

EXACT ALGEBRAIC CONDITIONS FOR INDIRECT CONTROLLABILITY OF QUANTUM SYSTEMS*

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Abstract. In several quantum control schemes, a target quantum system S is put in contact with an auxiliary system A and the coherent control can directly affect only A . The system S is controlled *indirectly* through the interaction with A . The system S is said to be *indirectly controllable* if every unitary transformation can be performed on the state of S with this scheme. The indirect controllability of S will depend on the *dynamical Lie algebra* \mathcal{L} characterizing the dynamics of the total system $S + A$ and on the initial state of the auxiliary system A . In this paper, we describe this characterization exactly. A natural assumption is that the auxiliary system A is *minimal*, which means that there is no part of A that is not coupled to S , and we denote by n_A the dimension of such a minimal A , which we assume to be fully controllable. We show that if n_A is greater than or equal to 3, indirect controllability of S is verified if and only if complete controllability of the total system $S + A$ is verified, i.e., $\mathcal{L} = su(n_S n_A)$ or $\mathcal{L} = u(n_S n_A)$, where n_S denotes the dimension of the system S . If $n_A = 2$, it is possible to have indirect controllability without having complete controllability. The exact condition for that to happen is given in terms of a Lie algebra \mathcal{L}_S which describes the evolution of the system S only. We prove that indirect controllability is verified if and only if $\mathcal{L}_S = u(n_S)$ and the initial state of the auxiliary system A is pure.

Key words. control of quantum systems, Lie algebraic methods, indirect control, interacting quantum systems

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1. Introduction. In the paper [18], Lloyd proposed a scheme for control of quantum systems where the controller itself was a quantum system that was affecting the target system via the interaction. This scheme, named *coherent feedback control*, was later expanded in several ways (see [26] for a recent review) and it is currently the object of intensive research. The consideration of this approach motivates the fundamental question of to what extent one can control a quantum system S indirectly through the interaction with an auxiliary system A . A further motivation comes from the fact that, in many experimental set-ups, the target system is not directly accessible for control or it is not advisable to control it directly as the influence of the environment might become too strong during the experiment, therefore destroying the peculiar (potentially useful) features of quantum dynamics (see, e.g., [13], [20]). Schemes of indirect control are, for example, given by spin chains where the control field can affect only a selected number of spins [12]. Several recent papers in quantum information have discussed the experimental need of implementing quantum operations by acting only on a selected number of qubits (see, e.g., [27] and the references therein). Further motivation comes from situations where one wants to effectively switch off the interaction of a system S with an additional system A such as in solid

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state implementations of quantum information (see, e.g., [9]). Indirect controllability gives a way to achieve this by controlling system A only.

Mathematical controllability studies for systems where only one subsystem (A) can be directly accessed, but another system (S) is the target of control, have been carried out in several papers (see, e.g., [3], [10]). However, conditions have always been given so that the complete controllability of the whole system $S + A$ (see Definition 2.1 below) is verified. In a recent paper [8], a study was started of indirect controllability (for a precise definition see Definition 2.2 below) and the case where both system S and A are two dimensional was treated in detail. It was shown that it is possible to have indirect controllability of system S without having complete controllability on the system $S + A$ (while the converse implication is obvious). It was, however, shown later in [7] that if the system A is assumed to be in a perfectly mixed state at the beginning of the control experiment, then complete controllability is necessary to have indirect controllability. In this paper, we solve the general problem to give exact conditions for indirect controllability for systems S and A of arbitrary dimensions.

It is well known in quantum control theory that the dynamical Lie algebra \mathcal{L} generated by the Hamiltonians available for the evolution of a finite dimensional system of dimension n describes the set of available evolutions for that system (see, e.g., [15], [17]). In particular, if $\mathcal{L} = su(n)$ (resp., $\mathcal{L} = u(n)$), every special unitary evolution (resp., every unitary evolution) is available for the system. This is called the *Lie algebra rank condition*. It is a result of practical use as it reduces a problem on a Lie group to a linear algebraic test, and it suggests the use of the theory of Lie algebras and Lie groups as a comprehensive approach to the analysis and control of quantum systems. In this approach, the dynamical Lie algebra is the fundamental object of study as it describes how “rich” the dynamics of the system is. In this paper, we shall use the dynamical Lie algebra to characterize the indirect controllability properties in indirect control schemes.

The paper is organized as follows. In the next section, we give the main definitions and state the main results, while deferring the proofs to the rest of the paper. Section 3 is devoted to some technical lemmas concerning the general structure of Lie subalgebras of the Lie algebra $u(n)$. In this section, we also recall some results proved in other papers by the authors (in particular [7] and [8]), which are used in the proof of the main results. In section 4, we prove the result for the case where the (minimal) dimension of the auxiliary system A is greater than or equal to three. In this case, indirect controllability and complete controllability are equivalent properties independently of the initial state of the system A . This equivalence does not hold in the case where this dimension is 2. The proof of the indirect controllability condition in this case is separated in two parts, presented in sections 5 and 6. This proof is, in fact, quite long and technical and is made up of the treatment of several special cases. In order to streamline the proof, we report some of the special cases in Appendix B. We give some concluding remarks in section 7.

2. Basic definitions and main results. Although the general definitions of *controllability* for quantum mechanical systems can be given for systems of infinite dimension, this property is much better understood in the finite dimensional case. We shall restrict ourselves to this case and denote by \mathcal{H}_S the Hilbert space of dimension n_S of the target system S and by \mathcal{H}_A the Hilbert space of dimension n_A of the auxiliary system A . The total system $S + A$ evolves on the Hilbert space $\mathcal{H} := \mathcal{H}_S \otimes \mathcal{H}_A$. The dimension of the total system $S + A$ is $n_{SA} := n_S \times n_A$. In assigning a dimension n_A to

A , we are making a natural *minimality assumption*, namely, we assume that A is fully coupled to S , that is, it does not contain any subsystem that is completely decoupled from S . It is clear, in fact, that the dimension of A could be made arbitrarily large by adding “dummy” subsystems or energy levels that are not coupled to S .

Recall that the state of a quantum mechanical system is described by a *density matrix* ρ (see, e.g., [23]), i.e., a Hermitian, trace 1, positive semidefinite operator (matrix) on the Hilbert space associated with the system. We shall denote by ρ_S , ρ_A , and ρ_{TOT} the states of the systems S , A , and $S + A$, respectively. We also shall make the assumption that the system $S + A$ has been prepared at the beginning of the control experiment in an uncorrelated state, i.e., at time $t = 0$,

$$(1) \quad \rho_{TOT} = \rho_S \otimes \rho_A.$$

In typical experimental set-ups, the dynamics of the total system $S + A$ is determined by a set \mathcal{F} of Hermitian operators on the Hilbert space \mathcal{H} . These are the *Hamiltonians* associated with the system. In the control theory setting, elements in \mathcal{F} are parametrized by a *control variable* u which is allowed to take values in a set \mathcal{U} , so that $\mathcal{F} := \{H_u \mid u \in \mathcal{U}\}$. Thus, the dynamics of the model is given by

$$(2) \quad \rho_{TOT}(t) = U(t)\rho_{TOT}(0)U^\dagger(t),$$

where the unitary operator $U(t)$ is the solution of the *Schrödinger operator equation*,

$$(3) \quad i\dot{U}(t) = H_u U(t), \quad U(0) = \mathbf{1}_{n_{SA}},^1$$

and the control parameter u varies with time in the set \mathcal{U} . In the Schrödinger equation (3), we have assumed to use units so that the Planck constant \hbar is equal to 1. A typical situation in experiments is when the H_u 's are linear in u , i.e., they have the form $H_u := H_0 + \sum_j H_j u_j$ for some finite number of Hamiltonians H_0 , H_j 's and control variables u_j . We shall assume in the following that all the Hamiltonians involved have zero trace. This is done without loss of generality because the introduction of the trace in the Hamiltonians only has the effect of introducing a phase factor in the evolution of the state which has no physical meaning.

The controllability of a finite dimensional quantum system (see, e.g., [4], [15], [17]) can be assessed by analyzing the (*dynamical*) *Lie algebra* \mathcal{L} generated by the Hamiltonians available for the evolution of the system. This is the smallest subalgebra of $su(n_{SA})$ containing $i\mathcal{F}$.² If $e^{\mathcal{L}}$ denotes the Lie group associated with \mathcal{L} , then the set of possible evolutions for the quantum system is dense in $e^{\mathcal{L}}$, and it is equal to $e^{\mathcal{L}}$ if $e^{\mathcal{L}}$ is compact.

DEFINITION 2.1. *A quantum system is said to be completely controllable if, for any special unitary transformation U_f , there exists a feasible evolution (i.e., a sequence of exponentials of the form $e^{-iH_u t}$ with $t \geq 0$ and H_u in \mathcal{F}) realizing that transformation (i.e., whose product is equal to U_f).*

THEOREM 1 (see [15], [17]). *A system $S + A$ is completely controllable if and only if $\mathcal{L} = su(n_{SA})$.*

In the case where \mathcal{L} is only a *proper* Lie subalgebra of $su(n_{SA})$, the knowledge of \mathcal{L} still gives information on the dynamics of the system. In particular, decompositions

¹ $\mathbf{1}_q$ ($\mathbf{0}_q$) denotes the identity (zero) operator on a Hilbert space of dimension q . We shall often omit this subscript when the dimension is obvious from the context. $\mathbf{1}_{r,s}$ denotes the square matrix of dimension $r + s$, $\begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & -\mathbf{1}_s \end{pmatrix}$.

²For a set or space \mathcal{F} of matrices, we shall often use the notation $i\mathcal{F}$ to indicate the set or space consisting of the elements in \mathcal{F} multiplied by the imaginary unit i . This allows us to go from Hermitian to skew-Hermitian matrices and vice versa.

of \mathcal{L} correspond to decompositions of the dynamics of the system, [5], [21], and there exists a fascinating interplay between symmetries in quantum dynamics and the structure of the dynamical Lie algebra \mathcal{L} [25].

In the indirect control setting, the Hilbert space associated to the system $S + A$ is the tensor product of the space associated with S and the space associated with A . In all operators expressed as tensor product in the following, the operator on the left acts on the Hilbert space associated with S while the operator on the right acts on the Hilbert space associated with A . In this setting, we make the following two assumptions on the dynamics of our model:

(A-a) The set \mathcal{F} contains at least one element with nonzero component on the space of operators

$$\text{span}\{S \otimes \sigma \mid S \in su(n_S), \sigma \in su(n_A)\}.$$

This is a natural assumption because it means that there exists an available Hamiltonian modeling the *interaction* between S and A . If that was not the case, then all the operators in \mathcal{F} would be of the form $F_S \otimes \mathbf{1}_{n_A}$ and of the form $\mathbf{1}_{n_S} \otimes F_A$, and systems S and A would evolve independently. (All elements in $e^{\mathcal{L}}$ would be of the form $U_S \otimes U_A$ (local transformations) with U_S unitary on the system S and U_A unitary on the system A .)

(A-b) The dynamical Lie algebra \mathcal{L} contains all matrices of the form $\mathbf{1}_{n_S} \otimes \sigma$ with $\sigma \in su(n_A)$. This fact means that we have *full unitary control* on the auxiliary system A . Whether this control is directly available in the experimental set-up or results from the back-action of the system S on A is irrelevant from a mathematical point of view.

We also recall that we have a standing minimality assumption on A in that n_A denotes the dimension of the part of A which is *fully coupled to S* and does not take into account possibly decoupled additional subsystems.

With initial condition $\rho_{TOT} = \rho_S \otimes \rho_A$, the set of available states for $S + A$ is (dense in)³

$$(4) \quad \mathcal{O} := \{U\rho_S \otimes \rho_A U^\dagger \mid U \in e^{\mathcal{L}}\}.$$

The set of possible values for ρ_S is obtained by taking the partial trace with respect to the system A of the elements in \mathcal{O} , i.e., it is the set of matrices

$$(5) \quad Tr_A(\mathcal{O}) := \{Tr_A(U\rho_S \otimes \rho_A U^\dagger) \mid U \in e^{\mathcal{L}}\}.$$

The topic of this paper is indirect controllability as described in the following definition.

DEFINITION 2.2. *A quantum system S is said to be indirectly controllable given ρ_A , initial state of the auxiliary system A , if for every $X \in SU(n_S)$, there exists a reachable evolution $U \in e^{\mathcal{L}}$ of the whole system $S + A$ such that*

$$(6) \quad Tr_A(U\rho_S \otimes \rho_A U^\dagger) = X\rho_S X^\dagger,$$

i.e., $X\rho_S X^\dagger \in Tr_A(\mathcal{O})$, in (4), (5), for every ρ_S , initial state of S . Equivalently, in terms of maps, the system S is indirectly controllable given ρ_A if, for every unitary X , there exists $U \in e^{\mathcal{L}}$ such that the map $\rho_S \rightarrow Tr_A(U\rho_S \otimes \rho_A U^\dagger)$ coincides with the map $\rho_S \rightarrow X\rho_S X^\dagger$.

³We shall neglect in the following this distinction and refer to the set \mathcal{O} as the set of available states for $S + A$. In fact all the Lie groups we will encounter will be compact so that equality holds.

Our goal is to give necessary and sufficient conditions for indirect controllability given ρ_A , in terms of the dynamical Lie algebra \mathcal{L} and ρ_A itself. The situation is different if $n_A \geq 3$ and if $n_A = 2$. Theorems 2 and 3 below are our main results.

THEOREM 2. *Assume $n_A \geq 3$, and let ρ_A be any initial state of the auxiliary system A . S is indirectly controllable given ρ_A if and only if $S + A$ is completely controllable, i.e., $\mathcal{L} = su(n_{SA})$.*

As a consequence of this result, for $n_A \geq 3$ indirect controllability does not depend on the initial state ρ_A of A .

In the case $n_A \geq n_S^2$, the condition of complete controllability gives much more than indirect controllability of S . An important result of quantum information theory (see, e.g., [19], the lecture notes [22, section 3.2] or the lecture notes [24, Theorem 2.5]) says that any trace preserving, completely positive linear map on ρ_S (a *quantum channel*) can be obtained as $\rho_S \rightarrow Tr_A(U\rho_S \otimes \rho_A U^\dagger)$ with an appropriate choice of a pure state ρ_A and choice of unitary U . Therefore, in this case, as the controlled dynamics can generate any unitary evolution on the system $S + A$, any completely positive map on S can be obtained, not just a unitary one.

In the case $n_A = 2$, the equivalence of Theorem 2 is false as shown in [1], [8]. In order to state the result in this case, we consider two subspaces of $su(n_S)$. We let

$$(7) \quad \begin{aligned} \mathcal{K} &= \{K \in su(n_S) \mid K \otimes \mathbf{1}_{n_A} \in \mathcal{L}\}, \\ \mathcal{P} &= \{P \in su(n_S) \mid \exists \sigma_1 \in su(n_A), \sigma_1 \neq 0, \text{ with } iP \otimes \sigma_1 \in \mathcal{L}\}. \end{aligned}$$

Notice that under assumption (A-b), \mathcal{P} contains at least $i\mathbf{1}_{n_S}$. Moreover, it follows from the *simplicity lemma* (Lemma 2.2 in [7]), under assumption (A-b), that if $iP \otimes \sigma_1 \in \mathcal{L}$, then $iP \otimes \sigma \in \mathcal{L}$ for every $\sigma \in su(n_A)$. Thus, in the definition of \mathcal{P} above, we may write \forall instead of \exists . It also follows from the *disintegration lemma* (Lemma 2.3 in [7]), again under assumption (A-b), that if an element in \mathcal{L} contains $iP \otimes \sigma$ as a summand,⁴ then $iP \otimes \sigma$ also belongs to \mathcal{L} . Therefore, \mathcal{L} is the (direct) sum of $\mathcal{K} \otimes \mathbf{1}_{n_A}$ and $i\mathcal{P} \otimes su(n_A)$, i.e.,

$$(8) \quad \mathcal{L} = \{\mathcal{K} \otimes \mathbf{1}_{n_A}\} + \{i\mathcal{P} \otimes su(n_A)\}.$$

We shall denote by \mathcal{L}_S the subspace (which is in fact a Lie algebra)

$$(9) \quad \mathcal{L}_S := \mathcal{K} + \mathcal{P}.$$

Notice that this definitions hold for any value of $n_A \geq 2$, and we shall, in fact, use it both for the case $n_A \geq 3$ and the case $n_A = 2$. In the case $n_A = 2$, the space \mathcal{L}_S is used in the statement of the next theorem to give the characterization of indirect controllability.

THEOREM 3. *Assume $n_A = 2$. System S is indirectly controllable given ρ_A if and only if one of the following two situations occurs:*

1. $\mathcal{L} = su(n_{SA})$, i.e., the system $S + A$ is completely controllable.
2. ρ_A is a pure state and $\mathcal{L}_S = u(n_S)$.

Example 2.3. Consider a Hamiltonian for two spin $\frac{1}{2}$ particles, S and A , interacting via Ising interaction. We assume a constant electromagnetic field on the spin S and full (time varying electromagnetic) control on A . Such a Hamiltonian may be given by

$$(10) \quad H_u = J\sigma_x \otimes \sigma_x + i\mathbf{1}_2 \otimes \sigma_x u_x(t) + i\mathbf{1}_2 \otimes \sigma_y u_y(t) + \omega_z i\sigma_z \otimes \mathbf{1}_2.$$

⁴Here, all the σ 's in the sum are assumed to be linearly independent.

Here, $\sigma_{x,y,z}$ are the Pauli matrices defined in (15) below, J is the coupling constant, u_x and u_y are the components of the (control) electromagnetic field in the x and y direction, and ω_z is the Larmor frequency. By setting $(u_x, u_y) = (0, 0)$, then $(u_x, u_y) = (1, 0)$, and then $(u_x, u_y) = (0, 1)$, we find that the dynamical Lie algebra \mathcal{L} contains the matrices $\{iJ\sigma_x \otimes \sigma_x - \omega_z \sigma_z \otimes \mathbf{1}_2, \mathbf{1}_2 \otimes \sigma_x, \mathbf{1} \otimes \sigma_y\}$. Using the commutation and anticommutation relations for Pauli matrices (see (16), (17) below),⁵ the Lie algebra generated by these matrices is given by

$$(11) \quad \mathcal{L} = \text{span} \{i\{\sigma_x, \sigma_y\} \otimes \{\sigma_x, \sigma_y, \sigma_z\}, \mathbf{1}_2 \otimes \{\sigma_x, \sigma_y, \sigma_z\}, \sigma_z \otimes \mathbf{1}_2\}.$$

This is in fact the dynamical Lie algebra associated with the system since iH_u in (10) for every $u \in \mathbb{R}$ that belongs to \mathcal{L} . A simple dimension count shows that $\mathcal{L} \neq su(4)$.⁶ Therefore, the system is *not* completely controllable. However, the subspaces \mathcal{K} and \mathcal{P} of Theorem 3 are given by $\mathcal{K} := \text{span}\{\sigma_z\}$ and $\mathcal{P} := \text{span}\{i\mathbf{1}_2, \sigma_x, \sigma_y\}$. Therefore, (see (9)) condition 2 of Theorem 3 is verified if the initial state ρ_A of A is a pure state. In this case, system S is indirectly controllable. A more complete analysis of this example along with a constructive algorithm for indirect control is presented in [1]. Some additional remarks will be given in section 7.

