Asymptotic Analysis -1 (2020) 1–31
DOI 10.3233/ASY-201605
IOS Press

Multiple eigenvalues for the Steklov problem in a domain with a small hole. A functional analytic approach

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Abstract. Let $\alpha \in]0,1[$. Let Ω^o be a bounded open domain of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let ν_{Ω^o} denote the outward unit normal to $\partial\Omega^o$. We assume that the Steklov problem $\Delta u = 0$ in Ω^o, $\frac{\partial u}{\partial \nu_{\Omega^o}} = \lambda u$ on $\partial\Omega^o$ has a multiple eigenvalue $\tilde{\lambda}$ of multiplicity r. Then we consider an annular domain $\Omega(\epsilon)$ obtained by removing from Ω^o a small cavity of class $C^{1,\alpha}$ and size $\epsilon > 0$, and we show that under appropriate assumptions each elementary symmetric function of r eigenvalues of the Steklov problem $\Delta u = 0$ in $\Omega(\epsilon)$, $\frac{\partial u}{\partial \nu_{\Omega(\epsilon)}} = \lambda u$ on $\partial\Omega(\epsilon)$ which converge to $\tilde{\lambda}$ as ϵ tend to zero, equals real a analytic function defined in an open neighborhood of $(0,0)$ in $\mathbb{R}^2$ and computed at the point $(\epsilon, \delta_{2,n}\epsilon \log \epsilon)$ for $\epsilon > 0$ small enough. Here $\nu_{\Omega(\epsilon)}$ denotes the outward unit normal to $\partial\Omega(\epsilon)$, and $\delta_{2,2} \equiv 1$ and $\delta_{2,n} \equiv 0$ if $n \geqslant 3$. Such a result is an extension to multiple eigenvalues of a previous result obtained for simple eigenvalues in collaboration with S. Gryshchuk.

Keywords: Multiple Steklov eigenvalues and eigenfunctions, singularly perturbed domain, Laplace operator, real analytic continuation

1. Introduction

In this paper we consider a Steklov eigenvalue problem in a domain perforated by a hole. First we introduce the problem with no hole, and then we consider the case with the hole.

We consider an open bounded connected subset of Ω^o of $\mathbb{R}^n$ of class $C^{1,\alpha}$, for some $\alpha \in]0,1[$, such that $0 \in \Omega^o$ and such that the complement in $\mathbb{R}^n$ of the closure $\mathrm{cl}\,\Omega^o$ is also connected. Then we consider the Steklov eigenvalue problem

$$\Delta u = 0 \quad \text{in } \Omega^o, \qquad \frac{\partial u}{\partial \nu_{\Omega^o}} = \lambda u \quad \text{on } \partial\Omega^o. \tag{1.1}$$

Here ν_{Ω^o} denotes the outward unit normal to $\partial\Omega^o$. By definition, a Steklov eigenvalue for Δ in Ω^o is a real number λ for which problem (1.1) has a nontrivial solution in $C^{1,\alpha}(\mathrm{cl}\,\Omega^o)$. As is well known, problem (1.1) has an increasing sequence of eigenvalues $\{\lambda_j[\Omega^o]\}_{j\in\mathbb{N}}$, and we write each eigenvalue as many times as its multiplicity (cf. *e.g.*, Henrot [13, p. 113]). Here $\mathbb{N}$ denotes the set of natural numbers including 0. In this paper, we assume that there exists $t \in \mathbb{N} \setminus \{0\}$ such that the eigenvalue $\tilde{\lambda} \equiv \lambda_t[\Omega^o]$

has multiplicity $r \in \mathbb{N} \setminus \{0\}$ and that

$$\lambda_{t-1}\big[\Omega^o\big] < \lambda_t\big[\Omega^o\big] = \cdots = \lambda_{t+r-1}\big[\Omega^o\big] < \lambda_{t+r}\big[\Omega^o\big].$$

Then problem (1.1) has an eigenspace of solutions $u \in C^{1,\alpha}(\mathrm{cl}\,\Omega^o)$ of dimension r.

Next we make a hole in the domain Ω^o. Namely, we consider another open bounded connected subset Ω^i of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $0 \in \Omega^i$ and such that the complement in $\mathbb{R}^n$ of $\mathrm{cl}\,\Omega^i$ is also connected, and we take $\epsilon_0 \in]0,1[$ such that $\epsilon\,\mathrm{cl}\,\Omega^i \subseteq \Omega^o$ for $|\epsilon| \leqslant \epsilon_0$, and we consider the annular (or perforated) domain $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\,\mathrm{cl}\,\Omega^i$. Obviously, $\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o$. For each $\epsilon \in]0, \epsilon_0[$, we consider the Steklov eigenvalue problem

$$\Delta u = 0 \quad \text{in } \Omega(\epsilon), \qquad \frac{\partial u}{\partial \nu_{\Omega(\epsilon)}} = \lambda u \quad \text{on } \partial\Omega(\epsilon). \tag{1.2}$$

Here $\nu_{\Omega(\epsilon)}$ denotes the outward unit normal to $\partial\Omega(\epsilon)$. By definition, a Steklov eigenvalue for Δ in $\Omega(\epsilon)$ is a real number λ for which problem (1.2) has a nontrivial solution. As is well known, problem (1.2) has an increasing sequence of Steklov eigenvalues $\{\lambda_j[\Omega(\epsilon)]\}_{j\in\mathbb{N}}$, and we write each Steklov eigenvalue as many times as its multiplicity. We are interested in the behaviour of the eigenvalues of (1.2) as ϵ tends to 0. This type of problem has been considered for a long time. We mention the explicit computations of Dittmar [6] in a circular annulus. Then we mention Nazarov [29], who has proved that the eigenvalues of $\{\lambda_j[\Omega(\epsilon)]\}_{j\in\mathbb{N}}$ tend to those of $\{\lambda_j[\Omega^o]\}_{j\in\mathbb{N}}$ as ϵ tends to zero and who has computed corresponding complete asymptotic expansions.

We also mention the asymptotic expansions for singularly perturbed domains with a peak of Nazarov [27, 28], and the paper of Nazarov and Taskinen [30] on the spectrum in a domain with a peak. We also mention the work of Chiado Piat, Nazarov, Piatnitski [3], Douanla [8], Mel'nik [24], Pastukhova [31], Vanninathan [36], which concern the case of periodic perforations and who aim at understanding the limiting behaviour of the eigenvalues and the existence of asymptotic expansions.

In this paper instead, we generalize the work of [12] with Gryshchuk in the case of simple eigenvalues and by following a pattern for problems with multiple eigenvalues of Lamberti and the author in [18], of Lamberti [17], and of Buoso and Lamberti [1] for regular perturbations, we show that there exist ϵ^*, $\iota* \in]0, +\infty[$ and real analytic functions $\Lambda_{t,1}(\cdot,\cdot), \dots, \Lambda_{t,r}(\cdot,\cdot)$ from $]-\epsilon^*, \epsilon^*[\times]-\iota^*, \iota^*[$ to $\mathbb{R}$ such that

$$\begin{aligned}
&\Lambda_{t,1}(\epsilon, \delta_{2,n}\epsilon\log\epsilon) = \sum_{j_1=1}^{r} \lambda_{t+j_1-1}\big[\Omega(\epsilon)\big],\\
&\Lambda_{t,2}(\epsilon, \delta_{2,n}\epsilon\log\epsilon) = \sum_{\substack{j_1,j_2=1,\dots,r\\ j_1<j_2}} \lambda_{t+j_1-1}\big[\Omega(\epsilon)\big]\cdot\lambda_{t+j_2-1}\big[\Omega(\epsilon)\big],\\
&\cdots\cdots\cdots\\
&\Lambda_{t,r-1}(\epsilon, \delta_{2,n}\epsilon\log\epsilon) = \sum_{\substack{j_1,\dots,j_r=1,\dots,r\\ j_1<\cdots<j_{r-1}}} \lambda_{t+j_1-1}\big[\Omega(\epsilon)\big]\cdot\dots\cdot\lambda_{t+j_{r-1}-1}\big[\Omega(\epsilon)\big],\\
&\Lambda_{t,r}(\epsilon, \delta_{2,n}\epsilon\log\epsilon) = \lambda_1\big[\Omega(\epsilon)\big]\cdot\dots\cdot\lambda_r\big[\Omega(\epsilon)\big],
\end{aligned} \tag{1.3}$$

for $\epsilon \in]0, \epsilon^*[$ (cf. Theorem 11.1). Here $\delta_{2,n} \equiv 1$ if $n = 2$, $\delta_{2,n} \equiv 0$ if $n \geqslant 3$.

By (1.3), if $n=2$, then (1.3) implies that the following convergent series expansion holds

$$\sum_{\substack{j_1,\dots,j_s=1,\dots,r\\ j_1<\dots<j_s}} \lambda_{t+j_1-1}\big[\Omega(\epsilon)\big]\cdot\dots\cdot\lambda_{t+j_s-1}\big[\Omega(\epsilon)\big] = \sum_{\beta\equiv(\beta_1,\beta_2)\in\mathbb{N}^2} \frac{1}{\beta!}D^{\beta}\Lambda_s[0,0]\epsilon^{\beta_1}(\epsilon\log\epsilon)^{\beta_2}$$

for $\epsilon>0$ sufficiently small and for all $s\in\{1,\dots,r\}$. Instead, if $n\geqslant 3$, then possibly shrinking ϵ^*, the s-th elementary symmetric functions of the eigenvalues $\lambda_{t+l-1}[\Omega(\epsilon)]$ for $l\in\{1,\dots,r\}$ can be continued real analytically in $\epsilon\in\left]-\epsilon^*,\epsilon^*\right[$, and the following convergent series expansion holds

$$\sum_{\substack{j_1,\dots,j_s=1,\dots,r\\ j_1<\dots<j_s}} \lambda_{t+j_1-1}\big[\Omega(\epsilon)\big]\cdot\dots\cdot\lambda_{t+j_s-1}\big[\Omega(\epsilon)\big] = \sum_{l=0}^{\infty}\frac{1}{l!}\partial_{\epsilon}^{l}\Lambda_s[0,0]\epsilon^{l}$$

for $\epsilon>0$ sufficiently small and for all $s\in\{1,\dots,r\}$. Here ∂_ϵ^l denotes a partial differentiation with respect to the first variable of Λ_s. In case $n\geqslant 3$, we also prove that each $\lambda_j[\Omega(\epsilon)]$ can be continued analytically to (small) negative values of ϵ.

In our analysis we reduce our problem to a system of integral equations, and we mention that the reduction of the Steklov problem to integral equations has also been exploited by Kuznetsov and Motygin [16], and by Shamma [35]. In this paper, we have considered the case of a single hole. By the same ideas one could consider the case with a finite number of holes, at the price of having to consider systems of integral equations with more equations. One could also consider our problem under weaker regularity assumptions on the domain as long as the spectrum is discrete as in the case of Lipschitz sets. The author believes that the ideas of the present paper could be applied as long as the corresponding integral equations still correspond to Fredholm operators.

