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A limiting absorption principle for the Helmholtz equation with variable coefficients

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Abstract. We prove a limiting absorption principle for a generalized Helmholtz equation on an exterior domain with Dirichlet boundary conditions

$$(L + \lambda)v = f, \quad \lambda \in \mathbb{R}$$

under a Sommerfeld radiation condition at infinity. The operator L is a second order elliptic operator with variable coefficients; the principal part is a small, long range perturbation of $-\Delta$, while lower order terms can be singular and large.

The main tool is a sharp uniform resolvent estimate, which has independent applications to the problem of embedded eigenvalues and to smoothing estimates for dispersive equations.

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Contents

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1. Introduction

The Helmholtz equation

$$\Delta v + \kappa^2 v = f(x), \quad \kappa \in \mathbb{R} \tag{1.1}$$

on an exterior domain $\Omega = \mathbb{R}^n \setminus \Sigma$, is used to model the scattering by a compact obstacle Σ of waves generated by a source $f(x)$. The operator $\Delta + \kappa^2$ has a nontrivial kernel and to properly select solutions of (1.1) additional conditions are needed. It is natural to require asymptotic conditions at infinity, and the standard one is the *Sommerfeld radiation condition*

$$|x|^{\frac{n-1}{2}} \nabla(e^{-i\kappa|x|}v) \longrightarrow 0 \quad \text{as } |x| \to \infty. \tag{1.2}$$

Condition (1.2) guarantees uniqueness for (1.1), but it can be substantially relaxed as discussed in the following.

The second part of the problem is the effective construction of solutions; this is usually done by taking $\kappa^2 = \lambda + i\epsilon$ complex valued and letting $\epsilon \to 0$. When the limit exists, one says that the *limiting absorption principle* holds. Note that for $\kappa^2 \notin \mathbb{R}$ equation (1.1) is the resolvent equation $v = R(\kappa^2) f$ for $R(z) = (z+\Delta)^{-1}$, which is a bounded operator on L^2 if and only if $z \notin \sigma(-\Delta)$. Thus the problem amounts to estimate the resolvent operator $R(z)$ uniformly in $z \notin \mathbb{R}$. As a byproduct, one obtains that the resolvent operator in the limits $\pm\Im z \to 0$ extends to operators $R(\lambda \pm i0)$ which are bounded between suitable weighted Sobolev spaces.

The Helmholtz equation with potential perturbations was studied in [1] and [2], where the correct functional setting for the problem was established, and in [20] and [21], where non decaying potentials were allowed. More general Schrödinger operators with electromagnetic potentials were considered in [3] [4], [5], [13], [14], [17], [27], and [28]. Uniform resolvent estimates in the case of variable coefficients were obtained in [19], [23], and [25] and the predecessor [8] of this paper, and estimates local in frequency for general elliptic operators were proved in Chapter 30 of [16]. We also mention the connection of resolvent estimates with smoothing and Strichartz estimates for the corresponding evolution equations (exploited first in [18], [26], and [22]; see also [11], [9], and the references in the papers mentioned above).

In recent years the problem of establishing sharp regularity and decay conditions on the potentials has attracted some attention, also in view of the applications to dispersive equations. The critical threshold for electric potentials is $\sim |x|^{-2}$

and for magnetic potentials $\sim |x|^{-1}$. Uniform resolvent estimates for singular potentials of critical decay were obtained in [6] and [15] (see also [12]), while the limiting absorption principle was studied in [27] and [5].

Our goal here is to study the interaction of singular potentials with a non-flat metric which is a long range, small perturbation of the euclidean metric. We consider the following *generalized Helmholtz equation*

$$(L + \lambda + i\epsilon)v = f, \quad \lambda, \epsilon \in \mathbb{R} \tag{1.3}$$

where L is an operator of the form

$$Lv = \nabla^b \cdot (a(x)\nabla^b v) + cv, \quad \nabla^b = \nabla + ib, \tag{1.4}$$

defined on the exterior $\Omega = \mathbb{R}^n \setminus \Sigma$ of a compact, possibly empty obstacle Σ with C^1 boundary, in dimension $n \geq 3$. Here $a(x) = [a_{jk}(x)]_{j,k=1}^n$ is a real valued, positive definite symmetric matrix, b takes vaues in $\mathbb{R}^n$ and c in $\mathbb{R}$. We shall always assume that

$$L \text{ is selfadjoint with domain } H^2(\Omega) \cap H_0^1(\Omega) \tag{1.5}$$

i.e., we restrict to Dirichlet boundary conditions. Note however that in the course of the paper we shall use the same notation for the selfadjoint operator L and the differential operator (1.4) (which operates also on functions outside $D(L)$, e.g. in weighted L^2 spaces). We shall assume that the metric $a(x)$ is a small perturbation of the flat metric, in an appropriate sense precised below, so that in particular trapping is excluded. Concerning the boundary $\partial\Omega$, we shall always assume that it is *starshaped with respect to the metric* $a(x)$: this means

$$a(x)x \cdot \vec{\nu}(x) \leq 0 \quad \text{for all } x \in \partial\Omega \tag{1.6}$$

where $\vec{\nu}(x)$ is the exterior normal to Ω at $x \in \partial\Omega$.

The assumptions on the magnetic potential $b(x) = (b_1, \dots, b_n)$ will be expressed in terms of the corresponding field

$$db = [\partial_j b_\ell - \partial_\ell b_j]_{j,\ell=1}^n$$

as it is physically natural; actually it is sufficient to impose bounds only on the *tangential part* of db for the metric $a(x)$, which is the vector $\widehat{db} = (\widehat{db}_1, \dots, \widehat{db}_n)$ defined by

$$\widehat{db}(x) = db(x)a(x)\hat{x} \quad \text{i.e.} \quad \widehat{db}_j = (\partial_j b_\ell - \partial_\ell b_j)a_{\ell m}\hat{x}_m, \quad \hat{x} = \frac{x}{|x|}.$$

This fact was already noted in [5] (see also [7]). Here and in the following we use the convention of implicit summation over repeated indices. Note that for a vector $w \in \mathbb{C}^n$ we define its radial part w_R and its tangential part w_T as

$$w_R := (\hat{x} \cdot w)\hat{x}, \quad w_T := w - w_R \tag{1.7}$$

respectively; we have of course $|w|^2 = |w_R|^2 + |w_T|^2$.

The relevant functional spaces for our problem are the space $\dot{Y}$ with norm

$$\|v\|_{\dot{Y}}^2 := \sup_{R>0} \frac{1}{R} \int_{\Omega \cap \{|x| \le R\}} |v|^2 dx \simeq \| |x|^{-1/2} v \|_{\ell^\infty L^2}^2$$

and its (pre)dual space $\dot{Y}^*$ with norm

$$\|v\|_{\dot{Y}^*} \simeq \| |x|^{1/2} v \|_{\ell^1 L^2};$$

the notation $\ell^p L^q$ refers to the dyadic norms

$$\|v\|_{\ell^p L^q} := \Big(\sum_{j \in \mathbb{Z}} \|v\|_{L^q(\Omega \cap \{2^j \le |x| < 2^{j+1}\})}^p \Big)^{1/p}, \tag{1.8}$$

with obvious modification when $p = \infty$. Note that $\dot{Y}^*$ is an homogeneous version of the Agmon–Hörmander space B (see [2]). An important role will be played also by the space $\dot{X}$ with norm

$$\|v\|_{\dot{X}}^2 := \sup_{R>0} \frac{1}{R^2} \int_{\Omega \cap \{|x| = R\}} |v|^2 dS$$

where dS is the surface measure on the sphere $|x| = R$. Our main result is the following; in the statement $|a(x)|$ denotes the operator norm of the matrix $a(x)$, and we use the shorthand notation $|a'(x)|$ to denote $\sum_{|\alpha|=1} |\partial^\alpha a(x)|$, and similarly for a'', a''', while $|b'(x)| = \sum_{|\alpha|=1} |\partial^\alpha b(x)|$.

Theorem 1.1 (limiting absorption principle). *Let $n \ge 3$, $\delta \in (0,1)$ and let L and Ω be as in (1.4)–(1.6). There exist two constants $\bar{\kappa} > 0$, $\bar{\sigma} > 0$ depending only on n, δ such that the following holds.*

Assume that for some $\kappa \in [0, \bar{\kappa}]$ and $K \ge 0$ the coefficients of L satisfy:

(i) $\| \langle x \rangle^\delta (|a - I| + |x||a'|) \|_{\ell^1 L^\infty} < \infty$ *and*

$$\| |a - I| + |x||a'| \|_{\ell^1 L^\infty} + |x|^2 |a''| + |x|^3 |a'''| \le \kappa.$$

(ii) *$b', b^2 \in L^{n,\infty}$ and $b = b_S + b_L$ with*

$$|x|^2 |\widehat{db}_S| \le \kappa, \quad \langle x \rangle^{\delta+1} |\widehat{db}_L| \le K.$$

When $n = 3$ we assume the stronger condition $\| |x|^2 \widehat{db}_S \|_{\ell^1 L^\infty} \le \kappa$.

(iii) *$c = c_S + c_L$ with $|x|^2 c_S$, $|x|^3 \nabla c_S \in L^\infty$ and*

$$c_S \ge -\frac{\kappa}{|x|^2}, \quad -\partial_r(|x| c_S) \ge -\frac{\kappa}{|x|^2}, \quad \langle x \rangle^\delta |c_L| \le K.$$

Then for $\lambda > \bar{\sigma} \cdot (K + K^2)$ and all f with $\int |x|^\delta \langle x \rangle |f|^2 < \infty$ the equation

$$(L + \lambda) v = f \tag{1.9}$$

has a unique solution $v \in \dot{Y} \cap H^2_{\text{loc}}(\Omega)$ satisfying $v|_{\partial\Omega} = 0$ and the radiation condition

$$\liminf_{R \to +\infty} \int_{|x|=R} |\nabla^b v - i \hat{x} \lambda^{1/2} v|^2 dS = 0. \tag{1.10}$$

In addition, the solution satisfies the smoothing estimate

$$\|v\|_{\dot{X}} + \lambda^{\frac{1}{2}} \|v\|_{\dot{Y}} + \|\nabla^b v\|_{\dot{Y}} + \|(a \nabla^b v)_T\|_{L^2} + (n-3) \left\| \frac{v}{|x|^{3/2}} \right\|_{L^2} \le c(n) \|f\|_{\dot{Y}^*} \tag{1.11}$$

and if $\epsilon_k \in \mathbb{R} \setminus \{0\}$ is an arbitrary sequence with $\epsilon_k \to 0$, then v is the limit in $H^1_{\text{loc}}(\Omega)$ of the solutions $v_k \in H^1_0(\Omega) \cap H^2(\Omega)$ of

$$(L + \lambda + i\epsilon_k) v_k = f.$$

When $K = 0$, i.e., when the long range components b_L, c_L of the potentials are absent, the previous result implies that the limiting absorption principle is valid for all values of λ and for (short range) potentials with critical singularities, provided suitable smallness conditions are assumed. When $K \neq 0$, i.e., if long range potentials are present, we obtain a similar result but only for large frequencies λ depending on the size of the potentials, which can be arbitrarily large.

The structure of the proof is the following.

- The main tool used in the Theorem is a smoothing estimate for the resolvent $R(z) = (L + z)^{-1}$ outside the spectrum, proved in Section 2 (Theorem 2.1). The estimate improves on earlier results, notably on a similar estimate in the predecessor of this paper [8]. Indeed, we admit large potentials with critical singularities and the estimate is uniform for $\Re z \gg 1$. In the short range case, if $\widehat{db}_S$ and the negative part of c_S satisfy suitable smallness conditions, the estimate is uniform for all $z \in \mathbb{C}$. A few applications include the non existence of embedded eigenvalues or resonances for L, and smoothing estimates for the Schrödinger and wave flows associated to L.

- The smoothing estimate alone is not sufficient to exclude functions in the kernel of $L+\lambda$. However, if the source term f has a slightly better decay, then the difference $\nabla^b v - i\hat{x}\sqrt{\lambda}v$ satisfies a stronger estimate, and this is enough to deduce a weak Sommerfeld radiation condition and hence uniqueness of the solution. The radiation estimate is proved in Theorem 3.2 in Section 3.
- In the last Section 4 we put together all the elements and prove the limiting absorption principle for L.

We conclude the Introduction by examining a few physically interesting singular potentials to which the previous result can be applied.

Remark 1.1 (Coulomb potential). We can handle potentials of the form

$$c(x) = \frac{C}{|x|^a}, \quad 0 < a \le 2$$

including in particular the Coulomb potential $a = 1$. In the critical case $a = 2$, we must require in addition that $C \ge -\bar{\kappa}$ for a suitable $\bar{\kappa} \ge 0$ depending on n, however in this case the result is valid without restrictions on the frequency.

Remark 1.2 (Aharonov–Bohm). Consider a magnetic potential $b(x)$ satisfying

$$x \cdot b(x) = 0 \quad \text{and} \quad b(tx) = t^{-1}b(x) \tag{1.12}$$

for all $x \in \Omega$ and $t > 0$ such that $tx \in \Omega$. The first condition is simply a choice of gauge, which is not restrictive, and the second one states that $b(x)$ is homogeneous of degree -1, which is precisely the critical scaling for magnetic potentials. Then one checks easily (see [5]) that

$$db(x)\hat{x} = 0 \quad \text{for all } x \in \Omega.$$

This implies

$$\widehat{db}(x) = db(x)a(x)\hat{x} = db(x)(a(x) - I)\hat{x}$$

and as a consequence

$$\||x|^2\widehat{db}(x)\|_{\ell^1 L^\infty} \le \|a - I\|_{\ell^1 L^\infty} \||x|^2 b\|_{L^\infty}.$$

Since by homogeneity we have also $\||x|^2 b\|_{L^\infty} < \infty$, recalling that $\|a - I\|_{\ell^1 L^\infty}$ is assumed to be sufficiently small, we conclude that any magnetic potential b satisfying (1.12) (or more generally, any potential $b = b_S + b_L$ with b_S satisfying (1.12) and b_L as in the Theorem) is covered by Theorem 1.1. Interesting examples in $\mathbb{R}^3$ include the so called *Aharonov–Bohm* potentials

$$b(x) = C\Big(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0\Big)$$

and potentials of the form

$$b(x) = C\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}, 0\Big).$$

In both cases the result is valid for all frequencies, independently of the size or sign of C.

2. The smoothing estimate

In this section we develop the key tool for Theorem 1.1: a smoothing estimate for the resolvent of L which is uniform on appropriate regions of $\mathbb{C}$. In order to get sharp results, we distinguish two situations.

(1) *Short range* perturbations of Δ with critical singularities (Assumption (A_0)). In this case we can prove a uniform smoothing estimate for all $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) *Long range* perturbations of Δ, with large electromagnetic potentials of milder decay at infinity (Assumption (A)). In this case the estimate is uniform on a region $\Re z > C$, where C is a suitable norm of the long range component of the potentials.

Moreover, from our analysis one can read precisely the influence of different components of the potentials b and c on the estimate.

The assumptions in the short range case are the following.

