \mathbb{I}\right\}} dI_s^\star \quad \textit{for all } t \geq 0,\ \mathbb{P}\textit{-a.s.} \tag{3.13}$$

Then we have

$$V(x, y) = w(x, y), \quad (x, y) \in \mathbb{R} \times [0, \bar{y}],$$

and $I^\star$ is optimal; that is, $V(x, y) = \mathcal{J}(x, y, I^\star)$.

4. Constructing an optimal solution to the installation problem. As mentioned in the introduction, we start to solve the problem by finding a classical solution w to the associated HJB equation (3.7) with boundary condition (3.8). To do so, we make the educated guess that the regions $\mathbb{W}$ and $\mathbb{I}$, defined in (3.9) and (3.10), are separated by a so-called free boundary; then we solve the HJB equation separately in these two regions, and we glue together the two solutions in the free boundary so that the resulting function w satisfies the global $C^{2,1}$ regularity required by Theorem 3.3. This free boundary is also unknown, and here it turns out to be characterized as the solution of an ODE, whose solution cannot be computed analytically but that we prove to admit a unique global solution. This is done in section 4.1, while in section 4.2 we build the candidate optimal strategy and verify the optimality by applying Theorem 3.3.

As described above, we now assume that there exists an injective function $F : [0,\bar{y}] \to \mathbb{R}$, called the *free boundary*, which separates the waiting region $\mathbb{W}$ and the installation region $\mathbb{I}$, defined respectively in (3.9) and (3.10), such that

$$\mathbb{W} = \{(x,y) \in \mathbb{R} \times [0,\bar{y}) : x < F(y)\}, \tag{4.1}$$

$$\mathbb{I} = \{(x,y) \in \mathbb{R} \times [0,\bar{y}) : x \geq F(y)\}. \tag{4.2}$$

For all $(x,y) \in \mathbb{W}$, the candidate value function w should satisfy (cf. (3.9))

$$\mathcal{L}^y w(x,y) - \rho w(x,y) + xy = 0. \tag{4.3}$$

Notice that the left-hand side of (4.3) only takes into account the derivatives with respect to x, and instead, y can be treated as a parameter here: thus, (4.3) is a second-order linear ODE in x, where the differential operator is parameterized by y. Hence, we can apply the standard theory of ODEs in order to solve (4.3).

It is straightforward to check that a particular solution to (4.3) is given by the function R defined in (3.4). Moreover, the homogeneous ODE

$$\mathcal{L}^y w(x,y) - \rho w(x,y) = 0 \tag{4.4}$$

admits two fundamental strictly positive solutions (see pp. 18–19 of [7]). These are given by $\phi(x + \beta y)$ and $\psi(x + \beta y)$, with $\phi(\cdot)$ strictly decreasing and $\psi(\cdot)$ strictly increasing; cf. Lemma A.1(1) and (5). Therefore, our candidate value function w takes the form

$$w(x,y) = A(y)\psi(x + \beta y) + B(y)\phi(x + \beta y) + R(x,y), \quad (x,y) \in \mathbb{W}, \tag{4.5}$$

for some functions $A, B : [0,\bar{y}] \mapsto \mathbb{R}$ to be found. Notice that, for $y \geq 0$ to be given and fixed, $\phi(x+\beta y)$ grows to $+\infty$ exponentially fast whenever $x \downarrow -\infty$; see Appendix 1 in [7]. In light of both the linear growth of V (cf. Proposition 3.2) and the structure of the waiting region $\mathbb{W}$ (cf. (4.1)), we must have $B(y) = 0$ for all $y \in [0,\bar{y}]$. Thus, we conjecture that

$$w(x,y) = A(y)\psi(x + \beta y) + R(x,y) \quad \text{for } (x,y) \in \mathbb{W} \tag{4.6}$$

with $A(\bar{y}) = 0$ from (3.8).

Equation (4.6) gives the following intuition: R represents the value of selling permanently y units of electricity (the initial level of installed power) in the market, while the product of A and ψ represents the value of the option to increase the solar production. The sum of both then gives the company's total expected profits.

Remark 4.1. In [13], the corresponding equation to (4.6) is solely the product of some function A and the increasing fundamental solution ψ where the latter is independent of the variable y; cf. (4.4) therein. In this way, the authors are able to derive an explicit expression for the free boundary upon relying on an application of the chain rule (cf. derivation after Lemma 4.2). Instead, our analysis is more involved as we cannot follow the same approach which is due to the dependence of the fundamental solution on y and the presence of R.

We move on to derive equations that characterize the function A appearing in the representation (4.6) and the free boundary F which separates $\mathbb{W}$ and $\mathbb{I}$ according to the assumptions in (4.1)–(4.2). With reference to (3.10), for all $(x, y) \in \mathbb{I}$, w should instead satisfy

$$w_y(x, y) - c = 0, \tag{4.7}$$

implying

$$w_{yx}(x, y) = 0. \tag{4.8}$$

Now, we impose the so-called smooth fit condition, i.e., we suppose that $w \in C^{2,1}(\mathbb{R} \times [0, \bar{y}])$, and therefore by (4.6), (4.7), and (4.8), we must have for all $(x, y) \in \overline{\mathbb{W}} \cap \mathbb{I}$ (that is, where $x = F(y)$)

$$A'(y)\psi\big(F(y) + \beta y\big) + \beta A(y)\psi'\big(F(y) + \beta y\big) + R_y(F(y), y) - c = 0, \tag{4.9}$$

$$A'(y)\psi'\big(F(y) + \beta y\big) + \beta A(y)\psi''\big(F(y) + \beta y\big) + R_{yx}(F(y), y) = 0. \tag{4.10}$$

Notice that the derivatives of R can be easily obtained from (3.4), which gives

$$R_y(x, y) = \frac{x}{\rho + \kappa} + \frac{\mu\kappa}{\rho(\rho + \kappa)} - \frac{2\kappa\beta y}{\rho(\rho + \kappa)} \quad \text{and} \quad R_{xy}(x, y) = (\rho + \kappa)^{-1}.$$

For the sake of simplicity, we introduce a function $\tilde{F}$ for a substitution, that is, we let

$$\tilde{F}(y) = F(y) + \beta y. \tag{4.11}$$

We have

$$R_y(F(y), y) = \frac{\rho F(y) + \mu\kappa - 2\kappa\beta y}{\rho(\rho + \kappa)} = \frac{\mu\kappa + \rho\tilde{F}(y) - \beta(\rho + 2\kappa)y}{\rho(\rho + \kappa)} = \tilde{R}(\tilde{F}(y), y),$$

where $\tilde{R} : \mathbb{R}^2 \mapsto \mathbb{R}$ is defined as

$$\tilde{R}(x, y) := \frac{\mu\kappa + \rho x - \beta(\rho + 2\kappa)y}{\rho(\rho + \kappa)}.$$

Notice that

$$\tilde{R}_x(\tilde{F}(y), y) = (\rho + \kappa)^{-1} = R_{yx}(F(y), y).$$

From now on, we will often use the functions $Q_k : \mathbb{R} \mapsto \mathbb{R}$, $k \in \mathbb{N}_0$, and their first derivatives, given by

$$\begin{aligned} Q_k(z) &:= \psi^{(k)}(z)\psi^{(k+2)}(z) - \psi^{(k+1)}(z)^2, \\ Q_k'(z) &= \psi^{(k)}(z)\psi^{(k+3)}(z) - \psi^{(k+1)}(z)\psi^{(k+2)}(z). \end{aligned} \tag{4.12}$$

Substituting F according to (4.11) in both (4.9) and (4.10), and solving for A and A', gives

$$A(y) = \beta^{-1} \times \frac{\psi'(\tilde{F}(y))\Big(c - \tilde{R}(\tilde{F}(y), y)\Big) + (\rho + \kappa)^{-1}\,\psi(\tilde{F}(y))}{-Q_0(\tilde{F}(y))} \tag{4.13}$$

and

$$A'(y) = \frac{\psi''(\tilde{F}(y))\Big(c - \tilde{R}(\tilde{F}(y), y)\Big) + (\rho+\kappa)^{-1}\,\psi'(\tilde{F}(y))}{Q_0(\tilde{F}(y))}. \tag{4.14}$$

Lemma A.1(3) ensures that Q_k is strictly positive for all $k \in \mathbb{N}_0$, and therefore the denominator on the right-hand side of both (4.13) and (4.14) is nonzero.

The following lemma provides essential properties of the function A and a lower bound for $\tilde{F}$ that are needed for results of sections 4.1 and 4.2. Its proof can be found in Appendix B.

Lemma 4.2. *The function A appearing in the representation formula* (4.6) *is strictly positive and strictly decreasing. Moreover, A admits the representation*

(4.15)

$$A(y) = (\beta\rho(\rho+\kappa))^{-1} \times \frac{(\rho+\kappa)\Big(c\rho + \frac{(\rho+2\kappa)\beta}{\rho+\kappa}y - \tilde{F}(y)\Big)\psi'(\tilde{F}(y)) + \frac{\sigma^2}{2}\psi''(\tilde{F}(y))}{-Q_0(\tilde{F}(y))},$$

where $\tilde{F}$ is defined in (4.11), *and we have*

$$\tilde{F}(y) \geq c\rho + \frac{(\rho+2\kappa)\beta}{\rho+\kappa}y \quad \textit{for all } y \in [0, \bar{y}]. \tag{4.16}$$

From (4.16), we therefore obtain the inequality

$$F(y) \geq c\rho + \frac{\kappa\beta}{\rho+\kappa}y \geq c\rho \quad \text{for all } y \in [0, \bar{y}], \tag{4.17}$$

which will be exploited in section 4.2.

4.1. The free boundary: Existence and characterization. In this section, we aim to prove the existence and a monotonicity property of $\tilde{F}$, satisfying (4.9) and (4.10) (with F being replaced according to (4.11)), so as to draw the implications for F after. The proofs of the following lemma and proposition can be found in Appendix B.