This paper is organized as follows. Section 2 is a section of preliminaries. In Section 3, we introduce some basic notation and results in potential theory. In Section 4, we formulate the Steklov eigenvalue problem on a domain as an eigenvalue problem for a compact selfadjoint operator in a Hilbert function space on the boundary. In Section 5, we formulate the Steklov eigenvalue problem in $\Omega(\epsilon)$ as an eigenvalue problem for a compact selfadjoint operator $\tilde{A}_\epsilon$ in the space $L^2(\partial\Omega^i)\times L^2(\partial\Omega^0)$ with a specific scalar product Q_ϵ. In Section 6, we show that the family of scalar products $\{Q_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ can be continued analytically for negative values of ϵ. In Section 7, we prove that the operator $\tilde{A}_\epsilon$ can be defined implicitly. In Section 8, we prove the real analytic representation formula for the family $\{\tilde{A}_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ of Theorem 8.1. In Section 9, we prove Theorem 9.5 on the representation of the elementary symmetric functions of the eigenvalues of the operators of the family $\{\tilde{A}_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ which split from a given multiple eigenvalue. In Section 10, we present the above mentioned continuity result for the Steklov eigenvalues of Nazarov [27, Thm. 2.1, p. 288]. In Section 11, we prove our main result Theorem 11.1 on the representation of the elementary symmetric functions of the Steklov eigenvalues.

2. Preliminaries and notation

We denote the norm on a normed space X by $\|\cdot\|_X$. Let X and Y be normed spaces. We endow the product space $X\times Y$ with the norm defined by $\|(x,y)\|_{X\times Y}\equiv\|x\|_X+\|y\|_Y\ \forall(x,y)\in X\times Y$, while we use the Euclidean norm for $\mathbb{R}^n$. We denote by $\mathcal{L}(X,Y)$ the normed space of linear and continuous maps from

X to Y, equipped with its usual norm of the uniform convergence on the unit sphere of X (and we set $\mathcal{L}(X) \equiv \mathcal{L}(X, X)$). If T is a linear map from X to Y, then we set $\mathrm{Im}\, T \equiv T(X)$. For standard definitions of Calculus in normed spaces, we refer to Cartan [2] and to Prodi and Ambrosetti [32]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper,

$$n \in \mathbb{N} \setminus \{0, 1\}.$$

A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\, \mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega, \mathbb{R})$, or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^r$. The s-th component of f is denoted f_s and the Jacobian matrix of f is denoted Df. Let $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f such that f and its derivatives $D^\eta f$ of order $|\eta| \leqslant m$ can be extended with continuity to $\mathrm{cl}\, \Omega$ is denoted $C^m(\mathrm{cl}\, \Omega)$. The subspace of $C^m(\mathrm{cl}\, \Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0, 1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\, \Omega)$ (cf. *e.g.* Gilbarg and Trudinger [11]). Now let Ω be a bounded open subset of $\mathbb{R}^n$. Then $C^m(\mathrm{cl}\, \Omega)$ endowed with the norm $\|f\|_{C^m(\mathrm{cl}\, \Omega)} \equiv \sum_{|\eta| \leqslant m} \sup_{\mathrm{cl}\, \Omega} |D^\eta f|$ is a Banach space. If $f \in C^{0,\alpha}(\mathrm{cl}\, \Omega)$, then its Hölder constant $|f : \Omega|_\alpha$ is defined as $\sup\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in \mathrm{cl}\, \Omega, x \neq y\}$. The space $C^{m,\alpha}(\mathrm{cl}\, \Omega)$, equipped with its usual norm $\|f\|_{C^{m,\alpha}(\mathrm{cl}\, \Omega)} = \|f\|_{C^m(\mathrm{cl}\, \Omega)} + \sum_{|\eta|=m} |D^\eta f : \Omega|_\alpha$, is well-known to be a Banach space. We say that a bounded open subset of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if its closure is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [11, §6.2]). For standard properties of the functions of class $C^{m,\alpha}$ both on a domain of $\mathbb{R}^n$ or on a manifold imbedded in $\mathbb{R}^n$ we refer to Gilbarg and Trudinger [11] (see also [20, §2, Lem. 3.1, 4.26, Thm. 4.28], [23, §2]). We retain the standard notation of L^p spaces and of corresponding norms. Also, we find convenient to set

$$\fint_{\partial\Omega} \equiv \frac{1}{m_{n-1}(\partial\Omega)} \int_{\partial\Omega}, \qquad Y_0 \equiv \left\{ f \in Y : \int_{\partial\Omega} f \, d\sigma = 0 \right\}, \qquad \chi_{\partial\Omega}(x) \equiv 1 \quad \forall x \in \partial\Omega,$$

where m_{n-1} is the $(n-1)$-dmensional measure of $\partial\Omega$ and Y is a vector subspace of $L^1(\partial\Omega)$. We note that throughout the paper 'analytic' means 'real analytic'. For the definition and properties of analytic operators, we refer to Deimling [5, p. 150] and to Prodi Ambrosetti [32, p. 89].

3. Some basic facts in potential theory

We denote by S_n the function of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_2(\xi) \equiv \frac{1}{s_2} \log |\xi| \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \qquad S_n(\xi) \equiv \frac{1}{(2-n)s_n} |\xi|^{2-n} \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \text{if } n > 2,$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n(0, 1)$ for $n \geqslant 2$. S_n is well-known to be the fundamental solution of the Laplace operator. Now let Ω be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$. Let ν_Ω denote the outward unit normal to $\partial\Omega$. Let

$$\Omega^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\,\Omega,$$

be the exterior of Ω. Then we set

$$v_\Omega[\mu](x) \equiv \int_{\partial\Omega} S_n(x-y)\mu(y)\,d\sigma_y \quad \forall x \in \mathbb{R}^n,$$
$$w_\Omega[\mu](x) \equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(y)}\big(S_n(x-y)\big)\mu(y)\,d\sigma_y \equiv -\int_{\partial\Omega} \nu_\Omega(y)\cdot DS_n(x-y)\mu(y)\,d\sigma_y \quad \forall x \in \mathbb{R}^n,$$

for all $\mu \in L^2(\partial\Omega)$, and we introduce a notaton for the corresponding boundary operators. Namely,

$$V_\Omega[\mu](x) \equiv v_\Omega[\mu](x), \qquad W_\Omega[\mu](x) \equiv w_\Omega[\mu](x),$$
$$W^t_\Omega[\mu](x) \equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(x)}\big(S_n(x-y)\big)\mu(y)\,d\sigma_y$$

for all $x \in \partial\Omega$ and $\mu \in L^2(\partial\Omega)$. As is well known (cf. *e.g.*, Folland [10, Prop. 3.25, p. 129]), if $\mu \in C^0(\partial\Omega)$, then $v_\Omega[\mu] \in C^0(\mathbb{R}^n)$, and we set

$$v^+_\Omega[\mu] \equiv v_\Omega[\mu]_{|\mathrm{cl}\,\Omega}, \qquad v^-_\Omega[\mu] \equiv v_\Omega[\mu]_{|\mathrm{cl}\,\Omega^-}.$$

Also, if $\mu \in C^0(\partial\Omega)$, then $w_\Omega[\mu]_{|\Omega}$ admits a unique continuous extension to $\mathrm{cl}\,\Omega$, which we denote by $w^+_\Omega[\mu]$, and $w_\Omega[\mu]_{|\Omega^-}$ admits a unique continuous extension to $\mathrm{cl}\,\Omega^-$, which we denote by $w^-_\Omega[\mu]$. Then we have the following result of classical potential theory (cf. Miranda [25, 26], see also [23, Thm. 3.1]).

Theorem 3.1. *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $R > 0$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Then the following statements hold.*

(i) *The map from $C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega)$ which takes μ to $w^+_\Omega[\mu]$ is linear and continuous. The map from $C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)$ which takes μ to $w^-_\Omega[\mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega}$ is linear and continuous. Furthermore, $w^\pm_\Omega[\mu] = \pm\frac{1}{2}\mu + W_\Omega[\mu]$ on $\partial\Omega$, and $\frac{\partial w^+_\Omega[\mu]}{\partial\nu_\Omega} = \frac{\partial w^-_\Omega[\mu]}{\partial\nu_\Omega}$ on $\partial\Omega$ for all $\mu \in C^{1,\alpha}(\partial\Omega)$, and $w^+_\Omega[1] = 1$ on $\mathrm{cl}\,\Omega$, and $W_\Omega[1] = 1/2$ on $\partial\Omega$.*

(ii) *The map from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega)$ which takes μ to $v^+_\Omega[\mu]$ is linear and continuous. The map from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega)$ which takes μ to $v^-_\Omega[\mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\Omega}$ is linear and continuous. Furthermore, $v^+_\Omega[\mu] = v^-_\Omega[\mu]$ on $\partial\Omega$, and $\frac{\partial v^\pm_\Omega[\mu]}{\partial\nu_\Omega} = \mp\frac{1}{2}\mu + W^t_\Omega[\mu]$ on $\partial\Omega$.*

(iii) *The map from $C^{0,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ which takes μ to $W^t_\Omega[\mu]$ is linear and continuous.*

Then we have the following two known elementary lemmas.

Lemma 3.2. *Let $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then*

(i) *The map Υ from $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to the subspace of $C^{1,\alpha}(\mathrm{cl}\,\Omega)$ of those functions which are harmonic in Ω and which takes (μ,c) to $\Upsilon[\mu,c] \equiv v^+_\Omega[\mu] + c$ is a linear homeomorphism.*

(ii) *The map $\Upsilon_{|\partial\Omega}$ from $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to $C^{1,\alpha}(\partial\Omega)$ which takes (μ, c) to $\Upsilon_{|\partial\Omega}[\mu, c] \equiv V_\Omega[\mu] + c$ is a linear homeomorphism.*

For a proof we refer for example to [12, Lem. 3.6], and to the mongraph [4, Thms. 6.41, 6.42] with Dalla Riva and Musolino. For a proof of the following lemma, we refer for example to paper [12, Lem. 3.10 (i)] with Gryshchuk together with the Open Mapping Theorem.

Lemma 3.3. *Let $\alpha \in]0, 1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that the exterior Ω^- is connected. Then the operator $\frac{1}{2}I + W^t_\Omega$ is an isomorphism of $L^2(\partial\Omega)$ onto itself.*

4. Formulation of the Steklov eigenvalue problem as an eigenvalue problem for a compact selfadjoint operator in a Hilbert function space on the boundary

We plan to write a formulation of the Steklov problem in an open subset Ω of $\mathbb{R}^n$ by means of integral equations on $\partial\Omega$. To do so, one can resort to the representation Lemma 3.2(i) for harmonic functions in Ω and obtain and integral equation (cf. paper [12, Cor. 3.7] with Gryshchuk). However, such an equation is not an eigenvalue equation for a self adjoint operator in $L^2(\partial\Omega)$. In order to obtain an eigenvalue equation for a self adjoint operator in $L^2(\partial\Omega)$, we need the following preliminary statement.

Lemma 4.1. *Let $\alpha \in]0, 1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the map $\Upsilon^\dagger$ from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)_0 \times \mathbb{R}$ which takes τ to $\Upsilon^\dagger[\tau] \equiv ((-\frac{1}{2}I + W^t_\Omega)[\tau], \int_{\partial\Omega} \tau\, d\sigma)$ is a linear homeomorphism. In particular, problem*

$$\left(-\frac{1}{2}I + W^t_\Omega\right)[\tau] = 0 \quad \text{on } \partial\Omega, \qquad \int_{\partial\Omega} \tau\, d\sigma = 1, \tag{4.1}$$

has a unique solution τ_Ω in $L^2(\partial\Omega)$. Moreover, $\tau_\Omega \in C^{0,\alpha}(\partial\Omega)$ and τ_Ω generates $\mathrm{Ker}(-\frac{1}{2}I + W^t_\Omega)$.

