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REGULARITY ESTIMATES FOR CONTINUOUS SOLUTIONS OF α-CONVEX BALANCE LAWS

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Abstract. This paper proves new regularity estimates for continuous solutions to the balance equation

$$\partial_t u + \partial_x f(u) = g \qquad g \text{ bounded, } f \in C^{2n}(\mathbb{R}),$$

when the flux f satisfies a convexity assumption that we denote as $2n$-*convexity*. The results are known in the case of the quadratic flux by very different arguments in [14, 10, 8]. We prove that the continuity of u must be in fact $1/2n$-Hölder continuity and that the distributional source term g is determined by the classical derivative of u along any characteristics; part of the proof consists in showing that this classical derivative is well defined at any 'Lebesgue point' of g for suitable coverings. These two regularity statements fail in general for $C^\infty(\mathbb{R})$, *strictly convex* fluxes, see [3].

1. **Introduction.** This paper is part of a series of papers concerning the interplay among the Lagrangian and Eulerian formulation for solutions of balance laws with a bounded source term

$$\partial_t u + \partial_x f(u) = g \qquad g \text{ bounded, } f \in C^{2n}(\mathbb{R}). \tag{1.1}$$

The fact of considering continuous solutions is motivated by the idea that the source term g might act as a control device preserving continuity, differently from the case of conservation laws. Indeed the analysis of continuous solutions might be an intermediate step even when more regularity is proved to hold in the end, see for instance [11, 16]. In the case of the quadratic flux, when considering the Cauchy problem with an Hölder continuous initial datum, one can indeed construct continuous solutions for all times, see [14]. There are interesting contributions, like the last one just mentioned, coming from sub-Riemannian geometry because relevant geometric objects in modeling surfaces have been related to balance laws, see for instance [5, 24]. Studying the structure and the regularity of continuous solutions when the source is bounded is also interesting as a toy problem for more complex situations, e.g. developing the physically relevant program of determining the structure of distributional solutions for unbounded sources, see [7] as a progress in this direction.

By 'Eulerian' solution we just mean here a distributional solution. Even if the entropy condition does not play any role in the analysis of this paper, we recall [2]

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the intuitive but nontrivial fact that continuous distributional solutions of (1.1) are indeed Kružkov entropy solutions [18]. We also recall [2] that continuous Eulerian solutions are 'Lagrangian' in the following sense: they satisfy a transliteration of the infinitely dimensional system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = f'(u(t,\gamma(t))) \qquad \gamma(0) = x_0 \in \mathbb{R} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}u(t,\gamma(t)) = g(t,\gamma(t)) \end{cases} \tag{1.2}$$

which in the smooth setting arises by reducing the balance law along characteristic curves, see [2, Definitions 5;12] for a precise definition. In this reduction, a point which was not fully clear in [2] is the correspondence among the source term g and its 'restriction on characteristics' which appears in the formulation in ordinary differential equations: as g is defined $\mathcal{L}^2$-a.e. this restriction on curves in the plane is fairly nontrivial. Under general assumptions, a compatibility statement ensures [3] that one can pointwise select a Borel function $\mathfrak{g}$ good for both the formulations. Nevertheless, g is not fully determined by the classical derivative of u along characteristics even if $f \in C^\infty$ is strictly convex: [3] contains a counterexample. We prove here that, if f is α-convex as defined below, the classical derivative of u along characteristics does determine g: (a) $\mathcal{L}^2$-a.e. points of the plane are Lebesgue points of g, for suitable coverings, and (b) at those Lebesgue points the classical derivative of u along characteristics exists and it is equal to that Lebesgue value. We thus extend the known case of the quadratic flux [14, 10, 8].

Aim of this paper is indeed proving new regularity estimates for continuous solutions to balance equations (1.1) when the flux f satisfies the following α-convexity assumption.

Definition 1.1. Let $\alpha > 1$. A function $f : \mathbb{R} \to \mathbb{R}$ is *α-convex at v* if there exist $\varepsilon, c > 0$ and $\ell \in \mathbb{R}$ such that for $|z - v| < \varepsilon$ one has

$$f(z) - f(v) - (z - v)\ell > c|z - v|^\alpha.$$

We say that f is *α-convex* if f is α-convex at every point of its domain, with ε, c independent of the point. In particular, an α-convex function is convex. When $\alpha = 2$ and $f \in C^1(\mathbb{R})$ we recover uniform convexity.

We state now the main theorem of this paper assuming α-convexity of the flux for $\alpha = 2n$, with $n \in \mathbb{N}$. It will be described in more detail in the next sections.

Theorem 1.2. *Let $\mathfrak{g}$ be a bounded Borel function on $\mathbb{R}^2$. Let $f \in C^{2n}(\mathbb{R})$ satisfy Definition 1.1 of $2n$-convexity and let u be a continuous distributional solution of*

$$\partial_t u(t,x) + \partial_x(f(u(t,x))) = \mathfrak{g}(t,x). \tag{1.3}$$

Then

1. *u is $\frac{1}{2n}$-Hölder continuous and*
2. *at $\mathcal{L}^2$-a.e. (t,x), u is differentiable along any characteristic curve through (t,x) with derivative $\mathfrak{g}(t,x)$.*

Property 2. in the above statement is of importance for the correspondence of the sources among the Eulerian formulation (1.1) and the Lagrangian formulation (1.2) of the balance law. We stress again that Point 2. is highly nontrivial: it fails in general if the flux is only *strictly convex* rather than $2n$-convex. If $f \in C^\infty(\mathbb{R})$ is strictly convex, indeed, but not $2n$-convex for any $n \in \mathbb{N}$, it might still happen [3] that there is a compact set $K \subset \mathbb{R}^2$, with $\mathcal{L}^2(K) > 0$, which is made of points (t,x)

where u is not differentiable along characteristics, whichever characteristic is chosen. When u is not differentiable along characteristics at points belonging to K, possible if f is *strictly convex*, then it is not evident how to determine the distributional source term for (1.1) given the derivative of u only along characteristics: on the non-negligible set K there is no obvious candidate value! Even when the flux is quadratic there might be [3] a continuous solution u which admits a non-negligible compact set K, $\mathcal{L}^2(K) > 0$, which intersects each characteristic curve in at most a single point. Nevertheless, Point (2) of Theorem (1.2) states that u is differentiable $\mathcal{L}^2$-a.e. along characteristics also at points belonging to K: the natural candidate for the distributional source term is thus the classical derivative along characteristics.

Theorem 1.2 is know for the quadratic flux in the context of sub-Riemannian geometry. Differentiability $\mathcal{L}^2$-a.e. was proved [14], and the strong approximation [10] is now available. The Hölder continuity was shown in [9, 8] for $f(z) = z^2$. We include the proof here, as well, for a broader presentation. Local Lipschitz-regularity can be proved where $f'(u(t,x))f''(u(t,x)) \neq 0$ only for autonomous sources [1].

In § 2 we discuss how continuity improves to $1/\alpha$-Hölder continuity when $f \in C^1(\mathbb{R})$ is α-convex. In § 3 we discuss the classical differentiability of u along characteristic curves and how it identifies g.

2. **Rough Hölder continuity estimate.** We derive in this section a rough Hölder continuity estimate. The estimate we derive is rough because it is known for the quadratic flux [14, 10, 8] that, for $\mathcal{L}^2$-a.e. (t,x) fixed, the quotient among the increment $|u(t,x) - u(t',x')|$ and the distance $\sqrt{|t-t'| + |x-x'|}$ converges to 0 as $(t',x') \to (t,x)$. Here we prove that, if u is a continuous solution to the balance law (1.1) in a connected open set O and the flux $f \in C^1(\mathbb{R})$ is α-convex, then u is $1/\alpha$-Hölder continuous on compact subsets of O. The proof follows [9, 8] relative to the quadratic flux. We do not have any pretense of establishing an optimal constant of Hölder continuity and we do not enter in a finer pointwise analysis.

