SHEAVES ON $\mathcal{T}$-TOPOLOGIES

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Abstract. The aim of this paper is to give a unifying description of various constructions of sites (subanalytic, semialgebraic, o-minimal) and consider the corresponding theory of sheaves. The method used applies to a more general context and gives new results in semialgebraic and o-minimal sheaf theory.

Contents

Introduction

Sheaf theory in some tame contexts such as semi-algebraic geometry ([10]), subanalytic geometry ([28, 35]) and o-minimal geometry ([19]) has had recently different applications in various fields of mathematics such as model theory [4, 5, 20], analysis [28, 30, 31, 36] and representation theory [1, 2, 37]. Each one of the above

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theories is very useful for the mentioned applications but has some elements which are missing in the other ones: the aim of this paper is to give a unifying description of all these various constructions (subanalytic, semialgebraic, o-minimal) using a modification of the notion of $\mathcal{T}$-topology introduced by Kashiwara and Schapira in [28].

The idea is the following: on a topological space X one chooses a subfamily $\mathcal{T}$ of open subsets of X satisfying some suitable hypothesis, and for each $U \in \mathcal{T}$ one defines the category of coverings of U as the topological coverings $\{U_i\}_{i\in I} \subset \mathcal{T}$ of U admitting a finite subcover. In this way one defines a site $X_\mathcal{T}$ and studies the category of sheaves on $X_\mathcal{T}$ (called $\mathrm{Mod}(k_\mathcal{T})$). This idea was already present in [28]. However in [28], the space X is assumed to be Hausdorff, locally compact and the elements of $\mathcal{T}$ are assumed to have finitely many connected components.

The exigence to treat in a unifying way all the previous constructions, to treat also some non Hausdorff cases (as conic subanalytic sheaves which are related to the extension of the Fourier-Sato transform [36]) and the non-standard setting which appears naturally in the o-minimal context (where the elements of $\mathcal{T}$ are totally disconnected and never locally compact), motivates a modification of the definition of [28]. In particular, in our definition we replace "connectedness" by the notion of $\mathcal{T}$-connectedness (which in the standard o-minimal context is connectedness). Remark that there are many important o-minimal expansions

$$\mathcal{M} = (\mathbb{R}, <, 0, 1, +, \cdot, (f)_{f\in\mathcal{F}})$$

of the ordered field of real numbers. For example $\mathbb{R}_{\mathrm{an}}$, $\mathbb{R}_{\mathrm{exp}}$, $\mathbb{R}_{\mathrm{an,\,exp}}$, $\mathbb{R}_{\mathrm{an}^*}$, $\mathbb{R}_{\mathrm{an}^*,\,\mathrm{exp}}$ see resp., [12, 40, 15, 17, 18]. For each such we have 2^κ many non-isomorphic non standard o-minimal models for each $\kappa >$ cardinality of the language. There is however a non-standard o-minimal structure

$$\mathcal{M} = \left(\bigcup_{n\in\mathbb{N}} \mathbb{R}((t^{\frac{1}{n}})), <, 0, 1, +, \cdot, (f_p)_{p\in\mathbb{R}[[\zeta_1,\ldots,\zeta_n]]}\right)$$

which does not came from a standard one ([32, 23]).

With this more general notion of $\mathcal{T}$-space X we study the category of sheaves on the site $X_\mathcal{T}$. The natural functor of sites $\rho : X \to X_\mathcal{T}$ induces relations between the categories of sheaves on X and $X_\mathcal{T}$, given by the functors ρ_* and ρ^{-1}. The functor ρ_* is fully faithful. Moreover when X is locally weakly quasi-compact there is a right adjoint to the functor ρ^{-1}, denoted by $\rho_!$. The functor $\rho_!$ is exact, commutes with $\varinjlim$ and $\otimes$ and is fully faithful. We introduce the category of $\mathcal{T}$-flabby sheaves (known as *sa*-flabby in [10] and as quasi-injective in [35]): $F \in \mathrm{Mod}(k_\mathcal{T})$ is $\mathcal{T}$-flabby if the restriction $\Gamma(U; F) \to \Gamma(V; F)$ is surjective for each $U, V \in \mathcal{T}$ with $U \supseteq V$. We prove that $\mathcal{T}$-flabby sheaves are stable under $\varinjlim$ and $\otimes$ and are acyclic with respect to the functor $\Gamma(U; \bullet)$, for $U \in \mathcal{T}$. More generally, if one introduces the category $\mathrm{Coh}(\mathcal{T}) \subset \mathrm{Mod}(k_X)$ of coherent sheaves (i.e. sheaves admitting a finite resolution consisting of finite sums of k_{U_i}, $U_i \in \mathcal{T}$), then $\mathcal{T}$-flabby sheaves are acyclic with respect to $\mathrm{Hom}_{k_\mathcal{T}}(\rho_* G, \bullet)$, for $G \in \mathrm{Coh}(\mathcal{T})$. Coherent sheaves also give a description of sheaves on $X_\mathcal{T}$: for each $F \in \mathrm{Mod}(k_\mathcal{T})$ there exists a filtrant inductive family $\{F_i\}_{i\in I}$ such that $F \simeq \varinjlim_i \rho_* F_i$. In fact, we have an equivalence between the categories $\mathrm{Mod}(k_{\mathcal{T}})$ and $\mathrm{Ind}(\mathrm{Coh}(\mathcal{T}))$ the indization of the category $\mathrm{Coh}(\mathcal{T})$.

All of the above results and methods are new in the o-minimal context and most of them are new even in the semialgebraic case as well. On the other hand, we also introduce a method for studying the category $\mathrm{Mod}(k_{\mathcal{T}})$ of sheaves on $\mathcal{T}$-spaces which is the fundamental tool in the semialgebraic and o-minimal case, namely, we prove that as in [19] the category of sheaves on $X_{\mathcal{T}}$ is equivalent to the category of sheaves on a locally quasi-compact space $\widetilde{X}_{\mathcal{T}}$, the $\mathcal{T}$-spectrum of X, which generalizes the notion of o-minimal spectrum as well as the real spectrum of commutative rings from real algebraic geometry. In particular, sheaves on the subanalytic site are sheaves on the $\mathcal{T}$-spectrum associated to the family of relatively compact subanalytic subsets. Such a result was not present in [28].

This theory can then be specialized to each of the examples we mentioned above: when $\mathcal{T}$ is the category of semialgebraic open subsets of a locally semialgebraic space X we obtain the constructions (and the generalizations) of results of [10], in particular, when X is a Nash manifold, we recover the setting of [37]. When $\mathcal{T}$ is the category of relatively compact subanalytic open subsets of a real analytic manifold X we obtain the constructions and results of [28, 35]. Moreover, when $\mathcal{T}$ is the category of conic subanalytic open subsets of a real analytic manifold X we obtain a suitable category of conic subanalytic sheaves considered in [36]. Finally, when $\mathcal{T}$ is the category of definable open subsets of a locally definable space X we obtain in the definable case the constructions of [19] and we obtain new results in the o-minimal context generalizing those of the two previous cases.

The paper is organized in the following way. In Section 1 we introduce the locally weakly quasi-compact spaces and study some properties of sheaves on such spaces. The results of this section will be used in two crucial ways on the theory of sheaves on $\mathcal{T}$-spaces, they are required to show that: (i) when a $\mathcal{T}$-space X is locally weakly quasi-compact, then there is a right adjoint $\rho_!$ to the functor ρ^{-1} induced by the natural functor of sites $\rho : X \to X_{\mathcal{T}}$; (ii) for a $\mathcal{T}$-space X, the category of sheaves on $X_{\mathcal{T}}$ is equivalent to the category of sheaves on a locally quasi-compact space $\widetilde{X}_{\mathcal{T}}$, the $\mathcal{T}$-spectrum of X. In Section 2 we introduce the $\mathcal{T}$-spaces and develop the theory of sheaves on such spaces as already described above.

1. Sheaves on locally weakly quasi-compact spaces

Let X be a non necessarily Hausdorff topological space. One denotes by $\mathrm{Op}(X)$ the category whose objects are the open subsets of X and the morphisms are the inclusions. In this section we generalize some classical results about sheaves on locally compact spaces. For classical sheaf theory our basic reference is [26]. We refer to [39] for an introduction to sheaves on Grothendieck topologies.

1.1. **Locally weakly quasi-compact spaces.**

Definition 1.1.1. *An open subset U of X is said to be relatively weakly quasi-compact in X if, for any covering $\{U_i\}_{i\in I}$ of X, there exists $J \subset I$ finite, such that $U \subset \bigcup_{i\in J} U_i$.*

We will write for short $U \subset\subset X$ to say that U is a relatively weakly quasi-compact open set in X, and we will call $\mathrm{Op}^c(U)$ the subcategory of $\mathrm{Op}(U)$ consisting of open sets $V \subset\subset U$. Note that, given $V, W \in \mathrm{Op}^c(U)$, then $V \cup W \in \mathrm{Op}^c(U)$.

Definition 1.1.2. *A topological space X is locally weakly quasi-compact if satisfies the following hypothesis for every $U, V \in \mathrm{Op}(X)$*

LWC1. *Every $x \in U$ has a fundamental neighborhood system $\{V_i\}$ with $V_i \in \mathrm{Op}^c(U)$.*
LWC2. *For every $U' \in \mathrm{Op}^c(U)$ and $V' \in \mathrm{Op}^c(V)$ one has $U' \cap V' \in \mathrm{Op}^c(U \cap V)$.*
LWC3. *For every $U' \in \mathrm{Op}^c(U)$ there exists $W \in \mathrm{Op}^c(U)$ such that $U' \subset\subset W$.*

Of course an open subset U of a locally weakly quasi-compact space X is also a locally weakly quasi-compact space. Let us consider some examples of locally weakly quasi-compact spaces:

Example 1.1.3. A locally compact topological space X is a locally weakly quasi-compact. In this case, for $U, V \in \mathrm{Op}(X)$ we have $V \subset\subset U$ if and only if V is relatively compact subset of U.

Example 1.1.4. Let X be a topological space with a basis of quasi-compact (i.e. each open covering admits a finite subcover) open subsets closed under taking finite intersections. Then X is locally weakly quasi-compact and, for $U, V \in \mathrm{Op}(X)$ we have $V \subset\subset U$ if and only if V is contained in a quasi-compact subset of U. In this situation we have the following particular cases:

(i) X is a Noetherian topological space (each open subset of X is quasi-compact). This includes in particular: (a) algebraic varieties over algebraically closed fields; (b) complex varieties (reduced, irreducible complex analytic spaces) with the Zariski topology.
(ii) X is a spectral topological space (in addition: (i) X is quasi-compact; (ii) T_0; (iii) every irreducible closed subset is the closure of a unique point). This includes in particular: (a) real algebraic varieties over real closed fields; (b) the o-minimal spectrum of a definable space in some o-minimal structure.
(iii) X is an increasing union of open spectral topological spaces X_i's, i.e. X is the space $\bigcup_{i \in I} X_i$. This space X has a basis of quasi-compact open subsets closed under taking finite intersections and in addition is: (i) not quasi-compact in general unless I is finite; (ii) T_0. This includes in particular: (a) the semialgebraic spectrum of locally semialgebraic space; (b) more generally, the o-minimal spectrum of a locally definable space in some o-minimal structure.

Example 1.1.5. Let E be a real vector bundle over a locally compact space Z endowed with the natural action μ of $\mathbb{R}^+$ (the multiplication on the fibers). Let $\dot{E} = E \setminus Z$, and for $U \in \mathrm{Op}(E)$ set $U_Z = U \cap Z$ and $\dot{U} = U \cap \dot{E}$. Let $E_{\mathbb{R}^+}$ denote the space E endowed with the conic topology i.e. open sets of $E_{\mathbb{R}^+}$ are open sets of E which are μ-invariant. With this topology $E_{\mathbb{R}^+}$ is a locally weakly quasi-compact space and, for $U, V \in \mathrm{Op}(E_{\mathbb{R}^+})$ we have $V \subset\subset U$ if and only if $V_Z \subset\subset U_Z$ in Z and $\dot{V} \subset\subset \dot{U}$ in $\dot{E}_{\mathbb{R}^+}$ (the later is $\dot{E}$ with the induced conic topology).

1.2. **Sheaves on locally weakly quasi-compact spaces.** Recall that X is a non necessarily Hausdorff topological space.

Definition 1.2.1. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{U}' = \{U'_j\}_{j\in J}$ be two families of open subsets of X. One says that $\mathcal{U}'$ is a refinement of $\mathcal{U}$ if for each $U_i \in \mathcal{U}$ there is $U'_j \in \mathcal{U}'$ with $U'_j \subseteq U_i$.*

One denotes by $\mathrm{Cov}(U)$ the category whose objects are the coverings of $U \in \mathrm{Op}(X)$ and the morphisms are the refinements, and by $\mathrm{Cov}^f(U)$ its full subcategory consisting of finite coverings of U.

Given $V \in \mathrm{Op}(U)$ and $S \in \mathrm{Cov}(U)$, one sets $S \cap V = \{U \cap V\}_{U\in S} \in \mathrm{Cov}(V)$.

Definition 1.2.2. *The site X^f on the topological space X is the category $\mathrm{Op}(X)$ endowed with the following topology: $S \subset \mathrm{Op}(U)$ is a covering of U if and only if it has a refinement $S^f \in \mathrm{Cov}^f(U)$.*

Definition 1.2.3. *Let $U, V \in \mathrm{Op}(X)$ with $V \subset U$. Given $S = \{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ and $T = \{V_j\}_{j\in J} \in \mathrm{Cov}(V)$, we write $T \subset\subset S$ if T is a refinement of $S \cap V$, and $V_j \subset U_i$ if and only if $V_j \subset\subset U_i$.*

Let us recall the definitions of presheaf and sheaf on a site.

Definition 1.2.4. *A presheaf of k-modules on X is a functor $\mathrm{Op}(X)^{op} \to \mathrm{Mod}(k)$. A morphism of presheaves is a morphism of such functors. One denotes by $\mathrm{Psh}(k_X)$ the category of presheaves of k-modules on X.*

Let $F \in \mathrm{Psh}(k_X)$, and let $S \in \mathrm{Cov}(U)$. One sets

$$F(S) = \ker\Big(\prod_{W\in S} F(W) \rightrightarrows \prod_{W',W''\in S} F(W' \cap W'')\Big).$$

Definition 1.2.5. *A presheaf F is separated (resp. is a sheaf) if for any $U \in \mathrm{Op}(X)$ and for any $S \in \mathrm{Cov}(U)$ the natural morphism $F(U) \to F(S)$ is a monomorphism (resp. an isomorphism). One denotes by $\mathrm{Mod}(k_X)$ the category of sheaves of k-modules on X.*

Let $F \in \mathrm{Psh}(k_X)$, one defines the presheaf F^+ by setting

$$F^+(U) = \varinjlim_{S\in\mathrm{Cov}(U)} F(S).$$

One can show that F^+ is a separated presheaf and if F is a separated presheaf, then F^+ is a sheaf. Let $F \in \mathrm{Psh}(k_X)$, the sheaf F^{++} is called the sheaf associated to the presheaf F.

Lemma 1.2.6. *For $F \in \mathrm{Psh}(k_X)$, and let $U \in \mathrm{Op}(X)$. If F is a sheaf on X^f, then for any $V \in \mathrm{Op}^c(U)$ the morphism*

$$F^+(U) \to F^+(V) \tag{1.1}$$

factors through $F(V)$.

