FRACTIONAL PERIMETERS FROM A FRACTAL PERSPECTIVE

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Abstract. The purpose of this paper consists in better understanding the fractional nature of the nonlocal perimeters introduced in [5]. Following [22], we exploit these fractional perimeters to introduce a definition of fractal dimension for the measure theoretic boundary of a set.

We calculate the fractal dimension of sets which can be defined in a recursive way and we give some examples of this kind of sets, explaining how to construct them starting from well known self-similar fractals. In particular, we show that in the case of the von Koch snowflake $S \subseteq \mathbb{R}^2$ this fractal dimension coincides with the Minkowski dimension.

We also obtain an optimal result for the asymptotics as $s \to 1^-$ of the fractional perimeter of a set having locally finite (classical) perimeter.

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1. Introduction and main results

The s-fractional perimeter and its minimizers, the s-minimal sets, were introduced in [5] in 2010 and since then they have attracted a lot of interest, especially concerning the regularity theory of the boundaries of the s-minimal sets, which are the so-called nonlocal minimal surfaces. We refer the interested reader to the recent survey [11] and the references cited therein.

Even if finding the optimal regularity of nonlocal minimal surfaces is still an engaging open problem, it is known that nonlocal minimal surfaces are $(n-1)$-rectifiable. More precisely, they are smooth, except possibly for a singular set of Hausdorff dimension at most equal to $n-3$ (see [5], [20] and [15]). In particular, an s-minimal set has (locally) finite perimeter (in the sense of De Giorgi and Caccioppoli).

On the other hand, the boundary of a generic set E having finite s-perimeter can be very irregular and indeed it can be "nowhere rectifiable", like in the case of the von Koch snowflake.

Actually, the s-perimeter can be used (following the seminal paper [22]) to define a "fractal dimension" for the measure theoretic boundary

$$\partial^- E := \{x \in \mathbb{R}^n \,|\, 0 < |E \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0\},$$

of a set $E \subseteq \mathbb{R}^n$.

Before going on, it is useful to recall the definition of the s-perimeter. Given a fractional parameter $s \in (0,1)$, we define the interaction

$$\mathcal{L}_s(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\,dx\,dy,$$

for every couple of disjoint sets A, $B \subseteq \mathbb{R}^n$. Then the s-fractional perimeter of a set $E \subseteq \mathbb{R}^n$ in an open set Ω is defined as

$$P_s(E,\Omega) := \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) + \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, \mathcal{C}E\cap\Omega).$$

We observe that we can rewrite the s-perimeter as

$$P_s(E,\Omega) = \frac{1}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy. \tag{1.1}$$

Formula (1.1) shows that the fractional perimeter is, roughly speaking, the Ω-contribution to the $W^{s,1}$-seminorm of the characteristic function χ_E.

This functional is nonlocal, in that we need to know the set E in the whole of $\mathbb{R}^n$ even to compute its s-perimeter in a small bounded domain Ω (contrary to what happens with the classical perimeter or the $\mathcal{H}^{n-1}$ measure, which are local functionals). Moreover, the s-perimeter is "fractional", in the sense that the $W^{s,1}$-seminorm measures a fractional order of regularity.

The main purpose of this paper consists in clarifying and better understanding the "fractional" nature of the s-perimeter.

In 1991, in the paper [22] the author suggested using the index s of the fractional seminorm $[\chi_E]_{W^{s,1}(\Omega)}$ (and more general continuous families of functionals satisfying appropriate generalized coarea formulas) as a way to measure the codimension of the measure theoretic boundary $\partial^- E$ of the set E in Ω. He proved that the fractal dimension obtained in this way,

$$\mathrm{Dim}_F(\partial^- E,\Omega) := n - \sup\{s \in (0,1) \,|\, [\chi_E]_{W^{s,1}(\Omega)} < \infty\}.$$

is less than or equal to the (upper) Minkowski dimension.

The relationship between the Minkowski dimension of the boundary of E and the fractional regularity (in the sense of Besov spaces) of the characteristic function χ_E was investigated also in [21], in 1999. In particular, in [21, Remark 3.10], the author proved that the dimension Dim_F of the von Koch snowflake S coincides with its Minkowski dimension, exploiting the fact that S is a John domain.

The Sobolev regularity of a characteristic function χ_E was further studied in [14], in 2013, where the authors consider the case in which the set E is a quasiball. Since the von Koch snowflake S is a typical example of quasiball, the authors were able to prove that the dimension Dim_F of S coincides with its Minkowski dimension.

In this paper we compute the dimension Dim_F of the von Koch snowflake S in an elementary way, using only the roto-translation invariance and the scaling property of the s-perimeter and the "self-similarity" of S.

The proof can be extended in a natural way to all sets which can be defined in a recursive way similar to that of the von Koch snowflake.
As a consequence, we compute the dimension Dim_F of all such sets, without having to require them to be John domains or quasiballs.

Furthermore, we show that we can easily obtain a lot of sets of this kind by appropriately modifying well known self-similar fractals like e.g. the von Koch snowflake, the Sierpinski triangle and the Menger sponge. An example is depicted in Figure 1.

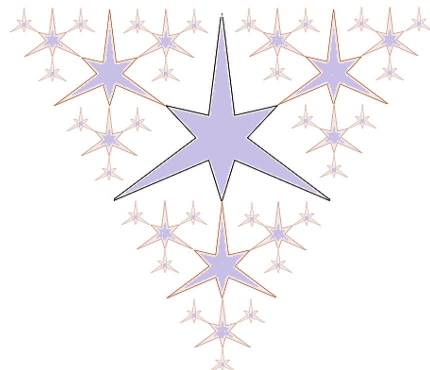

FIGURE 1. *Example of a "fractal" set constructed exploiting the structure of the Sierpinski triangle (seen at the fourth iterative step).*

The previous discussion shows that the s-perimeter of a set E with an irregular, eventually fractal, boundary can be finite for s below some threshold, $s < \sigma$, and infinite for $s \in (\sigma, 1)$. On the other hand, it is well known that sets with a regular boundary have finite s-perimeter for every s and actually their s-perimeter converges, as s tends to 1, to the classical perimeter, both in the classical sense (see [6]) and in the Γ-convergence sense (see [2] and also [19] for related results).

In this paper we exploit [7, Theorem 1] to prove an optimal version of this asymptotic property for a set E having finite classical perimeter in a bounded open set with Lipschitz boundary. More precisely, we prove that if E has finite classical perimeter in a neighborhood of Ω, then

$$\lim_{s\to 1}(1-s)P_s(E,\Omega) = \omega_{n-1}P(E,\overline{\Omega}).$$

We observe that we lower the regularity requested in [6], where the authors required the boundary ∂E to be $C^{1,\alpha}$, to the optimal regularity (asking E to have only finite perimeter). Moreover, we don't have to ask E to intersect $\partial\Omega$ "transversally", i.e. we don't require

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega) = 0,$$

with $\partial^* E$ denoting the reduced boundary of E.

Indeed, we prove that the "nonlocal part" of the s-perimeter converges to the perimeter on the boundary of Ω, i.e. we prove that

$$\lim_{s\to 1}(1-s)P_s^{NL}(E,\Omega)=\omega_{n-1}\mathcal{H}^{n-1}(\partial^* E\cap\partial\Omega),$$

which is, to the best of the author's knowledge, a new result.

Now we give precise statements of the results obtained, starting with the fractional analysis of fractal dimensions.

1.1. **Fractal boundaries.** We observe that we can split the fractional perimeter as the sum

$$P_s(E,\Omega)=P_s^L(E,\Omega)+P_s^{NL}(E,\Omega),$$

where

$$P_s^L(E,\Omega):=\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\cap\Omega)=\frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)},$$
$$P_s^{NL}(E,\Omega):=\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\setminus\Omega)+\mathcal{L}_s(E\setminus\Omega,\mathcal{C}E\cap\Omega).$$

We can think of $P_s^L(E,\Omega)$ as the local part of the fractional perimeter, in the sense that if $|(E\Delta F)\cap\Omega|=0$, then $P_s^L(F,\Omega)=P_s^L(E,\Omega)$.

We sometimes refer to $P_s^{NL}(E,\Omega)$ as the nonlocal part of the s-perimeter.

We say that a set E has locally finite s-perimeter if it has finite s-perimeter in every bounded open set $\Omega\subseteq\mathbb{R}^n$.

When $\Omega=\mathbb{R}^n$, we simply write

$$P_s(E):=P_s(E,\mathbb{R}^n)=\frac{1}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)}.$$

First of all, we prove in Section 3.1 that in some sense the measure theoretic boundary $\partial^- E$ is the "right definition" of boundary for working with the s-perimeter.

To be more precise, we show that

$$\partial^- E=\{x\in\mathbb{R}^n\,|\,P_s^L(E,B_r(x))>0,\,\forall\, r>0\},$$

and that if Ω is a connected open set, then

$$P_s^L(E,\Omega)>0\quad\iff\quad\partial^- E\cap\Omega\neq\emptyset.$$

This can be thought of as an analogue in the fractional framework of the fact that for a Caccioppoli set E we have $\partial^- E=\operatorname{supp}|D\chi_E|$.

Now the idea of the definition of the fractal dimension consists in using the index s of $P_s^L(E,\Omega)$ to measure the codimension of $\partial^- E\cap\Omega$,

$$\mathrm{Dim}_F(\partial^- E,\Omega):=n-\sup\{s\in(0,1)\,|\,P_s^L(E,\Omega)<\infty\}.$$

As shown in [22] (Proposition 11 and Proposition 13), the fractal dimension Dim_F defined in this way is related to the (upper) Minkowski dimension (whose precise definition we recall in Definition 3.4) by

$$\mathrm{Dim}_F(\partial^- E,\Omega)\leq\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E,\Omega).\tag{1.2}$$

For the convenience of the reader we provide a proof of inequality (1.2) in Proposition 3.6.

If Ω is a bounded open set with Lipschitz boundary, (1.2) means that

$$P_s(E,\Omega)<\infty\qquad\text{for every } s\in\big(0,n-\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E,\Omega)\big),$$

since the nonlocal part of the s-perimeter of any set $E \subseteq \mathbb{R}^n$ is

$$P_s^{NL}(E,\Omega) \leq 2P_s(\Omega) < \infty, \qquad \text{for every } s \in (0,1).$$

We show that for the von Koch snowflake (1.2) is actually an equality.

Namely, we prove the following:

Theorem 1.1 (Fractal dimension of the von Koch snowflake)**.** *Let $S \subseteq \mathbb{R}^2$ be the von Koch snowflake. Then*

$$P_s(S) < \infty, \qquad \forall\, s \in \Big(0, 2 - \frac{\log 4}{\log 3}\Big), \tag{1.3}$$

and

$$P_s(S) = \infty, \qquad \forall\, s \in \Big[2 - \frac{\log 4}{\log 3}, 1\Big). \tag{1.4}$$

Therefore

$$Dim_F(\partial S) = Dim_{\mathcal{M}}(\partial S) = \frac{\log 4}{\log 3}.$$

Actually, exploiting the self-similarity of the von Koch curve, we have

$$\mathrm{Dim}_F(\partial S, \Omega) = \frac{\log 4}{\log 3},$$

for every Ω such that $\partial S \cap \Omega \neq \emptyset$. In particular, this is true for every $\Omega = B_r(p)$ with $p \in \partial S$ and $r > 0$ as small as we want.

We remark that this represents a deep difference between the classical and the fractional perimeter.

Indeed, if a set E has (locally) finite perimeter, then by De Giorgi's structure Theorem we know that its reduced boundary $\partial^* E$ is locally $(n-1)$-rectifiable. Moreover $\overline{\partial^* E} = \partial^- E$, so the reduced boundary is, in some sense, a "big" portion of the measure theoretic boundary.

On the other hand, we have seen that there are (open) sets, like the von Koch snowflake, which have a "nowhere rectifiable" boundary (meaning that $\partial^- E \cap B_r(p)$ is not $(n-1)$-rectifiable for every $p \in \partial^- E$ and $r > 0$) and still have finite s-perimeter for every $s \in (0, \sigma_0)$.

1.1.1. *Self-similar fractal boundaries.* Our argument for the von Koch snowflake is quite general and can be adapted to compute the dimension Dim_F of all sets which can be constructed in a similar recursive way.

To be more precise, we start with a bounded open set $T_0 \subseteq \mathbb{R}^n$ with finite perimeter $P(T_0) < \infty$, which is, roughly speaking, our basic "building block".

Then we go on inductively by adding roto-translations of a scaling of the building block T_0, i.e. sets of the form

$$T_k^i = F_k^i(T_0) := \mathcal{R}_k^i\big(\lambda^{-k} T_0\big) + x_k^i,$$

where $\lambda > 1$, $k \in \mathbb{N}$, $1 \leq i \leq ab^{k-1}$, with a, $b \in \mathbb{N}$, $\mathcal{R}_k^i \in SO(n)$ and $x_k^i \in \mathbb{R}^n$. We ask that these sets do not overlap, i.e.

$$|T_k^i \cap T_h^j| = 0, \qquad \text{whenever } i \neq j \text{ or } k \neq h.$$

Then we define

$$T_k := \bigcup_{i=1}^{ab^{k-1}} T_k^i \qquad \text{and} \qquad T := \bigcup_{k=1}^{\infty} T_k. \tag{1.5}$$

The final set E is either

$$E := T_0 \cup \bigcup_{k\geq 1} \bigcup_{i=1}^{ab^{k-1}} T_k^i, \quad \text{or} \quad E := T_0 \setminus \Big(\bigcup_{k\geq 1} \bigcup_{i=1}^{ab^{k-1}} T_k^i \Big).$$

For example, the von Koch snowflake is obtained by adding pieces.
Examples obtained by removing the T_k^i's are the middle Cantor set $E \subseteq \mathbb{R}$, the Sierpinski triangle $E \subseteq \mathbb{R}^2$ and the Menger sponge $E \subseteq \mathbb{R}^3$.