As a preliminary observation, we shall indeed remark that the subject of this section is not completely local in the following sense. If one has that u is a continuous solution to (1.3) in a connected open set O, then one cannot in general obtain the estimate in Theorem 2.1 below with the same constant $\|g\|_{L^\infty}$ in the right hand side: the constant $\|g\|_{L^\infty}$ in the right hand side of Theorem 2.1 below is deduced when $O = \mathbb{R}^2$. It shall be replaced by a bigger constant close to boundaries—or it otherwise holds for x_1, x_2 sufficiently close to each other on compact subsets of O.

For keeping the exposition in this paper as simple as possible, we avoid to give the full definition [2, Definitions 5;12] of Lagrangian continuous solution of (1.1). We just stress that Lagrangian solutions, for fluxes whose set of inflection points is negligible in the sense of [2, Assumption (H) in § 1.2.2], satisfy the following property which in the general case defines the stronger notion of Broad solutions:

$$\text{if } \gamma \text{ is a characteristic curve of } u \text{ in (1.1)} \quad \exists \frac{\mathrm{d}}{\mathrm{d}t} u(t,\gamma(t)) = \hat{\mathfrak{g}}(t,\gamma(t)) \quad \text{in } \mathcal{D}(\mathbb{R})$$

for a suitable Borel function $\hat{\mathfrak{g}}$, independent of γ, which satisfies $\|\mathfrak{g}\|_\infty \leq \|g\|_\infty$. In Section 3 we prove that when f is α-convex then the above property necessarily defines the Borel function $\hat{\mathfrak{g}}$ $\mathcal{L}^2$-a.e. and it moreover defines precisely the distribution g; this is not the case in general even for C^∞, strictly convex fluxes [3]. For the rest of this section, we determine the Hölder continuity of u.

Theorem 2.1. *Assume $f \in C^1(\mathbb{R})$ is strictly convex. Then a continuous distributional solution u in $\mathbb{R}^2$ of the conservation law (1.3) satisfies for all x_1, x_2, t the*

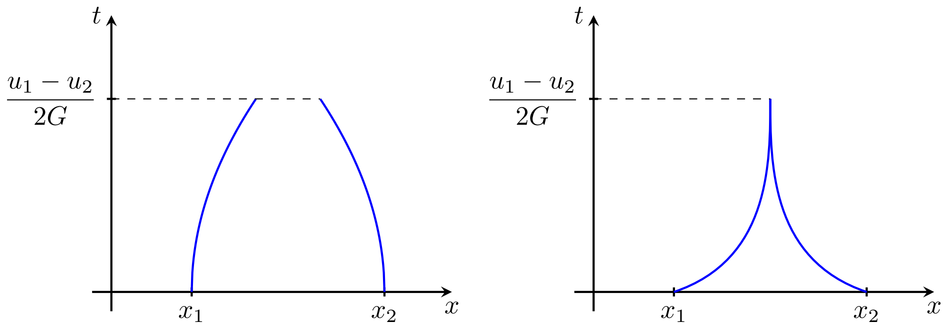

FIGURE 1. Proof of a rough Hölder continuity estimate of u

inequality

$$f(u(t,x_1)) + f(u(t,x_2)) - 2f\left(\frac{u(t,x_1)+u(t,x_2)}{2}\right) \leq \|g\|_{L^\infty}|x_2 - x_1|.$$

Proof. As in [9, 8] about Hölder continuity, the proof is based on the Taylor expansion of the characteristic curves, that however we apply here differently.

We fix the attention on the case

$$t = 0, \quad x_1 < x_2, \quad u_1 := u(0,x_1) > u(0,x_2) =: u_2, \quad G := \|g\|_{L^\infty}.$$

This assumption is not restrictive: the case when $u(0,x_1) < u(0,x_2)$ can be similarly obtained by considering negative rather than positive times, while the case when $x_1 > x_2$ follows by exchanging the indexes 1 and 2; finally, setting $t = 0$ is of course a convenient notation but it does not play any mathematical role. Let γ_1, γ_2 be characteristic curves respectively through $(0,x_1)$, $(0,x_2)$. Denote by $i_{\gamma_i}(s)$ the injection $(s,\gamma_i(s))$, for $i = 1,2$. The characteristic curves can be written in integral form as

$$\gamma_i(t) = x_i + \int_0^t \dot{\gamma}(s)ds = x_i + \int_0^t f'(u(i_{\gamma_i}(s))ds, \qquad \forall i = 1,2.$$

Remember moreover that, by the Lipschitz continuity along characteristics [2, Theorem 30],

$$u_i - Gs \leq u(i_{\gamma_i}(s)) \leq u_i + Gs \qquad \forall s \geq 0 \qquad \forall i = 1,2. \tag{2.1}$$

Applying then the monotonicity of $f'(z)$, which follows by the convexity hypothesis on f,

$$\begin{aligned} x_i - \frac{f(u_i - Gt) - f(u_i)}{G} &= x_i + \int_0^t f'(u_i - Gs)ds \\ &\leq \gamma_i(t) \\ &\leq x_i + \int_0^t f'(u_i + Gs)ds = x_i + \frac{f(u_i + Gt) - f(u_i)}{G}. \end{aligned} \tag{2.2}$$

The exploitation of monotonicity, as for Lagrangian parameterizations in [2, 3], is motivated by [6].

If the two curves γ_1, γ_2 do not intersect up to time t (Figure 1, on the left), then the lower bound for γ_1 must be less than the upper bound for γ_2 and we find the inequality

$$Gx_1 - [f(u_1 - Gt) - f(u_1)] < Gx_2 + f(u_2 + Gt) - f(u_2).$$

In particular, if they do not intersect before $t = \frac{u_1 - u_2}{2G}$ one has the thesis:

$$\begin{aligned} Gx_2 - Gx_1 &\geq f(u_1) + f(u_2) - f(u_1 - Gt) - f(u_2 + Gt) \\ &= f(u_1) + f(u_2) - 2f\left(\frac{u_1 + u_2}{2}\right). \end{aligned}$$

If instead γ_1, γ_2 intersect at some time t (Figure 1, on the right), then by the tangency condition, by (2.1) and by the monotonicity of f' one has

$$f'(u_1 - Gt) \leq f'(u(i_{\gamma_1}(t))) = f'(u(i_{\gamma_2}(t))) \leq f'(u_2 + Gt),$$

which by the *strict* convexity of f implies

$$u_1 - Gt \leq u_2 + Gt.$$

In particular, $t = \frac{u_1 - u_2}{2G}$ is the first possible time of intersection, achieving the thesis. □

Corollary 2.2. *Assume $f \in C^1(\mathbb{R})$ satisfies Definition 1.1 of α-convexity when restricted to the image of a continuous distributional solution u of (1.3). Then $u \in C^{1/\alpha}_{\text{loc}}(\mathbb{R}^2)$.*

Proof. The proof is divided into two steps similar to [9, 8]. We first prove that a continuous Lagrangian solution u of (1.3) is locally $1/\alpha$-Hölder continuous on t-sections. We then prove that it is $1/\alpha$-Hölder continuous as a function of two variables.

