A NOTE ON GEOMETRIC PROPERTIES FOR CURRENTS

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1. Introduction

The celebrated Frobenius Theorem states the equivalence between the complete integrability of a k-dimensional simple vectorfield $\xi = \tau_1 \wedge \ldots \wedge \tau_k \in C^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$ in $\mathbb{R}^d$ – that is, the existence of a local foliation of $\mathbb{R}^d$ by k-dimensional tangent submanifolds of class C^2 – and the involutivity condition for ξ, that is

$$[\tau_m, \tau_n](x) \in \operatorname{span}\{\tau_1(x) \ldots, \tau_k(x)\} \quad x \in \mathbb{R}^d, \ \forall\, m, n = 1, \ldots, k,$$

where $[\cdot,\cdot]$ denotes the Lie bracket (commutator) between vector fields in $\mathbb{R}^d$.

In this paper, we consider the following natural question: given a k-vectorfield ξ, is it possible to generalize the Frobenius Theorem to k-dimensional surface of a weaker type? In Section 4 we prove the following version of Frobenius Theorem for integral currents, where we use the standard notation $R = [\![\Sigma, \xi, \theta]\!]$ to denote the current R in $\mathbb{R}^d$ defined by

$$R(\omega) = \int_\Sigma \langle \omega(x), \xi(x)\rangle \theta(x)\, d\mathcal{H}^k(x), \qquad \forall \omega \in C_c^\infty(\mathbb{R}^d; \Lambda^k(\mathbb{R}^d))$$

for $\Sigma \subset \mathbb{R}^d$ a k-rectifiable set, ξ a k-vectorfield and θ an integer multiplicity.

Theorem 1.1. *Let $\xi = \tau_1 \wedge \ldots \wedge \tau_k$ be a k-dimensional simple vector field in $\mathbb{R}^d$, with $\tau_1, \ldots, \tau_k \in C^1(\mathbb{R}^d)$, and let $R = [\![\Sigma, \xi, \theta]\!]$ be a k-dimensional integral current in $\mathbb{R}^d$. Then, for every pair $m, n = 1, \ldots, k$ and for every x in the closure of the set of points of positive density of Σ, one has*

$$[\tau_m, \tau_n](x) \in \operatorname{span}\{\tau_1(x), \ldots, \tau_k(x)\}\,.$$

The proof of Theorem 1.1 builds upon two results. The first one, which is probably the main result of the paper, establishes an intuitive geometric property for the boundary of an integral current.

Theorem 1.2. *Let R be a k-dimensional integral current in $\mathbb{R}^d$ and let $\xi \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$ be a continuous k-vectorfield which is tangent to R. Then, the orientation $\eta(x) \in \Lambda_{k-1}(\mathbb{R}^d)$ of the boundary ∂R is a subspace of $\xi(x)$ for $\mathscr{H}^{k-1}$-almost every x in the support of ∂R.*

This result is studied in Section 3 and it is obtained by a blow-up technique.

The second main tool in the proof of Theorem 1.1 is the following Differential Geometry result.

Lemma 1.3. *If ξ is a smooth non-involutive simple k-vectorfield, then there exists a $(k-1)$-form α with the following properties:*

- (i) $\langle \xi, \mathrm{d}\alpha\rangle \neq 0$ *on a suitable open set;*
- (ii) *if η is a simple $(k-1)$-vector field representing a linear subspace of ξ, then* $\langle \eta, \alpha\rangle = 0$.

The proof of Theorem 1.1 follows quite readily once Theorem 1.2 and Lemma 1.3 are available. Indeed, assuming by contradiction that ξ is non-involutive and representing $\partial R = [\![\Sigma', \eta, \theta']\!]$, up to a localization argument one obtains

$$0 = \int_{\Sigma'} \langle \eta, \alpha \rangle \theta' \, d\mathscr{H}^{k-1} = \partial R(\alpha) = R(\mathrm{d}\alpha) = \int_{\Sigma} \langle \xi, \mathrm{d}\alpha \rangle \theta \, d\mathscr{H}^k \neq 0 \,.$$

Let us point out the Theorem 1.1 concerns only one of the two implications in Frobenius Theorem. One might wonder whether the converse is true in the following version: given a normal current $T = \xi\mu$, where μ is a finite measure on $\mathbb{R}^d$ and $\xi \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$ is an involutive vectorfield, is it possible to "foliate" T with a family of integral currents in a proper way (that is, without wasting any mass)? This question was formulated in a broader version by F. Morgan who asked in [1] whether, given a k-dimensional normal current T in $\mathbb{R}^d$, it is possible to find a family of integral currents $(R_\lambda)_{\lambda \in L}$ (where L is a suitable measure space) such that

(i) $T = \int_L R_\lambda \, d\lambda$;
(ii) $\mathbb{M}(T) = \int_L \mathbb{M}(R_\lambda) \, d\lambda$;
(iii) $\mathbb{M}(\partial T) = \int_L \mathbb{M}(\partial R_\lambda) \, d\lambda$.