Proof. Let $S \in \mathrm{Cov}(U)$, and set $S \cap V = \{W \cap V\}_{W\in S}$. Since $V \in \mathrm{Op}^c(U)$, there is a finite refinement $T^f \in \mathrm{Cov}^f(V)$ of $S \cap V$. Then the morphism (1.1) is defined by

$$\begin{aligned} F^+(U) &\simeq \varinjlim_{S \in \mathrm{Cov}(U)} F(S) \\ &\to \varinjlim_{S \in \mathrm{Cov}(U)} F(S \cap V) \\ &\to \varinjlim_{T^f \in \mathrm{Cov}^f(V)} F(T^f) \\ &\to \varinjlim_{T \in \mathrm{Cov}(V)} F(T) \\ &\simeq F^+(V). \end{aligned}$$

The result follows because $F(T^f) \simeq F(V)$. □

Corollary 1.2.7. *With the hypothesis of Lemma 1.2.6, we consider two coverings $S \in \mathrm{Cov}(U)$ and $T \in \mathrm{Cov}(V)$. If $T \subset\subset S$, then the morphism*

$$F^+(S) \to F^+(T) \tag{1.2}$$

factors through $F(T)$. In particular, if T is finite, then the morphism (1.2) *factors through $F(V)$.*

From now on we will assume the following hypothesis:

$$\text{the topological space } X \text{ is locally weakly quasi-compact.} \tag{1.3}$$

Lemma 1.2.8. *Let $U \in \mathrm{Op}(X)$, and consider a subset $V \subset\subset U$. Then for any $S^f \in \mathrm{Cov}^f(U)$ there exists $T^f \in \mathrm{Cov}^f(V)$ with $T^f \subset\subset S^f$.*

Proof. Let $S^f = \{U_i\}$. For each $x \in U$ and $U_i \ni x$, consider a $V_{x,i} \in \mathrm{Op}^c(U_i)$ containing x. Set $V_x = \bigcap_i V_{x,i}$, the family $\{V_x\}$ forms a covering of U. Then there exists a finite subfamily $\{V_j\}$ containing V. By construction $V_j \cap V \subset\subset U_i$ whenever $V_j \subset U_i$. □

Lemma 1.2.9. *Let $F \in \mathrm{Psh}(k_X)$, and let $U \in \mathrm{Op}(X)$. If F is a sheaf on X^f, then for any $V \in \mathrm{Op}^c(U)$ the morphism*

$$F^{++}(U) \to F^{++}(V) \tag{1.4}$$

factors through $F(V)$.

Proof. Since X is locally weakly quasi-compact, there exists $W \in \mathrm{Op}^c(U)$ with $V \subset\subset W$. As in Lemma 1.2.6 we obtain a diagram

$$\begin{array}{ccccc} F^{++}(U) & \longrightarrow & F^{++}(W) & \longrightarrow & F^{++}(V) \\ \downarrow & \nearrow & \downarrow & \nearrow & \\ \varinjlim_{S^f \in \mathrm{Cov}^f(W)} F^+(S^f) & \longrightarrow & \varinjlim_{T^f \in \mathrm{Cov}^f(V)} F^+(T^f). & & \end{array}$$

Since X is locally weakly quasi-compact then by Lemma 1.2.8 for any $S^f \in \mathrm{Cov}^f(W)$ there exists $T^f \in \mathrm{Cov}^f(V)$ with $T^f \subset\subset S^f$. By Corollary 1.2.7 the morphism

$$F^+(S^f) \to F^+(T^f)$$

factors through $F(T^f) \simeq F(V)$. Then the morphism

$$\varinjlim_{S^f \in \mathrm{Cov}^f(W)} F^+(S^f) \to \varinjlim_{T^f \in \mathrm{Cov}^f(V)} F^+(T^f)$$

factors through $F(V)$ and the result follows. $\square$

Corollary 1.2.10. *Let $F \in \mathrm{Psh}(k_X)$. If F is a sheaf on X^f, then:*

(i) *for any $V \in \mathrm{Op}^c(X)$ one has the isomorphism* $\varinjlim_{U \supset\supset V} F(U) \xrightarrow{\sim} \varinjlim_{U \supset\supset V} F^{++}(U)$.

(ii) *for any $U \in \mathrm{Op}(X)$ one has the isomorphism* $\varprojlim_{V \subset\subset U} F(V) \xrightarrow{\sim} \varprojlim_{V \subset\subset U} F^{++}(V)$.

Proof. (i) By Lemma 1.2.9 for each $U \in \mathrm{Op}(X)$ with $U \supset\supset V$ we have a commutative diagram

$$\begin{array}{ccc} F^{++}(U) & \longrightarrow & F^{++}(V) \\ \uparrow & \searrow & \uparrow \\ F(U) & \longrightarrow & F(V) \end{array}$$

This implies that the identity morphism of $\varinjlim_{U \supset\supset V} F(U)$ factors through $\varinjlim_{U \supset\supset V} F^{++}(U)$. On the other hand this also implies that the identity morphism of $\varinjlim_{U \supset\supset V} F^{++}(U)$ factors through $\varinjlim_{U \supset\supset V} F(U)$. Then $\varinjlim_{U \supset\supset V} F(U) \xrightarrow{\sim} \varinjlim_{U \supset\supset V} F^{++}(U)$.

The proof of (ii) is similar. $\square$

Corollary 1.2.11. *Let X be a quasi-compact and locally weakly quasi-compact space, and let $F \in \mathrm{Psh}(k_X)$. If F is a sheaf on X^f, then the natural morphism*

$$F(X) \to F^{++}(X) \tag{1.5}$$

is an isomorphism.

Proof. It follows immediately from Corollary 1.2.10 (i) with $V = X$. $\square$

Let $\{F_i\}_{i \in I}$ be a filtrant inductive system in $\mathrm{Mod}(k_X)$. One sets

$$\text{“}\varinjlim_i\text{”} F_i = \text{inductive limit in the category of presheaves},$$
$$\varinjlim_i F_i = \text{inductive limit in the category of sheaves}.$$

Recall that $\varinjlim_i F_i = (\text{“}\varinjlim_i\text{”} F_i)^{++}$.

Proposition 1.2.12. *Let $\{F_i\}_{i \in \mathrm{I}}$ be a filtrant inductive system in $\mathrm{Mod}(k_X)$ and let $U \in \mathrm{Op}(X)$. Then for any $V \in \mathrm{Op}^c(U)$ the morphism*

$$\Gamma(U; \varinjlim_i F_i) \to \Gamma(V; \varinjlim_i F_i)$$

factors through $\varinjlim_i \Gamma(V; F_i)$.

Proof. By Lemma 1.2.9 it is enough to show that "$\varinjlim_i$"F_i is a sheaf on X^f. Let $U \in \mathrm{Op}(X)$ and $S \in \mathrm{Cov}^f(U)$. Since $\varinjlim_i$ commutes with finite projective limits we obtain the isomorphism ("$\varinjlim_i$"$F_i)(S) \simeq \varinjlim_i F_i(S)$. The result follows because $F_i \in \mathrm{Mod}(k_X)$ for each $i \in I$. □

Corollary 1.2.13. *Let $\{F_i\}_{i\in I}$ be a filtrant inductive system in* $\mathrm{Mod}(k_X)$.

(i) *For any $V \in \mathrm{Op}^c(X)$ one has the isomorphism* $\varinjlim_{U\supset\supset V,i} \Gamma(U;F_i) \xrightarrow{\sim} \varinjlim_{U\supset\supset V} \Gamma(U;\varinjlim_i F_i)$.

(ii) *For any $U \in \mathrm{Op}(X)$ one has the isomorphism* $\varprojlim_{V\subset\subset U} \varinjlim_i \Gamma(V;F_i) \xrightarrow{\sim} \varprojlim_{V\subset\subset U} \Gamma(V;\varinjlim_i F_i)$.

Proof. It follows from Corollary 1.2.10 with $F =$ "$\varinjlim_i$"F_i. □

Corollary 1.2.14. *Let X be a quasi-compact and locally weakly quasi-compact space. Then the natural morphism*

$$\varinjlim_i \Gamma(X;F_i) \to \Gamma(X;\varinjlim_i F_i)$$

is an isomorphism.

Proof. It follows from Corollary 1.2.11 with $F =$ "$\varinjlim_i$"F_i. □

Example 1.2.15. *Let us consider the formula*

$$\varinjlim_{U\supset\supset V,i} \Gamma(U;F_i) \xrightarrow{\sim} \varinjlim_{U\supset\supset V} \Gamma(U;\varinjlim_i F_i) \tag{1.6}$$

(i) *Let X be a Noetherian space and let $V \in \mathrm{Op}(X)$. Then $\Gamma(V;F) \simeq \varinjlim_{U\supset\supset V} \Gamma(U;F)$, since every open set is quasi-compact and* (1.6) *becomes $\varinjlim_i \Gamma(V;F_i) \simeq \Gamma(V;\varinjlim_i F_i)$.*

(ii) *Assume that X has a basis of quasi-compact open subsets and let $V \in \mathrm{Op}^c(X)$. Then V is contained in a quasi-compact open subset of X and $\varinjlim_{U\supset\supset V} \Gamma(U;F) \simeq \varinjlim_{W\supset V} \Gamma(W;F)$, where W ranges through the family of quasi-compact subsets of X.*

(iii) *Let X be a locally compact space and let $V \in \mathrm{Op}^c(X)$. Then $\Gamma(\overline{V};F) \simeq \varinjlim_{U\supset\supset V} \Gamma(U;F)$, and* (1.6) *becomes $\varinjlim_i \Gamma(\overline{V};F_i) \simeq \Gamma(\overline{V};\varinjlim_i F_i)$.*

(iv) *Let $E_{\mathbb{R}^+}$ be a vector bundle endowed with the conic topology, and let $V \in \mathrm{Op}^c(E_{\mathbb{R}^+})$. Then $\varinjlim_{U\supset\supset V} \Gamma(U;F) \simeq \Gamma(K;F)$, where K is the union of the closures of V_Z in Z and $\dot{V}$ in $\dot{E}_{\mathbb{R}^+}$, and* (1.6) *becomes $\varinjlim_i \Gamma(K;F_i) \simeq \Gamma(K;\varinjlim_i F_i)$.*

Lemma 1.2.16. *Let $F \in \mathrm{Psh}(k_X)$. Then we have the isomorphism*

$$\varprojlim_{V\subset\subset X} \varinjlim_{V\subset\subset W} F(W) \xrightarrow{\sim} \varprojlim_{V\subset\subset X} F(V).$$

Proof. The result follows since for each $V \in \mathrm{Op}^c(X)$ there exists $W \in \mathrm{Op}^c(X)$ such that $V \subset\subset W$ since X is locally weakly compact. Let $U, V \subset\subset X$ such that $U \supset\supset V$. The restriction morphism $F(U) \to F(V)$ factors through $\varinjlim\limits_{W \supset\supset V} F(W)$. Taking the projective limit we obtain the result. □

Remark 1.2.17. *The notion of locally weakly quasi-compact can be extended to the case of a site, by generalizing the hypothesis LWC1-LWC3. For our purpose we are interested in the topological setting and we refer to* [34] *for this approach.*

1.3. **c-soft sheaves on locally weakly quasi-compact spaces.** Let X be a locally weakly quasi-compact space, and consider the category $\mathrm{Mod}(k_X)$.

Definition 1.3.1. *We say that a sheaf F on X is c-soft if the restriction morphism $\Gamma(W;F) \to \varinjlim\limits_{U \supset\supset V} \Gamma(U;F)$ is surjective for each $V, W \in \mathrm{Op}^c(X)$ with $V \subset\subset W$.*

It follows from the definition that injective sheaves and flabby sheaves are c-soft. Moreover, it follows from Corollary 1.2.13 that filtrant inductive limits of c-soft sheaves are c-soft.

Proposition 1.3.2. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(k_X)$, and assume that F' is c-soft. Then the sequence*

$$0 \to \varinjlim_{U \supset\supset V} \Gamma(U;F') \to \varinjlim_{U \supset\supset V} \Gamma(U;F) \to \varinjlim_{U \supset\supset V} \Gamma(U;F'') \to 0$$

is exact for any $V \in \mathrm{Op}^c(X)$.

Proof. Let $s'' \in \varinjlim\limits_{U \supset\supset V} \Gamma(U;F'')$. Then there exists $U \supset\supset V$ such that s'' is represented by $s''_U \in \Gamma(U;F'')$. Let $\{U_i\}_{i \in I} \in \mathrm{Cov}(U)$ such that there exists $s_i \in \Gamma(U_i;F)$ whose image is $s''_U|_{U_i}$ for each i. There exists $W \in \mathrm{Op}^c(U)$ with $W \supset\supset V$, a finite covering $\{W_j\}_{j=1}^n$ of W and a map $\varepsilon : J \to I$ of the index sets such that $W_j \subset\subset U_{\varepsilon(j)}$. We may argue by induction on n. If $n = 2$, set $U_i = U_{\varepsilon(i)}$, $i = 1, 2$. Then $(s_1 - s_2)|_{U_1 \cap U_2}$ belongs to $\Gamma(U_1 \cap U_2; F')$, and its restriction defines an element of $\varinjlim\limits_{W' \supset\supset W_1 \cap W_2} \Gamma(W';F')$, hence it extends to $s' \in \Gamma(U;F')$. By replacing s_1 with $s_1 - s'$ on W_1 we may assume that $s_1 = s_2$ on $W_1 \cap W_2$. Then there exists $s \in \Gamma(W_1 \cup W_2; F)$ with $s|_{W_i} = s_i$. Thus the induction proceeds. □

Proposition 1.3.3. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(k_X)$, and assume F', F c-soft. Then F'' is c-soft.*

Proof. Let $V, W \in \mathrm{Op}^c(X)$ with $V \subset\subset W$ and let us consider the diagram below

$$\begin{array}{ccc}
\Gamma(W;F) & \longrightarrow & \Gamma(W;F'') \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\
\varinjlim\limits_{U \supset\supset V} \Gamma(U;F) & \xrightarrow{\beta} & \varinjlim\limits_{U \supset\supset V} \Gamma(U;F'').
\end{array}$$

The morphism α is surjective since F is c-soft and β is surjective by Proposition 1.3.2. Then γ is surjective. □

Proposition 1.3.4. *The family of c-soft sheaves is injective respect to the functor* $\varinjlim_{U \supset\supset V} \Gamma(U; \bullet)$ *for each* $V \in \mathrm{Op}^c(X)$.

Proof. The family of c-soft sheaves contains injective sheaves, hence it is cogenerating. Then the result follows from Propositions 1.3.2 and 1.3.3. □

Assume the following hypothesis

$$X \text{ has a countable cover } \{U_n\}_{n \in \mathbb{N}} \text{ with } U_n \in \mathrm{Op}^c(X),\ \forall n \in \mathbb{N}. \tag{1.7}$$

Lemma 1.3.5. *Assume* (1.7). *Then there exists a covering* $\{V_n\}_{n\in\mathbb{N}}$ *of* X *such that* $V_n \subset\subset V_{n+1}$ *and* $V_n \in \mathrm{Op}^c(X)$ *for each* $n \in \mathbb{N}$.

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable cover of X with $U_n \in \mathrm{Op}^c(X)$ for each $n \in \mathbb{N}$. Set $V_1 = U_1$. Given $\{V_i\}_{i=1}^n$ with $V_{i+1} \supset\supset V_i$, $i = 1, \dots, n-1$, let us construct $V_{n+1} \supset\supset V_n$. Consider $x \notin V_n$. Up to take a permutation of $\mathbb{N}$ we may assume $x \in U_{n+1}$. Since X is locally weakly quasi-compact there exists $V_{n+1} \in \mathrm{Op}^c(X)$ such that $V_n \cup U_{n+1} \subset\subset V_{n+1}$. □

Proposition 1.3.6. *Assume* (1.7). *Then the category of c-soft sheaves is injective respect to the functor* $\Gamma(X; \bullet)$.