1: Study of t-sections. By the inequality of Theorem 2.1 one has that for all x_1, x_2, t

$$f(u(t, x_1)) + f(u(t, x_2)) - 2f\left(\frac{u(t, x_1) + u(t, x_2)}{2}\right) \leq \|g\|_{L^\infty} |x_2 - x_1|.$$

Being continuous, u is uniformly continuous on compact sets: for every compact set $K \subset \mathbb{R}^2$ and $\varepsilon > 0$, let $\omega(\varepsilon; K)$ be a ε-modulus of continuity for u in space, at any fixed time t, on the compact set K. We mean that

$$(t, x_1), (t, x_2) \in K, \ |x_2 - x_1| \leq \omega(\varepsilon; K) \quad \Rightarrow \quad |u(t, x_1) - u(t, x_2)| \leq \varepsilon.$$

By the α-convexity assumption of Definition 1.1, one has that if $u_1, u_2 \in \mathbb{R}$ belong to the image of a continuous Lagrangian solution u of (1.3) and if $|u_1 - u_2| \leq \varepsilon$ then

$$\begin{aligned} &f(u_1) + f(u_2) - 2f\left(\frac{u_1 + u_2}{2}\right) = \\ &\quad = f\left(\frac{u_1 + u_2}{2} + \frac{u_1 - u_2}{2}\right) - f\left(\frac{u_1 + u_2}{2}\right) - \cancel{\frac{u_1 - u_2}{2} f'\left(\frac{u_1 + u_2}{2}\right)} \\ &\quad\quad + f\left(\frac{u_1 + u_2}{2} + \frac{u_2 - u_1}{2}\right) - f\left(\frac{u_1 + u_2}{2}\right) - \cancel{\frac{u_2 - u_1}{2} f'\left(\frac{u_1 + u_2}{2}\right)} \\ &\quad \geq 2c \left|\frac{u_1 - u_2}{2}\right|^\alpha \end{aligned}$$

Collecting the information, we conclude that if

$$|x_2 - x_1| < \omega(\varepsilon; K) \quad \text{and} \quad (t, x_1), (t, x_2) \in K$$

then

$$f(u(t,x_1)) + f(u(t,x_2)) - 2f\left(\frac{u(t,x_1)+u(t,x_2)}{2}\right) \geq 2c\left|\frac{u(t,x_1)-u(t,x_2)}{2}\right|^{\alpha}.$$

Together with the inequality of Theorem 2.1 we obtain the local $1/\alpha$-Hölder continuity estimate of the t-sections on the compact K: if

$$(t,x_1),(t,x_2) \in K,\ |x_2 - x_1| \leq \omega(\varepsilon; K)$$

then

$$|u(t,x_1) - u(t,x_2)| \leq 2\sqrt[\alpha]{\frac{\|g\|_{L^\infty}}{2c}}\sqrt[\alpha]{|x_2 - x_1|}. \tag{2.3}$$

2: Study in $\mathbb{R}^2$. The Hölder continuity of u in the two variables follows as in [9, 8] combining estimate (2.3) on t-sections, which are horizontal lines, with the Lipschitz continuity estimate of u along characteristic curves, as characteristic curves cannot have horizontal tangent. In formulae, consider $(t_1,x_1),(t_2,x_2) \in [-M,M]^2$ for some $M > 0$. We term x_{12} the point where a characteristic curve γ through (t_1,x_1) intersects the coordinate line $t = t_2$: setting $\lambda := f'(u)$ we obtain

$$\gamma(t_1) = x_1, \qquad \gamma(t_2) = x_{12}, \qquad \dot\gamma(t) = \lambda(i_\gamma(t)).$$

Below, we now apply: • Hölder continuity on the t_2-section for estimating $|u(t_2,x_2) - u(t_2,x_{12})|$ and • Lipschitz continuity along γ for estimating $|u(t_1,x_1) - u(t_2,x_{12})|$.

We first need to check that the points $(t_2,x_2),(t_{12},x_{12})$ belong to a same compact $[-M-L,M+L]^2$, where $\|\lambda\|_\infty = L$, and that $|x_2 - x_{12}| \leq \omega(\varepsilon;[-M-L,M+L]^2)$. Since $|\lambda| \leq L$, the differential condition of being a characteristic curve implies

$$|x_{12}| = \left|x_1 + \int_{t_1}^{t_2}\dot\gamma\right| \leq |x_1| + L|t_2 - t_1|,$$
$$|x_2 - x_{12}| = \left|x_2 - x_1 - \int_{t_1}^{t_2}\dot\gamma\right| \leq |x_2 - x_1| + L|t_2 - t_1|.$$

In particular, if

$$|x_2 - x_1| + L|t_2 - t_1| \leq \omega(\varepsilon;[-M-L,M+L]^2), \qquad |t_1 - t_2| \leq 1\,,$$

then necessarily

$$|x_{12}| \leq |x_1| + |x_1 - x_{12}| \leq M + L|t_1 - t_2| \leq M + L\ ,$$
$$(t_2,x_{12}) \in [-M-L,M+L]^2\ ,$$
$$|x_2 - x_{12}| \leq |x_2 - x_1| + L|t_2 - t_1| \leq \omega(\varepsilon;[-M-L,M+L]^2)\ .$$

We recall from [2, Theorem 30] that $u \circ i_\gamma$ is G-Lipschitz continuous. The estimates we just made in this step show that we can apply the Hölder continuity on the t_2-sections, provided by previous step, with the choice of the compact set

$K = [-M-L, M+L]^2$: by the triangular inequality we get then

$$\begin{aligned}|u(t_1,x_1)-u(t_2,x_2)| &\leq |u(t_1,x_1)-u(t_2,x_{12})| + |u(t_2,x_{12})-u(t_2,x_2)| \\ &\leq G|t_2-t_1| + 2\sqrt[\alpha]{\frac{\|g\|_{L^\infty}}{2c}}\sqrt[\alpha]{|x_2-x_{12}|} \\ &\leq G|t_2-t_1| + 2\sqrt[\alpha]{\frac{\|g\|_{L^\infty}}{2c}}\sqrt[\alpha]{|x_2-x_1|+L|t_2-t_1|} \\ &\leq \left(\frac{G}{\sqrt[\alpha]{L}} + 2\sqrt[\alpha]{\frac{\|g\|_{L^\infty}}{2c}}\right)\sqrt[\alpha]{|x_2-x_1|+L|t_2-t_1|}.\end{aligned}$$

We therefore find the following. If $(t_1,x_1),(t_2,x_2) \in [-M,M]^2$ for some $M>0$ and if $\varepsilon>0$, then

$$|x_2-x_1|+L|t_2-t_1| \leq \omega(\varepsilon;[-M-L,M+L]^2) \leq 1, \qquad |t_1-t_2| \leq 1$$

implies that

$$|u(t_1,x_1)-u(t_2,x_2)| \leq \left(\frac{G}{\sqrt[\alpha]{L}} + 2\sqrt[\alpha]{\frac{\|g\|_{L^\infty}}{2c}}\right)\sqrt[\alpha]{|x_2-x_1|+L|t_2-t_1|}.$$

Notice that we are using as a modulus of continuity in the two variables on the compact set $[-M,M]^2$ precisely the modulus of continuity ω, introduced in the previous step for the t-sections, on the larger square $[-M-L,M+L]^2$. This concludes the proof of the local $1/\alpha$-Hölder continuity of u because $|x|+L|t|$ defines a norm on $\mathbb{R}^2$ equivalent to the Euclidean one. □

Example 2.3. We remind the well-known, classical example $u(t,x)=\sqrt{|x|}$, which satisfies

$$\partial_t u(t,x) + \partial_x(u^2(t,x)) = 1.$$

We notice that the Hölder estimate on the t-sections (2.3) cannot be lowered in this example if one of the two points is the origin, as in this example $c=\alpha=2$ and $\|g\|_{L^\infty}=1$: for the two points $x_2=x$, $x_1=0$ estimate (2.3) reduces precisely to $|u(t,x_2)-u(t,x_1)| = |u(t,x)-u(t,0)| = \sqrt{|x|} = 2\sqrt[\alpha]{\|g\|_{L^\infty}/2c}\sqrt[\alpha]{|x_2-x_1|}$.

3. **Lebesgue differentiation theorem with characteristic regions.** In this section we aim at showing that the distributional source term is the classical derivative along characteristics, when u is a continuous distributional solution to (1.1) with a smooth $2n$-convex flux as in Definition 1.1. More precisely, we provide countably many Vitali coverings of $\mathbb{R}^2$ such that if (t,x) is a Lebesgue point of the distributional source term for all these coverings, then u is differentiable at (t,x) along characteristic curves with derivative given by the Lebesgue value of the distributional source. For $f(z)=z^2/2$ an alternative proof is available [14, 10, 8], and a third line of proof was pointed out by Stefano Bianchini (Taipei, 2012).

The section is organized as follows:

§ 3.1: We make clear what we mean by 'Lebesgue value' and which coverings we consider.

§ 3.2: We prove a differentiation theorem at 'Lebesgue values' for the coverings specified in § 3.1. This differentiation theorem identifies Lagrangian and Eulerian source terms. We show that for uniformly convex fluxes $\mathcal{L}^2$-a.e. point is a 'Lebesgue value'.