Remark 2.4. The fact that the condition of indirect controllability is weaker in the case $n_A = 2$ than in the case $n_A \geq 3$ seems counterintuitive since, in the latter case, we have a larger auxiliary system to use in the control experiment. We could, in principle, use a subsystem of dimension 2 of A and limited dynamics to achieve the goal of indirect control. We recall, however, that we have assumed *full interaction* between S and A , and therefore there will be a back action of S onto A which cannot be neglected. Therefore, A has to be considered necessarily in its entirety. Moreover, we shall see in the proof of Theorem 2 that the rank condition on \mathcal{L}_S of part 2 of Theorem 3 is sufficient for indirect controllability in the case $n_A \geq 3$ as well. However, in this case, since we have assumed that A is fully controllable (assumption (A-b)), the set of generators of the Lie algebra is rich enough to generate the whole $su(n_S n_A)$, something that does not necessarily happen in the “*smaller*” case $n_A = 2$.

The remainder of the paper is devoted to proving Theorems 2 and 3.

3. Preliminary results. The following lemma, which was proved in [8] (cf. Theorem 1 in that paper), is going to be a basic tool to prove the necessity of the conditions for indirect controllability. Let $\rho_S \otimes \rho_A$ be the initial condition of the system $S + A$, and, given the dynamical Lie algebra \mathcal{L} , consider the space⁷

$$(12) \quad \mathcal{V} := \bigoplus_{k=0}^{\infty} ad_{\mathcal{L}}^k \text{span}\{i\rho_S \otimes \rho_A\}.$$

We have the following.

LEMMA 3.1. *Given $\rho_S \neq \frac{1}{n_S} \mathbf{1}_{n_S}$ and ρ_A , assume that for every $X \in SU(n_S)$, there exists a $U \in e^{\mathcal{L}}$ such that*

$$(13) \quad Tr_A (U\rho_S \otimes \rho_A U^\dagger) = X\rho_S X^\dagger.$$

⁵See also (18), (19).

⁶This is the case since $\dim(\mathcal{L}) = 10$ and $\dim(su(4)) = 15$.

⁷Recall that, for a Lie algebra \mathcal{L} and a vector space of matrices \mathcal{M} , the space $ad_{\mathcal{L}}^k \mathcal{M}$ is defined recursively as $ad_{\mathcal{L}}^0 \mathcal{M} = \mathcal{M}$, $ad_{\mathcal{L}}^{k+1} = ad_{\mathcal{L}}(ad_{\mathcal{L}}^k \mathcal{M})$, where $ad_{\mathcal{L}} \mathcal{M}$ is the span of all matrices of the form $[L, M]$ with $L \in \mathcal{L}$ and $M \in \mathcal{M}$.

Then,

$$(14) \quad \text{Tr}_A(\mathcal{V}) = u(n_S).$$

Notice that this necessary condition is given for “nonuniform” indirect controllability, which is a weaker property than the one defined in Definition 2.2. This means that the transformation U in (13) could, in principle, depend on ρ_S . The property (14) is therefore also necessary for indirect controllability as in Definition 2.2. It is known that this condition is, in general, not sufficient [6].

We now recall the definition of the Pauli matrices $\sigma_{x,y,z}$ in quantum mechanics,

$$(15) \quad \sigma_x := \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_y := \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z := \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

which satisfy the commutation relations,

$$(16) \quad [\sigma_x, \sigma_y] = \sigma_z, \quad [\sigma_y, \sigma_z] = \sigma_x, \quad [\sigma_z, \sigma_x] = \sigma_y,$$

and anticommutation relations,

$$(17) \quad \{\sigma_j, \sigma_k\} = -\frac{1}{2} \delta_{j,k} \mathbf{1}_2,$$

$j, k = x, y, z$.⁸

Much of the proof of our theorems will be based on understanding the nature of the subspaces \mathcal{K} and \mathcal{P} defined in (7). These spaces satisfy the commutation relations of a Riemannian symmetric space [11], i.e.,

$$(20) \quad [\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}.$$

The first two of these relations are obvious from the definition, while the third one is obtained by calculating, given P_1 and P_2 in \mathcal{P} ,

$$(21) \quad -\frac{1}{2} \sum_{j=1}^{n_A} [iP_1 \otimes \Sigma_j, iP_2 \otimes \Sigma_j] = [P_1, P_2] \otimes \mathbf{1}_{n_A} \in \mathcal{L}.$$

Here, $\Sigma_j, j = 1, \dots, n_A$, denotes the matrix in $su(n_A)$ with i and $-i$ in position j and $j + 1 \pmod{n_A}$, on the main diagonal, respectively, and zeros everywhere else.

We also have the anticommutation relation

$$(22) \quad i\{\mathcal{P}, \mathcal{P}\} \subseteq \mathcal{P}.$$

In order to see this, consider σ_x and σ_y the standard Pauli matrices in $su(2)$ which satisfy the commutation and anticommutation relations (16), (17). In $su(n_A)$, with

⁸In the following, we shall be interested in commutators and anticommutators of matrices that are tensor products of two matrices. The following relations will be repeatedly used without necessarily being explicitly mentioned:

$$(18) \quad [A \otimes B, C \otimes D] = \frac{1}{2} ([A, C] \otimes \{B, D\} + \{A, C\} \otimes [B, D]),$$

$$(19) \quad \{A \otimes B, C \otimes D\} = \frac{1}{2} (\{A, C\} \otimes \{B, D\} + [A, C] \otimes [B, D]).$$

$n_A \geq 3$, we denote in the following calculation (23), with some abuse in notation, by $\sigma_{x,y,z}$, matrices which have the corresponding Pauli matrix in the diagonal block corresponding to the first two rows and columns and zeros everywhere else. By extending naturally the commutation and anticommutation relations (16), (17), we have for any P_1 and P_2 in \mathcal{P}

$$(23) \quad [iP_1 \otimes \sigma_x, iP_2 \otimes \sigma_y] = \frac{i}{2}i\{P_1, P_2\} \otimes \sigma_z \in \mathcal{L}.$$

From this, (22) follows by definition.⁹

The following two lemmas are the first step to understand the structure of \mathcal{P} .

LEMMA 3.2. *Let \mathcal{A} be a maximal Abelian subalgebra of \mathcal{L}_S with $\mathcal{A} \subseteq \mathcal{P}$.¹⁰ After a possible change of coordinates on \mathcal{L}_S ,¹¹ a basis of \mathcal{A} is given by*

$$(24) \quad \begin{aligned} D_1 &:= \text{diag}\{i\mathbf{1}_{n_1}, \mathbf{0}_{n_2}, \dots, \mathbf{0}_{n_l}\}, \dots, \\ D_l &:= \text{diag}\{\mathbf{0}_{n_1}, \mathbf{0}_{n_2}, \dots, i\mathbf{1}_{n_l}\} \end{aligned}$$

for some integers n_1, \dots, n_l .

The proof is given in Appendix A.

To further investigate the subspace $\mathcal{P} \subseteq \mathcal{L}_S$, we introduce a partition of the row and the column indexes and a block structure in the matrices in \mathcal{P} according to (24). Each index $j = 1, \dots, l$, corresponds to a set of indices of the rows and the columns of matrices in \mathcal{P} , the set being of cardinality n_j . Let us introduce an auxiliary undirected graph $G_{\mathcal{P}}$ whose nodes correspond to the indices $\{1, 2, \dots, l\}$. There is an edge between the node j and the node k , ($j \neq k$) if and only if there is a matrix in \mathcal{P} such that the (j, k) th block (and therefore the (k, j) th block, since the matrix is skew-Hermitian) is different from zero. We have the following lemma on the structure of \mathcal{P} .

LEMMA 3.3. *Let $G_{\mathcal{P}}$ be the indirect graph defined above. Then we have the following:*

1. *If $G_{\mathcal{P}}$ is not connected, there exists a change of coordinates to put all matrices of \mathcal{P} in block diagonal form with the r th block, corresponding to the indices of the r th connected component of $G_{\mathcal{P}}$, I_r , having dimension $\sum_{j \in I_r} n_j$.*
2. *If $G_{\mathcal{P}}$ is connected, then $n_1 = n_2 = \dots = n_l$.*

The proof is given in Appendix A.

3.1. Some results on Lie subalgebras of $u(n)$ and symmetric spaces, with application to \mathcal{L}_S . In the attempt to understand the nature of \mathcal{L}_S , we shall use some results about general Lie algebras and, in particular, Lie subalgebras of $u(n)$. We recall here these results, refer to standard texts on Lie algebras, Lie groups, and symmetric spaces such as [11] for further details, and report (in Appendix A) some proofs that we were not able to find in the literature.

Given a Lie algebra \mathcal{L} , a *representation* of \mathcal{L} is a homomorphism $\Phi : \mathcal{L} \rightarrow \text{End}(\mathcal{V})$, i.e., a linear map from \mathcal{L} to the Lie algebra of endomorphisms of a vector space \mathcal{V} , satisfying for any $A, B \in \mathcal{L}$, $\Phi([A, B]) = [\Phi(A), \Phi(B)]$. In this equality, with some abuse of notation, the commutator $[\cdot, \cdot]$ on the left-hand side is the commutator

⁹See the comment on the simplicity lemma and the definition of \mathcal{P} after formula (7).

¹⁰The word *maximal* means that it is not a proper subalgebra of any Abelian subalgebra which is also contained in \mathcal{P} .

¹¹By a change of coordinates, we mean a transformation $\mathcal{L}_S \rightarrow T\mathcal{L}_S T^\dagger$ with $T \in U(n_S)$. Such a transformation does not affect the properties of indirect controllability of system S .

in the Lie algebra \mathcal{L} , while the commutator on the right-hand side is the standard matrix commutator, i.e., $[A, B] := AB - BA$. A particular representation is the *adjoint representation*, $A \rightarrow ad_A$, where the space \mathcal{V} is \mathcal{L} itself and $ad_A B := [A, B]$. The *Killing form* $\langle \cdot, \cdot \rangle_K$ on \mathcal{L} is defined as

$$(25) \quad \langle A, B \rangle_K = Tr(ad_A ad_B).$$

This form is *bilinear* and *symmetric* (i.e., $\langle A, B \rangle_K = \langle B, A \rangle_K$) as well as *invariant*, i.e.,

$$(26) \quad \langle [A, B], C \rangle_K = \langle [B, C], A \rangle_K.$$

Moreover, it is invariant under *automorphisms* θ of the Lie algebra, i.e., one to one and onto homomorphism of the Lie algebra to itself. This invariance property means that $\langle \theta(A), \theta(B) \rangle_K = \langle A, B \rangle_K$.

A Lie algebra \mathcal{L} is called *simple* if it has no ideals except the trivial ones, i.e., \mathcal{L} and zero, and its dimension is at least two. It is called *semisimple* if it is the direct sum of simple ideals.¹² A Lie algebra is called *reductive* if it is the direct sum of a semisimple Lie algebra and an Abelian Lie algebra. Subalgebras of $u(n)$ are always reductive. The Killing form is a very important tool in the analysis of Lie algebras. *Cartan's criterion* states that a Lie algebra \mathcal{L} is semisimple if and only if the corresponding Killing form is nondegenerate.¹³ Another equivalent condition of semisimplicity is that $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$. For a semisimple Lie subalgebra of $u(n)$, \mathcal{L} , the Killing form is *negative definite* and the corresponding Lie group, $e^{\mathcal{L}}$, is compact. Moreover, see, e.g., [14, (II 5.1)], if \mathcal{I} is an ideal of \mathcal{L} , the Killing form of \mathcal{I} is equal to the restriction to $\mathcal{I} \times \mathcal{I}$ of the Killing form on \mathcal{L} .

Let us now consider again the subspace \mathcal{L}_S of $u(n_S)$ defined in (9). From the commutation relations (20), it follows that \mathcal{L}_S is a Lie algebra and, in fact, a Lie subalgebra of $u(n_S)$, and therefore it is reductive. The following fact will be useful. (See Appendix A for the proof.)

LEMMA 3.4. *Assume $\mathcal{K} \cap \mathcal{P} = \{0\}$. The subalgebra \mathcal{K} of \mathcal{L}_S can be written as*

$$(27) \quad \mathcal{K} = [\mathcal{P}, \mathcal{P}] + \mathcal{R},$$

where \mathcal{R} commutes with \mathcal{P} and it is an ideal in \mathcal{K} (and therefore in \mathcal{L}_S).

The next corollary is a consequence of Lemma 3.4.

COROLLARY 3.5. *Assume that $su(n)$ has a decomposition $su(n) = \mathcal{K} + \mathcal{P}$ with $\mathcal{K} \cap \mathcal{P} = \{0\}$ and satisfying conditions (20). Then $[\mathcal{P}, \mathcal{P}] = \mathcal{K}$. Furthermore, $[\mathcal{K}, \mathcal{P}] = \mathcal{P}$.*

Proof. Since $su(n)$ is a simple Lie algebra, it does not have any nontrivial ideal, so in (27), the space \mathcal{R} must be zero, which implies $[\mathcal{P}, \mathcal{P}] = \mathcal{K}$. The fact that $[\mathcal{K}, \mathcal{P}] = \mathcal{P}$ can be seen as follows. Assume that $\hat{\mathcal{P}} := [\mathcal{K}, \mathcal{P}] \subsetneq \mathcal{P}$ and define $\hat{\mathcal{L}} := \hat{\mathcal{P}} + \mathcal{K}$, which is a proper subspace of $su(n)$. We have

$$(28) \quad [su(n), \hat{\mathcal{L}}] = [\mathcal{K} + \mathcal{P}, \mathcal{K} + \hat{\mathcal{P}}] \subseteq [\mathcal{K}, \mathcal{K}] + [\mathcal{K}, \hat{\mathcal{P}}] + [\mathcal{P}, \mathcal{K}] + [\mathcal{P}, \hat{\mathcal{P}}] \subseteq \hat{\mathcal{L}},$$

and it follows that $\hat{\mathcal{L}}$ is an ideal in $su(n)$ which contradicts the fact that $su(n)$ is a simple Lie algebra. \square

¹²That is, $\mathcal{L} = \mathcal{S}_1 + \mathcal{S}_2 + \dots + \mathcal{S}_m$ with ideals \mathcal{S}_j 's with $[\mathcal{S}_j, \mathcal{S}_k] = 0$ and $\mathcal{S}_j \cap \mathcal{S}_k = \{0\}$ if $j \neq k$.

¹³This means that the only $X \in \mathcal{L}$ such that $\langle X, Y \rangle_K = 0$ for every Y is $X = 0$.

Decompositions $su(n) = \mathcal{K} + \mathcal{P}$ with $\mathcal{K} \cap \mathcal{P} = \{0\}$ with (20), are also called *Cartan decompositions* of $su(n)$ and correspond to symmetric spaces of the corresponding Lie group $SU(n)$ [11]. According to Cartan classification, modulo a change of coordinates, there are only three types of such decompositions, which are denoted by **AI**, **AII**, and **AIII**. We shall use in the following the decompositions **AI** and a special case of decomposition **AIII**. In particular for the decomposition **AI**, \mathcal{K} is the space of (skew-Hermitian, zero trace) real matrices, $\mathcal{R}e$, and \mathcal{P} is the space of (skew-Hermitian, zero trace) purely imaginary matrices, $\mathcal{I}m$. Therefore, we write

$$(29) \quad su(n) = \mathcal{R}e + \mathcal{I}m.$$

If we take $\mathcal{R}e$ as \mathcal{K} and $\mathcal{I}m$ as \mathcal{P} , conditions (20) are verified. In the **AIII** Cartan decomposition, we collect two groups of row and column indices and decompose the matrices in $su(n)$ in terms of matrices that are block diagonal with respect to this decomposition, $\mathcal{D}i$, and antidiagonal with respect with respect to this decomposition, $\mathcal{A}n$. So that we have

$$(30) \quad su(n) = \mathcal{D}i + \mathcal{A}n.$$

If we take $\mathcal{D}i$ as \mathcal{K} and $\mathcal{A}n$ as \mathcal{P} , conditions (20) are verified. In this paper, we shall use the special case where the first group of row and column indexes contains only the first row and column and the other group contains the remaining indexes. By combining the two above Cartan decompositions, we can construct another one, by defining

$$(31) \quad \mathcal{L}_1 := (\mathcal{R}e \cap \mathcal{A}n) + (\mathcal{I}m \cap \mathcal{D}i)$$

and

$$(32) \quad \mathcal{L}_2 := (\mathcal{R}e \cap \mathcal{D}i) + (\mathcal{I}m \cap \mathcal{A}n).$$

It is easily verified that $su(n) = \mathcal{L}_2 + \mathcal{L}_1$ with \mathcal{L}_1 and \mathcal{L}_2 satisfying the relations (20) with $\mathcal{K} = \mathcal{L}_2$ and $\mathcal{P} = \mathcal{L}_1$, i.e.,

$$(33) \quad [\mathcal{L}_2, \mathcal{L}_2] \subseteq \mathcal{L}_2, \quad [\mathcal{L}_2, \mathcal{L}_1] \subseteq \mathcal{L}_1, \quad [\mathcal{L}_1, \mathcal{L}_1] \subseteq \mathcal{L}_2.$$

One can also readily verify the following anticommutation relations:

$$(34) \quad i\{\mathcal{L}_1, \mathcal{L}_1\} \subseteq \mathcal{L}_1 + \text{span}\{i\mathbf{1}_n\}, \quad i\{\mathcal{L}_2, \mathcal{L}_2\} \subseteq \mathcal{L}_1 + \text{span}\{i\mathbf{1}_n\}, \quad i\{\mathcal{L}_1, \mathcal{L}_2\} \subseteq \mathcal{L}_2.$$

If \mathcal{L} is a semisimple Lie subalgebra of $u(n)$ with a Cartan decomposition $\mathcal{L} = \mathcal{K} + \mathcal{P}$, $\mathcal{K} \cap \mathcal{P} = \{0\}$ with \mathcal{K} and \mathcal{P} satisfying (20), Cartan's theorem (cf. [11]) provides a way to parametrize the corresponding Lie group $e^{\mathcal{L}}$. In particular, if $\mathcal{A} \subseteq \mathcal{P}$ is a maximal Abelian subalgebra of \mathcal{P} , every element $Y \in e^{\mathcal{L}}$ can be written as $Y = K_1 e^{\tilde{A}} K_2$ with $K_1, K_2 \in e^{\mathcal{K}}$ and $\tilde{A} \in \mathcal{A}$. We shall use this representation of elements in $e^{\mathcal{L}}$ several times in the following.