Proof. Since the kernel of the integral operator W^t_Ω has a weak singularity, W^t_Ω is compact in $L^2(\partial\Omega)$. Since $\int_{\partial\Omega} -\frac{1}{2}\tau + W^t_\Omega[\tau]\, d\sigma = \int_{\partial\Omega} -\frac{1}{2}\tau + \tau W_\Omega[1]\, d\sigma = \int_{\partial\Omega} -\frac{1}{2}\tau + \tau\frac{1}{2}\, d\sigma = 0$, $\Upsilon^\dagger$ is linear and continuous from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)_0 \times \mathbb{R}$. By Folland [10, Prop. 3.34, 3.36], $\mathrm{Ker}(-\frac{1}{2}I + W^t_\Omega)$ has dimension one and thus it has a generator $\tau^\sharp$. If $\int_{\partial\Omega} \tau^\sharp\, d\sigma = 0$, then $\tau^\sharp$ is orthogonal to $\mathrm{Ker}(-\frac{1}{2}I + W_\Omega)$, which is generated by the characteristic function $\chi_{\partial\Omega}$ (cf. *e.g.*, Folland [10, Prop. 3.34, 3.36]). Then equality $(\mathrm{Ker}(-\frac{1}{2}I + W_\Omega))^\perp \cap \mathrm{Ker}(-\frac{1}{2}I + W^t_\Omega) = \{0\}$ implies that $\tau^\sharp = 0$, a contradiction (cf. *e.g.*, Folland [10, Prop. 3.38]). Hence, $\int_{\partial\Omega} \tau^\sharp\, d\sigma \neq 0$. By classical regularity theory, $\tau_\Omega \equiv \frac{\tau^\sharp}{\int_{\partial\Omega} \tau^\sharp\, d\sigma} \in C^{0,\alpha}(\partial\Omega)$ (cf. *e.g.*, [21, Thm. 5.1 (i)]). By equality $\mathrm{Ker}(-\frac{1}{2}I + W_\Omega) = \langle \chi_{\partial\Omega} \rangle$, we can apply Fredholm Alternative to the first component of $\Upsilon^\dagger$ and show that $\Upsilon^\dagger$ is surjective. If $\tau \in L^2(\partial\Omega)$ and $\Upsilon^\dagger[\tau] = 0$, then there exists $a \in \mathbb{R}$ such that $\tau = a\tau_\Omega$ and condition $\int_{\partial\Omega} a\tau_\Omega\, d\sigma = 0$ implies that $a = 0$ and $\Upsilon^\dagger$ is injective. Hence $\Upsilon^\dagger$ is a bijection and the Open Mapping Theorem implies that $\Upsilon^\dagger$ is a homeomorphism. □

By Lemma 4.1, $(-\frac{1}{2}I + W^t_\Omega)$ restricts to an isomorphism from $L^2(\partial\Omega)_0$ onto itself. Then we can prove the following.

Proposition 4.2. *Let $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. The set*

$$G_1(\Omega) \equiv \Big\{ (\mu,\gamma) \in L^2(\partial\Omega)_0 \times (\mathbb{R}\setminus\{0\}) : \\ \gamma\mu = \Big(-\frac{1}{2}I + W^t_\Omega\Big)^{(-1)} \Big(V_\Omega[\mu] - \fint_{\partial\Omega} V_\Omega[\mu]\, d\sigma \Big) \text{ on } \partial\Omega \Big\}, \tag{4.2}$$

is contained in $C^{0,\alpha}(\partial\Omega)_0 \times (\mathbb{R}\setminus\{0\})$ and the map from $G_1(\Omega)$ to the set of $(u,\lambda) \in C^{1,\alpha}(\mathrm{cl}\,\Omega)\times(\mathbb{R}\setminus\{0\})$ which satisfy the Steklov eigenvalue value problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu_\Omega} = \lambda u \quad \text{on } \partial\Omega, \tag{4.3}$$

which takes (μ,γ) to $(v^+_\Omega[\mu] - \fint_{\partial\Omega} V_\Omega[\mu]\, d\sigma, 1/\gamma)$ is a bijection. In particular, if $(\mu,\gamma) \in G_1(\Omega)$ and $\mu \neq 0$, then $\gamma > 0$.

Proof. We first assume that $(\mu,\gamma) \in G_1(\Omega)$. Then (μ,γ) satisfies equation

$$\Big(-\frac{1}{2}I + W^t_\Omega\Big)[\mu] = \frac{1}{\gamma}\Big(V_\Omega[\mu] - \fint_{\partial\Omega} V_\Omega[\mu]\, d\sigma \Big) \quad \text{on } \partial\Omega, \tag{4.4}$$

which we rewrite as

$$-\frac{1}{2}\mu(x) + \int_{\partial\Omega} \Big[\frac{\partial}{\partial\nu_\Omega(x)}(S_n(x-y)) - \frac{1}{\gamma} S_n(x-y) \Big] \mu(y)\, d\sigma_y = -\frac{1}{\gamma} \fint_{\partial\Omega} V_\Omega[\mu]\, d\sigma \quad \text{for a.a. } x \in \partial\Omega.$$

Since the kernel in brackets has a weak singularity, a classical regularization argument implies that $\mu \in C^0(\partial\Omega)$ (cf. *e.g.*, Folland [10, Prop. 3.13]). Then $V[\mu] \in C^{0,\alpha}(\partial\Omega)$ (cf. *e.g.*, Miranda [26], and paper [7, Thm. 7.2] with Dondi). Since μ satisfies equation (4.4), a classical regularity result implies that $\mu \in C^{0,\alpha}(\partial\Omega)$ (cf. *e.g.*, [21, Thm. 5.1 (i)]). Then the jump relations for the single layer of Theorem 3.1 imply that $(v^+_\Omega[\mu] - \fint_{\partial\Omega} V_\Omega[\mu]\, d\sigma, 1/\gamma)$ belongs to $C^{1,\alpha}(\mathrm{cl}\,\Omega) \times (\mathbb{R}\setminus\{0\})$ and satisfies the Steklov eigenvalue value problem (4.3). In particular, if (μ,γ) belongs to $G_1(\Omega)$, then its image by the map of the statement solves the Steklov eigenvalue problem. If (μ_1,γ_1), (μ_2,γ_2) belong to $G_1(\Omega)$ and if $v^+_\Omega[\mu_1] - \fint_{\partial\Omega} V_\Omega[\mu_1]\, d\sigma = v^+_\Omega[\mu_2] - \fint_{\partial\Omega} V_\Omega[\mu_2]\, d\sigma$, $\gamma_1 = \gamma_2$, then Lemma 3.2(i) implies that $\mu_1 = \mu_2$ and that $\fint_{\partial\Omega} V_\Omega[\mu_1]\, d\sigma = \fint_{\partial\Omega} V_\Omega[\mu_2]\, d\sigma$. Hence, the map of the statement is injective.

We now assume that $(u,\lambda) \in C^{1,\alpha}(\mathrm{cl}\,\Omega) \times (\mathbb{R}\setminus\{0\})$ satisfies the Steklov eigenvalue value problem (4.3). Then Lemma 3.2(i) and the jump relations for the single layer potential ensure that there exists $(\mu,c,\lambda) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R} \times (\mathbb{R}\setminus\{0\})$ such that $u = v^+_\Omega[\mu] + c$ and $-\frac{1}{2}\mu + W^t_\Omega[\mu] = \lambda(V_\Omega[\mu] + c)$. Then we can integrate on $\partial\Omega$ and deduce that $-\frac{1}{2}\int_{\partial\Omega} \mu\, d\sigma + \int_{\partial\Omega} W^t_\Omega[\mu]\, d\sigma = \lambda \int_{\partial\Omega} V_\Omega[\mu]\, d\sigma + \lambda c m_{n-1}(\partial\Omega)$. Since $\int_{\partial\Omega} W^t_\Omega[\mu]\, d\sigma = \int_{\partial\Omega} \mu W_\Omega[1]\, d\sigma = \int_{\partial\Omega} \mu \frac{1}{2}\, d\sigma$ and $\lambda \neq 0$, we deduce that $c = -\fint_{\partial\Omega} V_\Omega[\mu]\, d\sigma$, and that accordingly $(-\frac{1}{2}I + W^t_\Omega)[\mu] = \lambda(V_\Omega[\mu] - \fint_{\partial\Omega} V_\Omega[\mu]\, d\sigma)$ on $\partial\Omega$. Since the integral of the right hand side on $\partial\Omega$ equals 0, the right hand side belongs to $L^2(\partial\Omega)_0$ and thus Lemma 4.1 implies that the right hand side belongs to the image of $(-\frac{1}{2}I + W^t_\Omega)$ and thus we can write $\frac{1}{\lambda}\mu = (-\frac{1}{2}I + W^t_\Omega)^{(-1)}(V_\Omega[\mu] - \fint_{\partial\Omega} V_\Omega[\mu]\, d\sigma)$, and $(\mu, 1/\lambda)$ belongs to $G_1(\Omega)$. In particular, the map of the statement is surjective. □

Proposition 4.2 provides a formulation of our problem in $L^2(\partial\Omega)_0$. To obtain a formulation in $L^2(\partial\Omega)$, we exploit the following decomposition of $L^2(\partial\Omega)$, where $\chi_{\partial\Omega}$ is the constant function equal to 1 on $\partial\Omega$.

Lemma 4.3. *Let $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the map π from $L^2(\partial\Omega)$ onto $L^2(\partial\Omega)_0$ which takes μ to $\pi[\mu] \equiv \mu - \fint_{\partial\Omega} \mu \, d\sigma \chi_{\partial\Omega}$ is a projection onto $L^2(\partial\Omega)_0$ along the subspace of $L^2(\partial\Omega)$ generated by the function $\chi_{\partial\Omega}$. In particular, $L^2(\partial\Omega) = L^2(\partial\Omega)_0 \oplus \langle \chi_{\partial\Omega} \rangle$, where the direct sum is both topological and orthogonal in $L^2(\partial\Omega)$.*

We now consider the inclusion $J_{\partial\Omega}$ of $L^2(\partial\Omega)_0$ into $L^2(\partial\Omega)$, and we show that the map A_Ω defined by

$$A_\Omega[\mu] \equiv J_{\partial\Omega}[\mu] \circ \left(-\frac{1}{2}I + W_\Omega^t\right)^{(-1)} \circ \left(V_\Omega[\pi[\mu]] - \fint_{\partial\Omega} V_\Omega[\pi[\mu]] \, d\sigma\right) \quad \forall \mu \in L^2(\partial\Omega) \tag{4.5}$$

is selfadjoint in $L^2(\partial\Omega)$ endowed by a scalar product, which we introduce in the following proposition.

Proposition 4.4. *Let $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_\Omega$ from $L^2(\partial\Omega)^2$ to $\mathbb{R}$ defined by*

$$\begin{aligned} \langle\langle \mu_1, \mu_2 \rangle\rangle_\Omega \equiv & \int_{\partial\Omega} \left(-\frac{1}{2}I + W_\Omega^t\right)[\pi[\mu_1]] \left(-\frac{1}{2}I + W_\Omega^t\right)[\pi[\mu_2]] \, d\sigma \\ & + \int_{\partial\Omega} \left[\fint_{\partial\Omega} \mu_1 \, d\sigma \chi_{\partial\Omega} \fint_{\partial\Omega} \mu_2 \, d\sigma \chi_{\partial\Omega}\right] d\sigma \quad \forall (\mu_1, \mu_2) \in L^2(\partial\Omega)^2, \end{aligned} \tag{4.6}$$

is continuous and the following statements hold.