We will consider just the set T and exploit the same argument used for the von Koch snowflake to compute the fractal dimension related to the s-perimeter.
However, we observe that the Cantor set, the Sierpinski triangle and the Menger sponge are such that $|E| = 0$, i.e. $|T_0 \Delta T| = 0$.
Therefore neither the perimeter nor the s-perimeter can detect the fractal nature of the (topological) boundary of T and indeed, since

$$P(T) = P(T_0) < \infty,$$

we have $P_s(T) < \infty$ for every $s \in (0,1)$.

For example, in the case of the Sierpinski triangle, T_0 is an equilateral triangle and $\partial^- T = \partial T_0$, even if ∂T is a self-similar fractal.

The reason of this situation is that the fractal object is the topological boundary of T, while the s-perimeter "measures" the measure theoretic boundary, which is regular. Roughly speaking, the problem is that in these cases there is not room enough to find a small ball $B_k^i = F_k^i(B) \subseteq \mathcal{C}T$ near each piece T_k^i.

Therefore, we will make the additional assumption that

$$\exists\, S_0 \subseteq \mathcal{C}T \quad \text{s.t. } |S_0| > 0 \quad \text{and } S_k^i := F_k^i(S_0) \subseteq \mathcal{C}T \quad \forall\, k,\, i. \tag{1.6}$$

We remark that it is not necessary to ask that these sets do not overlap.

Theorem 1.2. *Let $T \subseteq \mathbb{R}^n$ be a set which can be written as in (1.5). If $\frac{\log b}{\log \lambda} \in (n-1, n)$ and (1.6) holds true, then*

$$P_s(T) < \infty, \qquad \forall\, s \in \Big(0, n - \frac{\log b}{\log \lambda}\Big)$$

and

$$P_s(T) = \infty, \qquad \forall\, s \in \Big[n - \frac{\log b}{\log \lambda}, 1\Big).$$

Thus

$$Dim_F(\partial^- T) = \frac{\log b}{\log \lambda}.$$

Furthermore, we show how to modify self-similar sets like the Sierpinski triangle, without altering their "structure", to obtain new sets which satisfy the hypothesis of Theorem 1.2 (see Remark 3.10 and the final part of Section 3.4). An example is given in Figure 1 above.

However, we also remark that the measure theoretic boundary of such a new set will look quite different from the original fractal (topological) boundary and in general it will be a mix of smooth parts and unrectifiable parts.

The most interesting examples of this kind of sets are probably represented by bounded sets, because in this case the measure theoretic boundary does indeed have, in some sense, a "fractal nature" (see Remark 3.11).

Indeed, if T is bounded, then its boundary $\partial^- T$ is compact. Nevertheless, it has infinite (classical) perimeter and actually $\partial^- T$ has Minkowski dimension strictly greater than $n-1$, thanks to (1.2).

However, even unbounded sets can have an interesting behavior. Indeed we obtain the following

Proposition 1.3. *Let $n \geq 2$. For every $\sigma \in (0,1)$ there exists a Caccioppoli set $E \subseteq \mathbb{R}^n$ such that*

$$P_s(E) < \infty \qquad \forall\, s \in (0,\sigma) \quad \textit{and} \quad P_s(E) = \infty \qquad \forall\, s \in [\sigma, 1).$$

Roughly speaking, the interesting thing about this Proposition is the following. Since E has locally finite perimeter, $\chi_E \in BV_{loc}(\mathbb{R}^n)$, it also has locally finite s-perimeter for every $s \in (0,1)$, but the global perimeter $P_s(E)$ is finite if and only if $s < \sigma < 1$.

1.2. **Asymptotics as $s \to 1^-$.** In Section 1.1 we have shown that sets with an irregular, eventually fractal, boundary can have finite s-perimeter.

On the other hand, if the set E is "regular", then it has finite s-perimeter for every $s \in (0,1)$. Indeed, if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary (or $\Omega = \mathbb{R}^n$), then $BV(\Omega) \hookrightarrow W^{s,1}(\Omega)$. As a consequence of this embedding, we find that

$$P(E,\Omega) < \infty \qquad \Longrightarrow \qquad P_s(E,\Omega) < \infty \quad \text{for every } s \in (0,1).$$

Actually we can be more precise and obtain a sort of converse, using only the local part of the s-perimeter and adding the condition

$$\liminf_{s \to 1^-} (1-s) P_s^L(E,\Omega) < \infty.$$

Indeed one has the following result, which is a combination of [4, Theorem 3'] and [7, Theorem 1], restricted to characteristic functions:

Theorem 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then $E \subseteq \mathbb{R}^n$ has finite perimeter in Ω if and only if $P_s^L(E,\Omega) < \infty$ for every $s \in (0,1)$, and*

$$\liminf_{s \to 1} (1-s) P_s^L(E,\Omega) < \infty. \tag{1.7}$$

In this case we have

$$\lim_{s \to 1} (1-s) P_s^L(E,\Omega) = \frac{n\omega_n}{2} K_{1,n} P(E,\Omega). \tag{1.8}$$

We briefly show how to get this result (and in particular why the constant looks like that) from the two Theorems cited above. Then we compute the constant $K_{1,n}$ in an elementary way, proving that

$$\frac{n\omega_n}{2} K_{1,n} = \omega_{n-1}.$$

Moreover we show the following:

Remark 1.5. Condition (1.7) is necessary. Indeed, there exist bounded sets (see Example 1.1) having finite s-perimeter for every $s \in (0,1)$ which do not have finite perimeter. This also shows that in general the inclusion

$$BV(\Omega) \subseteq \bigcap_{s \in (0,1)} W^{s,1}(\Omega)$$

is strict.

Example 1.1. Let $0 < a < 1$ and consider the open intervals $I_k := (a^{k+1}, a^k)$ for every $k \in \mathbb{N}$. Define $E := \bigcup_{k\in\mathbb{N}} I_{2k}$, which is a bounded (open) set. Due to the infinite number of jumps $\chi_E \notin BV(\mathbb{R})$. However it can be proved that E has finite s-perimeter for every $s \in (0,1)$. We postpone the proof to Appendix A.

Remark 1.6. For completeness, we also mention a related result contained in [9], where the authors provide an example (Example 2.10) of a bounded set $E \subseteq \mathbb{R}$ which does not have finite s-perimeter for any $s \in (0,1)$. In particular, this example proves that in general the inclusion

$$\bigcup_{s\in(0,1)} W^{s,1}(\Omega) \subseteq L^1(\Omega)$$

is strict.

The main result of Section 2 is the following Theorem, which extends the asymptotic convergence of (1.8) to the whole s-perimeter.

Theorem 1.7 (Asymptotics)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E \subseteq \mathbb{R}^n$. Then, E has locally finite perimeter in Ω if and only if E has locally finite s-perimeter in Ω for every $s \in (0,1)$ and*

$$\liminf_{s\to 1}(1-s)P_s^L(E,\Omega') < \infty, \qquad \forall\, \Omega' \Subset \Omega.$$

If E has locally finite perimeter in Ω, then

$$\lim_{s\to 1}(1-s)P_s(E,\mathcal{O}) = \omega_{n-1}P(E,\overline{\mathcal{O}}),$$

for every open set $\mathcal{O} \Subset \Omega$ with Lipschitz boundary. More precisely,

$$\lim_{s\to 1}(1-s)P_s^L(E,\mathcal{O}) = \omega_{n-1}P(E,\mathcal{O})$$

and

$$\lim_{s\to 1}(1-s)P_s^{NL}(E,\mathcal{O}) = \omega_{n-1}P(E,\partial\mathcal{O}) = \omega_{n-1}\mathcal{H}^{n-1}(\partial^* E \cap \partial\mathcal{O}). \tag{1.9}$$

The proof of Theorem 1.7 relies only on [4, Theorem 3'], [7, Theorem 1] and on an appropriate estimate of what happens in a neighborhood of $\partial\mathcal{O}$. The main improvement of the known asymptotics results is the convergence (1.9).

1.3. **Notation and assumptions.**

- We write $A \Subset B$ to mean that the closure of A is compact in $\mathbb{R}^n$ and $\overline{A} \subseteq B$.
- In $\mathbb{R}^n$ we will usually write $|E| = \mathcal{L}^n(E)$ for the n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$.
- We write $\mathcal{H}^d$ for the d-dimensional Hausdorff measure, for any $d \geq 0$.
- We define the dimensional constants

$$\omega_d := \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}, \qquad d \geq 0.$$

  In particular, we remark that $\omega_k = \mathcal{L}^k(B_1)$ is the volume of the k-dimensional unit ball $B_1 \subseteq \mathbb{R}^k$ and $k\,\omega_k = \mathcal{H}^{k-1}(\mathbb{S}^{k-1})$ is the surface area of the $(k-1)$-dimensional sphere

$$\mathbb{S}^{k-1} = \partial B_1 = \{x \in \mathbb{R}^k \,|\, |x| = 1\}.$$

- Since
$$|E\Delta F|=0 \quad\Longrightarrow\quad P(E,\Omega)=P(F,\Omega) \quad\text{and}\quad P_s(E,\Omega)=P_s(F,\Omega),$$
in Section 2 we implicitly identify sets up to sets of negligible Lebesgue measure. Moreover, whenever needed we can choose a particular representative for the class of χ_E in $L^1_{loc}(\mathbb{R}^n)$, as in Remark 1.8.
We will not make this assumption in Section 3, since the Minkowski content can be affected even by changes in sets of measure zero, that is, in general
$$|\Gamma_1\Delta\Gamma_2|=0 \quad\not\Rightarrow\quad \overline{\mathcal{M}}^r(\Gamma_1,\Omega)=\overline{\mathcal{M}}^r(\Gamma_2,\Omega)$$
(see Section 3 for a more detailed discussion).
- We consider the open tubular ϱ-neighborhood of $\partial\Omega$,
$$N_\varrho(\partial\Omega):=\{x\in\mathbb{R}^n\,|\,d(x,\partial\Omega)<\varrho\}=\{|\bar{d}_\Omega|<\varrho\}=\Omega_\varrho\setminus\overline{\Omega_{-\varrho}}$$
(see Appendix B).

Remark 1.8. Let $E\subseteq\mathbb{R}^n$. Up to modifying E on a set of measure zero, we can assume (see Appendix C) that
$$E_{int}\subseteq E,\qquad E\cap E_{ext}=\emptyset$$
$$\text{and}\quad \partial E=\partial^-E=\{x\in\mathbb{R}^n\,|\,0<|E\cap B_r(x)|<\omega_n r^n,\ \forall\, r>0\}.$$

2. Asymptotics as $s\to 1^-$

We say that an open set $\Omega\subseteq\mathbb{R}^n$ is an extension domain if there exists a constant $C=C(n,s,\Omega)>0$ such that for every $u\in W^{s,1}(\Omega)$ there exists $\tilde{u}\in W^{s,1}(\mathbb{R}^n)$ with $\tilde{u}_{|\Omega}=u$ and
$$\|\tilde{u}\|_{W^{s,1}(\mathbb{R}^n)}\le C\|u\|_{W^{s,1}(\Omega)}.$$
Every open set with bounded Lipschitz boundary is an extension domain (see [8] for a proof). By definition we consider $\mathbb{R}^n$ itself as an extension domain.

We begin with the following embedding.

Proposition 2.1. *Let $\Omega\subseteq\mathbb{R}^n$ be an extension domain. Then there exists a constant $C=C(n,s,\Omega)\ge 1$ such that for every $u:\Omega\longrightarrow\mathbb{R}$*
$$\|u\|_{W^{s,1}(\Omega)}\le C\|u\|_{BV(\Omega)}. \tag{2.1}$$
In particular we have the continuous embedding
$$BV(\Omega)\hookrightarrow W^{s,1}(\Omega).$$

Proof. The claim is trivially satisfied if the right hand side of (2.1) is infinite, so let $u\in BV(\Omega)$. Let $\{u_k\}\subseteq C^\infty(\Omega)\cap BV(\Omega)$ be an approximating sequence as in [16, Theorem 1.17], that is
$$\|u-u_k\|_{L^1(\Omega)}\longrightarrow 0 \qquad\text{and}\qquad \lim_{k\to\infty}\int_\Omega|\nabla u_k|\,dx=|Du|(\Omega).$$
We only need to check that the $W^{s,1}$-seminorm of u is bounded by its BV-norm.
Since Ω is an extension domain, we know (see [8, Proposition 2.2]) that $\exists\, C(n,s)\ge 1$ such that
$$\|v\|_{W^{s,1}(\Omega)}\le C\|v\|_{W^{1,1}(\Omega)}.$$

Then

$$[u_k]_{W^{s,1}(\Omega)} \le \|u_k\|_{W^{s,1}(\Omega)} \le C\|u_k\|_{W^{1,1}(\Omega)} = C\|u_k\|_{BV(\Omega)},$$

and hence, using Fatou's Lemma,

$$\begin{aligned}[u]_{W^{s,1}(\Omega)} &\le \liminf_{k\to\infty}[u_k]_{W^{s,1}(\Omega)} \le C\liminf_{k\to\infty}\|u_k\|_{BV(\Omega)} = C\lim_{k\to\infty}\|u_k\|_{BV(\Omega)}\\ &= C\|u\|_{BV(\Omega)},\end{aligned}$$

proving (2.1). □

Given a set $E \subseteq \mathbb{R}^n$ and $r \in \mathbb{R}$, we denote

$$E_r := \{x \in \mathbb{R}^n \,|\, \bar{d}_E(x) < r\},$$

where $\bar{d}_E$ is the signed distance function from E (see Appendix B).

Corollary 2.2. *(i) If $E \subseteq \mathbb{R}^n$ has finite perimeter, i.e. $\chi_E \in BV(\mathbb{R}^n)$, then E has also finite s-perimeter for every $s \in (0,1)$.*

(ii) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists $r_0 > 0$ such that

$$\sup_{|r|<r_0} P_s(\Omega_r) < \infty. \tag{2.2}$$

(iii) If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then

$$P_s^{NL}(E,\Omega) \le 2P_s(\Omega) < \infty$$

for every $E \subseteq \mathbb{R}^n$.