Proof. Take an exact sequence $0 \to F' \to F \to F'' \to 0$, and suppose F' c-soft. By Lemma 1.3.5 there exists a covering $\{V_n\}_{n\in\mathbb{N}}$ of X such that $V_n \subset\subset V_{n+1}$ (and $V_n \in \mathrm{Op}^c(X)$) for each $n \in \mathbb{N}$. All the sequences

$$0 \to \varinjlim_{U_n \supset\supset V_n} \Gamma(U_n; F') \to \varinjlim_{U_n \supset\supset V_n} \Gamma(U_n; F) \to \varinjlim_{U_n \supset\supset V_n} \Gamma(U_n; F'') \to 0$$

are exact by Proposition 1.3.2, and the morphism $\varinjlim_{U_{n+1} \supset\supset V_{n+1}} \Gamma(U_{n+1}; F') \to \varinjlim_{U_n \supset\supset V_n} \Gamma(U_n; F')$ is surjective for all n. Then by Proposition 1.12.3 of [26] the sequence

$$0 \to \varprojlim_n \varinjlim_{U_n \supset\supset V_n} \Gamma(U_n; F') \to \varprojlim_n \varinjlim_{U_n \supset\supset V_n} \Gamma(U_n; F) \to \varprojlim_n \varinjlim_{U_n \supset\supset V_n} \Gamma(U_n; F'') \to 0$$

is exact. By Lemma 1.2.16 $\varprojlim_n \varinjlim_{U_n \supset\supset V_n} \Gamma(U_n; G) \simeq \Gamma(X; G)$ for any $G \in \mathrm{Mod}(k_X)$ and the result follows. □

Example 1.3.7. Let us consider some particular cases

(i) When X is Noetherian c-soft sheaves are flabby sheaves.
(ii) When X has a basis of quasi-compact open subsets, then $F \in \mathrm{Mod}(k_X)$ is c-soft if the restriction morphism $\Gamma(U; F) \to \Gamma(V; F)$ is surjective, for any quasi-compact open subsets U, V of X with $U \supseteq V$.
(iii) When X is a locally compact space countable at infinity, then we recover c-soft sheaves as in chapter II of [26].
(iv) When $E_{\mathbb{R}^+}$ is a vector bundle endowed with the conic topology, then $F \in \mathrm{Mod}(k_{E_{\mathbb{R}^+}})$ is c-soft if the restriction morphism $\Gamma(E_{\mathbb{R}^+}; F) \to \Gamma(K; F)$ is surjective, where K is defined as in Example 1.2.15.

2. Sheaves on $\mathcal{T}$-spaces.

In the following we shall assume that k is a field and X is a topological space. Below we give the definition of $\mathcal{T}$-space, adapting the construction of Kashiwara and

Schapira [28]. We study the category of sheaves on $X_{\mathcal{T}}$ generalizing results already known in the case of subanalytic sheaves. Then we prove that as in [19] the category of sheaves on $X_{\mathcal{T}}$ is equivalent to the category of sheaves on a locally weakly-compact topological space $\widetilde{X}_{\mathcal{T}}$, the $\mathcal{T}$-spectrum, which generalizes the notion of o-minimal spectrum.

2.1. **$\mathcal{T}$-sheaves.** Let X be a topological space and let us consider a family $\mathcal{T}$ of open subsets of X.

Definition 2.1.1. *The topological space X is a $\mathcal{T}$-space if the family $\mathcal{T}$ satisfies the hypotheses below*

$$\begin{cases} \textit{(i) } \mathcal{T} \textit{ is a basis for the topology of } X, \textit{ and } \varnothing \in \mathcal{T}, \\ \textit{(ii) } \mathcal{T} \textit{ is closed under finite unions and intersections}, \\ \textit{(iii) every } U \in \mathcal{T} \textit{ has finitely many } \mathcal{T}\textit{-connected components,} \end{cases} \tag{2.1}$$

where we define:

- *a $\mathcal{T}$-subset is a finite Boolean combination of elements of $\mathcal{T}$;*
- *a closed (resp. open) $\mathcal{T}$-subset is a $\mathcal{T}$-subset which is closed (resp. open) in X;*
- *a $\mathcal{T}$-connected subset is a $\mathcal{T}$-subset which is not the disjoint union of two proper $\mathcal{T}$-subsets which are closed and open.*

Example 2.1.2. Let $R = (R, <, 0, 1, +, \cdot)$ be a real closed field. Let X be a locally semialgebraic space ([10, 11]) and consider the subfamily of $\mathrm{Op}(X)$ defined by $\mathcal{T} = \{U \in \mathrm{Op}(X) : U \text{ is semialgebraic}\}$. The family $\mathcal{T}$ satisfy (2.1). Note also that the $\mathcal{T}$-subsets of X are exactly the semialgebraic subsets of X ([7]).

Example 2.1.3. Let X be a real analytic manifold and consider the subfamily of $\mathrm{Op}(X)$ defined by $\mathcal{T} = \mathrm{Op}^c(X_{sa}) = \{U \in \mathrm{Op}(X_{sa}) : U \text{ is subanalytic relatively compact}\}$. The family $\mathcal{T}$ satisfies (2.1).

Example 2.1.4. Let X be a real analytic manifold endowed with a subanalytic action μ of $\mathbb{R}^+$. In other words we have a subanalytic map

$$\mu : X \times \mathbb{R}^+ \to X,$$

which satisfies, for each $t_1, t_2 \in \mathbb{R}^+$:

$$\begin{cases} \mu(x, t_1 t_2) = \mu(\mu(x, t_1), t_2), \\ \mu(x, 1) = x. \end{cases}$$

Denote by $X_{\mathbb{R}^+}$ the topological space X endowed with the conic topology, i.e. $U \in \mathrm{Op}(X_{\mathbb{R}^+})$ if it is open for the topology of X and invariant by the action of $\mathbb{R}^+$. We will denote by $\mathrm{Op}^c(X_{\mathbb{R}^+})$ the subcategory of $\mathrm{Op}(X_{\mathbb{R}^+})$ consisting of relatively weakly quasi-compact open subsets. Consider the subfamily of $\mathrm{Op}(X_{\mathbb{R}^+})$ defined by $\mathcal{T} = \mathrm{Op}^c(X_{sa,\mathbb{R}^+}) = \{U \in \mathrm{Op}^c(X_{\mathbb{R}^+}) : U \text{ is subanalytic}\}$. The family $\mathcal{T}$ satisfies (2.1).

Example 2.1.5. Let $\mathcal{M} = (M, <, (c)_{\in \mathcal{C}}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}})$ be an arbitrary o-minimal structure. Let X be a locally definable space ([3]) and consider the subfamily of $\mathrm{Op}(X)$ defined by $\mathcal{T} = \mathrm{Op}(X_{\mathrm{def}}) = \{U \in \mathrm{Op}(X) : U \text{ is definable}\}$. The family $\mathcal{T}$ satisfies (2.1). Note also that (i) the $\mathcal{T}$-subsets of X are exactly the definable subsets of X (by the cell decomposition theorem in [13], see [19] Proposition 2.1).

Let X be a $\mathcal{T}$-space. One can endow the category $\mathcal{T}$ with a Grothendieck topology, called the $\mathcal{T}$-topology, in the following way: a family $\{U_i\}_i$ in $\mathcal{T}$ is a covering of $U \in \mathcal{T}$ if it admits a finite subcover. We denote by $X_{\mathcal{T}}$ the associated site, write for short $k_{\mathcal{T}}$ instead of $k_{X_{\mathcal{T}}}$, and let $\rho : X \to X_{\mathcal{T}}$ be the natural morphism of sites. We have functors

$$\mathrm{Mod}(k_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(k_{\mathcal{T}}). \tag{2.2}$$

Proposition 2.1.6. *We have $\rho^{-1} \circ \rho_* \simeq \mathrm{id}$. Equivalently, the functor ρ_* is fully faithful.*

Proof. Let $V \in \mathrm{Op}(X)$ and let $G \in \mathrm{Mod}(k_{\mathcal{T}})$. Then $\rho^{-1}G = (\rho^{\leftarrow}F)^{++}$, where $\rho^{\leftarrow}G \in \mathrm{Psh}(k_X)$ is defined by

$$\mathrm{Op}(X) \ni V \mapsto \varinjlim_{U \supseteq V, U \in \mathcal{T}} G(U).$$

In particular, when $U \in \mathcal{T}$, $\rho^{\leftarrow}G(U) = G(U)$.

Let $F \in \mathrm{Mod}(k_X)$ and denote by $\iota : \mathrm{Mod}(k_X) \to \mathrm{Psh}(k_X)$ the forgetful functor. The adjunction morphism $\rho^{\leftarrow} \circ \rho_* \to \mathrm{id}$ in $\mathrm{Psh}(k_X)$ defines $\rho^{\leftarrow}\rho_*F \to \iota F$. This morphism is an isomorphism on $\mathcal{T}$, since $\rho^{\leftarrow}\rho_*F(U) \simeq \rho_*F(U) \simeq F(U) \simeq \iota F(U)$ when $U \in \mathcal{T}$. By (2.1) (i) $\mathcal{T}$ forms a basis for the topology of X, hence we get an isomorphism

$$\rho^{-1}\rho_*F \simeq (\rho^{\leftarrow}\rho_*F)^{++} \simeq (\iota F)^{++} \simeq F$$

and the result follows. $\square$

Proposition 2.1.7. *Let $\{F_i\}_{i \in I}$ be a filtrant inductive system in $\mathrm{Mod}(k_{\mathcal{T}})$ and let $U \in \mathcal{T}$. Then*

$$\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i).$$

Proof. Denote by "$\varinjlim_i$"F_i the presheaf $V \mapsto \varinjlim_i \Gamma(V; F_i)$ on $X_{\mathcal{T}}$. Let $U \in \mathcal{T}$ and let S be a finite covering of U. Since $\varinjlim_i$ commutes with finite projective limits we obtain the isomorphism ("$\varinjlim_i$"$F_i)(S) \xrightarrow{\sim} \varinjlim_i F_i(S)$ and $F_i(U) \xrightarrow{\sim} F_i(S)$ since $F_i \in \mathrm{Mod}(k_{\mathcal{T}})$ for each i. Moreover the family of finite coverings of U is cofinal in $\mathrm{Cov}(U)$. Hence "$\varinjlim_i$"$F_i \xrightarrow{\sim}$ ("$\varinjlim_i$"$F_i)^+$. Applying once again the functor $(\cdot)^+$ we get

$$\text{“}\varinjlim_i\text{”}F_i \simeq (\text{“}\varinjlim_i\text{”}F_i)^+ \simeq (\text{“}\varinjlim_i\text{”}F_i)^{++} \simeq \varinjlim_i F_i.$$

Hence applying the functor $\Gamma(U; \cdot)$ we obtain the isomorphism $\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i)$ for each $U \in \mathcal{T}$. $\square$

Proposition 2.1.8. *Let F be a presheaf on $X_{\mathcal{T}}$ and assume that*

(i) $F(\varnothing) = 0$,

(ii) *For any $U, V \in \mathcal{T}$ the sequence $0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$ is exact.*

Then $F \in \mathrm{Mod}(k_{\mathcal{T}})$.

Proof. Let $U \in \mathcal{T}$ and let $\{U_j\}_{j=1}^n$ be a finite covering of U. Set for short $U_{ij} = U_i \cap U_j$. We have to show the exactness of the sequence

$$0 \to F(U) \to \oplus_{1\leq k\leq n} F(U_k) \to \oplus_{1\leq i<j\leq n} F(U_{ij}),$$

where the second morphism sends $(s_k)_{1\leq k\leq n}$ to $(t_{ij})_{1\leq i<j\leq n}$ by $t_{ij} = s_i|_{U_{ij}} - s_j|_{U_{ij}}$. We shall argue by induction on n. For $n = 1$ the result is trivial, and $n = 2$ is the hypothesis. Suppose that the assertion is true for $j \leq n-1$ and set $U' = \bigcup_{1\leq k<n} U_k$. By the induction hypothesis the following commutative diagram is exact

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(U) & \longrightarrow & F(U') \oplus F(U_n) & \longrightarrow & F(U' \cap U_n) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \bigoplus_{i<n} F(U_i) \oplus F(U_n) & \longrightarrow & \bigoplus_{i<n} F(U_{in}) \\
 & & & & \downarrow & & \\
 & & & & \bigoplus_{i<j<n} F(U_{ij}). & &
\end{array}$$

Then the result follows. □

Example 2.1.9. Let us see some examples of sites associated to $\mathcal{T}$-topologies:

(i) When $\mathcal{T}$ is the family of Example 2.1.2 we obtain the semi-algebraic site of [10, 11].
(ii) When $\mathcal{T}$ is the family of Example 2.1.3 we obtain the subanalytic site X_{sa} of [28, 35].
(iii) When $\mathcal{T}$ is the family of Example 2.1.4 we obtain the conic subanalytic site of [36].
(iv) When $\mathcal{T}$ is the family of Example 2.1.5 we obtain the o-minimal site X_{def}. It is the one considered in [19] when X is a definable space.

2.2. **$\mathcal{T}$-coherent sheaves.** Let us consider the category $\mathrm{Mod}(k_X)$ of sheaves of k_X-modules on X, and denote by $\mathcal{K}$ the subcategory whose objects are the sheaves $F = \oplus_{i\in I} k_{U_i}$ with I finite and $U_i \in \mathcal{T}$ for each i. The following definition is extracted from [28].

Definition 2.2.1. *Let $\mathcal{T}$ be a subfamily of* $\mathrm{Op}(X)$ *satisfying* (2.1)*, and let $F \in$* $\mathrm{Mod}(k_X)$.

(i) *F is $\mathcal{T}$-finite if there exists an epimorphism $G \twoheadrightarrow F$ with $G \in \mathcal{K}$.*
(ii) *F is $\mathcal{T}$-pseudo-coherent if for any morphism $\psi : G \to F$ with $G \in \mathcal{K}$,* $\ker\psi$ *is $\mathcal{T}$-finite.*
(iii) *F is $\mathcal{T}$-coherent if it is both $\mathcal{T}$-finite and $\mathcal{T}$-pseudo-coherent.*

Remark that (ii) is equivalent to the same condition with "G is $\mathcal{T}$-finite" instead of "$G \in \mathcal{K}$". One denotes by $\mathrm{Coh}(\mathcal{T})$ the full subcategory of $\mathrm{Mod}(k_X)$ consisting of $\mathcal{T}$-coherent sheaves. It is easy (see [29], Exercise 8.23) to prove that $\mathrm{Coh}(\mathcal{T})$ is additive and stable by kernels.

Lemma 2.2.2. *Let $F, G \in \mathcal{K}$. Then, given $\varphi : F \to G$, we have* $\ker\varphi \in \mathcal{K}$.

Proof. We have $F = \oplus_{i=1}^{l} k_{W_i}$, $G = \oplus_{j=1}^{m} k_{W'_j}$. Composing with the projection p_j, $j = 1, ..., m$ on each factor of G, $\ker\varphi$ will be the intersection of the $\ker p_j \circ \varphi$ so that, if each one has the desired form, the same will happen to their intersection. Therefore it is sufficient to assume $m = 1$, let us say, $G = k_W$. A morphism $\varphi : F \to G$ is then defined by a sequence $v = (v_1, \dots, v_l)$, where v_i is the image by φ of the section of k_{W_i} defined by 1 on W_i, so $v_i = 0$ if $W_i \not\subset W$. More precisely, if $s = (s_1, ..., s_l)$ is a germ of F in y, we have $\varphi(s_1, ..., s_l) = \sum_{i=1}^{l} v_{iy} s_i$. So, given $s = (s_1, ..., s_l) \in \ker\varphi$, if, for a given i, we have $v_{iy} s_i \neq 0$, then s defines a germ of $H_i =: \oplus_{i' \neq i} k_{W_{i'} \cap W_i}$ in y.