Assumption (A_0). Let $n \geq 3$ and let L and Ω be as in (1.4)–(1.6), with $b', b^2 \in L^{n,\infty}$. We assume that, for some constant $\mu \geq 0$

$$C_a(x) := |a - I| + |x||a'| + |x|^2|a''| + |x|^3|a'''| \leq \mu, \quad \||x|a'\|_{\ell^1 L^\infty} \leq \mu.$$

The magnetic field in dimension $n \geq 4$ is of the form $b = b_{\mathrm{I}} + b_{\mathrm{II}}$ and in dimension $n = 3$ of the form $b = b_{\mathrm{I}}$, with

$$\||x|^2 \widehat{db}_{\mathrm{I}}\|_{\ell^1 L^\infty} + \||x|^2 |a - I| \widehat{db}_{\mathrm{II}}\|_{\ell^1 L^\infty} + \||x|^2 \widehat{db}_{\mathrm{II}}\|_{L^\infty} \leq \mu.$$

The electric field is of the form $c = c_{\mathrm{I}} + c_{\mathrm{II}}$ with

$$\||x|^2 c_{\mathrm{II}}\|_{\ell^1 L^\infty} \le \mu, \quad c_{\mathrm{I},-} \in L^\infty$$

and in dimension $n \ge 4$

$$|a - I| \cdot (|x|^2 |c_{\mathrm{I}}| + |x|^3 |\nabla c_{\mathrm{I}}|) + |x|^2 \cdot (c_{\mathrm{I},-} + [\partial_r (r c_{\mathrm{I}})]_+) \le \mu$$

while in dimension $n = 3$

$$\||a - I| \cdot (|x|^2 |c_{\mathrm{I}}| + |x|^3 |\nabla c_{\mathrm{I}}|) + |x|^2 \cdot (c_{\mathrm{I},-} + [\partial_r (r c_{\mathrm{I}})]_+)\|_{\ell^1 L^\infty} \le \mu.$$

In the long range case the assumptions are the following. Note that Assumption (A) reduces to (A_0) when $Z = 0$:

Assumption (A). We assume $b = b_{\mathrm{I}} + b_{\mathrm{II}} + b_{\mathrm{III}}$ and $c = c_{\mathrm{I}} + c_{\mathrm{II}} + c_{\mathrm{III}} + c_{\mathrm{IV}}$ with $b_{\mathrm{I}}, b_{\mathrm{II}}, c_{\mathrm{I}}, c_{\mathrm{II}}$ as in (A_0) while

$$\||x| \widehat{db}_{\mathrm{III}}\|_{\ell^1 L^\infty} \le Z, \quad \||x| \langle x \rangle^{-1} c_{\mathrm{IV}}\|_{\ell^1 L^\infty} \le Z,$$

$$\||a - I| \cdot (|c_{\mathrm{III}}| + |x||\nabla c_{\mathrm{III}}|) + |x|^2 \cdot (c_{\mathrm{III},-} + [\partial_r (r c_{\mathrm{III}})]_+\|_{\ell^1 L^\infty} \le Z.$$

Then we can prove the following result.

Theorem 2.1 (smoothing estimate). *There exist two constants $\mu_0(n)$ and $c_0(n)$ depending only on n such that the following holds.*

Let $v \in H^2_{\mathrm{loc}}(\Omega)$ with $v|_{\partial\Omega} = 0$ be such that

$$\liminf_{R \to \infty} \int_{|x|=R} (|\nabla^b v|^2 + |v|^2) dS = 0 \tag{2.1}$$

and define for some $\lambda, \epsilon \in \mathbb{R}$

$$f = (L + \lambda + i\epsilon) v.$$

If (A_0) holds with $\mu < \mu_0(n)$ then

$$7\|v\|_{\dot X} + (|\lambda| + |\epsilon|)^{1/2} \|v\|_{\dot Y} + \|\nabla^b v\|_{\dot Y} + \|(a \nabla^b v)_T\|_{L^2} + (n-3) \left\| \frac{v}{|x|^{3/2}} \right\|_{L^2}$$
$$\le c(n) \|f\|_{\dot Y^*}. \tag{2.2}$$

The same estimate is valid if (A) *holds with $\mu < \mu_0(n)$ and $\lambda \ge c_0(n)(Z + Z^2)$.*

Remark 2.1 (uniform resolvent estimate). Condition (2.1) is satisfied if v is in H^1. Thus the Theorem applies in particular to the solution v of

$$(L + \lambda + i\epsilon)v = f$$

for $\epsilon \neq 0$ and $f \in L^2(\Omega)$, which exists and belongs to $H_0^1(\Omega) \cap H^2(\Omega)$ by the assumptions on L. This gives the following estimate for the resolvent operator $R(z) = (z + L)^{-1}$, uniform in $z \notin \mathbb{R}$ or in $\Re z \geq c_0(n)(Z + Z^2), z \notin \mathbb{R}$ respectively:

$$\|\nabla^b R(z) f\|_{\dot{Y}} + |z|^{1/2}\|R(z) f\|_{\dot{Y}} + \|R(z) f\|_{\dot{X}} \lesssim \|f\|_{\dot{Y}^*}.$$

Remark 2.2 (absence of embedded eigenvalues or resonances). Suppose v is a solution of

$$(L + \lambda)v = 0, \quad v|_{\partial\Omega} = 0$$

for some $\lambda \geq c_0(n)(Z + Z^2)$. From the smoothing estimate, we see that if v satisfies condition (2.1) then $v \equiv 0$.

Since any function $v \in H_0^1(\Omega)$ satisfies condition (2.1), this implies that there is no eigenvalue $\lambda \geq c_n(Z + Z^2)$. In particular in the case $Z = 0$ (that is to say, under condition (A_0)) we obtain there are no *embedded eigenvalues* in the spectrum of L.

A similar argument gives a more general result about resonances. Writing $\Omega_{\leq R} = \Omega \cap \{|x| \leq R\}$, we say that a function v is a *resonance at* $z \in \mathbb{C}$ if

$$(L + z)v = 0, \quad v|_{\partial\Omega} = 0, \quad v \not\equiv 0, \quad \liminf_{R\to\infty} \frac{1}{R}\int_{\Omega_{\leq R}} |v|^2 = 0.$$

Note that the last condition is weaker than the usual one:

$$\langle x\rangle^{-s} v \in L^2 \text{ for some } s < \frac{1}{2} \implies \lim_{R\to\infty} \frac{1}{R}\int_{\Omega_{\leq R}} |v|^2 = 0.$$

Then we have the following result.

Corollary 2.2 (absence of resonances). *Assume* (*A*) *holds with* $\mu < \mu_0(n)$*, and let* $\lambda \geq c_0(n)(Z + Z^2)$*. Then no resonance exists at* λ*.*

Proof. We must only prove that v satisfies condition (2.1). For $|v|^2$ this follows immediately from the assumption $\liminf \frac{1}{R}\int_{\Omega_{\leq R}} |v|^2 = 0$. For $|\nabla^b v|^2$, we apply Lemma 4.2 from Section 4 which gives

$$\liminf \frac{1}{R}\int_{\Omega_{\leq R}} |\nabla^b v|^2 \lesssim \liminf \frac{1}{R}\int_{\Omega_{\leq R}} |v|^2 = 0. \qquad \square$$

Remark 2.3 (smoothing estimates for dispersive flows). A natural application of estimate (2.2) to dispersive equations is given by Kato's theory of smoothing operators. We recall the procedure in the simplest case. Assume (A_0) holds. Then, from (2.2) we deduce the (Hilbert space) estimate

$$\|\langle x\rangle^{-3/2-} v\|_{L^2} \lesssim \|\langle x\rangle^{1/2+} f\|_{L^2}$$

uniform in $\lambda + i\epsilon \notin \mathbb{R}$, which can be written as the resolvent estimate

$$\|\langle x\rangle^{-3/2-} R(z) f\|_{L^2} \lesssim \|\langle x\rangle^{1/2+} f\|_{L^2}$$

uniform in $z \notin \mathbb{R}$. By duality and interpolation we get

$$\|\langle x\rangle^{-1-} R(z) f\|_{L^2} \lesssim \|\langle x\rangle^{1+} f\|_{L^2} \quad \text{i.e.} \quad \|A^* R(z) A f\|_{L^2} \lesssim \|f\|_{L^2}$$

where $A = \langle x\rangle^{-1-}$ is the multiplication operator. In terms of Kato's theory, this means that A is *supersmoothing* for the operator L, and immediate consequences of the theory are the estimates for the Schrödinger group e^{itL}

$$\|\langle x\rangle^{-1-} e^{itL} f\|_{L^2_t L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$$

and the corresponding Duhamel term

$$\left\| \int_0^t \langle x\rangle^{-1-} e^{i(t-s)L} F(s)ds \right\|_{L^2_t L^2(\Omega)} \lesssim \|\langle x\rangle^{1+} F\|_{L^2_t L^2(\Omega)}.$$

Moreover, if L is nonnegative, we also get the estimate for the wave flow $e^{it\sqrt{L}}$

$$\|\langle x\rangle^{-1-} e^{it\sqrt{L}} f\|_{L^2_t L^2(\Omega)} \lesssim \|L^{1/4} f\|_{L^2(\Omega)}$$

and a similar one for the Duhamel term. With some more work, smoothing estimates with a weight $\langle x\rangle^{-1/2-}$ can be deduced for the flows $|D|^{1/2} e^{itL}$ and $|D|^{1/2} e^{it\sqrt{L}}$. For more details, and the extension of Kato's theory to the wave and Klein–Gordon groups, we refer to [10].

2.1. Notations. With the convention of implicit summation over repeated indices, we write

$$A^b v := \nabla^b \cdot (a(x) \nabla^b v) = \partial^b_j (a_{jk}(x) \partial^b_k v),$$

$$A v := \nabla \cdot (a(x) \nabla v) = \partial_j (a_{jk}(x) \partial_k v).$$

We use the notations

$$\hat{x} := \frac{x}{|x|} = (\hat{x}_1, \dots, \hat{x}_n), \quad a(w,z) := a_{jk}(x) w_k \bar{z}_j, \quad a_{jk;\ell} := \partial_\ell a_{jk}$$

and

$$\hat{a}(x) := a_{\ell m}(x)\hat{x}_\ell \hat{x}_m, \quad \bar{a}(x) := tr\, a(x) = a_{mm}(x), \quad \tilde{a} := a_{\ell m;\ell}\hat{x}_m.$$

If $a(x)$ is positive definite, we have

$$0 \le \hat{a} = a\hat{x} \cdot \hat{x} \le |a\hat{x}| \le \bar{a}.$$

We shall use frequently the following identity, valid for any radial function $\psi(x) = \psi(|x|)$:

$$A\psi(x) = \partial_\ell(a_{\ell m}\hat{x}_m \psi') = \hat{a}\psi'' + \frac{\bar{a} - \hat{a}}{|x|}\psi' + \tilde{a}\psi' \tag{2.3}$$

where ψ' denotes the derivative of $\psi(r)$ with respect to the radial variable.

In order to refine the scale of dyadic spaces $\ell^p L^q$, we introduce the mixed radial-angular $L^q L^r$ norms on an annulus $C = R_1 \le |x| \le R_2$

$$\begin{aligned} \|v\|_{L^q L^r(C)} = \|v\|_{L^q_{|x|} L^r_\theta(C)} &:= \left(\int_{R_1}^{R_2} \left(\int_{|x|=\rho} |v|^r dS \right)^{q/r} d\rho \right)^{1/q} \\ &= \big\| \|v\|_{L^r(|x|=\rho)} \big\|_{L^q(R_1,R_2;d\rho)}, \end{aligned}$$

and on $\Omega \cap C$ we define $\|v\|_{L^q L^r(\Omega\cap C)} = \|\mathbf{1}_\Omega v\|_{L^q L^r}$. When $q = r$ this definition reduces to the usual $L^q(C)$ norm. Then we define for all $p, q, r \in [1, \infty]$

$$\|v\|_{\ell^p L^q L^r} := \|\{\|v\|_{L^q L^r(\Omega_j)}\}_{j\in\mathbb{Z}}\|_{\ell^p}, \quad \Omega_j = \Omega \cap \{2^j \le |x| < 2^{j+1}\}. \tag{2.4}$$

In the case $q = r$ we reobtain the previous dyadic norms:

$$\|v\|_{\ell^p L^q} = \|v\|_{\ell^p L^q L^q}.$$

Both spaces $\dot{X}, \dot{Y}$ are included in this finer scale

$$\|v\|_{\dot{X}} \simeq \||x|^{-1}v\|_{\ell^\infty L^\infty L^2}, \quad \|v\|_{\dot{Y}} \simeq \||x|^{-1/2}v\|_{\ell^\infty L^2} \tag{2.5}$$

like the predual norm $\dot{Y}^*$, which is given by

$$\|v\|_{\dot{Y}^*} \simeq \||x|^{1/2}v\|_{\ell^1 L^2} \simeq \sum_{j\in\mathbb{Z}} 2^{j/2}\|v\|_{L^2(C_j\cap\Omega)}.$$

Remark 2.4 (magnetic Hardy inequality). We shall make frequent use of the *magnetic Hardy inequality*, valid for $s < n/2$ and $w \in H_0^1(\Omega)$:

$$\||x|^{-s} w\|_{L^2} \leq \frac{2}{n-2s} \||x|^{1-s} \nabla^b w\|_{L^2}. \tag{2.6}$$

This is proved as usual, starting from the identity

$$\begin{aligned}\nabla \cdot \left\{ \frac{\hat{x}}{|x|^{2s-1}} |w|^2 \right\} &= 2\Re \frac{w\, \hat{x} \cdot \overline{\nabla w}}{|x|^{2s-1}} + \frac{n-2s}{|x|^{2s}} |w|^2 \\ &= 2\Re \frac{w\, \hat{x} \cdot \overline{\nabla^b w}}{|x|^{2s-1}} + \frac{n-2s}{|x|^{2s}} |w|^2,\end{aligned}$$

then integrating on Ω, estimating with Cauchy–Schwartz

$$\int_\Omega \frac{n-2s}{|x|^{2s}} |w|^2 dx \leq \alpha \int_\Omega \frac{|w|^2}{|x|^{2s}} dx + \alpha^{-1} \int_\Omega \frac{|\nabla^b w|^2}{|x|^{2s-2}}$$

and finally optimizing the value of α.