(i) *$\langle\langle \cdot, \cdot \rangle\rangle_\Omega$ is a real scalar product on $L^2(\partial\Omega)$.*

(ii) *$\inf_{\mu \in L^2(\partial\Omega)\setminus\{0\}} \frac{\langle\langle \mu, \mu \rangle\rangle_\Omega}{\|\mu\|_{L^2(\partial\Omega)}} > 0$. In particular the norm $\langle\langle \cdot, \cdot \rangle\rangle_\Omega^{1/2}$ which is canonically associated to the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_\Omega$ is equivalent to the usual norm of $L^2(\partial\Omega)$.*

Proof. $\langle\langle \cdot, \cdot \rangle\rangle_\Omega$ is obviously bilinear and symmetric. The continuity of $\langle\langle \cdot, \cdot \rangle\rangle_\Omega$ in $L^2(\partial\Omega)^2$ follows by the continuity of π, by the continuity of $(-\frac{1}{2}I + W_\Omega^t)$, by the continuity of the ordinary scalar product in $L^2(\partial\Omega)$ and of the integral in $L^2(\partial\Omega)$. By Lemma 4.1, $(-\frac{1}{2}I + W_\Omega^t)$ is a linear homeomorphism in $L^2(\partial\Omega)_0$, and thus we have $a \equiv \inf_{\mu \in L^2(\partial\Omega)_0\setminus\{0\}} \frac{\|(-\frac{1}{2}I + W_\Omega^t)[\mu]\|_{L^2(\partial\Omega)}}{\|\mu\|_{L^2(\partial\Omega)}} > 0$. Hence,

$$\begin{aligned} \langle\langle \mu, \mu \rangle\rangle_\Omega &= \left\| \left(-\frac{1}{2}I + W_\Omega^t\right)[\pi[\mu]] \right\|_{L^2(\partial\Omega)}^2 + \left\| \fint_{\partial\Omega} \mu \, d\sigma \chi_{\partial\Omega} \right\|_{L^2(\partial\Omega)}^2 \\ &\geqslant a \|\pi[\mu]\|_{L^2(\partial\Omega)}^2 + \left\| \fint_{\partial\Omega} \mu \, d\sigma \chi_{\partial\Omega} \right\|_{L^2(\partial\Omega)}^2 \\ &\geqslant \min\{a, 1\} \left(\|\pi[\mu]\|_{L^2(\partial\Omega)}^2 + \left\| \fint_{\partial\Omega} \mu \, d\sigma \chi_{\partial\Omega} \right\|_{L^2(\partial\Omega)}^2 \right) \end{aligned}$$

for all $\mu \in L^2(\partial\Omega)$. Since the sum $L^2(\partial\Omega) = L^2(\partial\Omega)_0 \oplus \langle \chi_{\partial\Omega} \rangle$ is topological, the map from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)_0 \times \langle \chi_{\partial\Omega} \rangle$ which takes μ to $(\pi[\mu], (I-\pi)[\mu]) = (\pi[\mu], \fint_{\partial\Omega} \mu \, d\sigma \chi_{\partial\Omega})$ is a continuous isomorphism. Then the Open Mapping Theorem implies the existence of $c > 0$ such that

$$\left(\|\pi[\mu]\|^2_{L^2(\partial\Omega)} + \left\| \fint_{\partial\Omega} \mu \, d\sigma \chi_{\partial\Omega} \right\|^2_{L^2(\partial\Omega)} \right)^{1/2} \geqslant c \|\mu\|_{L^2(\partial\Omega)} \quad \forall \mu \in L^2(\partial\Omega),$$

and thus $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Omega$ is positive definite and accordingly a scalar product. By the same inequality statement (ii) holds true. $\square$

We are now ready to prove the following.

Proposition 4.5. *Let $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the operator A_Ω defined in (4.5) is compact and selfadjoint in the real Hilbert space $(L^2(\partial\Omega), \langle\!\langle \cdot, \cdot \rangle\!\rangle_\Omega)$. Moreover, the map from the set*

$$G_2(\Omega) \equiv \left\{ (\mu, \gamma) \in L^2(\partial\Omega) \times \left(\mathbb{R} \setminus \{0\}\right) : \gamma\mu = A_\Omega[\mu] \right\}, \tag{4.7}$$

to the set $G_1(\Omega)$ defined in (4.2) which takes (μ, γ) to $(\pi[\mu], \gamma)$ is a bijection. In particular, if (μ, γ) belongs to $G_2(\Omega)$ and $\mu \neq 0$, then $\gamma > 0$.

Proof. Since V_Ω has a weakly singular kernel, it defines a compact operator in $L^2(\partial\Omega)$. Since A_Ω is the composition of linear and continuous operators and of a compact operator, it is compact.

We now show that A_Ω is selfadjoint. Let $\mu_1, \mu_2 \in L^2(\partial\Omega)$. Since the image of $J_{\partial\Omega}$ is contained in $L^2(\partial\Omega)_0$, the π-projection equals the identity map on the image of $J_{\partial\Omega}$ and the integral of an element of the image of $J_{\partial\Omega}$ equals zero. Then the second Green identity implies that

$$\begin{aligned}
&\langle\!\langle A_\Omega[\mu_1], \mu_2 \rangle\!\rangle_\Omega \\
&= \int_{\partial\Omega} \left[V_\Omega[\pi[\mu_1]] - \fint_{\partial\Omega} V_\Omega[\pi[\mu_1]] \, d\sigma \right] \left[\left(-\frac{1}{2} I + W^t_\Omega \right) [\pi[\mu_2]] \right] d\sigma \\
&\quad + \fint_{\partial\Omega} A_\Omega[\mu_1] \, d\sigma \fint_{\partial\Omega} \mu_2 \, d\sigma \int_{\partial\Omega} \chi^2_{\partial\Omega} \, d\sigma \\
&= \int_{\partial\Omega} \left[V_\Omega[\pi[\mu_1]] - \fint_{\partial\Omega} V_\Omega[\pi[\mu_1]] \, d\sigma \right] \frac{\partial}{\partial\nu_\Omega} \left(v^+_\Omega[\pi[\mu_2]] - \fint_{\partial\Omega} V_\Omega[\pi[\mu_2]] \, d\sigma \right) d\sigma \\
&= \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} \left(v^+_\Omega[\pi[\mu_1]] - \fint_{\partial\Omega} V_\Omega[\pi[\mu_1]] \, d\sigma \right) \left[V_\Omega[\pi[\mu_2]] - \fint_{\partial\Omega} V_\Omega[\pi[\mu_2]] \, d\sigma \right] d\sigma.
\end{aligned}$$

Now by interchanging the roles of μ_1 and μ_2 in the first two equalities, the right hand side equals $\langle\!\langle A_\Omega[\mu_2], \mu_1 \rangle\!\rangle_\Omega$, and thus A_Ω is selfadjoint. The last part of the statement follows because the membership of (μ, γ) in $G_2(\Omega)$ implies that $\mu \in L^2(\partial\Omega)_0$, and accordingly that $\mu = \pi[\mu]$. $\square$

We conclude this section by showing that if $\mu \in L^2(\partial\Omega)$, then $A_\Omega[\mu]$ is uniquely determined by an implicit relation. Namely, we show the following.

Proposition 4.6. *Let $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in L^2(\partial\Omega)$. Then the problem*

$$\left(-\frac{1}{2}I + W^t_\Omega\right)[\zeta] = V_\Omega\left[\pi[\mu]\right] - \fint_{\partial\Omega} V_\Omega\left[\pi[\mu]\right]\,d\sigma, \qquad \int_{\partial\Omega} \zeta\,d\sigma = 0$$

has a unique solution $\zeta \in L^2(\partial\Omega)$ and $\zeta = A_\Omega[\mu]$.

Proof. Since $\int_{\partial\Omega}(V_\Omega[\pi[\mu]] - \fint_{\partial\Omega} V_\Omega[\pi[\mu]])\,d\sigma = 0$, the statement follows by Lemma 4.1. $\square$

5. A boundary Hilbert space formulation of the Steklov eigenvalue problem in the perforated domain $\Omega(\epsilon)$

We shall consider the following assumptions for some $\alpha \in]0,1[$.

$$\begin{aligned}&\text{Let } \Omega \text{ be a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{1,\alpha}.\\ &\text{Let } \mathbb{R}^n \setminus \mathrm{cl}\,\Omega \text{ be connected. Let } 0 \in \Omega.\end{aligned} \tag{5.1}$$

Now let Ω^i, Ω^o be as in (5.1). Then there exists

$$\epsilon_0 \in]0,1[\text{ such that } \epsilon\,\mathrm{cl}\,\Omega^i \subseteq \Omega^o \quad \forall \epsilon \in [-\epsilon_0, \epsilon_0]. \tag{5.2}$$

A simple topological argument shows that $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\,\mathrm{cl}\,\Omega^i$ is connected, and that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega(\epsilon)$ has exactly the two connected components $\epsilon\Omega^i$ and $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega^o$, and that $\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$. For brevity, we set

$$\nu^i \equiv \nu_{\Omega^i} \qquad \nu^o \equiv \nu_{\Omega^o} \qquad \nu_\epsilon \equiv \nu_{\Omega(\epsilon)}.$$

Obviously, $\nu_\epsilon(x) = -\nu^i(x/\epsilon)\,\mathrm{sgn}(\epsilon)\ \forall x \in \epsilon\partial\Omega^i$, and $\nu_\epsilon(x) = \nu^o(x)\ \forall x \in \partial\Omega^o$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, where $\mathrm{sgn}(\epsilon) = 1$ if $\epsilon > 0$, $\mathrm{sgn}(\epsilon) = -1$ if $\epsilon < 0$. In order to shorten our notation, we set

$$X_{io} \equiv L^2\left(\partial\Omega^i, \mathbb{R}\right) \times L^2\left(\partial\Omega^o, \mathbb{R}\right),$$

and we emphasize that X_{io} has a natural real Hilbert space structure (although not the only one that we will consider). We now convert the Steklov eigenvalue problem (1.2) in the perforated domain $\Omega(\epsilon)$ for $\epsilon > 0$ small enough into an eigenvalue problem for a selfadjoint operator in X_{io} by exploiting the results of the previous section. By Proposition 4.5, the nonzero Steklov eigenvalues of problem (1.2) are precisely the reciprocals of those of the compact selfadjoint operator $A_{\Omega(\epsilon)}$ in the Hilbert space $(L^2(\partial\Omega(\epsilon)), \langle\langle\cdot,\cdot\rangle\rangle_{\Omega(\epsilon)})$. So we now introduce an isomorphism Ψ_ϵ from X_{io} onto $L^2(\partial\Omega(\epsilon))$ and exploit it to define an ϵ-dependent map in X_{io} which corresponds to $A_{\Omega(\epsilon)}$ and an ϵ-dependent scalar product on X_{io} which corresponds to $\langle\langle\cdot,\cdot\rangle\rangle_{\Omega(\epsilon)}$. To do so, we define Ψ_ϵ to be the isomorphism from X_{io} onto $L^2(\partial\Omega(\epsilon))$ defined by

$$\Psi_\epsilon\left[\theta^i, \theta^o\right](x) \equiv \theta^i(x/\epsilon) \quad \forall x \in \epsilon\Omega^i, \qquad \Psi_\epsilon\left[\theta^i, \theta^o\right](x) \equiv \theta^o(x) \quad \forall x \in \partial\Omega^o,$$

for each $\epsilon \in]0, \epsilon_0[$ (cf. (5.2)). Then the operator

$$\tilde{A}_\epsilon \equiv \Psi_\epsilon^{(-1)} \circ A_{\Omega(\epsilon)} \circ \Psi_\epsilon$$