§ 3.3: We show that if the flux $f \in C^{2n}(\mathbb{R})$ is $2n$-convex, then there are at most countably many exceptional values $\{\bar{v}_k\}_{k\in\mathbb{N}}$ where f'' might vanish. We prove then the differentiation theorem for the covering of § 3.1 also of the set where f'' vanishes. In particular, we identify Lagrangian and Eulerian source terms also on this remaining set.

This brings us to the following conclusion, which will be better explained in the subsections.

Theorem 3.1. *Suppose $f \in C^{2n}$ is $2n$-convex in the sense of Definition 1.1 for some $n \in \mathbb{N}$. Let $g = [\mathfrak{g}] \in L^\infty(\mathbb{R}^2)$ be the equivalence class of functions $\mathcal{L}^2$-a.e. equal to the Borel function $\mathfrak{g}$. If u is a continuous solution to the balance law (1.1), then $\mathcal{L}^2$-a.e. point is a Lebesgue point of $\mathfrak{g}$ in the sense of (3.13) below. At those points $(\bar{t}, \bar{x})$ where (3.13) holds one has*

$$\exists \frac{\mathrm{d}}{\mathrm{d}t} u(t, \gamma(t))|_{t=\bar{t}} \equiv \mathfrak{g}(\bar{t}, \gamma(\bar{t})) \qquad \text{for all } \gamma \text{ characteristic curve of } u \text{ with } \gamma(\bar{t}) = \bar{x}.$$

The proof of the above Theorem is given in Lemma 3.5 below for $n = 1$ and in Lemma 3.8 below for $n > 1$.

3.1. **Lebesgue values.** We now specify the coverings of $\mathbb{R}^2$ that we consider when we talk later on of Lebesgue values of a function, and in particular of the source term in (1.1). The coverings are indexed by a parameter $\rho_n \downarrow 0$ with $n \in \mathbb{N}$, and they would degenerate to pieces of characteristic curves of (1.1) if we had rather fixed $\rho = 0$. Each covering is made by compact sets which are 'flow tubes': two edges are space segments, say of length $\rho_n\delta$, and the other two edges are given by any characteristic curve of (1.1) and its translation, with a fixed amount of time equal to $\delta^{1/\alpha}$ as the eight. These kind of regions resembles 'characteristic regions'. This covering is of course devised taking into account the α-convexity of f for passing to the limit when expressing the PDE (1.1) in an integral form over these regions: this limit is matter of Sections 3.2, 3.3.

For fixed $0 < \rho < 1 < \alpha$, consider the following family of subsets of $\mathbb{R}^2$ (Figure 2):

$$\mathcal{V}_\alpha^\rho := \Big\{ S_{\ell,\alpha}^{\pm;\rho}(\gamma, \varepsilon, \sigma),\ S_{r,\alpha}^{\pm;\rho}(\gamma, \varepsilon, \sigma) \ \Big| \ \gamma \in C^1(\mathbb{R}) : \dot{\gamma}(t) = f'(u(t, \gamma(t))),\ \varepsilon > 0,\ \sigma \in \mathbb{R} \Big\} , \tag{3.1}$$

$$S_{r,\alpha}^{+;\rho}(\gamma, \varepsilon, \sigma) := \{(t, x) \ : \ \sigma \le t \le \sigma + \varepsilon,\ \gamma(t) \le x \le \gamma(t) + \rho\varepsilon^\alpha\} , \tag{3.2}$$

$$S_{r,\alpha}^{-;\rho}(\gamma, \varepsilon, \sigma) := \{(t, x) \ : \ \sigma - \varepsilon \le t \le \sigma,\ \gamma(t) \le x \le \gamma(t) + \rho\varepsilon^\alpha\} , \tag{3.3}$$

$$S_{\ell,\alpha}^{+;\rho}(\gamma, \varepsilon, \sigma) := \{(t, x) \ : \ \sigma \le t \le \sigma + \varepsilon,\ \gamma(t) - \rho\varepsilon^\alpha \le x \le \gamma(t)\} , \tag{3.4}$$

$$S_{\ell,\alpha}^{-;\rho}(\gamma, \varepsilon, \sigma) := \{(t, x) \ : \ \sigma - \varepsilon \le t \le \sigma,\ \gamma(t) - \rho\varepsilon^\alpha \le x \le \gamma(t)\} , \tag{3.5}$$

$$S_\alpha^\rho(\gamma, \varepsilon, \sigma) := S_{r,\alpha}^{-;\rho}(\gamma, \varepsilon, \sigma) \cup S_{\ell,\alpha}^{-;\rho}(\gamma, \varepsilon, \sigma) \cup S_{r,\alpha}^{+;\rho}(\gamma, \varepsilon, \sigma) \cup S_{\ell,\alpha}^{+;\rho}(\gamma, \varepsilon, \sigma) . \tag{3.6}$$

For brevity, we will at some occurrences omit the indexes ρ, α from the above sets. Define also the following quasimetric: for all points $A, B \in \mathbb{R}^2$ set

$$d_\alpha(A, B) = \begin{cases} \min\left\{\varepsilon \in [0, 1] \mid \exists\sigma, \exists\gamma \text{ characteristic curve} : \ A, B \in S_\rho^\alpha(\gamma, \varepsilon/2, \sigma)\right\} \\ \text{ or} \\ 1 \text{ if the set in the above minimum is empty.} \end{cases}$$

We can indeed assume that the minimum exists in the first branch of the definition of d_α because we defined the regions S_ρ^α as compact sets.

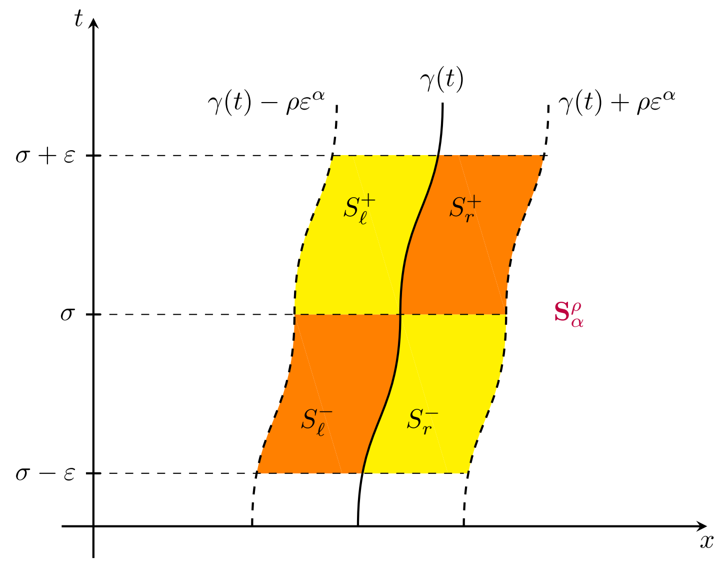

FIGURE 2. Balances on characteristic regions

Lemma 3.2. *Let $f \in W^{2,\infty}(\mathbb{R})$. Let $\alpha \in [1,2]$ and $\rho > 0$. If u is $1/\alpha$-Hölder continuous, the function d_α is a quasimetric in $\mathbb{R}^2$ in the sense that the following properties hold:*

1. $d_\alpha(A,B) = 0$ *if and only if* $A = B$,
2. $d_\alpha(A,B) = d_\alpha(B,A)$ *for all points* $A, B \in \mathbb{R}^2$,
3. *there exists* $q = q(\alpha,\rho) > 1$ *such that*

$$d_\alpha(A,B) \le q\,[d_\alpha(A,C) + d_\alpha(C,B)] \qquad \text{for all points } A,B,C \in \mathbb{R}^2.$$

Moreover, the quasimetric d_α is doubling: there exists $C = C(\alpha,\rho) > 0$ such that

$$\frac{\mathcal{L}^2(\{B \ : \ d_\alpha(A,B) \le 2\varepsilon\})}{\mathcal{L}^2(\{B \ : \ d_\alpha(A,B) \le \varepsilon\})} \le C \qquad \forall A \in \mathbb{R}^2,\ \forall \varepsilon \in \left(0, \frac{1}{2}\right)$$

and such that for every curve

$$\bar\gamma \quad \text{through a point} \quad A = (t_A, x_A), \quad \text{with} \quad \dot{\bar\gamma}(t) = f'(u(t,\bar\gamma(t))),$$

one has

$$S_\alpha^\rho\,(\bar\gamma, \varepsilon, t_A) \subset \{B \ : \ d_\alpha(A,B) \le 2\varepsilon\} \subset S_\alpha^\rho\,(\bar\gamma, C\,\varepsilon, t_A)\,. \tag{3.7}$$

We refer to the reference [15, § 14.1] concerning doubling measures on quasimetric spaces. Before proving the statement of Lemma 3.2, we point out below an elementary estimate that provides insights on the geometry induced on the plane by characteristic curves of (1.3). This estimate is indeed crucial in order to have that d_α provides a doubling quasimetric.