2.2. Basic identities and boundary terms.

Using the two multipliers

$$[A^b, \psi]\bar{v} = (A\psi)\bar{v} + 2a(\nabla\psi, \nabla^b v) \quad \text{and} \quad \phi\bar{v}$$

one obtains the following Morawetz type identities, proved in [8]:

text slightly changed to avoid an overfull

Theorem 2.3. *Let $v \in H^2_{\mathrm{loc}}(\Omega)$ on an open set $\Omega \subseteq \mathbb{R}^n$, $\lambda, \epsilon \in \mathbb{R}$, the map $a(x)\colon \Omega \to \mathbb{R}^{n\times n}$ symmetric, $b(x)\colon \Omega \to \mathbb{R}^n$ and the maps $c, \phi, \psi\colon \Omega \to \mathbb{R}$ sufficiently smooth, and let*

$$f := A^b v - c(x)v + (\lambda + i\epsilon)v. \tag{2.7}$$

Then the following identity holds:

$$I_{\nabla v} + I_v + I_\epsilon + I_b + I_f = \Re\partial_j\{Q_j + P_j\} \tag{2.8}$$

where

$$I_f = \Re[(A\psi + \phi)\bar{v}f + 2a(\nabla\psi, \nabla^b v) f], \tag{2.9}$$

$$I_{\nabla v} = \alpha_{\ell m} \Re(\partial_m^b v\, \overline{\partial_\ell^b v}) + a(\nabla^b v, \nabla^b v)\phi, \tag{2.10}$$

with

$$\alpha_{\ell m} := 2a_{jm}\partial_j(a_{\ell k}\partial_k\psi) - a_{jk}\partial_k\psi\partial_j a_{\ell m},$$

$$I_v = -\frac{1}{2}A(A\psi + \phi)|v|^2 - [a(\nabla\psi, \nabla c) - c\phi + \lambda\phi]|v|^2, \tag{2.11}$$

$$I_\epsilon = 2\epsilon\Im[a(\nabla\psi, \nabla^b v)v], \tag{2.12}$$

$$I_b = 2\Im[a_{jk}\partial_k^b v(\partial_j b_\ell - \partial_\ell b_j)a_{\ell m}\partial_m\psi\ \bar{v}] = 2\Im[(a\nabla^b v)\cdot(db\ a\nabla\psi)\bar{v}], \tag{2.13}$$

and

$$Q_j = a_{jk}\partial_k^b v\cdot[A^b, \psi]\bar{v} - \frac{1}{2}a_{jk}(\partial_k A\psi)|v|^2 - a_{jk}\partial_k\psi\left[(c-\lambda)|v|^2 + a(\nabla^b v, \nabla^b v)\right],$$

$$P_j = a_{jk}\partial_k^b v\phi\bar{v} - \frac{1}{2}a_{jk}\partial_k\phi|v|^2.$$

Moreover we have the identity

$$\partial_j P_j = a(\nabla^b v, \nabla^b v)\phi + (c-\lambda-i\epsilon)|v|^2\phi + f\bar{v}\phi - \frac{1}{2}A\phi|v|^2 + i\Im a(\nabla^b v, v\nabla\phi). \tag{2.14}$$

Remark 2.5 (boundary terms). In the next computations we shall integrate identities (2.8) and (2.14) on Ω, with various choices of real valued weights ϕ and ψ, with ψ radial, for a function $v \in H^2_{\mathrm{loc}}(\Omega)$ vanishing at $\partial\Omega$ and satisfying the asymptotic condition (2.1). The weights will always be piecewise smooth functions, with possible singularities only at 0 or along spheres $|x| = R$; the worst singularity at 0 appearing in all computations is dominated by $|x|^{-3}$ in dimension $n \geq 4$ and by $|x|^{-2}$ in dimension $n = 3$; concerning the singularity appearing along the sphere, in the worst case it will be a surface measure $\delta_{|x|=R}$ with a definite sign. Moreover, in our choice of ψ we have $\psi' \in L^\infty$ and $\psi' \geq 0$ (see (2.33) below).

In order to handle the boundary terms, some smoothness of the coefficients is necessary. We note that from our assumptions it follows that a, a', a'', a''', c are bounded for large x and

$$a, |x|a', |x|^2 a'', |x|^3 a''' \in L^\infty, \quad b \in L^{n/2,\infty}, \quad b', c \in L^{n,\infty}. \tag{2.15}$$

Then one checks easily that for $v \in H^2_{\mathrm{loc}}(\Omega)$ the quantities Q_j and P_j are in L^1_{loc}, using the Sobolev–Lorentz embedding $H^1 \hookrightarrow L^2 \cap L^{\frac{2n}{n-2},2}$ which implies $|v|^2 \in L^1 \cap L^{\frac{n}{n-2},1}$, and the Hölder–Lorentz inequality.

We integrate the identities (2.8) and (2.14) on a set $\Omega \cap \{|x| \leq M\}$ and let $M \to \infty$. At the boundary $\Omega \cap \{|x| = M\}$ we get the quantities

$\Omega_{=M}$ with index $= M$? Is it correct?

$$\int_{\Omega_{=M}} \nu_j Q_j\, dS, \quad \int_{\Omega_{=M}} \nu_j P_j\, dS,$$

where $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ is the exterior normal and dS is the surface measure on the sphere $\{|x| = M\}$. Letting $M \to \infty$ along a suitable subsequence, by condition (2.1) we see that both integrals tend to 0. Moreover, at the boundary $\partial\Omega$ one has directly $P_j|_{\partial\Omega} = 0$ since $v|_{\partial\Omega} = 0$. Concerning Q_j, after canceling the terms containing a factor v and noticing that $\nabla^b v = \nabla v + ibv = \nabla v$ on $\partial\Omega$, we are left with

$$\int_\Omega \partial_j Q_j = \int_{\partial\Omega} [2a(\nabla v, \vec{\nu}) \cdot a(\hat{x}, \nabla v) - a(\nabla v, \nabla v) \cdot a(\hat{x}, \vec{\nu})]\psi' dS \tag{2.16}$$

where $\vec{\nu}$ is the exterior unit normal to $\partial\Omega$. Dirichled boundary conditions imply that ∇v is normal to $\partial\Omega$ so that $\nabla v = (\vec{\nu} \cdot \nabla v)\vec{\nu}$ and hence

$$a(\nabla v, \vec{\nu}) = (\vec{\nu} \cdot \nabla v) a(\vec{\nu}, \vec{\nu}), \quad a(\hat{x}, \nabla v) = (\vec{\nu} \cdot \nabla \bar{v}) a(\hat{x}, \vec{\nu}),$$

$$a(\nabla v, \nabla v) = |\vec{\nu} \cdot \nabla \bar{v}|^2 a(\vec{\nu}, \vec{\nu}),$$

and

$$\int_\Omega \partial_j \Re Q_j = \int_{\partial\Omega} |\vec{\nu} \cdot \nabla v|^2 a(\vec{\nu}, \vec{\nu}) a(\hat{x}, \vec{\nu}) \psi' dS.$$

Now using the condition that $\partial\Omega$ is $a(x)$-starshaped and recalling that $\psi' \geq 0$ we conclude

$$\int_\Omega \partial_j \Re Q_j \leq 0. \tag{2.17}$$

2.3. Preliminary estimates. The first group of estimates is based on the identity (2.14).

Lemma 2.4 (I_ϵ). *We have the identities*

$$\epsilon \int_\Omega |v|^2 = \Im \int_\Omega f\bar{v}, \quad \int_\Omega a(\nabla^b v, \nabla^b v) = \lambda \int_\Omega |v|^2 - \Re \int_\Omega f\bar{v} - \int_\Omega c|v|^2. \tag{2.18}$$

Moreover if we assume $\|a - I\|_{L^\infty} \leq 1/2$ *and* $c = c_{\mathrm{I}} + c_{\mathrm{II}}$ *with* $c_{\mathrm{I},-} \in L^\infty$ *and* $\||x|^2 c_{\mathrm{II},-}\|_{L^\infty} \leq \frac{n-2}{8}$, *we have the following estimate of the quantity* $I_\epsilon := 2\epsilon\Im[a(v\nabla\psi, \nabla^b v)]$

$$\int_\Omega |I_\epsilon| \leq \sigma(|\lambda| + |\epsilon| + \|c_{\mathrm{I},-}\|_{L^\infty})\|v\|_{\dot{Y}}^2 + C\sigma^{-1}\|f\|_{\dot{Y}^*} \tag{2.19}$$

where $C = C(n, \|\nabla\psi\|_{L^\infty})$ *and* $\sigma \in (0, 1)$ *is arbitrary.*

Proof. Consider identity (2.14) with $\phi = 1$ and $c = 0$ and let $g = A^b v + (\lambda + i\epsilon)v$, so that $g = f + c(x)v$. Taking the imaginary part we get

$$\epsilon|v|^2 = \Im(g\bar{v}) - \Im\partial_j\{\bar{v}a_{jk}\partial_k^b v\} \tag{2.20}$$

and integrating on Ω we obtain the first identity in (2.18), since $\Im(f\bar{v}) = \Im(g\bar{v})$. Note that the identity implies

$$|\epsilon|\|v\|_{L^2}^2 \le \|f\bar{v}\|_{L^1}. \tag{2.21}$$

If instead we take the real part of (2.14) with $\phi = 1$ and $c = 0$ we get

$$a(\nabla^b v, \nabla^b v) = \lambda|v|^2 - \Re(g\bar{v}) + \Re\partial_j\{\bar{v}a_{jk}\partial_k^b v\}.$$

Integrating on Ω, the boundary term vanishes (see Remark 2.5), and we get the second identity (2.18), after replacing $g = f + c(x)v$.

We can now write

$$-\int_\Omega c|v|^2 \le \int_\Omega c_{\mathrm{I},-}|v|^2 + \int_\Omega c_{\mathrm{II},-}|v|^2 \le \int_\Omega c_{\mathrm{I},-}|v|^2 + \||x|^2 c_{\mathrm{II},-}\|_{L^\infty}\int_\Omega \frac{|v|^2}{|x|^2}$$

and by the magnetic Hardy inequality (2.6)

$$\||x|^2 c_{\mathrm{II},-}\|_{L^\infty}\int_\Omega \frac{|v|^2}{|x|^2} \le \frac{2\||x|^2 c_{\mathrm{II},-}\|_{L^\infty}}{(n-2)}\int_\Omega |\nabla^b v|^2 \le \frac{1}{2}\int_\Omega a(\nabla^b v, \nabla^b v)$$

provided $\|a - I\|_{L^\infty} \le 1/2$ and $\||x|^2 c_{\mathrm{II},-}\|_{L^\infty} \le \frac{n-2}{8}$. Absorbing the last term at the left hand side of (2.18) we have proved

$$\int_\Omega a(\nabla^b v, \nabla^b v) \le 2\lambda\int_\Omega |v|^2 - 2\Re\int_\Omega f\bar{v} + 2\int_\Omega c_{\mathrm{I},-}|v|^2. \tag{2.22}$$

Next, by Cauchy–Schwartz and $a \le NI$ we have

$$\begin{aligned}|I_\epsilon| &\le |v||\epsilon|a(\nabla\psi, \nabla\psi)^{1/2}a(\nabla^b v, \nabla^b v)^{1/2}\\ &\le N^{1/2}\|\nabla\psi\|_{L^\infty}|\epsilon||v|a(\nabla^b v, \nabla^b v)^{1/2}\end{aligned}$$

and using (2.18), (2.22), with $C = 2N^{1/2}\|\nabla\psi\|_{L^\infty}$,

$$\int_\Omega |I_\epsilon| \le C\left[(\operatorname{sgn}\epsilon)\Im\int_\Omega f\bar{v}\right]^{1/2}\left[|\epsilon|\lambda\int_\Omega |v|^2 - |\epsilon|\Re\int_\Omega f\bar{v} + |\epsilon|\int_\Omega c_{\mathrm{I},-}|v|^2\right]^{1/2}$$

(note that both quantities inside brackets are positive). Using again (2.21) we get

$$\int_\Omega |I_\epsilon| \leq C \left| \int_\Omega f\bar{v} \right|^{1/2} \left[(|\lambda| + |\epsilon|) \left| \int_\Omega f\bar{v} \right| + |\epsilon| \int_\Omega c_{\mathrm{I},-}|v|^2 \right]^{1/2}$$

which implies

$$\int_\Omega |I_\epsilon| \leq C(|\lambda| + |\epsilon|)^{1/2} \|f\bar{v}\|_{L^1} + C|\epsilon|^{1/2} \|f\bar{v}\|_{L^1}^{1/2} \|c_{\mathrm{I},-}^{1/2} v\|_{L^2}.$$

Using (2.21) we have

$$|\epsilon|^{1/2} \|c_{\mathrm{I},-}^{1/2} v\|_{L^2} \leq |\epsilon|^{1/2} \|c_{\mathrm{I},-}\|_{L^\infty}^{1/2} \|v\|_{L^2} \leq \|c_{\mathrm{I},-}\|_{L^\infty}^{1/2} \| \|f\bar{v}\|_{L^1}^{1/2};$$

plugging it into the previous inequality we get

$$\int_\Omega |I_\epsilon| \leq C(|\lambda| + |\epsilon| + \|c_{\mathrm{I},-}\|_{L^\infty})^{1/2} \|f\bar{v}\|_{L^1}$$

and using Cauchy–Schwartz we obtain (2.19). □

Lemma 2.5 (auxiliary estimate I). *We have*

$$|\epsilon|^{1/2} \|v\|_{\dot{Y}} \leq C\|a\|_{L^\infty} (\|\nabla^b v\|_{\dot{Y}} + \|v\|_{\dot{X}} + \|f\|_{\dot{Y}^*}) \tag{2.23}$$

for some universal constant C.

Proof. Take the imaginary part of (2.14) and choose ϕ as follows:

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq R, \\ 2 - \dfrac{|x|}{R} & \text{if } R \leq |x| \leq 2R, \\ 0 & \text{if } |x| \geq 2R. \end{cases} \tag{2.24}$$

Integrating on Ω the boundary term vanishes and we get

$$\begin{aligned} |\epsilon| \int_{\Omega_{\leq R}} |v|^2 \leq & \int_{\Omega_{\leq 2R}} |f\bar{v}| + \frac{N}{R} \int_{\Omega_{R\leq|x|\leq 2R}} |v||\nabla^b v| \\ \leq & 2R\|f\|_{\dot{Y}^*} \|v\|_{\dot{X}} + 3NR\|v\|_{\dot{X}} \|\nabla^b v\|_{\dot{Y}}. \end{aligned} \tag{2.25}$$

Dividing by R and taking the sup for $R > 0$ we obtain (2.23). □

Lemma 2.6 (auxiliary estimate II). *Assume*

$$\lambda = -\lambda_- \le 0 \quad and \|a - I\|_{L^\infty} + \||x|a'\|_{L^\infty} \le 1/8.$$

Then in dimension $n \ge 4$ we have

$$\lambda_- \|v\|_{\dot Y}^2 \le C\|c_-|x|^2\|_{L^\infty}\||x|^{-3/2}v\|_{L^2}^2 + \delta\|v\|_{\dot X}^2 + C\delta^{-1}\|f\|_{\dot Y^*} \tag{2.26}$$

and in all dimensions $n \ge 3$ we have

$$\lambda_- \||x|^{-1/2}v\|_{L^2}^2 + \||x|^{-1/2}\nabla^b v\|_{L^2}^2 \le (C\||x|^2 c_-\|_{\ell^1 L^\infty} + \delta)\|v\|_{\dot X}^2 + C\delta^{-1}\|f\|_{\dot Y^*}^2 \tag{2.27}$$

for some universal constant C and all $\delta \in (0, 1)$. Note also that

$$\|v\|_{\dot Y} \le \||x|^{-1/2}v\|_{L^2}.$$

Proof. Since $\lambda = -\lambda_- \le 0$, we can rewrite (the real part of) (2.14) in the form

$$(c_+ + \lambda_-)|v|^2\phi + a(\nabla^b v, \nabla^b v)\phi = \partial_j \Re P_j + c_-|v|^2\phi - \Re(f\bar v)\phi + \frac{1}{2}A\phi|v|^2. \tag{2.28}$$

We choose the radial weight

$$\phi = \frac{1}{|x| \vee R} \implies \phi' = -\frac{1}{|x|^2}\mathbf{1}_{|x|>R},\ \phi'' = -\frac{1}{R^2}\delta_{|x|=R} + \frac{2}{|x|^3}\mathbf{1}_{|x|>R}.$$

By the formula $A\phi = \hat a\phi'' + \frac{\bar a - \hat a}{|x|}\phi' + \tilde a\phi'$, writing $\hat a = 1 + (a - I)\hat x \cdot \hat x$ and $\bar a = n + \operatorname{tr}(a - I)$ and dropping a negative term, we get

$$\begin{aligned} A\phi &= -\frac{\hat a}{R^2}\delta_{|x|=R} + \frac{3\hat a - \bar a + |x|\tilde a}{|x|^3}\mathbf{1}_{|x|>R} \\ &\le \frac{3 - n + (n+3)(|a - I| + |x||a'|)}{|x|^3}\mathbf{1}_{|x|>R}. \end{aligned}$$

In dimension $n \ge 4$, if $\|a - I\|_{L^\infty} + \||x|a'\|_{L^\infty} \le 1/6$, we get $A\phi \le 0$; hence integrating (2.28) on Ω and estimating $a(\nabla^b v, \nabla^b v) \ge \nu|\nabla^b v|^2$, we get

$$\int_\Omega \frac{(c_+ + \lambda_-)|v|^2 + \nu|\nabla^b v|^2}{|x| \vee R} \le \int_\Omega \frac{c_-|v|^2 + |f\bar v|}{|x| \vee R}.$$

Taking the sup over $R > 0$ we conclude

$$\begin{aligned} &\|c_+^{1/2}|x|^{-1/2}v\|_{L^2}^2 + \lambda_-\||x|^{-1/2}v\|_{L^2}^2 + \nu\||x|^{-1/2}\nabla^b v\|_{L^2}^2 \\ &\le \|c_-^{1/2}|x|^{-1/2}v\|_{L^2}^2 + \int_\Omega \frac{|f\bar v|}{|x|}. \end{aligned}$$

Since $\|v\|_{\dot Y} \lesssim \||x|^{-1/2}v\|_{L^2}$, we have in particular

$$\lambda_- \|v\|_{\dot Y}^2 \le C\|c_-|x|^2\|_{L^\infty}\||x|^{-3/2}v\|_{L^2}^2 + \||x|^{-1}v\|_{\dot Y}\|f\|_{\dot Y^*}$$

and using the inequality $\||x|^{-1}v\|_{\dot Y} \lesssim \|v\|_{\dot X}$ we obtain (2.26).