Accordingly, $\ker\varphi \simeq \oplus_{i=1}^{l} H_i$. □

Therefore, according to the definition of $\mathrm{Coh}(\mathcal{T})$ and to Lemma 2.2.2, any $F \in \mathrm{Coh}(\mathcal{T})$ admits a finite resolution

$$K^\bullet := 0 \to K^1 \to \cdots \to K^n \to F \to 0$$

consisting of objects belonging to $\mathcal{K}$.

Proposition 2.2.3. *Let $U \in \mathcal{T}$ and consider the constant sheaf $k_{U_{X_\mathcal{T}}} \in \mathrm{Mod}(k_\mathcal{T})$. We have $k_{U_{X_\mathcal{T}}} \simeq \rho_* k_U$.*

Proof. Let F be the presheaf on $X_\mathcal{T}$ defined by $F(V) = k$ if $V \subset U$, $F(V) = 0$ otherwise. This is a separated presheaf and $k_{U_{X_\mathcal{T}}} = F^{++}$. Moreover there is an injective arrow $F(V) \hookrightarrow \rho_* k_U(V)$ for each $V \in \mathrm{Op}(X_\mathcal{T})$. Hence $F^{++} \hookrightarrow \rho_* k_U$ since the functor $(\cdot)^{++}$ is exact. Let $\mathcal{S} \subseteq \mathcal{T}$ be the sub-family of $\mathcal{T}$-connected elements. Then $\mathcal{S}$ forms a basis for the Grothendieck topology of $X_\mathcal{T}$. For each $W \in \mathcal{S}$ we have $F(W) \simeq \rho_* k_U(W) \simeq k$ if $W \subset U$ and $F(W) = 0$ otherwise. Then $F^{++} \simeq \rho_* k_U$. □

Proposition 2.2.4. *The restriction of ρ_* to $\mathrm{Coh}(\mathcal{T})$ is exact.*

Proof. Let us consider an epimorphism $G \twoheadrightarrow F$ in $\mathrm{Coh}(\mathcal{T})$, we have to prove that $\psi : \rho_* G \to \rho_* F$ is an epimorphism. Let $U \in \mathcal{T}$ and let $0 \neq s \in \Gamma(U; \rho_* F) \simeq \mathrm{Hom}_{k_X}(k_U, F)$ (by adjunction). Set $G' = G \times_F k_U = \ker(G \oplus k_U \rightrightarrows F)$. Then $G' \in \mathrm{Coh}(\mathcal{T})$ and moreover $G' \twoheadrightarrow k_U$. There exists a finite $\{U_i\}_{i \in I} \subset \mathcal{T}$ of $\mathcal{T}$-connected elements such that $\oplus_i k_{U_i} \twoheadrightarrow G'$. The composition $k_{U_i} \to G' \to k_U$ is given by the multiplication by $a_i \in k$. Set $I_0 = \{k_{U_i};\ a_i \neq 0\}$, we may assume $a_i = 1$. We get a diagram

$$\begin{array}{ccccc} \oplus_{i \in I_0} k_{U_i} & \longrightarrow & G' & \longrightarrow & G \\ & \searrow & \downarrow & & \downarrow \\ & & k_U & \xrightarrow{s} & F. \end{array}$$

The composition $k_{U_i} \to G' \to G$ defines $t_i \in \mathrm{Hom}_{k_X}(k_{U_i}, G) \simeq \Gamma(U_i; \rho_* G)$. Hence for each $s \in \Gamma(U; \rho_* F)$ there exists a finite covering $\{U_i\}$ of U and $t_i \in \Gamma(U_i; \rho_* G)$ such that $\psi(t_i) = s|_{U_i}$. This means that ψ is surjective. □

Notation 2.2.5. *Since the functor ρ_* is fully faithfull and exact on $\mathrm{Coh}(\mathcal{T})$, we will often identify $\mathrm{Coh}(\mathcal{T})$ with its image in $\mathrm{Mod}(k_\mathcal{T})$ and write F instead of $\rho_* F$ for $F \in \mathrm{Coh}(\mathcal{T})$.*

Theorem 2.2.6. *The following hold:*

(i) *The category* $\mathrm{Coh}(\mathcal{T})$ *is stable by finite sums, kernels, cokernels and extensions in* $\mathrm{Mod}(k_{\mathcal{T}})$.

(ii) *The category* $\mathrm{Coh}(\mathcal{T})$ *is stable by* $\bullet \otimes_{k_{\mathcal{T}}} \bullet$ *in* $\mathrm{Mod}(k_{\mathcal{T}})$.

Proof. (i) The result follows from a general result of homological algebra of [27], Appendix A.1. With the notations of [27] let $\mathbf{P}$ be the set of finite families of elements of $\mathcal{T}$, for $\mathcal{U} = \{U_i\}_{i\in I} \in \mathbf{P}$ set

$$L(\mathcal{U}) = \oplus_i k_{U_i},$$

for $\mathcal{V} = \{V_j\}_{j\in J} \in \mathbf{P}$ set

$$\mathrm{Hom}_{\mathbf{P}}(\mathcal{U}, \mathcal{V}) = \mathrm{Hom}_{k_{\mathcal{T}}}(L(\mathcal{U}), L(\mathcal{V})) = \oplus_i \oplus_j \mathrm{Hom}_{k_{\mathcal{T}}}(k_{U_i}, k_{V_j})$$

and for $F \in \mathrm{Mod}(k_{\mathcal{T}})$ set

$$H(\mathcal{U}, F) = \mathrm{Hom}_{k_{\mathcal{T}}}(L(\mathcal{U}), F) = \oplus_i \mathrm{Hom}_{k_{\mathcal{T}}}(k_{U_i}, F).$$

By Proposition A.1 of [27] in order to prove (i) it is enough to prove the properties (A.1)-(A.4) below:

(A.1) For any $\mathcal{U} = \{U_i\} \in \mathbf{P}$ the functor $H(\mathcal{U}, \bullet)$ is left exact in $\mathrm{Mod}(k_{\mathcal{T}})$.

(A.2) For any morphism $g : \mathcal{V} \to \mathcal{W}$ in $\mathbf{P}$, there exists a morphism $f : \mathcal{U} \to \mathcal{V}$ in $\mathbf{P}$ such that $\mathcal{U} \xrightarrow{f} \mathcal{V} \xrightarrow{g} \mathcal{W}$ is exact.

(A.3) For any epimorphism $f : F \to G$ in $\mathrm{Mod}(k_{\mathcal{T}})$, $\mathcal{U} \in \mathbf{P}$ and $\psi \in H(\mathcal{U}, G)$, there exists $\mathcal{V} \in \mathbf{P}$ and an epimorphism $g \in \mathrm{Hom}_{\mathbf{P}}(\mathcal{V}, \mathcal{U})$ and $\varphi \in H(\mathcal{V}, F)$ such that $\psi \circ g = f \circ \varphi$.

(A.4) For any $\mathcal{U}, \mathcal{V} \in \mathbf{P}$ and $\psi \in H(\mathcal{U}, L(\mathcal{V}))$ there exists $\mathcal{W} \in \mathbf{P}$ and an epimorphism $f \in \mathrm{Hom}_{\mathbf{P}}(\mathcal{W}, \mathcal{U})$ and a morphism $g \in \mathrm{Hom}_{\mathbf{P}}(\mathcal{W}, \mathcal{U})$ such that $L(g) = \psi \circ f$ in $\mathrm{Hom}_{k_{\mathcal{T}}}(L(\mathcal{W}), L(\mathcal{V}))$.

It is easy to check that the axioms (A.1)-(A.4) are satisfied.

(ii) Let $F \in \mathrm{Coh}(\mathcal{T})$. Then F has a resolution

$$\oplus_{j\in J} k_{U_j} \to \oplus_{i\in I} k_{U_i} \to F \to 0$$

with I and J finite. Let $V \in \mathcal{T}$. The sequence

$$\oplus_{j\in J} k_{V\cap U_j} \to \oplus_{i\in T} k_{V\cap U_i} \to F_V \to 0$$

is exact. Then it follows from (i) that F_V is coherent. Let $G \in \mathrm{Coh}(\mathcal{T})$. The sequence

$$\oplus_{j\in J} G_{U_j} \to \oplus_{i\in I} G_{U_i} \to G \otimes_{k_{\mathcal{T}}} F \to 0$$

is exact. The sheaves G_{U_i} and G_{U_j} are coherent for each $i \in I$ and each $j \in J$. Hence it follows by (i) that $G \otimes_{k_{\mathcal{T}}} F$ is coherent as required. □

Corollary 2.2.7. *The following hold:*

(i) *The category* $\mathrm{Coh}(\mathcal{T})$ *is stable by finite sums, kernels, cokernels in* $\mathrm{Mod}(k_X)$.

(ii) *The category* $\mathrm{Coh}(\mathcal{T})$ *is stable by* $\bullet \otimes_{k_X} \bullet$ *in* $\mathrm{Mod}(k_X)$.

Proof. (i) The stability under finite sums and kernels is easy, see [29], Exercise 8.23. Let $F, G \in \mathrm{Coh}(\mathcal{T})$ and let $\varphi : F \to G$ be a morphism in $\mathrm{Mod}(k_X)$. Then $\rho_*(\varphi)$ is a morphism in $\mathrm{Mod}(k_{\mathcal{T}})$ and $\mathrm{coker}(\rho_*\varphi) \in \mathrm{Coh}(\mathcal{T})$ by Theorem 2.2.6. We have $\mathrm{coker}(\rho_*\varphi) \simeq \rho_* \mathrm{coker}\varphi$ since ρ_* is exact on $\mathrm{Coh}(\mathcal{T})$ by Proposition 2.2.4. Composing with ρ^{-1} and applying Proposition 2.1.6 we obtain $\mathrm{coker}\varphi \in \mathrm{Coh}(\mathcal{T})$.

(ii) The proof of the stability by $\bullet \otimes_{k_X} \bullet$ is similar to that of Theorem 2.2.6. □

Theorem 2.2.8. *(i) Let $G \in \mathrm{Coh}(\mathcal{T})$ and let $\{F_i\}$ be a filtrant inductive system in $\mathrm{Mod}(k_{\mathcal{T}})$. Then we have the isomorphism*

$$\varinjlim_i \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_* G, F_i) \xrightarrow{\sim} \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_* G, \varinjlim_i F_i).$$

(ii) Let $F \in \mathrm{Mod}(k_{\mathcal{T}})$. There exists a small filtrant inductive system $\{F_i\}_{i \in I}$ in $\mathrm{Coh}(\mathcal{T})$ such that $F \simeq \varinjlim_i \rho_* F_i$.

Proof. (i) There exists an exact sequence $G_1 \to G_0 \to G \to 0$ with G_1, G_0 finite direct sums of constant sheaves k_U with $U \in \mathcal{T}$. Since ρ_* is exact on $\mathrm{Coh}(\mathcal{T})$ and commutes with finite sums, by Proposition 2.2.3 we are reduced to prove the isomorphism $\varinjlim_i \Gamma(U; F_i) \xrightarrow{\sim} \Gamma(U; \varinjlim_i F_i)$. Then the result follows from Proposition 2.1.7.

(ii) Let $F \in \mathrm{Mod}(k_{\mathcal{T}})$, and define

$$\begin{aligned} I_0 &:= \{(U, s) :\ U \in \mathcal{T},\ s \in \Gamma(U; F)\} \\ G_0 &:= \oplus_{(U,s) \in I_0} \rho_* k_U \end{aligned}$$

The morphism $\rho_* k_U \to F$, where the section $1 \in \Gamma(U; k_U)$ is sent to $s \in \Gamma(U; F)$ defines un epimorphism $\varphi : G_0 \to F$. Replacing F by $\ker \varphi$ we construct a sheaf $G_1 = \oplus_{(V,t) \in I_1} \rho_* k_V$ and an epimorphism $G_1 \twoheadrightarrow \ker \varphi$. Hence we get an exact sequence $G_1 \to G_0 \to F \to 0$. For $J_0 \subset I_0$ set for short $G_{J_0} = \oplus_{(U,s) \in J_0} \rho_* k_U$ and define similarly G_{J_1}. Set

$$J = \{(J_1, J_0);\ J_k \subset I_k,\ J_k \text{ is finite and } \mathrm{im}\varphi|_{G_{J_1}} \subset G_{J_0}\}.$$

The category J is filtrant and $F \simeq \varinjlim_{(J_1, J_0) \in J} \mathrm{coker}(G_{J_1} \to G_{J_0})$. □

Corollary 2.2.9. *Let $G \in \mathrm{Coh}(\mathcal{T})$ and let $\{F_i\}$ be a filtrant inductive system in $\mathrm{Mod}(k_{\mathcal{T}})$. Then we have an isomorphism*

$$\varinjlim_i \mathcal{H}om_{k_{\mathcal{T}}}(G, F_i) \xrightarrow{\sim} \mathcal{H}om_{k_{\mathcal{T}}}(G, \varinjlim_i F_i).$$

Proof. Let $U \in \mathcal{T}$. We have the chain of isomorphisms

$$\begin{aligned} \Gamma(U; \varinjlim_i \mathcal{H}om_{k_{\mathcal{T}}}(G, F_i)) &\simeq \varinjlim_i \Gamma(U; \mathcal{H}om_{k_{\mathcal{T}}}(G, F_i)) \\ &\simeq \varinjlim_i \mathrm{Hom}_{k_{\mathcal{T}}}(G_U, F_i)) \\ &\simeq \mathrm{Hom}_{k_{\mathcal{T}}}(G_U, \varinjlim_i F_i)) \\ &\simeq \Gamma(U; \mathcal{H}om_{k_{\mathcal{T}}}(G, \varinjlim_i F_i)), \end{aligned}$$

where the first and the third isomorphism follow from Theorem 2.2.8 (i). the fact that $G_U \in \mathrm{Coh}(\mathcal{T})$ follows from Theorem 2.2.6 (ii). □

As in [28], we can define the indization of the category $\mathrm{Coh}(\mathcal{T})$. Recall that the category $\mathrm{Ind}(\mathrm{Coh}(\mathcal{T}))$, of ind-$\mathcal{T}$-coherent sheaves is the category whose objects are filtrant inductive limits of functors

$$\varinjlim_i \mathrm{Hom}_{\mathrm{Coh}(\mathcal{T})}(\bullet, F_i) \qquad (\text{“}\varinjlim_i\text{”} F_i \text{ for short}),$$

where $F_i \in \mathrm{Coh}(\mathcal{T})$, and the morphisms are the natural transformations of such functors. Note that since $\mathrm{Coh}(\mathcal{T})$ is a small category, $\mathrm{Ind}(\mathrm{Coh}(\mathcal{T}))$ is equivalent to the category of k-additive left exact contravariant functors from $\mathrm{Coh}(\mathcal{T})$ to $\mathrm{Mod}(k)$. See [29] for a complete exposition on indizations of categories. We can extend the functor $\rho_* : \mathrm{Coh}(\mathcal{T}) \to \mathrm{Mod}(k_{\mathcal{T}})$ to $\lambda : \mathrm{Ind}(\mathrm{Coh}(\mathcal{T})) \to \mathrm{Mod}(k_{\mathcal{T}})$ by setting $\lambda(\text{“}\varinjlim_i\text{”} F_i) := \varinjlim_i \rho_* F_i$.