Lemma 3.3. *Let $1 < \alpha \le 2$. Let $\bar\omega$ be a modulus of $1/\alpha$-Hölder continuity of a distributional solution u to (1.3) and set $M = \|f''(u)\|_{L^\infty}$. Consider two integral curves $\gamma_1 \le \gamma_2$ of $f'(u)$: the distance $d = \gamma_2 - \gamma_1$ satisfies for $|t| \le \bar t$ in (3.9)*

$$\left(d(0)^{\frac{\alpha-1}{\alpha}} - \left(\frac{\alpha-1}{\alpha}\right) M\bar\omega t\right)^{\frac{\alpha}{\alpha-1}} \le d(t) \le \left(d(0)^{\frac{\alpha-1}{\alpha}} + \left(\frac{\alpha-1}{\alpha}\right) M\bar\omega t\right)^{\frac{\alpha}{\alpha-1}}.$$

Proof. By the definition of characteristic curves and the Hölder continuity in Corollary 2.2,

$$|\dot{d}(t)| = |f'(u(t,\gamma_2(t))) - f'(u(t,\gamma_1(t)))| \leq M\bar{\omega}[d(t)]^{\frac{1}{\alpha}}. \tag{3.8}$$

Integrating the ordinary differential equation (3.8) one proves the statement. If $\alpha \in (1,2]$ the two curves cannot collide before

$$\bar{t} = \frac{\alpha d(0)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)M\bar{\omega}} > \frac{1}{(\alpha-1)M\bar{\omega}} d(0)^{\frac{1}{\alpha}} \qquad \text{when } d(0) < 1. \tag{3.9}$$

□

Proof of Lemma 3.2. Properties (1), (2) are immediate. We prove (3) and the doubling estimate. Set $A = (t_A, x_A)$, $B = (t_B, x_B)$ and $C = (t_C, x_C)$.
1: Proof of (3). If $d_\alpha(A,C) + d_\alpha(C,B) \geq 1$ the thesis holds trivially: we only study the other case. Define

$$d_\alpha(A,C) = \varepsilon > 0, \qquad d_\alpha(C,B) = \eta > 0, \qquad \varepsilon + \eta = \xi < 1. \tag{3.10}$$

Since $\varepsilon < 1$ and $\eta < 1$, there exist $\sigma_1, \sigma_2 > 0$ and integral curves γ_1, γ_2 of $f'(u)$ such that

$$A, C \in S^\rho_\alpha(\gamma_1, \varepsilon/2, \sigma_1), \qquad C, B \in S^\rho_\alpha(\gamma_2, \eta/2, \sigma_2).$$

We directly suppose $\gamma_1 \leq \gamma_2$, as one can easily reduce to this case. Necessarily

$$|t_C - t_A| \leq \varepsilon, \quad |t_B - t_C| \leq \eta \quad \Rightarrow \quad |t_B - t_A| \leq \varepsilon + \eta. \tag{3.11}$$

Comparing the x-variable is trickier and we apply the estimate from Lemma 3.3 above. By definition of $S^\rho_\alpha(\gamma_1, \varepsilon/2, \sigma_1)$ and $S^\rho_\alpha(\gamma_2, \eta/2, \sigma_2)$, at time t_C one has

$$|\gamma_1(t_C) - x_C| \leq \rho \left(\frac{\varepsilon}{2}\right)^\alpha, \quad |\gamma_2(t_C) - x_C| \leq \rho \left(\frac{\eta}{2}\right)^\alpha$$

from which we obtain, since we defined $\xi = \varepsilon + \eta = d_\alpha(A,C) + d_\alpha(C,B)$ in (3.10), the estimate

$$|\gamma_1(t_C) - \gamma_2(t_C)| \leq \rho \frac{\varepsilon^\alpha + \eta^\alpha}{2^\alpha} \leq \rho \left(\frac{\xi}{2}\right)^\alpha.$$

By Lemma 3.3 above and estimates (3.11) computed among times t_A, t_B and since $\xi \leq 1$, we get

$$|\gamma_1(t_A) - \gamma_2(t_A)| \leq \left(\rho^{\frac{\alpha-1}{\alpha}} \cdot \left(\frac{\xi}{2}\right)^{\alpha-1} + \left(\frac{\alpha-1}{\alpha}\right) M\bar{\omega}\xi\right)^{\frac{\alpha}{\alpha-1}} \leq \rho \cdot \left(\frac{D_1\xi}{2}\right)^\alpha$$

$$|\gamma_1(t_B) - \gamma_2(t_B)| \leq \rho \cdot \left(\frac{D_1\xi}{2}\right)^\alpha$$

where $D_1 = 2\left(1 + \frac{\alpha-1}{\alpha\rho^{1-\frac{1}{\alpha}}} M\bar{\omega}\right)^{\frac{1}{\alpha-1}} > 1$. Being also, by definition of $S^\rho_\alpha(\gamma_1, \varepsilon/2, \sigma_1)$ and $S^\rho_\alpha(\gamma_2, \varepsilon/2, \sigma_2)$,

$$|x_A - \gamma_1(t_A)| \leq 2^{-\alpha}\rho\varepsilon^\alpha, \qquad |x_B - \gamma_2(t_B)| \leq 2^{-\alpha}\rho\eta^\alpha$$

one arrives to

$$|x_A - \gamma_2(t_A)| \leq \rho[D_1\xi]^\alpha, \quad |x_B - \gamma_1(t_B)| \leq \rho[D_1\xi]^\alpha.$$

Owing to (3.11), the last inequalities prove that $d_\alpha(A,B) \leq 2D_1[d_\alpha(A,C) + d_\alpha(C,B)]$ because

$$A, B \in S^\rho_\alpha(\gamma_1, D_1\xi, t_C), \qquad A, B \in S^\rho_\alpha(\gamma_2, D_1\xi, t_C).$$

2: Proof of the doubling estimate: estimate from above on $\mathcal{L}^2(\{B \; : \; d_\alpha(A,B) \leq$

$2\varepsilon\})$. Fix an integral curve $\bar\gamma$ of $f'(u)$ through A. If $d_\alpha(A,B)\leq 2\varepsilon$, there exists an integral curve γ of $f'(u)$ such that

$$|\bar\gamma(t_A)-\gamma(t_A)|=|x_A-\gamma(t_A)|\leq\rho\varepsilon^\alpha,\qquad |x_B-\gamma(t_B)|\leq\rho\varepsilon^\alpha,\qquad |t_B-t_A|\leq 2\varepsilon.$$

Applying the estimate from above in Lemma 3.3 above one has

$$|\bar\gamma(t_B)-\gamma(t_B)|\leq\left([\rho\varepsilon^\alpha]^{\frac{\alpha-1}{\alpha}}+\left(\frac{\alpha-1}{\alpha}\right)M\bar\omega\cdot 2\varepsilon\right)^{\frac{\alpha}{\alpha-1}}\leq\rho\left(D_2\varepsilon\right)^\alpha.$$

for a positive constant $D_2=D_2(\rho,\alpha)>1$. In particular we obtain that $B\in S^\rho_\alpha(\bar\gamma,2D_2\varepsilon,t_A)$ because

$$|x_B-\bar\gamma(t_B)|\leq|x_B-\gamma(t_B)|+|\gamma(t_B)-\bar\gamma(t_B)|\leq\rho\varepsilon^\alpha+\rho\left(D_2\varepsilon\right)^\alpha\leq\rho\left(2D_2\varepsilon\right)^\alpha.$$