3.2. Normal vector spaces. When analyzing the structure of the Lie algebra \mathcal{L}_S and in particular using the property (22), we will have to consider subspaces of the algebra of $n \times n$ complex matrices which satisfy a “normalization” condition. We will say that a vector space \mathcal{N}_n of $n \times n$ complex matrices over the field of real numbers is *normal* if for every pair of matrices A and B in \mathcal{N}_n , it holds that

$$(35) \quad A^\dagger B + B^\dagger A = BA^\dagger + AB^\dagger = \alpha \mathbf{1}_n$$

for some real number α . In particular, any matrix $A \in \mathcal{N}_n$ is normal since

$$(36) \quad AA^\dagger = A^\dagger A = \alpha \mathbf{1}_n$$

for some real α .

Normal vector spaces can be mapped isomorphically one to the other by *doubly unitary conjugacy transformations* (DUCT) determined by a pair of unitary matrices \mathbf{U} and \mathbf{V} and defined as $A \in \mathcal{N}_n \rightarrow \mathbf{UAV}$. Notice in particular that a DUCT transformation does not modify the defining relation (35). A normal vector space can be defined recursively up to a DUCT transformation, as described in the following Proposition.

PROPOSITION 3.6. *Modulo a DUCT transformation, a normal vector space \mathcal{N}_n of $n \times n$ matrices is spanned by the following matrices:*

1. $\mathbf{0}_n$, or
2. $\mathbf{1}_n$, or
3. $\mathbf{1}_n, i\mathbf{1}_{r,s}$ with $r, s \geq 0$ and $r + s = n$, or
4. $\mathbf{1}_n, i\mathbf{1}_{r,s}$ with $r = s = \frac{n}{2}$ and matrices of the form $C := \begin{pmatrix} 0 & C_{1,2} \\ -C_{1,2}^\dagger & 0 \end{pmatrix}$, where the matrices $C_{1,2}$ span a normal vector space of $\frac{n}{2} \times \frac{n}{2}$ matrices, $\mathcal{N}_{\frac{n}{2}}$.

The proof can be found in Appendix A.

Using DUCT transformations, it is always possible to put the matrices of a basis of \mathcal{N}_n in a canonical form in which all matrices in \mathcal{N}_n and the following vector spaces $\mathcal{N}_j, j = \frac{n}{2}, \frac{n}{4}, \dots$, are the identity $\mathbf{1}_j, j = n, \frac{n}{2}, \dots$ or the matrix $i\mathbf{1}_{r,s}$, with $r_j + s_j = j$, according to the algorithm described in the proof of the proposition.

In the following, we shall also be interested in cases where the normal vector space of matrices \mathcal{N}_n is not only a vector space but also a Lie algebra when equipped with the standard matrix Lie bracket ($[A, B] := AB - BA$). We first notice that this property is *not* invariant anymore under DUCT transformation. However, it will be enough for us to consider the case where the basis of \mathcal{N}_n is in the canonical form described in Proposition 3.6.¹⁴ In this case, there is only a finite number of possible cases, as we shall see in the following lemma.

LEMMA 3.7. *Consider a normal vector space \mathcal{N}_n with a basis in canonical form. If \mathcal{N}_n is a Lie algebra, there are only the following possibilities:*

1. $\mathcal{N}_n = \{\mathbf{0}\}$.
2. $\mathcal{N}_n = \text{span}\{\mathbf{1}_n\}$.
3. $\mathcal{N}_n = \text{span}\{\mathbf{1}_n, i\mathbf{1}_{r,s}\}$.
- 4.

$$(37) \quad \mathcal{N}_n = \text{span} \left\{ \mathbf{1}_n, i\mathbf{1}_{\frac{n}{2}, \frac{n}{2}}, \begin{pmatrix} 0 & \mathbf{1}_{\frac{n}{2}} \\ -\mathbf{1}_{\frac{n}{2}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & i\mathbf{1}_{\frac{n}{2}} \\ i\mathbf{1}_{\frac{n}{2}} & 0 \end{pmatrix} \right\}.$$

Proof. Cases 1–3 correspond to the first three cases in the construction of Proposition 3.6 in Appendix A. The intermediate case between case 3 and 4 is not possible because the Lie bracket between the second and third term of the right-hand side of (37) gives the fourth term, which therefore has to belong to \mathcal{N}_n . However, as noted in the proof of Proposition 3.6, the presence of this matrix implies that no other linearly independent matrix can be found. Therefore, these four cases are the only admissible ones. \square

¹⁴The normal spaces \mathcal{N}_n that we will consider are a factor in a tensor product space and are obtained after a change of coordinates on this space. We shall be able to assume that this change of coordinates puts \mathcal{N}_n in canonical form.

Remark 3.8. In the last case of the above list, we can write the basis of \mathcal{N}_n in terms of the Pauli matrices so that

$$(38) \quad \mathcal{N}_n = \text{span}\{\mathbf{1}, \sigma_z \otimes \mathbf{1}_{\frac{n}{2}}, \sigma_y \otimes \mathbf{1}_{\frac{n}{2}} \sigma_x \otimes \mathbf{1}_{\frac{n}{2}}\}.$$

4. Proof of Theorem 2. In Theorem 2, we consider the case $n_A \geq 3$. However, the first lemma holds for any value of n_A .

LEMMA 4.1. *Assume that \mathcal{P} in (9) is Abelian. Then system S is not indirectly controllable (independently of ρ_A).*

Proof. If \mathcal{P} is Abelian, we can assume that all the matrices in \mathcal{P} are linear combinations of the elements in the basis (24) of Lemma 3.2 and we can take the basis of \mathcal{P} as in (24). Partition any $K \in \mathcal{K}$ according to the partition in the basis of \mathcal{P} . Since $[\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}$, every matrix in $[\mathcal{K}, \mathcal{P}]$ must be a linear combination of the elements in (24). From this fact, it is easy to see that the matrices $K \in \mathcal{K}$ must have the following block diagonal structure:

$$(39) \quad K = \begin{pmatrix} K_{1,1} & 0 & 0 & 0 \\ 0 & K_{2,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & K_{l,l} \end{pmatrix}$$

with $K_{j,j} \in su(n_j)$. Matrices in the Lie algebra \mathcal{L}_S also have this block diagonal structure,¹⁵ and matrices in \mathcal{L} also have a block diagonal structure induced by this structure. Thus, a matrix $U \in e^{\mathcal{L}}$ is of the form

$$(40) \quad U = \begin{pmatrix} U_1 & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & U_l \end{pmatrix},$$

where the blocks U_j , $j = 1, \dots, l$ have dimension $n_j n_A$.¹⁶ Choose any initial state $\rho_S \otimes \rho_A$ where ρ_S has the same block structure as in (39). Then for any $U \in e^{\mathcal{L}}$, the matrix $\text{Tr}_A(U\rho_S \otimes \rho_A U^\dagger)$ will have the same block diagonal structure as in (39). Since not all the matrices unitarily equivalent to ρ_S have this diagonal structure, the model is not indirectly controllable. \square

LEMMA 4.2. *Assume that $n_A \geq 3$ and let ρ_A be any initial state of the auxiliary system A . If S is indirectly controllable given ρ_A , then $\mathcal{L}_S = u(n_S)$*

Proof. Assume that S is not indirectly controllable and assume by contradiction that $\mathcal{L}_S \neq u(n_S)$. If \mathcal{P} is Abelian, from the previous lemma we already know that indirect controllability is not verified. Therefore, we can assume that \mathcal{P} is not Abelian. We prove the lemma in two steps:

(a) If \mathcal{P} is not Abelian, then $\mathcal{K} \cap \mathcal{P} \neq \emptyset$.

(b) If $\mathcal{K} \cap \mathcal{P} \neq \{0\}$ and $\mathcal{L}_S \neq u(n_S)$, then indirect controllability is not verified.

For step (a), assume that \mathcal{P} is not Abelian and let $P_1, P_2 \in \mathcal{P}$ such that $[P_1, P_2] = K \neq 0$ with $K \in \mathcal{K}$. Since $n_A \geq 3$, there exist $\sigma_1, \sigma_2 \in su(n_A)$ such that

$$(41) \quad [\sigma_1, \sigma_2] = 0, \quad \{\sigma_1, \sigma_2\} = \mathbf{1} + i\hat{\sigma},$$

¹⁵Notice that this structure assumes a particular system of coordinates, but the transformation to get in these coordinates is a local transformation acting on S only, so it does not affect the indirect controllability properties of system S .

¹⁶Recall that $l \geq 2$ since \mathcal{P} is not the span of multiples of the identity because of assumption (A-a). It has dimension at least 2, and it contains multiples of the identity.

with $\hat{\sigma} \in su(n_A)$, different from zero. We have

$$(42) \quad \begin{aligned} [iP_1 \otimes \sigma_1, iP_2 \otimes \sigma_2] &= -1/2[P_1, P_2] \otimes (\mathbf{1} + i\hat{\sigma}) \\ &= -1/2K \otimes (\mathbf{1} + i\hat{\sigma}) \in \mathcal{L}. \end{aligned}$$

Since $K \otimes \mathbf{1} \in \mathcal{L}$, it follows that $iK \otimes \hat{\sigma} \in \mathcal{L}$. Thus, $K \in \mathcal{K} \cap \mathcal{P}$, which shows that $\mathcal{K} \cap \mathcal{P} \neq \{0\}$. Now we show part (b). It will follow from the proof that part (b) holds for any value of n_A . If $\mathcal{K} \cap \mathcal{P} \neq \{0\}$, then given any matrix $B \neq 0$ such that $B \in \mathcal{K} \cap \mathcal{P}$, we choose as initial state $\rho_S = \frac{1}{n_S}\mathbf{1} + \alpha iB$ with $\alpha \neq 0$ and sufficiently small so that ρ_S is an admissible density matrix.¹⁷ Given any $\rho_A = \frac{1}{n_A}\mathbf{1} + i\sigma$, for $\sigma \in su(n_A)$, we have that $i\rho_S \otimes \rho_A$ belongs to $\tilde{\mathcal{L}} = \text{span}\{i\mathbf{1} \otimes \mathbf{1}\} + \mathcal{L}$, which is invariant under \mathcal{L} . Therefore, \mathcal{V} defined in (12) of Lemma 3.1 is such that $\mathcal{V} \subseteq \tilde{\mathcal{L}}$, and we have $Tr_A(\mathcal{V}) \subseteq Tr_A(\tilde{\mathcal{L}}) = \mathcal{K} + \text{span}\{i\mathbf{1}\} \subseteq \mathcal{L}_S \subsetneq u(n_S)$, which contradicts Lemma 3.1. \square

The proof of Theorem 2 is now a consequence of the previous two lemmas.

Proof of the theorem. We only need to prove that indirect controllability (for a fixed ρ_A) implies $\mathcal{L} = su(n_S n_A)$. The converse implication is obvious. Assume indirect controllability. From Lemmas 4.1 and 4.2, we know that $\mathcal{L}_S = u(n_S)$. Let $\tilde{\mathcal{P}}$ be the subspace of matrices in \mathcal{P} with zero trace. We will establish that all the matrices of the type $iK \otimes \sigma$ and $P \otimes \mathbf{1}$ with $K \in \mathcal{K}$, $P \in \tilde{\mathcal{P}}$, and $\sigma \in su(n_A)$ are in \mathcal{L} . This implies that $\mathcal{L} = su(n_S n_A)$. Notice that since $\tilde{\mathcal{L}}_S := \mathcal{K} + \tilde{\mathcal{P}} = su(n_S)$, from Corollary 3.5 (or from Lemma 4.1) it follows that $\tilde{\mathcal{P}}$ cannot be Abelian.

Since $n_A \geq 3$, we may take $\sigma_1, \sigma_2 \in su(n_A)$ such that (41) is satisfied. Then, as computed in (42), given $P_1, P_2 \in \mathcal{P}$ we have

$$(43) \quad [iP_1 \otimes \sigma_1, iP_2 \otimes \sigma_2] = -1/2K \otimes (\mathbf{1} + i\hat{\sigma})$$

for $K \in \mathcal{K}$. We can assume $K \neq 0$ since $\tilde{\mathcal{P}}$ cannot be Abelian. In fact, since $[\tilde{\mathcal{P}}, \tilde{\mathcal{P}}] = \mathcal{K}$ from Corollary 3.5, we have

$$iK \otimes \hat{\sigma} \in \mathcal{L} \text{ for all } K \in \mathcal{K}.$$

Since $\mathbf{1} \otimes su(n_A) \in \mathcal{L}$, from the previous equation we get that¹⁸

$$(44) \quad iK \otimes \sigma \in \mathcal{L} \text{ for all } K \in \mathcal{K}, \text{ and } \sigma \in su(n_A).$$

Now calculate

$$(45) \quad -\frac{1}{2} \sum_{j=1}^{n_A} [iK \otimes \sigma_j, iP \otimes \sigma_j] = [K, P] \otimes \mathbf{1} \in \mathcal{L}.$$

Here, $\sigma_j, j = 1, \dots, n_A$, denotes the matrix in $su(n_A)$ with i and $-i$ in position j and $j + 1 \pmod{n_A}$, on the main diagonal, respectively, and zeros everywhere else, while K and P are general matrices in \mathcal{K} and $\tilde{\mathcal{P}}$. Since $[\mathcal{K}, \tilde{\mathcal{P}}] = \tilde{\mathcal{P}}$ from Corollary 3.5, we have

$$(46) \quad P \otimes \mathbf{1} \in \mathcal{L} \text{ for all } P \in \tilde{\mathcal{P}}.$$

From (44) and (46), the statement follows. \square

¹⁷Note that B cannot be a multiple of the identity because we have assumed at the beginning that all Hamiltonians involved in the dynamics have zero trace, that is, \mathcal{L} is a subalgebra of $su(n_S n_A)$.

¹⁸This is from the simplicity lemma in [7].

5. Proof of Theorem 3, Part I. From this point on, $n_A = 2$. In the next subsection, we prove the sufficiency of conditions 1 and 2 of Theorem 3. In fact, condition 1 being obviously sufficient, we need to treat only the sufficiency of condition 2. Then the proof of necessity is divided in two parts: one in subsection 5.2 and one in section 6. Much of the proof of necessity is carried out by looking at the various possibilities for the Lie algebra \mathcal{L}_S . From this analysis, there are several special cases to be treated. Some special cases are presented in Appendix B.

5.1. $\mathcal{L}_S = u(n_S)$ and ρ_A pure imply indirect controllability of S .

Proof. The argument is a generalization to $n_S \geq 2$ of the one given in [8]. Assume that we want to steer any ρ_S to the unitarily equivalent $X\rho_S X^\dagger$ with $X \in SU(n_S)$, i.e., we need to find a reachable evolution $U \in e^{\mathcal{L}}$, such that

$$(47) \quad \text{Tr}_A(U\rho_S \otimes \rho_A U^\dagger) = X\rho_S X^\dagger$$

for every ρ_S . Since $\mathcal{L}_S = u(n_S)$, if we define $\tilde{\mathcal{P}}$ the subspace of \mathcal{P} of matrices with zero trace, we have from (9) $su(n_S) = \mathcal{K} + \tilde{\mathcal{P}}$, where \mathcal{K} and $\tilde{\mathcal{P}}$ provide a Cartan decomposition of $su(n_S)$ (see (20) with $\tilde{\mathcal{P}}$ replacing \mathcal{P}).¹⁹ Thus, we can write X as

$$(48) \quad X = K_1 e^{\tilde{A}} K_2,$$

where $K_{1,2} \in e^{\mathcal{K}}$ and $\tilde{A} \in \mathcal{A}$, where \mathcal{A} is a maximal Abelian subalgebra (Cartan subalgebra) in $\tilde{\mathcal{P}}$ [11] (cf. the discussion at the end of subsection 3.1).

Let $\bar{\sigma} := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. The Lie group $e^{\mathcal{L}}$ contains all elements of the form $K \otimes \mathbf{1}$, $\mathbf{1} \otimes B$, $e^{it\tilde{A} \otimes \bar{\sigma}}$ with $K \in e^{\mathcal{K}}$, $B \in SU(2)$, and $\tilde{A} \in \mathcal{A}$. Since ρ_A is a pure state, there exists a unitary T such that $T\rho_A T^\dagger = E_1$, where E_1 is the 2×2 matrix with 1 in the (1,1) position and zero elsewhere. With this T , we choose $U \in e^{\mathcal{L}}$ given by (cf (48))

$$(49) \quad U := (K_1 \otimes \mathbf{1})(e^{i\tilde{A} \otimes \bar{\sigma}})(K_2 \otimes \mathbf{1})(\mathbf{1} \otimes T).$$

We verify that

$$(50) \quad U(\rho_S \otimes \rho_A)U^\dagger = X\rho_S X^\dagger \otimes E_1.$$

This follows from the definitions of U and X in (49) and (48) and from the observation that since $i\bar{\sigma}E_1 = iE_1\bar{\sigma} = E_1$, for a general matrix ρ we have

$$(51) \quad e^{it\tilde{A} \otimes \bar{\sigma}}(\rho \otimes E_1)e^{-it\tilde{A} \otimes \bar{\sigma}} = \left(e^{t\tilde{A}}\rho e^{-t\tilde{A}}\right) \otimes E_1.$$

Taking the partial trace with respect to the system A of (50), we get (47), as desired. \square

5.2. If $\mathcal{L} \neq su(n_S n_A)$, ρ_A pure is necessary for indirect controllability.

We use the main result of [7], i.e., the following theorem (which we state for $n_A = 2$).

THEOREM 4. *Assume $\rho_A = \frac{1}{2}\mathbf{1}$. If for each $X \in SU(n_S)$ there exists $U \in e^{\mathcal{L}}$ which verifies (47) for every density matrix ρ_S , then $\mathcal{L} = su(n_S n_A)$, i.e., complete controllability is verified.*

In other terms, indirect controllability with $\rho_A = \frac{1}{2}\mathbf{1}$ implies complete controllability.

¹⁹Notice that $\mathcal{K} \cap \tilde{\mathcal{P}} = \{0\}$ because if this was not the case (from (20)), $\mathcal{K} \cap \tilde{\mathcal{P}}$ would be an ideal of $su(n_S)$ which is excluded since $su(n_S)$ is simple, unless $su(n_S) = \mathcal{K} = \tilde{\mathcal{P}}$, which would imply $\mathcal{L} = su(n_S n_A)$, which gives complete controllability and therefore obviously indirect controllability.