In light of the boundary condition $w(x, \bar{y}) = R(x, \bar{y})$ (cf. Theorem 3.3), we impose

$$A(\bar{y}) = 0. \tag{4.18}$$

Due to (4.13) and (4.18), we must have that there exists a point $\tilde{x} = \tilde{F}(\bar{y}) \in \mathbb{R}$ such that

$$\psi'(\tilde{x})\left(c - \tilde{R}(\tilde{x}, \bar{y})\right) + (\rho+\kappa)^{-1}\,\psi(\tilde{x}) = 0. \tag{4.19}$$

Lemma 4.3. *There exists a unique $\tilde{x} \in \mathbb{R}$ such that* (4.19) *holds.*

Remark 4.4. An analytically tractable expression for R is necessary (when imitating the proofs we follow) for getting a (full) characterization of the free boundary of F in terms of an ODE that we are able to study. A more general case seems to be when the running integral in $\mathcal{J}$ in (2.5) is equal to $e^{-\rho t}X_tY_t^{\gamma}$, with $\gamma \in (0,1]$ (in this paper $\gamma = 1$), with the cost c of the singular control unchanged (we conjecture that, in this more general case, one could obtain results analogous to those which follow). However, we would have a value function with derivatives in y which explode for $y = 0$, so the treatment of this more general case should be more delicate. Here, the linear-quadratic structure of R provides analytically tractable functions A and A' (cf. (4.13) and (4.14)). These functions, for instance, make it possible to prove Lemma 4.3 and thus obtain the boundary condition of the ODE (4.24).

Differentiating (4.13), we find after some algebra

$$A'(y) = (\beta(\rho+\kappa)Q_0(\tilde{F}(y))^2)^{-1} \times \left(\tilde{F}'(y)D(y,\tilde{F}(y)) - \frac{\beta(\rho+2\kappa)}{\rho}\psi'(\tilde{F}(y))Q_0(\tilde{F}(y))\right), \tag{4.20}$$

where $D : \mathbb{R}^2 \mapsto \mathbb{R}$ is defined as

$$D(y,z) = \psi(z)[(\rho+\kappa)(c - \tilde{R}(z,y))Q_1(z) + Q_0'(z)]. \tag{4.21}$$

Now, equating both expressions (4.14) and (4.20), we get

$$\tilde{F}'(y)D(y,\tilde{F}(y)) = \beta N(y,\tilde{F}(y)), \tag{4.22}$$

where $N : \mathbb{R}^2 \mapsto \mathbb{R}$ is defined as

$$N(y,z) = Q_0(z)\left(\frac{\rho+2\kappa}{\rho}\psi'(z) + \left((\rho+\kappa)\left(c - \tilde{R}(z,y)\right)\psi''(z) + \psi'(z)\right)\right). \tag{4.23}$$

We then obtain from (4.22) the ODE

$$\tilde{F}'(y) = \mathcal{G}(y,\tilde{F}(y)), \tag{4.24}$$

with boundary condition $\tilde{F}(\bar{y}) = \tilde{x}$ (cf. Lemma 4.3), and where $\mathcal{G} : (\mathbb{R}\times\mathbb{R}) \setminus \{(y,z) \in \mathbb{R}^2 : D(y,z) = 0\} \mapsto \mathbb{R}$ is such that

$$\mathcal{G}(y,z) = \beta \times \frac{N(y,z)}{D(y,z)}. \tag{4.25}$$

The next result guarantees the existence and uniqueness of a solution $\tilde{F}$ on $[0,\bar{y}]$ of (4.24) which is such that $\tilde{F}'(y) > \beta$ for all $y \in [0,\bar{y}]$. Consequently, we then obtain the existence and uniqueness of a strictly increasing free boundary F on $[0,\bar{y}]$ (cf. (4.11)).

PROPOSITION 4.5. *There exists a unique solution $\tilde{F}$ on $[0,\bar{y}]$ of the ODE* (4.24) *with boundary condition $\tilde{F}(\bar{y}) = \tilde{x}$. Moreover, $\tilde{F}'(y) \geq \beta$ for all $y \in [0,\bar{y}]$.*

COROLLARY 4.6. *The free boundary F as in* (4.1) *and* (4.2) *is well defined. Moreover, it is strictly increasing and given by*

$$F(y) = \tilde{F}(y) - \beta y \quad \text{for all } y \in [0,\bar{y}].$$

Proof. The existence and uniqueness is an implication of Proposition 4.5. It also ensures that $F'(y) = \tilde{F}'(y) - \beta > 0$ for all $y \in [0,\bar{y}]$. ∎

4.2. The optimal strategy and the value function: Verification. In the following, the initial price level at which the company starts to install solar panels is denoted by $x_0 := F(0)$, and we define $\bar{x} := F(\bar{y}) = \tilde{x} - \beta\bar{y}$ (cf. (4.11)). Since F is strictly increasing, its inverse function exists on $[x_0,\bar{x}]$ and is denoted by F^{-1}.

We divide the (candidate) *installation region* $\mathbb{I}$ into

$$\mathbb{I}_1 := \{(x,y) \in \mathbb{R}\times[0,\bar{y}) : x \in [F(y),\bar{x})\} \quad \text{and} \quad \mathbb{I}_2 := \{(x,y) \in \mathbb{R}\times[0,\bar{y}) : x \geq \bar{x}\}.$$

An optimal installation strategy can be described as follows: in $\mathbb{W}$ (cf. (4.1)), that is, when the current price x is sufficiently low such that $x < F(y)$, the company does not increase the level of installed power. Whenever the price crosses $F(y)$, then the company makes infinitesimal installations so as to keep the state process (X,Y) inside $\overline{\mathbb{W}}$. Conversely, if the current price x is sufficiently large such that $x \geq F(y)$ (i.e., in $\mathbb{I}$; cf. (4.2)), then the company makes an instantaneous lump sum installation. In particular, on the one hand, whenever the maximum level of installed power $\bar{y}$, that the firm is able to reach, is sufficiently high (that is, $(x,y) \in \mathbb{I}_1$), then the company pushes the state process (X,Y) immediately to the locus of points $\{(x,y) \in \mathbb{R}\times[0,\bar{y}] : x = F(y)\}$ in direction $(0,1)$, so as to increase the level of installed power by $F^{-1}(x) - y$ units. The associated payoff to this action is then the difference of the continuation value starting from the new state $(x, F^{-1}(x))$ and the costs associated to the installation of additional solar panels, that is, $c(F^{-1}(x) - y)$. On the other hand, whenever the firm has to restrict its actions due to the upper bound $\bar{y}$ (that is, $(x,y) \in \mathbb{I}_2$), then the company immediately installs the maximum number of panels, so as to increase the level of installed power up to $\bar{y}$ units, and the associated payoff to such a strategy is $R(x,\bar{y}) - c(\bar{y} - y)$.

In light of the previous discussion, we now define our candidate value function $w : \mathbb{R}\times[0,\bar{y}] \mapsto \mathbb{R}$ as

(4.26)
$$w(x,y) = \begin{cases} A(y)\psi\big(x+\beta y\big) + R(x,y) & \text{if } x \in \mathbb{W} \cup ((-\infty,\bar{x})\times\{\bar{y}\}), \\ A(F^{-1}(x))\psi\big(x+\beta F^{-1}(x)\big) & \\ \quad +R(x,F^{-1}(x)) - c(F^{-1}(x) - y) & \text{if } (x,y) \in \mathbb{I}_1, \\ R(x,\bar{y}) - c(\bar{y} - y) & \text{if } (x,y) \in \mathbb{I}_2 \cup ([\bar{x},\infty)\times\{\bar{y}\}). \end{cases}$$

The next two results verify that w is a classical solution to the HJB equation (3.7).

LEMMA 4.7. *The function w is $C^{2,1}(\mathbb{R}\times[0,\bar{y}])$.*

Proof. In the following, we denote by $\mathrm{Int}(\cdot)$ the interior of a set. Clearly, by (4.26) it holds for all $(x,y) \in \mathrm{Int}(\mathbb{W})$ that

$$w_x(x,y) = A(y)\psi'(x+\beta y) + R_x(x,y), \tag{4.27}$$
$$w_{xx}(x,y) = A(y)\psi''(x+\beta y), \tag{4.28}$$
$$w_y(x,y) = A'(y)\psi(x+\beta y) + \beta A(y)\psi'(x+\beta y) + R_y(x,y), \tag{4.29}$$

and moreover,

(4.30)
$$w_x(x,y) = R_x(x,\bar{y}), \quad w_{xx}(x,y) = 0, \quad w_y(x,y) = c, \quad \text{for all } (x,y) \in \mathrm{Int}(\mathbb{I}_2).$$

To evaluate w_x, w_{xx}, and w_y inside $\mathbb{I}_1$, we need some more work. We find for all $(x,y) \in \mathrm{Int}(\mathbb{I}_1)$

$$
\begin{aligned}
w_x(x,y) &= (F^{-1})'(x)\big[A'\big(F^{-1}(x)\big)\psi\big(x+\beta F^{-1}(x)\big)+\beta A\big(F^{-1}(x)\big)\psi'\big(x+\beta F^{-1}(x)\big)\\
&\quad + R_y\big(x,F^{-1}(x)\big)-c\big]+A(F^{-1}(x))\psi'\big(x+\beta F^{-1}(x)\big)+R_x\big(x,F^{-1}(x)\big)\\
&= A(F^{-1}(x))\psi'\big(x+\beta F^{-1}(x)\big)+R_x\big(x,F^{-1}(x)\big),
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
w_{xx}(x,y) &= A(F^{-1}(x))\psi''(x+\beta F^{-1}(x))+(F^{-1})'(x)\Big[A'(F^{-1}(x))\psi'\big(x+\beta F^{-1}(x)\big)\\
&\quad +\beta A(F^{-1}(x))\psi''\big(x+\beta F^{-1}(x)\big)+R_{yx}(x,F^{-1}(x))\Big]\\
&= A(F^{-1}(x))\psi''(x+\beta F^{-1}(x)),
\end{aligned} \tag{4.32}
$$

$$
w_y(x,y) = c, \tag{4.33}
$$

where we have used (4.9) in (4.31), and (4.10) in (4.32). Notice that the functions A, F^{-1}, ψ, ψ', R_y, and R_x are continuous. The previous equations and (4.9) easily provide the continuity of the derivatives on $\mathbb{R}\times\{\bar{y}\}$. Letting $(x_n,y_n)_n\subset\mathbb{I}_1$ be any sequence converging to $(F(y),y)$, $y\in[0,\bar{y})$, we find the required continuity results along $\overline{\mathbb{W}}\cap\overline{\mathbb{I}}_1$ upon employing (4.9). Moreover, (4.18) ensures the continuity of w_x and w_{xx} along $\overline{\mathbb{I}}_1\cap\overline{\mathbb{I}}_2$, and we clearly have the continuity of w_y along $\overline{\mathbb{I}}_1\cap\overline{\mathbb{I}}_2$. □

Proposition 4.8. *The function w from* (4.26) *is a $C^{2,1}(\mathbb{R}\times[0,\bar{y}])$ solution of the HJB equation* (3.7)*, with boundary condition given in* (3.8).