This yields the above inclusion in (3.7) and the estimate from above

$$\mathcal{L}^2(\{B\ :\ d_\alpha(A,B)\leq 2\varepsilon\})\leq\mathcal{L}^2\left(S^\rho_\alpha(\bar\gamma,2D_2\varepsilon,t_A)\right)=4(2D_2)^{1+\alpha}\rho\varepsilon^{1+\alpha}.\tag{3.12}$$

3: Proof of the doubling estimate: estimate from below on $\mathcal{L}^2(\{B\ :\ d_\alpha(A,B)\leq\varepsilon\})$. Fix an integral curve $\bar\gamma$ of $f'(u)$ through A. Notice that by definition of d_α the whole set $S^\rho_\alpha(\bar\gamma,\frac{\varepsilon}{2},t_A)$ is contained in $\mathcal{L}^2(\{B\ :\ d_\alpha(A,B)\leq\varepsilon\})$: we get thus the lower inclusion in (3.7) and

$$\mathcal{L}^2(\{B\ :\ d_\alpha(A,B)\leq\varepsilon\})\geq\mathcal{L}^2\left(S^\rho_\alpha\left(\bar\gamma,\frac{\varepsilon}{2},t_A\right)\right)=(2)^{1-\alpha}\rho\varepsilon^{1+\alpha}.$$

4: Conclusion. Finally, one can conclude, owing to the estimate from above, that

$$\frac{\mathcal{L}^2(\{B\ :\ d_\alpha(A,B)\leq 2\varepsilon\})}{\mathcal{L}^2(\{B\ :\ d_\alpha(A,B)\leq\varepsilon\})}\leq\frac{\mathcal{L}^2(S^\rho_\alpha(\bar\gamma,2D_2\varepsilon,t_A)}{\mathcal{L}^2(S^\rho_\alpha(\bar\gamma,\frac{\varepsilon}{2},t_A))}\leq\frac{4(2D_2)^{1+\alpha}\cancel{\rho\varepsilon^{1+\alpha}}}{(2)^{1-\alpha}\cancel{\rho\varepsilon^{1+\alpha}}}=(4D_2)^{1+\alpha}.$$

□

Corollary 3.4. *Let $f\in W^{2,\infty}(\mathbb{R})$, $\alpha\in[1,2]$ and fix any sequence $\rho_n\downarrow 0$. At fixed n, the d_α-balls of radius $0<\varepsilon<1$ are a Vitali covering of any open set O where u is $\frac{1}{\alpha}$-Hölder continuous. Let $g\in L^\infty(\mathbb{R}^2)$ be the equivalence class of functions $\mathcal{L}^2$-a.e. equal to a Borel function $\mathfrak{g}$. Then on O*

$$\exists\lim_{\substack{(t,x)\in S\in\mathcal{V}^{\rho_n}_\alpha\\ \mathrm{diam}(S)\downarrow 0}}\frac{1}{\mathcal{L}^2(S)}\int_S|g-\mathfrak{g}(t,x)|=0\qquad\textit{for } \mathcal{L}^2\textit{-a.e. } (t,x),\ \textit{for } n\in\mathbb{N}.\tag{3.13}$$

Proof. Owing to Lemma 3.2, when n is fixed one can directly deduce by [23, § 1.3] that the family of ε-balls of d_α is a Vitali covering of O. The differentiation theorem follows then at $\mathcal{L}^2$-a.e. point of O directly from [13, Theorem 2.9.8] for this covering. This implies the differentiation theorem also for the covering $\mathcal{V}^{\rho_n}_\alpha$ by (3.7). Denote by N_n the $\mathcal{L}^2$-negligible set where the limit (3.13) does not hold for the fixed family $\mathcal{V}^{\rho_n}_\alpha$. The limit (3.13) therefore holds for any fixed n when $(t,x)\in O$ does not belong to the $\mathcal{L}^2$-negligible set $N=\cup_{n\in\mathbb{N}}N_n$. □

3.2. **Differentiation theorem at 'Lebesgue points'.** We derive now consequences of Lebesgue differentiation theorem on all the families $\mathcal{V}^{\rho_n}_\alpha$, $n\in\mathbb{N}$, defined in § 3.1 for a sequence $\rho_n\downarrow 0$: we identify in particular Eulerian and Lagrangian source terms when f is uniformly convex.

Lemma 3.5. *Let $g=[\mathfrak{g}]\in L^\infty(\mathbb{R}^2)$ be the equivalence class of functions $\mathcal{L}^2$-a.e. equal to the Borel function $\mathfrak{g}$. If f is uniformly convex, then any continuous distributional solution u of*

$$\partial_t\,u(t,x)+\partial_x(f(u(t,x)))=g(t,x)$$

is differentiable along any characteristic curve with derivative $\mathfrak{g}(t,x)$ *at any point* (t,x) *satisfying* (3.13) *for some* $\rho_n \downarrow 0$. *In particular, the thesis holds for* $\mathcal{L}^2$-*a.e.* (t,x).

Proof. We prove the thesis when $f \in W^{2,\infty}(\mathbb{R})$ satisfies Definition 1.1 of α-convexity: the meaningful case is the one of uniformly convex functions when $\alpha = 2$. Fix ρ belonging to a sequence $\rho_n \downarrow 0$. We integrate the balance law on domains belonging to the covering $\mathcal{V}^\rho_\alpha$ defined in (3.1), which are regions in the plane in the time-strip between σ and $\tau = \sigma+\varepsilon$ obtained translating of $\rho\varepsilon^\alpha$ a characteristic curve γ through $(\bar{t},\bar{x}) = (\sigma,\gamma(\sigma))$. The convexity of f provides [11] (or, more explicitly, [2, (3.1)]) the one sided estimate, if $\sigma < \tau = \sigma + \varepsilon$,

$$\int_{\gamma(\tau)}^{\gamma(\tau)+\rho\varepsilon^\alpha} u(\tau,x)dx - \int_{\gamma(\sigma)}^{\gamma(\sigma)+\rho\varepsilon^\alpha} u(\sigma,x)dx \leq \int_\sigma^\tau \int_{\gamma(t)}^{\gamma(t)+\rho\varepsilon^\alpha} g(t,x)dxdt.$$

This estimate implies, denoting by C the $1/\alpha$-Hölder continuity constant given by Corollary 2.2, that

$$\rho\varepsilon^\alpha[u(\tau,\gamma(\tau)) - u(\sigma,\gamma(\sigma))] - 2C\int_0^{\rho\varepsilon^\alpha} \sqrt[\alpha]{x}dx \leq \int_\sigma^\tau \int_{\gamma(t)}^{\gamma(t)+\rho\varepsilon^\alpha} g(t,x)dxdt.$$

Divide by $\rho\varepsilon^\alpha|\tau-\sigma| = \rho\varepsilon^{\alpha+1}$. Corollary 3.4 assures that at $\mathcal{L}^2$-a.e. $(\bar{t},\bar{x})$ the right hand side converges to the Lebesgue value $\mathfrak{g}(\bar{t},\bar{x})$. At those points $(\bar{t},\bar{x})$, one has therefore for some constant $\widetilde{C} = \widetilde{C}(\alpha) > 0$

$$\limsup_{\tau\downarrow\sigma} \frac{u(\tau,\gamma(\tau)) - u(\sigma,\gamma(\sigma))}{\tau-\sigma} \leq \mathfrak{g}(\bar{t},\bar{x}) + \widetilde{C}\lim_{\varepsilon\downarrow 0}\frac{(\rho\varepsilon^\alpha)^{1+\frac{1}{\alpha}}}{\rho\varepsilon^{\alpha+1}} = \mathfrak{g}(\bar{t},\bar{x}) + \widetilde{C}\sqrt[\alpha]{\rho}.$$