belongs to $\mathcal{L}(X_{io})$ and can be considered to be the operator in $\mathcal{L}(X_{io})$ which corresponds to $A_{\Omega(\epsilon)}$ and the bilinear map

$$Q_\epsilon\big[(\theta_1^i, \theta_1^o), (\theta_2^i, \theta_2^o)\big] \equiv \big\langle\!\big\langle \Psi_\epsilon\big[\theta_1^i, \theta_1^o\big], \Psi_\epsilon\big[\theta_2^i, \theta_2^o\big]\big\rangle\!\big\rangle_{\Omega(\epsilon)} \quad \forall\big(\theta_1^i, \theta_1^o\big), \big(\theta_2^i, \theta_2^o\big) \in X_{io}, \tag{5.3}$$

is a scalar product in X_{io} which can be considered to be the scalar product in X_{io} which corresponds to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\Omega(\epsilon)}$. Next we note that the equalities

$$\begin{aligned}
\big\|\big(\theta^i, \theta^o\big)\big\|_{Q_\epsilon} &\equiv Q_\epsilon\big[\big(\theta^i, \theta^o\big), \big(\theta^i, \theta^o\big)\big]^{1/2} \\
&= \big\langle\!\big\langle \Psi_\epsilon\big[\theta^i, \theta^o\big], \Psi_\epsilon\big[\theta^i, \theta^o\big]\big\rangle\!\big\rangle_{\Omega(\epsilon)}^{1/2} \equiv \big\|\Psi_\epsilon\big[\theta^i, \theta^o\big]\big\|_{\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\Omega(\epsilon)}^{1/2}}, \\
\big\|\big(\theta^i, \theta^o\big)\big\|_{X_{io}} &\equiv \big(\theta^i, \theta^i\big)_{L^2(\partial\Omega^i)}^{1/2} + \big(\theta^o, \theta^o\big)_{L^2(\partial\Omega^o)}^{1/2} \\
&= \epsilon^{(1-n)/2}\big(\Psi_\epsilon\big[\theta^i, \theta^o\big]_{|\epsilon\partial\Omega^i}, \Psi_\epsilon\big[\theta^i, \theta^o\big]_{|\epsilon\partial\Omega^i}\big)_{L^2(\epsilon\partial\Omega^i)}^{1/2} \\
&\quad + \big(\Psi_\epsilon\big[\theta^i, \theta^o\big]_{|\partial\Omega^o}, \Psi_\epsilon\big[\theta^i, \theta^o\big]_{|\partial\Omega^o}\big)_{L^2(\partial\Omega^o)}^{1/2} \quad \forall\big(\theta^i, \theta^o\big) \in X_{io},
\end{aligned}$$

and the equivalence of the norm

$$\epsilon^{(1-n)/2}\big(\mu_{|\epsilon\partial\Omega^i}, \mu_{|\epsilon\partial\Omega^i}\big)_{L^2(\epsilon\partial\Omega^i)}^{1/2} + \big(\mu_{|\partial\Omega^o}, \mu_{|\partial\Omega^o}\big)_{L^2(\partial\Omega^o)}^{1/2} \quad \forall \mu \in L^2\big(\partial\Omega(\epsilon)\big),$$

and of the usual norm $\|\mu\|_{L^2(\partial\Omega(\epsilon))}$ of $L^2(\partial\Omega(\epsilon))$ and Proposition 4.4(ii) applied to $\Omega = \Omega(\epsilon)$ imply that the norm $\|\cdot\|_{Q_\epsilon}$ is equivalent to the usual norm

$$\big\|\big(\theta^i, \theta^o\big)\big\|_{X_{io}} \equiv \big(\theta^i, \theta^i\big)_{L^2(\partial\Omega^i)}^{1/2} + \big(\theta^o, \theta^o\big)_{L^2(\partial\Omega^o)}^{1/2} \quad \forall\big(\theta^i, \theta^o\big) \in X_{io},$$

of X_{io} for all $\epsilon \in]0, \epsilon_0[$. Then we have the following consequence of Propositions 4.2, 4.5

Proposition 5.1. *Let $\alpha \in]0, 1[$. Let Ω^i, Ω^o be as in (5.1). If $\epsilon \in]0, \epsilon_0[$, then the linear operator $\tilde{A}_\epsilon$ is compact and selfadjoint in the real Hilbert space (X_{io}, Q_ϵ). Moreover, the map from the set*

$$G_3(X_{io}, \epsilon) \equiv \big\{\big(\theta^i, \theta^o, \gamma\big) \in X_{io} \times \big(\mathbb{R} \setminus \{0\}\big) : \gamma\big(\theta^i, \theta^o\big) = \tilde{A}_\epsilon\big(\theta^i, \theta^o\big)\big\}$$

to the set of pairs $(u, \lambda) \in C^{1,\alpha}(\mathrm{cl}\,\Omega(\epsilon)) \times (\mathbb{R} \setminus \{0\})$ which satisfy the Steklov eigenvalue problem (1.2) in $\Omega(\epsilon)$, which takes $(\theta^i, \theta^o, \gamma)$ to $(v_\Omega^+[\pi[\Psi_\epsilon[\theta^i, \theta^o]]] - \fint_{\partial\Omega} V_\Omega[\pi[\Psi_\epsilon[\theta^i, \theta^o]]]\, d\sigma, 1/\gamma)$, is a bijection. In particular, $\gamma \in \mathbb{R} \setminus \{0\}$ is an eigenvalue of multiplicity $r \in \mathbb{N} \setminus \{0\}$ of $\tilde{A}_\epsilon$ if and only if $1/\gamma$ is an eigenvalue of multiplicity r of the Steklov eigenvalue value problem (1.2) and if so, we must necessarily have $\gamma > 0$.

We now devote the next three sections to the behavior of $\{Q_\epsilon\}_{\epsilon \in]0,\epsilon_0[}$ and $\{\tilde{A}_\epsilon\}_{\epsilon \in]0,\epsilon_0[}$ as ϵ is close to 0.

6. Real analytic continuation for the family of scalar products $\{Q_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$

We are now ready to show the existence of an analytic continuation of the family $\{Q_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ to negative values of ϵ. To do so, we write out Q_ϵ explicitly by exploiting formula (5.3).

Theorem 6.1. *Let $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (5.1), $\epsilon_1 \equiv \min\{\epsilon_0, (m_{n-1}(\partial\Omega^o)/m_{n-1}(\partial\Omega^i))^{1/(n-1)}\}$. Let*

$$
\begin{aligned}
m^i\big[\epsilon,\theta^i,\theta^o\big] &\equiv \theta^i_j - \frac{\int_{\partial\Omega^i}\theta^i_j\,d\sigma\epsilon^{n-1} + \int_{\partial\Omega^o}\theta^o_j\,d\sigma}{\epsilon^{n-1}m_{n-1}(\partial\Omega^i)+m_{n-1}(\partial\Omega^o)}\chi_{\partial\Omega^i} \quad \text{on } \partial\Omega^i,\\
m^o\big[\epsilon,\theta^i,\theta^o\big] &\equiv \theta^o_j - \frac{\int_{\partial\Omega^i}\theta^i_j\,d\sigma\epsilon^{n-1} + \int_{\partial\Omega^o}\theta^o_j\,d\sigma}{\epsilon^{n-1}m_{n-1}(\partial\Omega^i)+m_{n-1}(\partial\Omega^o)}\chi_{\partial\Omega^o} \quad \text{on } \partial\Omega^o,
\end{aligned}
\tag{6.1}
$$

for all $(\epsilon,\theta^i,\theta^o)\in]-\epsilon_1,\epsilon_1[\times X_{io}$. If $\epsilon\in]-\epsilon_1,\epsilon_1[$, then the bilinear and symmetric map $\mathcal{Q}_\epsilon$ defined by

$$
\begin{aligned}
&\mathcal{Q}_\epsilon\big[(\theta^i_1,\theta^o_1),(\theta^i_2,\theta^o_2)\big]\\
&\quad\equiv \epsilon^{(n-1)}\int_{\partial\Omega^i}\prod_{j=1,2}\bigg\{-\frac{1}{2}m^i\big[\epsilon,\theta^i_j,\theta^o_j\big](\xi) - W^t_{\Omega^i}\big[m^i\big[\epsilon,\theta^i_j,\theta^o_j\big]\big](\xi)\\
&\qquad - \int_{\partial\Omega^o}\nu^i(\xi)DS_n(\epsilon\xi-y)m^o\big[\epsilon,\theta^i_j,\theta^o_j\big](y)\,d\sigma_y\bigg\}\,d\sigma_\xi\\
&\qquad + \int_{\partial\Omega^o}\prod_{j=1,2}\bigg\{-\frac{1}{2}m^o\big[\epsilon,\theta^i_j,\theta^o_j\big](x) + W^t_{\Omega^o}\big[m^o\big[\epsilon,\theta^i_j,\theta^o_j\big]\big](x)\\
&\qquad + \int_{\partial\Omega^i}\nu^o(x)\cdot DS_n(x-\epsilon\eta)m^i\big[\epsilon,\theta^i_j,\theta^o_j\big](\eta)\,d\sigma_\eta\epsilon^{(n-1)}\bigg\}\,d\sigma_x\\
&\qquad + \big(\epsilon^{n-1}m_{n-1}(\partial\Omega^i)+m_{n-1}(\partial\Omega^o)\big)^{-1}\prod_{j=1,2}\bigg(\int_{\partial\Omega^i}\theta^i_j\,d\sigma\epsilon^{n-1} + \int_{\partial\Omega^o}\theta^o_j\,d\sigma\bigg),
\end{aligned}
\tag{6.2}
$$

for all (θ^i_1,θ^o_1), $(\theta^i_2,\theta^o_2)\in X_{io}$ is continuous. Moreover, the following statements hold.

(i) *$\mathcal{Q}_\epsilon = Q_\epsilon$ for all $\epsilon\in]0,\epsilon_1[$.*
(ii) *$\mathcal{Q}_0[(\theta^i_1,\theta^o_1),(\theta^i_2,\theta^o_2)] = \langle\langle\theta^o_1,\theta^o_2\rangle\rangle_{\Omega^o}$ for all (θ^i_1,θ^o_1), $(\theta^i_2,\theta^o_2)\in X_{io}$. In particular, the continuous symmetric bilinear form $\mathcal{Q}_0$ is positive semidefinite.*
(iii) *The map from $]-\epsilon_1,\epsilon_1[\times X^2_{io}$ to $\mathbb{R}$ which takes $(\epsilon,\theta^i_1,\theta^o_1,\theta^i_2,\theta^o_2)$ to $\mathcal{Q}_\epsilon[(\theta^i_1,\theta^o_1),(\theta^i_2,\theta^o_2)]$, is analytic, and $\{\mathcal{Q}_\epsilon\}_{\epsilon\in]-\epsilon_1,\epsilon_1[}$ is a real analytic family in the space $\mathcal{B}_s(X_{io}\times X_{io},\mathbb{R})$ of real valued symmetric bilinear and continuous maps on X^2_{io}.*

Proof. Since the integral is a linear and continuous functional on $L^2(\partial\Omega^i)$ and $L^2(\partial\Omega^o)$, the operator $m^i[\epsilon,\cdot,\cdot]$ is linear and continuous from X^2_{io} to $L^2(\partial\Omega^i)$ and the operator $m^o[\epsilon,\cdot,\cdot]$ is linear and continuous from X^2_{io} to $L^2(\partial\Omega^o)$. Since the pointwise product is continuous from $L^2(\partial\Omega^i)^2$ to $L^1(\partial\Omega^i)$ and from $L^2(\partial\Omega^o)^2$ to $L^1(\partial\Omega^o)$ and the terms in braces are compositions of linear and continuous operators in $L^2(\partial\Omega^i)$ and in $L^2(\partial\Omega^o)$, we conclude that $\mathcal{Q}_\epsilon$ is bilinear and continuous in X^2_{io}.