If the dimension is $n \ge 3$ we choose a different weight, for $\sigma > 0$ arbitrary:

$$\phi = \frac{1}{\sigma + |x|} \implies \frac{1}{2}A\phi = \frac{\hat a}{(\sigma+|x|)^3} - \frac{\bar a - \hat a + |x|\tilde a}{(\sigma+|x|)^2|c|}.$$

By the estimates $\hat a \le 1 + C_a'$, $|x||\tilde a| \le C_a'$, $\bar a \ge n(1-C_a')$ with $C_a' = |a-I| + |x||a'|$, we have

$$\frac{1}{2}A\phi \le -\frac{|x|(n-2-(n+1)C_a') - \epsilon(n-1-nC_a')}{|x|(\sigma+|x|)^3} \le -\frac{1}{2|x|(\sigma+|x|)^2}$$

provided we choose e.g. $C_a' \le 1/8$. Hence integrating (2.28) on Ω and using again that $a(\nabla^b v, \nabla^b v) \ge \nu|\nabla^b v|^2$, $\nu \ge \frac{1}{2}$, we get for some universal constant C

$$\begin{aligned}\int_\Omega \frac{\lambda_-|v|^2 + \nu|\nabla^b v|^2}{\sigma+|x|} + \int_\Omega \frac{|v|^2}{|x|(\sigma+|x|)^2}\\ \le C\||x|^{-1/2}c_-^{1/2}v\|_{L^2}^2 + C\||x|^{-1}f\bar v\|_{L^1}\\ \le C\||x|^2c_-\|_{\ell^1L^\infty}\|v\|_{\dot X}^2 + C\|f\|_{\dot Y^*}\|v\|_{\dot X}.\end{aligned}$$

Letting $\sigma \to 0$ we obtain (2.27). □

Lemma 2.7 (auxiliary estimate III). *Let $n \ge 4$. Assume $\|C_a\|_{L^\infty} + \||x|^2c_-\|_{L^\infty} \le 1/16$. Then*

$$\||x|^{-1/2}\nabla^b v\|_{L^2(|x|\le 1)}^2 \le \lambda_+\||x|^{-1/2}v\|_{L^2(|x|\le 2)}^2 + c(n)\|v\|_{\dot X}^2 + c(n)\|f\|_{\dot Y^*}^2. \tag{2.29}$$

Note also that $\lambda_+\||x|^{-1/2}v\|_{L^2(|x|\le 2)} \le 2\lambda_+\||x|^{-3/2}v\|_{L^2(|x|\le 2)}$.

Proof. Choose a smooth nonnegative weight of the form

$$\phi = |x|^{-1} \text{ for } |x| \le 1, \quad 0 \le \phi \le |x|^{-1} \text{ for } 1 \le |x| \le 2, \quad \phi = 0 \text{ for } |x| \ge 2$$

in (2.14), take the real part and integrate on Ω. We get

$$\begin{aligned}\int_{\Omega_{|x|\le 1}} \frac{a(\nabla^b v, \nabla^b v)}{|x|} \le \int_{\Omega_{|x|\le 2}} \Big((\lambda_+ + c_-)\frac{|v|^2}{|x|} - \frac{\bar a - 3\hat a + |x|\tilde a}{|x|^3}|v|^2 + \frac{1}{|x|}|f\bar v|\Big)\\ + C\|v\|_{L^2(1\le|x|\le 2)}^2\end{aligned}$$

for some $C = C(n, \|C_a\|_{L^\infty})$. Since

$$\bar{a} - 3\hat{a} + |x|\tilde{a} \geq n - 3 - (n+4)\|C_a\|_{L^\infty} \geq \frac{1}{2} \tag{2.30}$$

if e.g. $\|C_a\|_{L^\infty} \leq 1/16$, and moreover

$$\int\limits_{\Omega_{|x|\leq 2}} \frac{|f||v|}{|x|} \leq \delta \||x|^{-3/2}v\|^2_{L^2(|x|\leq 2)} + \delta^{-1}\|f\|^2_{\dot{Y}^*}, \quad \|v\|_{L^2(1\leq|x|\leq 2)} \leq 2\|v\|_{\dot{X}},$$

we have

$$\begin{aligned}\int\limits_{\Omega_{|x|\leq 1}} \frac{a(\nabla^b v, \nabla^b v)}{|x|} \leq \lambda_+ &\left\| \frac{v}{|x|^{1/2}} \right\|^2_{L^2(|x|\leq 2)} \\ &+ \left(\delta + \||x|^2 c_-\|_{L^\infty} - \frac{1}{4}\right) \left\| \frac{v}{|x|^{3/2}} \right\|^2_{L^2(|x|\leq 2)} \\ &+ C(n, \|C_a\|_{L^\infty})(\|v\|^2_{\dot{X}} + \delta^{-1}\|f\|^2_{\dot{Y}^*})\end{aligned}$$

by taking δ sufficiently small we get the claim. □

We recall the notations

$$(a\nabla^b v)_R = (\hat{x} \cdot a\nabla^b v)\hat{x}, \quad (a\nabla^b v)_T = a\nabla^b v - (a\nabla^b v)_R$$

for the radial and the tangential part of $a\nabla^b v$. Note that in case the weight $\psi = \psi(|x|)$ is a radial function, the term I_b takes the form

$$I_b = 2\Im[(a\nabla^b v) \cdot (db\ a\hat{x})\ \bar{v}]\psi' = 2\Im[(a\nabla^b v) \cdot \widehat{db}\ \bar{v}]\psi'$$

where $\widehat{db} := db\ a\hat{x}$ is the tangential part of the magnetic field.

Lemma 2.8 (I_b). *Assume ψ is a radial function, $b = b_{\mathrm{I}} + b_{\mathrm{II}} + b_{\mathrm{III}}$. Then,*

$$\begin{aligned}\int\limits_{\Omega} |I_b| \leq\ & C\beta_1 \|\nabla^b v\|_{\dot{Y}} \|v\|_{\dot{X}} \\ &+ C\beta_2 \Bigg(\int\limits_{\Omega} \frac{|v|^2}{|x|^3} \Bigg)^{1/2} \Bigg(\int\limits_{\Omega} \frac{|(a\nabla^b v)_T|^2}{|x|} \Bigg)^{1/2} + C\beta_3 \|\nabla^b v\|_{\dot{Y}} \|v\|_{\dot{Y}},\end{aligned}$$

where $C = 2\|a\|_{L^\infty}\|\nabla\psi\|_{L^\infty}$ and

$$\beta_1 = \||x|^{3/2}\widehat{db}_{\mathrm{I}}\|_{\ell^1 L^2 L^\infty} + \||x|^{3/2}|a - I|\widehat{db}_{\mathrm{II}}\|_{\ell^1 L^2 L^\infty}, \tag{2.31}$$

$$\beta_2 = \||x|^2\widehat{db}_{\mathrm{II}}\|_{L^\infty}, \quad \beta_3 = \||x|\widehat{db}_{\mathrm{III}}\|_{\ell^1 L^\infty}. \tag{2.32}$$

Proof. We split $I_b = I_{b_{\mathrm{I}}} + I_{b_{\mathrm{II}}} + I_{b_{\mathrm{III}}}$ with $I_{b_{\mathrm{I}}} = 2\Im[(a\nabla^b v) \cdot \widehat{db}_{\mathrm{I}}\bar{v}]\psi'$ and so on. Then

$$\int_\Omega |I_{b_{\mathrm{I}}}| \le C\|\nabla^b v\|_{\dot{Y}} \||x|^{1/2}\widehat{db}_{\mathrm{I}} v\|_{\ell^1 L^2 L^2} \\ \le C\|\nabla^b v\|_{\dot{Y}}\|v\|_{\dot{X}}\||x|^{3/2}\widehat{db}_{\mathrm{I}}\|_{\ell^1 L^2 L^\infty}$$

where $C = 2N\|\nabla\psi\|_{L^\infty}$, and similarly

$$\int_\Omega |I_{b_{\mathrm{III}}}| \le C\|\nabla^b v\|_{\dot{Y}} \||x|^{1/2}\widehat{db}_{\mathrm{III}} v\|_{\ell^1 L^2 L^2} \\ \le C\|\nabla^b v\|_{\dot{Y}}\|v\|_{\dot{Y}}\||x|\widehat{db}_{\mathrm{III}}\|_{\ell^1 L^\infty L^\infty}.$$

Next we note that db_{II} is antisymmetric, hence $(a\hat{x}) \cdot \widehat{db}_{\mathrm{II}} = (a\hat{x}) \cdot (db_{\mathrm{II}}\, a\hat{x}) = 0$, and for any $\gamma \in \mathbb{C}$ we can rewrite $I_{b_{\mathrm{II}}}$ as

$$I_{b_{\mathrm{II}}} = 2\Im[(a\nabla^b v - \gamma\hat{x} + \gamma\hat{x} - \gamma a\hat{x}) \cdot \widehat{db}_{\mathrm{II}}\bar{v}]\psi'.$$

If we choose $\gamma = \hat{x} \cdot a\nabla^b v$ we obtain

$$I_{b_{\mathrm{II}}} = 2\Im[(a\nabla^b v)_T \cdot \widehat{db}_{\mathrm{II}}\bar{v}]\psi' + 2\Im[(\hat{x} \cdot a\nabla^b v)((I-a)\hat{x}) \cdot \widehat{db}_{\mathrm{II}}\bar{v}]\psi' =: I'_{b_{\mathrm{II}}} + I''_{b_{\mathrm{II}}}.$$

We estimate $I''_{b_{\mathrm{II}}}$ like $I_{b_{\mathrm{I}}}$:

$$\int_\Omega |I''_{b_{\mathrm{II}}}| \le C\|\nabla^b v\|_{\dot{Y}}\|v\|_{\dot{X}}\||x|^{3/2}|a-I|\widehat{db}_{\mathrm{II}}\|_{\ell^1 L^2 L^\infty}.$$

Finally we have

$$\int_\Omega |I'_{b_{\mathrm{II}}}| \le C\||x|^{-1/2}(a\nabla^b v)_T\|_{L^2}\||x|^{-3/2}v\|_{L^2}\||x|^2\widehat{db}_{\mathrm{II}}\|_{L^\infty}. \qquad \square$$

2.4. Choice of the weights and main terms. We choose, for arbitrary $R > 0$,

$$\psi = \frac{1}{2R}|x|^2\mathbf{1}_{|x|\le R} + |x|\mathbf{1}_{|x|>R}, \quad \phi = -\frac{\hat{a}}{R}\mathbf{1}_{|x|\le R}. \tag{2.33}$$

Note that ϕ is not radial. We have then

$$\psi' = \frac{|x|}{|x| \vee R}, \quad \psi'' = \frac{1}{R}\mathbf{1}_{|x|\le R}, \quad A\psi + \phi = \frac{\bar{a} - \hat{a} + |x|\tilde{a}}{|x| \vee R}. \tag{2.34}$$

since $A\psi = \hat{a}\psi'' + \frac{\bar{a}-\hat{a}}{|x|}\psi' + \tilde{a}\psi'$. Recalling the notation

$$C_a(x) = |a(x) - I| + |x||a'(x)| + |x|^2|a''(x)| + |x|^3|a'''(x)|$$

we have, after a long but easy computation,

$$|x|^2(|A\bar{a}|+|A\hat{a}|+|x||A\tilde{a}|+|\nabla\tilde{a}|)+|x|(|\nabla\bar{a}|+|\nabla\hat{a}|+|\tilde{a}|) \leq C(n, \|C_a\|_{L^\infty})\cdot C_a(x). \tag{2.35}$$

Then for $|x| > R$ we find that

$$A(A\psi + \phi) = -\frac{(\bar{a}-\hat{a})(\bar{a}-3\hat{a})}{|x|^3} + R(x)$$

where

$$R(x) = -\frac{2a(\nabla\bar{a} - \nabla\hat{a}, \hat{x}) + \tilde{a}(\bar{a}-\hat{a})}{|x|^2} + \frac{A(\bar{a}-\hat{a})}{|x|} + A\tilde{a}$$

and by (2.35)

$$|R(x)| \leq C(n, \|C_a\|_{L^\infty}) \cdot \frac{C_a(x)}{|x|^3}, \quad |x| > R. \tag{2.36}$$

In the region $|x| < R$ we have instead

$$A(A\psi + \phi) = R(x) = \frac{A(\bar{a}-\hat{a}) + \tilde{a}^2 + |x|A\tilde{a} + 2a(\nabla\tilde{a}, \hat{x})}{R} + \frac{\tilde{a}(\bar{a}-\hat{a})}{R|x|}$$

and again by (2.35)

$$|R(x)| \leq C(n, \|C_a\|_{L^\infty}) \cdot \frac{C_a(x)}{R|x|^2}, \quad |x| \leq R. \tag{2.37}$$

Finally, along the sphere $|x| = R$ there is a singularity of delta type, originated by the term

$$\hat{a}\left(\frac{\bar{a}-\hat{a}+|x|\tilde{a}}{|x| \vee R}\right)''$$

and therefore the singular term has the form

$$-\frac{\hat{a}(\bar{a}-\hat{a}+R\tilde{a})}{R^2}\delta_{|x|=R}.$$

Summing up we have

$$A(A\psi + \phi) = -\frac{(\bar{a}-\hat{a})(\bar{a}-3\hat{a})}{|x|^3}\mathbf{1}_{|x|>R} - \frac{\hat{a}(\bar{a}-\hat{a}+R\tilde{a})}{R^2}\delta_{|x|=R} + R(x) \tag{2.38}$$

where $R(x)$ satisfies (2.36), (2.37). Further, we note that

$$|\hat{a}-1| + |x||\tilde{a}| \leq C_a(x), \quad |\bar{a}-n| \leq nC_a(x) \tag{2.39}$$

so that

$$\hat{a}(\bar{a}-\hat{a}+R\tilde{a}) \geq 1$$

provided e.g $\|C_a\|_{L^\infty} \leq 1/6$. Moreover we have

$$(\bar{a} - \hat{a})(\bar{a} - 3\hat{a}) \geq (n-1)(n-3) - 2(n+2)C_a$$

and in conclusion we have proved the inequality

$$-A(A\psi + \phi) \geq \frac{(n-1)(n-3)}{|x|^3}\mathbf{1}_{|x|>R} + \frac{1}{R^2}\delta_{|x|=R} + R_1(x) \tag{2.40}$$

where $R_1(x)$ satisfies for all x

$$|R_1(x)| \leq C(n) \cdot \frac{C_a(x)}{|x|^2(|x| \vee R)}$$

with a constant $C(n)$ depending only on n (polynomially).