Proof. Lemma 4.7 guarantees the claimed regularity of w. Moreover, from (4.26) we see that $w(x,\bar{y})=R(x,\bar{y})$ since $A(\bar{y})=0$, and by construction, we clearly have $\mathcal{L}^y w(x,y)-\rho w(x,y)+xy=0$ for all $(x,y)\in\mathbb{W}$, and $w_y(x,y)-c=0$ for all $(x,y)\in\mathbb{I}_1\cup\mathbb{I}_2$. We prove the inequalities $\mathcal{L}^y w(x,y)-\rho w(x,y)+xy\le 0$ for all $(x,y)\in\mathbb{I}$, and $w_y(x,y)-c\le 0$ for all $(x,y)\in\mathbb{W}$ in the following three steps separately. It is worth bearing in mind that $R_x(x,y)=\frac{y}{\rho+\kappa}$ by (3.4).

Step 1. Let $(x,y)\in\mathbb{I}_1$ be fixed. From the second line of (4.26), (4.31), and (4.32), we find

$$
\begin{aligned}
\mathcal{L}^y w(x,y)-\rho w(x,y)+xy &= \mathcal{L}^{F^{-1}(x)}w(x,F^{-1}(x))-\rho w(x,F^{-1}(x))+xF^{-1}(x)\\
&\quad +\kappa\beta w_x(x,F^{-1}(x))(F^{-1}(x)-y)+(c\rho-x)(F^{-1}(x)-y)\\
&= (F^{-1}(x)-y)\left(c\rho+\kappa\beta w_x(x,F^{-1}(x))-x\right),
\end{aligned} \tag{4.34}
$$

where we have employed that $w(x,F^{-1}(x))$ solves

$$
\mathcal{L}^{F^{-1}(x)}w(x,F^{-1}(x))-\rho w(x,F^{-1}(x))+xF^{-1}(x)=0.
$$

For any $(x,y)\in\mathbb{I}_1$, we have $x\ge F(y)$ implying $F^{-1}(x)\ge y$ because F, and hence F^{-1}, is strictly increasing (cf. Corollary 4.6). Thus, in order to show that (4.34) is negative on $\mathbb{I}_1$, it suffices to prove that the function

$$
Z(x,F^{-1}(x)) := c\rho+\kappa\beta w_x(x,F^{-1}(x))-x \tag{4.35}
$$

is negative for any $x\in[x_0,\bar{x}]$. Due to the regularity of w, we can use (4.31), and the fact that $A(F^{-1}(\bar{x}))=A(\bar{y})=0$, to obtain

$$Z(\bar{x}, F^{-1}(\bar{x})) = c\rho + R_x(\bar{x}, \bar{y}) - \bar{x} < 0, \tag{4.36}$$

where the inequality holds by (4.17) with $y = \bar{y}$. Taking the total derivative of $Z(x, F^{-1}(x))$ with respect to x gives

$$\begin{aligned}\frac{dZ(x, F^{-1}(x))}{dx} &= \kappa\beta w_{xx}(x, F^{-1}(x)) - 1 = \kappa\beta A(F^{-1}(x))\psi''(x + \beta F^{-1}(x)) - 1 \\ &= \left[\rho\left(\psi(x + \beta F^{-1}(x))\psi''(x + \beta F^{-1}(x)) - \psi'(x + \beta F^{-1}(x))^2\right)\right]^{-1} \\ &\quad \times \Bigg[\rho\left(\psi'(x + \beta F^{-1}(x))^2 - \psi(x + \beta F^{-1}(x))\psi''(x + \beta F^{-1}(x))\right) \\ &\quad - \kappa\psi'(x + \beta F^{-1}(x))\psi''(x + \beta F^{-1}(x))\left(c\rho + \frac{\kappa\beta}{\rho + \kappa}F^{-1}(x) - x\right) \\ &\quad - \frac{\sigma^2}{2}\kappa\psi''(x + \beta F^{-1}(x))^2 R_{xy}(x, F^{-1}(x))\Bigg],\end{aligned} \tag{4.37}$$

where we have employed $w_{xy}(x, F^{-1}(x)) = 0$ (cf. (4.8)) for the first equality, and (4.15) with $\tilde{F}$ being replaced according to (4.11) for the last equality (after rearranging terms). Now, suppose that there exists a point $x^\star \in [x_0, \bar{x})$ such that $Z(x^\star, F^{-1}(x^\star)) = 0$. It follows from (4.35), together with (4.15) and (4.31), that $(x^\star, F^{-1}(x^\star))$ satisfies

$$\begin{aligned}&c\rho + \frac{\kappa\beta}{\rho + \kappa}F^{-1}(x^\star) - x^\star \\ &= \frac{-\frac{\sigma^2}{2}\kappa\psi'(x^\star + \beta F^{-1}(x^\star))\psi''(x^\star + \beta F^{-1}(x^\star))R_{xy}(x^\star, F^{-1}(x^\star))}{(\rho + \kappa)\psi'(x^\star + \beta F^{-1}(x^\star))^2 - \rho\psi(x^\star + \beta F^{-1}(x^\star))\psi''(x^\star + \beta F^{-1}(x^\star))}.\end{aligned} \tag{4.38}$$

Then, exploiting the latter, one can find with (4.37) that

$$\left.\frac{dZ(x, F^{-1}(x))}{dx}\right|_{x = x^\star} = \frac{\sigma^2}{2}Q_1(x^\star + \beta F^{-1}(x^\star))^{-1}Q_2(x^\star + \beta F^{-1}(x^\star)) > 0 \tag{4.39}$$

after using (A-3) with $k = 0, 1, 2$, and some simple algebra. We conclude from both (4.36) and (4.39) that there cannot exist a point $x^\star \in [x_0, \bar{x})$ such that $Z(x^\star, F^{-1}(x^\star)) = 0$. Therefore, we have $\mathcal{L}^y w(x, y) - \rho w(x, y) + xy \le 0$ for all $(x, y) \in \mathbb{I}_1$.

Step 2. For all $(x, y) \in \mathbb{I}_2$ we find from the third line of (4.26) and (4.30)

$$\begin{aligned}&\mathcal{L}^y w(x, y) - \rho w(x, y) + xy \\ &\quad = \mathcal{L}^{\bar{y}} R(x, \bar{y}) - \rho R(x, \bar{y}) + x\bar{y} + \kappa\beta R_x(x, \bar{y})(\bar{y} - y) + (c\rho - x)(\bar{y} - y) \\ &\quad = (\bar{y} - y)\left(\frac{\kappa\beta}{\rho + \kappa}\bar{y} + c\rho - x\right) \le (\bar{y} - y)\left(\frac{\kappa\beta}{\rho + \kappa}\bar{y} + c\rho - \bar{x}\right) \le 0,\end{aligned}$$

where we have used that $R(x, \bar{y})$ solves $(\mathcal{L}^{\bar{y}} - \rho)R(x, \bar{y}) + x\bar{y} = 0$ for the second equality, $x \ge \bar{x}$ for any $(x, y) \in \mathbb{I}_2$ for the first inequality, and (4.17) with $y = \bar{y}$ and $F(\bar{y}) = \bar{x}$ for the last inequality.

Step 3. Let $(x, y) \in \mathbb{W}$ be fixed. We define

$$S(x, y) := w_y(x, y) - c = A'(y)\psi(x + \beta y) + \beta A(y)\psi'(x + \beta y) + R_y(x, y) - c,$$

where the last equality holds true by (4.29). We clearly have $S(F(y), y) = 0$ by (4.9). Hence, it suffices to show that $S_x(x, y) \ge 0$ because $x < F(y)$ for all $(x, y) \in \mathbb{W}$. Computing the derivative of S with respect to x gives

$$S_x(x, y) = A'(y)\psi'(x + \beta y) + \beta A(y)\psi''(x + \beta y) + R_{xy}(x, y),$$

and from (4.10) we observe that $S_x(F(y),y) = 0$. Moreover, we have

$$S_{xx}(x,y) = A'(y)\psi''(x+\beta y) + \beta A(y)\psi'''(x+\beta y). \tag{4.40}$$

Recall (4.11) and (4.21). Lemma A.3 and Proposition 4.5 imply that

$$D(y, F(y)+\beta y) > 0 \quad \text{for all } y \in [0,\bar{y}]. \tag{4.41}$$

Now, exploiting (4.13) and (4.14), we find

$$S_{xx}(F(y),y) = -\left[(\rho+\kappa)\psi(F(y)+\beta y)Q_0(F(y)+\beta y)\right]^{-1} D(y,F(y)+\beta y) < 0 \tag{4.42}$$

for all $y \in [0,\bar{y}]$, where the inequality is due to (4.41) and the fact that Q_0 is (strictly) positive. Since $\frac{\psi'''(\cdot)}{\psi''(\cdot)}$ is increasing by Lemma A.1(3), and $A(y)$ is positive for all $y \in [0,\bar{y}]$ by Lemma 4.2, we have for all $x \le F(y)$

$$A'(y) + \frac{\psi'''(x+\beta y)}{\psi''(x+\beta y)}\beta A(y) < A'(y) + \frac{\psi'''(F(y)+\beta y)}{\psi''(F(y)+\beta y)}\beta A(y) < 0,$$

where we have employed both (4.40) and (4.42) for the last inequality. Thus, we have $S_{xx}(x,y) < 0$, and therefore $S_x(x,y) > 0$ for all $(x,y) \in \mathbb{W}$. This completes the proof. ∎

We conclude that w identifies with the value function.