In Section 5 we study this problem and we show that the normal current $T = \xi \mathscr{L}^d$ does not admit such a decomposition if ξ is a non-involutive vectorfield. Indeed, conditions (i) and (ii) imply that $R_\lambda = [\![\Sigma_\lambda, \xi, \theta_\lambda]\!]$ for almost every $\lambda \in L$, thus λ-almost every R_λ should violate Theorem 1.1.

2. Notation

In the following, we work with a fixed orthonormal basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$ and a dual basis $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_d\}$ such that

$$\langle e_i ; \mathrm{d}x_j \rangle = \delta_{ij} \,.$$

A basis for the space of alternating vectors $\Lambda_k(\mathbb{R}^d)$ is given by

$$\{ e_{i_1} \wedge \ldots \wedge e_{i_k} : 1 \le i_1 < \ldots < i_k \le d \} \,.$$

Correspondingly, a basis for the space of alternating covectors $\Lambda^h(\mathbb{R}^d)$ is given by

$$\{ \mathrm{d}x_{i_1} \wedge \ldots \mathrm{d}x_{i_h} : 1 \le i_1 < \ldots < i_h \le d \} \,.$$

Definition 2.1. Given a h-covector $w \in \Lambda^h(\mathbb{R}^d)$ and a k-vector $v \in \Lambda_k(\mathbb{R}^d)$, with $h \ge k$, the *interior product* is defined as

$$\langle \hat{v}; v \lrcorner w \rangle := \langle \hat{v} \wedge v; w \rangle \quad \forall\, \hat{v} \in \Lambda_{h-k}(\mathbb{R}^d) \,.$$

Vice versa, if $h \le k$, we can define

$$\langle v \llcorner w; \hat{w} \rangle := \langle v; w \wedge \hat{w} \rangle \quad \forall\, \hat{w} \in \Lambda^{k-h}(\mathbb{R}^d) \,.$$

Definition 2.2. Given a k-vector $v \in \Lambda_k(\mathbb{R}^d)$, we define the *span* of v as the smallest linear subspace V of $\mathbb{R}^d$ such that $v \in \Lambda_k(V)$, that is

$$\mathrm{span}(v) := \bigcap_{\substack{V \subset \mathbb{R}^d \text{ l.s.} \\ v \in \Lambda_k(V)}} V \,.$$

Sometimes, when we have $v \in \Lambda_k(\mathbb{R}^d)$, $v' \in \Lambda_{k'}(\mathbb{R}^d)$ and $\mathrm{span}(v') \subset \mathrm{span}(v)$, we abbreviate $v' \subset v$.

Finally we recall the definition of the *Lie bracket* in a convenient coordinate version.

Definition 2.3. Given a vectorfield $X : U \subset \mathbb{R}^d \to \mathbb{R}^d$ of class $\mathscr{C}^1$, we represent its action on a smooth function $f : U \to \mathbb{R}$ as $X(f) = \nabla_X f := \langle \nabla f; X\rangle = \sum_{i=1}^d \frac{\partial f}{\partial x_i} X_i$. If $X \equiv e_i$ for some $i \in \{1, \dots, d\}$, then we also write $\nabla_i f = \frac{\partial f}{\partial x_i}$. Given two $\mathscr{C}^1$-vectorfields $X, Y : U \to \mathbb{R}^d$ their Lie bracket is $[X, Y] := XY - YX$ and $[X, Y](f) = \nabla_X \nabla_Y f - \nabla_Y \nabla_X f$. In coordinates

$$[X, Y]_h = \langle \nabla Y_h; X\rangle - \langle \nabla X_h; Y\rangle = \sum_{i=1}^d \left(\frac{\partial Y_h}{\partial x_i} X_i - \frac{\partial X_h}{\partial x_i} Y_i \right), \quad \text{with } h = 1, \dots, d\,.$$