Lemma 2.9 (I_v). *Let $c = c_1 + c_2 + c_3 + c_4 + c_5$, with c_5 supported in $|x| \leq 1$, and ϕ, ψ as in (2.24). If $n \geq 4$ we have, for all $\delta \in (0,1)$,*

$$\begin{aligned}\sup_{R>0} \int_\Omega I_v \geq{}& (\mu_n - \gamma_1 - c(n)(\gamma_2 + \gamma_5 + \|C_a\|_{L^\infty}))\||x|^{-3/2}v\|_{L^2}^2 + \|v\|_{\dot{X}}^2 \\ &+ (\lambda - \Gamma_3 - c(n)\delta^{-1}\Gamma_4)\|v\|_{\dot{Y}}^2 \\ &- (\gamma_2 + \delta)\|\nabla^b v\|_{\dot{Y}}^2 - \gamma_5\||x|^{-1/2}\nabla^b v\|_{L^2(|x|\leq 1)}\end{aligned} \tag{2.41}$$

where $\mu_n = (n-1)(n-3)/2$ and ($\partial_r := \hat{x} \cdot \nabla$)

$$\gamma_1 = \||x|^2([\partial_r(|x|c_1)]_+ + c_{1,-} + |a - I|(|x||\nabla c_1| + |c_1|)\|_{L^\infty},$$

$$\gamma_2 = \||x|^2 c_2\|_{\ell^2 L^\infty}, \quad \gamma_5 = \||x|^2 c_5\|_{L^\infty},$$

$$\Gamma_3 = \|[\partial_r(|x|c_3)]_+ + c_{3,-} + |a - I|(|x||\nabla c_3| + |c_3|\|_{\ell^2 L^\infty},$$

$$\Gamma_4 = \|c_{4,-}\|_{\ell^1 L^\infty} + \|c_4\|_{\ell^1 L^\infty}^2.$$

In dimension $n = 3$, provided $c_5 = 0$, we have instead

$$\begin{aligned}\sup_{R>0} \int_\Omega I_v \geq{}& (1 - \gamma_1 - c(n)\gamma_2 - c(n)\|C_a\|_{L^\infty})\|v\|_{\dot{X}}^2 \\ &+ (\lambda - \Gamma_3 - c(n)\Gamma_4)\|v\|_{\dot{Y}}^2 - (\gamma_2 + \delta)\|\nabla^b v\|_{\dot{Y}}^2\end{aligned} \tag{2.42}$$

where the definition of Γ_3, Γ_4 is the same, while

$$\gamma_1 = \||x|([\partial_r(|x|c_1)]_+ + c_{1,-} + |a - I|(|x||\nabla c_1| + |c_1|)\|_{\ell^1 L^1 L^\infty},$$

$$\gamma_2 = \||x|^{3/2} c_2\|_{\ell^1 L^2 L^\infty}.$$

Proof. Integrating I_v on Ω and using (2.40) we obtain

$$\begin{aligned}\int_\Omega I_v \geq \int_{\Omega_{\geq R}} \frac{\mu_n|v|^2}{|x|^3} + \int_{\Omega_{=R}} \frac{|v|^2}{R^2}dS - \int_\Omega \frac{c(n)C_a(x)|v|^2}{|x|^2(|x|\vee R)} \\ + \int_{\Omega_{\leq R}} \frac{\lambda|v|^2}{R} - \int_\Omega \Big(\frac{|x|(a\hat{x})\cdot\nabla c}{|x|\vee R} + \frac{\hat{a}c}{R}\mathbf{1}_{|x|\leq R}\Big)|v|^2.\end{aligned} \tag{2.43}$$

Consider first the case $n \geq 4$. We estimate the term

$$I_c = \Big(\frac{|x|(a\hat{x})\cdot\nabla c}{|x|\vee R} + \frac{\hat{a}c}{R}\mathbf{1}_{|x|\leq R}\Big)|v|^2$$

in two different ways for c_1, c_3 and for c_2, c_4. For c_1, writing $r = |x|$ and $\partial_r = \hat{x}\cdot\nabla$, we have

$$\begin{aligned}\frac{|x|(a\hat{x})\cdot\nabla c_1}{|x|\vee R} + \frac{\hat{a}c_1}{R}\mathbf{1}_{|x|\leq R} &= \partial_r\Big(\frac{|x|c_1}{|x|\vee R}\Big) + (a-I)\hat{x}\cdot\nabla\Big(\frac{|x|c_1}{|x|\vee R}\Big) \\ &\leq \frac{1}{|x|}([\partial_r(rc_1)]_+ + c_{1,-} + |a-I|(|x||\nabla c_1| + |c_1|))\end{aligned}$$

so that

$$\begin{aligned}\sup_{R>0}\int_\Omega I_{c_1} &\leq \||x|^{-1}\big([\partial_r(rc_1)]_+ + c_{1,-} + |a-I|(|x||\nabla c_1| + |c_1|)\big)|v|^2\|_{L^1} \\ &\leq \gamma_1\||x|^{-3/2}v\|_{L^2}^2.\end{aligned}$$

A similar computation for I_{c_3} gives (also in the case $n = 3$)

$$\sup_{R>0}\int_\Omega I_{c_3} \leq \Gamma_3\|v\|_{\dot{Y}}^2.$$

On the other hand for c_2 we write

$$I_{c_2} = \nabla\cdot\Big\{\frac{a\hat{x}|x|c_2|v|^2}{|x|\vee R}\Big\} - \frac{\bar{a}-\hat{a}+|x|\tilde{a}}{|x|\vee R}c_2|v|^2 - 2\frac{|x|c_2}{|x|\vee R}\Re a(\nabla^b v, \hat{x}v)$$

and if e.g. $\|C_a\|_{L^\infty} \leq 1/4$, recalling also (2.30), we get

$$I_{c_2} \leq \nabla\cdot\Big\{\frac{a\hat{x}|x|c_2|v|^2}{|x|\vee R}\Big\} + c(n)\frac{c_{2,-}}{|x|}|v|^2 + 4|c_2||v||\nabla^b v|$$

so that

$$\begin{aligned}\sup_{R>0}\int_\Omega I_{c_2} &\leq c(n)\||x|^{-1/2}(c_{2,-})^{1/2}v\|_{L^2}^2 + 4\|c_2 v|\nabla^b v|\|_{L^1} \\ &\leq c(n)\||x|^2 c_{2,-}\|_{L^\infty}\||x|^{-3/2}v\|_{L^2}^2 \\ &\quad + 4\||x|^2 c_2\|_{\ell^2 L^\infty}\||x|^{-3/2}v\|_{L^2}\|\nabla^b v\|_{\dot{Y}} \\ &\leq c(n)\gamma_2\||x|^{-3/2}v\|_{L^2}^2 + \gamma_2\|\nabla^b v\|_{\dot{Y}}^2.\end{aligned}$$

Using the same identity for c_4 we can estimate

$$\sup_{R>0}\int_\Omega I_{c_4} \le c(n)\|c_{4,-}\|_{\ell^1 L^\infty}\|v\|_{\dot Y}^2 + 4\|c_4\|_{\ell^1 L^\infty}\|v\|_{\dot Y}\|\nabla^b v\|_{\dot Y}$$

and this implies

$$\sup_{R>0}\int_\Omega I_{c_4} \le \delta\|\nabla^b v\|_{\dot Y}^2 + c(n)\delta^{-1}\Gamma_4\|v\|_{\dot Y}^2.$$

The same identity for c_5 can be estimated as follows, with $C = c(n)$:

$$\sup_{R>0}\int_\Omega I_{c_5} \le C\gamma_5\|\frac{v}{|x|^{3/2}}\|_{L^2}^2 + \gamma_5\|\frac{\nabla^b v}{|x|^{1/2}}\|_{L^2(|x|\le 1)}^2.$$

Hence taking the sup in $R > 0$ of (2.43) and using the previous estimates we get (2.41).

In the case $n = 3$ we have $\mu_3 = 0$ and the weighted L^2 norm of v is unavailable. We use the $\dot X$ norm instead and we obtain

$$\sup_{R>0}\int_\Omega I_{c_1} \le \gamma_1\|v\|_{\dot X}^2,$$

$$\sup_{R>0}\int_\Omega I_{c_2} \le c(n)\|c_{2,-}|x|\|_{\ell^1 L^1 L^\infty}\|v\|_{\dot X}^2 + 4\||x|^{3/2}c_2\|_{\ell^1 L^2 L^\infty}\|v\|_{\dot X}\|\nabla^b v\|_{\dot Y}$$

with the new definition of γ_1, γ_2, and this gives (2.42). □

Lemma 2.10 ($I_{\nabla v}$). *With ψ as in (2.34), we have*

$$\sup_{R>0}\int_\Omega I_{\nabla v} \ge (1 - 6\|a - I\|_{L^\infty} - c(n)\||x|a'\|_{\ell^1 L^\infty})\|\nabla^b v\|_{\dot Y}^2 + \int_\Omega \frac{|(a\nabla^b v)_T|^2}{|x|}. \tag{2.44}$$

Proof. By separating the terms in $\alpha_{\ell m}$ which contain derivatives of a_{jk} we have

$$I_{\nabla v} = s_{\ell m}\cdot \Re(\partial_\ell^b v\overline{\partial_m^b v}) + r_{\ell m}\cdot \Re(\partial_\ell^b v\overline{\partial_m^b v}) + a(\nabla^b v, \nabla^b v)\phi$$

where

$$s_{\ell m}(x) = 2a_{jm}a_{\ell k}\hat x_j\hat x_k\psi'' + 2[a_{jm}a_{j\ell} - a_{jm}a_{\ell k}\hat x_j\hat x_k]\frac{\psi'}{|x|},$$

$$r_{\ell m}(x) = [2a_{jm}a_{\ell k;j} - a_{jk}a_{\ell m;j}]\hat x_k\psi'.$$

With our choice of ψ we get

$$|r_{\ell m}(x)\Re(\partial_\ell^b v\overline{\partial_m^b v})| \le c(n)\frac{|x||a'|}{|x|\vee R}|\nabla^b v|^2 \le c(n)|a'||\nabla^b v|^2,$$

$$a(\nabla^b v,\nabla^b v)\phi = -\frac{\hat{a}}{R}\mathbf{1}_{|x|\le R}a(\nabla^b v,\nabla^b v) \ge -\frac{N^2}{R}\mathbf{1}_{|x|\le R}|\nabla^b v|^2.$$

Moreover

$$s_{\ell m}\Re(\partial_\ell^b v\overline{\partial_m^b v}) = 2|(a\nabla^b v)_R|^2\psi'' + 2|(a\nabla^b v)_T|^2\frac{\psi'}{|x|}$$

which gives, using $|w_R|^2+|w_T|^2=|w|^2$ (we recall notation (1.7))

$$s_{\ell m}\Re(\partial_\ell^b v\overline{\partial_m^b v}) = \frac{2}{R}|a\nabla^b v|^2\mathbf{1}_{|x|\le R} + \frac{2}{|x|}|(a\nabla^b v)_T|^2\mathbf{1}_{|x|>R}.$$

Summing up we obtain

$$I_{\nabla v} \ge \frac{(2\nu^2-N^2)}{R}\cdot|\nabla^b v|^2\mathbf{1}_{|x|\le R} + \frac{2}{|x|}|(a\nabla^b v)_T|^2\mathbf{1}_{|x|>R} - c(n)|a'||\nabla^b v|^2.$$

Note that we can assume $\nu \ge 1-\|a-I\|_{L^\infty}$ and $N \le 1+\|a-I\|_{L^\infty}$ so that

$$2\nu^2 - N^2 \ge 1 - 6\|a-I\|_{L^\infty}.$$

Integrating on Ω and taking the sup over $R>0$ we obtain

$$\sup_{R>0}\int_\Omega I_{\nabla v} \ge (1-6\|a-I\|_{L^\infty})\|\nabla^b v\|_{\dot{Y}}^2 + \int_\Omega \frac{|(a\nabla^b v)_T|^2}{|x|} - c(n)\||a'||\nabla^b v|^2\|_{L^1}$$

and this implies the claim. □

Lemma 2.11 (I_f). *With ϕ,ψ as in (2.33), we have for all $\delta\in(0,1)$*

$$\int_\Omega I_f \le \delta\|v\|_{\dot{X}}^2 + \delta\|\nabla^b v\|_{\dot{Y}}^2 + C(n,\|C_a\|_{L^\infty})\delta^{-1}\|f\|_{\dot{Y}^*}^2. \tag{2.45}$$

Proof. By (2.34)

$$I_f = \frac{\bar{a}-\hat{a}+|x|\tilde{a}}{|x|\vee R}\Re(\bar{v}f) + \frac{2|x|a\hat{x}}{|x|\vee R}\Re(\overline{\nabla^b v}f) \le C(n,\|C_a\|_{L^\infty})(\frac{|v|}{|x|}+|\nabla^b v|)|f|$$

and hence

$$\int_\Omega I_f \le C(n,\|C_a\|_{L^\infty})(\||x|^{-1}v\|_{\dot{Y}} + \|\nabla^b v\|_{\dot{Y}})\|f\|_{\dot{Y}^*}.$$

The claim follows recalling that $\||x|^{-1}v\|_{\dot{Y}} \le \|v\|_{\dot{X}}$. □

2.5. Conclusion of the proof. We are ready to complete the proof of Theorem 2.1. We integrate (2.8) on Ω with the choice of weights (2.33) and we take the supremum over $R > 0$. We then apply the previous Lemmas to estimate the individual terms.

We consider first the case (A_0). One checks easily that the assumptions on b, c imply the following: for $b = b_{\mathrm{I}} + b_{\mathrm{II}}$ we have

$$\||x|^{3/2}\widehat{db}_{\mathrm{I}}\|_{\ell^1 L^2 L^\infty} + \||x|^{3/2}|a - I|\widehat{db}_{\mathrm{II}}\|_{\ell^1 L^2 L^\infty} + \||x|^2\widehat{db}_{\mathrm{II}}\|_{L^\infty} < \mu,$$

with $b_{\mathrm{II}} = 0$ in $n = 3$, while the electric potential can be written $c = c_1 + c_2 + c_f$ with

$$\||x|^{3/2}c_f\|_{\ell^1 L^2 L^\infty} < \mu, \quad c_{1,-} \in L^\infty$$

and in dimension $n \geq 4$

$$|a-I|\cdot(|x|^2|c_1|+|x|^3|\nabla c_1|)+|x|^2\cdot(c_{1,-}+[\partial_r(rc_1)]_+ + c_{2,-}) + \||x|c_2\|_{\ell^1 L^\infty} < \mu$$

while in dimension $n = 3$

$$\begin{aligned}\||a - I| \cdot (|x|^2|c_1| + |x|^3|\nabla c_1|) + |x|^2 \cdot (c_{1,-} + [\partial_r(rc_1)]_+ + c_{2,-})\|_{\ell^1 L^\infty} \\ + \||x|c_2\|_{\ell^1 L^\infty} \\ < \mu.\end{aligned}$$

Indeed, it is sufficient to take $c_1 = c_{\mathrm{I}}$ and, for a fixed cutoff $0 \leq \chi(x) \leq 1$ supported near 0, $c_2 = (1-\chi)\cdot c_{\mathrm{II}}$ and $c_f = \chi \cdot c_{\mathrm{II}}$.