THEOREM 4.9. *Recall w from (4.26), and let $\Delta := (\bar{y}-y)\mathbb{1}_{\{x \ge \bar{x}\}} + (F^{-1}(x) - y)\mathbb{1}_{\{\bar{x}>x>F(y)\}}$. The function w identifies with the value function V from (2.4), and the optimal installation strategy, denoted by $I^\star$, is given by*

$$I^\star_{0-} = 0, \qquad I^\star_t = \Delta + K_{t\wedge\tau}, \quad t \ge 0, \tag{4.43}$$

where $\tau := \inf\{t \ge 0 : K_t = \bar{y} - (y+\Delta)\}$, and where (X,K) is the unique $\mathbb{F}$-adapted process on $[0,\tau]$ with increasing K and starting point $(X_0,K_0) = (x,0)$ such that

$$\begin{aligned} &X_t \le F(y+\Delta+K_t),\\ &dX_t = \kappa\Big((\mu - \beta(y+\Delta+K_t)) - X_t\Big)dt + \sigma dW_t,\\ &dK_t = \mathbb{1}_{\{X_t = F(y+\Delta+K_t)\}}dK_t. \end{aligned} \tag{4.44}$$

Proof. To prove the claim, we aim at applying Theorem 3.3. We already know that $w \in C^{2,1}(\mathbb{R}\times[0,\bar{y}])$ is a solution to the HJB equation (3.7) by Proposition 4.8. Moreover, the function w satisfies the growth condition in (3.11) upon exploiting the facts that A is continuous, ψ is continuous and increasing, and $|R(x,y)| \le K\big(1+|x|\big)$ for any $y \in [0,\bar{y}]$ and some constant $K > 0$.

In a next step, we show the existence of (X,K) satisfying the stochastic differential equation (4.44). To do so, we borrow ideas from [10]; cf. section 5 therein. Fix an arbitrary $T > 0$. We let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{Q})$ be a filtered probability space with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and let B be a $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion under $\mathbb{Q}$. Define the process (X,K) such that, for all $t \in [0,T]$,

$$dX_t = \kappa\Big((\mu - \beta(y+\Delta)) - X_t\Big)dt + \sigma dB_t, \tag{4.45}$$

$$K_t = \min\left\{\sup_{0\le s\le t}\{\bar{F}^{-1}(X_s)\}, \bar{y} - (y+\Delta)\right\}, \tag{4.46}$$

with starting point $(X_0, K_0) = (x, 0)$, and where $\bar{F}^{-1}$ is such that

$$\bar{F}^{-1}(x) := \begin{cases} 0 & \text{if } x < x_0, \\ F^{-1}(x) & \text{if } x \in [x_0, \bar{x}], \\ \bar{y} & \text{if } x > \bar{x}. \end{cases} \tag{4.47}$$

Notice that the pair (X, K) satisfies

$$X_t \le F(y + \Delta + K_t),$$
$$dK_t = \mathbb{1}_{\{X_t = F(y+\Delta+K_t)\}} dK_t$$

for any $t \le \tau \wedge T$. Since K is increasing and $K_t \le \bar{y} - (y + \Delta)$ for any $t \le \tau \wedge T$, we apply Girsanov's theorem (cf. section 3.5 in [19]), so as to obtain an equivalent probability measure $\mathbb{P}$ with respect to $\mathbb{Q}$ such that

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_T} = \exp\left(-\int_0^T \frac{\kappa\beta}{\sigma} K_s dB_s - \frac{1}{2} \int_0^T \left(\frac{\kappa\beta}{\sigma} K_s \right)^2 ds \right)$$

and

$$W_t = B_t + \int_0^t \frac{\kappa\beta}{\sigma} K_s ds$$

is a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The pair (X, K) constructed in this way is a weak solution to (4.44) on $[0, T]$. We will prove in the following that, indeed, (4.44) admits a pathwise unique solution, hence a strong solution. Recall (4.11). We obtain

$$0 < \left(F^{-1}\right)'(x) \le \max_{x_0 \le x' \le \bar{x}} \beta^{-1} \frac{D(F^{-1}(x'), x')}{N(F^{-1}(x'), x') - D(F^{-1}(x'), x')} \quad \text{for all } x \in [x_0, \bar{x}],$$

where the first inequality is due to the monotonicity of F^{-1} and the last inequality is due to (4.24) and (4.25). The continuity of the functions N and D and the fact that

$$N(F^{-1}(x), x) - D(F^{-1}(x), x) > 0 \quad \text{for any } x \in [x_0, \bar{x}],$$

which is due to Lemma A.3, Proposition 4.5, and Lemma A.2, imply $\left(F^{-1}\right)'(x) < \infty$ for all $x \in [x_0, \bar{x}]$. The previous results show that $\bar{F}^{-1}$ is (globally) Lipschitz continuous. Now, let $(\tilde{X}, \tilde{K})$ and $(\hat{X}, \hat{K})$ be two weak solutions of (4.44), defined on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with respect to the same Brownian motion W, and starting from $(x, 0)$, such that both $\tilde{K}$ and $\hat{K}$ are nondecreasing. The (global) Lipschitz continuity of $\bar{F}^{-1}$ implies

$$\begin{aligned} &\left|\tilde{K}_t - \hat{K}_t\right| \\ &\quad = \left| \sup_{0\le s\le t} \left\{F^{-1}(\tilde{X}_s) - (\bar{y} - (y + \Delta))\right\}^+ - \sup_{0\le s\le t} \left\{F^{-1}(\hat{X}_s) - (\bar{y} - (y + \Delta))\right\}^+ \right| \\ &\quad \le \sup_{0\le s\le t} \left\{ \left|F^{-1}(\tilde{X}_s) - F^{-1}(\hat{X}_s)\right| \right\} \le \sup_{0\le s\le t} \bar{K} \left|\tilde{X}_s - \hat{X}_s\right|. \end{aligned} \tag{4.48}$$

The second line of (4.44) written in integral form, together with $\tilde{X}_0 = \hat{X}_0 = x$, implies (recall that now we are working under the new probability measure $\mathbb{P}$)

$$\mathbb{E}\left[\left|\tilde{X}_t - \hat{X}_t\right|\right] \le C_0 \int_0^t \mathbb{E}\left[\left|\tilde{X}_s - \hat{X}_s\right| + \left|\tilde{K}_s - \hat{K}_s\right|\right] ds$$

for some constant $C_0 > 0$. This, combined with (4.48), gives for some constant $C_1 > 0$ the estimate

$$\mathbb{E}\left[\left|\tilde{X}_t - \hat{X}_t\right| + \left|\tilde{K}_t - \hat{K}_t\right|\right] \leq C_1 \int_0^t \mathbb{E}\left[\left|\tilde{X}_s - \hat{X}_s\right| + \left|\tilde{K}_s - \hat{K}_s\right|\right] ds. \tag{4.49}$$

Now, Grönwall's inequality applied on $t \mapsto \mathbb{E}[|\tilde{X}_t - \hat{X}_t| + |\tilde{K}_t - \hat{K}_t|]$ yields

$$0 \leq \mathbb{E}\left[\left|\tilde{X}_t - \hat{X}_t\right| + \left|\tilde{K}_t - \hat{K}_t\right|\right] \leq 0 \tag{4.50}$$

upon recalling that $t \mapsto X_t$ and $t \mapsto K_t$ are continuous for any solution of (4.44). Thus, by (4.50), pathwise uniqueness holds. By [19, Corollary 5.3.23], we find that (4.44) admits a unique strong solution on $[0, T]$. However, since $T > 0$ was chosen to be arbitrary, (4.44) admits a unique strong solution on the whole time interval $[0, +\infty)$.

Finally, since $I^\star$ from (4.43) satisfies (3.12) and (3.13), we conclude that w identifies with V, and $I^\star$ is an optimal installation strategy by Theorem 3.3. ◻

5. Numerical illustrations. The ODE (4.24) cannot be solved analytically, but we are able to solve it numerically with MATLAB. Figure 1 displays a plot of the inverse of the free boundary F with parameters' values given in Table 1.

Recalling the discussion at the beginning of section 4.2, the waiting region $\mathbb{W}$ lies to the left of the free boundary F, while the installation region $\mathbb{I}$ lies to the right. Inside the installation region $\mathbb{I}$, it is optimal to make a lump installation sufficient to arrive to the boundary of $\mathbb{I}$, i.e., to $(x, \min(F^{-1}(x), \bar{y}))$.

5.1. Comparative statics. In this section, we study the sensitivity of the free boundary on the model parameters numerically. The baseline parameters' values are given as in Table 1, and in the following we let σ, β, κ, and $\bar{y}$ vary within a particular set. The sensitivity behaviors of μ, ρ, and c do not show interesting new patterns as the studied ones and are not shown explicitly here. The numerical results can be observed in Figure 2.

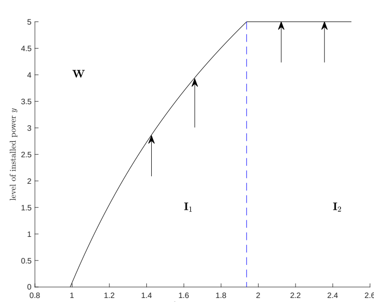
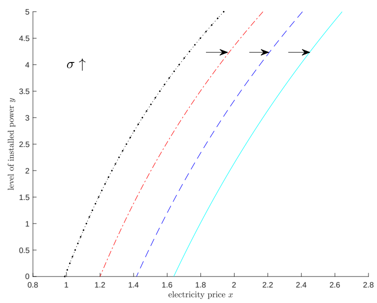


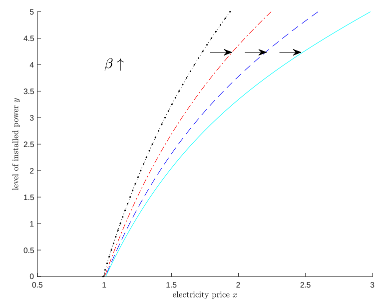
FIG. 1. *Plot of $x \mapsto F^{-1}(x)$ with parameters' values provided in Table 1 and with the waiting region $\mathbb{W}$ (as in (4.1)) and the installation regions $\mathbb{I}_1$ and $\mathbb{I}_2$ (as defined in section 4.2).*

TABLE 1
Parameters' values.