3. Geometric structure of the boundary

Definition 3.1. Given a k-dimensional normal current $T = \xi\mu$, with orientation $\xi \in C(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$ and μ Radon measure in $\mathbb{R}^d$, and its boundary $\partial T = \eta\mu'$, with orientation η and μ' Radon measure in $\mathbb{R}^d$, we say that T has the *geometric boundary property* if

$$\operatorname{span}\eta(x) \subset \operatorname{span}\xi(x) \quad \mu'\text{-a.e. } x\,. \tag{3.1}$$

Remark 3.2. If ξ is a simple vectorfield, then the geometric boundary property is equivalent to the existence, μ'-a.e. x, of a $w \in \Lambda^1(\mathbb{R}^d)$ such that

$$\eta(x) = \xi(x) \llcorner w\,. \tag{3.2}$$

In general, (3.2) is stronger than (3.1).

Remark 3.3. The geometric boundary property does not hold for every normal current T. For instance, if $T = \xi\mathscr{L}^d$, where ξ is a compactly supported non-involutive vectorfield in the sense of Theorem 4.2 below, then ∂T has orientation $\operatorname{div}\xi$ and the geometric boundary property does not hold.

Theorem 3.4. *Let ξ be a continuous k-dimensional vectorfield on $\mathbb{R}^d$ and let $R = [\![\Sigma, \xi, \theta]\!]$ be a k-dimensional integral current, then T has the geometric boundary property.*

The proof of Theorem 3.4 is based on the blow up technique and can be essentially split in two parts:

(1) for an integral current R with boundary $\partial R = [\![\Sigma', \eta, \theta']\!]$ it is possible to do a blow up at $\mathscr{H}^{k-1}$-a.e. point of Σ' and this blow up commutes with the boundary;
(2) if $x_0 \in \Sigma'$ is one of the points where we could perform such a blow up and $\xi(x_0)$ is the constant orientation of the blown up current and $\eta(x_0)$ is the constant orientation of the blow up of the boundary of the current, then $\operatorname{span}\eta(x_0) \subset \operatorname{span}\xi(x_0)$, because

$$\partial R_0 \llcorner \nu(\varphi) = \partial R_0(\nu) = -R_0(\nu \wedge \mathrm{d}\varphi) = -R_0 \llcorner \nu(\mathrm{d}\varphi) = 0$$

for every $\nu \in \Lambda^1(\mathbb{R}^d)$ such that $\xi(x_0) \llcorner \nu = 0$.

4. Frobenius-type result for integral currents

One of the most important theorems in Differential Geometry fully answers to this question: under which conditions a given k-dimensional simple vectorfield $\xi = \tau_1 \wedge \ldots \wedge \tau_k \in \mathscr{C}^\infty(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$ represents the tangent vectorfield of a smooth manifold M?

Definition 4.1. A k-dimensional simple vectorfield $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$, is *integrable* if, for every point $x_0 \in \mathbb{R}^d$, there exist an open neighborhood $U \ni x_0$ and a k-dimensional submanifold $M \ni x_0$ such that

$$T_x M = \operatorname{span}\{\tau_1(x), \ldots, \tau_k(x)\}$$

for every $x \in U \cap M$. We will say that ξ is *completely integrable* if, for every $x_0 \in \mathbb{R}^d$, there exist an open neighborhood $U \ni x_0$ and a $\mathscr{C}^2$-function[1] $F : U \to \mathbb{R}^{d-k}$ such that its level sets $\{F = p\}$ are k-submanifolds with $\operatorname{span}\{\tau_1, \ldots, \tau_k\}$ as tangent space.

Roughly speaking, a completely integrable vectorfield is a tangent field for a local foliation of $\mathbb{R}^d$ in $\mathscr{C}^2$-submanifolds. Obviously, a completely integrable vectorfield is integrable.

The proof of the following theorem characterizing integrable vectorfields can be found in [4] or in any other book about the basics of smooth manifolds.

Theorem 4.2 (Frobenius Theorem)**.** *A k-dimensional simple vectorfield $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$, is completely integrable if and only if*

$$[\tau_m, \tau_n](x) \in \operatorname{span}\{\tau_1(x), \ldots, \tau_k(x)\} \tag{4.1}$$

for every $x \in \mathbb{R}^d$ and for every $m, n = 1, \ldots, k$.

Let us remark that condition (4.1) does not depend on the choice of $\tau_1, \ldots, \tau_k$, but only on their product $\xi = \tau_1 \wedge \ldots \wedge \tau_k$.