Consider the case $n \geq 4$. Write $\tilde{c} = c_1 + c_2$ and

$$\tilde{f} = (A^b - \tilde{c} + \lambda + i\epsilon)v, \quad \tilde{f} = f + c_f v.$$

Then all the assumptions of Lemmas 2.4, 2.5, 2.6, 2.8, 2.9, 2.10, and 2.11 are satisfied by a, b and $\tilde{c}$. As a consequence we have

$$\begin{aligned}\sup_{R>0} \int_\Omega |I_\epsilon| + |I_b| + |I_{\tilde{f}}| \\ \leq C\cdot(\delta + \mu)(\|v\|^2_{\dot{X}} + \|\nabla^b v\|^2_{\dot{Y}} + \||x|^{-3/2}v\|^2_{L^2}) \\ + [\mu + \delta(|\lambda| + |\epsilon| + \|c_{\mathrm{I},-}\|_{L^\infty})]\|v\|^2_{\dot{Y}} + C\delta^{-4}\|\tilde{f}\|^2_{\dot{Y}^*},\end{aligned}$$

$$\begin{aligned}\sup_{R>0} \int_\Omega (I_v + I_{\nabla v}) \\ \geq \Big(\frac{1}{2} - C\mu\Big)\||x|^{-3/2}v\|^2_{L^2} + (1 - C(\mu + \delta))\|\nabla^b v\|^2_{\dot{Y}} + \|v\|^2_{\dot{X}} + \lambda\|v\|^2_{\dot{Y}}, \\ \lambda_-\|v\|^2_{\dot{Y}} \leq C\mu\||x|^{-3/2}v\|^2_{L^2} + \delta\|v\|^2_{\dot{X}} + C\delta^{-1}\|\tilde{f}\|^2_{\dot{Y}^*}.\end{aligned}$$

Thus integrating (2.8) on Ω and dropping the boundary terms, which give a negative contribution as proved in Remark 2.5, from the previous inequalities taking δ and μ sufficiently small we obtain (2.2), with $\tilde{f} = f + c_f v$ in place of f. More precisely, we use Lemma 2.5 to get rid of the ϵ term at the right hand side, so that we obtain (2.2) with $\epsilon = 0$. To reinclude the ϵ term, we can use again (2.23) combined with the local smoothing just obtained, which gives

$$|\epsilon|^{1/2}\|v\|_{\dot{Y}} \leq C\|a\|_{L^\infty}(\|\nabla^b v\|_{\dot{Y}} + \|v\|_{\dot{X}} + \|f\|_{\dot{Y}^*}) \leq C(a)(1 + c(n))\|f\|_{\dot{Y}^*}.$$

Now it remains to estimate

$$\|f + c_f v\|_{\dot{Y}^*} \leq \|f\|_{\dot{Y}^*} + \||x|^{3/2}c_f\|_{\ell^1 L^2 L^\infty}\|v\|_{\dot{X}} < \|f\|_{\dot{Y}^*} + \mu\|v\|_{\dot{X}}$$

and absorb the last term at the left hand side, provided μ is small enough. The proof for $n = 3$ is completely analogous.

In the case of the weaker condition (A) the argument is almost the same. We split $c = c_1 + c_2 + c_3 + c_4 + c_f$ with $c_1 = c_{\mathrm{I}}$, $c_2 = (1 - \chi)c_{\mathrm{II}}$, $c_3 = c_{\mathrm{III}}$, $c_4 = (1 - \chi)c_{\mathrm{IV}}$ and $c_f = \chi \cdot (c_{\mathrm{II}} + c_{\mathrm{IV}})$, and we write $\tilde{c} = c_1 + c_2 + c_3 + c_4$ and

$$\tilde{f} = (A^b - \tilde{c} + \lambda + i\epsilon)v, \quad \tilde{f} = f + c_f v$$

as before. Note that in the estimate of I_ϵ we get an additional term $\|c_{3,-} + c_{4,-}\|_{L^\infty}\|v\|^2_{\dot{Y}}$, while in estimate (2.41) we must take $\frac{1}{2}\lambda > c(n)(Z + Z^2) \geq \Gamma_3 + c(n)\Gamma_4$ in order to obtain positive terms. Then we can apply the lemmas as above, and in the final step we estimate $\tilde{f}$ as follows:

$$\begin{aligned}\|\tilde{f}\|_{\dot{Y}^*} &\leq \|f\|_{\dot{Y}^*} + \||x|^{3/2}\chi c_{\mathrm{II}}\|_{\ell^1 L^2 L^\infty}\|v\|_{\dot{X}} + \||x|\chi c_{\mathrm{IV}}\|_{\ell^1 L^\infty}\|v\|_{\dot{Y}} \\ &\leq \|f\|_{\dot{Y}^*} + \mu\|v\|_{\dot{X}} + Z\|v\|_{\dot{Y}}.\end{aligned}$$

In conclusion, we arrive at an estimate of the form

$$\|v\|_{\dot{X}} + (\lambda + |\epsilon|)^{1/2}\|v\|_{\dot{Y}} + \|\nabla^b v\|_{\dot{Y}} + \|(a\nabla^b v)_T\|_{L^2} \leq c(n)\|f\|_{\dot{Y}^*} + c(n)Z\|v\|_{\dot{Y}}$$

and the additional term $\|v\|_{\dot{Y}}$ can be absorbed at the left hand side, provided λ is large enough. We omit the details.

Remark 2.6 (Inverse square potentials). Note that in dimension $n \geq 4$ and for $\lambda > 0$ we can add to the electric potential c a further term c_V satisfying

$$\gamma_5 := \||x|^2 c_V\|_{L^\infty} \ll 1 \quad c_V \text{ supported in } \{|x| \leq 1\}.$$

Indeed, taking $c_5 = c_V$ in Lemma 2.9, we obtain an additional term at the right hand side of the estimate:

$$\begin{aligned}&\|v\|_{\dot{X}} + \||x|^{-3/2}v\|_{L^2} + (\lambda_+ + |\epsilon|)^{1/2}\|v\|_{\dot{Y}} + \|\nabla^b v\|_{\dot{Y}} + \|(a\nabla^b v)_T\|_{L^2} \\ &\leq c(n)(\|f\|_{\dot{Y}^*} + \gamma_5^{1/2}\||x|^{-1/2}\nabla^b v\|_{L^2(|x|\leq 1)}).\end{aligned}$$

We can estimate the additional term using Lemma 2.7:

$$\||x|^{-1/2}\nabla^b v\|_{L^2(|x|\le 1)} \le 2\lambda_+ \||x|^{-3/2}v\|^2_{L^2(|x|\le 2)} + c(n)\|v\|^2_{\dot X} + c(n)\|f\|^2_{\dot Y^*}$$

and if μ is small enough we can absorb the $\|v\|_{\dot X}$ term at the right hand side:

$$\begin{aligned}\|v\|_{\dot X} + \||x|^{-3/2}v\|_{L^2} &+ (|\lambda|+|\epsilon|)^{1/2}\|v\|_{\dot Y} + \|\nabla^b v\|_{\dot Y} + \|(a\nabla^b v)_T\|_{L^2} \\ &\le c(n)\|f\|_{\dot Y^*} + c(n)(\gamma_5\lambda_+)^{1/2}\||x|^{-3/2}v\|_{L^2}.\end{aligned}$$

In conclusion, if we assume

$$\||x|^2 c_V\|_{L^\infty}\cdot\lambda_+ < \epsilon(n) \tag{2.46}$$

for a suitable constant $\epsilon(n)$ depending only on n, we can absorb also the remaining term and we obtain that the estimate (2.2) continues to hold. However in this case the condition on c_V is not independent of λ and actually becomes worse as λ_+ grows.

3. The radiation estimate

The goal of this Section is to prove an estimate for the difference

$$\nabla^b_S v := \nabla^b v - i\hat{x}\sqrt{\lambda}v$$

(S stands for Sommerfeld) in a norm slightly stronger than $\|\cdot\|_{\dot Y}$; to this purpose we use the weighted L^2 norms, for some $\delta > 0$,

$$\int_\Omega |x|^{\delta-1}|\nabla^b_S v|^2 dx. \tag{3.1}$$

This is enough to rule out functions in the kernel of $L+\lambda$ and hence to get uniqueness for the Helmholtz equation. Indeed, if the previous norm is finite then condition (1.10) is satisfied. The value of δ is connected to the asymptotic behaviour of the metric $a(x)$ (see the statement of Theorem 3.2), a fact already observed in [23].

Note that we can only estimate (3.1) in terms of the $\dot Y$ norms of v and its derivative; in order to get an actual estimate, this result must be combined with the smoothing estimate of Section 2.

Since we are interested in the behaviour of solutions in the limit $\lambda + i\epsilon \to \lambda > 0$, it is actually sufficient to prove an estimate in the quarter plane $|\epsilon| < \lambda$. However, the estimate in the case $\lambda \le |\epsilon|$ is elementary (and actually stronger), and we give it here for the sake of completeness.

Proposition 3.1 (radiation estimate, case $\lambda \le |\epsilon|$). *Let $\epsilon \in \mathbb{R}$, $0 < \lambda \le |\epsilon|$. Assume $C_a < 1/2$ and $c = c_{\mathrm{I}} + c_{\mathrm{II}}$ with*

$$\||x|^2 c_{\mathrm{I},-}\|_{L^\infty} \le \kappa, \quad \|c_{\mathrm{II},-}\|_{L^\infty} \le K.$$

If κ is sufficiently small with respect to n, we have

$$\|\nabla^b v\|_{L^2}^2 + \lambda \|v\|_{L^2}^2 \lesssim (1 + K\lambda^{-1})[\|v\|_{\dot Y}^2 + \|f\|_{\dot Y^*}^2]. \tag{3.2}$$

Proof. We can assume $\epsilon > 0$. By $\lambda \le \epsilon$ and (2.18) we have

$$\lambda \int |v|^2 \le \epsilon \int |v|^2 \le \int |f\bar v| \le \|v\|_{\dot Y}^2 + \|f\|_{\dot Y^*}^2.$$

Also by (2.18), we can write for all $\delta > 0$

$$\int a(\nabla^b v, \nabla^b v) \le \lambda \int |v|^2 + \int c_- |v|^2 + \int |f\bar v|$$
$$\le (\lambda + K) \int |v|^2 + \kappa \int \frac{|v|^2}{|x|^2} + \int |f\bar v|.$$

By the magnetic Hardy inequality (2.6) and the previous inequality we have then

$$\le (2 + K\lambda^{-1}) \int |f\bar v| + \kappa c(n) \|\nabla^b v\|_{L^2}^2$$

and if κ is sufficiently small we deduce

$$\|\nabla^b v\|_{L^2}^2 \lesssim (1 + K\lambda^{-1}) \int |f\bar v|.$$

Appying the Cauchy–Schwartz inequality we obtain (3.2). □

Theorem 3.2 (radiation estimate, case $\lambda > |\epsilon|$). *Let $\delta \in (0, 1]$, $b = b_{\mathrm{I}} + b_{\mathrm{II}}$, $c = c_{\mathrm{I}} + c_{\mathrm{II}}$ and assume that $|x|^3 \nabla c_{\mathrm{I}} \in L^\infty$ and for some constants κ, K*

$$\|C_a\|_{L^\infty} + \||x|^2 \widehat{db}_{\mathrm{I}}\|_{L^\infty} + \||x|^2 (\partial_r(|x|c_{\mathrm{I}}))_+\|_{L^\infty} + \||x|^2 c_{\mathrm{I},-}\|_{L^\infty} \le \kappa,$$
$$\||x|^\delta (|a - I| + |x||a'|)\|_{\ell^1 L^\infty} + \||x|^{\delta+1} \widehat{db}_{\mathrm{II}}\|_{\ell^1 L^\infty} + \||x|^\delta c_{\mathrm{II}}\|_{\ell^1 L^\infty} \le K.$$

If κ is sufficiently small with respect to n, δ, then we have for $\delta < 1$

$$\begin{aligned}(1-\delta)\||x|^{\delta-1}(a\nabla^b v)_T\|_{L^2}^2 &+ \int \left(|x|^{\delta-1} + \frac{\epsilon}{\sqrt{\lambda}}|x|^\delta\right)|\nabla_S^b v|^2 \\ &\lesssim (1+K)\left[(1+\lambda)\|v\|_{\dot Y}^2 + \|\nabla^b v\|_{\dot Y}^2 + \int |x|^\delta \langle x\rangle |f|^2\right] \\ &\quad + K\lambda^{-1} \int |x|^\delta \langle x\rangle |f|^2,\end{aligned} \tag{3.3}$$

while for $\delta = 1$ *we have*

$$\begin{aligned}\int \Big(1 + \frac{\epsilon}{\sqrt{\lambda}}|x|\Big)|\nabla_S^b v|^2 \\ \lesssim (1+K)\Big[(1+\lambda)\|v\|_{\dot{Y}}^2 + \|\nabla^b v\|_{\dot{Y}}^2 + \lambda^{-1}\|f\|_{\dot{Y}^*}^2 + \int |x|^2|f|^2\Big].\end{aligned} \tag{3.4}$$

Proof. In the proof we shall use the shorthand notation

$$a(w) := a(w, w) = a(x)w \cdot \bar{w}, \quad w \in \mathbb{C}^n$$

for the quadratic form associated to the matrix a. We can assume $\epsilon \geq 0$, the other case being similar.

For later use we write the computations in terms of a generic weight function χ as far as possible. We consider again identity (2.8) with the choices

$$\psi' = \chi \quad \text{i.e.} \quad \psi(|x|) = \int_0^{|x|} \chi(s)ds, \quad \phi = -\chi' + \frac{\epsilon}{\sqrt{\lambda}}\chi$$

where χ is a smooth radial function with $\chi, \chi' \geq 0$, and we add to it the imaginary part of identity (2.14) with the choice $\phi = -2\sqrt{\lambda}\chi$. We also rearrange the terms using the identities

$$I_\epsilon = 2\epsilon\Im[a(\nabla\psi, \nabla^b v)v] = [a(\nabla^b v - i\hat{x}\sqrt{\lambda}v) - a(\nabla^b v) - \hat{a}\lambda|v|^2]\frac{\epsilon}{\sqrt{\lambda}}\chi$$

and

$$\Im a(\nabla^b v, v\nabla(-2\sqrt{\lambda}\chi)) = [a(\nabla^b v - i\hat{x}\sqrt{\lambda}v) - a(\nabla^b v) - \hat{a}\lambda|v|^2]\chi'. \tag{3.5}$$

We obtain the following identity:

$$I_S + I_{\nabla v} + I_v + I_c + I_b + I_f = \partial_j\{\Re Q_j + \Re P_j + \Im \widetilde{P}_j\} \tag{3.6}$$

where

$$I_S = \Big[\chi' + \frac{\epsilon}{\sqrt{\lambda}}\chi\Big]a(\nabla^b v - i\sqrt{\lambda}\hat{x}v)$$

$$I_{\nabla v} = 2|(a\nabla^b v)_R|^2\chi' + 2|(a\nabla^b v)_T|^2\frac{\chi}{|x|} - 2a(\nabla^b v)\chi' + r_{\ell m}\Re(\partial_\ell^b v\overline{\partial_m^b v})$$

with $r_{\ell m}(x) = [2a_{jm}a_{\ell k;j} - a_{jk}a_{\ell m;j}]\hat{x}_k\chi$ and using notation (1.7),

$$I_v = \Big[-\frac{1}{2}A(A\psi + \phi) + (1-\hat{a})(\epsilon\sqrt{\lambda}\chi + \lambda\chi')\Big]|v|^2$$

$$I_c = \Big[\frac{\epsilon}{\sqrt{\lambda}}\chi c - \chi' c - a(\hat{x}, \nabla c)\chi\Big]|v|^2$$

$$I_b = 2\Im[(a\nabla^b v)\cdot(db)_T)\bar{v}]\chi$$

$$I_f = \Re[(A\psi + \phi)\bar{v}f + 2a(\hat{x}, \nabla^b v)\chi f] - 2\sqrt{\lambda}\Im(\bar{v}f\chi)$$

where

$$Q_j = a_{jk}\partial_k^b v\cdot[A^b, \psi]\bar{v} - \frac{1}{2}a_{jk}(\partial_k A\psi)|v|^2 - a_{jk}\hat{x}_k\chi[(c-\lambda)|v|^2 + a(\nabla^b v)]$$

with $\psi' = \chi$, and

$$P_j = a_{jk}\partial_k^b v\bar{v}\Big[\frac{\epsilon}{\sqrt{\lambda}}\chi - \chi'\Big] - \frac{1}{2}a_{jk}\hat{x}_k|v|^2\Big[\frac{\epsilon}{\sqrt{\lambda}}\chi' - \chi''\Big], \quad \widetilde{P}_j = a_{jk}\bar{v}\partial_k^b v\chi.$$

Note that at $\partial\Omega$ the boundary terms $P_j, \widetilde{P}_j$ vanish, while Q_j give a negative contribution as proved in Remark 2.5; on the other hand, the integrals of $P_j, \widetilde{P}_j, Q_j$ on the sphere $\{|x| = M\}$ tend to zero as $M \to \infty$ by the conditions imposed on the growth of χ. Hence by integrating (3.6) on $\Omega \cap \{|x| \le M\}$ and letting $M \to \infty$ we can neglect the boundary terms and we obtain

$$\int_\Omega (I_S + I_{\nabla v} + I_v + I_c + I_b + I_f) \le 0.$$

We shall also use the magnetic Hardy inequality (2.6) for different choices of s. Note that with the substitution $w = e^{-i\sqrt{\lambda}|x|}v$ we have also

$$\||x|^{-s}v\|_{L^2} \le \frac{2}{n-2s}\||x|^{1-s}\nabla_S^b v\|_{L^2} \tag{3.7}$$

where we used the notation $\nabla_S^b = \nabla^b - i\hat{x}\sqrt{\lambda}$.