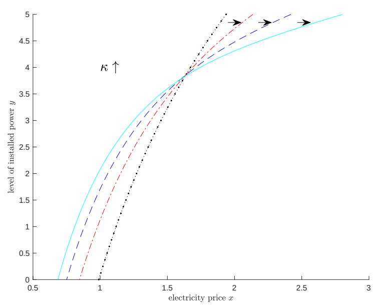
κ	μ	σ	ρ	c	β	$\bar{y}$
0.10	1.00	0.50	0.05	0.30	0.15	5



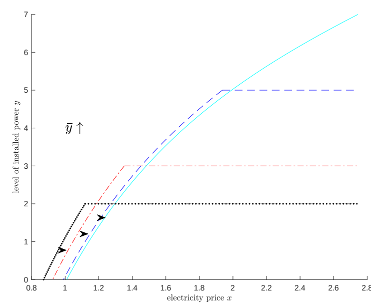
(a) The function F^{-1} with $\sigma = 0.5$ (dotted black), $\sigma = 0.6$ (dash-dot red), $\sigma = 0.7$ (dashed blue), $\sigma = 0.8$ (solid cyan).



(b) The function F^{-1} with $\beta = 0.15$ (dotted black), $\beta = 0.175$ (dash-dot red), $\beta = 0.2$ (dashed blue), $\beta = 0.225$ (solid cyan).



(c) The function F^{-1} with $\kappa = 0.1$ (dotted black), $\kappa = 0.15$ (dash-dot red), $\kappa = 0.20$ (dashed blue), $\kappa = 0.25$ (solid cyan).



(d) The function F^{-1} with $\bar{y} = 2$ (dotted black), $\bar{y} = 3$ (dash-dot red), $\bar{y} = 5$ (dashed blue), $\bar{y} = 7$ (solid cyan)

FIG. 2. *Sensitivity of the function $x \mapsto F^{-1}(x)$ with respect to the model parameters. In each subfigure, the parameter values which are not varied are those provided in Table* 1.

We first study the behavior of the free boundary with respect to the volatility displayed in Figure 2(a). Here the volatility parameter σ takes values in $\{0.5; 0.6; 0.7; 0.8\}$, and we can observe that F^{-1} is shifted to the right as σ increases, that is, the installation of additional panels is undertaken at higher prices. Though monotone, the shift is not parallel, differently from what the figure suggests. The firm might be afraid of receiving lower (possibly even negative) future prices due to higher uncertainty. This behavior is in line with the real options literature: when uncertainty increases, the agent is more reluctant to act; see, for example, [23].

In Figure 2(b), the impact parameter β takes values in $\{0.15; 0.175; 0.2; 0.225\}$, and as a consequence we find that F^{-1} is shifted to the right as β increases. We explain this observation by the fact that the impact of a higher electricity production on the future electricity prices is higher as β increases. Therefore, the company starts to produce more electricity at higher prices, so as to avoid lower (and possibly negative) electricity prices in the short run.

The dependence on κ can be observed in Figure 2(c) and is maybe the most peculiar of the observed dependencies, acting in a nonmonotone way. Here, we let κ take values in $\{0.1; 0.15; 0.2; 0.25\}$. We find that higher values for the mean reversion speed κ lead the company to start installing solar panels at lower prices, but after some point, the company becomes instead more reluctant. This behavior can be explained by the fact that two effects play a role: on one hand, a higher mean reversion speed reduces its ratio with respect to σ, the uncertainty is decreased, and hence a converse behavior with respect to Figure 2(a) can be observed. On the other hand, a higher mean reversion speed also intensifies the impact of the company's actions on the price dynamics. Therefore, it behaves as in 2(b).

Finally, we let $\bar{y}$ vary in $\{0.5; 1; 2; 5\}$, and we observe that F^{-1} moves to the right as $\bar{y}$ increases. Consequently, the possibility to increase the level of installed power up to a higher level makes the company more reluctant to act.

As concerns the sensitivity with respect to the other parameters μ, ρ, and c, we do not provide graphical illustrations explicitly here, as they are similar to other plots shown in Figure 2. In more detail, the dependence of the free boundary on c is similar to that of σ, while instead the dependence on μ and ρ are "symmetric" to that on β, i.e., the free boundary is displaced toward negative values of x as μ or ρ increases.

Appendix A. Auxiliary results.

LEMMA A.1. *Let $\mathcal{L}^y$, for $y \geq 0$, be the generator from* (3.6). *Then the following holds true.*

(1) *The strictly increasing positive fundamental solution $\psi(\cdot)$ and the strictly decreasing positive fundamental solution $\phi(\cdot)$ to the ODE $(\mathcal{L}^0 - \rho)u = 0$ are given by*

$$\text{(A-1)} \qquad \begin{aligned} \psi(x) &= e^{\frac{\kappa(x-\mu)^2}{2\sigma^2}} D_{-\frac{\rho}{\kappa}}\left(-\frac{x-\mu}{\sigma}\sqrt{2\kappa}\right) \quad \textit{and} \\ \phi(x) &= e^{\frac{\kappa(x-\mu)^2}{2\sigma^2}} D_{-\frac{\rho}{\kappa}}\left(\frac{x-\mu}{\sigma}\sqrt{2\kappa}\right), \end{aligned}$$

where

$$\text{(A-2)} \qquad D_\alpha(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\frac{t^2}{2} - xt} dt, \quad \alpha < 0,$$

is the cylinder function of order α and $\Gamma(\,\cdot\,)$ is the Euler's Gamma function.

(2) *Denoting by $\psi^{(k)}$ and $\phi^{(k)}$ the kth derivative of ψ and ϕ, $k \in \mathbb{N}_0$, one has that $\psi^{(k)}$ and $\phi^{(k)}$ are strictly convex and $\psi^{(k)}$ ($\phi^{(k)}$, respectively) identifies with the strictly increasing positive (strictly decreasing positive, respectively) fundamental solution (up to a positive constant) to $(\mathcal{L}^0 - (\rho + k\kappa))u = 0$. In particular, it holds that*

$$\text{(A-3)} \qquad \begin{aligned} &\frac{\sigma^2}{2}\psi^{(k+2)}(x+\beta y) + \kappa\big((\mu - \beta y) - x\big)\psi^{(k+1)}(x+\beta y) \\ &\quad - (\rho + k\kappa)\psi^{(k)}(x+\beta y) = 0 \end{aligned}$$

for any $x \in \mathbb{R}$ and $y \geq 0$.

(3) *For any $k \in \mathbb{N}_0$, $\psi^{(k)}(x)\psi^{(k+2)}(x) - \psi^{(k+1)}(x)^2 > 0$ for all $x \in \mathbb{R}$.*

(4) *For any $k \in \mathbb{N}_0$, the function $\Psi_k : \mathbb{R} \mapsto \mathbb{R}$ defined as*

$$\Psi_k(x) = \frac{\psi^{(k+1)}(x)^2}{\psi^{(k)}(x)\psi^{(k+2)}(x)}$$

is strictly increasing.

(5) *Denote by* $\psi(\,\cdot\,;y)$ ($\phi(\,\cdot\,;y)$, *respectively) the strictly increasing (strictly decreasing, respectively) positive fundamental solution to* $(\mathcal{L}^y-\rho)u=0$ *for* $y\geq 0$. *Then, one can identify*

$$\psi(x;y)=\psi(x+\beta y),\qquad \phi(x;y)=\phi(x+\beta y).$$

Proof. The proofs of (1)–(3) can be found in [13]; cf. Lemma 4.3 therein. In [20], the author provides bounds and monotonicity properties of ratios involving a class of Hermite and parabolic cylinder functions. These results are helpful for problems that deal with the eigenfunctions of the Ornstein–Uhlenbeck generators as they are connected to Hermite and parabolic cylinder functions. Here, we exploit the result from Step 1 in the proof of Theorem 3.1 in [20] in order to obtain (4). Moreover, (5) follows from (2), and in particular from equation (A-3) with $k=0$. ∎

LEMMA A.2. *Recall the functions* D *and* N *from* (4.21) *and* (4.23). *For any* $(y,z)\in\mathbb{R}\times\mathbb{R}$ *such that* $D(y,z)\geq 0$, *we have*

$$N(y,z)>D(y,z).$$

Proof. Let $(y,z)\in\mathbb{R}\times\mathbb{R}$ be such that $D(y,z)\geq 0$. The previous inequality implies

$$(\rho+\kappa)\left(c-\tilde{R}(z,y)\right)\geq -\frac{Q_0'(z)}{Q_1(z)}, \tag{A-4}$$

as Q_1 is strictly positive.

In order to proceed, we introduce the function $\Theta:\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$ such that

$$\Theta(z):=\psi''(z)Q_0(z)-\psi(z)Q_1(z).$$

Exploiting Lemma A.1(4) with $k=0$, we find that Θ is strictly positive. Now, we use both (A-4) and the positivity of Θ to get

$$\begin{aligned} N(y,z)-D(y,z)&=(\rho+\kappa)\left(c-\tilde{R}(z,y)\right)\Theta(z)+\frac{2(\rho+\kappa)}{\rho}\psi'(z)Q_0(z)-\psi(z)Q_0'(z)\\ &\geq -\frac{Q_0'(z)}{Q_1(z)}\Theta(z)+\frac{2(\rho+\kappa)}{\rho}\psi'(z)Q_0(z)-\psi(z)Q_0'(z)\\ &=(\rho Q_1(z))^{-1}Q_0(z)\left[-\rho\psi''(z)Q_0'(z)+2(\rho+\kappa)\psi'(z)Q_1(z)\right], \end{aligned} \tag{A-5}$$

where we have rearranged terms after the equality. To finish the proof, we employ (A-3) with $k=0,1,2$ for (A-5), to obtain

$$N(y,z)-D(y,z)\geq\frac{\sigma^2}{2}\left(\rho Q_1(z)\right)^{-1}Q_0(z)\Big[\psi'''(z)Q_1(z)-\psi'(z)Q_2(z)\Big]>0, \tag{A-6}$$

where the last inequality holds true upon recalling $Q_k>0$ and by the fact $\psi'''(z)Q_1(z)-\psi'(z)Q_2(z)>0$ which is due to Lemma A.1(4) with $k=1$. ∎

LEMMA A.3. *We have* $D(\bar{y},\tilde{F}(\bar{y}))>0$, *and it holds that* $\tilde{F}'(\bar{y})>\beta$.