We now verify the formula of statement (i). To do so, we compute the pull back of the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\Omega(\epsilon)}$ (cf. (4.6), (5.3)). Clearly,

$$\Psi_\epsilon^{(-1)}[\chi_{\partial\Omega(\epsilon)}] = (\chi_{\partial\Omega^i}, \chi_{\partial\Omega^o}) \tag{6.3}$$

for all $\epsilon \in]0, \epsilon_0[$. Let $j \in \{1, 2\}$. By the rule of change of variables in integrals, we have

$$\int_{\partial\Omega(\epsilon)} \Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\, d\sigma = \bigg(\int_{\partial\Omega^i} \theta^i_j\, d\sigma \epsilon^{n-1} + \int_{\partial\Omega^o} \theta^o_j\, d\sigma\bigg), \tag{6.4}$$

$$m_{n-1}\big(\partial\Omega(\epsilon)\big) = \epsilon^{n-1} m_{n-1}\big(\partial\Omega^i\big) + m_{n-1}\big(\partial\Omega^o\big), \tag{6.5}$$

$$\begin{aligned}
\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](\epsilon\xi) &= \Psi_\epsilon\big[\theta^i_j, \theta^o_j\big](\epsilon\xi) - \fint_{\partial\Omega(\epsilon)} \Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\, d\sigma \chi_{\partial\Omega(\epsilon)}(\epsilon\xi) \\
&= \theta^i_j(\xi) - \frac{\int_{\partial\Omega^i} \theta^i_j\, d\sigma \epsilon^{n-1} + \int_{\partial\Omega^o} \theta^o_j\, d\sigma}{\epsilon^{n-1} m_{n-1}(\partial\Omega^i) + m_{n-1}(\partial\Omega^o)} \chi_{\partial\Omega^i}(\xi) \\
&= m^i\big[\epsilon, \theta^i_j, \theta^o_j\big](\xi) \quad \forall \xi \in \partial\Omega^i,
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](x) &= \Psi_\epsilon\big[\theta^i_j, \theta^o_j\big](x) - \fint_{\partial\Omega(\epsilon)} \Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\, d\sigma \chi_{\partial\Omega(\epsilon)}(x) \\
&= \theta^o_j(x) - \frac{\int_{\partial\Omega^i} \theta^i_j\, d\sigma \epsilon^{n-1} + \int_{\partial\Omega^o} \theta^o_j\, d\sigma}{\epsilon^{n-1} m_{n-1}(\partial\Omega^i) + m_{n-1}(\partial\Omega^o)} \chi_{\partial\Omega^o}(x) \\
&= m^o\big[\epsilon, \theta^i_j, \theta^o_j\big](x) \quad \forall x \in \partial\Omega^o,
\end{aligned} \tag{6.7}$$

for all $\epsilon \in]0, \epsilon_1[$, and

$$\begin{aligned}
W^t_{\Omega(\epsilon)}\big[\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big]\big](\epsilon\xi) &= \int_{\epsilon\partial\Omega^i} \nu_{\Omega(\epsilon)}(\epsilon\xi) \cdot DS_n(\epsilon\xi - y)\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](y)\, d\sigma_y \\
&\quad + \int_{\partial\Omega^o} \nu_{\Omega(\epsilon)}(\epsilon\xi) \cdot DS_n(\epsilon\xi - y)\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](y)\, d\sigma_y \\
&= -\int_{\partial\Omega^i} \nu^i(\xi) \cdot DS_n(\xi - \eta)\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](\epsilon\eta)\, d\sigma_\eta \\
&\quad - \int_{\partial\Omega^o} \nu^i(\xi) \cdot DS_n(\epsilon\xi - y)\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](y)\, d\sigma_y
\end{aligned} \tag{6.8}$$

for all $\xi \in \partial\Omega^i$ and for all $\epsilon \in]0, \epsilon_1[$, and

$$\begin{aligned}
&W^t_{\Omega(\epsilon)}\big[\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big]\big](x) \\
&\quad = \int_{\epsilon\partial\Omega^i} \nu_{\Omega(\epsilon)}(x) \cdot DS_n(x - y)\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](y)\, d\sigma_y \\
&\qquad + \int_{\partial\Omega^o} \nu_{\Omega(\epsilon)}(x) \cdot DS_n(x - y)\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](y)\, d\sigma_y
\end{aligned}$$

$$= \int_{\partial\Omega^i} \nu^o(x) DS_n(x - \epsilon\eta)\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](\epsilon\eta)\, d\sigma_\eta \epsilon^{n-1}$$

$$+ \int_{\partial\Omega^o} \nu_{\Omega^o}(x) \cdot DS_n(x - y)\pi\big[\Psi_\epsilon\big[\theta^i_j, \theta^o_j\big]\big](y)\, d\sigma_y \quad \forall x \in \partial\Omega^o, \tag{6.9}$$

for all $\epsilon \in]0, \epsilon_1[$. By combining formulas (4.6), (5.3) and (6.3)–(6.9), and by the rule of change of variables in integrals over $\partial\Omega(\epsilon) = \epsilon\partial\Omega^i \cup \Omega^o$, we deduce the validity of the formula of statement (i).

(ii) By setting $\epsilon = 0$, we obtain

$$m^i\big[0, \theta^i_j, \theta^o_j\big] = \theta^i_j - \fint_{\partial\Omega^o} \theta^o_j\, d\sigma \chi_{\partial\Omega^i}, \qquad m^o\big[0, \theta^i_j, \theta^o_j\big] = \theta^o_j - \fint_{\partial\Omega^o} \theta^o_j\, d\sigma \chi_{\partial\Omega^o},$$

and

$$\mathcal{Q}_0\big[(\theta^i_1, \theta^o_1), (\theta^i_2, \theta^o_2)\big] = \int_{\partial\Omega^o} \prod_{j=1,2} \Bigg\{ -\frac{1}{2}\Bigg[\theta^o_j - \fint_{\partial\Omega^o} \theta^o_j\, d\sigma \chi_{\partial\Omega^o}\Bigg]$$

$$+ W^t_{\Omega^o}\Bigg[\theta^o_j - \fint_{\partial\Omega^o} \theta^o_j\, d\sigma \chi_{\partial\Omega^o}\Bigg]\Bigg\}\, d\sigma + \int_{\partial\Omega^o} \chi^2_{\partial\Omega^o}\, d\sigma \prod_{j=1,2} \Bigg(\fint_{\partial\Omega^o} \theta^o_j\, d\sigma\Bigg),$$

and the right hand side equals $\langle\langle \theta^o_1, \theta^o_2 \rangle\rangle_{\Omega^o}$ (cf. (4.6)).

(iii) Next we show that $\mathcal{Q}_\epsilon[(\theta^i_1, \theta^o_1), (\theta^i_2, \theta^o_2)]$ is analytic in $(\epsilon, \theta^i_1, \theta^o_1, \theta^i_2, \theta^o_2) \in]-\epsilon_1, \epsilon_1[\times X^2_{io}$. Since the integral is a linear and continuous functional on $L^2(\partial\Omega^i)$ and on $L^2(\partial\Omega^o)$, the last addendum in the right hand side of the definition (6.2) of $\mathcal{Q}_\epsilon[(\theta^i_1, \theta^o_1), (\theta^i_2, \theta^o_2)]$ is analytic in $]-\epsilon_1, \epsilon_1[\times X^2_{io}$. Since the pointwise product is continuous from $L^2(\partial\Omega^i)^2$ to $L^1(\partial\Omega^i)$ and from $L^2(\partial\Omega^o)^2$ to $L^1(\partial\Omega^o)$, it suffices to show that the terms in braces in the right hand side of (6.2) define real analytic maps from $]-\epsilon_1, \epsilon_1[\times X^2_{io}$ to $L^2(\partial\Omega^i)$ in case of the first addendum and to $L^2(\partial\Omega^o)$ in case of the second addendum. We consider the first term in braces. Since the integral is a linear and continuous functional, the maps m^i and m^o are real analytic in $]-\epsilon_1, \epsilon_1[\times X^2_{io}$. Since $W^t_{\Omega^i}$ is linear and continuous in $L^2(\partial\Omega^i)$, we conclude that $-\frac{1}{2}m^i[\epsilon, \theta^i_j, \theta^o_j] - W^t_{\Omega^i}[m^i[\epsilon, \theta^i_j, \theta^o_j]]$ is analytic from $]-\epsilon_1, \epsilon_1[\times X^2_{io}$ to $L^2(\partial\Omega^i)$. By an analyticity result for integral operators with real analytic kernel, the map from $]-\epsilon_1, \epsilon_1[\times L^1(\partial\Omega^o)$ to $C^1(\partial\Omega^i)$ which takes (ϵ, f^o) to the function $\int_{\partial\Omega^o} \nu^i(\xi) \cdot DS_n(\epsilon\xi - y) f^o(y)\, d\sigma_y$ of the variable $\xi \in \partial\Omega^i$ is analytic (cf. [22, Prop. 4.1 (ii)]). Then the map from $]-\epsilon_1, \epsilon_1[\times X^2_{io}$ to $L^2(\partial\Omega^i)$ which takes $(\epsilon, \theta^i_1, \theta^o_1, \theta^i_2, \theta^o_2)$ to the function $\int_{\partial\Omega^o} \nu^i(\xi) \cdot DS_n(\epsilon\xi - y) m^o[\epsilon, \theta^i_j, \theta^o_j](y)\, d\sigma_y$ of the variable $\xi \in \partial\Omega^i$ is analytic for $j \in \{1, 2\}$. Hence, the first term in braces in the right hand side of (6.2) defines a real analytic map from $]-\epsilon_1, \epsilon_1[\times X^2_{io}$ to $L^2(\partial\Omega^i)$. The proof for the second term in braces is similar. Hence the proof of the analyticity of $\mathcal{Q}_\epsilon[(\theta^i_1, \theta^o_1), (\theta^i_2, \theta^o_2)]$ in the variable $(\epsilon, \theta^i_1, \theta^o_1, \theta^i_2, \theta^o_2) \in]-\epsilon_1, \epsilon_1[\times X^2_{io}$ is complete.

Next we prove that the map from $]-\epsilon_1, \epsilon_1[\times X^2_{io}$ to $\mathcal{B}_s(X_{io} \times X_{io}, \mathbb{R})$, which takes ϵ to the bilinear map $\mathcal{Q}_\epsilon$ is analytic. By the formula for the second order differential of the monomial associated to the symmetric bilinear and continuous map $\mathcal{Q}_\epsilon$, we have $\partial^2_{(\theta^i, \theta^o)} \mathcal{Q}_\epsilon[(\theta^i, \theta^o), (\theta^i, \theta^o)][\cdot, \cdot] = 2!\mathcal{Q}_\epsilon[\cdot, \cdot]$ for all $(\theta^i, \theta^o) \in X_{io}$, for all $\epsilon \in]-\epsilon_1, \epsilon_1[$ (cf. *e.g.*, Prodi and Ambrosetti [32, (10.1)]). By the proof above, the map $\mathcal{Q}_\epsilon[(\theta^i, \theta^o), (\theta^i, \theta^o)]$ in the variable $(\epsilon, \theta^i, \theta^o) \in]-\epsilon_1, \epsilon_1[\times X_{io}$ is analytic, and accordingly its second order partial differential $\partial^2_{(\theta^i, \theta^o)} \mathcal{Q}_\epsilon[(\theta^i, \theta^o), (\theta^i, \theta^o)]$ is analytic from $]-\epsilon_1, \epsilon_1[\times X_{io}$ to $\mathcal{B}_s(X_{io} \times X_{io}, \mathbb{R})$. Hence, the map from $]-\epsilon_1, \epsilon_1[\times X_{io}$ to $\mathcal{B}_s(X_{io} \times X_{io}, \mathbb{R})$, which takes $(\epsilon, \theta^i, \theta^o)$ to $\mathcal{Q}_\epsilon[\cdot, \cdot]$ is analytic.