We estimate each term separately. We can write

$$I_{\nabla v} = 2\chi' a(\nabla^b v, (a-I)\nabla^b v) + 2\Big(\frac{\chi}{|x|} - \chi'\Big)|(a\nabla^b v)_T|^2 + r_{\ell m}\Re(\partial_\ell^b v\overline{\partial_m^b v})$$

and noticing that $\chi \ge |x|\chi'$ for $\chi = |x|^\delta$, $\delta \le 1$, we obtain

$$\begin{aligned}\int_\Omega I_{\nabla v} \ge & -c(n)\||x|^\delta(|a-I| + |x||a'|)\|_{\ell^1 L^\infty}\cdot\|\nabla^b v\|_{\dot{Y}}^2 \\ & + (1-\delta)\||x|^{\frac{\delta-1}{2}}(a\nabla^b v)_T\|_{L^2}^2.\end{aligned} \tag{3.8}$$

In order to estimate I_v we first compute

$$A\psi + \phi = (\hat{a} - 1)\chi' + \frac{\bar{a} - \hat{a} + |x|\tilde{a}}{|x|}\chi + \frac{\epsilon}{\sqrt{\lambda}}\chi.$$

Recalling (2.35) we have easily

$$|A\psi + \phi| \leq c(n)\Big(\frac{\chi}{|x|} + \chi'\Big) + \frac{\epsilon}{\sqrt{\lambda}}\chi \leq c(n)|x|^{\delta-1} + \frac{\epsilon}{\sqrt{\lambda}}|x|^{\delta}, \tag{3.9}$$

while a straightforward computation gives, with $\mu_n = (n-1)(n-3)$,

$$\begin{aligned} A(A\psi + \phi) \leq & -\frac{\mu_n}{|x|^2}\Big(\frac{\chi}{|x|} - \chi'\Big) + \frac{n-1}{|x|}\chi'' + c(n)\frac{C_a C_\chi}{|x|^2} \\ & + \frac{\epsilon}{\sqrt{\lambda}}\Big(\hat{a}\chi'' + \frac{n-1}{|x|}\chi' + c(n)\frac{C_a\chi'}{|x|}\Big) \end{aligned}$$

where

$$C_\chi(x) := |x|^{-1}\chi + \chi' + |x||\chi''| + |x|^2|\chi'''|.$$

With the choice $\chi = |x|^\delta$, and dropping a negative term, this reduces to

$$A(A\psi + \phi) \leq -\frac{(1-\delta)(n-3+\delta)}{|x|^{3-\delta}} + \frac{c(n)C_a}{|x|^{3-\delta}} + \frac{\epsilon\delta}{\sqrt{\lambda}}\frac{n-1+C_a c(n)}{|x|^{2-\delta}}.$$

We shall drop also the first term at the right, although it gives a positive contribution, since it can be recovered from the final estimate. Thus we have

$$I_v \geq -\frac{c(n)C_a}{|x|^{3-\delta}}|v|^2 - \frac{\epsilon\delta}{\sqrt{\lambda}}\frac{n-1+C_a c(n)}{|x|^{2-\delta}}|v|^2 - |a - I|(\epsilon\sqrt{\lambda}\chi + \lambda\chi')|v|^2.$$

We now integrate I_v on Ω. Thanks to the magnetic Hardy inequality (3.7) with $s = (3-\delta)/2$ and using the previous estimate for $A(A\psi + \phi)$, we have

$$c(n)\int C_a|x|^{\delta-3}|v|^2 \leq \frac{4c(n)\|C_a\|_{L^\infty}}{(n-3+\delta)^2}\int |x|^{\delta-1}|\nabla_S^b v|^2 \leq \sigma\int I_S$$

(note that in 3D the constant $\to\infty$ as $\delta\to 0$) provided

$$\frac{4c(n)\|C_a\|_{L^\infty}}{\nu(n-3+\delta)^2} \leq \sigma\cdot\delta.$$

Here σ is a universal constant (it will be chosen equal to $1/10$) which we keep around to track the smallness assumptions on the coefficients. In a similar way, with $s = (2-\delta)/2$,

$$\begin{aligned} \frac{\epsilon}{\sqrt{\lambda}}\delta\int\frac{n-1+C_a c(n)}{|x|^{2-\delta}}|v|^2 \leq & \frac{4\delta(n-1+c(n)\|C_a\|_{L^\infty})}{(n-2-\delta)^2}\int\frac{\epsilon}{\sqrt{\lambda}}|x|^\delta|\nabla_S^b v|^2 \\ & \leq \sigma\int I_S \end{aligned} \tag{3.10}$$

provided

$$\frac{4(n-1+c(n)\|C_a\|_{L^\infty})}{\nu(n-2-\delta)^2}\cdot\delta\leq\sigma.$$

Note that the last condition restricts δ to an interval $(0,\delta_n]$ which covers $(0,1]$ only for n sufficiently large. To get around this difficulty we give an alternative estimate of the ϵ term. Fix $\alpha>0$ and split the integral in the regions $|x|\leq\alpha$ and $|x|\geq\alpha$:

$$\epsilon\int\frac{|v|^2}{|x|^{2-\delta}}\leq\epsilon\alpha\int\frac{|v|^2}{|x|^{3-\delta}}+\epsilon\alpha^{\delta-2}\int|v|^2\leq\frac{4\epsilon\alpha}{\nu(n-3+\delta)^2}\int I_S+\alpha^{\delta-2}\int|f\bar{v}|$$

where we used again (3.7) and the inequality $\epsilon\int_\Omega|v|^2\leq\int_\Omega|f\bar{v}|$ (recall the first identity (2.18)). Hence we obtain

$$\frac{\epsilon}{\sqrt{\lambda}}\delta\int\frac{n-1+C_ac(n)}{|x|^{2-\delta}}|v|^2\leq C_1\frac{\epsilon\delta\alpha}{\sqrt{\lambda}}\int I_S+C_2\frac{\delta\alpha^{\delta-2}}{\sqrt{\lambda}}\int|f\bar{v}|$$

where

$$C_1=\frac{4(n-1+c(n)\|C_a\|_{L^\infty})}{\nu(n-3+\delta)^2},\qquad C_2=n-1+c(n)\|C_a\|_{L^\infty}.$$

We choose now

$$\alpha=\frac{\sigma}{C_1\delta\sqrt{\lambda}}$$

and we arrive at the following inequality, which is valid for all $\delta\in(0,1]$:

$$\frac{\epsilon}{\sqrt{\lambda}}\delta\int\frac{n-1+C_ac(n)}{|x|^{2-\delta}}|v|^2\leq\frac{\epsilon}{\lambda}\sigma\int I_S+C_3\sqrt{\lambda}^{1-\delta}\int|f\bar{v}|$$

where

$$C_3=\frac{4^{2-\delta}[\delta(n-1+c(n)\|C_a\|_{L^\infty})]^{3-\delta}}{(\sigma\nu(n-3+\delta)^2)^{2-\delta}}$$

and we can estimate the coefficient ϵ/λ with 1 since $\lambda\geq\epsilon$. Thus we get

$$\frac{\epsilon}{\sqrt{\lambda}}\delta\int\frac{n-1+C_ac(n)}{|x|^{2-\delta}}|v|^2\leq\sigma\int I_S+c(n,\delta)(1+\lambda)^{1-\delta}\|v\|_{\dot{Y}}^2+\|f\|_{\dot{Y}*}^2.$$

Moreover we have

$$\int|a-I|\lambda\chi'|v|^2\leq\||a-I||x|^\delta\|_{\ell^1L^\infty}\lambda\|v\|_{\dot{Y}}^2$$

$$\int|a-I|\epsilon\sqrt{\lambda}\chi|v|^2\leq\||a-I||x|^\delta\|_{L^\infty}\sqrt{\lambda}\int|f\bar{v}|$$

where we used the estimate $\epsilon \int_\Omega |v|^2 \le \int_\Omega |f\bar{v}|$ which follows from (2.18). Summing up, we obtain, as $\delta \in (0,1]$,

$$\int I_v \ge -2\sigma \int I_S - c(n,\delta)(1+K)((1+\lambda)\|v\|_{\dot Y}^2 + \|f\|_{\dot Y^*}^2). \tag{3.11}$$

The term I_b can be estimated as follows. We note that

$$a\nabla^b v \cdot (db)_T = a\nabla_S^b v \cdot (db)_T$$

so that, with the choice $\chi = |x|^\delta$,

$$\int I_{b_{\mathrm{I}}} \ge -c(n)\||x|^{\frac{\delta-1}{2}}\nabla_S^{b_{\mathrm{I}}} v\|_{L^2}\||x|^2\widehat{db}_{\mathrm{I}}\|_{L^\infty}\||x|^{\frac{\delta-3}{2}}v\|_{L^2}$$

and using the magnetic Hardy inequality

$$\int I_{b_{\mathrm{I}}} \ge -\frac{2c(n)}{(n-3+\delta)}\||x|^2\widehat{db}_{\mathrm{I}}\|_{L^\infty}\||x|^{\delta-1}\nabla_S^b v\|_{L^2}^2 \ge -\sigma\int I_S$$

provided

$$\frac{2c(n)}{\nu(n-3+\delta)}\||x|^2\widehat{db}_{\mathrm{I}}\|_{L^\infty} \le \frac{2c(n)}{\nu(n-3+\delta)}\cdot\kappa \le \sigma\cdot\delta.$$

For the second piece $I_{b_{\mathrm{II}}}$ we have simply

$$\int I_{b_{\mathrm{II}}} \ge -c(n)\|\nabla^b v\|_{\dot Y}\|v\|_{\dot Y}\||x|^{1+\delta}\widehat{db}_{\mathrm{II}}\|_{\ell^1 L^\infty} \ge -c(n)K\|\nabla^b v\|_{\dot Y}\|v\|_{\dot Y}$$

and in conclusion

$$\int I_b \ge -\sigma\int I_S - c(n)K\|\nabla^b v\|_{\dot Y}\|v\|_{\dot Y}. \tag{3.12}$$

To estimate I_c we begin by writing, with $\chi = |x|^\delta$,

$$\int I_{c_{\mathrm{I}}} \ge -\frac{\epsilon}{\sqrt{\lambda}}\||x|^2 c_{\mathrm{I},-}\|_{L^\infty}\int ||x|^{\delta-2}|v|^2 - \int[\delta|x|^{\delta-1}c_{\mathrm{I}} + |x|^\delta a(\hat{x},\nabla c_{\mathrm{I}})]|v|^2$$

and the first term can be handled again using Hardy's inequality:

$$\ge -\sigma\int I_S - \int[\delta|x|^{\delta-1}c_{\mathrm{I}} + |x|^\delta a(\hat{x},\nabla c_{\mathrm{I}})]|v|^2$$

provided

$$\frac{4}{\nu(n-2+\delta)^2}\||x|^2 c_{\mathrm{I},-}\|_{L^\infty} \le \frac{4}{\nu(n-2+\delta)^2}\cdot\kappa \le \sigma\cdot\delta.$$

To bound the second integral we write, with ∂_r denoting the radial derivative,

$$\begin{aligned}
&\delta|x|^{\delta-1}c_{\mathrm{I}} + |x|^{\delta}a(\hat{x},\nabla c_{\mathrm{I}})]|v|^2\\
&\quad = \partial_r(|x|^{\delta}c_{\mathrm{I}}) + (a-I)\hat{x}\cdot\nabla c_{\mathrm{I}}|x|^{\delta}\\
&\quad = ((\delta-1)|x|^2c_{\mathrm{I}} + |x|^2\partial_r(|x|c_{\mathrm{I}}))|x|^{\delta-3} + (a-I)\hat{x}\cdot\nabla c_{\mathrm{I}}|x|^{\delta}\\
&\quad \le \kappa\cdot(1+\||x|^3\nabla c_{\mathrm{I}}\|_{L^\infty})\cdot|x|^{\delta-3}
\end{aligned}$$

and hence, using Hardy's inequality,

$$\int[\delta|x|^{\delta-1}c_{\mathrm{I}} + a(\hat{x},\nabla c_{\mathrm{I}})\chi]|v|^2 \le \sigma\int I_S$$

provided

$$\frac{4}{\nu(n-3+\delta)^2}(1+\||x|^3\nabla c_{\mathrm{I}}\|_{L^\infty})\cdot\kappa \le \sigma\cdot\delta.$$

Thus we have proved, for κ small enough,

$$\int I_{c_{\mathrm{I}}} \ge -2\sigma\int I_S.$$

For the second piece $I_{c_{\mathrm{II}}}$ we use again (2.18): with $\chi = |x|^{\delta}$, we have

$$\int I_{c_{\mathrm{II}}} \ge -\lambda^{-1/2}\||x|^{\delta}c_{\mathrm{II},-}\|_{L^\infty}\int|f\bar{v}| - \int[\chi' c_{\mathrm{II}} + a(\hat{x},\nabla c_{\mathrm{II}})\chi]|v|^2.$$

Using the identity ($c = c_{\mathrm{II}}$)

$$\begin{aligned}
a(\hat{x},\nabla c)\chi|v|^2 = \partial_j\{a_{jk}\hat{x}_k c\chi|v|^2\} - \frac{\bar{a}-\hat{a}+|x|\tilde{a}}{|x|}c\chi|v|^2\\
- \hat{a}c\chi'|v|^2 - 2\Re a(\nabla^b v,\hat{x}v)c\chi
\end{aligned}$$

we obtain

$$\int[\chi' c_{\mathrm{II}} + a(\hat{x},\nabla c_{\mathrm{II}})\chi]|v|^2 \le c(n)\||x|^{\delta}c_{\mathrm{II}}\|_{\ell^1L^\infty}(\|v\|^2_{\dot{Y}} + \|\nabla^b v\|^2_{\dot{Y}}).$$