Proof. Recall (4.19) and Lemma 4.3 accordingly. Since $\tilde{F}(\bar{y})=\tilde{x}$, the point $\bar{y}$ satisfies

$$(\rho+\kappa)\left(c-\tilde{R}(\tilde{F}(\bar{y}),\bar{y})\right)=-\frac{\psi(\tilde{F}(\bar{y}))}{\psi'(\tilde{F}(\bar{y}))}. \tag{A-7}$$

We get from (4.21) and (A-7) that

$$D(\bar{y},\tilde{F}(\bar{y})) = \frac{Q_0(\tilde{F}(\bar{y}))\psi(\tilde{F}(\bar{y}))\psi''(\tilde{F}(\bar{y}))}{\psi'(\tilde{F}(\bar{y}))} > 0 \tag{A-8}$$

upon recalling that $Q_0 > 0$. Now, Lemma A.2 implies $N(\bar{y},\tilde{F}(\bar{y})) - D(\bar{y},\tilde{F}(\bar{y})) > 0$. Hence, we find

$$\tilde{F}'(\bar{y}) = \mathcal{G}(\bar{y},\tilde{F}(\bar{y})) = \beta \times \frac{N(\bar{y},\tilde{F}(\bar{y}))}{D(\bar{y},\tilde{F}(\bar{y}))} > \beta. \tag{A-9}$$

□

Appendix B. Proofs of results from sections 3 and 4.

Proof of Lemma 3.1. To prove (3.1), first notice that $X_t^{x,y,I} \le X_t^x$ $\mathbb{P}$-a.s. for all $t \ge 0$, and therefore

$$\begin{aligned} X_t^{x,y,I} &= x + \int_0^t \kappa\big((\mu - \beta Y_t^{y,I}) - X_s^{x,y,I}\big)ds + \sigma W_t \\ &\ge x + \int_0^t \kappa\big(\mu - X_s^x\big)ds + \sigma W_t - \kappa\beta\bar{y}t \\ &= X_t^x - \kappa\beta\bar{y}t \ge -|X_t^x| - \kappa\beta\bar{y}t, \end{aligned}$$

where we have used that $Y_t^{y,I} \le \bar{y}$ $\mathbb{P}$-a.s. for all $t \ge 0$. Also, one clearly has $X_t^{x,y,I} \le X_t^x \le |X_t^x| + \kappa\beta\bar{y}t$. Hence, (3.1) follows. Equation (3.2) follows from elementary properties of Ornstein–Uhlenbeck processes. Finally, for all $t > 0$ we have

$$\begin{aligned} \mathbb{E}\left[|X_t^{x,y}|\right] &\le \big(\mathbb{E}[(X_t^{x,y})^2]\big)^{1/2} = \big(\mathrm{Var}[X_t^{x,y}] + (\mathbb{E}[X_t^{x,y}])^2\big)^{1/2} \\ &\le \left(\frac{\sigma^2}{2\kappa} + 2x^2e^{-2\kappa t} + 2(\mu - \beta y)^2(1 - e^{-\kappa t})^2\right)^{1/2} \\ &\le \left(2x^2 + 2(\mu-\beta y)^2 + \frac{\sigma^2}{2\kappa}\right)^{1/2}. \end{aligned}$$

Since $\sqrt{a^2 + x^2} \le a + |x|$ for all $a > 0$ and $x \in \mathbb{R}$, we obtain (3.3) by letting $K := \max(\sqrt{2}, \sqrt{2\mu^2 + \frac{\sigma^2}{2\kappa}}, \sqrt{2(\mu - \beta\bar{y})^2 + \frac{\sigma^2}{2\kappa}})$. □

Proof of Proposition 3.2. The proof employs arguments from the proof of Proposition 3.1 in [13] that are adjusted to our setting. In a first step we prove that (3.5) holds true, and then in a second step we show the monotonicity property of V.

Step 1. Let $(x,y) \in \mathbb{R} \times [0,\bar{y}]$ be given and fixed. In order to prove the lower bound of V, we take the admissible (non-)installation strategy I^0, and since $y \in [0,\bar{y}]$, we obtain

$$V(x,y) \ge R(x,y) > -K_1\big(1 + |x|\big) \tag{B-1}$$

for some $K_1 > 0$.

To determine the upper bound of V, recall that we indicate the uncontrolled price process with X^x and notice that, for any $I \in \mathcal{I}^{\bar{y}}(y)$, upon observing that $X^{x,y,I} \le X^x$ $\mathbb{P}$-a.s. for any $I \in \mathcal{I}^{\bar{y}}(y)$ we find by (3.3)

$$\begin{aligned} \mathcal{J}(x,y,I) &\le \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} X_t^{x,y,I} Y_t^{y,I} dt\bigg] \le \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} X_t^x Y_t^{y,I} dt\bigg] \\ &\le \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big|X_t^x\big|Y_t^{y,I} dt\bigg] \le \bar{y}\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}|X_t^x| dt\bigg] \le K_2\big(1 + |x|\big) \end{aligned} \tag{B-2}$$

for $K_2 := K_1/\rho > 0$, where K_1 is the constant in (3.3). Finally, from (B-1) and (B-2), we have that (3.5) holds with $K = \max(K_1, K_2)$.

Step 2. If $y = \bar{y}$, then the only admissible strategy is I^0, thus $V(x, \bar{y}) = R(x, \bar{y})$. In order to show that $x \mapsto V(x, y)$ is increasing, let $x_2 > x_1$, and notice that one has $X_t^{x_2,y,I} \geq X_t^{x_1,y,I}$ $\mathbb{P}$-a.s. for any $t \geq 0$ and $I \in \mathcal{I}^{\bar{y}}(y)$. Thus $\mathcal{J}(x_2, y, I) \geq \mathcal{J}(x_1, y, I)$ which implies $V(x_2, y) \geq V(x_1, y)$. ∎

Proof of Theorem 3.3. Since we have $w(x, \bar{y}) = R(x, \bar{y}) = V(x, \bar{y})$ by assumption, we let $y < \bar{y}$. In a first step, we prove that $w \geq v$ on $\mathbb{R} \times [0, \bar{y})$, and then in a second step, we show that $w \leq v$ on $\mathbb{R} \times [0, \bar{y})$ and the optimality of $I^\star$ satisfying (3.12) and (3.13).

Step 1. Let $(x, y) \in \mathbb{R} \times [0, \bar{y})$ be given and fixed, and $I \in \mathcal{I}^{\bar{y}}(y)$. For $N > 0$ we set $\tau_{R,N} := \tau_R \wedge N$, where $\tau_R := \inf\{s > 0 : X_s^{x,y,I} \notin (-R, R)\}$. In the following, we write $\Delta I_s := I_s - I_{s-}$, $s \geq 0$, and I^c denotes the continuous part of $I \in \mathcal{I}^{\bar{y}}(y)$. By an application of Itô's formula, we have

$$
\begin{aligned}
&e^{-\rho\tau_{R,N}} w\left(X^{x,y,I}_{\tau_{R,N}}, Y^{y,I}_{\tau_{R,N}}\right) - w(x,y) \\
&= \int_0^{\tau_{R,N}} e^{-\rho s}\Big(\mathcal{L}^y w\left(X_s^{x,y,I}, Y_s^{y,I}\right) - \rho w\left(X_s^{x,y,I}, Y_s^{y,I}\right)\Big) ds \\
&\quad + \underbrace{\sigma \int_0^{\tau_{R,N}} e^{-\rho s} w_x\left(X_s^{x,y,I}, Y_s^{y,I}\right) dW_s}_{=:M_{\tau_{R,N}}} \\
&\quad + \sum_{0 \leq s \leq \tau_{R,N}} e^{-\rho s}\left[w\left(X_s^{x,y,I}, Y_s^{y,I}\right) - w\left(X_s^{x,y,I}, Y_{s-}^{y,I}\right)\right] \\
&\quad + \int_0^{\tau_{R,N}} e^{-\rho s} w_y\left(X_s^{x,y,I}, Y_s^{y,I}\right) dI_s^c
\end{aligned}
\tag{B-3}
$$

upon noticing that $t \mapsto X_t^{x,y,I}$ is continuous almost surely for any $I \in \mathcal{I}^{\bar{y}}(y)$. Now, we find

$$
\begin{aligned}
w\left(X_s^{x,y,I}, Y_s^{y,I}\right) - w\left(X_s^{x,y,I}, Y_{s-}^{y,I}\right) &= w\left(X_s^{x,y,I}, Y_{s-}^{y,I} + \Delta I_s\right) - w\left(X_s^{x,y,I}, Y_{s-}^{y,I}\right) \\
&= \int_0^{\Delta I_s} w_y\left(X_s^{x,y,I}, Y_{s-}^{y,I} + u\right) du,
\end{aligned}
$$

which substituted back into (B-3) gives the equivalence

$$
\begin{aligned}
&\int_0^{\tau_{R,N}} e^{-\rho s} X_s^{x,y,I} Y_s^{y,I} ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI_s = w(x,y) - e^{-\rho\tau_{R,N}} w\left(X^{x,y,I}_{\tau_{R,N}}, Y^{y,I}_{\tau_{R,N}}\right) \\
&\quad + \int_0^{\tau_{R,N}} e^{-\rho s}\Big(\mathcal{L}^y w\left(X_s^{x,y,I}, Y_s^{y,I}\right) - \rho w\left(X_s^{x,y,I}, Y_s^{y,I}\right) + X_s^{x,y,I} Y_s^{y,I}\Big) ds + M_{\tau_{R,N}} \\
&\quad + \sum_{0 \leq s \leq \tau_{R,N}} e^{-\rho s} \int_0^{\Delta I_s} \left[w_y\left(X_s^{x,y,I}, Y_{s-}^{y,I} + u\right) - c\right] du \\
&\quad + \int_0^{\tau_{R,N}} e^{-\rho s}\left[w_y\left(X_s^{x,y,I}, Y_s^{y,I}\right) - c\right] dI_s^c
\end{aligned}
$$

upon adding $\int_0^{\tau_{R,N}} e^{-\rho s} X_s^{x,y,I} Y_s^{y,I} ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI_s$ on both sides of (B-3). Since w satisfies (3.7) and (3.11), by taking expectations on both sides of the latter equation, and using that $\mathbb{E}[M_{\tau_{R,N}}] = 0$, we have

$$\text{(B-4)}\qquad \begin{aligned}&\mathbb{E}\left[\int_0^{\tau_{R,N}} e^{-\rho s} X_s^{x,y,I} Y_s^{y,I} ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI_s\right]\\ &\quad \leq w(x,y) + K\mathbb{E}\Big[e^{-\rho\tau_{R,N}}\Big(1+|X^{x,y,I}_{\tau_{R,N}}|\Big)\Big].\end{aligned}$$

In order to apply the dominated convergence theorem in (B-4), we use (3.1) and we find that $\mathbb{P}$-a.s.