Corollary 2.2.10. *The functor* $\lambda : \mathrm{Ind}(\mathrm{Coh}(\mathcal{T})) \to \mathrm{Mod}(k_{\mathcal{T}})$ *is an equivalence of categories.*

Proof. Let $F = \text{“}\varinjlim_j\text{”} F_j, G = \text{“}\varinjlim_i\text{”} G_i \in \mathrm{I}(\mathrm{Coh}(\mathcal{T}))$. By Theorem 2.2.8 (i) and the fact that the functor ρ_* is fully faithfull on $\mathrm{Coh}(\mathcal{T})$ we have

$$\begin{aligned}
\mathrm{Hom}_{k_{\mathcal{T}}}(\lambda(F), \lambda(G)) &\simeq \mathrm{Hom}_{k_{\mathcal{T}}}(\varinjlim_j \rho_* F_j, \varinjlim_i \rho_* G_i) \\
&\simeq \varprojlim_j \varinjlim_i \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_* F_j, \rho_* G_i) \\
&\simeq \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathrm{Coh}(\mathcal{T})}(F_j, G_i) \\
&\simeq \mathrm{Hom}_{\mathrm{Ind}(\mathrm{Coh}(\mathcal{T}))}(F, G),
\end{aligned}$$

hence λ is fully faithful. By Theorem 2.2.8 (ii) for each $F \in \mathrm{Mod}(k_{\mathcal{T}})$ there exists $G = \text{“}\varinjlim_i\text{”} F_i \in \mathrm{Ind}(\mathrm{Coh}(\mathcal{T}))$ such that $\lambda(G) = \varinjlim_i \rho_* F_i \simeq F$, hence λ is essentially surjective. $\square$

2.3. $\mathcal{T}$-flabby sheaves.

Definition 2.3.1. *We say that an object* $F \in \mathrm{Mod}(k_{\mathcal{T}})$ *is $\mathcal{T}$-flabby if for each* $U, V \in \mathcal{T}$ *with* $V \supseteq U$ *the restriction morphism* $\Gamma(V; F) \to \Gamma(U; F)$ *is surjective.*

Remark 2.3.2. *Remark that the category* $\mathrm{Mod}(k_{\mathcal{T}})$ *is a Grothendieck category, hence it has enough injectives. It follows from the definition that injective sheaves are $\mathcal{T}$-flabby. This implies that the family of $\mathcal{T}$-flabby objects is cogenerating in* $\mathrm{Mod}(k_{\mathcal{T}})$.

Example 2.3.3. Let us see some examples of $\mathcal{T}$-flabby sheaves:

(i) When $\mathcal{T}$ is the family of Example 2.1.2 we obtain the family of *sa*-flabby objects of [10].
(ii) When $\mathcal{T}$ is the family of Example 2.1.3 we obtain the family of quasi-injective objects of [35].

Proposition 2.3.4. *The following hold:*

(i) *Let* F_i *be a filtrant inductive system of $\mathcal{T}$-flabby sheaves. Then* $\varinjlim_i F_i$ *is $\mathcal{T}$-flabby.*
(ii) *Products of $\mathcal{T}$-flabby objects are $\mathcal{T}$-flabby.*

Proof. We will only prove (i) since the proof of (ii) is similar since taking products is exact and commutes with taking sections. Let $U \in \mathcal{T}$. Then for each i

the restriction morphism $\Gamma(V;F_i) \to \Gamma(U;F_i)$ is surjective. Applying the exact $\varinjlim_i$ and using Proposition 2.1.7, the morphism

$$\Gamma(V;\varinjlim_i F_i) \simeq \varinjlim_i \Gamma(V;F_i) \to \varinjlim_i \Gamma(U;F_i) \simeq \Gamma(U;\varinjlim_i F_i)$$

is surjective. □

Proposition 2.3.5. *The full additive subcategory of* $\mathrm{Mod}(k_{\mathcal{T}})$ *of* $\mathcal{T}$*-flabby object is* $\Gamma(U;\bullet)$*-injective for every* $U \in \mathcal{T}$*, i.e.:*

(i) *For every* $F \in \mathrm{Mod}(k_{\mathcal{T}})$ *there exists a* $\mathcal{T}$*-flabby object* $F' \in \mathrm{Mod}(k_{\mathcal{T}})$ *and an exact sequence* $0 \to F \to F'$.

(ii) *Let* $0 \to F' \to F \to F'' \to 0$ *be an exact sequence in* $\mathrm{Mod}(k_{\mathcal{T}})$ *and assume that* F' *is* $\mathcal{T}$*-flabby. Then the sequence*

$$0 \to \Gamma(U;F') \to \Gamma(U;F) \to \Gamma(U;F'') \to 0$$

is exact.

(iii) *Let* $F', F, F'' \in \mathrm{Mod}(k_{\mathcal{T}})$*, and consider the exact sequence*

$$0 \to F' \to F \to F'' \to 0.$$

Suppose that F' *is* $\mathcal{T}$*-flabby. Then* F *is* $\mathcal{T}$*-flabby if and only if* F'' *is* $\mathcal{T}$*-flabby.*

Proof. (i) It follows from the definition that injective sheaves are $\mathcal{T}$-flabby. So (i) holds since it is true for injective sheaves. Indeed, as a Grothendieck category, $\mathrm{Mod}(k_{\mathcal{T}})$ admits enough injectives.

(ii) Let $s'' \in \Gamma(U;F'')$, and let $\{V_i\}_{i=1}^n \in \mathrm{Cov}(U)$ be such that there exists $s_i \in \Gamma(V_i;F)$ whose image is $s''|_{V_i}$. For $n \geq 2$ on $V_1 \cap V_2$ $s_1 - s_2$ defines a section of $\Gamma(V_1 \cap V_2; F')$ which extends to $s' \in \Gamma(U;F')$ since F' is $\mathcal{T}$-flabby. Replace s_1 with $s_1 - s'$ (identifying s' with it's image in F). We may suppose that $s_1 = s_2$ on $V_1 \cap V_2$. Then there exists $t \in \Gamma(V_1 \cup V_2, F)$ such that $t|_{V_i} = s_i$, $i = 1, 2$. Thus the induction proceeds.

(iii) Let $U, V \in \mathcal{T}$ with $V \supseteq U$ and let us consider the diagram below

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(V;F') & \longrightarrow & \Gamma(V;F) & \longrightarrow & \Gamma(V;F'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & \Gamma(U;F') & \longrightarrow & \Gamma(U;F) & \longrightarrow & \Gamma(U;F'') & \longrightarrow & 0
\end{array}$$

where the row are exact by (ii) and the morphism α is surjective since F' is $\mathcal{T}$-flabby. It follows from the five lemma that β is surjective if and only if γ is surjective. □

Theorem 2.3.6. *Let* $F \in \mathrm{Mod}(k_{\mathcal{T}})$*. Then the following hold:*

(i) F *is* $\mathcal{T}$*-flabby if and only if the functor* $\mathrm{Hom}_{k_{\mathcal{T}}}(\bullet, F)$ *is exact on* $\mathrm{Coh}(\mathcal{T})$.

(ii) *If* F *is* $\mathcal{T}$*-flabby then the functor* $\mathcal{H}om_{k_{\mathcal{T}}}(\bullet, F)$ *is exact on* $\mathrm{Coh}(\mathcal{T})$.

Proof. (i) is a consequence of a general result of homological algebra (see Theorem 8.7.2 of [29]). For (ii), let $F \in \mathrm{Mod}(k_{\mathcal{T}})$ be $\mathcal{T}$-flabby. There is an isomorphism of functors

$$\Gamma(U;\mathcal{H}om_{k_{\mathcal{T}}}(\bullet, F)) \simeq \mathrm{Hom}_{k_{\mathcal{T}}}((\bullet)_U, F)$$

for each $U \in \mathcal{T}$. By Theorem 2.2.6 and (i) the functor $\mathrm{Hom}_{k_{\mathcal{T}}}((\bullet)_U, F)$ is exact on $\mathrm{Coh}(\mathcal{T})$ and so the functor $\mathcal{H}om_{k_{\mathcal{T}}}(\bullet, F)$ is also exact on $\mathrm{Coh}(\mathcal{T})$. □

Theorem 2.3.7. *Let $G \in \mathrm{Coh}(\mathcal{T})$. Then the following hold:*

(i) *The family of $\mathcal{T}$-flabby sheaves is injective with respect to the functor $\mathrm{Hom}_{k_\mathcal{T}}(G, \bullet)$.*
(ii) *The family of $\mathcal{T}$-flabby sheaves is injective with respect to the functor $\mathcal{H}om_{k_\mathcal{T}}(G, \bullet)$.*

Proof. (i) Let $G \in \mathrm{Coh}(\mathcal{T})$. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(k_\mathcal{T})$ and assume that F' is $\mathcal{T}$-flabby. We have to show that the sequence

$$0 \to \mathrm{Hom}_{k_\mathcal{T}}(G, F') \to \mathrm{Hom}_{k_\mathcal{T}}(G, F) \to \mathrm{Hom}_{k_\mathcal{T}}(G, F'') \to 0$$

is exact.

There is an epimorphism $\varphi : \oplus_{i\in I} k_{U_i} \to G$ where I is finite and $U_i \in \mathcal{T}$ for each $i \in I$. The sequence $0 \to \ker\varphi \to \oplus_{i\in I} k_{U_i} \to G \to 0$ is exact. We set for short $G_1 = \ker\varphi$ and $G_2 = \oplus_{i\in I} k_{U_i}$. We get the following diagram where the first column is exact by Theorem 2.3.6 (i)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{k_\mathcal{T}}(G, F') & \to & \mathrm{Hom}_{k_\mathcal{T}}(G, F) & \to & \mathrm{Hom}_{k_\mathcal{T}}(G, F'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{k_\mathcal{T}}(G_2, F') & \to & \mathrm{Hom}_{k_\mathcal{T}}(G_2, F) & \to & \mathrm{Hom}_{k_\mathcal{T}}(G_2, F'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{k_\mathcal{T}}(G_1, F') & \to & \mathrm{Hom}_{k_\mathcal{T}}(G_1, F) & \to & \mathrm{Hom}_{k_\mathcal{T}}(G_1, F'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The second row is exact by Proposition 2.3.5 (ii), hence the top row is exact by the snake lemma.

(ii) Let $G \in \mathrm{Coh}(\mathcal{T})$. It is enough to check that for each $U \in \mathcal{T}$ and each exact sequence $0 \to F' \to F \to F'' \to 0$ with F' $\mathcal{T}$-flabby, the sequence

$$0 \to \Gamma(U; \mathcal{H}om_{k_\mathcal{T}}(G, F')) \to \Gamma(U; \mathcal{H}om_{k_\mathcal{T}}(G, F)) \to \Gamma(U; \mathcal{H}om_{k_\mathcal{T}}(G, F'')) \to 0$$

is exact. We have

$$\Gamma(U, \mathcal{H}om_{k_\mathcal{T}}(G, \bullet)) \simeq \mathrm{Hom}_{k_\mathcal{T}}(G_U, \bullet),$$

and, by (i) and the fact that $G_U \in \mathrm{Coh}(\mathcal{T})$ (Theorem 2.2.6 (ii)), $\mathcal{T}$-flabby objects are injective with respect to the functor $\mathrm{Hom}_{k_\mathcal{T}}(G_U, \bullet)$ for each $G \in \mathrm{Coh}(\mathcal{T})$, and for each $U \in \mathcal{T}$. □

Proposition 2.3.8. *Let $F \in \mathrm{Mod}(k_\mathcal{T})$. Then F is $\mathcal{T}$-flabby if and only if $\mathcal{H}om_{k_\mathcal{T}}(G, F)$ is $\mathcal{T}$-flabby for each $G \in \mathrm{Coh}(\mathcal{T})$.*

Proof. Suppose that F is $\mathcal{T}$-flabby, and let $G \in \mathrm{Coh}(\mathcal{T})$. We have

$$\mathrm{Hom}_{k_\mathcal{T}}(\bullet, \mathcal{H}om_{k_\mathcal{T}}(G, F)) \simeq \mathrm{Hom}_{k_\mathcal{T}}(\bullet \otimes_{k_\mathcal{T}} G, F)$$

and $\mathrm{Hom}_{k_\mathcal{T}}(\bullet \otimes_{k_\mathcal{T}} G, F)$ is exact on $\mathrm{Coh}(\mathcal{T})$ by Theorems 2.2.6 (ii) and 2.3.6 (i).

Suppose that $\mathcal{H}om_{k_\mathcal{T}}(G,F)$ is $\mathcal{T}$-flabby for each $G \in \mathrm{Coh}(\mathcal{T})$. Let $U, V \in \mathcal{T}$ with $V \supseteq U$. For each $W \in \mathcal{T}$ the morphism $\Gamma(V;\Gamma_W F) \to \Gamma(U;\Gamma_W F)$ is surjective. Hence the morphism

$$\begin{array}{rcl} \Gamma(V;F) & \simeq & \Gamma(V;\Gamma_V F) \\ & \to & \Gamma(U;\Gamma_V F) \\ & \simeq & \Gamma(U;F) \end{array}$$

is surjective. □

Let us consider the following subcategory of $\mathrm{Mod}(k_\mathcal{T})$:

$$\mathcal{P}_{X_\mathcal{T}} := \{G \in \mathrm{Mod}(k_\mathcal{T});\ G \text{ is } \mathrm{Hom}_{k_\mathcal{T}}(\bullet, F)\text{-acyclic for each } F \in \mathcal{F}_{X_\mathcal{T}}\},$$

where $\mathcal{F}_{X_\mathcal{T}}$ is the family of $\mathcal{T}$-flabby objects of $\mathrm{Mod}(k_\mathcal{T})$.

This category is generating. In fact if $\{U_j\}_{j\in J} \in \mathcal{T}$, then $\oplus_{j\in J} k_{U_j} \in \mathcal{P}_{X_\mathcal{T}}$ by Theorem 2.3.7 (and the fact that

$$\Pi \mathrm{Hom}_{k_\mathcal{T}}(\bullet, \bullet) \simeq \mathrm{Hom}_{k_\mathcal{T}}(\oplus \bullet, \bullet)$$

and products are exact). Moreover $\mathcal{P}_{X_\mathcal{T}}$ is stable by $\bullet \otimes_{k_\mathcal{T}} K$, where $K \in \mathrm{Coh}(\mathcal{T})$. In fact if $G \in \mathcal{P}_{X_\mathcal{T}}$ and $F \in \mathcal{F}_{X_\mathcal{T}}$ we have

$$\mathrm{Hom}_{k_\mathcal{T}}(G \otimes_{k_\mathcal{T}} K, F) \simeq \mathrm{Hom}_{k_\mathcal{T}}(G, \mathcal{H}om_{k_\mathcal{T}}(K,F))$$

and $\mathcal{H}om_{k_\mathcal{T}}(K,F)$ is $\mathcal{T}$-flabby by Proposition 2.3.8. In particular, if $G \in \mathcal{P}_{X_\mathcal{T}}$ then $G_U \in \mathcal{P}_{X_\mathcal{T}}$ for every $U \in \mathrm{Op}(X_\mathcal{T})$.

Theorem 2.3.9. *The category $(\mathcal{P}^{op}_{X_\mathcal{T}}, \mathcal{F}_{X_\mathcal{T}})$ is injective with respect to the functors $\mathrm{Hom}_{k_\mathcal{T}}(\bullet,\bullet)$ and $\mathcal{H}om_{k_\mathcal{T}}(\bullet,\bullet)$.*

Proof. (i) Let $G \in \mathcal{P}_{X_\mathcal{T}}$ and consider an exact sequence $0 \to F' \to F \to F'' \to 0$ with F' $\mathcal{T}$-flabby. We have to prove that the sequence

$$0 \to \mathrm{Hom}_{k_\mathcal{T}}(G,F') \to \mathrm{Hom}_{k_\mathcal{T}}(G,F) \to \mathrm{Hom}_{k_\mathcal{T}}(G,F'') \to 0$$

is exact. Since the functor $\mathrm{Hom}_{k_\mathcal{T}}(G,\bullet)$ is acyclic on $\mathcal{T}$-flabby sheaves we obtain the result.