(iv) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$P(E,\Omega) < \infty \qquad \Longrightarrow \qquad P_s(E,\Omega) < \infty \quad \text{for every } s \in (0,1).$$

Proof. Claim (i) follows from

$$P_s(E) = \frac{1}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)}$$

and Proposition 2.1 with $\Omega = \mathbb{R}^n$.

(ii) Let r_0 be as in Proposition B.1 and notice that

$$P(\Omega_r) = \mathcal{H}^{n-1}\big(\{\bar{d}_\Omega = r\}\big),$$

so that

$$\|\chi_{\Omega_r}\|_{BV(\mathbb{R}^n)} = |\Omega_r| + \mathcal{H}^{n-1}\big(\{\bar{d}_\Omega = r\}\big).$$

Thus

$$\sup_{|r|<r_0} P_s(\Omega_r) \le C\Big(|\Omega_{r_0}| + \sup_{|r|<r_0}\mathcal{H}^{n-1}\big(\{\bar{d}_\Omega = r\}\big)\Big) < \infty.$$

(iii) Notice that

$$\begin{aligned}\mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) &\le \mathcal{L}_s(\Omega,\mathcal{C}\Omega) = P_s(\Omega),\\ \mathcal{L}_s(\mathcal{C}E\cap\Omega, E\setminus\Omega) &\le \mathcal{L}_s(\Omega,\mathcal{C}\Omega) = P_s(\Omega),\end{aligned}$$

and use (2.2) (with $\Omega_0 = \Omega$).

(iv) The nonlocal part of the s-perimeter is finite thanks to (iii). As for the local part, recall that

$$P(E,\Omega) = |D\chi_E|(\Omega) \qquad \text{and} \qquad P_s^L(E,\Omega) = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)},$$

then use Proposition 2.1. □

2.1. **Asymptotics of the local part of the s-perimeter.** We recall the results of [4] and [7], which straightforwardly give Theorem 1.4.

Theorem 2.3 (Theorem 3' of [4])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain. Let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if*

$$\liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \varrho_n(x-y)\,dxdy < \infty,$$

and then

$$\begin{aligned} C_1|Du|(\Omega) &\leq \liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \varrho_n(x-y)\,dxdy \\ &\leq \limsup_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \varrho_n(x-y)\,dxdy \leq C_2|Du|(\Omega), \end{aligned}$$

for some constants C_1, C_2 depending only on Ω.

This result was refined by Dávila:

Theorem 2.4 (Theorem 1 of [7])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $u \in BV(\Omega)$. Then*

$$\lim_{k\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \varrho_k(x-y)\,dxdy = K_{1,n}|Du|(\Omega),$$

where

$$K_{1,n} = \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} |v \cdot e|\,d\sigma(v),$$

with $e \in \mathbb{R}^n$ any unit vector.

In the above Theorems ϱ_k is any sequence of radial mollifiers i.e. of functions satisfying

$$\varrho_k(x) \geq 0, \quad \varrho_k(x) = \varrho_k(|x|), \quad \int_{\mathbb{R}^n} \varrho_k(x)\,dx = 1 \tag{2.3}$$

and

$$\lim_{k\to\infty} \int_\delta^\infty \varrho_k(r) r^{n-1} dr = 0 \quad \text{for all } \delta > 0. \tag{2.4}$$

In particular, for R big enough, $R > \operatorname{diam}(\Omega)$, we can consider

$$\varrho(x) := \chi_{[0,R]}(|x|) \frac{1}{|x|^{n-1}}$$

and define for any sequence $\{s_k\} \subseteq (0,1)$, $s_k \nearrow 1$,

$$\varrho_k(x) := (1-s_k)\varrho(x) c_{s_k} \frac{1}{|x|^{s_k}},$$

where the c_{s_k} are normalizing constants. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \varrho_k(x)\,dx &= (1-s_k)c_{s_k} n\omega_n \int_0^R \frac{1}{r^{n-1+s_k}} r^{n-1}\,dr \\ &= (1-s_k)c_{s_k} n\omega_n \int_0^R \frac{1}{r^{s_k}}\,dr = c_{s_k} n\omega_n R^{1-s_k}, \end{aligned}$$

and hence taking $c_{s_k} := \frac{1}{n\omega_n} R^{s_k-1}$ gives (2.3); notice that $c_{s_k} \to \frac{1}{n\omega_n}$.
Also

$$\begin{aligned}\lim_{k\to\infty} \int_\delta^\infty \varrho_k(r) r^{n-1}\, dr &= \lim_{k\to\infty} (1-s_k) c_{s_k} \int_\delta^R \frac{1}{r^{s_k}}\, dr \\ &= \lim_{k\to\infty} c_{s_k}(R^{1-s_k} - \delta^{1-s_k}) = 0,\end{aligned}$$

giving (2.4). With this choice we obtain

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \varrho_k(x-y)\, dxdy = c_{s_k}(1-s_k)[u]_{W^{s_k,1}(\Omega)}.$$

Then, if $u \in BV(\Omega)$, Dávila's Theorem gives

$$\begin{aligned}\lim_{s\to 1}(1-s)[u]_{W^{s,1}(\Omega)} &= \lim_{s\to 1} \frac{1}{c_s}(c_s(1-s)[u]_{W^{s,1}(\Omega)}) \\ &= n\omega_n K_{1,n} |Du|(\Omega).\end{aligned} \tag{2.5}$$

2.2. **Proof of Theorem 1.7.** We split the proof of Theorem 1.7 into several steps, which we believe are interesting on their own.

2.2.1. *The constant* ω_{n-1}. We need to compute the constant $K_{1,n}$. Notice that we can choose e in such a way that $v \cdot e = v_n$.
Then using spheric coordinates for $\mathbb{S}^{n-1}$ we obtain $|v \cdot e| = |\cos\theta_{n-1}|$ and

$$d\sigma = \sin\theta_2 (\sin\theta_3)^2 \dots (\sin\theta_{n-1})^{n-2} d\theta_1 \dots d\theta_{n-1},$$

with $\theta_1 \in [0, 2\pi)$ and $\theta_j \in [0,\pi)$ for $j = 2, \dots, n-1$. Notice that

$$\begin{aligned}\mathcal{H}^k(\mathbb{S}^k) &= \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2\, d\theta_2 \dots \int_0^\pi (\sin\theta_{k-1})^{k-2}\, d\theta_{k-1} \\ &= \mathcal{H}^{k-1}(\mathbb{S}^{k-1}) \int_0^\pi (\sin t)^{k-2}\, dt.\end{aligned}$$

Then we get

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} |v\cdot e|\, d\sigma(v) &= \mathcal{H}^{n-2}(\mathbb{S}^{n-2}) \int_0^\pi (\sin t)^{n-2} |\cos t|\, dt \\ &= \mathcal{H}^{n-2}(\mathbb{S}^{n-2}) \Big(\int_0^{\frac{\pi}{2}} (\sin t)^{n-2} \cos t\, dt - \int_{\frac{\pi}{2}}^\pi (\sin t)^{n-2}\cos t\, dt \Big) \\ &= \frac{\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{n-1} \Big(\int_0^{\frac{\pi}{2}} \frac{d}{dt}(\sin t)^{n-1}\, dt - \int_{\frac{\pi}{2}}^\pi \frac{d}{dt}(\sin t)^{n-1}\, dt \Big) \\ &= \frac{2\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{n-1}.\end{aligned}$$

Therefore

$$n\omega_n K_{1,n} = 2\frac{\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{n-1} = 2\mathcal{L}^{n-1}(B_1(0)) = 2\omega_{n-1},$$

and hence (2.5) becomes

$$\lim_{s\to 1}(1-s)[u]_{W^{s,1}(\Omega)} = 2\omega_{n-1}|Du|(\Omega),$$

for any $u \in BV(\Omega)$.

2.2.2. *Estimating the nonlocal part of the s-perimeter.* The aim of this subsection consists in proving that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary and $E \subseteq \mathbb{R}^n$ has finite perimeter in Ω_β, for some $\beta \in (0, r_0)$ and r_0 as in Proposition B.1, then

$$\limsup_{s \to 1} (1-s) P_s^{NL}(E, \Omega) \leq 2\omega_{n-1} \lim_{\varrho \to 0^+} P(E, N_\varrho(\partial\Omega)). \tag{2.6}$$

Actually, we prove something slightly more general than (2.6). Namely, that to estimate the nonlocal part of the s-perimeter we do not necessarily need to use the sets Ω_ϱ: any "regular" approximation of Ω will do.

More precisely, let A_k, $D_k \subseteq \mathbb{R}^n$ be two sequences of bounded open sets with Lipschitz boundary strictly approximating Ω respectively from the inside and from the outside, that is

(i) $A_k \subseteq A_{k+1} \Subset \Omega$ and $A_k \nearrow \Omega$, i.e. $\bigcup_k A_k = \Omega$,
(ii) $\Omega \Subset D_{k+1} \subseteq D_k$ and $D_k \searrow \overline{\Omega}$, i.e. $\bigcap_k D_k = \overline{\Omega}$.

We define for every k

$$\Omega_k^+ := D_k \setminus \overline{\Omega}, \qquad \Omega_k^- := \Omega \setminus \overline{A_k} \qquad T_k := \Omega_k^+ \cup \partial\Omega \cup \Omega_k^-,$$
$$d_k := \min\{d(A_k, \partial\Omega),\, d(D_k, \partial\Omega)\} > 0.$$

In particular, we observe that we can consider Ω_ϱ with $\varrho < 0$ in place of A_k and with $\varrho > 0$ in place of D_k. Then T_k would be $N_\varrho(\partial\Omega)$ and $d_k = \varrho$.

Proposition 2.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $E \subseteq \mathbb{R}^n$ be a set having finite perimeter in D_1. Then*

$$\limsup_{s \to 1} (1-s) P_s^{NL}(E, \Omega) \leq 2\omega_{n-1} \lim_{k \to \infty} P(E, T_k).$$

In particular, if $P(E, \partial\Omega) = 0$, then

$$\lim_{s \to 1} (1-s) P_s(E, \Omega) = \omega_{n-1} P(E, \Omega).$$

Proof. Since Ω is regular and $P(E, \Omega) < \infty$, we already know that

$$\lim_{s \to 1} (1-s) P_s^L(E, \Omega) = \omega_{n-1} P(E, \Omega).$$

Notice that, since $|D\chi_E|$ is a finite Radon measure on D_1 and $T_k \searrow \partial\Omega$ as $k \nearrow \infty$, we have that

$$\exists \lim_{k \to \infty} P(E, T_k) = P(E, \partial\Omega).$$

Consider the nonlocal part of the fractional perimeter,

$$P_s^{NL}(E, \Omega) = \mathcal{L}_s(E \cap \Omega, \mathcal{C}E \setminus \Omega) + \mathcal{L}_s(\mathcal{C}E \cap \Omega, E \setminus \Omega),$$

and take any k. Then

$$\begin{aligned}\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\setminus\Omega)&=\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\cap\Omega_k^+)+\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\cap(\mathcal{C}\Omega\setminus D_k))\\&\leq\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\cap\Omega_k^+)+\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}\\&\leq\mathcal{L}_s(E\cap\Omega_k^-,\mathcal{C}E\cap\Omega_k^+)+2\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}\\&\leq\mathcal{L}_s(E\cap(\Omega_k^-\cup\Omega_k^+),\mathcal{C}E\cap(\Omega_k^-\cup\Omega_k^+))+2\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}\\&=P_s^L(E,T_k)+2\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}.\end{aligned}$$

Since we can bound the other term in the same way, we get

$$P_s^{NL}(E,\Omega)\leq 2P_s^L(E,T_k)+4\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}.$$

By hypothesis we know that T_k is a bounded open set with Lipschitz boundary

$$\partial T_k=\partial A_k\cup\partial D_k.$$

Therefore using (1.8) we have

$$\lim_{s\to 1}(1-s)P_s^L(E,T_k)=\omega_{n-1}P(E,T_k),$$

and hence

$$\limsup_{s\to 1}(1-s)P_s^{NL}(E,\Omega)\leq 2\omega_{n-1}P(E,T_k).$$

Since this holds true for any k, we get the claim. □

2.2.3. *Convergence in almost every Ω_ϱ.* Having a "continuous" approximating sequence (the Ω_ϱ) rather than numerable ones allows us to improve Proposition 2.5.

We first recall that if E has finite perimeter, then De Giorgi's structure Theorem (see, e.g., [17, Theorem 15.9]) guarantees in particular that

$$|D\chi_E|=\mathcal{H}^{n-1}\llcorner\partial^*E$$

and hence

$$P(E,B)=\mathcal{H}^{n-1}(\partial^*E\cap B)\qquad\text{for every Borel set } B\subseteq\mathbb{R}^n,$$

where ∂^*E is the reduced boundary of E.

Corollary 2.6. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let r_0 be as in Proposition B.1. Let $E\subseteq\mathbb{R}^n$ be a set having finite perimeter in Ω_β, for some $\beta\in(0,r_0)$, and define*

$$S:=\{\delta\in(-r_0,\beta)\,|\,P(E,\partial\Omega_\delta)>0\}.$$

Then the set S is at most countable. Moreover

$$\lim_{s\to 1}(1-s)P_s(E,\Omega_\delta)=\omega_{n-1}P(E,\Omega_\delta),\tag{2.7}$$

for every $\delta\in(-r_0,\beta)\setminus S$.