(B-5)

$$\left|\int_0^{\tau_{R,N}} e^{-\rho s} X_s^{x,y,I} Y_s^{y,I} ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI_s\right| \leq \bar{y}\int_0^{\infty} e^{-\rho s}\Big(|X_s^x| + \kappa\beta\bar{y}s\Big)ds + c\bar{y},$$

and the first expression on the right-hand side of (B-5) is integrable by (3.3). On the other hand, to take care of the expectation on the right-hand side of (B-4), we employ again (3.1) to get for some constant $C_1 > 0$

$$\text{(B-6)}\qquad \begin{aligned}&\mathbb{E}\Big[e^{-\rho\tau_{R,N}}\Big(1+|X^{x,y,I}_{\tau_{R,N}}|\Big)\Big]\\ &\quad\leq C_1\mathbb{E}\left[e^{-\rho\tau_{R,N}}\left(1+\tau_{R,N}\right)\right] + \mathbb{E}\left[e^{-\frac{\rho}{2}\tau_{R,N}}\sup_{t\geq 0} e^{-\frac{\rho}{2}t}|X_t^x|\right]\\ &\quad\leq C_1\mathbb{E}\left[e^{-\rho\tau_{R,N}}\left(1+\tau_{R,N}\right)\right] + \mathbb{E}\left[e^{-\rho\tau_{R,N}}\right]^{\frac{1}{2}}\mathbb{E}\left[\sup_{t\geq 0} e^{-\rho t}(X_t^x)^2\right]^{\frac{1}{2}},\end{aligned}$$

where we have used Hölder's inequality in the last step. As for the last expectation in (B-6), observe that by Itô's formula we find

(B-7)

$$\begin{aligned}e^{-\rho t}(X_t^x)^2 &\leq x^2 + \int_0^t e^{-\rho u}\Big[\rho(X_u^x)^2 + \sigma^2\Big]du\\ &\quad + \int_0^t 2e^{-\rho u}|X_u^x|(\kappa(|\mu| + |X_u^x|))du + 2\sigma\sup_{t\geq 0}\left|\int_0^t e^{-\rho u}X_u^x dW_u\right|.\end{aligned}$$

By an application of the Burkholder–Davis–Gundy inequality (cf. Theorem 3.28 in [19]), we find that

$$\text{(B-8)}\qquad \mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^t e^{-\rho u}\sigma X_u^x dW_u\right|\right] \leq C_2(1+|x|)$$

for some constant $C_2 > 0$. Then, since standard calculations show that $\mathbb{E}\big[|X_u^x|^q\big] \leq \tilde{C}(1+|x|^q)$ for $q \in \{1,2\}$ and some $\tilde{C} > 0$, we obtain from (B-7) and (B-8)

$$\text{(B-9)}\qquad \mathbb{E}\left[\sup_{t\geq 0} e^{-\rho t}(X_t^x)^2\right] \leq C_3(1+x^2)$$

for some constant $C_3 > 0$, and therefore, it follows with (B-6)

$$\text{(B-10)}\qquad \lim_{N\uparrow\infty}\lim_{R\uparrow\infty}\mathbb{E}\Big[e^{-\rho\tau_{R,N}}\Big(1+|X^{x,y,I}_{\tau_{R,N}}|\Big)\Big] = 0.$$

Hence, we can invoke the dominated convergence theorem in order to take limits as $R \to \infty$ and then as $N \to \infty$, so to get $\mathcal{J}(x,y,I) \leq w(x,y)$. Since $I \in \mathcal{I}^{\bar{y}}(y)$ is arbitrary, we have $V(x,y) \leq w(x,y)$, which yields $V \leq w$ by arbitrariness of (x,y) in $\mathbb{R}\times[0,\bar{y})$.

Step 2. Let $I^\star \in \mathcal{I}^{\bar{y}}(y)$ satisfying (3.12) and (3.13), and $\tau^\star_{R,N} := \inf\{t \geq 0 : X^{x,y,I^\star}_t \notin (-R,R)\} \wedge N$. Arguing in the same way as in Step 1 all the inequalities become equalities and we obtain

$$\mathbb{E}\left[\int_0^{\tau_{R,N}} e^{-\rho s} X^{x,y,I^\star}_s Y^{y,I^\star}_s ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI^\star_s\right] + \mathbb{E}\Big[e^{-\rho\tau^\star_{R,N}} w\left(X^{x,y,I^\star}_{\tau^\star_{R,N}}, I^\star_{\tau^\star_{R,N}}\right)\Big] = w(x,y). \tag{B-11}$$

Now, because $I^\star$ is admissible and upon employing (3.11) and (B-10), we proceed as in Step 1 and take limits as $R \uparrow \infty$ and $N \uparrow \infty$ in (B-11) to find $\mathcal{J}(x,y,I^\star) \geq w(x,y)$. Since clearly $V(x,y) \geq \mathcal{J}(x,y,I^\star)$, then $V(x,y) \geq w(x,y)$ for all $(x,y) \in \mathbb{R} \times [0,\bar{y})$. Hence, using Step 1, $V = w$ on $\mathbb{R} \times [0,\bar{y})$ and $I^\star$ is optimal. ∎

Proof of Lemma 4.2. In the following, Step 1 proves the positivity and the monotonicity property of the function A, while Step 2 provides both the representation of A and the lower bound of F.

Step 1. Recalling that $R_{yx}(x,y) = (\rho+\kappa)^{-1}$ for all $(x,y) \in \mathbb{R} \times [0,\bar{y}]$, we find from (4.10) that

$$A'(y) = -\beta\frac{\psi''(\tilde{F}(y))}{\psi'(\tilde{F}(y))}A(y) - \Big((\rho+\kappa)\psi'(\tilde{F}(y))\Big)^{-1} = \mathcal{H}(\tilde{F}(y), A(y)), \tag{B-12}$$

where $\mathcal{H} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is such that

$$\begin{aligned}\mathcal{H}(\bar{F}, A) &= -\beta\frac{\psi''(\bar{F})}{\psi'(\bar{F})}A - \left((\rho+\kappa)\psi'(\bar{F})\right)^{-1} \\ &= -\left((\rho+\kappa)\psi'(\bar{F})\right)^{-1}\left(\beta(\rho+\kappa)\psi''(\bar{F})A + 1\right).\end{aligned}$$

In light of the boundary condition $w(x,\bar{y}) = R(x,\bar{y})$ (cf. Theorem 3.3), we must have that

$$A(\bar{y}) = 0. \tag{B-13}$$

Due to (B-13) and the fact that $\mathcal{H}|_{\mathbb{R}\times[0,\infty)}$ is strictly negative as $\psi^{(k)}$ is strictly positive for any $k \in \mathbb{N}_0$ (cf. Lemma A.1(2)), we conclude that A is both strictly positive and strictly decreasing.

Step 2. Recall (4.13), that is,

$$A(y) = \beta^{-1} \times \frac{\psi'(\tilde{F}(y))\Big(c - \tilde{R}(\tilde{F}(y),y)\Big) + (\rho+\kappa)^{-1}\psi(\tilde{F}(y))}{-Q_0(\tilde{F}(y))}. \tag{B-14}$$

Now, the numerator on the right-hand side of (B-14) reads as

$$\begin{aligned}&(\rho(\rho+\kappa))^{-1}\Big[\rho(\rho+\kappa)\psi'(\tilde{F}(y))\left(c - \tilde{R}(\tilde{F}(y),y)\right) + \rho\psi(\tilde{F}(y))\Big] \\ &\quad = (\rho(\rho+\kappa))^{-1}\left[(\rho+\kappa)\Big(c\rho + \frac{(\rho+2\kappa)\beta}{\rho+\kappa}y - \tilde{F}(y)\Big)\psi'(\tilde{F}(y)) + \frac{\sigma^2}{2}\psi''(\tilde{F}(y))\right]\end{aligned}$$

upon using (A-3) with $k = 0$. Hence,

$$A(y) = (\beta\rho(\rho+\kappa))^{-1} \times \frac{(\rho+\kappa)\Big(c\rho + \frac{(\rho+2\kappa)\beta}{\rho+\kappa}y - \tilde{F}(y)\Big)\psi'(\tilde{F}(y)) + \frac{\sigma^2}{2}\psi''(\tilde{F}(y))}{-Q_0(\tilde{F}(y))}. \tag{B-15}$$

Due to the facts that the denominator on the right-hand side of (B-15) is strictly negative by Lemma A.1(3) and that A is strictly positive by Step 1, the numerator on the right-hand side of (B-15) must be strictly negative: this is possible only if

$$c\rho + \frac{(\rho+2\kappa)\beta}{\rho+\kappa}y - \tilde{F}(y) < 0$$

as $\psi^{(k)}$ is strictly positive for any $k \in \mathbb{N}$. Hence, $\tilde{F}$ satisfies

$$\tilde{F}(y) > c\rho + \frac{(\rho+2\kappa)\beta}{\rho+\kappa}y \quad \text{for all } y \in [0,\bar{y}]. \tag{B-16}$$

∎

Proof of Lemma 4.3. We rewrite the left-hand side of (4.19) as follows:

$$\begin{aligned}&\psi'(\tilde{x})\left(c - \tilde{R}(\tilde{x},\bar{y})\right) + (\rho+\kappa)^{-1}\psi(\tilde{x})\\ &\quad = -(\rho+\kappa)^{-1}\left(\psi'(\tilde{x})\left((\rho+\kappa)\tilde{R}(\tilde{x},\bar{y}) - c(\rho+\kappa)\right) - \psi(\tilde{x})\right).\end{aligned}$$

Now, the proof is a slight modification of the proof of Lemma 4.4 in [13] upon adjusting the cost factor in [13] by $c(\rho+\kappa) - \frac{\mu\kappa - \beta(\rho+2\kappa)\bar{y}}{\rho}$. ∎

Proof of Proposition 4.5. The proof is organized into two steps: in a first step, we provide a representation of the function D that is used after. Then, in Step 2, we show the existence and uniqueness of a strictly increasing maximal solution $\tilde{F}$ of the ODE (4.24) and prove (by a contradiction) that $\tilde{F}$ in fact exists on the interval $[0,\bar{y}]$.