Proof. Assume that $\mathcal{L} \neq su(n_S n_A)$ and ρ_A has the property that for each $X \in SU(n_S)$, there exists $U \in e^{\mathcal{L}}$ with (47) for every density matrix ρ_S . From Theorem 4, it follows that ρ_A cannot be the perfectly mixed state, i.e., $\rho_A \neq \frac{1}{2}\mathbf{1}$. We want to prove that ρ_A is necessarily a pure state. Assume this is not the case. Therefore, $\rho_A = c_1\rho_{A,1} + c_2\rho_{A,2}$ with $c_1 > 0$, $c_2 > 0$, $c_1 + c_2 = 1$, and $\rho_{A,1}$ and $\rho_{A,2}$ are two projection matrices with $\rho_{A,1} + \rho_{A,2} = \mathbf{1}_2$. From (47), we have

$$(52) \quad c_1\gamma_1[\rho_S] + c_2\gamma_2[\rho_S] = X\rho_S X^\dagger,$$

where we have used the definitions of the two trace-preserving completely positive maps (cf., e.g., [2]) γ_1 and γ_2 , $\gamma_1[\rho_S] := Tr_A(U\rho_S \otimes \rho_{A,1}U^\dagger)$ and $\gamma_2[\rho_S] := Tr_A(U\rho_S \otimes \rho_{A,2}U^\dagger)$. Their convex combination can be a unitary map if and only if both of them realize the same unitary transformation, that is, for every ρ_S ,

$$(53) \quad \gamma_1[\rho_S] = \gamma_2[\rho_S] = X\rho_S X^\dagger.$$

This follows from the *Choi–Jamiołkowski isomorphism* between trace-preserving completely positive maps and states [16]. According to this isomorphism, given a trace-preserving completely positive map γ , acting on the density operators on the Hilbert space \mathcal{H} , the corresponding state is a density operator Γ acting on the space $\mathcal{H} \otimes \mathcal{H}$. For our purposes, it is not necessary to describe the exact form of Γ .²⁰ This can be found, along with the proof of the one-to-one correspondence between γ and Γ in [16]. In [16], it is also shown that there is a one-to-one correspondence between unitary maps acting on the space of density matrices (a special case of completely positive maps) and pure states in $\mathcal{H} \otimes \mathcal{H}$. Therefore, the state corresponding to the unitary transformation in the right-hand side of (52), denoted here by Γ_X , is a pure state. If we call Γ_1 and Γ_2 the states corresponding to γ_1 and γ_2 in (52), we have, because of the isomorphism,

$$(54) \quad \Gamma_X = c_1\Gamma_1 + c_2\Gamma_2.$$

Since Γ_X is pure, this implies $\Gamma_1 = \Gamma_2 = \Gamma_X$, which, from the isomorphism, implies (53). From (53), we obtain

$$(55) \quad \begin{aligned} \frac{1}{2}(\gamma_1[\rho_S] + \gamma_2[\rho_S]) &= \frac{1}{2}Tr_A(U\rho_S \otimes \rho_{A,1}U^\dagger) + \frac{1}{2}Tr_A(U\rho_S \otimes \rho_{A,2}U^\dagger) \\ &= Tr_A\left(U\rho_S \otimes \left(\frac{1}{2}\mathbf{1}\right)U^\dagger\right) = X\rho_S X^\dagger \end{aligned}$$

for every ρ_S . Therefore, $\frac{1}{2}\mathbf{1}$ has the indirect controllability property. However, this, from Theorem 4, implies $\mathcal{L} = su(n_S n_A)$, which is not verified. Therefore, if $\mathcal{L} \neq su(n_S n_A)$, the only possibility to have indirect controllability given ρ_A is when ρ_A is a *pure state*. \square

6. Proof of Theorem 3, Part II: If the system is indirectly controllable given ρ_A , then $\mathcal{L}_S = u(n_S)$. This is the longest part of the proof. We have to analyze the Lie algebra \mathcal{L}_S in (9) under the assumption that there is indirect

²⁰ Γ has the form $\Gamma := (\mathbf{1} \otimes \gamma)\rho_0$, where ρ_0 is a given maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$ and $\mathbf{1}$ is the identity operator. In other words, in the Choi–Jamiołkowski isomorphism, the state Γ associated to the map γ is obtained by acting with γ on a single subsystem of a maximally entangled pair. More details can be found in [16].

controllability given ρ_A . We know that \mathcal{L}_S is a subalgebra of $u(n_S)$, and therefore it is a *reductive* Lie algebra.

We can assume $\mathcal{K} \cap \mathcal{P} = \{\mathbf{0}\}$. In fact, if $\mathcal{K} \cap \mathcal{P} \neq \{\mathbf{0}\}$, then indirect controllability implies $\mathcal{L}_S = u(n_S)$ from statement (b) in the proof of Lemma 4.2, which is independent of the assumption $n_A \geq 3$.

We can also assume that the graph $G_{\mathcal{P}}$ of Lemma 3.3 is connected. If this is not the case, in appropriate coordinates, \mathcal{P} will have a block diagonal form and \mathcal{K} will have a corresponding block diagonal form.²¹ This structure is incompatible with the assumption of indirect controllability because it implies a corresponding block diagonal structure on the matrices in \mathcal{L} . If ρ_S is chosen having this block-diagonal structure, this structure will be preserved after evolution and partial trace. Therefore, ρ_S cannot be transformed into every matrix which is unitarily equivalent to itself. A similar argument was used in the proof of Lemma 4.1 to show that \mathcal{P} cannot be Abelian, and the same argument which was independent of the dimension n_A shows that \mathcal{P} cannot be Abelian in this case either.

Given that the graph $G_{\mathcal{P}}$ is connected, we denote by n_0 the dimension $n_1 = n_2 = \dots = n_l$ of Lemmas 3.3 and 3.2. Notice that l is always ≥ 2 . $l = 1$ would mean that \mathcal{P} only contains the identity which is incompatible with the assumption $(A - a)$, since in this case there would be no interaction between S and A .

The proof can therefore be carried out under the above assumptions, which, as we have seen, can be considered without loss of generality. We assume, therefore, \mathcal{P} not Abelian, $\mathcal{K} \cap \mathcal{P} = \{\mathbf{0}\}$, $G_{\mathcal{P}}$ of Lemma 3.3 connected, and l of Lemmas 3.2, and 3.3, $l \geq 2$.

6.1. Overview of the proof. In almost all cases, the proof is carried out showing that if $\mathcal{L}_S \neq u(n_S)$, then for a suitable initial state ρ_S , the necessary condition of Lemma 3.1 is not satisfied. In order to do this, we construct subspaces \mathcal{V} invariant under the Lie algebra \mathcal{L} which contain an appropriate tensor product state $i\rho_S \otimes \rho_E$ and such that when the partial trace is taken, it does not give all of $u(n_S)$. The first step is to describe the structure of the subspace \mathcal{P} . We arrive at formula (57) of Lemma 6.2, where \mathcal{P} is described in terms of the spaces \mathcal{L}_1 and \mathcal{L}_2 of (31) and (32) and a normal vector space $\tilde{\mathcal{N}}_{n_0}$. In the case $l > 2$, $\tilde{\mathcal{N}}_{n_0}$ is not only a normal vector space but also a Lie algebra (Lemma 6.3), and therefore we prove violation of Lemma 3.1 in the cases enumerated in Lemma 3.7. In the case $l = 2$, this is not true anymore. Therefore, $\tilde{\mathcal{N}}_{n_0}$ only has the general (recursive) structure described in Proposition 3.6. According to how the recursion of Proposition 3.6 ends, \mathcal{P} has the structure (67) or (69). In both cases, it is easy to find the above invariant subspace \mathcal{V} unless n_S is a power of 2, i.e., $n_S = 2^p$. This is done in subsection 6.4. Therefore, the remainder of subsection 6.4 is devoted to treating the various cases of $n_S = 2^p$ for $p = 0, 1, 2, 3, \dots$. The cases $p = 0$ and $p = 1$ are treated in subsection 6.4.1, while the case $p > 2$ is treated in subsection 6.4.3 by induction after having proved some properties of the Lie algebra \mathcal{L}_S for the case $p = 2$. We in detail study the commutation and anticommutation of subspaces \mathcal{K} and \mathcal{P} , from which, introducing additional vector spaces, we are able to construct the invariant subspace \mathcal{V} to be used in the application of Lemma 3.1. For the case $p = 2$, we were not able to construct such invariant vector space. Therefore, the proof uses a different technique. It is given in Appendix B. We work here at the level of the group $e^{\mathcal{L}}$ rather than the Lie algebra

²¹Any element with nonzero off diagonal in \mathcal{K} would give from $[\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}$ a corresponding element with nonzero off diagonal in \mathcal{P} , using Lie brackets with elements in the basis (24) of Lemma 3.2.

\mathcal{L} and explicitly display a transformation which cannot be performed on every initial state. This part of the proof is reported in Appendix B.

6.2. Structure of \mathcal{P} . We examine the proof of Lemma 3.3 in Appendix A and the graph $G_{\mathcal{P}}$ and notice that \mathcal{P} is spanned by the matrices D_1, \dots, D_l (with $n_1 = \dots = n_l = n_0$) in (24) of Lemma 3.2 as well as matrices $P_{j,k}$ ($j < k, j, k \in \{1, \dots, l\}$), which are zero in every block except for the (j, k) th (and (k, j) th) block. These blocks are occupied by matrices $R_{j,k}$ (and $-R_{j,k}^\dagger$), which are different from zero and in fact nonsingular for all pairs $j < k$ for which there is an edge in the graph $G_{\mathcal{P}}$. In fact, there is a nonzero $P_{j,k}$ for every pair $j < k \in \{1, \dots, l\}$. In order to see this, fix j and k and, since $G_{\mathcal{P}}$ is connected, fix a path joining j and k . Let j_0, j_1 , and j_2 be three consecutive nodes on this path so that there is an edge connecting j_0 and j_1 and an edge connecting j_1 and j_2 . Taking the Lie bracket $[D_{j_0}, [P_{j_0, j_1}, P_{j_1, j_2}]]$, which is in \mathcal{P} , we obtain a matrix which has a nonzero (and nonsingular) block in the position (j_0, j_2) (and (j_2, j_0)). Therefore, j_0 and j_2 also are connected in $G_{\mathcal{P}}$. Repeating this argument and by induction, we see that j and k are also connected. Therefore, in \mathcal{P} there exists an element with all blocks zero except the (j, k) th and (k, j) th for every $j < k$. Now denote by $\mathcal{N}_{j,k}$ ($j < k, j, k \in \{1, 2, \dots, l\}$) the space of $n_0 \times n_0$ matrices occupying the (j, k) th positions in the matrices in \mathcal{P} . Because of property (22), for every pair (j, k) , $\mathcal{N}_{j,k}$ forms a *normal vector space* because any two matrices A and B in $\mathcal{N}_{j,k}$ satisfy the defining property (35). The following property considerably simplifies our analysis.

PROPOSITION 6.1. *Modulo a change of coordinates on \mathcal{L}_S ,*

$$(56) \quad \mathcal{N}_{1,2} = \mathcal{N}_{1,3} = \dots = \mathcal{N}_{1,l} := \mathcal{N}_{n_0}$$

and if $j \neq 1, j < k, \mathcal{N}_{j,k} = i\mathcal{N}_{n_0}$. Here, \mathcal{N}_{n_0} (and therefore $i\mathcal{N}_{n_0}$) is a normal vector space of $n_0 \times n_0$ matrices.

Proof. Consider given normal matrices $R_{1,k} \in \mathcal{N}_{1,k}, k = 2, \dots, l$, satisfying (cf. formula (36)) $R_{1,k}R_{1,k}^\dagger = R_{1,k}^\dagger R_{1,k} = \alpha_{1,k}\mathbf{1}_{n_0}$. After renormalization $R_{1,k} \rightarrow \frac{1}{\sqrt{\alpha_{1,k}}}R_{1,k}$ (recall that $\mathcal{N}_{1,k}$ is a vector space), we can assume that $R_{1,k}$'s are unitary. Perform a change of coordinates on \mathcal{L}_S and therefore $\mathcal{P}, \mathcal{L}_S \rightarrow T\mathcal{L}_S T^\dagger$ with $T = \text{diag}(\mathbf{1}_{n_0}, R_{1,2}^\dagger, \dots, R_{1,l}^\dagger)$, so that the matrix $\mathbf{1}_{n_0}$ belongs to each of the $\mathcal{N}_{1,k}$. This corresponds to DUCT transformations (cf. subsection 3.2) on the subspaces $\mathcal{N}_{1,k}$. Every $\mathcal{N}_{1,k}$ space is spanned by the identity and possibly (because of formula (36)) by skew-Hermitian matrices. Let us denote by $E_{1,k}$ the matrix in \mathcal{P} with the identity $\mathbf{1}_{n_0}$ in the $(1, k)$ th block (and $-\mathbf{1}_{n_0}$ in the $(k, 1)$ th block) and zeros everywhere else. Let $R_{1,j}$ be a matrix in $\mathcal{N}_{1,j}$ which we can assume skew-Hermitian, and let $\hat{R}_{1,j}$ the corresponding matrix in \mathcal{P} which has zero blocks everywhere except in the blocks $(1, j)$ th and $(j, 1)$ th which are occupied by $R_{1,j}$ and $-R_{1,j}^\dagger = R_{j,1}$, respectively. By calculating $[E_{1,j}, [E_{1,k}, \hat{R}_{1,j}]] \in \mathcal{P}$, we obtain a matrix which has zeros in every block except for the $(1, k)$ th block which is (proportional to) $R_{1,j}$ (and accordingly for the $(k, 1)$ th block). This shows $\mathcal{N}_{1,j} \subseteq \mathcal{N}_{1,k}$, and since j and k are arbitrary, equality holds for all j and k 's. Now using the definitions in (24), we calculate, for a given $R_{1,j} \in \mathcal{N}_{1,j}$, and corresponding $\hat{R}_{1,j} \in \mathcal{P}, [D_j, [E_{1,k}, \hat{R}_{1,j}]] \in \mathcal{P}$. This gives a matrix which has zeros in all blocks except $iR_{1,j}^\dagger$ in the (j, k) th position (and accordingly in the (k, j) th position). This shows that $i\mathcal{N}_{1,j} \subseteq \mathcal{N}_{j,k}$. To show the converse inclusion, calculate $[[\hat{R}_{j,k}, E_{1,k}], D_1] \in \mathcal{P}$. \square

It follows from the proof of the previous proposition that a change of coordinates on the Lie algebra \mathcal{L}_S can be performed in order to achieve a DUCT transformation

on the normal space \mathcal{N}_{n_0} to put it in the canonical form described in Proposition 3.6. We shall assume this to be the case in the following. Let $\tilde{\mathcal{N}}_{n_0}$ be the subspace of \mathcal{N}_{n_0} of skew-Hermitian matrices. Therefore, $\mathcal{N}_{n_0} = \tilde{\mathcal{N}}_{n_0} + \text{span}\{\mathbf{1}_{n_0}\}$. From Proposition 6.1, we know that a basis for \mathcal{P} can be taken as made up of the following:

- (1) the matrices D_1, \dots, D_l in (24) and the matrices that have $i\mathbf{1}_{n_0}$ in blocks (j, k) (and (k, j)) with $j, k = 2, \dots, l, j < k$;
- (2) the matrices which have the identity $\mathbf{1}_{n_0}$ in the blocks corresponding to the first row (and $-\mathbf{1}_{n_0}$ in the blocks corresponding to the first column) (except the diagonal block);
- (3) the matrices which have elements in a basis of $\tilde{\mathcal{N}}_{n_0}$ in the blocks corresponding to the first row (and first column) (except the diagonal block);
- (4) the matrices which have elements in a basis of $i\tilde{\mathcal{N}}_{n_0}$ in blocks (j, k) (and accordingly in (k, j)), with $j, k = 2, \dots, l, j < k$.

This basis can be conveniently expressed using the subspaces defined in (29)–(32), considering Cartan decompositions of $su(l)$.²² In particular, the matrices of point (1) above are the matrices of $(\mathcal{I}m \cap \mathcal{D}i) \otimes \mathbf{1}_{n_0} + \text{span}\{i\mathbf{1}_l \otimes \mathbf{1}_{n_0}\}$; the matrices of point (2) are the ones in $(\mathcal{R}e \cap \mathcal{A}n) \otimes \mathbf{1}_{n_0}$; the matrices of point (3) are the ones in $i(\mathcal{I}m \cap \mathcal{A}n) \otimes \tilde{\mathcal{N}}_{n_0}$; and the matrices of point (4) are the ones in $i(\mathcal{R}e \cap \mathcal{D}i) \otimes \tilde{\mathcal{N}}_{n_0}$. Therefore, we have the following.

LEMMA 6.2. *With the definitions (31), (32),*

$$(57) \quad \mathcal{P} = (\mathcal{L}_1 \otimes \mathbf{1}_{n_0}) + (i\mathcal{L}_2 \otimes \tilde{\mathcal{N}}_{n_0}) + \text{span}\{i\mathbf{1}_l \otimes \mathbf{1}_{n_0}\}.$$

The two cases $l > 2$ and $l = 2$ have to be treated separately, and this is done in the following two subsections.

6.3. Case $l > 2$.

LEMMA 6.3. *Assume $l > 2$. Then $\tilde{\mathcal{N}}_{n_0}$ (and therefore \mathcal{N}_{n_0}) is a Lie algebra.*

Proof. Assume without loss of generality $l = 3$, since if $l > 3$, we can assume in the following argument that all elements which are not at the intersection of the first three rows and columns, in the $l \times l$ matrices on the left of the tensor products in $i\mathcal{L}_2 \otimes \tilde{\mathcal{N}}_{n_0}$ of (57), are zero. Denote by $I_{j,k}$ and $R_{j,k}$ with $j < k \in \{1, 2, 3\}$ the matrix with all zeros except in the (j, k) th position which is occupied by i or 1 , respectively. (Correspondingly, the (k, j) th position is given.) For any pair of elements N_1, N_2 in $\tilde{\mathcal{N}}_{n_0}$, we calculate $[iI_{1,2} \otimes N_1, iI_{1,3} \otimes N_2]$, which is in $[\mathcal{P}, \mathcal{P}]$ because of (57). Since elements in $\tilde{\mathcal{N}}_{n_0}$ are skew-Hermitian and satisfy property (35), we have

$$(58) \quad \begin{aligned} Z &:= [iI_{1,2} \otimes N_1, iI_{1,3} \otimes N_2] \\ &= -\frac{1}{2} (\{I_{1,2}, I_{1,3}\} \otimes [N_1, N_2] + [I_{1,2}, I_{1,3}] \otimes \{N_1, N_2\}) \\ &= -\frac{1}{2} (iI_{2,3} \otimes [N_1, N_2] + \alpha R_{2,3} \otimes \mathbf{1}) \end{aligned}$$

for some real α . By taking the Lie bracket with $R_{1,2} \otimes \mathbf{1}$ which is in \mathcal{P} , we obtain an element in \mathcal{P} , which is given by

$$(59) \quad \begin{aligned} [R_{1,2} \otimes \mathbf{1}, Z] &= -\frac{1}{2} ([R_{1,2}, I_{2,3}] \otimes [N_1, N_2] + \alpha [R_{1,2}, R_{2,3}] \otimes \mathbf{1}) \\ &= -\frac{1}{2} (iI_{1,3} \otimes [N_1, N_2] + \alpha R_{1,3} \otimes \mathbf{1}). \end{aligned}$$

²²Recall that, in the Cartan decomposition AIII, we have chosen to partition the matrices of $su(l)$ in block diagonal and antidiagonal parts so that the diagonal blocks have dimensions 1×1 and $(l - 1) \times (l - 1)$.