Let F be $\mathcal{T}$-flabby, and let $0 \to G' \to G \to G'' \to 0$ be an exact sequence on $\mathcal{P}_{X_\mathcal{T}}$. Since the objects of $\mathcal{P}_{X_\mathcal{T}}$ are $\mathrm{Hom}_{k_\mathcal{T}}(\bullet,F)$-acyclic the sequence

$$0 \to \mathrm{Hom}_{k_\mathcal{T}}(G'',F) \to \mathrm{Hom}_{k_\mathcal{T}}(G,F) \to \mathrm{Hom}_{k_\mathcal{T}}(G',F) \to 0$$

is exact.

(ii) Let $G \in \mathcal{P}_{X_\mathcal{T}}$, and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence with F' $\mathcal{T}$-flabby. We shall show that for each $U \in \mathcal{T}$ the sequence

$$0 \to \Gamma(U;\mathcal{H}om_{k_\mathcal{T}}(G,F')) \to \Gamma(U;\mathcal{H}om_{k_\mathcal{T}}(G,F)) \to \Gamma(U;\mathcal{H}om_{k_\mathcal{T}}(G,F'')) \to 0$$

is exact. This is equivalent to show that for each $U \in \mathcal{T}$ the sequence

$$0 \to \mathrm{Hom}_{k_\mathcal{T}}(G_U,F') \to \mathrm{Hom}_{k_\mathcal{T}}(G_U,F) \to \mathrm{Hom}_{k_\mathcal{T}}(G_U,F'') \to 0$$

is exact. This follows since $G_U \in \mathcal{P}_{X_\mathcal{T}}$ as we saw above. The proof of the exactness in $\mathcal{P}^{op}_{X_\mathcal{T}}$ is similar. □

Proposition 2.3.10. *Let $F \in \mathrm{Mod}(k_\mathcal{T})$. The following assumptions are equivalent*

(i) *F is $\mathcal{T}$-flabby,*
(ii) *F is $\mathrm{Hom}_{k_\mathcal{T}}(G,\bullet)$-acyclic for each $G \in \mathrm{Coh}(\mathcal{T})$,*
(iii) *$R^1\mathrm{Hom}_{k_\mathcal{T}}(k_{V\setminus U},F) = 0$ for each $U, V \in \mathcal{T}$.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 2.3.7, (ii) $\Rightarrow$ (iii) setting $G = k_{V\setminus U}$ with $U, V \in \mathcal{T}$, (iii) $\Rightarrow$ (i) since if $R^1\mathrm{Hom}_{k_\mathcal{T}}(k_{V\setminus U}, F) = 0$ for each $U, V \in \mathcal{T}$ with $V \supseteq U$, then the restriction $\Gamma(V; F) \to \Gamma(U; F)$ is surjective. □

Let X, Y be two topological spaces and let $\mathcal{T} \subset \mathrm{Op}(X)$, $\mathcal{T}' \subset \mathrm{Op}(Y)$ satisfy (2.1). Let $f : X \to Y$ be a continuous map. If $f^{-1}(\mathcal{T}') \subset \mathcal{T}$ then f defines a morphism of sites $f : X_\mathcal{T} \to Y_{\mathcal{T}'}$.

Proposition 2.3.11. *Let $f : X_\mathcal{T} \to Y_{\mathcal{T}'}$ be a morphism of sites. $\mathcal{T}$-flabby sheaves are injective with respect to the functor f_*. The functor f_* sends $\mathcal{T}$-flabby sheaves to $\mathcal{T}'$-flabby sheaves.*

Proof. Let us consider $V \in \mathcal{T}'$. There is an isomorphism of functors $\Gamma(V; f_*\bullet) \simeq \Gamma(f^{-1}(V); \bullet)$. It follows from Proposition 2.3.5 that $\mathcal{T}$-flabby are injective with respect to the functor $\Gamma(f^{-1}(V); \bullet)$ for any $V \in \mathcal{T}'$.

Let F be $\mathcal{T}$-flabby and let $U, V \in \mathcal{T}'$ with $V \supset U$. Then the morphism

$$\Gamma(V; f_*F) = \Gamma(f^{-1}(V); F) \to \Gamma(f^{-1}(U); F) = \Gamma(U; f_*F)$$

is surjective. □

2.4. **$\mathcal{T}$-sheaves on locally weakly quasi-compact spaces.** Assume that X is a locally weakly quasi-compact space.

Lemma 2.4.1. *For each $U \in \mathrm{Op}^c(X)$ there exists $V \in \mathcal{T}$ such that $U \subset\subset V \subset\subset X$.*

Proof. Since X is locally weakly quasi-compact we may find $W \in \mathrm{Op}^c(X)$ such that $U \subset\subset W$. By (2.1) (i) we may find a covering $\{W_i\}_{i\in I}$ of X with $W_i \in \mathcal{T}$ and $W_i \subset\subset X$ for each $i \in I$. Then there exists a finite family $\{W_j\}_{j=1}^{\ell}$ whose union $V = \bigcup_{j=1}^{\ell} W_j$ contains W. Then $V \in \mathcal{T}$ and $U \subset\subset V \subset\subset X$. □

When X is locally weakly quasi-compact we can construct a left adjoint to the functor ρ^{-1}.

Proposition 2.4.2. *Let $F \in \mathrm{Mod}(k_\mathcal{T})$, and let $U \in \mathrm{Op}(X)$. Then*

$$\Gamma(U; \rho^{-1}F) \simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \Gamma(V; F)$$

Proof. By Theorem 2.2.8 we may assume $F = \varinjlim_i \rho_* F_i$, with $F_i \in \mathrm{Coh}(\mathcal{T})$. Then $\rho^{-1}F \simeq \varinjlim_i \rho^{-1}\rho_* F_i \simeq \varinjlim_i F_i$. We have the chain of isomorphisms

$$\begin{aligned}
\Gamma(U; \rho^{-1}F) &\simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \varinjlim_{V\subset\subset W} \Gamma(W; \rho^{-1}F) \simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \varinjlim_{V\subset\subset W} \Gamma(W; \varinjlim_i \rho^{-1}\rho_* F_i) \\
&\simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \varinjlim_{V\subset\subset W, i} \Gamma(W; \rho^{-1}\rho_* F_i) \simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \varinjlim_i \Gamma(V; \rho^{-1}\rho_* F_i) \\
&\simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \varinjlim_i \Gamma(V; \rho_* F_i) \simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \Gamma(V; F),
\end{aligned}$$

where the first and the fourth isomorphisms follow from Lemma 1.2.16, the third isomorphism is a consequence of Corollary 1.2.13, and the last isomorphism follows from Proposition 2.1.7. □

Proposition 2.4.3. *The functor ρ^{-1} admits a left adjoint, denoted by $\rho_!$. It satisfies*

(i) *for* $F \in \mathrm{Mod}(k_X)$ *and* $U \in \mathcal{T}$, $\rho_! F$ *is the sheaf associated to the presheaf* $U \mapsto \varinjlim_{U \subset\subset V} \Gamma(V; F)$,

(ii) *For* $U \in \mathrm{Op}(X)$ *one has* $\rho_! k_U \simeq \varinjlim_{V \subset\subset U, V \in \mathcal{T}} k_V$.

Proof. Let $\widetilde{F} \in \mathrm{Psh}(k_{\mathcal{T}})$ be the presheaf $U \mapsto \varinjlim_{U \subset\subset V} \Gamma(V; F)$, and let $G \in \mathrm{Mod}(k_{\mathcal{T}})$. We will construct morphisms

$$\mathrm{Hom}_{\mathrm{Psh}(k_{\mathcal{T}})}(\widetilde{F}, G) \underset{\vartheta}{\overset{\xi}{\rightleftarrows}} \mathrm{Hom}_{k_X}(F, \rho^{-1}G) \ .$$

To define ξ, let $\varphi : \widetilde{F} \to G$ and $U \in \mathrm{Op}(X)$. Then the morphism $\xi(\varphi)(U) : F(U) \to \rho^{-1}G(U)$ is defined as follows

$$F(U) \simeq \varprojlim_{V \subset\subset U, V \in \mathcal{T}} \varinjlim_{V \subset\subset W} F(W) \xrightarrow{\varphi} \varprojlim_{V \subset\subset U, V \in \mathcal{T}} G(V) \simeq \rho^{-1}G(U).$$

On the other hand, let $\psi : F \to \rho^{-1}G$ and $U \in \mathcal{T}$. Then the morphism $\vartheta(\psi)(U) : \widetilde{F}(U) \to G(U)$ is defined as follows

$$\widetilde{F}(U) \simeq \varinjlim_{U \subset\subset V \in \mathcal{T}} F(V) \xrightarrow{\psi} \varinjlim_{U \subset\subset V \in \mathcal{T}} \rho^{-1}G(V) \to G(U).$$

By construction one can check that the morphism ξ and ϑ are inverse to each others. Then (i) follows from the chain of isomorphisms

$$\mathrm{Hom}_{\mathrm{Psh}(k_{\mathcal{T}})}(\widetilde{F}, G) \simeq \mathrm{Hom}_{k_{\mathcal{T}}}(\widetilde{F}^{++}, G) \simeq \mathrm{Hom}_{k_{\mathcal{T}}}(\widetilde{F}^{++}, G).$$

To show (ii), consider the following sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_! k_U, F) &\simeq \mathrm{Hom}_{k_X}(k_U, \rho^{-1}F) \\ &\simeq \varprojlim_{V \subset\subset U, V \in \mathcal{T}} \mathrm{Hom}_{k_{\mathcal{T}}}(k_V, F) \\ &\simeq \mathrm{Hom}_{k_{\mathcal{T}}}(\varinjlim_{V \subset\subset U, V \in \mathcal{T}} k_V, F), \end{aligned}$$

where the second isomorphism follows from Proposition 2.4.2. □

Proposition 2.4.4. *The functor* $\rho_!$ *is exact and commutes with* $\varinjlim$ *and* $\otimes$.

Proof. It follows by adjunction that $\rho_!$ is right exact and commutes with $\varinjlim$, so let us show that it is also left exact. With the notations of Proposition 2.4.3, let $F \in \mathrm{Mod}(k_X)$, and let $\widetilde{F} \in \mathrm{Psh}(k_{\mathcal{T}})$ be the presheaf $U \mapsto \varinjlim_{U \subset\subset V} \Gamma(V; F)$. Then $\rho_! F \simeq \widetilde{F}^{++}$, and the functors $F \mapsto \widetilde{F}$ and $G \mapsto G^{++}$ are left exact.
Let us show that $\rho_!$ commutes with $\otimes$. Let $F, G \in \mathrm{Mod}(k_X)$, the morphism

$$\varinjlim_{U \subset\subset V} F(V) \otimes_k \varinjlim_{U \subset\subset V} G(V) \to \varinjlim_{U \subset\subset V} (F(V) \otimes_k G(V))$$

defines a morphism in $\mathrm{Mod}(k_{\mathcal{T}})$

$$\rho_! F \otimes_{k_{\mathcal{T}}} \rho_! G \to \rho_!(F \otimes_{k_X} G)$$

by Proposition 2.4.3 (i). Since $\rho_!$ commutes with $\varinjlim$ we may suppose that $F = k_U$ and $G = k_V$ and the result follows from Proposition 2.4.3 (ii). □

Proposition 2.4.5. *The functor $\rho_!$ is fully faithful. In particular one has $\rho^{-1}\circ\rho_! \simeq \mathrm{id}$. Moreover, for $F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_{\mathcal{T}})$ one has*

$$\rho^{-1}\mathcal{H}om_{k_{\mathcal{T}}}(\rho_! F, G) \simeq \mathcal{H}om_{k_X}(F, \rho^{-1}G).$$

Proof. For $F, G \in \mathrm{Mod}(k_X)$ by adjunction we have

$$\mathrm{Hom}_{k_X}(\rho^{-1}\rho_! F, G) \simeq \mathrm{Hom}_{k_X}(F, \rho^{-1}\rho_* G) \simeq \mathrm{Hom}_{k_X}(F, G).$$

This also implies that $\rho_!$ is fully faithful, in fact

$$\mathrm{Hom}_{k_{\mathcal{T}}}(\rho_! F, \rho_! G) \simeq \mathrm{Hom}_{k_X}(F, \rho^{-1}\rho_! G) \simeq \mathrm{Hom}_{k_X}(F, G).$$

Now let $K, F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_{\mathcal{T}})$, we have

$$\begin{aligned}
\mathrm{Hom}_{k_X}(K, \rho^{-1}\mathcal{H}om_{k_{\mathcal{T}}}(\rho_! F, G)) &\simeq \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_! K, \mathcal{H}om_{k_{\mathcal{T}}}(\rho_! F, G)) \\
&\simeq \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_! K \otimes_{k_{\mathcal{T}}} \rho_! F, G) \\
&\simeq \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_!(K \otimes_{k_X} F), G) \\
&\simeq \mathrm{Hom}_{k_X}(K \otimes_{k_X} F, \rho^{-1}G) \\
&\simeq \mathrm{Hom}_{k_X}(K, \mathcal{H}om_{k_X}(F, \rho^{-1}G)).
\end{aligned}$$

□

Finally let us consider sheaves of rings in $\mathrm{Mod}(k_{\mathcal{T}})$. If $\mathcal{A}$ is a sheaf of rings in $\mathrm{Mod}(k_X)$, then $\rho_*\mathcal{A}$ and $\rho_!\mathcal{A}$ are sheaves of rings in $\mathrm{Mod}(k_{\mathcal{T}})$.

Let $\mathcal{A}$ be a sheaf of unitary k-algebras on X, and let $\widetilde{\mathcal{A}} \in \mathrm{Psh}(k_{\mathcal{T}})$ be the presheaf defined by the correspondence $\mathcal{T} \ni U \mapsto \varinjlim_{U \subset\subset V} \Gamma(V; \mathcal{A})$. Let $F \in \mathrm{Psh}(k_{\mathcal{T}})$, and assume that, for $V \subset U$, with $U, V \in \mathcal{T}$, the following diagram is commutative:

$$\begin{array}{ccc}
\Gamma(U; \widetilde{\mathcal{A}}) \otimes_k \Gamma(U; F) & \longrightarrow & \Gamma(U; F) \\
\downarrow & & \downarrow \\
\Gamma(V; \widetilde{\mathcal{A}}) \otimes_k \Gamma(V; F) & \longrightarrow & \Gamma(V; F).
\end{array}$$

In this case one says that F is a presheaf of $\widetilde{\mathcal{A}}$-modules on $\mathcal{T}$.