The same upper limit can be obtained for $\tau \uparrow \sigma$ just considering the region $S^{-,\rho}_{r,\alpha}$ instead of $S^{+,\rho}_{r,\alpha}$. By analogous considerations on the other regions $S^{+,\rho}_{\ell,\alpha}$, $S^{-,\rho}_{\ell,\alpha}$ one gets the opposite inequality

$$\liminf_{\tau\to\sigma} \frac{u(\tau,\gamma(\tau)) - u(\sigma,\gamma(\sigma))}{\tau-\sigma} \geq \mathfrak{g}(\bar{t},\bar{x}) - \widetilde{C}\sqrt[\alpha]{\rho}.$$

Since we proved that at those points where (3.13) holds for every $\rho \in \{\rho_n\}_{n\in\mathbb{N}}$, the inequalities

$$\begin{aligned}\mathfrak{g}(\bar{t},\bar{x}) - \widetilde{C}\sqrt[\alpha]{\rho} &\leq \liminf_{|\tau-\sigma|\downarrow 0} \frac{u(\tau,\gamma(\tau)) - u(\sigma,\gamma(\sigma))}{\tau-\sigma} \\ &\leq \limsup_{|\tau-\sigma|\downarrow 0} \frac{u(\tau,\gamma(\tau)) - u(\sigma,\gamma(\sigma))}{\tau-\sigma} \leq \mathfrak{g}(\bar{t},\bar{x}) + \widetilde{C}\sqrt[\alpha]{\rho}\end{aligned}$$

also hold for each $\rho \in \{\rho_n\}_{n\in\mathbb{N}}$, and since $\rho_n \downarrow 0$, then the limit exists with value $\mathfrak{g}(\bar{t},\bar{x})$. □

3.3. **The case** $n > 1$**.** In this section we conclude the identification of the Lagrangian and Eulerian source terms when $f \in C^{2n}$ is α-convex for $\alpha = 2n > 2$. By Lemma 3.6 below the set of real values $\{\bar{v}_k\}_{\in\mathbb{N}}$ where f'' vanishes can be enumerated: in particular one has the partition

$$\mathbb{R}^2 = \bigcup_{k\in\mathbb{N}} \{(t,x) \ : \ u(t,x) = \bar{v}_k\} \cup \{(t,x) \ \ f''(t,x) \neq 0\}.$$

We already identified by Lemma 3.5 above the Lagrangian and Eulerian source terms $\mathcal{L}^2$-a.e. on the open set where f'' does not vanish. The only points (t,x) where we still need to identify Lagrangian and Eulerian source terms $\mathcal{L}^2$-a.e. correspond to the at most countably many values $\{\bar v_k\}_{\in\mathbb{N}}$ where f'' vanishes and where (3.14) below holds with $n>1$. In order to conclude we have to identify source terms on each set

$$B_{\bar v_k} := u^{-1}(\bar v_k).$$

Lemma 3.6. *If $f\in C^2(\mathbb{R})$ is α-convex with $\alpha\geq 2$ then the set of zeros of f'' are isolated.*

Proof. Suppose by contradiction that there exists $z_j\to\bar z$ where $f''(\bar z)=f''(z_j)=0$ for all $j\in\mathbb{N}$. Then in particular, as a simple consequence of de L'Hôpital's theorem, one has

$$\begin{aligned}\lim_{j\to\infty}\frac{f(z_j)-f(\bar z)-f'(\bar z)(z_j-\bar z)}{(z_j-\bar z)^\alpha} &= \lim_{j\to\infty}\frac{f''(z_j)}{(z_j-\bar z)^{\alpha-2}(\alpha-1)\alpha}\\ &= \lim_{j\to\infty}\frac{0}{(z_j-\bar z)^{\alpha-2}(\alpha-1)\alpha}=0\ .\end{aligned}$$

The fact that

$$f(z_j)-f(\bar z)-f'(\bar z)(z_j-\bar z)=o\left(|z_j-\bar z|^\alpha\right)\quad \text{for a sequence } z_j\to\bar z$$

contradicts Definition 1.1 of α-convexity. □

For the rest of the section, we fix a single value $\bar v$ where $f''(\bar v)$ vanishes and we study the set

$$B_{\bar v}=\{(t,x)\ :\ u(t,x)=\bar v\}\ .$$

Due to smoothness and α-convexity, where $\alpha=2\bar n>2$ since we consider $f''(\bar v)=0$, the following expansion holds at $\bar v$: $f^{(2n)}(\bar v)\neq 0$ for some $n\in\{2,\dots,\bar n\}$ and

$$f(z)-f(\bar v)-(z-\bar v)f'(\bar v)=\frac{f^{(2n)}(\bar v)}{(2n)!}(z-\bar v)^{2n}+o(|z-\bar v|^{2n})\qquad \text{when } z\to\bar v. \tag{3.14}$$

We now provide Vitali coverings ad hoc for the set $B_{\bar v}$. We then show in Lemma 3.8 how this implies that the differentiation theorem holds on $B_{\bar v}$ also for $\mathcal{V}^\rho_\alpha$ defined at (3.1), thus we identify sources as in §3.2.

Let $\rho>0$, set $\bar\lambda:=f'(\bar v)$. Consider the covering of the closed set $B_{\bar v}$ given by

$$\mathcal{V}^\rho_{2n}(\bar v):=\left\{P^{+;\rho}_{r,2n}(\bar t,\bar x,\varepsilon),\ P^{+;\rho}_{\ell,2n}(\bar t,\bar x,\varepsilon)\ |\ 1>\varepsilon>0,\ (\bar t,\bar x)\in B_{\bar v}\right\}, \tag{3.15}$$

$$\begin{aligned}P^{+;\rho}_{r,2n}(\bar t,\bar x,\varepsilon) &:= \{(t,x)\ :\\ &\qquad \bar t\leq t\leq \bar t+\varepsilon,\quad \bar x+\bar\lambda(t-\bar t)\leq x\leq \bar x+\bar\lambda(t-\bar t)+\rho\varepsilon^{2n}\}\ ,\end{aligned} \tag{3.16}$$

$$\begin{aligned}P^{+;\rho}_{\ell,2n}(\bar t,\bar x,\varepsilon) &:= \{(t,x)\ :\\ &\qquad \bar t\leq t\leq \bar t+\varepsilon,\quad \bar x+\bar\lambda(t-\bar t)-\rho\varepsilon^{2n}\leq x\leq \bar x+\bar\lambda(t-\bar t)\}\ .\end{aligned} \tag{3.17}$$

The above regions are analogous to the ones of the covering $\mathcal{V}^\rho_\alpha$ in Figure 2 but the lateral edges have the fixed slope $\bar\lambda$ rather than being characteristic curves.

Lemma 3.7. *$\mathcal{V}^\rho_{2n}(\bar v)$ is a Vitali covering of $B_{\bar v}$. In particular, given a Borel function $\mathfrak{g}$, $\mathcal{L}^2$-a.e. point $(\bar t,\bar x)$ of $B_{\bar v}$ is a $\mathcal{V}^\rho_{2n}(\bar v)$-Lebesgue point of $\mathfrak{g}$ in the sense that*

$$\lim_{\substack{(\bar t,\bar x)\in P\in\mathcal{V}^\rho_{2n}(\bar v)\\ \operatorname{diam}(P)\downarrow 0}}\frac{1}{\mathcal{L}^2(P)}\int_P|\mathfrak{g}(t,x)-\mathfrak{g}(\bar t,\bar x)|dtdx=0. \tag{3.18}$$

Proof. We apply [13, Theorem 2.8.17] with the covering relation $\mathcal{V}^{\rho}_{2n}(\bar v)$, the non-negative function $\delta(P^{+,\rho}_{r,2n}(\bar t,\bar x,\varepsilon))=\delta(P^{+,\rho}_{\ell,2n}(\bar t,\bar x,\varepsilon))=\varepsilon$, $\phi=\mathcal{L}^2$, $\tau=2$: indeed, being parallelograms, the τ-enlargement $\widehat P$ of $P\in\mathcal{V}^{\rho}_{2n}(\bar v)$ is a parallelogram with sides of slope $\bar\lambda$, base of length $\rho(1+2^{2n+1})\varepsilon^{2n}$, eight equal to 5ε. In particular,

$$\lim_{\substack{(\bar t,\bar x)\in P\in\mathcal{V}^{\rho}_{2n}(\bar v)\\ \mathrm{diam}(P)\downarrow 0}}\left[\delta(P)+\frac{\mathcal{L}^2(\widehat P)}{\mathcal{L}^2(P)}\right]=5(1+2^{2n+1}).$$