Since the last term in (59) is already in \mathcal{P} , in order for $[R_{1,2} \otimes \mathbf{1}, Z]$ to be in \mathcal{P} , we must have $[N_1, N_2] \in \tilde{\mathcal{N}}_{n_0}$, that is, $\tilde{\mathcal{N}}_{n_0}$ is closed under commutation. \square

We have from Lemma 6.3 that \mathcal{N}_{n_0} must be one of the Lie algebras listed in Lemma 3.7.²³ We can eliminate the first case which cannot be verified²⁴ and the case where $n_0 = 1$, which would mean that $\mathcal{L}_S = \mathcal{L}_1 + \mathcal{L}_2 + \text{span}\{i\mathbf{1}\} = u(l) = u(n_S)$, which we have excluded. In case 2 of Lemma 3.7, $\tilde{\mathcal{N}}_{n_0} = 0$, so that $\mathcal{P} = \mathcal{L}_1 \otimes \mathbf{1}_{n_0} + \text{span}\{i\mathbf{1}_{n_S}\} \otimes \mathbf{1}_{n_0}$. Since $[\mathcal{L}_1, \mathcal{L}_1] = \mathcal{L}_2$, from Corollary 3.5 (or by direct computation) we have $[\mathcal{P}, \mathcal{P}] = \mathcal{L}_2 \otimes \mathbf{1}_{n_0} \subseteq \mathcal{K}$. Since \mathcal{L}_S is reductive, from Lemma 3.4, we write \mathcal{K} as $\mathcal{K} = (\mathcal{L}_2 \otimes \mathbf{1}_{n_0}) + \mathcal{R}$, where \mathcal{R} commutes with \mathcal{P} and it is an ideal in \mathcal{L}_S . If we write a general element of \mathcal{R} as $\sum_j R_j \otimes \sigma_j$ with $R_j \in u(l)$ and $\sigma_j, n_0 \times n_0$, Hermitian, linearly independent matrices, we find that $[R_j, \mathcal{L}_1] = 0$, which also (using $[\mathcal{L}_1, \mathcal{L}_1] = \mathcal{L}_2$ and the Jacobi identity) implies $[R_j, \mathcal{L}_2] = 0$, and therefore $[R_j, u(l)] = 0$, which implies that R_j is a multiple of the identity. Therefore, $\mathcal{P} = \mathcal{L}_1 \otimes \mathbf{1}_{n_0} + \text{span}\{i\mathbf{1}_{n_S}\} \otimes \mathbf{1}_{n_0}$ and $\mathcal{K} = (\mathcal{L}_2 \otimes \mathbf{1}_{n_0}) + (\mathbf{1}_l \otimes \tilde{\mathcal{R}})$ for some subalgebra $\tilde{\mathcal{R}}$ of $u(n_0)$. Consider now the vector space

$$(60) \quad \mathcal{V} := (i\mathcal{L}_1 \otimes \mathbf{1}_{n_0} \otimes su(2)) + (\mathcal{L}_1 \otimes \mathbf{1}_{n_0} \otimes \mathbf{1}_2) + (\mathcal{L}_2 \otimes \mathbf{1}_{n_0} \otimes \mathbf{1}_2) + (i\mathcal{L}_2 \otimes \mathbf{1}_{n_0} \otimes su(2)) + (\mathbf{1}_l \otimes \mathbf{1}_{n_0} \otimes su(2)).$$

By using formulas (33) and (34), we can verify that $ad_{\mathcal{L}}\mathcal{V} \subseteq \mathcal{V}$. Now consider initial states (recall that $ln_0 = n_S$), $\rho_S = \frac{1}{ln_0}\mathbf{1}_{ln_0} + iL_1 \otimes \mathbf{1}_{n_0}$ for some $L_1 \in \mathcal{L}_1, L_1 \neq 0$, and arbitrary initial state for A , $\rho_A := \frac{1}{2}\mathbf{1} + i\sigma$, for some $\sigma \in su(2)$. The matrix

$$(61) \quad i\rho_S \otimes \rho_A = \frac{1}{2n_0l} (i\mathbf{1}_{ln_0} \otimes \mathbf{1}_2 - L_1 \otimes \mathbf{1}_{n_0} \otimes \mathbf{1}_2 - \mathbf{1}_{ln_0} \otimes \sigma - iL_1 \otimes \mathbf{1} \otimes \sigma)$$

belongs to $\mathcal{V} + \text{span}\{i\mathbf{1}_{ln_0} \otimes \mathbf{1}_2\}$, which is also invariant under $ad_{\mathcal{L}}$. Via direct computation, we get

$$Tr_A(\mathcal{V} + \text{span}\{i\mathbf{1}_{ln_0} \otimes \mathbf{1}_2\}) = \mathcal{L}_1 \otimes \mathbf{1}_{n_0} + \mathcal{L}_2 \otimes \mathbf{1}_{n_0} + \text{span}\{i\mathbf{1}_{n_S}\} \neq u(n_S),$$

which contradicts Lemma 3.1.

The cases 3 and 4 of Lemma 3.7 are treated with a similar technique. In case 3, \mathcal{P} is given by

$$(62) \quad \mathcal{P} = (\mathcal{L}_1 \otimes \mathbf{1}_{n_0}) + (\mathcal{L}_2 \otimes \mathbf{1}_{r,s}) + \text{span}\{i\mathbf{1}_l \otimes \mathbf{1}_{n_0}\}.$$

Calculating $[\mathcal{P}, \mathcal{P}]$ using (33) and Corollary 3.5, we obtain

$$(63) \quad [\mathcal{P}, \mathcal{P}] = \mathcal{L}_2 \otimes \mathbf{1}_{n_0} + \mathcal{L}_1 \otimes \mathbf{1}_{r,s}.$$

The ideal \mathcal{R} of Proposition 3.4 has again the form $\mathbf{1}_l \otimes \tilde{\mathcal{R}}$, where now $\tilde{\mathcal{R}}$ is a subalgebra of $u(n_0)$ which commutes with $\mathbf{1}_{r,s}$ (and therefore spanned by block diagonal matrices). If we consider the vector space,

$$(64) \quad \mathcal{V} := (u(l) \otimes \mathbf{1}_{n_0} \otimes \mathbf{1}_2) + (u(l) \otimes \mathbf{1}_{r,s} \otimes \mathbf{1}_2) + (iu(l) \otimes \mathbf{1}_{n_0} \otimes su(2)) + (iu(l) \otimes \mathbf{1}_{r,s} \otimes su(2)),$$

²³Recall that we are assuming that we have performed a change of coordinates so that \mathcal{N}_{n_0} has the canonical form of Proposition 3.6 and Lemma 3.7.

²⁴Recall that we are assuming \mathcal{P} nonAbelian.

it is easy to check that this space is invariant under $ad_{\mathcal{L}}$. By considering the initial condition

$$(65) \quad \rho_S \otimes \rho_A := \left(\frac{1}{n_0 l} \mathbf{1}_{n_0 l} + iL \otimes \mathbf{1}_{n_0} \right) \otimes \left(\frac{1}{2} \mathbf{1}_2 + i\sigma \right)$$

for some $L \in iu(l)$, $L \neq 0$, and any $\sigma \in su(2)$, since $i\rho_S \otimes \rho_A \in \mathcal{V}$ and $Tr_A(\mathcal{V}) \neq u(n_S) = u(n_0 l)$, we find a contradiction with Lemma 3.1. In case 4, we must assume n_0 even and at least equal to 4, since if n_0 is equal to 2, using (37) and Lemma 6.2, $\mathcal{K} + \mathcal{P} = u(n_S) = u(2l)$, which we have excluded. If $n_0 \geq 4$, the ideal \mathcal{R} of Lemma 3.4 has (using (38) of Remark 3.8) the form $\mathcal{R} = \mathbf{1}_l \otimes \mathbf{1}_2 \times \tilde{\mathcal{R}}$, where $\tilde{\mathcal{R}}$ is a Lie subalgebra of $u(\frac{n_0}{2})$. We consider a vector space

$$(66) \quad \mathcal{V} := \left(u(2l) \otimes \mathbf{1}_{\frac{n_0}{2}} \otimes \mathbf{1}_2 \right) + i \left(u(2l) \otimes \mathbf{1}_{\frac{n_0}{2}} \otimes su(2) \right),$$

which is invariant under $ad_{\mathcal{L}}$ and such that $Tr_A(\mathcal{V}) \neq u(n_S) = u(n_0 l)$. By taking an initial state $i\rho_S \otimes \rho_A \in \mathcal{V}$, we find again a contradiction with Lemma 3.1.

6.4. Case $l = 2$. Recall Lemma 6.2 and Proposition 3.6. Aside from the trivial case 1 of Proposition 3.6,²⁵ the recursion described in this proposition ends with case 2 or 3 for some appropriate n . If the recursion ends with case 2, \mathcal{P} is given by

$$(67) \quad \mathcal{P} := +\mathop{\text{span}}_{j=0}^{j_{\max}} \{ (i)^j (\sigma_x)^{\otimes j} \otimes \{ \sigma_z, \sigma_y \} \otimes \mathbf{1}_{n_j} \} + \mathop{\text{span}} \{ i\mathbf{1}_{n_S} \},$$

where $n_j := n_S 2^{-(j+1)}$.²⁶ The number j_{\max} is an integer number with $j_{\max} \leq \log_2 n_S - 1$, which gives the number of iterations, i.e., how many times we return to step 2. In order to see this,²⁷ assume first that we reach step 2 and never come back. Then, in Lemma 6.2, we only have $\mathcal{L}_1 \otimes \mathbf{1}_{n_0}$, and $j_{\max} = 0$ and the only linearly independent matrices to be included in a basis of \mathcal{P} are (beside the $i\mathbf{1}_{n_S}$) $\sigma_y \otimes \mathbf{1}_{\frac{n_S}{2}}$ and $\sigma_z \otimes \mathbf{1}_{\frac{n_S}{2}}$. However, if $\tilde{\mathcal{N}}_{n_0} \neq 0$, we move on to step 3 and have to add the matrix $i\sigma_x \otimes i\mathbf{1}_{\frac{n_S}{4}, \frac{n_S}{4}} = i\sigma_x \otimes \sigma_z \otimes \mathbf{1}_{\frac{n_S}{4}}$ and, since we are supposed to go back to 2, the matrix

$$(68) \quad i\sigma_x \otimes \begin{pmatrix} \mathbf{0} & \mathbf{1}_{\frac{n_S}{4}} \\ -\mathbf{1}_{\frac{n_S}{4}} & \mathbf{0} \end{pmatrix} = i\sigma_x \otimes \sigma_y \otimes \mathbf{1}_{\frac{n_S}{4}}.$$

Continuing this way we obtain the basis in (67).

Analogously, in the case where the iteration ends with step 3, we obtain for \mathcal{P}

$$(69) \quad \mathcal{P} := +\mathop{\text{span}}_{j=0}^{j_{\max}} \{ (i)^j \sigma_x^{\otimes j} \otimes \{ \sigma_y, \sigma_z \} \otimes \mathbf{1}_{n_j} \} + \mathop{\text{span}} \{ (i)^{j_{\max}} \sigma_x^{j_{\max}+1} \otimes \mathbf{1}_{r,s} \} + \mathop{\text{span}} \{ i\mathbf{1}_{n_S} \},$$

where j_{\max} is some nonnegative integer number with $j_{\max} \leq \log_2(n_S - 2) - 1$ and r and s are two nonnegative integer numbers with $r + s = n_S 2^{-(j_{\max}+1)}$.

²⁵This case would imply \mathcal{P} Abelian, which we have excluded.

²⁶With some abuse of notation, we are using the notation n_j , here again as in Lemma 3.2. However, the meaning of n_j for $j = 1, \dots, j_{\max}$ is different here than in that lemma. In fact, we are already in the situation where all the n_j 's of Lemma 3.2 are equal to n_0 . In formula (67), however, n_0 coincides with the one previously defined.

²⁷We neglect here the factor $\frac{1}{2}$ in the definition of the Pauli matrices (15), which has no effect on the vector spaces we are describing.

Consider the case (67) first. If $n_{j_{\max}} \geq 2$, we have²⁸

$$[\mathcal{P}, \mathcal{P}] \subseteq u\left(\frac{n_S}{n_j}\right) \otimes \mathbf{1}_{n_{j_{\max}}}.$$

Moreover, similar to what was described in the previous subsection, the ideal $\mathcal{R} \subseteq \mathcal{K}$ of (27) has the form $\mathbf{1}_{\frac{n_S}{n_{j_{\max}}}} \otimes \tilde{\mathcal{R}}$ for some subalgebra $\tilde{\mathcal{R}} \subseteq u(n_{j_{\max}})$. The subspace

$$(70) \quad \mathcal{V} := \left(u\left(\frac{n_S}{n_{j_{\max}}}\right) \otimes \mathbf{1}_{n_{j_{\max}}} \otimes \mathbf{1}_2\right) + i\left(u\left(\frac{n_S}{n_{j_{\max}}}\right) \otimes \mathbf{1}_{n_{j_{\max}}} \otimes su(2)\right)$$

is invariant under $ad_{\mathcal{L}}$, and by taking an initial condition $\rho_S \otimes \rho_A$ of the form

$$(71) \quad \rho_S \otimes \rho_A = \left(\frac{1}{2}\mathbf{1}_{n_S} + iL \otimes \mathbf{1}_{n_{j_{\max}}}\right) \otimes \left(\frac{1}{2}\mathbf{1}_2 + i\sigma\right)$$

with L a nonzero matrix in $su\left(\frac{n_S}{n_{j_{\max}}}\right)$ and σ any matrix in $su(2)$, we find a contradiction with Lemma 3.1. Therefore, $n_{j_{\max}}$ must be 1 in this case. The same thing can be proved in case (69). If $n_{j_{\max}} \geq 2$, then \mathcal{P} and $[\mathcal{P}, \mathcal{P}]$ are subspaces of $u\left(\frac{n_S}{n_{j_{\max}}}\right) \otimes \{\text{span}\{\mathbf{1}_{n_{j_{\max}}}, \mathbf{1}_{r,s}\}\}$, and the ideal \mathcal{R} of \mathcal{K} defined in (27) has the form $\mathbf{1}_{\frac{n_S}{n_{j_{\max}}}} \otimes \tilde{\mathcal{R}}$, where now $\tilde{\mathcal{R}}$ has to commute with $\mathbf{1}_{r,s}$. The space

$$(72) \quad \mathcal{V} := \left(u\left(\frac{n_S}{n_{j_{\max}}}\right) \otimes \{\mathbf{1}_{n_{j_{\max}}}, \mathbf{1}_{r,s}\} \otimes \mathbf{1}_2\right) + i\left(u\left(\frac{n_S}{n_{j_{\max}}}\right) \otimes \{\mathbf{1}_{n_{j_{\max}}}, \mathbf{1}_{r,s}\} \otimes su(2)\right)$$

is invariant under $ad_{\mathcal{L}}$ and, once again, we find a contradiction with Lemma 3.1.

In conclusion, we have to study only the cases (67) and (69) only for $n_{j_{\max}} = 1$, which is $j_{\max} = \log_2 n_S - 1 := p$, assumed integer. The dimension n_S is equal to 2^{p+1} for some integer $p \geq 0$. In case (67), we have

$$(73) \quad \mathcal{P} = +_{j=0}^p \text{span}\{(i)^j (\sigma_x)^{\otimes j} \otimes \{\sigma_z, \sigma_y\} \otimes \mathbf{1}_{n_j}\} + \text{span}\{i\mathbf{1}_{n_S}\},$$

and in case (69),

$$(74) \quad \mathcal{P} = +_{j=0}^p \text{span}\{(i)^j \sigma_x^{\otimes j} \otimes \{\sigma_y, \sigma_z\} \otimes \mathbf{1}_{n_j}\} + \text{span}(i)^p \sigma_x^{j_{\max}+1} \text{span}\{i\mathbf{1}_{n_S}\}.$$

We have therefore reduced the problem to the case where n_S is equal to $n_S = 2^{p+1}$, and $p+1$ is the number of factors in the tensor products of 2×2 matrices which span \mathcal{L}_S . Recall that we denote by $\tilde{\mathcal{P}}$ the subspace of \mathcal{P} of matrices with zero trace. To be more explicit in case (73), we have that $\tilde{\mathcal{P}}$ is the span of the matrices

$$(75) \quad \begin{aligned} &\{\sigma_y, \sigma_z\} \otimes \mathbf{1}, \\ &i\sigma_x \otimes \{\sigma_y, \sigma_z\} \otimes \mathbf{1}, \\ &\sigma_x \otimes \sigma_x \otimes \{\sigma_y, \sigma_z\} \otimes \mathbf{1}, \\ &\vdots \\ &(i)^{p+1} \sigma_x \otimes \sigma_x \cdots \sigma_x \otimes \sigma_x \otimes \{\sigma_y, \sigma_z\}, \end{aligned}$$

²⁸This is trivially true even if $n_{j_{\max}} = 1$, but this case will be treated later.

while in case (74), we have that $\tilde{\mathcal{P}}$ is the span of the matrices

$$\begin{aligned}
 (76) \quad & \{\sigma_y, \sigma_z\} \otimes \mathbf{1}, \\
 & i\sigma_x \otimes \{\sigma_y, \sigma_z\} \otimes \mathbf{1}, \\
 & \sigma_x \otimes \sigma_x \otimes \{\sigma_y, \sigma_z\} \otimes \mathbf{1}, \\
 & \vdots \\
 & (i)^{p+1} \sigma_x \otimes \sigma_x \cdots \sigma_x \otimes \sigma_x \otimes \{\sigma_y, \sigma_z, \sigma_x\}.
 \end{aligned}$$

The proof can be carried out by considering separately the cases $p = 0, 1, 2$ and then by induction for $p > 2$. The case $p = 2$ is quite long, and it is postponed to Appendix B. The other cases are treated below.