Proposition 2.4.6. *Let $\mathcal{A}$ be a sheaf of k-algebras on X, and let F be a presheaf of $\widetilde{\mathcal{A}}$-modules on $X_{\mathcal{T}}$. Then $F^{++} \in \mathrm{Mod}(\rho_!\mathcal{A})$.*

Proof. Let $U \in \mathcal{T}$, and let $r \in \varinjlim_{U \subset\subset V} \Gamma(V; \mathcal{A})$. Then r defines a morphism $\varinjlim_{U \subset\subset V} \Gamma(V; \mathcal{A}) \otimes_k \Gamma(W; F) \to \Gamma(W; F)$ for each $W \subseteq U$, $W \in \mathcal{T}$, hence an endomorphism of $(F^{++})|_{U_{X_{\mathcal{T}}}} \simeq (F|_{U_{X_{\mathcal{T}}}})^{++}$. This morphism defines a morphism of presheaves $\widetilde{\mathcal{A}} \to \mathcal{E}nd(F^{++})$ and $\widetilde{\mathcal{A}}^{++} \simeq \rho_!\mathcal{A}$ by Proposition 2.4.3. Then $F^{++} \in \mathrm{Mod}(\rho_!\mathcal{A})$. □

Proposition 2.4.7. *Assume that X is locally weakly quasi-compact. Let $F \in \mathrm{Mod}(k_{\mathcal{T}})$ be $\mathcal{T}$-flabby. Then $\rho^{-1}F$ is c-soft.*

Proof. Recall that if $U \in \mathrm{Op}(X)$ then $\Gamma(U; \rho^{-1}F) \simeq \varprojlim_{V \subset\subset U} \Gamma(V; F)$, where $V \in \mathcal{T}$. Let $W \in \mathrm{Op}(X)$, $W \subset\subset X$. It follows from Lemma 2.4.1 that every

$U' \supset\supset W$, $U' \in \mathrm{Op}(X)$ contains $U \in \mathcal{T}$ such that $U \supset\supset W$. Hence

$$\varinjlim_{U'} \Gamma(U';F) \simeq \varinjlim_{U} \Gamma(U;F),$$

where $U' \supset\supset W$, $U' \in \mathrm{Op}(X)$ and $U \in \mathcal{T}$ such that $U \supset\supset W$. We have the chain of isomorphisms

$$\begin{aligned} \varinjlim_{U} \Gamma(U;\rho^{-1}F) &\simeq \varinjlim_{U} \varprojlim_{V \subset\subset U} \Gamma(V;F) \\ &\simeq \varinjlim_{U} \Gamma(U;F) \end{aligned}$$

where $U \in \mathcal{T}$, $U \supset\supset W$ and $V \in \mathcal{T}$. The first isomorphism follows from Proposition 2.4.2 and second one follows since for each $U \supset\supset W$, $U \in \mathcal{T}$, there exists $V \in \mathcal{T}$ such that $U \supset\supset V \supset\supset W$.

Let $V, W \in \mathrm{Op}^c(X)$ with $V \subset\subset W$. Since F is $\mathcal{T}$-flabby and filtrant inductive limits are exact, the morphism $\varinjlim_{W'} \Gamma(W';\rho^{-1}F) \simeq \varinjlim_{W'} \Gamma(W';F) \to \varinjlim_{U} \Gamma(U;F) \simeq \varinjlim_{U} \Gamma(U;\rho^{-1}F)$, where $W', U \in \mathcal{T}$, $W' \supset\supset W$, $U \supset\supset V$, is surjective. Hence $\Gamma(W;\rho^{-1}F) \to \varinjlim_{U \supset\supset V} \Gamma(U;\rho^{-1}F)$ is surjective. □

2.5. **$\mathcal{T}_{loc}$-sheaves.** Let X be a $\mathcal{T}$-space and let

$$\mathcal{T}_{loc} = \{U \in \mathrm{Op}(X) : U \cap W \in \mathcal{T} \text{ for every } W \in \mathcal{T}\}. \tag{2.3}$$

Clearly, $\varnothing, X \in \mathcal{T}_{loc}$, $\mathcal{T} \subseteq \mathcal{T}_{loc}$ and $\mathcal{T}_{loc}$ is closed under finite intersections.

Definition 2.5.1. *We make the following definitions:*

- *a subset S of X is a $\mathcal{T}_{loc}$-subset if and only if $S \cap V$ is a $\mathcal{T}$-subset for every $V \in \mathcal{T}$;*
- *a closed (resp. open) $\mathcal{T}_{loc}$-subset is a $\mathcal{T}_{loc}$-subset which is closed (resp. open) in X;*
- *a $\mathcal{T}_{loc}$-connected subset is a $\mathcal{T}_{loc}$-subset which is not the disjoint union of two proper clopen $\mathcal{T}_{loc}$-subsets.*

Observe that if $\{S_i\}_i$ is a family of $\mathcal{T}_{loc}$-subsets such that $\{i : S_i \cap W \neq \emptyset\}$ is finite for every $W \in \mathcal{T}$, then the union and the intersection of the family $\{S_i\}_i$ is a $\mathcal{T}_{loc}$-subset. Also the complement of a $\mathcal{T}_{loc}$-subset is a $\mathcal{T}_{loc}$-subset. Therefore the $\mathcal{T}_{loc}$-subsets form a Boolean algebra.

Example 2.5.2. Let us see some examples of $\mathcal{T}_{loc}$ subsets:

(i) Let $\mathcal{T}$ be the family of Example 2.1.2. Then the $\mathcal{T}_{loc}$ subsets are the locally semi-algebraic subsets of X.
(ii) Let $\mathcal{T}$ be the family of Example 2.1.3. Then the $\mathcal{T}_{loc}$ subsets are the subanalytic subsets of X.
(iii) Let $\mathcal{T}$ be the family of Example 2.1.4. Then the $\mathcal{T}_{loc}$ subsets are the conic subanalytic subsets of X.
(iv) Let $\mathcal{T}$ be the family of Example 2.1.5. Then the $\mathcal{T}_{loc}$ subsets are the locally definable subsets of X.

One can endow $\mathcal{T}_{loc}$ with a Grothendieck topology in the following way: a family $\{U_i\}_i$ in $\mathcal{T}_{loc}$ is a covering of $U \in \mathcal{T}_{loc}$ if for any $V \in \mathcal{T}$, there exists a finite subfamily covering $U \cap V$. We denote by $X_{\mathcal{T}_{loc}}$ the associated site, write for short $k_{\mathcal{T}_{loc}}$ instead of $k_{X_{\mathcal{T}_{loc}}}$, and let

$$\begin{array}{ccc} & X & \\ {}^{\rho_{loc}}\swarrow & & \searrow^{\rho} \\ X_{\mathcal{T}_{loc}} & \longrightarrow & X_{\mathcal{T}} \end{array}$$

be the natural morphisms of sites.

Remark 2.5.3. *The forgetful functor, induced by the natural morphism of sites $X_{\mathcal{T}_{loc}} \to X_{\mathcal{T}}$, gives an equivalence of categories*

$$\mathrm{Mod}(k_{\mathcal{T}_{loc}}) \xrightarrow{\sim} \mathrm{Mod}(k_{\mathcal{T}}).$$

The quasi-inverse to the forgetful functor sends $F \in \mathrm{Mod}(k_{\mathcal{T}})$ to $F_{loc} \in \mathrm{Mod}(k_{\mathcal{T}_{loc}})$ given by $F_{loc}(U) = \varprojlim_{V \in \mathcal{T}} F(U \cap V)$ for every $U \in \mathcal{T}_{loc}$.

Therefore, we can and will identify $\mathrm{Mod}(k_{\mathcal{T}_{loc}})$ with $\mathrm{Mod}(k_{\mathcal{T}})$ and apply the previous results for $\mathrm{Mod}(k_{\mathcal{T}})$ to obtain analogues results for $\mathrm{Mod}(k_{\mathcal{T}_{loc}})$.

Recall that $F \in \mathrm{Mod}(k_{\mathcal{T}})$ is $\mathcal{T}$-flabby if the restriction $\Gamma(V;F) \to \Gamma(U;F)$ is surjective for any $U, V \in \mathcal{T}$ with $V \supseteq U$. Assume that

(2.4) $\quad X_{\mathcal{T}_{loc}}$ has a countable cover $\{V_n\}_{n\in\mathbb{N}}$ with $V_n \in \mathcal{T}$, $\forall n \in \mathbb{N}$.

Proposition 2.5.4. *Let $F \in \mathrm{Mod}(k_{\mathcal{T}})$. Then F is $\mathcal{T}$-flabby if and only if the restriction $\Gamma(X;F) \to \Gamma(U;F)$ is surjective for any $U \in \mathcal{T}_{loc}$.*

Proof. Suppose that F is $\mathcal{T}$-flabby. Consider a covering $\{V_n\}_{n\in\mathbb{N}}$ of $X_{\mathcal{T}_{loc}}$ satisfying (2.4). Set $U_n = U \cap V_n$ and $S_n = V_n \setminus U_n$. All the sequences

$$0 \to k_{U_n} \to k_{V_n} \to k_{S_n} \to 0$$

are exact. Since F is $\mathcal{T}$-flabby the sequence

$$0 \to \mathrm{Hom}_{k_{\mathcal{T}}}(k_{S_n}, F) \to \mathrm{Hom}_{k_{\mathcal{T}}}(k_{V_n}, F) \to \mathrm{Hom}_{k_{\mathcal{T}}}(k_{U_n}, F) \to 0$$

is exact. Moreover the morphism $\mathrm{Hom}_{k_{\mathcal{T}}}(k_{S_{n+1}}, F) \to \mathrm{Hom}_{k_{\mathcal{T}}}(k_{S_n}, F)$ is surjective for all n since $S_n = S_{n+1} \cap V_n$ is open in S_{n+1}. Then by Proposition 1.12.3 of [26] the sequence

$$0 \to \varprojlim_n \mathrm{Hom}_{k_{\mathcal{T}}}(k_{S_n}, F) \to \varprojlim_n \mathrm{Hom}_{k_{\mathcal{T}}}(k_{V_n}, F) \to \varprojlim_n \mathrm{Hom}_{k_{\mathcal{T}}}(k_{U_n}, F) \to 0$$

is exact. The result follows since $\varprojlim_n \Gamma(U_n;G) \simeq \Gamma(U;G)$ for any $G \in \mathrm{Mod}(k_{\mathcal{T}})$ and $U \in \mathcal{T}_{loc}$. The converse is obvious. □

Proposition 2.5.5. *The full additive subcategory of $\mathrm{Mod}(k_{\mathcal{T}})$ of $\mathcal{T}$-flabby object is $\Gamma(U;\bullet)$-injective for every $U \in \mathcal{T}_{loc}$.*

Proof. Take an exact sequence $0 \to F' \to F \to F'' \to 0$, and suppose that F' is $\mathcal{T}$-flabby. Consider a covering $\{V_n\}_{n\in\mathbb{N}}$ of $X_{\mathcal{T}_{loc}}$ satisfying (2.4). Set $U_n = U \cap V_n$. All the sequences

$$0 \to \Gamma(U_n;F') \to \Gamma(U_n;F) \to \Gamma(U_n;F'') \to 0$$

are exact by Proposition 2.3.5, and the morphism $\Gamma(U_{n+1}; F') \to \Gamma(U_n; F')$ is surjective for all n. Then by Proposition 1.12.3 of [26] the sequence

$$0 \to \varprojlim_n \Gamma(U_n; F') \to \varprojlim_n \Gamma(U_n; F) \to \varprojlim_n \Gamma(U_n; F'') \to 0$$

is exact. Since $\varprojlim_n \Gamma(U_n; G) \simeq \Gamma(U; G)$ for any $G \in \mathrm{Mod}(k_{\mathcal{T}})$ the result follows. □

Let X, Y be two topological spaces and let $\mathcal{T} \subset \mathrm{Op}(X)$, $\mathcal{T}' \subset \mathrm{Op}(Y)$ satisfy (2.1). Let $f : X \to Y$ be a continuous map. If $f^{-1}(\mathcal{T}'_{loc}) \subseteq \mathcal{T}_{loc}$ then f defines a morphism of sites $f : X_{\mathcal{T}_{loc}} \to Y_{\mathcal{T}'_{loc}}$.

Corollary 2.5.6. *Let $f : X_{\mathcal{T}_{loc}} \to Y_{\mathcal{T}'_{loc}}$ be a morphism of sites. $\mathcal{T}$-flabby sheaves are injective with respect to the functor f_*. The functor f_* sends $\mathcal{T}$-flabby sheaves to $\mathcal{T}'$-flabby sheaves.*

Proof. Let us consider $V \in \mathcal{T}'_{loc}$. There is an isomorphism of functors $\Gamma(V; f_*\bullet) \simeq \Gamma(f^{-1}(V); \bullet)$. It follows from Proposition 2.5.5 that $\mathcal{T}$-flabby are injective with respect to the functor $\Gamma(f^{-1}(V); \bullet)$ for any $V \in \mathcal{T}'_{loc}$.

Let F be $\mathcal{T}$-flabby and let $U, V \in \mathcal{T}'$ with $V \supset U$. Then the morphism

$$\Gamma(V; f_*F) = \Gamma(f^{-1}(V); F) \to \Gamma(f^{-1}(U); F) = \Gamma(U; f_*F)$$

is surjective by Proposition 2.5.4. □

Remark 2.5.7. *An interesting case is when X is a locally weakly quasi-compact space and there exists $\mathcal{S} \subseteq \mathrm{Op}(X)$ with $\mathcal{T} = \{U \in \mathcal{S} : U \subset\subset X\}$ satisfying* (2.1).

Assume that X satisfies (1.7). *Then X has a covering $\{V_n\}_{n \in \mathbb{N}}$ of X such that $V_n \in \mathcal{T}$ and $V_n \subset\subset V_{n+1}$ for each $n \in \mathbb{N}$. By Lemma 1.3.5 we may find a covering $\{U_n\}_{n \in \mathbb{N}}$ of X such that $U_n \in \mathrm{Op}^c(X)$ and $U_n \subset\subset U_{n+1}$ for each $n \in \mathbb{N}$. By Lemma 2.4.1 for each $n \in \mathbb{N}$ there exists $V_n \in \mathcal{T}$ such that $U_n \subset\subset V_n \subset\subset U_{n+1}$.*

In this situation Proposition 2.5.4 and 2.5.5 are satisfied.

2.6. **$\mathcal{T}$-spectrum.** Let X be a topological space and let $\mathcal{P}(X)$ be the power set of X. Consider a subalgebra $\mathcal{F}$ of the power set Boolean algebra $\langle \mathcal{P}(X), \subseteq \rangle$. Then $\mathcal{F}$ is closed under finite unions, intersections and complements. We refer to [25] for an introduction to this subject.

The Boolean algebra $\mathcal{F}$ has an associated topological space, that we denote by $S(\mathcal{F})$, called its Stone space. The points in $S(\mathcal{F})$ are the ultrafilters α on $\mathcal{F}$. The topology on $S(\mathcal{F})$ is generated by a basis of open and closed sets consisting of all sets of the form

$$\widetilde{A} = \{\alpha \in S(\mathcal{F}) : A \in \alpha\},$$

where $A \in \mathcal{F}$. The space $S(\mathcal{F})$ is a compact totally disconnected Hausdorff space. Moreover, for each $A \in \mathcal{F}$, the subspace $\widetilde{A}$ is Hausdorff and compact.

Definition 2.6.1. *Let X be a $\mathcal{T}$-space and let $\mathcal{F}$ be the Boolean algebra of $\mathcal{T}_{loc}$-subsets of X (i.e. Boolean combinations of elements of $\mathcal{T}_{loc}$). The topological space $\widetilde{X}_{\mathcal{T}}$ is the data of:*

- *the points of $S(\mathcal{F})$ such that $U \in \alpha$ for some $U \in \mathcal{T}$,*
- *a basis for the topology is given by the family of subsets $\{\widetilde{U} : U \in \mathcal{T}\}$.*

We call $\widetilde{X}_{\mathcal{T}}$ the $\mathcal{T}$-spectrum of X.

With this topology, for $U \in \mathcal{T}$, the set $\widetilde{U}$ is quasi-compact in $\widetilde{X}_{\mathcal{T}}$ since it is quasi-compact in $S(\mathcal{F})$. Hence $\widetilde{X}_{\mathcal{T}}$ is locally weakly quasi-compact with a basis of quasi-compact open subsets given by $\{\widetilde{U} : U \in \mathcal{T}\}$. Note that if $X \in \mathcal{T}$, then $\widetilde{X}_{\mathcal{T}} = \widetilde{X}$ which is a spectral topological space.

Remark 2.6.2. *We may also define $\widetilde{X}_{\mathcal{T}}$ by means of prime filters of elements of $\mathcal{T}$. This is because $\mathcal{T}$-subsets can be written as finite unions and intersections of $\mathcal{T}$-open and $\mathcal{T}$-closed subsets. In this situation an ultrafilter is determined by the prime filter contained in it.*

Proposition 2.6.3. *Let X be a $\mathcal{T}$-space. Then there is an equivalence of categories* $\mathrm{Mod}(k_{\mathcal{T}}) \simeq \mathrm{Mod}(k_{\widetilde{X}_{\mathcal{T}}})$.