Definition 4.3. Consider a k-dimensional simple vectorfield $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, such that $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$. We say that ξ is involutive[2] if (4.1) holds.

Lemma 4.4. *Given a non-involutive simple k-vectorfield $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$, there exist an open subset $U \subset \mathbb{R}^d$ and a $(k-1)$-form α such that we have*

$$\langle \xi(x), \mathrm{d}\alpha(x) \rangle \neq 0 \qquad \forall\, x \in U \tag{4.2}$$

and

$$\langle \eta(x), \alpha(x) \rangle = 0 \qquad \forall\, x \in \mathbb{R}^d \tag{4.3}$$

whenever $\eta(x)$ is a simple $(k-1)$-vectorfield representing a linear subspace of $\xi(x)$.

Consider a non-involutive k-vectorfield ξ in $\mathbb{R}^d$, as above. We may wonder if the non-involutivity property is strong enough to prevent not only the existence of a surface with tangent field ξ, but also the existence of an integral current with such

[1] From this perspective, it is clear that this problem and the problem of the existence of a potential $F : U \subset \mathbb{R}^d \to \mathbb{R}$ for a given map $f : U \to \mathbb{R}^d$, with $\nabla F = f$, are related and we refer to them as integrability problems.

[2] Actually, a more general definition of involutivity can be given, after noticing that $\operatorname{div}\xi \wedge \tau_m \wedge \tau_n = (-1)^{k-1}\xi \wedge [\tau_m, \tau_n]$. Indeed, one can say that $\xi = \tau_1 \wedge \ldots \wedge \tau_k$ is involutive if and only if $\operatorname{div}\xi \wedge \tau_m \wedge \tau_n = 0$ for every $m, n = 1, \ldots, d$. This definition applies when $\tau_1, \ldots, \tau_k$ are barely continuous, provided $\operatorname{div}\xi$ is a measure.

a tangent field. The answer is affirmative, as we state in Theorem 4.5. We begin with the key theorem.

Theorem 4.5. *Let $\xi = \tau_1 \wedge \ldots \wedge \tau_k$ be a k-dimensional simple vectorfield on $\mathbb{R}^d$, with $\tau_1, \ldots, \tau_k \in \mathscr{C}^1(\mathbb{R}^d)$, and let $T \in \mathscr{I}_k(\mathbb{R}^d)$ be a k-dimensional integral current with $R = [\![\Sigma, \xi, \theta]\!]$, then*

$$[\tau_m, \tau_n](x) \in \operatorname{span}\{\tau_1(x), \ldots, \tau_k(x)\} \tag{4.4}$$

for every pair $m, n = 1, \ldots, k$ and for every x in the closure of the set of points of positive density of Σ.

Proof. Choose a $(k-1)$-form α satisfying (4.2) and (4.3) of Lemma 4.4 for some open set $U \subset \mathbb{R}^d$. Therefore,

$$0 \neq \langle T \llcorner U; \mathrm{d}\alpha \rangle = \langle \partial T \llcorner U; \alpha \rangle\,,$$

because of (4.2). But then Theorem 3.4 and condition (4.3) imply

$$\langle \partial T \llcorner U; \alpha \rangle = 0$$

and this is a contradiction. □

5. Decomposition of normal currents

An interesting problem in the theory of currents concerns the *decomposition of a normal current* by means of a family of integral currents. This problem firstly appeared in [1], formulated by F. Morgan. More precisely, given a normal current $T \in \mathbb{N}_k(\mathbb{R}^d)$, we ask whether there exists a family of integral currents $(R_\lambda)_{\lambda \in L}$, where L is a suitable measure space, such that

(i) $T = \int_L R_\lambda \, d\lambda$, i.e., for every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$, we can write
$$T(\omega) = \int_L R_\lambda(\omega)\, d\lambda\,;$$
(ii) $\mathbb{M}(T) = \int_L \mathbb{M}(R_\lambda)\, d\lambda$;
(iii) $\mathbb{M}(\partial T) = \int_L \mathbb{M}(\partial R_\lambda)\, d\lambda$.

Condition (ii) and (iii) express the requirement of a decomposition where no mass is wasted. In the analysis below, we will discuss also weaker versions of the problem: we can drop condition (iii), and we can also change the type of "decomposing" currents, saying we are satisfied with a family of rectifiable currents, instead of the family of integral ones. We agree that, when we do not specify the type of currents to which the "decomposing" family belongs, we will always be looking for a decomposition into integral currents; otherwise, we will always specify the type of the decomposing currents.