6.4.1. $p = 0$ and $p = 1$. If $p = 0$, then both in case (75) and in case (76), $\mathcal{L}_S := \mathcal{P} + \mathcal{K} = u(n_S)$.²⁹ Therefore, the condition we want to prove is automatically satisfied. If $p = 1$, then calculating $[\mathcal{P}, \mathcal{P}] = \mathcal{K}$, we find that in case (76), $\mathcal{L}_S = u(n_S)$, and therefore the theorem is automatically satisfied. In case (75),

$$\begin{aligned}
 (77) \quad \mathcal{P} &:= \text{span}\{\{\sigma_y, \sigma_z\} \otimes \mathbf{1}_2\} + \{\text{span}\{i\{\sigma_x\} \otimes \{\sigma_y, \sigma_z\}\}\} + \text{span}\{i\mathbf{1}_4\}, \\
 (78) \quad \mathcal{K} &= [\mathcal{P}, \mathcal{P}] = \text{span}\{i\{\sigma_y, \sigma_z\} \otimes \{\sigma_y, \sigma_z\}\} + \text{span}\{\sigma_x \otimes \mathbf{1}_2\} + \text{span}\{\mathbf{1}_2 \otimes \sigma_x\},
 \end{aligned}$$

which is 10-dimensional. Therefore, the condition $\mathcal{L}_S = u(n_S)$ is not verified. We want to show that indirect controllability cannot be verified in this case. Once again, consider the definition of $\tilde{\mathcal{P}}$, the subspace of \mathcal{P} spanned by matrices with zero trace. Moreover, define

$$(79) \quad \mathcal{L}_{S,1}^\perp := \text{span}\{i\sigma_x \otimes \sigma_x\}, \quad \mathcal{L}_{S,2}^\perp := \text{span}\{\mathbf{1} \otimes \sigma_z, \mathbf{1} \otimes \sigma_y, i\sigma_z \otimes \sigma_x, i\sigma_y \otimes \sigma_x\}.$$

By using (16), (17),³⁰ it is straightforward to verify the commutation relations

$$\begin{aligned}
 [\tilde{\mathcal{P}}, \tilde{\mathcal{P}}] &= \mathcal{K}, \\
 [\tilde{\mathcal{P}}, \mathcal{K}] &= \tilde{\mathcal{P}}, \\
 [\tilde{\mathcal{P}}, \mathcal{L}_{S,1}^\perp] &= \mathcal{L}_{S,2}^\perp, \\
 [\tilde{\mathcal{P}}, \mathcal{L}_{S,2}^\perp] &= \mathcal{L}_{S,1}^\perp, \\
 [\mathcal{K}, \mathcal{K}] &= \mathcal{K}, \\
 [\mathcal{K}, \mathcal{L}_{S,1}^\perp] &= 0, \\
 [\mathcal{K}, \mathcal{L}_{S,2}^\perp] &= \mathcal{L}_{S,2}^\perp, \\
 [\mathcal{L}_{S,1}^\perp, \mathcal{L}_{S,1}^\perp] &= 0, \\
 [\mathcal{L}_{S,1}^\perp, \mathcal{L}_{S,2}^\perp] &= \tilde{\mathcal{P}}, \\
 [\mathcal{L}_{S,2}^\perp, \mathcal{L}_{S,2}^\perp] &= \mathcal{K}
 \end{aligned}$$

²⁹In this case, $n_0 = 1$ and $n_S = 2$, and in the case (75), $\tilde{\mathcal{P}} = \text{span}\{\sigma_y, \sigma_z\}$, while in case (76), $\tilde{\mathcal{P}} = \text{span}\{\sigma_x, \sigma_y, \sigma_z\}$. By using $[\tilde{\mathcal{P}}, \tilde{\mathcal{P}}] \subseteq \mathcal{K}$, we obtain that $\mathcal{L}_S = u(2)$.

³⁰Equations (18), (19) are also used.

and the anticommutation relations

$$\begin{aligned}
 i\{\tilde{\mathcal{P}}, \tilde{\mathcal{P}}\} &= \text{span}\{i\mathbf{1}\}, \\
 i\{\tilde{\mathcal{P}}, \mathcal{K}\} &= \mathcal{L}_{S,2}, \\
 i\{\tilde{\mathcal{P}}, \mathcal{L}_{S,1}^\perp\} &= 0, \\
 i\{\tilde{\mathcal{P}}, \mathcal{L}_{S,2}^\perp\} &= \mathcal{K}, \\
 i\{\mathcal{K}, \mathcal{K}\} &= \mathcal{L}_{S,1}^\perp + \text{span}\{i\mathbf{1}\}, \\
 i\{\mathcal{K}, \mathcal{L}_{S,1}^\perp\} &= \mathcal{K}, \\
 i\{\mathcal{K}, \mathcal{L}_{S,2}^\perp\} &= \tilde{\mathcal{P}}, \\
 i\{\mathcal{L}_{S,1}^\perp, \mathcal{L}_{S,1}^\perp\} &= \text{span}\{i\mathbf{1}\}, \\
 i\{\mathcal{L}_{S,1}^\perp, \mathcal{L}_{S,2}^\perp\} &= 0 \\
 i\{\mathcal{L}_{S,2}^\perp, \mathcal{L}_{S,2}^\perp\} &= \text{span}\{i\mathbf{1}\}.
 \end{aligned}$$

Consider now the vector space

$$\begin{aligned}
 (80) \quad \bar{\mathcal{V}} &:= \mathcal{L} + \{\mathcal{K} \otimes (i \text{span}\{\sigma_x, \sigma_y, \sigma_z\})\} + \{\mathcal{P} \otimes \mathbf{1}_2\} \\
 &\quad + \{\mathcal{L}_{S,2}^\perp \otimes (i \text{span}\{\sigma_x, \sigma_y, \sigma_z\})\} + \{\mathcal{L}_{S,1}^\perp \otimes \mathbf{1}_2\}.
 \end{aligned}$$

From the fact that \mathcal{L} is spanned by matrices of the form $K \otimes \mathbf{1}$ with $K \in \mathcal{K}$ and $iP \otimes \sigma$ with $P \in \mathcal{P}$ and σ any Pauli matrix, using the above commutation and anticommutation relations, we verify that $\bar{\mathcal{V}}$ is invariant under \mathcal{L} , i.e., $[\mathcal{L}, \bar{\mathcal{V}}] \subseteq \bar{\mathcal{V}}$.

Consider now Lemma 3.1 and pick initial conditions ρ_S and ρ_A of the form $\rho_S = \frac{1}{4}\mathbf{1} + K$ for a $K \in i\mathcal{K}$, $K \neq 0$, and $\rho_A = \frac{1}{2}\mathbf{1} + \sigma$ with $\sigma \in isu(2)$. With this choice, $i\rho_S \otimes \rho_A \in \bar{\mathcal{V}}$, and from invariance \mathcal{V} of Lemma 3.1, is such that $\mathcal{V} \subseteq \bar{\mathcal{V}}$. Since $Tr_A(\bar{\mathcal{V}}) \neq u(4)$, the necessary condition of Lemma 3.1 is not satisfied, and therefore indirect controllability cannot be verified.

6.4.2. $p = 2$. See Appendix B.

6.4.3. $p > 2$. \mathcal{P} in (75) is a subspace of \mathcal{P} in (76), and a straightforward computation shows that, for case (76), $[\mathcal{P}, \mathcal{P}] = \mathcal{K}$ (namely, the ideal \mathcal{R} of Lemma 3.4 is $\{0\}$ ³¹ and $\mathcal{L}_S \neq u(8)$). Therefore, it is enough to prove that indirect controllability cannot be verified in case (76).

Consider first the slightly more general case $p \geq 2$. To simplify the notation, we make a change of coordinates local on each one of the first p positions so as to change the span of σ_x into the span of σ_z and vice versa and leave the span of σ_y unchanged. We denote by $\mathcal{P}_n, \tilde{\mathcal{P}}$ for the case of $n := p + 1$ positions and $\mathcal{K}_n, \mathcal{K}$ in that case. By defining $Y := \text{span}\{i\sigma_x, i\sigma_y\}$, $Z := \text{span}\{i\sigma_z\}$, $\sigma := \text{span}\{i\sigma_x, i\sigma_y, i\sigma_z\}$, we have in particular³²

$$(81) \quad i\tilde{\mathcal{P}} := i\mathcal{P}_3 := Y \otimes \mathbf{1} \otimes \mathbf{1} + Z \otimes Y \otimes \mathbf{1} + Z \otimes Z \otimes \sigma,$$

$$\begin{aligned}
 (82) \quad i\mathcal{K} &:= i\mathcal{K}_3 = i[\mathcal{P}_3, \mathcal{P}_3] = \mathbf{1} \otimes Z \otimes \mathbf{1} + Z \otimes \mathbf{1} \otimes \mathbf{1} \\
 &\quad + Y \otimes Y \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \sigma + \mathbf{1} \otimes Y \otimes \sigma + Y \otimes Z \otimes \sigma,
 \end{aligned}$$

³¹This can be seen in both cases (75) and (76), imposing the fact that \mathcal{R} commutes with \mathcal{P} as from Lemma 3.4.

³²See formulas (84) and (85) below for a recursive expression of \mathcal{P}_n and \mathcal{K}_n .

so that \mathcal{L}_S in the case $p = 2$ can be taken equal to $\mathcal{L}_S = \mathcal{K}_3 + \mathcal{P}_3 + \text{span}\{i\mathbf{1}_8\}$. Define the subspace of $su(8)$

$$(83) \quad \mathcal{B}_3 = iY \otimes Z \otimes \mathbf{1} + i\mathbf{1} \otimes Y \otimes \mathbf{1} + Y \otimes \mathbf{1} \otimes su(2) + Y \otimes Y \otimes su(2) + \mathbf{1} \otimes Z \otimes su(2) + Z \otimes Y \otimes su(2) + Z \otimes \mathbf{1} \otimes su(2) + iZ \otimes Z \otimes \mathbf{1}.$$

We have that

$$su(8) = \mathcal{K}_3 + \mathcal{P}_3 + \mathcal{B}_3.$$

We can verify the following commutation and anticommutation relations:

- (B1) $i\{\mathcal{K}_3, \mathcal{P}_3\} = \mathcal{B}_3,$
- (B2) $i\{\mathcal{B}_3, \mathcal{P}_3\} = \mathcal{K}_3,$
- (B3) $[\mathcal{K}_3, \mathcal{B}_3] = \mathcal{B}_3,$
- (B4) $[\mathcal{P}_3, \mathcal{B}_3] = \mathcal{B}_3,$
- (B5) $i\{\mathcal{P}_3, \mathcal{P}_3\} = \text{span}\{i\mathbf{1}\}.$

From (76) (and after the local change of coordinates defined above), it is straightforward to verify the following recursive relations:

$$(84) \quad \mathcal{P}_{n+1} = Z \otimes \mathcal{P}_n + Y \otimes \mathbf{1}_2,$$

$$(85) \quad \mathcal{K}_{n+1} = \mathbf{1}_2 \otimes \mathcal{K}_n + Y \otimes \mathcal{P}_n + Z \otimes \mathbf{1}_2.$$

Using (B5) and (84) above, by induction on n we find, for every n ,

$$(86) \quad i\{\mathcal{P}_n, \mathcal{P}_n\} = \text{span}\{i\mathbf{1}\}.$$

LEMMA 6.4. *For any $n \geq 4$, there exist disjoint subspaces \mathcal{B}_n and \mathcal{C}_n such that*

- (A1) $i\{\mathcal{K}_n, \mathcal{P}_n\} = \mathcal{B}_n,$
- (A2) $i\{\mathcal{P}_n, \mathcal{B}_n\} = \mathcal{K}_n,$
- (A3) $[\mathcal{K}_n, \mathcal{B}_n] = \mathcal{B}_n,$
- (A4) $[\mathcal{K}_n, \mathcal{C}_n] = \mathcal{C}_n,$
- (A5) $[\mathcal{P}_n, \mathcal{B}_n] = \mathcal{C}_n,$
- (A6) $[\mathcal{P}_n, \mathcal{C}_n] = \mathcal{B}_n.$

See Appendix C.

Given the above set-up, the proof of the Theorem for the case $p > 2$ is based on the following observation.

LEMMA 6.5. *Consider the Lie algebra $\mathcal{L}_S = \mathcal{K}_n + \mathcal{P}_n + \text{span}\{i\mathbf{1}\}$ and the disjoint subspaces of $2^n \times 2^n$ matrices, \mathcal{B}_n and \mathcal{C}_n , defined above, so that for every $n \geq 4$, the four disjoint subspaces \mathcal{K}_n , \mathcal{P}_n , \mathcal{B}_n , and \mathcal{C}_n satisfy conditions (A1)–(A6) (besides (86)). Then the following space \mathcal{V} is invariant under \mathcal{L} :*

$$(87) \quad \mathcal{V} = \mathcal{K}_n \otimes \mathbf{1}_2 + i\mathcal{K}_n \otimes su(2) + \mathcal{P}_n \otimes \mathbf{1}_2 + i\mathcal{P}_n \otimes su(2) + i\mathcal{B}_n \otimes su(2) + \mathcal{C}_n \otimes \mathbf{1}_2.$$

Proof. Using properties (A1)–(A6) and the definition (8), we verify that $[\mathcal{K}_n \otimes \mathbf{1}, \mathcal{V}] \subseteq \mathcal{V}$, $[i\mathcal{P}_n \otimes su(2), \mathcal{V}] \subseteq \mathcal{V}$, and $[\mathbf{1}_{2^n} \otimes su(2), \mathcal{V}] \subseteq \mathcal{V}$. \square

This lemma allows us to conclude the proof for any $p > 2$ ($n \geq 4$). Take an initial state $\rho_S = \mathbf{1} + K$, with $K \in i\mathcal{K}_n$, and $\rho_A = \mathbf{1} + \tilde{\sigma}$, with $\tilde{\sigma} \in isu(2)$. Then

$$\rho_S \otimes \rho_A = \mathbf{1}_{2^n} \otimes \mathbf{1}_2 + \mathbf{1}_{2^n} \otimes \sigma + K \otimes \mathbf{1}_2 + K \otimes \sigma \in \mathcal{V} + \text{span}i\{\mathbf{1}\}_{2^{n+1}},$$

where \mathcal{V} is the subspace defined in (87). Since, from Lemma 6.5, $\mathcal{V} + \text{span}\{i\mathbf{1}_{2^{n+1}}\}$ is invariant under \mathcal{L} , we have that

$$\text{Tr}_A(\mathcal{V} + \text{span}\{i\mathbf{1}_{2^{n+1}}\}) = \mathbf{1} + \mathcal{K}_n + \mathcal{P}_n + \mathcal{C}_n.$$

This is not equal to $u(n_S)$ since \mathcal{B}_n is missing, thus contradicting the necessary condition of Lemma 3.1. Therefore, the model is not indirectly controllable.

7. Concluding remarks. It is possible to have full unitary control on a target system by controlling it indirectly via an auxiliary system without having full controllability on the total system. The necessary and sufficient conditions for this to happen have been given in this paper. These conditions are given in terms of the dynamical Lie algebra associated with the total system and the initial state of the auxiliary system. Further research is needed to design protocols for *constructive* indirect control, to investigate indirect controllability in cases where there exists a *network* of quantum systems in between the auxiliary (fully controlled) system and the target system, and to investigate *more general notions of indirect controllability*. These notions may be given in terms not only of unitary maps but of more general completely positive maps. We believe that the results and the framework developed here will be useful for the treatment of these problems as well. In order to give an illustration of this, we go back to Example 2.3 and investigate the possibility of driving ρ_S to an arbitrary state, not necessarily unitarily equivalent to ρ_S .

According to Theorem 3, we assume that ρ_A is a pure state. From the proof of the theorem, it follows that we can transform ρ_S into any unitarily equivalent matrix by keeping A in the state $\rho_A = E_1$. Therefore, there is no loss of generality in assuming that ρ_S is diagonal, i.e., $\rho_S = \text{diag}(x, 1 - x)$. Using a Cartan decomposition of the Lie algebra \mathcal{L} in (11), the possible transformations of the initial state of the total system $S + A$ can be written as $\rho_{TOT} \rightarrow X\rho_{TOT}X^\dagger$ with $X = (e^{i\sigma_z t_1} \otimes Y_1)e^{i\sigma_x \otimes \sigma_x \theta_1} e^{i\sigma_y \otimes \sigma_y \theta_2} (e^{i\sigma_z t_2} \otimes Y_2)$ for parameters $t_1, t_2, \theta_1, \theta_2 \in \mathbb{R}$ and general $Y_1, Y_2 \in SU(2)$. Using this and the fact that ρ_S is diagonal, we obtain

$$\begin{aligned} (88) \quad & Tr_A(X\rho_S \otimes \rho_A X^\dagger) \\ &= Tr_A(X\rho_S \otimes E_1 X^\dagger) \\ &= e^{\sigma_z t_1} Tr_A \left(e^{i\sigma_x \otimes \sigma_x \theta_1} e^{i\sigma_y \otimes \sigma_y \theta_2} (\rho_S \otimes Y_2 E_1 Y_2^\dagger) e^{-i\sigma_y \otimes \sigma_y \theta_2} e^{-i\sigma_x \otimes \sigma_x \theta_1} \right) e^{-\sigma_z t_1}. \end{aligned}$$

We can choose t_1 to adjust to an arbitrary value the phase of the resulting density matrix. Moreover, a direct computation using (88) shows that there exists free real parameters α, β, ψ , and ϕ so that the (1,1) entry of the resulting ρ_S matrix is

$$(89) \quad \rho_{1,1} := x(\cos^2(\psi) \cos^2(\alpha) + \sin^2(\psi) \cos^2(\beta)) + (1-x)(\cos^2(\psi) \sin^2(\beta) + \sin^2(\psi) \sin^2(\alpha)),$$

while the (1,2) entry is, up to a phase,

$$(90) \quad \rho_{1,2} := \sin(\psi) \cos(\psi)(1 - 2x) (e^{i\phi} \cos(\alpha) \sin(\beta) + e^{-i\phi} \sin(\alpha) \cos(\beta)).$$

From these formulas, it is easily seen that it is not possible to go from a perfectly mixed state ($x = \frac{1}{2}$) to a state with off diagonal element different from zero. It is, however, possible to go from a pure state ($x = 1$) to any arbitrary mixed state (choose $\cos^2(\alpha) = \cos^2(\beta)$ equal to the desired diagonal (1,1) element, $\phi = 0$ and then determine ψ according to the desired absolute value of the off diagonal element). It is also possible to go from an arbitrary state to a pure state (*purification*) by choosing $\psi = 0, \cos^2(\alpha) = \sin^2(\beta) = 1$.

Appendix A. Some proofs of the results in section 3.

Proof of Lemma 3.2.

Proof. All matrices in \mathcal{A} can be simultaneously diagonalized via a change of coordinates. So we can assume that all matrices in \mathcal{A} are diagonal. Consider a basis

of \mathcal{A} , $\mathcal{B}_S := \{A_1, \dots, A_l\}$. Take any element A_j in the basis \mathcal{B}_S . We have that A_j can be written as

$$(91) \quad A_j := \sum_k i\lambda_{j,k}\Pi_k^j,$$

where Π_k^j are diagonal projections and $\lambda_{j,k}$ are all distinct eigenvalues. Since from (22) and the fact that \mathcal{A} is maximal, we have $i\{\mathcal{A}, \mathcal{A}\} \subseteq \mathcal{A}$, and it follows that if $i^{m-1}A_j^m \in \mathcal{A}$, $i^m A_j^{m+1}$ is also in \mathcal{A} , since $i\{A_j, i^{m-1}A_j^m\} \in \mathcal{A}$. Therefore, $i\sum_k \lambda_{j,k}^m \Pi_k^j \in \mathcal{A}$ for every $m \geq 0$ (since \mathcal{A} also contains multiples of the identity). A Vandermonde determinant argument, using the fact that the $\lambda_{j,k}$'s are all different, shows that the diagonal projections Π_k^j also belong to \mathcal{A} . Repeating this argument for all A_j 's, we find a set of diagonal projections which (multiplied by i) span \mathcal{A} . In this set, choose a maximal linearly independent set $\{i\Pi_1, \dots, i\Pi_l\}$. Starting from the set $\{i\Pi_1, \dots, i\Pi_l\}$, it is possible to construct another spanning set for \mathcal{A} of the form $\{i\tilde{\Pi}_1, \dots, i\tilde{\Pi}_s\}$ with $s \geq l$, and $\tilde{\Pi}_j$ all diagonal projections, with the property that

$$(92) \quad \tilde{\Pi}_j \tilde{\Pi}_k = \delta_{j,k} \tilde{\Pi}_j.$$

This is done recursively starting from the set $\{\Pi_1, \dots, \Pi_l\}$. Given two projections, say, Π_1 and Π_2 , we can replace them in the set with three projections $\Pi_1\Pi_2$, $\Pi_1 - \Pi_1\Pi_2$, and $\Pi_2 - \Pi_1\Pi_2$, which still span the subspace spanned by Π_1 and Π_2 , which are such that when multiplied by i they belong to \mathcal{A} because of the property $i\{\mathcal{A}, \mathcal{A}\} \subseteq \mathcal{A}$, and which have the property that the product of any pair of them give zero. Repeating this process recursively, we obtain a spanning set, $\{i\tilde{\Pi}_1, \dots, i\tilde{\Pi}_s\}$, for \mathcal{A} , with all the products between different projections equal to zero. A basis is obtained choosing a minimal spanning set in this set. The basis (24) is obtained after a change of coordinates which groups together the 1's in the same matrix. \square

Proof of Lemma 3.3.