Since such a map is constant with respect to (θ^i, θ^o), the map from $]-\epsilon_1, \epsilon_1[$ to $\mathcal{B}_s(X_{io} \times X_{io}, \mathbb{R})$, which takes ϵ to $\mathcal{Q}_\epsilon[\cdot, \cdot]$ is analytic and the proof is complete. $\square$

7. An implicit definition of the operators of the family $\{\tilde{A}_\epsilon\}_{\epsilon \in]0,\epsilon_0[}$

The proof of a real analytic representation formula for the family $\{\tilde{A}_\epsilon\}_{\epsilon \in]0,\epsilon_0[}$ is based upon an implicit relation satisfied by the operators $\tilde{A}_\epsilon$ and which we derive by pulling back on X_{io} the characterization of $A_{\Omega(\epsilon)}$ of Proposition 4.6. We do so in the following statement.

Proposition 7.1. *Let $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (5.1). Let ϵ_1 be as in Theorem 6.1. Let $\epsilon \in]0, \epsilon_1[$.*

If $(\theta^i, \theta^o) \in X_{io}$, then the pair $(\psi^i, \psi^o) \equiv \tilde{A}_\epsilon[\theta^i, \theta^o]$ is the only solution in X_{io} of the system of the following three equations

$$
\begin{aligned}
&-\frac{1}{2}\psi^i(\xi) - W^t_{\Omega^i}\left[\psi^i\right](\xi) - \int_{\partial\Omega^o} \nu^i(\xi) \cdot DS_n(\epsilon\xi - y)\psi^o(y)\, d\sigma_y \\
&= \epsilon V_{\Omega^i}\left[m^i\left[\epsilon, \theta^i, \theta^o\right]\right](\xi) + \frac{(\delta_{2,n}\epsilon \log \epsilon)}{2\pi} \int_{\partial\Omega^i} m^i\left[\epsilon, \theta^i, \theta^o\right] d\sigma \\
&\quad + \int_{\partial\Omega^o} S_n(\epsilon\xi - y) m^o\left[\epsilon, \theta^i, \theta^o\right](y)\, d\sigma_y \\
&\quad - \frac{1}{m_{n-1}(\partial\Omega^i)\epsilon^{n-1} + m_{n-1}(\partial\Omega^o)} \int_{\partial\Omega^i} \Big\{ \epsilon V_{\Omega^i}\left[m^i\left[\epsilon, \theta^i, \theta^o\right]\right](\xi) \\
&\quad + \frac{(\delta_{2,n}\epsilon \log \epsilon)}{2\pi} \int_{\partial\Omega^i} m^i\left[\epsilon, \theta^i, \theta^o\right] d\sigma \\
&\quad + \int_{\partial\Omega^o} S_n(\epsilon\xi - y) m^o\left[\epsilon, \theta^i, \theta^o\right](y)\, d\sigma_y \Big\} d\sigma \epsilon^{n-1} \\
&\quad - \frac{1}{m_{n-1}(\partial\Omega^i)\epsilon^{n-1} + m_{n-1}(\partial\Omega^o)} \int_{\partial\Omega^o} \Big\{ \int_{\partial\Omega^i} S_n(x - \epsilon\eta) m^i\left[\epsilon, \theta^i, \theta^o\right](\eta)\, d\sigma_\eta \epsilon^{n-1} \\
&\quad + V_{\Omega^o}\left[m^o\left[\epsilon, \theta^i, \theta^o\right]\right](y)\, d\sigma_y \Big\} d\sigma_x \quad \forall \xi \in \partial\Omega^i,
\end{aligned} \tag{7.1}
$$

$$
\begin{aligned}
&-\frac{1}{2}\psi^o(x) + W^t_{\Omega^o}\left[\psi^o\right](x) + \int_{\partial\Omega^i} \nu^o(x) \cdot DS_n(x - \epsilon\eta)\psi^i(\eta)\, d\sigma_\eta \epsilon^{n-1} \\
&= \Big\{ \int_{\partial\Omega^i} S_n(x - \epsilon\eta) m^i\left[\epsilon, \theta^i, \theta^o\right](\eta)\, d\sigma_\eta \epsilon^{n-1} + V_{\Omega^o}\left[m^o\left[\epsilon, \theta^i, \theta^o\right]\right](y)\, d\sigma_y \Big\} \\
&\quad - \frac{1}{m_{n-1}(\partial\Omega^i)\epsilon^{n-1} + m_{n-1}(\partial\Omega^o)} \int_{\partial\Omega^i} \Big\{ \epsilon V_{\Omega^i}\left[m^i\left[\epsilon, \theta^i, \theta^o\right]\right](\xi) \\
&\quad + \frac{(\delta_{2,n}\epsilon \log \epsilon)}{2\pi} \int_{\partial\Omega^i} m^i\left[\epsilon, \theta^i, \theta^o\right] d\sigma + \int_{\partial\Omega^o} S_n(\epsilon\xi - y) m^o\left[\epsilon, \theta^i, \theta^o\right](y)\, d\sigma_y \Big\} d\sigma_\xi \epsilon^{n-1}
\end{aligned}
$$

$$- \frac{1}{m_{n-1}(\partial\Omega^i)\epsilon^{n-1} + m_{n-1}(\partial\Omega^o)} \int_{\partial\Omega^o} \left\{ \int_{\partial\Omega^i} S_n(x - \epsilon\eta) m^i\left[\epsilon, \theta^i, \theta^o\right](\eta)\, d\sigma_\eta \epsilon^{n-1} \right.$$
$$\left. + V_{\Omega^o}\left[m^o\left[\epsilon, \theta^i, \theta^o\right]\right](y)\, d\sigma_y \right\} d\sigma_x \quad \forall x \in \partial\Omega^o, \tag{7.2}$$

$$\int_{\partial\Omega^i} \psi^i\, d\sigma \epsilon^{n-1} + \int_{\partial\Omega^o} \psi^o\, d\sigma = 0. \tag{7.3}$$

Proof. By Proposition 4.6, $\zeta_\epsilon \equiv \Psi_\epsilon[\psi^i, \psi^o] = \Psi_\epsilon \circ \tilde{A}_\epsilon[\theta^i, \theta^o] = A_{\Omega(\epsilon)}[\Psi_\epsilon[\theta^i, \theta^o]]$, is characterized to be the only solution of the following problem

$$\begin{aligned} &\left(-\frac{1}{2} I + W^t_{\Omega(\epsilon)}\right)[\zeta_\epsilon] = V_{\Omega(\epsilon)}\left[\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right]\right] - \fint_{\partial\Omega(\epsilon)} V_{\Omega(\epsilon)}\left[\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right]\right], \\ &\int_{\partial\Omega(\epsilon)} \zeta_\epsilon\, d\sigma = 0. \end{aligned} \tag{7.4}$$

We have already computed the terms in the left hand side in terms of ψ^i, ψ^o (see the computations of (6.4), (6.8), (6.9) where we can replace (θ^i_j, θ^o_j) by (ψ^i, ψ^o) and $\pi[\Psi_\epsilon[\cdot, \cdot]]$ by $\Psi_\epsilon[\cdot, \cdot]$). By formulas (6.6), (6.7), we have

$$\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](\epsilon\xi) = m^i\left[\epsilon, \theta^i, \theta^o\right](\xi) \quad \forall \xi \in \partial\Omega^i,$$
$$\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](x) = m^o\left[\epsilon, \theta^i, \theta^o\right](x) \quad \forall x \in \partial\Omega^o.$$

Hence,

$$\begin{aligned} V_{\Omega(\epsilon)}\left[\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right]\right](\epsilon\xi) &= \int_{\partial\Omega^i} S_n(\epsilon\xi - \epsilon\eta)\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](\epsilon\eta)\, d\sigma_\eta \epsilon^{n-1} \\ &\quad + \int_{\partial\Omega^o} S_n(\epsilon\xi - y)\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](y)\, d\sigma_y \\ &= \epsilon \int_{\partial\Omega^i} S_n(\xi - \eta)\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](\epsilon\eta)\, d\sigma_\eta \\ &\quad + \frac{\delta_{2,n}\epsilon \log \epsilon}{2\pi} \int_{\partial\Omega^i} \pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](\epsilon\eta)\, d\sigma_\eta \\ &\quad + \int_{\partial\Omega^o} S_n(\epsilon\xi - y)\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](y)\, d\sigma_y \quad \forall \xi \in \partial\Omega^i, \\ V_{\Omega(\epsilon)}\left[\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right]\right](x) &= \int_{\partial\Omega^i} S_n(x - \epsilon\eta)\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](\epsilon\eta)\, d\sigma_\eta \epsilon^{n-1} \\ &\quad + \int_{\partial\Omega^o} S_n(x - y)\pi\left[\Psi_\epsilon\left[\theta^i, \theta^o\right]\right](y)\, d\sigma_y \quad \forall x \in \partial\Omega^o. \end{aligned}$$

Then we can compute $W^t_{\Omega(\epsilon)}[\zeta_\epsilon](\epsilon\xi) = W^t_{\Omega(\epsilon)}[\Psi_\epsilon[\psi^i, \psi^o]](\epsilon\xi)$ for $\xi \in \partial\Omega^i$ and $W^t_{\Omega(\epsilon)}[\zeta_\epsilon](x) = W^t_{\Omega(\epsilon)}[\Psi_\epsilon[\psi^i, \psi^o]](x)$ for $x \in \partial\Omega^o$ by exploiting the rule of change of variables in integrals, and we can invoke formula (6.4) in which we replace (θ^i_j, θ^o_j) by (ψ^i, ψ^o) and rewrite system (7.4) in the form of the system of the three equations (7.1)–(7.3). $\square$

Next we note that by letting ϵ tend to 0 in (7.1)–(7.3), we obtain the following 'limiting system'

$$
\begin{aligned}
&-\frac{1}{2}\psi^i(\xi) - W^t_{\Omega^i}[\psi^i](\xi) + \nu^i(\xi)\cdot\int_{\partial\Omega^o} DS_n(y)\psi^o(y)\,d\sigma_y \\
&\quad = \int_{\partial\Omega^o} S_n(-y)\left[\theta^o(y) - \fint_{\partial\Omega^o}\theta^o\,d\sigma\chi_{\partial\Omega^o}(y)\right]d\sigma_y \\
&\qquad - \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o}\theta^o\,d\sigma\chi_{\partial\Omega^o}\right]d\sigma \quad \forall\xi\in\partial\Omega^i, \\
&-\frac{1}{2}\psi^o(x) + W^t_{\Omega^o}[\psi^o](x) \\
&\quad = V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o}\theta^o\,d\sigma\chi_{\partial\Omega^o}\right](x) - \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o}\theta^o\,d\sigma\chi_{\partial\Omega^o}\right]d\sigma \quad \forall x\in\partial\Omega^o, \\
&\int_{\partial\Omega^o}\psi^o\,d\sigma = 0.
\end{aligned}
\tag{7.5}
$$

Now assume that (ψ^i, ψ^o), $(\theta^i, \theta^o) \in X_{io}$ satisfy the limiting system. Then Proposition 4.6 with $\Omega = \Omega^o$ and the last two equations of the limiting system imply that

$$\psi^o = A_{\Omega^o}[\theta^o] \tag{7.6}$$

(cf. (4.5)). Since the exterior Ω^{i-} is connected, Lemma 3.3 implies that $\frac{1}{2}I + W^t_{\Omega^i}$ is an isomorphism in $L^2(\partial\Omega^i)$ and accordingly, the first equation of the limiting system implies that

$$
\begin{aligned}
\psi^i = -\left(\frac{1}{2}I + W^t_{\Omega^i}\right)^{(-1)}&\Bigg[-\nu^i(\cdot)\cdot\int_{\partial\Omega^o} DS_n(y)A_{\Omega^o}[\theta^o](y)\,d\sigma_y \\
&+ v^+_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o}\theta^o\,d\sigma\chi_{\partial\Omega^o}\right](0) - \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o}\theta^o\,d\sigma\chi_{\partial\Omega^o}\right]d\sigma\Bigg],
\end{aligned}
\tag{7.7}
$$

on $\partial\Omega^i$. Then we introduce the following.