Proof. We observe that

$$P(E, \partial\Omega_\delta) = \mathcal{H}^{n-1}(\partial^* E \cap \{\bar{d}_\Omega = \delta\}),$$

for every $\delta \in (-r_0, \beta)$, and

$$M := \mathcal{H}^{n-1}(\partial^* E \cap (\Omega_\beta \setminus \overline{\Omega_{-r_0}})) \leq P(E, \Omega_\beta) < \infty. \tag{2.8}$$

Then we define the sets

$$S_k := \Big\{\delta \in (-r_0, \beta) \,|\, \mathcal{H}^{n-1}(\partial^* E \cap \{\bar{d}_\Omega = \delta\}) > \frac{1}{k}\Big\},$$

for every $k \in \mathbb{N}$ and we remark that

$$S = \bigcup_{k\in\mathbb{N}} S_k.$$

Since by (2.8) we have

$$\mathcal{H}^{n-1}\Big(\bigcup_{-r_0<\delta<\beta} (\partial^* E \cap \{\bar{d}_\Omega = \delta\})\Big) = M,$$

the number of elements in each S_k is at most

$$\sharp S_k \leq M\, k.$$

As a consequence the set S is at most countable, as claimed.

Finally, since Ω_δ is a bounded open set with Lipschitz boundary for every $\delta \in (-r_0, r_0)$ (see Proposition B.1), we obtain (2.7) by Proposition 2.5. □

2.2.4. *Conclusion.* We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. We begin by observing that if $E \subseteq \mathbb{R}^n$ and we have two open sets $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then

$$P_s(E, \mathcal{O}_1) \leq P_s(E, \mathcal{O}_2).$$

More precisely, we have

$$\begin{aligned} P_s(E, \mathcal{O}_2) = P_s(E, \mathcal{O}_1) &+ \mathcal{L}_s\big(E \cap (\mathcal{O}_2 \setminus \mathcal{O}_1), \mathcal{C}E \cap (\mathcal{O}_2 \setminus \mathcal{O}_1)\big) \\ &+ \mathcal{L}_s\big(E \cap (\mathcal{O}_2 \setminus \mathcal{O}_1), \mathcal{C}E \setminus \mathcal{O}_2\big) + \mathcal{L}_s\big(\mathcal{C}E \cap (\mathcal{O}_2 \setminus \mathcal{O}_1), E \setminus \mathcal{O}_2\big). \end{aligned} \tag{2.9}$$

Moreover, we also have

$$P_s^L(E, \mathcal{O}_1) \leq P_s(E, \mathcal{O}_2) \quad \text{and} \quad P(E, \mathcal{O}_1) \leq P(E, \mathcal{O}_2).$$

Now suppose that E has locally finite perimeter in Ω and let $\Omega' \Subset \Omega$. Notice that we can find a bounded open set $\mathcal{O}$ with Lipschitz boundary, such that

$$\Omega' \Subset \mathcal{O} \Subset \Omega.$$

Since E has finite perimeter in $\mathcal{O}$, by point (iv) of Corollary 2.2, we know that E has finite s-perimeter in $\mathcal{O}$ (and hence also in $\Omega' \Subset \mathcal{O}$) for every $s \in (0, 1)$. Moreover, by Theorem 1.4 we obtain

$$\liminf_{s\to 1}(1-s)P_s^L(E, \Omega') \leq \liminf_{s\to 1}(1-s)P_s^L(E, \mathcal{O}) < \infty.$$

The converse implication is proved similarly.

Now suppose that E has locally finite perimeter in Ω and let $\mathcal{O} \Subset \Omega$ have Lipschitz boundary. Let $r_0 = r_0(\mathcal{O}) > 0$ be as in Proposition B.1. Since $\mathcal{O} \Subset \Omega$, we can find

$\beta \in (0, r_0)$ small enough such that $\mathcal{O}_\beta \Subset \Omega$. Moreover, since E has locally finite perimeter in Ω, E has finite perimeter in $\mathcal{O}_\beta$.

Then, by Corollary 2.6, we can find $\delta \in (0, \beta)$ such that $P(E, \partial\mathcal{O}_\delta) = 0$ and we have

$$\lim_{s\to 1}(1-s)P_s(E, \mathcal{O}_\delta) = \omega_{n-1}P(E, \mathcal{O}_\delta). \tag{2.10}$$

We also remark that, since $|\partial\mathcal{O}| = 0$, we can rewrite (2.9) as

$$\begin{aligned} P_s(E, \mathcal{O}_\delta) &= P_s(E, \mathcal{O}) + P_s^L(E, \mathcal{O}_\delta \setminus \overline{\mathcal{O}}) \\ &\qquad + \mathcal{L}_s\big(E \cap (\mathcal{O}_\delta \setminus \overline{\mathcal{O}}), \mathcal{C}E \setminus \mathcal{O}_\delta\big) + \mathcal{L}_s\big(\mathcal{C}E \cap (\mathcal{O}_\delta \setminus \overline{\mathcal{O}}), E \setminus \mathcal{O}_\delta\big). \end{aligned} \tag{2.11}$$

Let

$$I_s := \mathcal{L}_s\big(E \cap (\mathcal{O}_\delta \setminus \overline{\mathcal{O}}), \mathcal{C}E \setminus \mathcal{O}_\delta\big) + \mathcal{L}_s\big(\mathcal{C}E \cap (\mathcal{O}_\delta \setminus \overline{\mathcal{O}}), E \setminus \mathcal{O}_\delta\big)$$

and notice that

$$I_s \le P_s^{NL}(E, \mathcal{O}_\delta). \tag{2.12}$$

Hence, since $P(E, \partial\mathcal{O}_\delta) = 0$, by (2.12) and Proposition 2.5 we obtain

$$\lim_{s\to 1}(1-s)I_s = 0. \tag{2.13}$$

Furthermore, since E has finite perimeter in $\mathcal{O}_\delta \setminus \overline{\mathcal{O}}$, which is a bounded open set with Lipschitz boundary, by (1.8) of Theorem 1.4, we find

$$\lim_{s\to 1}(1-s)P_s^L(E, \mathcal{O}_\delta \setminus \overline{\mathcal{O}}) = \omega_{n-1}P(E, \mathcal{O}_\delta \setminus \overline{\mathcal{O}}). \tag{2.14}$$

Therefore, by (2.11), (2.10), (2.13) and (2.14), and exploiting the fact that $P(E, \cdot\,)$ is a measure, we get

$$\begin{aligned} \lim_{s\to 1}(1-s)P(E, \mathcal{O}) &= \omega_{n-1}\big(P(E, \mathcal{O}_\delta) - P(E, \mathcal{O}_\delta \setminus \overline{\mathcal{O}})\big) \\ &= \omega_{n-1}P(E, \overline{\mathcal{O}}). \end{aligned} \tag{2.15}$$

Finally, since by (1.8) we know that

$$\lim_{s\to 1}(1-s)P_s^L(E, \mathcal{O}) = \omega_{n-1}P(E, \mathcal{O}), \tag{2.16}$$

by (2.15) and (2.16) we obtain

$$\lim_{s\to 1}(1-s)P_s^{NL}(E, \mathcal{O}) = \omega_{n-1}P(E, \partial\mathcal{O}),$$

concluding the proof of the Theorem. □

3. Irregularity of the boundary

3.1. The measure theoretic boundary as "support" of the local part of the s-perimeter.

First of all we show that the (local part of the) s-perimeter does indeed measure a quantity related to the measure theoretic boundary.

Lemma 3.1. *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite s-perimeter. Then*

$$\partial^- E = \{x \in \mathbb{R}^n \mid P_s^L(E, B_r(x)) > 0 \text{ for every } r > 0\}.$$

Proof. The claim follows from the following observation. Let A, $B \subseteq \mathbb{R}^n$ such that $A \cap B = \emptyset$; then

$$\mathcal{L}_s(A,B) = 0 \quad \Longleftrightarrow \quad |A| = 0 \quad \text{or} \quad |B| = 0.$$

Therefore

$$\begin{aligned} x \in \partial^- E \quad &\Longleftrightarrow \quad |E \cap B_r(x)| > 0 \text{ and } |\mathcal{C}E \cap B_r(x)| > 0 \quad \forall\, r > 0 \\ &\Longleftrightarrow \quad \mathcal{L}_s(E \cap B_r(x), \mathcal{C}E \cap B_r(x)) > 0 \quad \forall\, r > 0, \end{aligned}$$

concluding the proof $\square$

This characterization of $\partial^- E$ can be thought of as a fractional analogue of (C.7). However we can not really think of $\partial^- E$ as the support of

$$P_s^L(E,\,\cdot\,) : \Omega \longmapsto P_s^L(E,\Omega),$$

in the sense that, in general

$$\partial^- E \cap \Omega = \emptyset \quad \not\Rightarrow \quad P_s^L(E,\Omega) = 0.$$

For example, consider $E := \{x_n \leq 0\} \subseteq \mathbb{R}^n$ and notice that $\partial^- E = \{x_n = 0\}$. Let $\Omega := B_1(2e_n) \cup B_1(-2e_n)$. Then $\partial^- E \cap \Omega = \emptyset$, but

$$P_s^L(E,\Omega) = \mathcal{L}_s(B_1(2e_n), B_1(-2e_n)) > 0.$$

On the other hand, the only obstacle is the non connectedness of the set Ω and indeed we obtain the following

Proposition 3.2. *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite s-perimeter and let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then*

$$\partial^- E \cap \Omega \neq \emptyset \quad \Longrightarrow \quad P_s^L(E,\Omega) > 0.$$

Moreover, if Ω is connected

$$\partial^- E \cap \Omega = \emptyset \quad \Longrightarrow \quad P_s^L(E,\Omega) = 0.$$

Therefore, if $\widehat{\mathcal{O}}(\mathbb{R}^n)$ denotes the family of bounded and connected open sets, then $\partial^- E$ can be considered as the "support" of

$$\begin{aligned} P_s^L(E,\,\cdot\,) : \widehat{\mathcal{O}}(\mathbb{R}^n) &\longrightarrow [0,\infty) \\ \Omega &\longmapsto P_s^L(E,\Omega), \end{aligned}$$

in the sense that, if $\Omega \in \widehat{\mathcal{O}}(\mathbb{R}^n)$, then

$$P_s^L(E,\Omega) > 0 \quad \Longleftrightarrow \quad \partial^- E \cap \Omega \neq \emptyset.$$

Proof. Let $x \in \partial^- E \cap \Omega$. Since Ω is open, we have $B_r(x) \subseteq \Omega$ for some $r > 0$ and hence

$$P_s^L(E,\Omega) \geq P_s^L(E, B_r(x)) > 0.$$

Let Ω be connected and suppose $\partial^- E \cap \Omega = \emptyset$. Notice that we have the partition of $\mathbb{R}^n$ as $\mathbb{R}^n = E_{ext} \cup \partial^- E \cup E_{int}$ (see Appendix C). Thus we can write Ω as the disjoint union

$$\Omega = (E_{ext} \cap \Omega) \cup (E_{int} \cap \Omega).$$

However, since Ω is connected and both E_{ext} and E_{int} are open, we must have $E_{ext} \cap \Omega = \emptyset$ or $E_{int} \cap \Omega = \emptyset$. Now, if $E_{ext} \cap \Omega = \emptyset$ (the other case is analogous), then $\Omega \subseteq E_{int}$ and hence $|\mathcal{C}E \cap \Omega| = 0$. Thus

$$P_s^L(E,\Omega) = \mathcal{L}_s(E \cap \Omega, \mathcal{C}E \cap \Omega) = 0,$$

concluding the proof. $\square$

3.2. **A notion of fractal dimension.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then

$$t > s \qquad \Longrightarrow \qquad W^{t,1}(\Omega) \hookrightarrow W^{s,1}(\Omega),$$

(see, e.g., [8, Proposition 2.1]). As a consequence, for every $u \in L^1(\Omega)$ there exists a unique $R(u) \in [0,1]$ such that

$$[u]_{W^{s,1}(\Omega)} \quad \begin{cases} < \infty, & \forall\, s \in (0, R(u)) \\ = \infty, & \forall\, s \in (R(u), 1) \end{cases}$$

that is

$$\begin{aligned} R(u) &= \sup \left\{ s \in (0,1) \,\middle|\, [u]_{W^{s,1}(\Omega)} < \infty \right\} \\ &= \inf \left\{ s \in (0,1) \,\middle|\, [u]_{W^{s,1}(\Omega)} = \infty \right\}. \end{aligned} \tag{3.1}$$

In particular, exploiting this result for characteristic functions, in [22] the author suggested the following definition of fractal dimension.

Definition 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E \subseteq \mathbb{R}^n$ such that $|E \cap \Omega| < \infty$. If $\partial^- E \cap \Omega \neq \emptyset$, we define*

$$Dim_F(\partial^- E, \Omega) := n - R(\chi_E),$$

the fractal dimension of $\partial^- E$ in Ω, relative to the fractional perimeter. If $\Omega = \mathbb{R}^n$, we drop it in the formulas.

Notice that in the case of sets (3.1) becomes

$$\begin{aligned} R(\chi_E) &= \sup \left\{ s \in (0,1) \,\middle|\, P_s^L(E, \Omega) < \infty \right\} \\ &= \inf \left\{ s \in (0,1) \,\middle|\, P_s^L(E, \Omega) = \infty \right\}. \end{aligned} \tag{3.2}$$

We observe that, since $P_s^L(\mathcal{C}E, \Omega) = P_s^L(E, \Omega)$, in order to define the fractal dimension of $\partial^- E$ in Ω, it is actually enough to require that either $|E \cap \Omega| < \infty$ or $|\mathcal{C}E \cap \Omega| < \infty$. Clearly, when the open set Ω is bounded, such assumptions are trivially satisfied.

In particular we can consider Ω to be the whole of $\mathbb{R}^n$, or a bounded open set with Lipschitz boundary. In the first case the local part of the fractional perimeter coincides with the whole fractional perimeter, while in the second case we know that we can bound the nonlocal part with $2P_s(\Omega) < \infty$ for every $s \in (0,1)$. Therefore, in both cases in (3.2) we can as well take the whole fractional perimeter $P_s(E, \Omega)$ instead of just the local part.

Now we recall the definition of Minkowski dimension, given in terms of the Minkowski contents. For equivalent definitions of the Minkowski dimension and for the main properties, we refer to [18] and [13] and the references cited therein.