Proof. The first statement of the lemma is a consequence of the definition of the graph $G_{\mathcal{P}}$. Perform a change of coordinates which puts together indexes corresponding to the same connected component of the graph. If j and k are two block indices corresponding to different components, each block at the intersection of the j th and k th row and column block for every matrix in \mathcal{P} is zero, by definition. So the matrices in \mathcal{P} have the corresponding block diagonal structure.

To show the second point of the lemma, denote by $P_{j,k}$ a matrix different from zero at the intersection of the j th and k th row and column block. Let $R_{j,k}$ be the block different from zero at the intersection of the j th and k th index in $P_{j,k}$ (with the block at the intersection of the k th and j th position equal to $-R_{j,k}^\dagger$). Using the basis matrices D_1, \dots, D_l , defined in (24), we calculate $\hat{P}_{j,k} := [D_j, [D_k, P_{j,k}]]$, which is in \mathcal{P} because of (20). The matrix $\hat{P}_{j,k}$ contains only zeros except in the (j, k) th and (k, j) th block, which are occupied by $R_{j,k}$ and $-R_{j,k}^\dagger$, respectively. Consider the matrix $\hat{P}_{j,k} \in \mathcal{P}$ with $j < k$, as defined above. Calculating $i\{\hat{P}_{j,k}, \hat{P}_{j,k}\} \in \mathcal{P}$, we see that this matrix is zero except for the (j, j) th and (k, k) th block that are equal to $-2iR_{j,k}R_{j,k}^\dagger$ and $-2iR_{j,k}^\dagger R_{j,k}$, respectively. Since this new matrix commutes with the maximal Abelian algebra in \mathcal{A} defined in Lemma 3.2, both matrices must be multiples of the identity in dimensions n_j and n_k , respectively, from which we get

$$(93) \quad R_{j,k}R_{j,k}^\dagger = \alpha\mathbf{1}_{n_j}, \quad R_{j,k}^\dagger R_{j,k} = \beta\mathbf{1}_{n_k}.$$

Since $R_{j,k} \neq 0$, both α and β must be different from zero, and we have

$$(94) \quad n_j = \mathbf{rank}(R_{j,k}R_{j,k}^\dagger) = \mathbf{rank}(R_{j,k}^\dagger R_{j,k}) = n_k.$$

Since the graph $G_{\mathcal{P}}$ is connected, taking a path between any two nodes and repeating this argument between neighboring nodes, it follows that $n_1 = n_2 = \dots = n_l$. \square

Proof of Lemma 3.4.

Proof. First, write \mathcal{L}_S as $\mathcal{L}_S = [\mathcal{L}_S, \mathcal{L}_S] + \mathcal{A}_S$, where $[\mathcal{L}_S, \mathcal{L}_S]$ is the semisimple part of \mathcal{L}_S and \mathcal{A}_S the Abelian part. Observe that if $Y \in [\mathcal{L}_S, \mathcal{L}_S]$ and $\langle Y, Y \rangle_K = 0$, then $Y = 0$ since the restriction of the Killing form on $[\mathcal{L}_S, \mathcal{L}_S]$ is equal to the Killing form on this semisimple Lie algebra, which is (negative) definite.

Now, given a basis in $[\mathcal{P}, \mathcal{P}]$, complete it in \mathcal{K} with matrices $\{R_1, \dots, R_r\}$ which are orthogonal to $[\mathcal{P}, \mathcal{P}]$ with respect to the Killing form and set $\mathcal{R} := \text{span}\{R_1, \dots, R_r\}$. Let $R \in \mathcal{R}$ and $P_1, P_2 \in \mathcal{P}$. We have

$$(95) \quad \langle [R, P_1], P_2 \rangle_K = \langle [P_1, P_2], R \rangle_K = 0,$$

which says that $[R, P_1] \in \mathcal{P}^\perp$.³³ However, $[R, P_1] \in [\mathcal{R}, \mathcal{P}] \subseteq [\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}$. Therefore, $[R, P_1] \in \mathcal{P} \cap \mathcal{P}^\perp$. Since $[R, P_1] \in [\mathcal{L}_S, \mathcal{L}_S]$ and the Killing form is negative definite in $[\mathcal{L}_S, \mathcal{L}_S]$, necessarily $[R, P_1] = 0$. Thus, \mathcal{R} commutes with \mathcal{P} .

Now we show that \mathcal{R} is also an ideal in \mathcal{K} . Let R be an arbitrary element in \mathcal{R} , K an arbitrary element in \mathcal{K} , and P_1 and P_2 arbitrary elements in \mathcal{P} . Using the invariance property (26) and the Jacobi identity for Lie algebras, we have

$$(96) \quad \begin{aligned} \langle [K, R], [P_1, P_2] \rangle_K &= \langle [[P_1, P_2], K], R \rangle_K \\ &= -\langle [[P_2, K], P_1], R \rangle_K - \langle [[K, P_1], P_2], R \rangle_K = 0, \end{aligned}$$

where the last equality follows from $[\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}$ and the fact that \mathcal{R} is orthogonal with respect to the Killing form to $[\mathcal{P}, \mathcal{P}]$. Therefore, $[K, R]$ not only belongs to \mathcal{K} but is also orthogonal to $[\mathcal{P}, \mathcal{P}]$. Now write $[K, R]$ as $[K, R] = Y + \tilde{R}$ with $Y \in [\mathcal{P}, \mathcal{P}]$ and $\tilde{R} \in \mathcal{R}$. Since $\tilde{R} \in [\mathcal{P}, \mathcal{P}]^\perp$ and $[K, R] \in [\mathcal{P}, \mathcal{P}]^\perp$, $Y \in [\mathcal{P}, \mathcal{P}]^\perp$ as well. Therefore, we have $Y \in [\mathcal{L}_S, \mathcal{L}_S]$ and $Y \in [\mathcal{P}, \mathcal{P}] \cap [\mathcal{P}, \mathcal{P}]^\perp$, which as before implies $Y = 0$. Therefore, $[K, R] = \tilde{R} \in \mathcal{R}$ and \mathcal{R} is an ideal in \mathcal{K} . \square

Proof of Proposition 3.6.

Proof. If \mathcal{N}_n is not zero, consider a matrix $A \neq 0$ in \mathcal{N}_n . Because of (36), we can replace A with $\frac{1}{\sqrt{\alpha}}A$ and assume that A is unitary. Moreover, by applying a DUCT transformation on \mathcal{N}_n , $A \in \mathcal{N}_n \rightarrow \mathbf{U}A\mathbf{V}$, with \mathbf{U} equal to the identity and $\mathbf{V} := A^\dagger$, we can assume that A is the identity matrix $\mathbf{1}_n$. Using $A = \mathbf{1}_n$ in (35), we find that the Hermitian part of every matrix $B \in \mathcal{N}_n$ is a multiple of the identity, which means that (modulo a DUCT transformation) \mathcal{N}_n is spanned by the identity and skew-Hermitian matrices (if any). If \mathcal{N}_n has dimension ≥ 2 , let us consider a nonzero skew-Hermitian matrix B . We apply a DUCT transformation of a special form with $\mathbf{V} = \mathbf{U}^\dagger$ above (that is, a single unitary conjugacy transformation) $B \rightarrow \mathbf{U}B\mathbf{U}^\dagger$ which does not modify the identity matrix and diagonalizes B . With these coordinates, $B = \text{diag}(ia_1, ia_2, \dots, ia_n)$, and from the fact that BB^\dagger is a multiple of the identity, it follows that $a_1^2 = a_2^2 = \dots = a_n^2$. By scaling B , we can assume that all of the a_j 's are either 1 or -1 , so that, modulo a reordering of row and column indexes, $B = i\mathbf{1}_{r,s}$. In the special case where $s = 0$ and $r = n$, applying relation (35) with a skew-Hermitian

³³Orthogonality is meant with respect to the Killing form.

A and $B = i\mathbf{1}_n$, we see that A must necessarily be a multiple of $i\mathbf{1}_n$. So there is no other skew-Hermitian matrix in \mathcal{N}_n except for multiples of $i\mathbf{1}_n$ in this case. If \mathcal{N}_n has dimension ≥ 3 , we must have that $1 \leq r \leq n - 1$. We decompose one extra (not a multiple of $i\mathbf{1}_{r,s}$) skew-Hermitian matrix C in a basis of \mathcal{N}_n as

$$(97) \quad C := \begin{pmatrix} C_{1,1} & C_{1,2} \\ -C_{1,2}^\dagger & C_{2,2} \end{pmatrix},$$

where $C_{1,1}$ and $C_{2,2}$ are skew-Hermitian and of dimension $r \times r$ and $s \times s$, respectively. By using (35) with $A = C$ and $B = i\mathbf{1}_{r,s}$, we discover that $-2iC_{1,1} = \alpha\mathbf{1}_r$ and $2iC_{2,2} = \alpha\mathbf{1}_s$, so that the block diagonal part of C is a multiple of $i\mathbf{1}_{r,s}$. Therefore, in the basis of \mathcal{N}_n , we can take C of the form (97) with $C_{1,1}$ and $C_{2,2}$ equal to the $r \times r$ and $s \times s$ zero matrix, respectively. The possible matrices C_{12} form themselves a vector space. Moreover, take any possible matrix C . By applying (35) with both A and B equal to C , we find that $C_{1,2}C_{1,2}^\dagger = \alpha\mathbf{1}_r$ and $C_{1,2}^\dagger C_{1,2} = \alpha\mathbf{1}_s$ for some real number α . This shows that $C_{1,2}$ is either zero or it has full rank, and in that case $r = s$. Therefore, the only case where we can have normal vector space of dimensions ≥ 3 is when $r = s = \frac{n}{2}$. In particular, n must be even. Moreover, the space of all matrices $C_{1,2}$ is such that if we apply (35) with the corresponding matrices C , we obtain relation (35) again for matrices of dimension $\frac{n}{2}$. Therefore, the matrices $C_{1,2}$ span a normal space, $\mathcal{N}_{\frac{n}{2}}$, of $\frac{n}{2} \times \frac{n}{2}$ matrices. Moreover, notice that a DUCT transformation on this space $A_{\frac{n}{2}} \rightarrow \mathbf{U}A_{\frac{n}{2}}\mathbf{V}$ can be obtained by a single unitary conjugacy transformation on \mathcal{N}_n of the form

$$(98) \quad A_n \rightarrow \begin{pmatrix} \mathbf{U} & 0 \\ 0 & \mathbf{V}^\dagger \end{pmatrix} \begin{pmatrix} 0 & A_{\frac{n}{2}} \\ -A_{\frac{n}{2}}^\dagger & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}^\dagger & 0 \\ 0 & \mathbf{V} \end{pmatrix},$$

which does not affect the first two matrices that we have found in the basis of \mathcal{N}_n . This gives the recursive construction described in the statement of the theorem. \square

Appendix B. Case $p = 2$ in subsection 6.4.

Proof. The same considerations done at the beginning of subsection 6.4.3 hold for the case $p = 2$ to argue that it is enough to prove that indirect controllability is not verified in the case (76). We also use the notation and the change of coordinates described at the beginning of subsection 6.4.3.

Observe that the dynamical Lie algebra \mathcal{L} admits a Cartan decomposition,³⁴

$$(99) \quad \mathcal{L} := \hat{\mathcal{K}} + \hat{\mathcal{P}},$$

with $\hat{\mathcal{K}} := (\mathcal{K}_3 \otimes \mathbf{1}) + (\mathbf{1}_8 \otimes su(2))$ and $\hat{\mathcal{P}} = i\mathcal{P}_3 \otimes su(2)$. Using such a Cartan decomposition, a general transformation U in $e^{\mathcal{L}}$ can be parametrized as³⁵ $U = T_2 \otimes V_2 e^{\tilde{A}} T_1 \otimes V_1$, where T_1 and T_2 are general unitary transformations in $e^{\mathcal{K}_3}$, V_1 and V_2 are general matrices in $SU(2)$, and \tilde{A} is a matrix in a Cartan subalgebra \mathcal{A} (maximal Abelian subalgebra) in $\hat{\mathcal{P}}$. A general unitary matrix U in $e^{\mathcal{L}}$ gives a transformation on the state ρ_S of the form

$$(100) \quad \begin{aligned} \rho_S &\rightarrow Tr_A(U\rho_S \otimes \rho_A U^\dagger) \\ &= Tr_A \left(T_2 \otimes V_2 e^{\tilde{A}} T_1 \otimes V_1 \rho_S \otimes \rho_A T_1^\dagger \otimes V_1^\dagger e^{-\tilde{A}} T_2^\dagger \otimes V_2^\dagger \right) \\ &= T_2 Tr_A \left(e^{\tilde{A}} (T_1 \rho_S T_1^\dagger \otimes \tilde{\rho}_A) e^{-\tilde{A}} \right) T_2^\dagger \end{aligned}$$

³⁴Recall that we are including now the auxiliary system A in the analysis. The matrices in \mathcal{L} are 16×16 .

³⁵See the discussion at the end of subsection 3.1.

with $\tilde{\rho}_A := V_1 \rho_A V_1^\dagger$. In this case, the Cartan subalgebra \mathcal{A} is three-dimensional. We define (cf. (15))

$$(101) \quad \tilde{\sigma}_x := -2i\sigma_x, \quad \tilde{\sigma}_y := 2i\sigma_y, \quad \tilde{\sigma}_z := -2i\sigma_z,$$

and we take as a basis of \mathcal{A} , $\{i\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x \otimes \tilde{\sigma}_x, i\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y \otimes \tilde{\sigma}_y, i\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_z\}$, so that \tilde{A} in (100) is written as

$$(102) \quad \tilde{A} := ix\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x \otimes \tilde{\sigma}_x + iy\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y \otimes \tilde{\sigma}_y + iz\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_z$$

for real parameters x, y , and z . Moreover, since $\tilde{\rho}_A$ is assumed pure according to what was proved in subsection 5.2, we can write $\tilde{\rho}_A$ as

$$(103) \quad \tilde{\rho}_A := \begin{pmatrix} \cos^2(\theta) & -\frac{1}{2}\sin(2\theta)e^{it} \\ -\frac{1}{2}\sin(2\theta)e^{-it} & \sin^2(\theta) \end{pmatrix}$$

for parameters θ and t in \mathbb{R} . Formula (100) describes the set of available transformations on ρ_S . Each of these transformations can be seen as the cascade of three transformations:

1. a unitary transformation $\rho \rightarrow T_1 \rho T_1^\dagger$ with $T_1 \in e^{\mathcal{K}_3}$ and therefore depending on $21 = \dim \mathcal{K}_3$ parameters;
2. a, not necessarily unitary, transformation $\rho \rightarrow Tr_A(e^{\tilde{A}} \rho \otimes \tilde{\rho}_A e^{-\tilde{A}})$, which depends on five parameters, i.e., x, y, z in (102) and θ and t in (103);
3. another unitary transformation $\rho \rightarrow T_2 \rho T_2^\dagger$, with $T_2 \in e^{\mathcal{K}_3}$ and therefore depending on $21 = \dim \mathcal{K}_3$ parameters.

To prove the claim, it is enough to show that there is a unitary similarity transformation $X_f, \rho_S \rightarrow X_f \rho_S X_f^\dagger$, which cannot be obtained as the cascade of the above three transformations, no matter what parameters are chosen in the various steps. We shall show that this is the case for the transformation $X_f = X_{1-2}$, which switches the first and second position in a tensor product of three 2×2 Hermitian matrices $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$, i.e.,

$$(104) \quad X_{1-2} \tilde{\sigma}_1 \otimes \tilde{\sigma}_2 \otimes \tilde{\sigma}_3 X_{1-2}^\dagger = \tilde{\sigma}_2 \otimes \tilde{\sigma}_1 \otimes \tilde{\sigma}_3 \text{ for all } \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3 \in iu(2).$$

Let us set up a few more definitions.