Let us briefly sketch the state of the art for this problem.

(1) When the dimension of the normal current is 1, it is known that there exists a decomposition satisfying (i) and (ii): see Proposition 4.4 in [5] and [8] or [7] for the proof (while a decomposition satisfying (i), (ii) and (iii) may not always exist).
(2) In the special case of codimension 1 with an integer rectifiable boundary, there exists a decomposition satisfying (i), (ii) and (iii), thanks to an observation by M. Zworski in [9]. The core of this argument is the Hardt-Pitts decomposition proved in [3].

Moreover, one can prove the following theorems.

Theorem 5.1. *When the codimension of the normal current is* 1*, there exists a decomposition in rectifiable currents satisfying* (i) *and* (ii)*.*

Theorem 5.2. *If $T = \xi\mu$ is a k-dimensional normal current and $\xi \in C^1(\Lambda_k(\mathbb{R}^d))$ is involutive, then there exists a decomposition in rectifiable currents satisfying* (i)*,* (ii) *and* (iii)*.*

The existence of a decomposition for normal currents under suitable assumptions is an isolated result. Indeed, in general the search for a decomposition by means of integral currents is actually too strong: we claim that any normal current of the form $\xi\mathscr{L}^d$ cannot be decomposed into integral currents satisfying (i) and (ii), provided ξ is non-involutive. This claim follows from Lemma 5.3 and Theorem 4.5.

In [9], M. Zworski exhibited this very same counterexample, claiming that, in general, a normal current has no decomposition satisfying (i) and (ii), even if we allow the decomposing currents to be rectifiable only. However the proof in [9] does not work, as pointed out by Alberti (see Section 4.5 of [6]). There is a gap in the argument, possibly due to a misunderstanding when referring to the Federer Flatness Theorem 4.1.15 in [2]. We propose the same counterexample for the problem of decomposing a normal current with a family of integral currents satisfying (i) and (ii) (at page 66) and the results we got in Section 4 fill the aforementioned gap.

Lemma 5.3. *Consider a vector-valued measure $\mu = \xi|\mu|$, where*

$$\xi \in \mathscr{C}^\infty(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$$

is a smooth k-vectorfield and assume that

(i) $\mu = \int_L \mu_\lambda \, d\lambda$, *where (L, λ) is a measure space;*
(ii) $\|\mu\| = \int_L \|\mu_\lambda\| \, d\lambda$.

Then, for λ-a.e. μ_λ, we have that

$$\mu_\lambda = \xi|\mu_\lambda| \, .$$

Theorem 5.4. *Consider the normal current $T := \xi\mathscr{L}^d \in \mathbb{N}_k(\mathbb{R}^d)$ given by the smooth non-involutive vectorfield $\xi \in \mathscr{C}^\infty(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$. Then there exist no measure space L and no family of integral currents $(R_\lambda)_{\lambda\in L}$ such that* (i) *and* (ii) *at page 66 hold.*

The proof of this theorem is a consequence of Lemma 5.3 and Theorem 4.5.

References

[1] Some open problems in geometric measure theory and its applications suggested by participants of the 1984 AMS summer institute. In J. E. Brothers, editor, *Geometric measure theory and the calculus of variations (Arcata, Calif., 1984)*, volume 44 of *Proc. Sympos. Pure Math.*, pages 441–464. Amer. Math. Soc., Providence, RI, 1986.

[2] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[3] Robert M. Hardt and Jon T. Pitts. Solving Plateau's problem for hypersurfaces without the compactness theorem for integral currents. In *Geometric measure theory and the calculus of variations (Arcata, Calif., 1984)*, volume 44 of *Proc. Sympos. Pure Math.*, pages 255–259. Amer. Math. Soc., Providence, RI, 1986.

[4] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[5] Andrea Marchese and Annalisa Massaccesi. The steiner tree problem revisited through rectifiable g-currents. *Preprint*, 2013.

[6] Frank Morgan. *Geometric measure theory*. Elsevier/Academic Press, Amsterdam, fourth edition, 2009. A beginner's guide.

[7] E. Paolini and E. Stepanov. Decomposition of acyclic normal currents in a metric space. *Preprint*, 2012.

[8] S. K. Smirnov. Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. *Algebra i Analiz*, 5(4):206–238, 1993.

[9] Maciej Zworski. Decomposition of normal currents. *Proc. Amer. Math. Soc.*, 102(4):831–839, 1988.

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