For simplicity, given $\Gamma \subseteq \mathbb{R}^n$ we set

$$\bar{N}_\varrho^\Omega(\Gamma) := \overline{N_\varrho(\Gamma)} \cap \Omega = \{x \in \Omega \,|\, d(x, \Gamma) \le \varrho\},$$

for any $\varrho > 0$.

Definition 3.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For any $\Gamma \subseteq \mathbb{R}^n$ and $r \in [0, n]$ we define the inferior and superior r-dimensional Minkowski contents of Γ relative to the set Ω as, respectively*

$$\underline{\mathcal{M}}^r(\Gamma, \Omega) := \liminf_{\varrho \to 0} \frac{|\bar{N}_\varrho^\Omega(\Gamma)|}{\varrho^{n-r}}, \qquad \overline{\mathcal{M}}^r(\Gamma, \Omega) := \limsup_{\varrho \to 0} \frac{|\bar{N}_\varrho^\Omega(\Gamma)|}{\varrho^{n-r}}.$$

Then we define the lower and upper Minkowski dimensions of Γ *in* Ω *as*

$$\begin{aligned}\underline{Dim}_{\mathcal{M}}(\Gamma,\Omega) &:= \inf\big\{r\in[0,n]\,|\,\underline{\mathcal{M}}^r(\Gamma,\Omega)=0\big\}\\ &= n-\sup\big\{r\in[0,n]\,|\,\underline{\mathcal{M}}^{n-r}(\Gamma,\Omega)=0\big\},\\ \overline{Dim}_{\mathcal{M}}(\Gamma,\Omega) &:= \sup\big\{r\in[0,n]\,|\,\overline{\mathcal{M}}^r(\Gamma,\Omega)=\infty\big\}\\ &= n-\inf\big\{r\in[0,n]\,|\,\overline{\mathcal{M}}^{n-r}(\Gamma,\Omega)=\infty\big\}.\end{aligned}$$

If they agree, we write

$$Dim_{\mathcal{M}}(\Gamma,\Omega)$$

for the common value and call it the Minkowski dimension of Γ *in* Ω*. If* $\Omega=\mathbb{R}^n$ *or* $\Gamma\Subset\Omega$*, we drop the* Ω *in the formulas.*

Remark 3.5. Let $\mathrm{Dim}_{\mathcal{H}}$ denote the Hausdorff dimension. In general one has

$$\mathrm{Dim}_{\mathcal{H}}(\Gamma)\le\underline{\mathrm{Dim}}_{\mathcal{M}}(\Gamma)\le\overline{\mathrm{Dim}}_{\mathcal{M}}(\Gamma),$$

and all the inequalities might be strict (for some examples, see, e.g., [18, Section 5.3]). However for some sets, like self-similar sets which satisfy appropriate symmetric and regularity conditions, they are all equal (see, e.g., [18, Corollary 5.8]).

Now we give a proof of the relation (1.2) (obtained in [22]). For related results, see also [21] and [14].

Proposition 3.6. *Let* $\Omega\subseteq\mathbb{R}^n$ *be a bounded open set. Then for every* $E\subseteq\mathbb{R}^n$ *such that* $\partial^-E\cap\Omega\neq\emptyset$ *and* $\overline{Dim}_{\mathcal{M}}(\partial^-E,\Omega)\ge n-1$ *we have*

$$Dim_F(\partial^-E,\Omega)\le\overline{Dim}_{\mathcal{M}}(\partial^-E,\Omega).$$

Proof. By hypothesis we have

$$\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^-E,\Omega)=n-\inf\big\{r\in(0,1)\,|\,\overline{\mathcal{M}}^{n-r}(\partial^-E,\Omega)=\infty\big\},$$

and we need to show that

$$\inf\big\{r\in(0,1)\,|\,\overline{\mathcal{M}}^{n-r}(\partial^-E,\Omega)=\infty\big\}\le\sup\{s\in(0,1)\,|\,P^L_s(E,\Omega)<\infty\}.$$

Up to modifying E on a set of Lebesgue measure zero we can suppose that $\partial E=\partial^-E$, as in Remark 1.8. Notice that this does not affect the s-perimeter.

Now for any $s\in(0,1)$

$$\begin{aligned}2P^L_s(E,\Omega)&=\int_\Omega dx\int_\Omega\frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dy\\ &=\int_\Omega dx\int_0^\infty d\varrho\int_{\partial B_\varrho(x)\cap\Omega}\frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}(y)\\ &=\int_\Omega dx\int_0^\infty\frac{d\varrho}{\varrho^{n+s}}\int_{\partial B_\varrho(x)\cap\Omega}|\chi_E(x)-\chi_E(y)|\,d\mathcal{H}^{n-1}(y).\end{aligned}$$

Notice that

$$d(x,\partial E)>\varrho\quad\Longrightarrow\quad\chi_E(y)=\chi_E(x),\quad\forall\,y\in\overline{B_\varrho(x)},$$

and hence

$$\begin{aligned}\int_{\partial B_\varrho(x)\cap\Omega}|\chi_E(x)-\chi_E(y)|\,d\mathcal{H}^{n-1}(y)&\le\int_{\partial B_\varrho(x)\cap\Omega}\chi_{\bar N_\varrho(\partial E)}(x)\,d\mathcal{H}^{n-1}(y)\\ &\le n\omega_n\varrho^{n-1}\chi_{\bar N_\varrho(\partial E)}(x).\end{aligned}$$

Therefore

$$2P_s^L(E,\Omega)\le n\omega_n\int_0^\infty\frac{d\varrho}{\varrho^{1+s}}\int_\Omega\chi_{\bar N_\varrho(\partial E)}(x)=n\omega_n\int_0^\infty\frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^{1+s}}\,d\varrho.$$

We claim that

$$\overline{\mathcal{M}}^{n-r}(\partial E,\Omega)<\infty\quad\Longrightarrow\quad P_s^L(E,\Omega)<\infty,\quad\forall\, s\in(0,r).\tag{3.3}$$

Indeed

$$\limsup_{\varrho\to0}\frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^r}<\infty\quad\Longrightarrow\quad\exists\, C>0\text{ s.t. }\sup_{\varrho\in(0,C]}\frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^r}\le M<\infty.$$

Hence

$$\begin{aligned}2P_s^L(E,\Omega)&\le n\omega_n\Big\{\int_0^C\frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^{1-(r-s)+r}}\,d\varrho+\int_C^\infty\frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^{1+s}}\,d\varrho\Big\}\\&\le n\omega_n\Big\{M\int_0^C\frac{1}{\varrho^{1-(r-s)}}\,d\varrho+|\Omega|\int_C^\infty\frac{1}{\varrho^{1+s}}\,d\varrho\Big\}\\&=n\omega_n\Big\{\frac{M}{r-s}C^{r-s}+\frac{|\Omega|}{sC^s}\Big\}<\infty,\end{aligned}$$

proving (3.3). This implies that

$$r\le\sup\{s\in(0,1)\,|\,P_s^L(E,\Omega)<\infty\},$$

for every $r\in(0,1)$ such that $\overline{\mathcal{M}}^{n-r}(\partial E,\Omega)<\infty$.

Thus, for $\varepsilon>0$ very small, we have

$$\inf\big\{r\in(0,1)\,|\,\overline{\mathcal{M}}^{n-r}(\partial^-E,\Omega)=\infty\big\}-\varepsilon\le\sup\{s\in(0,1)\,|\,P_s^L(E,\Omega)<\infty\}.$$

Letting ε tend to zero, we conclude the proof. □

In particular, if Ω has Lipschitz boundary we obtain:

Corollary 3.7. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $E\subseteq\mathbb{R}^n$ such that $\partial^-E\cap\Omega\neq\emptyset$ and $\overline{Dim}_{\mathcal{M}}(\partial^-E,\Omega)\in[n-1,n)$. Then*

$$P_s(E,\Omega)<\infty\qquad\textit{for every } s\in\big(0,n-\overline{Dim}_{\mathcal{M}}(\partial^-E,\Omega)\big).$$

Remark 3.8. Actually, Proposition 3.6 and Corollary 3.7 still remain true when $\Omega=\mathbb{R}^n$, provided the set E we are considering is bounded. Indeed, if E is bounded, we can apply the previous results with $\Omega=B_R$ such that $E\Subset\Omega$. Moreover, since Ω has a regular boundary, as remarked above we can take the whole s-perimeter in (3.2), instead of just the local part. But then, since $P_s(E,\Omega)=P_s(E)$, we see that

$$\mathrm{Dim}_F(\partial^-E,\Omega)=\mathrm{Dim}_F(\partial^-E,\mathbb{R}^n).$$

3.2.1. *Remarks about the Minkowski content of ∂^-E.* In the beginning of the proof of Proposition 3.6 we chose a particular representative for the class of E in order to have $\partial E=\partial^-E$. This can be done since it does not affect the s-perimeter and we are already considering the Minkowski dimension of ∂^-E.

On the other hand, if we consider a set F such that $|E\Delta F|=0$, we can use the same proof to obtain the inequality

$$\mathrm{Dim}_F(\partial^-E,\Omega)\le\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial F,\Omega).$$

It is then natural to ask whether we can find a "better" representative F, whose (topological) boundary ∂F has Minkowski dimension strictly smaller than that of $\partial^- E$.

First of all, we remark that the Minkowski content can be influenced by changes in sets of measure zero. Roughly speaking, this is because the Minkowski content is not a purely measure theoretic notion, but rather a combination of metric and measure.

For example, let $\Gamma \subseteq \mathbb{R}^n$ and define $\Gamma' := \Gamma \cup \mathbb{Q}^n$. Then $|\Gamma \Delta \Gamma'| = 0$, but $N_\delta(\Gamma') = \mathbb{R}^n$ for every $\delta > 0$.

In particular, considering different representatives for E we will get different topological boundaries and hence different Minkowski dimensions.

However, since the measure theoretic boundary minimizes the size of the topological boundary, that is

$$\partial^- E = \bigcap_{|F\Delta E|=0} \partial F,$$

(see Appendix C), it minimizes also the Minkowski dimension.
Indeed, for every F such that $|F\Delta E| = 0$ we have

$$\begin{aligned} \partial^- E \subseteq \partial F \quad &\Longrightarrow \quad \bar{N}^\Omega_\varrho(\partial^- E) \subseteq \bar{N}^\Omega_\varrho(\partial F) \\ &\Longrightarrow \quad \overline{\mathcal{M}}^r(\partial^- E, \Omega) \leq \overline{\mathcal{M}}^r(\partial F, \Omega) \\ &\Longrightarrow \quad \overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E, \Omega) \leq \overline{\mathrm{Dim}}_{\mathcal{M}}(\partial F, \Omega). \end{aligned}$$

3.3. **Fractal dimension of the von Koch snowflake.** The von Koch snowflake $S \subseteq \mathbb{R}^2$ is an example of a bounded open set with fractal boundary, for which the Minkowski dimension and the fractal dimension introduced above coincide.

Moreover its boundary is "nowhere rectifiable", in the sense that $\partial S \cap B_r(p)$ is not $(n-1)$-rectifiable for any $r > 0$ and $p \in \partial S$.

First of all we recall how to construct the von Koch curve. Then the snowflake is made of three von Koch curves.

Let Γ_0 be a line segment of unit length. The set Γ_1 consists of the four segments obtained by removing the middle third of Γ_0 and replacing it by the other two sides of the equilateral triangle based on the removed segment.
We construct Γ_2 by applying the same procedure to each of the segments in Γ_1 and so on. Thus Γ_k comes from replacing the middle third of each straight line segment of Γ_{k-1} by the other two sides of an equilateral triangle.

As k tends to infinity, the sequence of polygonal curves Γ_k approaches a limiting curve Γ, called the von Koch curve.
If we start with an equilateral triangle with unit length side and perform the same construction on all three sides, we obtain the von Koch snowflake Σ (see Figure 2). Let S be the bounded region enclosed by Σ, so that S is open and $\partial S = \Sigma$. We still call S the von Koch snowflake.

It can be shown (see, e.g., [13]) that the Hausdorff dimension of the von Koch snowflake is equal to its Minkowski dimension and

$$\mathrm{Dim}_{\mathcal{H}}(\Sigma) = \mathrm{Dim}_{\mathcal{M}}(\Sigma) = \frac{\log 4}{\log 3}$$

Now we explain how to construct S in a recursive way and we observe that

$$\partial^- S = \partial S = \Sigma.$$

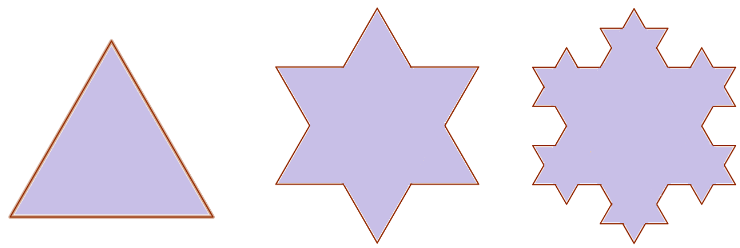

FIGURE 2. *The first three steps of the construction of the von Koch snowflake*

As starting point for the snowflake take the equilateral triangle T of side 1, with barycenter in the origin and a vertex on the y-axis, $P = (0, t)$ with $t > 0$.
Then T_1 is made of three triangles of side $1/3$, T_2 of $3 \cdot 4$ triangles of side $1/3^2$ and so on. In general T_k is made of $3 \cdot 4^{k-1}$ triangles of side $1/3^k$, call them $T_k^1, \dots, T_k^{3\cdot 4^{k-1}}$. Let x_k^i be the baricenter of T_k^i and P_k^i the vertex which does not touch T_{k-1}.

Then $S = T \cup \bigcup T_k$. Also notice that T_k and T_{k-1} touch only on a set of measure zero.