Step 1. Recall (4.21), and let $\tilde{D} : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ be a function which is given by

$$\tilde{D}(y,z) = \left[(\rho+\kappa)\psi(z)Q_0(z)\right]^{-1} D(y,z). \tag{B-17}$$

Then, where $\tilde{F}$ exists, we find upon employing (4.13) and (4.14)

$$\tilde{D}(y,\tilde{F}(y)) = -\beta\psi'''(\tilde{F}(y))A(y) - \psi''(\tilde{F}(y))A'(y). \tag{B-18}$$

Now, Lemma A.1(2) gives for any $k \in \mathbb{N}_0$

$$\frac{\sigma^2}{2}\psi^{(k+2)}(x) + \kappa(\mu - x)\psi^{(k+1)}(x) - (\rho + k\kappa)\psi^{(k)}(x) = 0, \quad x \in \mathbb{R}, \tag{B-19}$$

and therefore we have

(B-20)

$$\psi^{(k+2)}(\tilde{F}(y)) = -\frac{2\kappa}{\sigma^2}\left(\mu - \tilde{F}(y)\right)\psi^{(k+1)}(\tilde{F}(y)) + \frac{2(\rho+k\kappa)}{\sigma^2}\psi^{(k)}(\tilde{F}(y)), \quad k \in \mathbb{N}_0.$$

Using (B-18) and the latter equation (B-20) with $k = 0, 1$, we obtain

$$\begin{aligned}\tilde{D}(y,\tilde{F}(y)) =& \frac{2}{\sigma^2}\Big[\kappa\left(\mu - \tilde{F}(y)\right)\left(\beta\psi''(\tilde{F}(y))A(y) + \psi'(\tilde{F}(y))A'(y)\right)\\ &- \rho\big(\beta\psi'(\tilde{F}(y))A(y) + \psi(\tilde{F}(y))A'(y)\big) - \kappa\beta\psi'(\tilde{F}(y))A(y)\Big]\\ =& \frac{2}{\sigma^2}\Big[\tilde{F}(y) - c\rho - \frac{(\rho+2\kappa)\beta}{\rho+\kappa}y - \kappa\beta\psi'(\tilde{F}(y))A(y)\Big],\end{aligned} \tag{B-21}$$

where we have employed (4.9) and (4.10) (with F being replaced according to (4.11)) for the last equality.

Step 2. Recall (4.24) and (4.25). In the following, we denote by $\mathcal{D}_{\mathcal{G}}$ the domain of $\mathcal{G}$, that is, $\mathcal{D}_{\mathcal{G}} = (\mathbb{R} \times \mathbb{R}) \setminus \{(y, z) \in \mathbb{R}^2 : D(y, z) = 0\}$. Since $\psi^{(k)}$ is continuously differentiable for any $k \in \mathbb{N}$, the functions N and D are continuously differentiable, respectively. Therefore, $\mathcal{G}(y, \cdot)$ is locally Lipschitz-continuous on its domain $\mathcal{D}_{\mathcal{G}}$ which is an open set. Hence, we find that the ODE (4.24) with the boundary condition $\tilde{F}(\bar{y}) = \tilde{x}$ admits a unique maximal solution $\tilde{F}$ on an interval $I_{\max} = (y_-, y_+)$ with $\bar{y} \in I_{\max}$. Since we want to show the existence and uniqueness of a solution on $[0, \bar{y}]$, it is enough to prove that $y_- < 0$. Following, for example, Theorem 2.10 in [3], $y_- < \bar{y}$ is such that

$$\text{either (i)} \lim_{y \downarrow y_-} \Big(||(y, \tilde{F}(y))|| \Big)^{-1} = 0, \text{ or (ii)} \lim_{y \downarrow y_-} \inf_{w \in \partial \mathcal{D}_{\mathcal{G}}} ||(y, \tilde{F}(y)) - w|| = 0,$$

where $\partial \mathcal{D}_{\mathcal{G}} = \{(y, z) \in \mathbb{R}^2 : D(y, z) = 0\}$ is the boundary of the domain of $\mathcal{G}$, and $|| \cdot ||$ is a norm in $\mathbb{R}^2$.

Now, suppose $y_- \geq 0$. Notice that $N(y, \tilde{F}(y)) > D(y, \tilde{F}(y)) > 0$ for all $y \in I_{\max}$ by Lemmas A.3 and A.2, and therefore we have $\tilde{F}' > \beta > 0$ on $I_{\max}$. Lemma 4.2 guarantees that $\tilde{F}$ is bounded from below on $(y_-, \bar{y}]$, and together with its monotonicity property, we must have that $\lim_{y \downarrow y_-} (||(y, \tilde{F}(y))||)^{-1} > K$ for some $K > 0$. Thus, in order to derive a contradiction, it is left to prove that condition (ii) above is not satisfied, so as to show $\lim_{y \downarrow y_-} D(y, \tilde{F}(y)) \neq 0$. Again, due to the boundedness of $\tilde{F}$ and the fact that both Q_0 and ψ are strictly positive, we find

$$\psi(\tilde{F}(y)) Q_0(\tilde{F}(y)) > K_1 \quad \text{for all } y \in (y_-, \bar{y}]$$

for some $K_1 > 0$. Therefore, upon recalling (B-17), we can complete the proof by showing that $\lim_{y \downarrow y_-} \tilde{D}(y, \tilde{F}(y)) \neq 0$. Lemma A.3 implies

$$\tilde{D}(\bar{y}, \tilde{F}(\bar{y})) > 0. \tag{B-22}$$

Computing the total derivative of $\tilde{D}(y, \tilde{F}(y))$ with respect to $y \in I_{\max}$, upon using (B-21), gives

(B-23)
$$\begin{aligned} \frac{d}{dy} \tilde{D}(y, \tilde{F}(y)) &= \frac{2}{\sigma^2} \left[\tilde{F}'(y) \Big(1 - \kappa \beta \psi''(\tilde{F}(y)) A(y) \Big) - \frac{(\rho + 2\kappa)\beta}{\rho + \kappa} - \kappa \beta \psi'(\tilde{F}(y)) A'(y) \right] \\ &= \frac{2}{\sigma^2} \Big(\tilde{F}'(y) - \beta \Big) \Big(1 - \kappa \beta \psi''(\tilde{F}(y)) A(y) \Big), \end{aligned}$$

where the last equality holds by an application of (4.10) (again, with F being replaced according to (4.11)). Next, we write the last coefficient in (B-23), that is, $1 - \kappa \beta \psi''(\tilde{F}(y)) A(y)$, as a function of $G : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined as

$$\begin{aligned} G(y, z) = ((\rho + \kappa) Q_0(z))^{-1} \Big[&(\rho + 2\kappa) \psi(z) \psi''(z) - (\rho + \kappa) \psi'(z)^2 \\ &+ \kappa(\rho + \kappa) \Big(c - \tilde{R}(z, y) \Big) \psi'(z) \psi''(z) \Big]. \end{aligned}$$

Employing (4.13), we get $1 - \kappa \beta \psi''(\tilde{F}(y)) A(y) = G(y, \tilde{F}(y))$, and thus we have

$$\frac{d}{dy} \tilde{D}(y, \tilde{F}(y)) = \frac{2}{\sigma^2} \Big(\tilde{F}'(y) - \beta \Big) G(y, \tilde{F}(y)). \tag{B-24}$$

Now, let $(y^\star, z^\star) \in \mathbb{R} \times \mathbb{R}$ be such that $\tilde{D}(y^\star, z^\star) = 0$. We find from (B-17) that $D(y^\star, z^\star) = 0$. Hence, upon recalling (4.21), it holds that

$$(\rho + \kappa)\left(c - \tilde{R}(z^\star, y^\star)\right) = -\frac{Q_0'(z^\star)}{Q_1(z^\star)}. \tag{B-25}$$

Then, exploiting (B-25), we obtain

$$\begin{aligned} G(y^\star, z^\star) &= \Big((\rho+\kappa)Q_0(z^\star)Q_1(z^\star)\Big)^{-1} \\ &\quad\times \Big[(\rho+\kappa)\psi(z^\star)\psi'(z^\star)\psi''(z^\star)\psi'''(z^\star) - (\rho+2\kappa)\psi(z^\star)\psi''(z^\star)^3 \\ &\qquad + (\rho+2\kappa)\psi'(z^\star)^2\psi''(z^\star)^2 - (\rho+\kappa)\psi'(z^\star)^3\psi'''(z^\star)\Big] \\ &= -\frac{\sigma^2}{2}\Big((\rho+\kappa)Q_1(z^\star)\Big)^{-1} Q_2(z^\star) < 0. \end{aligned} \tag{B-26}$$

In (B-26) we have used (B-19) with $k = 0, 1, 2$ for the last equality, and the fact that Q_1 and Q_2 are strictly positive for the strict inequality.

Recalling that $\tilde{F}' - \beta > 0$ on $I_{\max}$, we conclude from (B-22), (B-24), and (B-26) that $\tilde{D}(y, \tilde{F}(y))$ cannot tend to zero as $y \downarrow y_-$. This completes the proof. ∎

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