Since the assumptions are verified, [13, Theorem 2.8.17] gives us that $\mathcal{V}^{\rho}_{2n}(\bar v)$ is a Vitali covering of $B_{\bar v}$. The differentiation theorem follows then from [13, Theorem 2.9.8]. □

Lemma 3.8. *Let $g=[\mathfrak{g}]\in L^\infty(\mathbb{R}^2)$ be the equivalence class of functions $\mathcal{L}^2$-a.e. equal to the Borel function $\mathfrak{g}$ and u a continuous solution to (1.1) with a $2n$-convex flux $f\in C^{2n}(\mathbb{R})$. If $(\bar t,\bar x)\in B_{\bar v}$ is a $\mathcal{V}^{\rho}_{2n}(\bar v)$-Lebesgue point of g in the sense of (3.18) and γ is a characteristic curve through $(\bar t,\bar x)$, then*

$$\gamma(t)=\bar x+\bar\lambda(t-\bar t)+\frac{f^{(2n)}(\bar v)}{(2n)!}[\mathfrak{g}(\bar t,\bar x)]^{2n-1}(t-\bar t)^{2n}+o((t-\bar t)^{2n}).$$

Moreover, also (3.13) holds and $u\circ\gamma$ is differentiable at $\bar t$ with derivative $\mathfrak{g}(\bar t,\bar x)$.

Proof. Simplify notations setting $\bar t=\bar x=0$.
1: Expansion of f' in $\bar z$. We claim that assumption (3.14) implies

$$f'(z)-\bar\lambda=\frac{f^{(2n)}(\bar v)}{(2n-1)!}(z-\bar v)^{2n-1}+o(|z-\bar v|^{2n-1}) \tag{3.19}$$

in a neighborhood of $\bar v$. Setting $h(w)=f(\bar v+w)-f(\bar v)-wf'(\bar v)-\frac{f^{(2n)}(\bar v)}{(2n)!}w^{2n}$, the statement follows by (3.14) and the following observation: if $h\in C^1(\mathbb{R})$ and $h(w)=o(w^{2n})$ as $w\to 0$, then $h(w)=k(w)w^{2n}$ with $k\in C^1(\mathbb{R}\setminus\{0\})$ and $k(w)\to 0$ as $w\to 0$, thus by differentiation of the product $k(w)w^{2n}$ one obtains

$$h\in C^1(\mathbb{R}),\quad h(w)=o(w^{2n})\text{ for }w\to 0\quad\Rightarrow\quad h'(w)=o(w^{2n-1})\text{ for }w\to 0.$$

2: Expansion of γ. Let us denote $i_\gamma(t)=(t,\gamma(t))$. By (3.19), the characteristic γ satisfies

$$\begin{aligned}\gamma(t)&=\bar x+\int_0^t f'(u(i_\gamma(s)))\,ds \\ &=\bar x+\bar\lambda t+\int_0^t\left[\frac{f^{(2n)}(\bar v)}{(2n-1)!}(u(i_\gamma(s))-\bar v)^{2n-1}+o(|u(i_\gamma(s))-\bar v|^{2n-1})\right]ds.\end{aligned}\tag{3.20}$$

Recall that u is G-Lipschitz continuous along characteristics [2, Theorem 30]: then

$$|u(i_\gamma(s))-\bar v|\le Gs$$

and therefore the distance among $\gamma(t)$ and the linearization $\bar x+\bar\lambda t$ is estimated by

$$\begin{aligned}\left|\gamma(t)-\bar x-\bar\lambda t\right|&\le\frac{f^{(2n)}(\bar v)}{(2n-1)!}G^{2n-1}\int_0^t\left[s^{2n-1}+o(s^{2n-1})\right]ds\\ &=\frac{f^{(2n)}(\bar v)}{(2n)!}G^{2n-1}t^{2n}+o(t^{2n}).\end{aligned}\tag{3.21}$$

3: Lebesgue points of g: differentiation theorem for $\mathcal{V}^{\rho_n}_{\alpha}$ on $B_{\bar v}$. Set

$$M = \rho^{-2n}\left(2 + \frac{f^{(2n)}(\bar v)}{(2n)!}G^{2n-1}\right).$$

Estimate (3.21) proves that both sets $S^{+,\rho}_{r,2n}(\gamma,\varepsilon,\sigma)$ and $S^{+,\rho}_{\ell,2n}(\gamma,\varepsilon,\sigma)$ defined in (3.1) are contained in

$$P^{+,\rho}_{2n} := P^{+,\rho}_{r,2n}(\bar t,\bar x, M\varepsilon)\cup P^{+,\rho}_{\ell,2n}(\bar t,\bar x, M\varepsilon)\,,$$

definitively as $\varepsilon\downarrow 0$ when ρ, $\alpha = 2n$ are fixed. Being

$$\frac{\mathcal{L}^2(P^{+,\rho}_{2n})}{\mathcal{L}^2(S^{+,\rho}_{r,2n}(\gamma,\varepsilon,\sigma))} = \frac{2\cancel{\rho}M^{2n+1}\cancel{\varepsilon^{2n+1}}}{\cancel{\rho}\cancel{\varepsilon^{2n+1}}} = 2M^{2n+1}$$

then by (3.18) necessarily, when ρ is fixed,

$$\begin{aligned}
0 &\le \lim_{\varepsilon\downarrow 0}\frac{1}{\mathcal{L}^2(S^{+,\rho}_{r,2n}(\gamma,\varepsilon,\sigma))}\int_{S^{+,\rho}_{r,2n}}|\mathfrak{g}(t,x)-\mathfrak{g}(\bar t,\bar x)|dtdx\\
&\le \lim_{\varepsilon\downarrow 0}\frac{1}{\mathcal{L}^2(S^{+,\rho}_{r,2n}(\gamma,\varepsilon,\sigma))}\int_{P^{+,\rho}_{2n}}|\mathfrak{g}(t,x)-\mathfrak{g}(\bar t,\bar x)|dtdx\\
&= \lim_{\varepsilon\downarrow 0}\frac{2M^{2n+1}}{\mathcal{L}^2(P^{+,\rho}_{2n})}\int_{\mathcal{L}^2(P^{+,\rho}_{2n})}|\mathfrak{g}(t,x)-\mathfrak{g}(\bar t,\bar x)|dtdx = 0.
\end{aligned}$$

The same limit holds as well for $S^{-,\rho}_{\varepsilon,2n}$. Since $S^{+,\rho}_{\varepsilon,2n}$ and $S^{-,\rho}_{\varepsilon,2n}$ are right and left domains bounded by a characteristic curve through $(0,0)$, and the analogous domains $S^{-,\rho}_{\ell,2n}$, $S^{-,\rho}_{r,2n}$ are similarly admissible, one can repeat the proof of Lemma 3.5 above thanks to convexity: one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u\circ i_\gamma\right)(0) = \mathfrak{g}(0,0).$$

4: Conclusion. Plugging this expansion into (3.20) one finds by elementary calculus the formula in the statement since

$$\begin{aligned}
&u(i_\gamma(s)) - \bar v = \mathfrak{g}(0,0)s + o(s)\\
&(u(i_\gamma(s)) - \bar v)^{2n-1} = [\mathfrak{g}(0,0)s + o(s)]^{2n-1} = [\mathfrak{g}(0,0)]^{2n-1}s^{2n-1} + o(s^{2n-1}). \qquad \square
\end{aligned}$$

Remark 1. Lemma 3.8 can be similarly adapted to the case when $f\in W^{2n,\infty}$ is $2n$-convex.

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