For each triangle T_k^i there exists a rotation $\mathcal{R}_k^i \in SO(n)$ such that

$$T_k^i = F_k^i(T) := \mathcal{R}_k^i\Big(\frac{1}{3^k}T\Big) + x_k^i.$$

We choose the rotations so that $F_k^i(P) = P_k^i$.

Notice that for each triangle T_k^i we can find a small ball which is contained in the complementary of the snowflake, $B_k^i \subseteq \mathcal{C}S$, and touches the triangle in the vertex P_k^i. Actually these balls can be obtained as the images of the affine transformations F_k^i of a fixed ball B.

To be more precise, fix a small ball contained in the complementary of T, which has the center on the y-axis and touches T in the vertex P, say $B := B_{1/1000}(0, t + 1/1000)$. Then

$$B_k^i := F_k^i(B) \subseteq \mathcal{C}S \tag{3.4}$$

for every i, k. To see this, imagine constructing the snowflake S using the same affine transformations F_k^i but starting with $T \cup B$ in place of T.

We know that $\partial^- S \subseteq \partial S$ (see Appendix C).
On the other hand, let $p \in \partial S$. Then every ball $B_\delta(p)$ contains at least a triangle $T_k^i \subseteq S$ and its corresponding ball $B_k^i \subseteq \mathcal{C}S$ (and actually infinitely many). Therefore

$$0 < |B_\delta(p) \cap S| < \omega_n \delta^n$$

for every $\delta > 0$ and hence $p \in \partial^- S$.

Proof of Theorem 1.1. Since S is bounded, its boundary is $\partial^- S = \Sigma$, and $\mathrm{Dim}_{\mathcal{M}}(\Sigma) = \frac{\log 4}{\log 3}$, we obtain (1.3) from Corollary 3.7 and Remark 3.8.

Exploiting the construction of S given above and (3.4) we prove (1.4).
We have

$$
\begin{aligned}
P_s(S) = \mathcal{L}_s(S, \mathcal{C}S) &= \mathcal{L}_s(T, \mathcal{C}S) + \sum_{k=1}^{\infty} \mathcal{L}_s(T_k, \mathcal{C}S) \\
&= \mathcal{L}_s(T, \mathcal{C}S) + \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \mathcal{L}_s(T_k^i, \mathcal{C}S) \geq \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \mathcal{L}_s(T_k^i, \mathcal{C}S) \\
&\geq \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \mathcal{L}_s(T_k^i, B_k^i) \qquad \text{(by (3.4))} \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \mathcal{L}_s(F_k^i(T), F_k^i(B)) \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \left(\frac{1}{3^k}\right)^{2-s} \mathcal{L}_s(T, B) \qquad \text{(by Proposition 3.12)} \\
&= \frac{3}{3^{2-s}} \mathcal{L}_s(T, B) \sum_{k=0}^{\infty} \left(\frac{4}{3^{2-s}}\right)^k.
\end{aligned}
$$

We remark that

$$
\mathcal{L}_s(T, B) \leq \mathcal{L}_s(T, \mathcal{C}T) = P_s(T) < \infty,
$$

for every $s \in (0, 1)$.

To conclude, notice that the last series is divergent if $s \geq 2 - \frac{\log 4}{\log 3}$. □

Exploiting the self-similarity of the von Koch curve, we show that the fractal dimension of S is the same in every open set which contains a point of ∂S.

Corollary 3.9. *Let $S \subseteq \mathbb{R}^2$ be the von Koch snowflake. Then*

$$
Dim_F(\partial S, \Omega) = \frac{\log 4}{\log 3}
$$

for every open set Ω such that $\partial S \cap \Omega \neq \emptyset$.

Proof. Since $P_s(S, \Omega) \leq P_s(S)$, we have

$$
P_s(S, \Omega) < \infty, \qquad \forall\, s \in \left(0, 2 - \frac{\log 4}{\log 3}\right).
$$

On the other hand, if $p \in \partial S \cap \Omega$, then $B_r(p) \subseteq \Omega$ for some $r > 0$. Now notice that $B_r(p)$ contains a rescaled version of the von Koch curve, including all the triangles T_k^i which constitute it and the relative balls B_k^i. We can thus repeat the argument above to obtain

$$
P_s(S, \Omega) \geq P_s(S, B_r(p)) = \infty, \qquad \forall\, s \in \left[2 - \frac{\log 4}{\log 3}, 1\right),
$$

concluding the proof. □

3.4. **Self-similar fractal boundaries.**

Proof of Theorem 1.2. Arguing as we did with the von Koch snowflake, we show that $P_s(T)$ is bounded both from above and from below by the series

$$\sum_{k=0}^{\infty}\Big(\frac{b}{\lambda^{n-s}}\Big)^k,$$

which converges if and only if $s < n - \frac{\log b}{\log \lambda}$.

Indeed

$$\begin{aligned}
P_s(T) &= \mathcal{L}_s(T, \mathcal{C}T) = \sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}} \mathcal{L}_s(T_k^i, \mathcal{C}T) \\
&\le \sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}} \mathcal{L}_s(T_k^i, \mathcal{C}T_k^i) = \sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}} \mathcal{L}_s(F_k^i(T_0), F_k^i(\mathcal{C}T_0)) \\
&= \frac{a}{\lambda^{n-s}}\mathcal{L}_s(T_0, \mathcal{C}T_0)\sum_{k=0}^{\infty}\Big(\frac{b}{\lambda^{n-s}}\Big)^k,
\end{aligned}$$

and

$$\begin{aligned}
P_s(T) &= \mathcal{L}_s(T, \mathcal{C}T) = \sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}} \mathcal{L}_s(T_k^i, \mathcal{C}T) \\
&\ge \sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}} \mathcal{L}_s(T_k^i, S_k^i) = \sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}} \mathcal{L}_s(F_k^i(T_0), F_k^i(S_0)) \\
&= \frac{a}{\lambda^{n-s}}\mathcal{L}_s(T_0, S_0)\sum_{k=0}^{\infty}\Big(\frac{b}{\lambda^{n-s}}\Big)^k.
\end{aligned}$$

Also notice that, since $P(T_0) < \infty$, we have

$$\mathcal{L}_s(T_0, S_0) \le \mathcal{L}_s(T_0, \mathcal{C}T_0) = P_s(T_0) < \infty,$$

for every $s \in (0,1)$. $\square$

Now suppose that T does not satisfy (1.6). Then we can obtain a set T' which does, simply by removing a portion S_0 from the building block T_0.
To be more precise, let $S_0 \subseteq T_0$ be such that

$$|S_0| > 0, \quad |T_0 \setminus S_0| > 0 \quad \text{and} \quad P(T_0 \setminus S_0) < \infty.$$

Then define a new building block $T_0' := T_0 \setminus S_0$ and the set

$$T' := \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{ab^{k-1}} F_k^i(T_0').$$

This new set has exactly the same structure of T, since we are using the same collection $\{F_k^i\}$ of affine maps.

Notice that

$$S_0 \subseteq T_0 \quad \Longrightarrow \quad F_k^i(S_0) \subseteq F_k^i(T_0),$$

and

$$F_k^i(T_0') = F_k^i(T_0) \setminus F_k^i(S_0),$$

for every k, i. Thus

$$T' = T \setminus \Big(\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{ab^{k-1}} F_k^i(S_0) \Big)$$

satisfies (1.6).

Remark 3.10. Roughly speaking, what matters in order to obtain a set which satisfies the hypothesis of Theorem 1.2 is that there exists a bounded open set T_0 such that

$$|F_k^i(T_0) \cap F_h^j(T_0)| = 0, \qquad \text{if } i \neq j \text{ or } k \neq h.$$

This can be thought of as a compatibility criterion for the family of affine maps $\{F_k^i\}$. We also need to ask that the ratio of the logarithms of the growth factor and the scaling factor is $\frac{\log b}{\log \lambda} \in (n-1, n)$. Then we are free to choose as building block any set $T_0' \subseteq T_0$ such that

$$|T_0'| > 0, \qquad |T_0 \setminus T_0'| > 0 \qquad \text{and } P(T_0') < \infty,$$

and the set

$$T' := \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{ab^{k-1}} F_k^i(T_0').$$

satisfies the hypothesis of Theorem 1.2.

Therefore, even if the Sierpinski triangle and the Menger sponge do not satisfy (1.6), we can exploit their structure to construct new sets which do.

However, we remark that the new boundary $\partial^- T'$ will look very different from the original fractal. Actually, in general it will be a mix of unrectifiable pieces and smooth pieces. In particular, we can not hope to get an analogue of Corollary 3.9. Still, the following Remark shows that the new (measure theoretic) boundary retains at least some of the "fractal nature" of the original set.

Remark 3.11. If the set T of Theorem 1.2 is bounded, exploiting Proposition 3.6 and Remark 3.8 we obtain

$$\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- T) \geq \frac{\log b}{\log \lambda} > n - 1.$$

Moreover, notice that if Ω is a bounded open set with Lipschitz boundary, then

$$P(E, \Omega) < \infty \quad \Longrightarrow \quad \mathrm{Dim}_F(E, \Omega) = n - 1.$$

Therefore, if $T \Subset B_R$, then

$$P(T) = P(T, B_R) = \infty,$$

even if T is bounded (and hence $\partial^- T$ is compact).

3.4.1. *Sponge-like sets.* The simplest way to construct the set T' consists in simply removing a small ball $S_0 := B \Subset T_0$ from T_0.

In particular, suppose that $|T_0 \Delta T| = 0$, as with the Sierpinski triangle.
Define

$$S := \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{ab^{k-1}} F_k^i(B) \quad \text{and} \quad T' := \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{ab^{k-1}} F_k^i(T_0 \setminus B) = T \setminus S.$$

Then

$$|T_0 \Delta T| = 0 \quad \Longrightarrow \quad |T' \Delta (T_0 \setminus S)| = 0. \tag{3.5}$$

Now the set $E := T_0 \setminus S$ looks like a sponge, in the sense that it is a bounded open set with an infinite number of holes (each one at a positive, but non-fixed distance from the others).

From (3.5) we get $P_s(E) = P_s(T')$. Thus, since T' satisfies the hypothesis of Theorem 1.2, we obtain

$$\mathrm{Dim}_F(\partial^- E) = \frac{\log b}{\log \lambda}.$$

3.4.2. *Dendrite-like sets.* Depending on the form of the set T_0 and on the affine maps $\{F_k^i\}$, we can define more intricate sets T'.

As an example we consider the Sierpinski triangle $E \subseteq \mathbb{R}^2$.
It is of the form $E = T_0 \setminus T$, where the building block T_0 is an equilateral triangle, say with side length one, a vertex on the y-axis and baricenter in 0. The pieces T_k^i are obtained with a scaling factor $\lambda = 2$ and the growth factor is $b = 3$ (see, e.g., [13] for the construction). As usual, we consider the set

$$T = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{3^{k-1}} T_k^i.$$

However, as remarked above, we have $|T \Delta T_0| = 0$.

Starting from $k = 2$ each triangle T_k^i touches with (at least) a vertex (at least) another triangle T_h^j. Moreover, each triangle T_k^i gets touched in the middle point of each side (and actually it gets touched in infinitely many points).

Exploiting this situation, we can remove from T_0 six smaller triangles, so that the new building block T_0' is a star polygon centered in 0, with six vertices, one in each vertex of T_0 and one in each middle point of the sides of T_0.

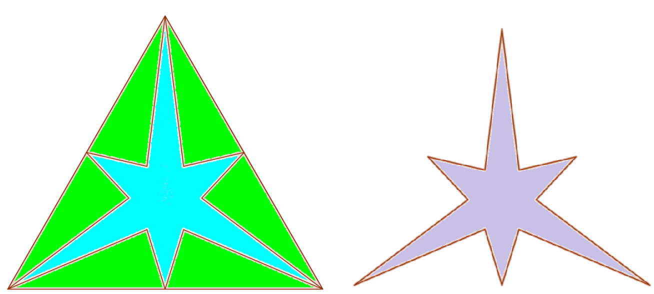

FIGURE 3. *Removing the six triangles (in green) to obtain the new "building block" T_0' (on the right)*

The resulting set

$$T' = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{3^{k-1}} F_k^i(T'_0)$$

will have an infinite number of ramifications.

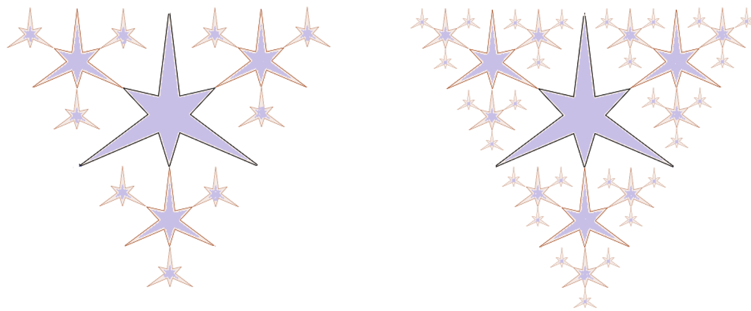

FIGURE 4. *The third and fourth steps of the iterative construction of the set T'*

Since T' satisfies the hypothesis of Theorem 1.2, we obtain

$$\mathrm{Dim}_F(\partial^- T') = \frac{\log 3}{\log 2}.$$

3.4.3. *"Exploded" fractals.* In all the previous examples, the sets T_k^i are accumulated in a bounded region.

On the other hand, imagine making a fractal like the von Koch snowflake or the Sierpinski triangle "explode" and then rearrange the pieces T_k^i in such a way that $d(T_k^i, T_h^j) \geq d$, for some fixed $d > 0$.

Since the shape of the building block is not important, we can consider $T_0 := B_{1/4}(0) \subseteq \mathbb{R}^n$, with $n \geq 2$. Moreover, since the parameter a does not influence the dimension, we can fix $a = 1$.