Summing up, we have proved

$$\int I_c \ge -2\sigma\int I_S - \lambda^{-1}K\|f\|^2_{\dot{Y}^*} - c(n)K(\|\nabla^b v\|^2_{\dot{Y}} + \|v\|^2_{\dot{Y}}). \tag{3.13}$$

Finally for I_f we can write

$$2\Re a(\hat{x}, \nabla^b v)\chi f - 2\sqrt{\lambda}\Im(\bar{v} f \chi) = 2\Re(a - I)\hat{x} \cdot \overline{\nabla^b v}\chi f + 2\Re\hat{x} \cdot \overline{\nabla^b_S v}\chi f$$

and recalling (3.9)

$$I_f \geq -(c(n)|x|^{\delta-1} + \frac{\epsilon}{\sqrt{\lambda}}|x|^{\delta})|f\bar{v}| - |a - I||x|^{\delta}|\nabla^b v||f| - 2|x|^{\delta}|\nabla^b_S v||f|.$$

The integral of the first term is estimated by Cauchy–Schwartz

$$\int |x|^{\delta-1}|f\bar{v}| \leq \alpha\delta \int |x|^{\delta-3}|v|^2 + \frac{1}{\alpha\delta}\int |x|^{\delta+1}|f|^2$$

and then by Hardy's inequality

$$\alpha\delta \int |x|^{\delta-3}|v|^2 \leq \frac{4\alpha\delta}{(n-3+\delta)^2}\int |x|^{\delta-1}|\nabla^b_S v|^2 \leq \sigma \int I_S,$$

with $4\alpha = \sigma(n-3+\delta)^2\nu$, and we conclude

$$\int |x|^{\delta-1}|f\bar{v}| \leq \sigma \int I_S + c(n,\delta)\int |x|^{\delta+1}|f|^2.$$

For the second term we use the condition $\epsilon \leq \lambda$ and we obtain

$$\frac{\epsilon}{\sqrt{\lambda}}\int |x|^{\delta}|f\bar{v}| \leq \epsilon \int |v|^2 + \int |x|^{2\delta}|f|^2 \leq \int |f\bar{v}| + \int |x|^{2\delta}|f|^2$$

Next we have

$$\int |a - I||x|^{\delta}|\nabla^b v||f| \leq \||x|^{\delta}(a - I)\|_{L^\infty}\|\nabla^b v\|_{\dot{Y}}\|f\|_{\dot{Y}^*}.$$

The integral of remaining term can be estimated as follows:

$$\begin{aligned}\int |x|^{\delta}|\nabla^b_S v||f| &\leq \sigma\delta\nu \int |x|^{\delta-1}|\nabla^b_S v|^2 + \frac{1}{\sigma\delta\nu}\int |x|^{\delta+1}|f|^2 \\ &\leq \sigma \int I_S + \frac{1}{\sigma\delta\nu}\int |x|^{\delta+1}|f|^2.\end{aligned}$$

Summing up, we have proved

$$\int_{\Omega} I_f \geq -2\sigma \int I_S - c\int (|x|^{\delta+1} + |x|^{2\delta})|f|^2 - \int |f\bar{v}| - K\|\nabla^b v\|_{\dot{Y}}\|f\|_{\dot{Y}^*} \tag{3.14}$$

for some $c = c(n, \sigma, \delta)$.

We collect (3.8), (3.11), (3.12), (3.13), and (3.14) to obtain

$$
\begin{aligned}
&(1-7\sigma)\int I_S + (1-\delta)\||x|^{\frac{\delta-1}{2}}(a\nabla^b v)_T\|_{L^2}^2 \\
&\quad \le c(n,\delta)\int (|x|^{\delta+1}+|x|^{2\delta})|f|^2 \\
&\qquad + c(n,\delta)(1+K)((1+\lambda)\|v\|_{\dot Y}^2 + \|\nabla^b v\|_{\dot Y}^2 + \lambda^{-1}\|f\|_{\dot Y^*}^2).
\end{aligned}
$$

We now choose $\sigma = 1/10$ so that $1-7\sigma > 0$. Moreover, in the case $\delta < 1$ we have easily

$$
\int (|x|^{\delta+1}+|x|^{2\delta})|f|^2 + \|f\|_{\dot Y^*}^2 \lesssim \int |x|^{\delta}\langle x\rangle |f|^2
$$

and this gives (3.3), while for $\delta = 1$ we leave the two norms of f separate, and we obtain (3.4). □

4. Proof of Theorem 1.1

We first prove that the only solution satisfying the Sommerfeld condition is 0.

Corollary 4.1 (uniqueness). *Assume* (*A*) *holds,*

$$
\mu < \mu_0(n) \quad \text{and} \quad \lambda \ge c_0(n)(Z+Z^2).
$$

Let $v \in H^1_{\mathrm{loc}}(\Omega)$ *with* $v|_{\partial\Omega} = 0$ *be a solution of*

$$
(L+\lambda)v = 0
$$

satisfying the Sommerfeld radiation condition

$$
\liminf_{R\to\infty} \int_{|x|=R} |\nabla^b v - i\sqrt{\lambda}\hat{x}v|^2 dS = 0. \tag{4.1}
$$

Then $v \equiv 0$. *If in particular*

$$
\int_{|x|\gg 1} |x|^{\delta-1}|\nabla^b v - i\sqrt{\lambda}\hat{x}v|^2 dx < \infty \tag{4.2}
$$

for some $\delta > 0$, *then* (4.1) *is satisfied and the same conclusion holds.*

Proof of the corollary. By the assumptions on L we have $v \in H^2_{\mathrm{loc}}$. Moreover, multiplying the equation by $\bar{v}$ and taking the imaginary part we obtain the identity

$$\Im \partial_j \{a_{jk} \partial^b_k v \bar{v}\} = 0$$

and integrating on $\Omega \cap \{|x| < R\}$, thanks to the Dirichlet boundary conditions we get, for R large enough,

$$\int_{|x|=R} \Im(\bar{v}\hat{x} \cdot \nabla^b v) dS = 0.$$

This implies

$$\int_{|x|=R} (|\nabla^b v|^2 + \lambda |v|^2) dS = \int_{|x|=R} \left|\nabla^b v - i\sqrt{\lambda}\hat{x} v\right|^2 dS$$

and hence condition (2.1) is satisfied. Then applying the previous estimate with $f = 0$, $\epsilon = 0$, we obtain that $v \equiv 0$. The last claim is proved by contradiction: if $\int_{|x|=R} |\nabla^b v - i\sqrt{\lambda}\hat{x} v|^2 dS \geq \sigma$ for some constant $\sigma > 0$, then multiplying by $|x|^{\delta-1}$ and integrating in the radial variable we obtain that the quantity (4.2) can not be finite. $\square$

Lemma 4.2. *Assume* (A)*, with* μ, λ *arbitrary, and let*

$$\Gamma = \|a - I\|_{L^\infty} + \||x|^2 c_-\|_{L^\infty(|x| \leq 2)}.$$

Let $v \in H^2_{\mathrm{loc}}(\Omega)$ *with* $v|_{\partial\Omega} = 0$*,* $\lambda, \epsilon \in \mathbb{R}$ *and let* $f = (L + \lambda + i\epsilon)v$*. Then, if* Γ *is sufficiently small with respect to* n*, for all* $R > 0$ *we have*

$$\int_{\Omega \cap \{|x| \leq R\}} |\nabla^b v|^2 \leq C \int_{\Omega \cap \{|x| \leq R+1\}} |v|^2 \quad + \int_{\Omega \cap \{|x| \leq R+1\}} |f|^2 \tag{4.3}$$

where $C = c(n)(1 + \lambda_+ + \|c_-\|_{L^\infty(|x| \geq 1)})$.

Proof. For any real valued test function ψ we can write

$$(L + \lambda + i\varepsilon)(\psi v) = \psi f + (A\psi)v + 2a(\nabla^b v, \nabla \psi)$$

and multiplying by $\psi \bar{v}$ and rearranging the terms we get

$$\begin{aligned} &\partial_j \{\psi \bar{v} a_{jk} \partial^b_k (\psi v)\} - a(\nabla^b(\psi v), \nabla^b(\psi v)) + (\lambda + i\epsilon - c)|\psi v|^2 \\ &\quad = f\psi^2 \bar{v} + (A\psi)\psi |v|^2 + 2a(\nabla^b v, \nabla \psi)\psi \bar{v}. \end{aligned}$$

Now we take the real part and use the fact that

$$
\begin{aligned}
2\Re a(\nabla^b v, \nabla\psi)\psi\bar{v} &= 2\Re a(\nabla v, \nabla\psi)\psi\bar{v} \\
&= -\frac{1}{2}a(\nabla|\psi|^2, \nabla|v|^2) \\
&= -\frac{1}{2}(A|\psi|^2)|v|^2 - \partial_j\left\{\frac{1}{2}a_{jk}|v|^2\partial_k|\psi|^2\right\}
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
\partial_j\left\{\Re\psi\bar{v}a_{jk}\partial_k^b(\psi v) + \frac{1}{2}a_{jk}|v|^2\partial_k|\psi|^2\right\} &= a(\nabla^b(\psi v), \nabla^b(\psi v)) + (c-\lambda)|\psi v|^2 \\
&\quad + \Re f\psi^2\bar{v} + (A\psi)\psi|v|^2 \\
&\quad - \frac{1}{2}(A|\psi|^2)|v|^2.
\end{aligned}
$$

Integrating on Ω and using $A|\psi|^2 = 2\psi A\psi + 2a(\nabla\psi, \nabla\psi)$ and the Dirichlet boundary conditions, we arrive at

$$
\int_\Omega a(\nabla^b(\psi v), \nabla^b(\psi v)) = \int_\Omega(\lambda - c)|\psi v|^2 - \int_\Omega \Re f\psi^2\bar{v} + \int_\Omega a(\nabla\psi, \nabla\psi)|v|^2. \tag{4.4}
$$

It is clear that this identity holds for any compactly supported, piecewise C^1 weight function ψ.

We introduce now a cutoff function χ equal to 1 in $|x| \leq 1$, equal to 0 for $|x| \geq 2$, and such that $0 \leq \chi \leq 1$. Then we can write

$$
-\int c|\psi v|^2 \leq \int(1-\chi)c_-|\psi v|^2 + \int \chi c_-|\psi v|^2.
$$

We estimate the first term simply as follows:

$$
\int_\Omega (1-\chi)c_-|\psi v|^2 \leq \|c_-\|_{L^\infty(|x|\geq 1)}\int_\Omega |\psi v|^2.
$$

On the other hand, for the second term we use the magnetic Hardy inequality:

$$
\int_\Omega \chi c_-|\psi v|^2 \leq \||x|^2 c_-\|_{L^\infty(|x|\leq 2)}\int_\Omega |x|^{-2}|\psi v|^2 \leq c(n)\Gamma\int_\Omega |\nabla^b(\psi v)|^2.
$$

Since $a \geq (1-\Gamma)I$, if Γ is sufficiently small with respect to n we can absorb the last term at the left hand side of (4.4) and we obtain the estimate

$$
\begin{aligned}
&\int_\Omega |\nabla^b(\psi v)|^2 \\
&\quad \leq c(n)(1 + \lambda_+ + \|c_-\|_{L^\infty(|x|\geq 1)})\int_\Omega |\psi v|^2 + \int_\Omega a(\nabla\psi, \nabla\psi)|v|^2 + \int_\Omega |\psi f|^2.
\end{aligned}
$$

Finally, we choose ψ as follows: for a given $R > 0$,

$$\psi = \begin{cases} 1 & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq R+1, \\ R+1-|x| & \text{elsewhere.} \end{cases}$$

Plugging ψ in the previous estimate we obtain the claim. □

We are ready to conclude the proof of Theorem 1.1. Given f with

$$\int |x|^{\delta} \langle x \rangle |f|^2 < \infty,$$

we consider a sequence $\epsilon_k > 0$ with $\epsilon_k \to 0$ and define v_k as the unique solution $v_k \in H_0^1(\Omega) \cap H^2(\Omega)$ of

$$(L + \lambda + i\epsilon_k)v_k = f.$$

We now remark that under the assumptions of Theorem 1.1, if κ is sufficiently small, all the conditions in both Theorems 2.1 and 3.2 are satisfied. Then, introducing the norm

$$\|w\|_{\dot{Z}} := \|w\|_{\dot{X}} + |\lambda| \|w\|_{\dot{Y}} + \|\nabla^b w\|_{\dot{Y}} + (n-3) \left\| \frac{|w|^2}{|x|^{3/2}} \right\| + \left(\int |x|^{\delta-1} |\nabla_S^b v|^2 dx \right)^{1/2},$$

we get the bound (uniform in $|\epsilon| < \lambda$ for fixed λ)

$$\|v_k\|_{\dot{Z}}^2 \lesssim \int |x|^{\delta} \langle x \rangle |f|^2 \tag{4.5}$$

since the last norm controls $\|f\|_{\dot{Y}^*}$. Note on the other hand that the smoothing estimate

$$\|v_k\|_{\dot{X}} + |\lambda|^{1/2} \|v_k\|_{\dot{Y}} + \|\nabla^b v_k\|_{\dot{Y}} + \|(a\nabla^b v_k)_T\|_{L^2} + (n-3) \left\| \frac{v_k}{|x|^{3/2}} \right\|_{L^2} \leq c(n) \|f\|_{\dot{Y}^*} \tag{4.6}$$

is uniform for all $\lambda > \bar{\sigma} \cdot (K + K^2)$ and all ϵ.

From (4.5) we deduce that v_k is a bounded sequence in $H^1(\Omega \cap \{|x| < R\})$ for all $R > 0$; by a diagonal procedure and the compact embedding of H^1 into L^2 we can extract a subsequence, which we denote again by v_k, strongly convergent in $L^2(\Omega \cap \{|x| < R\})$ for all $R > 0$. Moreover, the difference $v_k - v_h$ of two solutions satisfies the equation

$$(L + \lambda + i\epsilon_k)(v_k - v_h) = (\epsilon_k - \epsilon_h)v_h,$$

hence by Lemma 4.2 we see that v_k is a Cauchy sequence in $H^1(\Omega \cap \{|x| < R\})$, and in conclusion v_k converges strongly in $H^1(\Omega \cap \{|x| < R\})$ for all $R > 0$ to a limit v. Clearly $v \in H^1_{\mathrm{loc}}(\Omega)$, $v|_{\partial\Omega} = 0$, and v is a solution of

$$(L + \lambda)v = f.$$

We note that by (4.5) the sequence v_k is bounded in $\dot{Z}$ which is the dual of a separable space, hence it admits a weakly-* convergent subsequence whose limit satisfies the same bound. This means that $v \in \dot{Z}$ with

$$\|v\|^2_{\dot{Z}} \lesssim \int |x|^\delta \langle x \rangle |f|^2,$$

and that v satisfies also the smoothing estimate (4.6).

Finally, if we apply the same procedure to any subsequence of the original sequence, we can extract from it a subsequence which converges in H^1_{loc} strongly and in $\dot{Z}$ weakly-* to a solution $\tilde{v}$ of the Helmholtz equation satisfying the same bounds, and by Corollary 4.1 we must have $\tilde{v} = v$. This implies that the entire original sequence converges to v both in H^1_{loc} strongly and in $\dot{Z}$ weakly-*, and the proof is concluded.

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