With spaces of 4×4 Hermitian matrices

$$(105) \quad \mathbf{L} := \{\mathbf{1} \otimes Z\} + \{Z \otimes \mathbf{1}\} + \{Y \otimes Y\} \quad \text{and} \quad \mathbf{R} := \{\mathbf{1} \otimes Y\} + \{Y \otimes Z\},$$

we can rewrite $i\mathcal{K}_3$ as

$$(106) \quad i\mathcal{K}_3 := \{\mathbf{L} \otimes \mathbf{1}\} + \{\mathbf{R} \otimes (isu(2))\} + \{\mathbf{1} \otimes (isu(2))\}.$$

Consider now a general matrix $\tilde{\rho}_S$ of the form $\tilde{\rho}_S = \frac{1}{8}\mathbf{1}_8 + S$ with $S \in i\mathcal{K}_3$. Such a matrix can be written as

$$(107) \quad \tilde{\rho}_S := \frac{1}{8}\mathbf{1}_8 + L \otimes \mathbf{1}_2 + \sum_{j=x,y,z} R_j \otimes \tilde{\sigma}_j + \sum_{k=x,y,z} a_k \mathbf{1}_4 \otimes \tilde{\sigma}_k$$

with $L \in \mathbf{L}$, $R_{x,y,z} \in \mathbf{R}$, and $a_{x,y,z}$ real numbers. For such a type of matrix, we calculate explicitly $Tr_A(e^{\tilde{A}} \tilde{\rho}_S \otimes \tilde{\rho}_A e^{-\tilde{A}})$. Using the definition where $E_{j,k}$ is the 4×4

matrix with all zeros except for the entries j and k on the diagonal which are occupied by 1, we obtain

(108)

$$\begin{aligned}
 & Tr_A(e^{\tilde{A}}\tilde{\rho}_S \otimes \tilde{\rho}_A e^{-\tilde{A}}) \\
 &= \frac{1}{8}\mathbf{1}_8 + (\sin^2(y) - \sin^2(x)) \cos(2\theta)\mathbf{1}_4 \otimes \tilde{\sigma}_z \\
 &\quad + \sin(2\theta) \sin(2z) \cos(t) \sin(x - y)\mathbf{1}_4 \otimes \tilde{\sigma}_x \\
 &\quad + \sin(2\theta) \sin(2z) \sin(t) \sin(x + y)\mathbf{1}_4 \otimes \tilde{\sigma}_y + L \otimes \mathbf{1}_2 \\
 &\quad + (\sin^2(y) - \sin^2(x)) \cos(2\theta)L \otimes \tilde{\sigma}_z + \sin(2\theta) \sin(2z) \cos(t) \sin(x - y)L \otimes \tilde{\sigma}_x \\
 &\quad + \sin(2\theta) \sin(2z) \sin(t) \sin(x + y)L \otimes \tilde{\sigma}_y \\
 &\quad + \frac{1}{2} \cos(2\theta) \cos(2z)(\cos(2x) - \cos(2y))R_z \otimes \mathbf{1}_2 \\
 &\quad + \cos(2z)R_z \otimes \tilde{\sigma}_z + \frac{1}{2} \cos(2\theta) \sin(2z)(\cos(2x) \\
 &\quad + \cos(2y)) (iE_{1,4}R_zE_{2,3} - iE_{2,3}R_zE_{1,4}) \otimes \mathbf{1}_2 \\
 &\quad + \frac{1}{2} \sin(2z) \sin(2\theta) \cos(t) (\sin(2x) - \sin(2y)) R_x \otimes \mathbf{1}_2 + \cos(x + y)R_x \otimes \tilde{\sigma}_x \\
 &\quad + -\frac{1}{2} \cos(2z) \sin(2\theta) \cos(t)(\sin(2x) + \sin(2y)) (iE_{1,4}R_xE_{2,3} - iE_{2,3}R_xE_{1,4}) \otimes \mathbf{1}_2 \\
 &\quad + \frac{1}{2} \sin(2\theta) \sin(t) \sin(2z) (\sin(2x) + \sin(2y)) R_y \otimes \mathbf{1}_2 + \cos(x - y)R_y \otimes \tilde{\sigma}_y \\
 &\quad - \frac{1}{2} \sin(2\theta) \sin(t) \cos(2z) (\sin(2x) - \sin(2y)) (iE_{1,4}R_yE_{2,3} - iE_{2,3}R_yE_{1,4}) \otimes \mathbf{1}_2 \\
 &\quad + a_z (\cos^2(x) - \sin^2(y)) \mathbf{1}_4 \otimes \tilde{\sigma}_z - a_z \sin(2\theta) \cos(2z) \sin(t) \sin(y - x)\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x \\
 &\quad + a_z \sin(2\theta) \cos(2z) \cos(t) \sin(x + y)\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y + a_x \cos(2z) \cos(x - y)\mathbf{1}_4 \otimes \tilde{\sigma}_x \\
 &\quad - \frac{a_x}{2} (\sin(2x) - \sin(2y)) \sin(2\theta) \sin(t)\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_z \\
 &\quad - a_x \sin(2z) \cos(x + y) \cos(2\theta)\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y + a_y \cos(2z) \cos(x + y)\mathbf{1}_4 \otimes \tilde{\sigma}_y \\
 &\quad - \frac{a_y}{2} \cos(t) \sin(2\theta)(\sin(2y) + \sin(2x))\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_z \\
 &\quad + a_y \sin(2z) \cos(x - y) \cos(2\theta)\tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x.
 \end{aligned}$$

Since X_{1-2} is unitary, the transformation in (108) must be unitary and in particular it must leave multiples of the identity unchanged. This implies that we have in (108)

(109) $(\sin^2(y) - \sin^2(x)) \cos(2\theta) = 0,$

(110) $\sin(2\theta) \sin(2z) \cos(t) \sin(x - y) = 0,$

(111) $\sin(2\theta) \sin(2z) \sin(t) \sin(x + y) = 0.$

This also implies that if $\tilde{\rho}_S$ is of the form $\tilde{\rho}_S = \frac{1}{8}\mathbf{1}_8 + L \otimes \mathbf{1}_2$ with $L \in \mathbf{L}$, it is left unchanged by the transformation (108). From this, we try to obtain information on the form of T_1 . Denote by $\rho_{ini} := \rho_S - \frac{1}{8}\mathbf{1}_8$ and by $\rho_{fin} = X_{1-2}\rho_S X_{1-2}^\dagger - \frac{1}{8}\mathbf{1}_8$. Assume that $\rho_{ini} \in T_1^\dagger \mathbf{L} \otimes \mathbf{1}_2 T_1$. Since $T_1 \in e^{\mathcal{K}_3}$, ρ_{ini} belongs to $i\mathcal{K}_3$. Moreover, since $T_2 \in e^{\mathcal{K}_3}$, $\rho_{fin} \in i\mathcal{K}_3$ as well. Since ρ_{fin} is obtained from ρ_{ini} by switching the first and second position in the tensor products, ρ_{ini} must belong to the subspace of $i\mathcal{K}_3$, which remains in $i\mathcal{K}_3$ once we permute the first two positions. This subspace is given

by $\{\mathbf{L} \otimes \mathbf{1}_2\} + \{\mathbf{1}_4 \otimes isu(2)\}$. This reasoning shows that T_1 is such that

$$(112) \quad T_1^\dagger \mathbf{L} \otimes \mathbf{1} T_1 \subseteq \{\mathbf{L} \otimes \mathbf{1}_2\} + \{\mathbf{1}_4 \otimes isu(2)\}.$$

Now we proceed to a parametrization of T_1 according to a Cartan decomposition of \mathcal{K}_3 . Let $\mathcal{K}_3 := \mathcal{D} + \mathcal{Q}$, with $\mathcal{D} := \{i\mathbf{L} \otimes \mathbf{1}_2\} + \{\mathbf{1}_4 \otimes su(2)\}$, and $\mathcal{Q} := \mathbf{R} \otimes su(2)$, with (cf. (20))

$$(113) \quad [\mathcal{D}, \mathcal{D}] \subseteq \mathcal{D}, \quad [\mathcal{Q}, \mathcal{D}] \subseteq \mathcal{Q}, \quad [\mathcal{Q}, \mathcal{Q}] \subseteq \mathcal{D}.$$

Choosing a basis of a Cartan subalgebra in \mathcal{Q} given by $\{i\tilde{\sigma}_x \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x, i\tilde{\sigma}_y \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y, i\mathbf{1}_2 \otimes \tilde{\sigma}_x \otimes \tilde{\sigma}_z\}$, we write $T_1 \in e^{\mathcal{K}_3}$ as

$$(114) \quad T_1 := P_1 \otimes \tilde{V}_1 e^B P_2 \otimes \tilde{V}_2$$

with $P_1, P_2 \in e^{i\mathbf{L}}$, $\tilde{V}_1, \tilde{V}_2 \in SU(2)$, and

$$(115) \quad B := ia\tilde{\sigma}_x \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x + ib\tilde{\sigma}_y \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y + ic\mathbf{1}_2 \otimes \tilde{\sigma}_x \otimes \tilde{\sigma}_z$$

for real parameters a, b , and c . With the structure of T_1 in (114), condition (112) implies that, for every $L \in \mathbf{L}$,

$$(116) \quad e^{-B} L \otimes \mathbf{1}_2 e^B = M \otimes \mathbf{1}_2 + \mathbf{1}_4 \otimes \tilde{\sigma}$$

for some $M \in \mathbf{L}$ and $\tilde{\sigma} \in su(2)$. Imposing this for a basis of \mathbf{L} , we find, for the parameters a, b , and c in (115),

$$(117) \quad \sin(2a) = \sin(2b) = \sin(2c) = 0.$$

This, by writing e^B as

$$(118) \quad e^B = e^{ia\tilde{\sigma}_x \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x} e^{ib\tilde{\sigma}_y \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y} e^{ic\mathbf{1}_2 \otimes \tilde{\sigma}_x \otimes \tilde{\sigma}_z},$$

implies that the first factor is equal to $\pm \mathbf{1}_8$ or $\pm i\tilde{\sigma}_x \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x$, the second factor is equal to $\pm \mathbf{1}_8$ or $\pm i\tilde{\sigma}_y \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y$, and the third factor is equal to $\pm \mathbf{1}_8$ or $\pm i\mathbf{1}_2 \otimes \tilde{\sigma}_x \otimes \tilde{\sigma}_z$. In particular, in every case, e^B has the form of a ‘‘local’’ transformation

$$(119) \quad e^B = C_1 \otimes C_2 \otimes C_3$$

with unitary 2×2 transformations C_1, C_2, C_3 . This, combined with (114), shows that T_1 must be of the form

$$(120) \quad T_1 := Q_1 \otimes \tilde{V}_1 C_3 \tilde{V}_2$$

with $Q_1 := P_1(C_1 \otimes C_2)P_2$.

Let us now apply the full cascade of the three transformations in points 1, 2, and 3 above, which, by assumption, gives X_{1-2} , to $\mathbf{1}_4 \otimes (\tilde{V}_2^\dagger C_3^\dagger \tilde{V}_1^\dagger \tilde{\sigma}_z \tilde{V}_1 C_3 \tilde{V}_2)$. Application of the transformation $\rho \rightarrow T_1 \rho T_1^\dagger$ gives $\mathbf{1} \otimes \tilde{\sigma}_z$. By applying (108) to $\mathbf{1} \otimes \tilde{\sigma}_z$ with the conditions (109), (110), (111), we have

$$(121) \quad \begin{aligned} Tr_A(e^{\tilde{A}} \mathbf{1}_4 \otimes \tilde{\sigma}_z \otimes \tilde{\rho}_A e^{-\tilde{A}}) &= (\cos^2(x) - \sin^2(y)) \mathbf{1}_4 \otimes \tilde{\sigma}_z \\ &\quad - \sin(2\theta) \cos(2z) \sin(t) \sin(y-x) \tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x \\ &\quad + \sin(2\theta) \cos(2z) \cos(t) \sin(x+y) \tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y. \end{aligned}$$

Since the result of this transformation must be in $i\mathcal{K}_3$, because the third transformation $T_2 \in e^{\mathcal{K}_3}$ and $X_{1-2}\mathbf{1}_4 \otimes \tilde{\sigma}_z X_{1-2}^\dagger = \mathbf{1}_4 \otimes \tilde{\sigma}_z \in i\mathcal{K}_3$, we must have $\sin(2\theta) \cos(2z) \sin(t) \sin(y-x) = 0$, and $\sin(2\theta) \cos(2z) \cos(t) \sin(x+y) = 0$. These imply in (121), since the norm has to be preserved (because the total transformation must be unitary),

$$(122) \quad (\cos^2(x) - \sin^2(y))^2 = 1.$$

This condition along with (109), (110), (111) gives the following simplification of (108)

$$(123) \quad \begin{aligned} & Tr_A(e^{\tilde{A}} \tilde{\rho}_S \otimes \tilde{\rho}_A e^{-\tilde{A}}) \\ &= \frac{1}{8} \mathbf{1}_8 + L \otimes \mathbf{1}_2 + \cos(2z) R_z \otimes \tilde{\sigma}_z \\ &\quad \pm \cos(2\theta) \sin(2z) (iE_{1,4} R_z E_{2,3} - iE_{2,3} R_z E_{1,4}) \otimes \mathbf{1}_2 \pm R_x \otimes \tilde{\sigma}_x \\ &\quad \pm R_y \otimes \tilde{\sigma}_y \pm a_z \mathbf{1}_4 \otimes \tilde{\sigma}_z \pm a_x \cos(2z) \mathbf{1}_4 \otimes \tilde{\sigma}_x \\ &\quad \pm a_x \sin(2z) \cos(2\theta) \tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y \pm a_y \cos(2z) \mathbf{1}_4 \otimes \tilde{\sigma}_y \\ &\quad \pm a_y \sin(2z) \cos(2\theta) \tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_x. \end{aligned}$$

Now assume we start from $\mathbf{1}_4 \otimes (\tilde{V}_2^\dagger C_3^\dagger \tilde{V}_1^\dagger \tilde{\sigma}_x \tilde{V}_1 C_3 \tilde{V}_2)$. Application of the transformation $\rho \rightarrow T_1 \rho T_1^\dagger$ gives $\mathbf{1} \otimes \tilde{\sigma}_x$. Using (123) with $\tilde{\rho}_S = \frac{1}{8} \mathbf{1}_8 + \mathbf{1}_4 \otimes \tilde{\sigma}_x$ gives

$$(124) \quad Tr_A(e^{\tilde{A}} (\mathbf{1}_4 \otimes \tilde{\sigma}_x \otimes \tilde{\rho}_A) e^{-\tilde{A}}) = \pm \cos(2z) \mathbf{1}_4 \otimes \tilde{\sigma}_x \pm \sin(2z) \cos(2\theta) \tilde{\sigma}_z \otimes \tilde{\sigma}_z \otimes \tilde{\sigma}_y.$$

Imposing that this belongs to $i\mathcal{K}_3$ gives

$$(125) \quad \sin(2z) \cos(2\theta) = 0.$$

Moreover, norm preservation gives $\cos^2(2z) = 1$. Using this in (123), we get

$$(126) \quad \begin{aligned} & Tr_A(e^{\tilde{A}} \tilde{\rho}_S \otimes \tilde{\rho}_A e^{-\tilde{A}}) = \frac{1}{8} \mathbf{1}_8 + L \otimes \mathbf{1}_2 \\ &\quad \pm R_z \otimes \tilde{\sigma}_z \pm R_x \otimes \tilde{\sigma}_x \pm R_y \otimes \tilde{\sigma}_y \pm a_z \mathbf{1}_4 \otimes \tilde{\sigma}_z \\ &\quad \pm a_x \mathbf{1}_4 \otimes \tilde{\sigma}_x \pm a_y \mathbf{1}_4 \otimes \tilde{\sigma}_y. \end{aligned}$$

Therefore, $\tilde{\rho}_S \rightarrow Tr_A(e^{\tilde{A}} \tilde{\rho}_S \otimes \tilde{\rho}_A e^{-\tilde{A}})$ does not modify $\tilde{\rho}_S$ except for possibly some changes in the sign of the coefficients. It follows that if $\tilde{\rho}_S = \frac{1}{8} \mathbf{1}_8 + S$ with $S \in i\mathcal{K}_3$, the transformed also can be written as $\mathbf{1}_8 + \tilde{S}$ with $\tilde{S} \in i\mathcal{K}_3$. It follows that if the initial ρ_S has the property that $\rho_S - \frac{1}{8} \mathbf{1}_8 \in i\mathcal{K}_3$, the final value of the density matrix has this property as well (since the similarity transformations by T_1 and T_2 do not modify the property of a matrix to belong to $i\mathcal{K}_3$). However, this is incompatible with the form of X_{1-2} since the transformation $\rho \rightarrow X_{1-2} \rho X_{1-2}^\dagger$ does *not* leave $i\mathcal{K}_3$ invariant. This concludes the proof of this part of the theorem. \square

Appendix C.

Proof. We use induction on n . We first verify that (A1)–(A6) are satisfied for $n = 4$. This can be done using (84), (85), and (B1)–(B5) and defining

$$(127) \quad i\{\mathcal{K}_4, \mathcal{P}_4\} = Z \otimes \mathcal{B}_3 + Y \otimes \mathcal{K}_3 + \mathbf{1} \otimes \mathcal{P}_3 := \mathcal{B}_4$$

and

$$(128) \quad [\mathcal{P}_4, \mathcal{B}_4] = \mathbf{1} \otimes \mathcal{B}_3 + Y \otimes \mathcal{B}_3 + Z \otimes \mathcal{K}_3 := \mathcal{C}_4.$$

Then we show that, if (A1)–(A6) hold for a certain n , they hold for $n + 1$, which completes the proof by induction. In order to do that, define

$$(129) \quad \mathcal{B}_{n+1} := i \{ \mathcal{K}_n, \mathcal{P}_n \} = Z \otimes \mathcal{B}_n + Y \otimes \mathcal{K}_n + \mathbf{1} \otimes \mathcal{P}_n$$

and

$$(130) \quad \mathcal{C}_{n+1} := [\mathcal{P}_{n+1}, \mathcal{B}_{n+1}] = \mathbf{1} \otimes [\mathcal{P}_n, \mathcal{B}_n] + Y \otimes \mathcal{B}_n + Z \otimes \mathcal{K}_n = \mathbf{1} \otimes \mathcal{C}_n + Y \otimes \mathcal{B}_n + Z \otimes \mathcal{K}_n.$$

So both (A1) and (A5) are automatically satisfied. Using (86), we have

$$i \{ \mathcal{P}_{n+1}, \mathcal{B}_{n+1} \} = i \mathbf{1} \otimes \{ \mathcal{P}_n, \mathcal{B}_n \} + Y \otimes \mathcal{P}_n + i Z \otimes \mathbf{1} = \mathbf{1} \otimes \mathcal{K}_n + Y \otimes \mathcal{P}_n + i Z \otimes \mathbf{1} = \mathcal{K}_{n+1}.$$

Therefore, (A2) holds. Now we verify (A3):

$$\begin{aligned} [\mathcal{K}_{n+1}, \mathcal{B}_{n+1}] &= Z \otimes [\mathcal{K}_n, \mathcal{B}_n] + Y \otimes [\mathcal{K}_n, \mathcal{K}_n] + \mathbf{1} \otimes [\mathcal{K}_n, \mathcal{P}_n] \\ &\quad + i Y \otimes \{ \mathcal{P}_n, \mathcal{B}_n \} + i Z \otimes \{ \mathcal{K}_n, \mathcal{P}_n \} \\ &\quad + \mathbf{1} \otimes [\mathcal{P}_n, \mathcal{K}_n] + Y \otimes [\mathcal{P}_n, \mathcal{P}_n] + Y \otimes \mathcal{K}_n \\ &= Z \otimes \mathcal{B}_n + Y \otimes \mathcal{K}_n + \mathbf{1} \otimes \mathcal{P}_n = \mathcal{B}_{n+1}. \end{aligned}$$

Moreover, we have

$$[\mathcal{K}_{n+1}, \mathcal{C}_{n+1}] = \mathbf{1} \otimes \mathcal{C}_n + Y \otimes \mathcal{B}_n + Z \otimes \mathcal{K}_n := \mathcal{C}_{n+1}.$$

Thus, (A4) holds. Next, we verify that (A6) holds:

$$\begin{aligned} [\mathcal{P}_{n+1}, \mathcal{C}_{n+1}] &= Z \otimes [\mathcal{P}_n, \mathcal{C}_n] + Y \otimes i \{ \mathcal{P}_{n+1}, \mathcal{B}_{n+1} \} + \mathbf{1} \otimes [\mathcal{P}_n, \mathcal{K}_n] + Z \otimes \mathcal{B}_n + Y \otimes \mathcal{K}_n \\ &= Z \otimes \mathcal{B}_n + Y \otimes \mathcal{K}_n + \mathbf{1} \otimes \mathcal{P}_n := \mathcal{B}_{n+1}. \quad \square \end{aligned}$$

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