Then we rearrange the pieces obtaining

$$E := \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{b^{k-1}} B_{\frac{1}{4\lambda^k}}(k, 0, \ldots, 0, i). \tag{3.6}$$

Define for simplicity

$$B_k^i := B_{\frac{1}{4\lambda^k}}(k, 0, \ldots, 0, i) \quad \text{and} \quad x_k^i := k\, e_1 + i\, e_n,$$

and notice that

$$B_k^i = \lambda^{-k} B_{\frac{1}{4}}(0) + x_k^i.$$

Since for every k, h and every $i \neq j$ we have

$$d(B_k^i, B_h^j) \geq \frac{1}{2},$$

the boundary of the set E is the disjoint union of $(n-1)$-dimensional spheres

$$\partial^- E = \partial E = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{b^{k-1}} \partial B_k^i,$$

and in particular is smooth.

The (global) perimeter of E is

$$P(E) = \sum_{k=1}^{\infty} \sum_{i=1}^{b^{k-1}} P(B_k^i) = \frac{1}{\lambda} P(B_{1/4}(0)) \sum_{k=0}^{\infty} \Big(\frac{b}{\lambda^{n-1}}\Big)^k = +\infty,$$

since $\frac{\log b}{\log \lambda} > n-1$.

However E has locally finite perimeter, since its boundary is smooth and every ball B_R intersects only finitely many B_k^i's,

$$P(E, B_R) < \infty, \qquad \forall\, R > 0.$$

Therefore it also has locally finite s-perimeter for every $s \in (0,1)$

$$P_s(E, B_R) < \infty, \qquad \forall\, R > 0, \qquad \forall\, s \in (0,1).$$

What is interesting is that the set E satisfies the hypothesis of Theorem 1.2 and hence it also has finite global s-perimeter for every $s < \sigma_0 := n - \frac{\log b}{\log \lambda}$,

$$P_s(E) < \infty \qquad \forall\, s \in (0, \sigma_0) \quad \text{and} \quad P_s(E) = \infty \qquad \forall\, s \in [\sigma_0, 1).$$

Thus we obtain Proposition 1.3.

Proof of Proposition 1.3. It is enough to choose a natural number $b \geq 2$ and take $\lambda := b^{\frac{1}{n-\sigma}}$. Notice that $\lambda > 1$ and

$$\frac{\log b}{\log \lambda} = n - \sigma \in (n-1, n).$$

Then we can define E as in (3.6) and we are done. □

3.5. **Elementary properties of the s-perimeter.** In the following Proposition we collect some elementary but useful properties of the fractional perimeter which we have exploited throughout the paper.

Proposition 3.12. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set.*

(i) (Subadditivity) Let E, $F \subseteq \mathbb{R}^n$ be such that $|E \cap F| = 0$. Then

$$P_s(E \cup F, \Omega) \leq P_s(E, \Omega) + P_s(F, \Omega).$$

(ii) (Translation invariance) Let $E \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then

$$P_s(E + x, \Omega + x) = P_s(E, \Omega).$$

(iii) (Rotation invariance) Let $E \subseteq \mathbb{R}^n$ and $\mathcal{R} \in SO(n)$ a rotation. Then

$$P_s(\mathcal{R}E, \mathcal{R}\Omega) = P_s(E, \Omega).$$

(iv) (Scaling) Let $E \subseteq \mathbb{R}^n$ and $\lambda > 0$. Then

$$P_s(\lambda E, \lambda \Omega) = \lambda^{n-s} P_s(E, \Omega).$$

Proof. (i) follows from the following observations. Let A_1, A_2, $B \subseteq \mathbb{R}^n$. If $|A_1 \cap A_2| = 0$, then

$$\mathcal{L}_s(A_1 \cup A_2, B) = \mathcal{L}_s(A_1, B) + \mathcal{L}_s(A_2, B).$$

Moreover

$$A_1 \subseteq A_2 \quad \Longrightarrow \quad \mathcal{L}_s(A_1, B) \le \mathcal{L}_s(A_2, B),$$

and

$$\mathcal{L}_s(A, B) = \mathcal{L}_s(B, A).$$

Therefore

$$\begin{aligned} P_s(E \cup F, \Omega) &= \mathcal{L}_s((E \cup F) \cap \Omega, \mathcal{C}(E \cup F)) + \mathcal{L}_s((E \cup F) \setminus \Omega, \mathcal{C}(E \cup F) \cap \Omega) \\ &= \mathcal{L}_s(E \cap \Omega, \mathcal{C}(E \cup F)) + \mathcal{L}_s(F \cap \Omega, \mathcal{C}(E \cup F)) \\ &\qquad + \mathcal{L}_s(E \setminus \Omega, \mathcal{C}(E \cup F) \cap \Omega) + \mathcal{L}_s(F \setminus \Omega, \mathcal{C}(E \cup F) \cap \Omega) \\ &\le \mathcal{L}_s(E \cap \Omega, \mathcal{C}E) + \mathcal{L}_s(F \cap \Omega, \mathcal{C}F) \\ &\qquad + \mathcal{L}_s(E \setminus \Omega, \mathcal{C}E \cap \Omega) + \mathcal{L}_s(F \setminus \Omega, \mathcal{C}F \cap \Omega) \\ &= P_s(E, \Omega) + P_s(F, \Omega). \end{aligned}$$

(ii), (iii) and (iv) follow simply by changing variables in $\mathcal{L}_s$ and the following observations:

$$\begin{aligned} &(x + A_1) \cap (x + A_2) = x + A_1 \cap A_2, \qquad x + \mathcal{C}A = \mathcal{C}(x + A), \\ &\mathcal{R}A_1 \cap \mathcal{R}A_2 = \mathcal{R}(A_1 \cap A_2), \qquad \mathcal{R}(\mathcal{C}A) = \mathcal{C}(\mathcal{R}A), \\ &(\lambda A_1) \cap (\lambda A_2) = \lambda(A_1 \cap A_2), \qquad \lambda(\mathcal{C}A) = \mathcal{C}(\lambda A). \end{aligned}$$

For example, for claim (iv) we have

$$\begin{aligned} \mathcal{L}_s(\lambda A, \lambda B) &= \int_{\lambda A} \int_{\lambda B} \frac{dx\, dy}{|x - y|^{n+s}} = \int_A \lambda^n \, dx \int_B \frac{\lambda^n \, dy}{\lambda^{n+s} |x - y|^{n+s}} \\ &= \lambda^{n-s} \mathcal{L}_s(A, B). \end{aligned}$$

Then

$$\begin{aligned} P_s(\lambda E, \lambda \Omega) &= \mathcal{L}_s(\lambda E \cap \lambda \Omega, \mathcal{C}(\lambda E)) + \mathcal{L}_s(\lambda E \cap \mathcal{C}(\lambda \Omega), \mathcal{C}(\lambda E) \cap \lambda \Omega) \\ &= \mathcal{L}_s(\lambda(E \cap \Omega), \lambda \mathcal{C}E) + \mathcal{L}_s(\lambda(E \setminus \Omega), \lambda(\mathcal{C}E \cap \Omega)) \\ &= \lambda^{n-s} \left(\mathcal{L}_s(E \cap \Omega, \mathcal{C}E) + \mathcal{L}_s(E \setminus \Omega, \mathcal{C}E \cap \Omega) \right) \\ &= \lambda^{n-s} P_s(E, \Omega). \end{aligned}$$

This concludes the proof of the Proposition. □

Appendix A. Proof of Example 1.1

Note that $E \subseteq (0, a^2]$. Let $\Omega := (-1, 1) \subseteq \mathbb{R}$. Then $E \Subset \Omega$ and $\text{dist}(E, \partial\Omega) = 1 - a^2 =: d > 0$. Now

$$P_s(E) = \int_E \int_{\mathcal{C}E \cap \Omega} \frac{dxdy}{|x - y|^{1+s}} + \int_E \int_{\mathcal{C}\Omega} \frac{dxdy}{|x - y|^{1+s}}.$$

As for the second term, we have

$$\int_E \int_{\mathcal{C}\Omega} \frac{dxdy}{|x - y|^{1+s}} \le \frac{2|E|}{sd^s} < \infty.$$

We split the first term into three pieces

$$\begin{aligned}
&\int_E\int_{\mathcal{C}E\cap\Omega}\frac{dxdy}{|x-y|^{1+s}}\\
&\quad=\int_E\int_{-1}^0\frac{dxdy}{|x-y|^{1+s}}+\int_E\int_{\mathcal{C}E\cap(0,a)}\frac{dxdy}{|x-y|^{1+s}}+\int_E\int_a^1\frac{dxdy}{|x-y|^{1+s}}\\
&\quad=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.
\end{aligned}$$

Note that $\mathcal{C}E\cap(0,a)=\bigcup_{k\in\mathbb{N}}I_{2k-1}=\bigcup_{k\in\mathbb{N}}(a^{2k},a^{2k-1})$.
A simple calculation shows that, if $a<b\le c<d$, then

$$\begin{aligned}
&\int_a^b\int_c^d\frac{dxdy}{|x-y|^{1+s}}=\\
&\qquad\frac{1}{s(1-s)}\big[(c-a)^{1-s}+(d-b)^{1-s}-(c-b)^{1-s}-(d-a)^{1-s}\big].
\end{aligned}\tag{A.1}$$

Also note that, if $n>m\ge1$, then

$$\begin{aligned}
(1-a^n)^{1-s}-(1-a^m)^{1-s}&=\int_m^n\frac{d}{dt}(1-a^t)^{1-s}\,dt\\
&=(s-1)\log a\int_m^n\frac{a^t}{(1-a^t)^s}\,dt\\
&\le a^m(s-1)\log a\int_m^n\frac{1}{(1-a^t)^s}\,dt\\
&\le(n-m)a^m\frac{(s-1)\log a}{(1-a)^s}.
\end{aligned}\tag{A.2}$$

Now consider the first term

$$\mathcal{I}_1=\sum_{k=1}^{\infty}\int_{a^{2k+1}}^{a^{2k}}\int_{-1}^0\frac{dxdy}{|x-y|^{1+s}}.$$

Use (A.1) and notice that $(c-a)^{1-s}-(d-a)^{1-s}\le0$ to get

$$\int_{-1}^0\int_{a^{2k+1}}^{a^{2k}}\frac{dxdy}{|x-y|^{1+s}}\le\frac{1}{s(1-s)}\big[(a^{2k})^{1-s}-(a^{2k+1})^{1-s}\big]\le\frac{1}{s(1-s)}(a^{2(1-s)})^k.$$

Then, as $a^{2(1-s)}<1$ we get

$$\mathcal{I}_1\le\frac{1}{s(1-s)}\sum_{k=1}^{\infty}(a^{2(1-s)})^k<\infty.$$

As for the last term

$$\mathcal{I}_3=\sum_{k=1}^{\infty}\int_{a^{2k+1}}^{a^{2k}}\int_a^1\frac{dxdy}{|x-y|^{1+s}},$$

use (A.1) and notice that $(d-b)^{1-s}-(d-a)^{1-s}\le0$ to get

$$\begin{aligned}
\int_{a^{2k+1}}^{a^{2k}}\int_a^1\frac{dxdy}{|x-y|^{1+s}}&\le\frac{1}{s(1-s)}\big[(1-a^{2k+1})^{1-s}-(1-a^{2k})^{1-s}\big]\\
&\le\frac{-\log a}{s(1-a)^s}a^{2k}\quad\text{by (A.2)}.
\end{aligned}$$

Thus

$$\mathcal{I}_3 \leq \frac{-\log a}{s(1-a)^s} \sum_{k=1}^{\infty} (a^2)^k < \infty.$$

Finally we split the second term

$$\mathcal{I}_2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{a^{2k+1}}^{a^{2k}} \int_{a^{2j}}^{a^{2j-1}} \frac{dxdy}{|x-y|^{1+s}}$$

into three pieces according to the cases $j > k$, $j = k$ and $j < k$.

If $j = k$, using (A.1) we get

$$\begin{aligned}
&\int_{a^{2k+1}}^{a^{2k}} \int_{a^{2k}}^{a^{2k-1}} \frac{dxdy}{|x-y|^{1+s}} = \\
&\qquad = \frac{1}{s(1-s)} \big[(a^{2k} - a^{2k+1})^{1-s} + (a^{2k-1} - a^{2k})^{1-s} - (a^{2k-1} - a^{2k+1})^{1-s} \big] \\
&\qquad = \frac{1}{s(1-s)} \big[a^{2k(1-s)} (1-a)^{1-s} + a^{(2k-1)(1-s)} (1-a)^{1-s} \\
&\qquad\qquad\qquad - a^{(2k-1)(1-s)} (1-a^2)^{1-s} \big] \\
&\qquad = \frac{1}{s(1-s)} (a^{2(1-s)})^k \Big[(1-a)^{1-s} + \frac{(1-a)^{1-s}}{a^{1-s}} - \frac{(1-a^2)^{1-s}}{a^{1-s}} \Big].
\end{aligned}$$

Summing over $k \in \mathbb{N}$ we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_{a^{2k+1}}^{a^{2k}} \int_{a^{2k}}^{a^{2k-1}} \frac{dxdy}{|x-y|^{1+s}} = \\
&\qquad = \frac{1}{s(1-s)} \frac{a^{2(1-s)}}{1-a^{2(1-s)}} \Big[(1-a)^{1-s} + \frac{(1-a)^{1-s}}{a^{1-s}} - \frac{(1-a^2)^{1-s}}{a^{1-s}} \Big] < \infty.
\end{aligned}$$

In particular note that

$$\begin{aligned}
(1-s)P_s(E) &\geq (1-s)\mathcal{I}_2 \\
&\geq \frac{1}{s(1-a^{2(1-s)})} \big[a^{2(1-s)}(1-a)^{1-s} + a^{1-s}(1-a)^{1-s} - a^{1-s}(1-a^2)^{1-s} \big],
\end{aligned}$$

which tends to $+\infty$ when $s \to 1$. This shows that E cannot have finite perimeter.