Proof. Let us consider the functor

$$\begin{array}{rcl} \zeta^t : \mathcal{T} & \to & \mathrm{Op}(\widetilde{X}_{\mathcal{T}}) \\ U & \mapsto & \widetilde{U}. \end{array}$$

This defines a morphism of sites $\zeta : \widetilde{X}_{\mathcal{T}} \to X_{\mathcal{T}}$. Indeed, if $V \in \mathcal{T}$, $S \in \mathrm{Cov}(V)$, then $\widetilde{S} = \{\widetilde{V}_i : V_i \in S\} \in \mathrm{Cov}(\widetilde{V})$. Let $F \in \mathrm{Mod}(k_{\mathcal{T}})$ and consider the presheaf $\zeta^{\leftarrow} F \in \mathrm{Psh}(k_{\widetilde{X}_{\mathcal{T}}})$ defined by $\zeta^{\leftarrow} F(U) = \varinjlim_{U \subseteq \widetilde{V}} F(V)$. In particular, if $U = \widetilde{V}$, $V \in \mathcal{T}$, $\zeta^{\leftarrow} F(U) \simeq F(V)$. In this case, by Corollary 1.2.11 we have the isomorphisms

$$\zeta^{-1} F(\widetilde{V}) = (\zeta^{\leftarrow} F)^{++}(\widetilde{V}) \simeq \zeta^{\leftarrow} F(\widetilde{V}) \simeq F(V).$$

Then for $V \in \mathcal{T}$ we have

$$\zeta_* \zeta^{-1} F(V) \simeq \zeta^{-1} F(\widetilde{V}) \simeq F(V).$$

This implies $\zeta_* \circ \zeta^{-1} \simeq \mathrm{id}$. On the other hand, given $\alpha \in \widetilde{X}_{\mathcal{T}}$ and $G \in \mathrm{Mod}(k_{\widetilde{X}_{\mathcal{T}}})$,

$$\begin{array}{rcl} (\zeta^{-1} \zeta_* G)_\alpha & \simeq & \varinjlim_{\widetilde{U} \ni \alpha, U \in \mathcal{T}} \zeta^{-1} \zeta_* G(\widetilde{U}) \\ & \simeq & \varinjlim_{\widetilde{U} \ni \alpha, U \in \mathcal{T}} \zeta_* G(U) \\ & \simeq & \varinjlim_{\widetilde{U} \ni \alpha, U \in \mathcal{T}} G(\widetilde{U}) \\ & \simeq & G_\alpha \end{array}$$

since $\{\widetilde{U} : U \in \mathcal{T}\}$ forms a basis for the topology of $\widetilde{X}_{\mathcal{T}}$. This implies $\zeta^{-1} \circ \zeta_* \simeq \mathrm{id}$. □

Example 2.6.4. Let us see some examples of $\mathcal{T}$-spectra.

(i) When $\mathcal{T}$ is the family of Example 2.1.2 the $\mathcal{T}$-spectrum $\widetilde{X}_{\mathcal{T}}$ of X is the semilagebraic spectrum of X ([10]). When X is semialgebraic, then $\widetilde{X}_{\mathcal{T}} = \widetilde{X}$, the semialgebraic spectrum of X from [9].

(ii) When $\mathcal{T}$ is the family of Example 2.1.3 the $\mathcal{T}$-spectrum $\widetilde{X}_{\mathcal{T}}$ of X is the subanalytic spectrum of X. The equivalence $\mathrm{Mod}(k_{\widetilde{X}_{sa}}) \simeq \mathrm{Mod}(k_{X_{sa}})$ was used in [38] to bound the homological dimension of subanalytic sheaves.

(iii) When $\mathcal{T}$ is the family of Example 2.1.5 the $\mathcal{T}$-spectrum $\widetilde{X}_{\mathcal{T}}$ of X is the o-minimal spectrum of X. When X is a definable space, then $\widetilde{X}_{\mathcal{T}} = \widetilde{X}$, the o-minimal spectrum of X from [33, 19].

3. Examples

In this section we recall our main examples of $\mathcal{T}$-sheaves. Good references on o-minimality are, for example, the book [13] by van den Dries and the notes [8] by Coste. For semialgebraic geometry relevant to this paper the reader should consult the work by Delfs [10], Delfs and Knebusch [11] and the book [7] by Bochnak, Coste and Roy. For subanalytic geometry we refer to the work [6] by Bierstone and Milmann.

3.1. **The semialgebraic site.** Let $R = (R, <, 0, 1, +, \cdot)$ be a real closed field. Let X be a locally semialgebraic space and consider the subfamily of $\mathrm{Op}(X)$ defined by $\mathcal{T} = \{U \in \mathrm{Op}(X) : U \text{ is semialgebraic}\}$. The family $\mathcal{T}$ satisfies (2.1) and the associated site $X_{\mathcal{T}}$ is the semialgebraic site on X of [10, 11]. Note also that: (i) the $\mathcal{T}$-subsets of X are exactly the semialgebraic subsets of X ([7]); (ii) $\mathcal{T}_{loc} = \{U \in \mathrm{Op}(X) : U \text{ is locally semialgebraic}\}$ and (iii) the $\mathcal{T}_{loc}$-subsets of X are exactly the locally semialgebraic subsets of X ([11]).

One can show (using triangulation of semialgebraic sets, as in [26]) that the family $\mathrm{Coh}(\mathcal{T})$ corresponds to the family of sheaves which are locally constant on a locally semi-algebraic stratification of X. For each $F \in \mathrm{Mod}(k_{\mathcal{T}})$ there exists a filtrant inductive system $\{F_i\}_{i \in I}$ in $\mathrm{Coh}(\mathcal{T})$ such that $F \simeq \varinjlim_i \rho_* F_i$.

The subcategory of $\mathcal{T}$-flabby sheaves corresponds to the subcategory of *sa*-flabby sheaves of [10] and it is injective with respect to $\Gamma(U; \bullet)$, $U \in \mathrm{Op}(X_{\mathcal{T}})$ and $\mathrm{Hom}_{k_{\mathcal{T}}}(G, \bullet)$, $G \in \mathrm{Coh}(\mathcal{T})$. Our results on $\mathcal{T}$-flabby sheaves generalize those for *sa*-flabby sheaves from [10].

We call in this case the $\mathcal{T}$-spectrum $\widetilde{X}_{\mathcal{T}}$ of X the semialgebraic spectrum of X. The points of $\widetilde{X}_{\mathcal{T}}$ are the ultrafilters α of locally semialgebraic subsets of X such that $U \in \alpha$ for some $U \in \mathrm{Op}(X_{\mathcal{T}})$. This is a locally weakly quasi-compact space with basis of quasi-compact open subsets given by $\{\widetilde{U} : U \in \mathrm{Op}(X_{\mathcal{T}})\}$ and there is an equivalence of categories $\mathrm{Mod}(k_{\mathcal{T}}) \simeq \mathrm{Mod}(k_{\widetilde{X}_{\mathcal{T}}})$. When X is semialgebraic, then $\widetilde{X}_{\mathcal{T}} = \widetilde{X}$, the semialgebraic spectrum of X from [9], and there is an equivalence of categories $\mathrm{Mod}(k_{\mathcal{T}}) \simeq \mathrm{Mod}(k_{\widetilde{X}})$ ([10]).

3.2. **The subanalytic site.** Let X be a real analytic manifold and consider the subfamily of $\mathrm{Op}(X)$ defined by $\mathcal{T} = \mathrm{Op}^c(X_{sa}) = \{U \in \mathrm{Op}(X_{sa}) : U \text{ is subanalytic relatively compact}\}$. The family $\mathcal{T}$ satisfies (2.1) and the associated site $X_{\mathcal{T}}$ is the subanalytic site X_{sa} of [28, 35]. In this case the $\mathcal{T}_{loc}$-subsets are the subanalytic subsets of X.

The family $\mathrm{Coh}(\mathcal{T})$ corresponds to the family $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ of $\mathbb{R}$-constructible sheaves with compact support, and for each $F \in \mathrm{Mod}(k_{X_{sa}})$ there exists a filtrant inductive system $\{F_i\}_{i \in I}$ in $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ such that $F \simeq \varinjlim_i \rho_* F_i$.

The subcategory of $\mathcal{T}$-flabby sheaves corresponds to quasi-injective sheaves and it is injective with respect to $\Gamma(U; \bullet)$, $U \in \mathrm{Op}(X_{sa})$ and $\mathrm{Hom}_{k_{X_{sa}}}(G, \bullet)$, $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$.

We call in this case the $\mathcal{T}$-spectrum $\widetilde{X}_{\mathcal{T}}$ of X the subanalytic spectrum of X and denote it by $\widetilde{X}_{sa}$. The points of $\widetilde{X}_{sa}$ are the ultrafilters of subanalytic subsets of X such that $U \in \alpha$ for some $U \in \mathrm{Op}^c(X_{sa})$. Then there is an equivalence of categories $\mathrm{Mod}(k_{X_{sa}}) \simeq \mathrm{Mod}(k_{\widetilde{X}_{sa}})$.

Let $U \in \mathrm{Op}(X_{sa})$ and denote by $U_{X_{sa}}$ the site with the topology induced by X_{sa}. This corresponds to the site $X_{\mathcal{T}}$, where $\mathcal{T} = \mathrm{Op}^c(X_{sa}) \cap U$. In this situation (2.1) is satisfied.

3.3. **The conic subanalytic site.** Let X be a real analytic manifold endowed with a subanalytic action μ of $\mathbb{R}^+$. In other words we have a subanalytic map

$$\mu : X \times \mathbb{R}^+ \to X,$$

which satisfies, for each $t_1, t_2 \in \mathbb{R}^+$:

$$\begin{cases} \mu(x, t_1 t_2) = \mu(\mu(x, t_1), t_2), \\ \mu(x, 1) = x. \end{cases}$$

Denote by $X_{\mathbb{R}^+}$ the topological space X endowed with the conic topology, i.e. $U \in \mathrm{Op}(X_{\mathbb{R}^+})$ if it is open for the topology of X and invariant by the action of $\mathbb{R}^+$. We will denote by $\mathrm{Op}^c(X_{\mathbb{R}^+})$ the subcategory of $\mathrm{Op}(X_{\mathbb{R}^+})$ consisting of relatively weakly quasi-compact open subsets.

Consider the subfamily of $\mathrm{Op}(X_{\mathbb{R}^+})$ defined by $\mathcal{T} = \mathrm{Op}^c(X_{sa,\mathbb{R}^+}) = \{U \in \mathrm{Op}^c(X_{\mathbb{R}^+}) : U \text{ is subanalytic}\}$. The family $\mathcal{T}$ satisfies (2.1) and the associated site $X_{\mathcal{T}}$ is the conic subanalytic site $X_{sa,\mathbb{R}^+}$. In this case the $\mathcal{T}_{loc}$-subsets are the conic subanalytic subsets.

Set $\mathrm{Coh}(X_{sa,\mathbb{R}^+}) = \mathrm{Coh}(\mathcal{T})$. For each $F \in \mathrm{Mod}(k_{X_{sa,\mathbb{R}^+}})$ there exists a filtrant inductive system $\{F_i\}_{i \in I}$ in $\mathrm{Coh}(X_{sa,\mathbb{R}^+})$ such that $F \simeq \varinjlim_i \rho_* F_i$.

The subcategory of $\mathcal{T}$-flabby sheaves is injective with respect to $\Gamma(U; \bullet)$, $U \in \mathrm{Op}(X_{sa,\mathbb{R}^+})$ and $\mathrm{Hom}_{k_{X_{sa,\mathbb{R}^+}}}(G, \bullet)$, $G \in \mathrm{Coh}(X_{sa,\mathbb{R}^+})$.

We call in this case the $\mathcal{T}$-spectrum $\widetilde{X}_{\mathcal{T}}$ of X the conic subanalytic spectrum of X and denote it by $\widetilde{X}_{sa,\mathbb{R}^+}$. The points of $\widetilde{X}_{sa,\mathbb{R}^+}$ are the ultrafilters α of conic subanalytic subsets of X such that $U \in \alpha$ for some $U \in \mathrm{Op}^c(X_{sa,\mathbb{R}^+})$. Then there is an equivalence of categories $\mathrm{Mod}(k_{X_{sa,\mathbb{R}^+}}) \simeq \mathrm{Mod}(k_{\widetilde{X}_{sa,\mathbb{R}^+}})$.

3.4. **The o-minimal site.** Let $\mathcal{M} = (M, <, (c)_{\in \mathcal{C}}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}})$ be an arbitrary o-minimal structure. Let X be a locally definable space and consider the subfamily of $\mathrm{Op}(X)$ defined by $\mathcal{T} = \mathrm{Op}(X_{\mathrm{def}}) = \{U \in \mathrm{Op}(X) : U \text{ is definable}\}$. The family $\mathcal{T}$ satisfies (2.1) and the associated site $X_{\mathcal{T}}$ is the o-minimal site X_{def} of [19]. Note also that: (i) the $\mathcal{T}$-subsets of X are exactly the definable subsets of X (by the cell decomposition theorem in [13], see [19] Proposition 2.1); (ii) $\mathcal{T}_{loc} = \{U \in \mathrm{Op}(X) : U \text{ is locally definable}\}$ and (iii) the $\mathcal{T}_{loc}$-subsets of X are exactly the locally definable subsets of X.

Set $\mathrm{Coh}(X_{\mathrm{def}}) = \mathrm{Coh}(\mathcal{T})$. For each $F \in \mathrm{Mod}(k_{X_{\mathrm{def}}})$ there exists a filtrant inductive system $\{F_i\}_{i\in I}$ in $\mathrm{Coh}(X_{\mathrm{def}})$ such that $F \simeq \varinjlim_i \rho_* F_i$.

The subcategory of $\mathcal{T}$-flabby sheaves (or definably flabby sheaves) is injective with respect to $\Gamma(U;\bullet)$, $U \in \mathrm{Op}(X_{\mathrm{def}})$ and $\mathrm{Hom}_{k_{X_{\mathrm{def}}}}(G,\bullet)$, $G \in \mathrm{Coh}(X_{\mathrm{def}})$.

We call in this case the $\mathcal{T}$-spectrum $\widetilde{X}_{\mathcal{T}}$ of X the definable or o-minimal spectrum of X and denote it by $\widetilde{X}_{\mathrm{def}}$. The points of $\widetilde{X}_{\mathrm{def}}$ are the ultrafilters α of the Boolean algebra of locally definable subsets of X such that $U \in \alpha$ for some $U \in \mathrm{Op}(X_{\mathrm{def}})$. This is a locally weakly quasi-compact space with basis of quasi-compact open subsets given by $\{\widetilde{U} : U \in \mathrm{Op}(X_{\mathrm{def}})\}$ and there is an equivalence of categories $\mathrm{Mod}(k_{X_{\mathrm{def}}}) \simeq \mathrm{Mod}(k_{\widetilde{X}_{\mathrm{def}}})$. When X is definable, then $\widetilde{X}_{\mathrm{def}} = \widetilde{X}$, the o-minimal spectrum of X from [33, 19], and there is an equivalence of categories $\mathrm{Mod}(k_{X_{\mathrm{def}}}) \simeq \mathrm{Mod}(k_{\widetilde{X}})$ ([19]).

Finally observe that since locally semialgebraic spaces are locally definable spaces in a real closed field and real closed fields are o-minimal structures and, relatively compact subanalytic sets are definable sets in the o-minimal expansion of the field of real numbers by restricted globally analytic functions, both the semialgebraic and subanalytic sheaf theory are special cases of the o-minimal sheaf theory.

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