To conclude let $j > k$, the case $j < k$ being similar, and consider

$$\sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} \int_{a^{2j}}^{a^{2j-1}} \int_{a^{2k+1}}^{a^{2k}} \frac{dxdy}{|x-y|^{1+s}}.$$

Again, using (A.1) and $(d-b)^{1-s}-(d-a)^{1-s}\le 0$, we get

$$\begin{aligned}\int_{a^{2j}}^{a^{2j-1}}\int_{a^{2k+1}}^{a^{2k}}&\frac{dxdy}{|x-y|^{1+s}}\\&\le\frac{1}{s(1-s)}\big[(a^{2k+1}-a^{2j})^{1-s}-(a^{2k+1}-a^{2j-1})^{1-s}\big]\\&=\frac{a^{1-s}}{s(1-s)}(a^{2(1-s)})^k\big[(1-a^{2(j-k)-1})^{1-s}-(1-a^{2(j-k)-2})^{1-s}\big]\\&\le\frac{a^{1-s}}{s(1-s)}(a^{2(1-s)})^k\frac{(s-1)\log a}{(1-a)^s}a^{2(j-k)-2}\qquad\text{by (A.2)}\\&=\frac{-\log a}{s(1-a^s)a^{s+1}}(a^{2(1-s)})^k(a^2)^{j-k},\end{aligned}$$

for $j\ge k+2$. Then

$$\begin{aligned}\sum_{k=1}^\infty\sum_{j=k+2}^\infty&\int_{a^{2j}}^{a^{2j-1}}\int_{a^{2k+1}}^{a^{2k}}\frac{dxdy}{|x-y|^{1+s}}\\&\le\frac{-\log a}{s(1-a^s)a^{s+1}}\sum_{k=1}^\infty(a^{2(1-s)})^k\sum_{h=2}^\infty(a^2)^h<\infty.\end{aligned}$$

If $j=k+1$ we get

$$\begin{aligned}\sum_{k=1}^\infty\int_{a^{2k+2}}^{a^{2k+1}}\int_{a^{2k+1}}^{a^{2k}}\frac{dxdy}{|x-y|^{1+s}}&\le\frac{1}{s(1-s)}\sum_{k=1}^\infty(a^{2k+1}-a^{2k+2})^{1-s}\\&=\frac{a^{1-s}(1-a)^{1-s}}{s(1-s)}\sum_{k=1}^\infty(a^{2(1-s)})^k<\infty.\end{aligned}$$

This shows that also $\mathcal{I}_2<\infty$, so that $P_s(E)<\infty$ for every $s\in(0,1)$ as claimed.

Appendix B. Signed distance function

Given $\emptyset\neq E\subseteq\mathbb{R}^n$, the distance function from E is defined as

$$d_E(x)=d(x,E):=\inf_{y\in E}|x-y|,\qquad\text{for }x\in\mathbb{R}^n.$$

The signed distance function from ∂E, negative inside E, is then defined as

$$\bar d_E(x)=\bar d(x,E):=d(x,E)-d(x,\mathcal{C}E).$$

For the details of the main properties we refer e.g. to [1] and [3].

We also define the sets

$$E_r:=\{x\in\mathbb{R}^n\,|\,\bar d_E(x)<r\}.$$

Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary. By definition we can locally describe Ω near its boundary as the subgraph of appropriate Lipschitz functions. To be more precise, we can find a finite open covering $\{C_{\varrho_i}\}_{i=1}^m$ of $\partial\Omega$ made of cylinders, and Lipschitz functions $\varphi_i:B'_{\varrho_i}\longrightarrow\mathbb{R}$ such that $\Omega\cap C_{\varrho_i}$ is the subgraph of φ_i. That is, up to rotations and translations,

$$C_{\varrho_i}=\{(x',x_n)\in\mathbb{R}^n\,|\,|x'|<\varrho_i,\,|x_n|<\varrho_i\},$$

and

$$\Omega \cap C_{\varrho_i} = \{(x', x_n) \in \mathbb{R}^n \,|\, x' \in B'_{\varrho_i},\, -\varrho_i < x_n < \varphi_i(x')\},$$
$$\partial\Omega \cap C_{\varrho_i} = \{(x', \varphi_i(x')) \in \mathbb{R}^n \,|\, x' \in B'_{\varrho_i}\}.$$

Let L be the sup of the Lipschitz constants of the functions φ_i.

We observe that [12, Theorem 4.1] guarantees that also the bounded open sets Ω_r have Lipschitz boundary, when r is small enough, say $|r| < r_0$.
Moreover these sets Ω_r can locally be described, in the same cylinders C_{ϱ_i} used for Ω, as subgraphs of Lipschitz functions φ_i^r which approximate φ_i (see [12] for the precise statement) and whose Lipschitz constants are less than or equal to L.
Notice that

$$\partial\Omega_r = \{\bar{d}_\Omega = r\}.$$

Now, since in C_{ϱ_i} the set Ω_r coincides with the subgraph of φ_i^r, we have

$$\mathcal{H}^{n-1}(\partial\Omega_r \cap C_{\varrho_i}) = \int_{B'_{\varrho_i}} \sqrt{1 + |\nabla\varphi_i^r|^2}\, dx' \leq M_i,$$

with M_i depending on ϱ_i and L but not on r.
Therefore

$$\mathcal{H}^{n-1}(\{\bar{d}_\Omega = r\}) \leq \sum_{i=1}^m \mathcal{H}^{n-1}(\partial\Omega_r \cap C_{\varrho_i}) \leq \sum_{i=1}^m M_i$$

independently on r, proving the following

Proposition B.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists $r_0 = r_0(\Omega) > 0$ such that Ω_r is a bounded open set with Lipschitz boundary for every $r \in (-r_0, r_0)$ and*

$$\sup_{|r|<r_0} \mathcal{H}^{n-1}(\{\bar{d}_\Omega = r\}) < \infty.$$

Appendix C. Measure theoretic boundary

Since

$$|E\Delta F| = 0 \quad\Longrightarrow\quad P(E,\Omega) = P(F,\Omega) \quad \text{and} \quad P_s(E,\Omega) = P_s(F,\Omega), \tag{C.1}$$

we can modify a set making its topological boundary as big as we want, without changing its (fractional) perimeter.
For example, let $E \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then, if we set

$$F := (E \setminus \mathbb{Q}^n) \cup (\mathbb{Q}^n \setminus E),$$

we have $|E\Delta F| = 0$ and hence we get (C.1). However $\partial F = \mathbb{R}^n$.

For this reason one considers measure theoretic notions of interior, exterior and boundary, which solely depend on the class of χ_E in $L^1_{loc}(\mathbb{R}^n)$.
In some sense, by considering the measure theoretic boundary $\partial^- E$ defined below we can also minimize the size of the topological boundary (see (C.6)). Moreover, this measure theoretic boundary is actually the topological boundary of a set which is equivalent to E. Thus we obtain a "good" representative for the class of E.

We refer to [22, Section 3.2] (see also [16, Proposition 3.1]). For some details about the good representative of an s-minimal set, see the Appendix of [10].

Definition C.1. *Let $E \subseteq \mathbb{R}^n$. For every $t \in [0,1]$ we define the set*

$$E^{(t)} := \left\{ x \in \mathbb{R}^n \,\middle|\, \exists \lim_{r\to 0} \frac{|E \cap B_r(x)|}{\omega_n r^n} = t \right\}, \tag{C.2}$$

of points density t of E. We also define the essential boundary of E as

$$\partial_e E := \mathbb{R}^n \setminus \big(E^{(0)} \cup E^{(1)}\big).$$

Using the Lebesgue's points Theorem for the characteristic function χ_E, we see that the limit in (C.2) exists for a.e. $x \in \mathbb{R}^n$ and

$$\lim_{r\to 0} \frac{|E \cap B_r(x)|}{\omega_n r^n} = \begin{cases} 1 & \text{for a.e. } x \in E, \\ 0 & \text{for a.e. } x \in \mathcal{C}E. \end{cases}$$

So

$$|E \Delta E^{(1)}| = 0, \qquad |\mathcal{C}E \Delta E^{(0)}| = 0 \qquad \text{and } |\partial_e E| = 0.$$

In particular a set E is equivalent to the set $E^{(1)}$ of its points of density 1.
Roughly speaking, the sets $E^{(0)}$ and $E^{(1)}$ can be thought of as a measure theoretic version of, respectively, the exterior and the interior of the set E. However, notice that both $E^{(1)}$ and $E^{(0)}$ in general are not open.

We have another natural way to define measure theoretic versions of interior, exterior and boundary.

Definition C.2. *Given a set $E \subseteq \mathbb{R}^n$, we define the measure theoretic interior and exterior of E by*

$$E_{int} := \{x \in \mathbb{R}^n \,|\, \exists\, r > 0,\ |E \cap B_r(x)| = \omega_n r^n\}$$

and

$$E_{ext} := \{x \in \mathbb{R}^n \,|\, \exists\, r > 0,\ |E \cap B_r(x)| = 0\},$$

respectively. Then we define the measure theoretic boundary of E as

$$\begin{aligned} \partial^- E &:= \mathbb{R}^n \setminus (E_{ext} \cup E_{int}) \\ &= \{x \in \mathbb{R}^n \,|\, 0 < |E \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0\}. \end{aligned}$$

Notice that E_{ext} and E_{int} are open sets and hence $\partial^- E$ is closed. Moreover, since

$$E_{ext} \subseteq E^{(0)} \qquad \text{and} \qquad E_{int} \subseteq E^{(1)}, \tag{C.3}$$

we get

$$\partial_e E \subseteq \partial^- E.$$

We observe that

$$F \subseteq \mathbb{R}^n \text{ s.t. } |E \Delta F| = 0 \quad \Longrightarrow \quad \partial^- E \subseteq \partial F. \tag{C.4}$$

Indeed, if $|E\Delta F| = 0$, then $|F \cap B_r(x)| = |E \cap B_r(x)|$ for every $r > 0$. Thus for any $x \in \partial^- E$ we have

$$0 < |F \cap B_r(x)| < \omega_n r^n,$$

which implies

$$F \cap B_r(x) \neq \emptyset \quad \text{and} \quad \mathcal{C}F \cap B_r(x) \neq \emptyset \quad \text{for every } r > 0,$$

and hence $x \in \partial F$.

In particular, $\partial^- E \subseteq \partial E$.

Moreover

$$\partial^- E = \partial E^{(1)}. \tag{C.5}$$

Indeed, since $|E\Delta E^{(1)}| = 0$, we already know that $\partial^- E \subseteq \partial E^{(1)}$. The converse inclusion follows from (C.3) and the fact that both E_{ext} and E_{int} are open.
From (C.4) and (C.5) we obtain

$$\partial^- E = \bigcap_{F\sim E} \partial F, \tag{C.6}$$

where the intersection is taken over all sets $F \subseteq \mathbb{R}^n$ such that $|E\Delta F| = 0$, so we can think of $\partial^- E$ as a way to minimize the size of the topological boundary of E. In particular

$$F \subseteq \mathbb{R}^n \text{ s.t. } |E\Delta F| = 0 \quad \Longrightarrow \quad \partial^- F = \partial^- E.$$

From (C.3) and (C.5) we see that we can take $E^{(1)}$ as "good" representative for E, obtaining Remark 1.8.

Recall that the support of a Radon measure μ on $\mathbb{R}^n$ is defined as the set

$$\operatorname{supp} \mu := \{x \in \mathbb{R}^n \,|\, \mu(B_r(x)) > 0 \text{ for every } r > 0\}.$$

Notice that, being the complementary of the union of all open sets of measure zero, it is a closed set. In particular, if E is a Caccioppoli set, we have

$$\operatorname{supp} |D\chi_E| = \{x \in \mathbb{R}^n \,|\, P(E, B_r(x)) > 0 \text{ for every } r > 0\}, \tag{C.7}$$

and it is easy to verify that

$$\partial^- E = \operatorname{supp} |D\chi_E| = \overline{\partial^* E},$$

where $\partial^* E$ denotes the reduced boundary (see, e.g., [17, Chapter 15]). Moreover, $\partial^* E \subseteq \partial_e E$ and by Federer's Theorem (see, e.g., [17, Theorem 16.2]) we have

$$\mathcal{H}^{n-1}(\partial_e E \setminus \partial^* E) = 0.$$

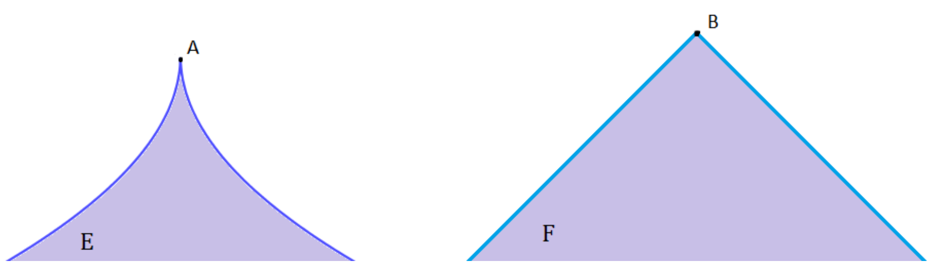

FIGURE 5. *The point A belongs to $\partial^- E$ but $A \notin \partial_e E$. The point B belongs to $\partial_e F$ but $B \notin \partial^* F$.*

We remark that in general the inclusions

$$\partial^* E \subseteq \partial_e E \subseteq \partial^- E \subseteq \partial E$$

are all strict. Indeed, we have already observed in the previous discussion that in general $\partial^- E$ is much smaller than the topological boundary ∂E. In order to have an example of a point $p \in \partial^- E \setminus \partial_e E$ it is enough to consider sublinear cusps. For example, if $E := \{(x, y) \in \mathbb{R}^2 \,|\, y < -|x|^{\frac{1}{2}}\}$ and $p := (0, 0)$, then it is easy to verify that $p \in E^{(0)}$ and hence $p \notin \partial_e E$. On the other hand, $p \in \partial^- E$. Finally, the vertex of an angle is an example of a point $p \in \partial_e E \setminus \partial^* E$ (see, e.g., [17, Example 15.4]).

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