Definition 7.2. Let $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (5.1). We define the limiting operator $\tilde{A}_0 \equiv (\tilde{A}^i_0, \tilde{A}^o_0)$ to be the operator from X_{io} to itself which maps (θ^i, θ^o) of X_{io} to the pair (ψ^i, ψ^o) of X_{io} defined by the right hand sides of (7.6), (7.7).

We note that $\tilde{A}_0$ is independent of its first functional variable θ^i and that the following holds.

Proposition 7.3. *Let $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (5.1). The positive semidefinite bilinear and symmetric form $\mathcal{Q}_0$ is positive definite on the subspace $\operatorname{Im}\tilde{A}_0$ of X_{io}. In particular, $\mathcal{Q}_0$ is positive definite on all the eigenspaces of $\tilde{A}_0$.*

Proof. By Theorem 6.1(ii), $\mathcal{Q}_0$ is positive semidefinite and if (ψ^i,ψ^o) belongs to $\mathrm{Im}\,\tilde{A}_0$, then $\mathcal{Q}_0[(\psi^i,\psi^o),(\psi^i,\psi^o)] = \langle\langle \psi^o,\psi^o\rangle\rangle_{\Omega^o}$. Thus if $\mathcal{Q}_0[(\psi^i,\psi^o),(\psi^i,\psi^o)] = 0$, then we have $\psi^o = 0$. Since $(\psi^i,\psi^o) \in \mathrm{Im}\,\tilde{A}_0$, there exists $(\theta^i,\theta^o) \in X_{io}$ such that $(\psi^i,\psi^o) = \tilde{A}_0[\theta^i,\theta^o]$. Since $0 = \psi^o = A_{\Omega^o}[\theta^o]$, Proposition 4.6 implies that $0 = \psi^o$ and θ^o must satisfy the second and the third equation of the limiting system (7.5) and accordingly

$$V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma\chi_{\partial\Omega^o}\right](x) = \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma\chi_{\partial\Omega^o}\right] d\sigma \quad \forall x \in \partial\Omega^o.$$

In particular, $v^+_{\Omega^o}[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma\chi_{\partial\Omega^o}]$ is constant on $\partial\Omega^o$ and thus the Maximum Principle implies that $v^+_{\Omega^o}[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma\chi_{\partial\Omega^o}]$ is constant on the whole of $\mathrm{cl}\,\Omega^o$. In particular,

$$v^+_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma\chi_{\partial\Omega^o}\right](0) = \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma\chi_{\partial\Omega^o}\right] d\sigma \quad \forall x \in \partial\Omega^o.$$

Since ψ^i satisfies (7.7), we have $\psi^i = 0$ and thus $(\psi^i,\psi^o) = (0,0)$ and $\mathcal{Q}_0$ is positive definite on $\mathrm{Im}\,\tilde{A}_0$. □

Since $\tilde{A}_0$ is entirely determined by its first component A_{Ω^o}, which is a selfadjoint and compact operator in $(L^2(\partial\Omega^o), \langle\langle\cdot,\cdot\rangle\rangle_{\Omega^o})$ whose nonzero eigenvalues coincide precisely with the reciprocals of those of the Steklov problem (1.1) in Ω^o, we can prove the following, which corresponds to Proposition 5.1 in case $\epsilon = 0$.

Proposition 7.4. *Let $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (5.1). Then the linear operator $\tilde{A}_0$ is compact and the following equality holds $\mathcal{Q}_0[\tilde{A}_0[\theta^i_1,\theta^o_1],(\theta^i_2,\theta^o_2)] = \mathcal{Q}_0[(\theta^i_1,\theta^o_1),\tilde{A}_0[\theta^i_2,\theta^o_2]]$ for all (θ^i_1,θ^o_1), $(\theta^i_2,\theta^o_2) \in X_{io}$. Moreover, the map Ξ_0 from the set $G_2(\Omega^o)$ defined in (4.7) with $\Omega = \Omega^o$ to the set*

$$G_3(X_{io},0) \equiv \left\{(\theta^i,\theta^o,\gamma) \in X_{io} \times (\mathbb{R}\setminus\{0\}) : \gamma(\theta^i,\theta^o) = \tilde{A}_0[\theta^i,\theta^o]\right\},$$

which takes (θ^o,γ) to $(\theta^i,\theta^o,\gamma)$, where

$$\theta^i = -\left(\frac{1}{2}I + W^t_{\Omega^i}\right)^{(-1)}\Bigg[-\nu^i(\cdot)\cdot\int_{\partial\Omega^o} DS_n(y)\theta^o(y)\, d\sigma_y + \frac{1}{\gamma}v^+_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma\chi_{\partial\Omega^o}\right](0) - \frac{1}{\gamma}\fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma\chi_{\partial\Omega^o}\right] d\sigma\Bigg], \tag{7.8}$$

on $\partial\Omega^i$, is a bijection. In particular, if $(\theta^i,\theta^o,\gamma) \in G_3(X_{io},0)$ and $(\theta^i,\theta^o) \neq (0,0)$, then $\gamma > 0$. Finally, if $\gamma \in \mathbb{R}\setminus\{0\}$, then the map $\Xi_0(\cdot,\gamma)$ is an isomorphism from the space $\{\theta^o \in L^2(\partial\Omega^o) : (\theta^o,\gamma) \in G_3(\Omega^o)\}$ onto the space $\{(\theta^i,\theta^o) \in X_{io} : (\theta^i,\theta^o,\gamma) \in G_3(X_{io},0)\}$.

Proof. Since the integral operator associated to a single layer potential has a weakly singular kernel, it is compact. Then the operator $\tilde{A}^i_0$ delivered by the right hand side of (7.7) equals the composition of the bounded operator $(\frac{1}{2}I + W^t_{\Omega^i})^{(-1)}$ and of a compact operator and is accordingly compact. Then we already know that $\tilde{A}^o_0$ is compact and we can conclude that $\tilde{A}_0 = (\tilde{A}^i_0, \tilde{A}^o_0)$ is compact. By Theorem 6.1(ii) and by the Definition 7.2 of $\tilde{A}_0$, we have $\mathcal{Q}_0[\tilde{A}_0[\theta^i_1,\theta^o_1],(\theta^i_2,\theta^o_2)] = \langle\langle A_{\Omega^o}[\theta^o_1],\theta^o_2\rangle\rangle_{\Omega^o}$. Since A_{Ω^o} is selfadjoint, we have $\langle\langle A_{\Omega^o}[\theta^o_1],\theta^o_2\rangle\rangle_{\Omega^o} = \langle\langle \theta^o_1, A_{\Omega^o}[\theta^o_2]\rangle\rangle_{\Omega^o}$, and by switching the roles of (θ^i_1,θ^o_1) and (θ^i_2,θ^o_2), we conclude that $\langle\langle \theta_1^o, A_{\Omega^o}[\theta_2^o]\rangle\rangle_{\Omega^o} = \mathcal{Q}_0[(\theta_1^i,\theta_1^o), \tilde{A}_0[\theta_2^i,\theta_2^o]]$. By Definition 7.2 of $\tilde{A}_0$, we have $(\theta^i,\theta^o,\gamma) \in G_3(X_{io},0)$ if and only if $\gamma\theta^o = \tilde{A}_0^o[\theta^i,\theta^o] = A_{\Omega^o}[\theta^o]$, *i.e.*, $(\theta^o,\gamma) \in G_2(\Omega^o)$, and

$$\gamma\theta^i = -\left(\frac{1}{2}I + W^t_{\Omega^i}\right)^{(-1)}\Bigg[-\nu^i(\cdot)\cdot\int_{\partial\Omega^o} DS_n(y)A_{\Omega^o}\big[\theta^o\big](y)\,d\sigma_y$$
$$+ v^+_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o}\theta^o\,d\sigma\chi_{\partial\Omega^o}\right](0) - \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o}\theta^o\,d\sigma\chi_{\partial\Omega^o}\right]d\sigma\Bigg].$$

Hence, the second part of the statement holds. Then the last part follows by the linearity of $\Xi_0(\cdot,\gamma)$. □

8. A real analytic representation formula for the family $\{\tilde{A}_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$

We now turn to show a real analytic representation formula for the family $\{\tilde{A}_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ in a sense which we clarify below. The starting point is that $\{\tilde{A}_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ is implicitly defined by equations (7.1)–(7.3). Then we observe that if we replace the term $(\delta_{2,n}\epsilon\log\epsilon)$ in equations (7.1)–(7.3) by a new variable ι, we obtain a system of equations whose terms are analytic in all functional variables and in ϵ, ι and that can be analyzed by an appropriate application of the Implicit Function Theorem in Banach spaces.

Theorem 8.1. *Let $\alpha\in]0,1[$. Let Ω^i, Ω^o be as in (5.1). Let ϵ_1 be as in Theorem 6.1. Then the following statements hold.*

(i) *There exists $\epsilon_2\in]0,\epsilon_1[$, $\iota_2\in]0,+\infty[$ and a real analytic family*

$$\{\mathcal{A}_{(\epsilon,\iota)}\}_{(\epsilon,\iota)\in]-\epsilon_2,\epsilon_2[\times]-\iota_2,\iota_2[} \tag{8.1}$$

in $\mathcal{L}(X_{io})$ such that $\delta_{2,n}\epsilon\log\epsilon\in]-\iota_2,\iota_2[$ and $\mathcal{A}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)} = \tilde{A}_\epsilon$ for all $\epsilon\in]0,\epsilon_2[$.

(ii) *$\mathcal{A}_{(0,0)} = \tilde{A}_0$ (cf. Definition 7.2).*

Proof. By Proposition 7.1, $\tilde{A}_\epsilon[\theta^i,\theta^o]$ is implicitly defined by system (7.1)–(7.3) for all (θ^i,θ^o) in X_{io}. Thus we now recast system (7.1)–(7.3) into an abstract functional equation in Banach space, which we analyze by means of the Implicit Function Theorem.

To do so, we introduce a map $\mathcal{F}\equiv(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$ from $]-\epsilon_1,\epsilon_1[\times X_{io}^2$ to $X_{io}\times\mathbb{R}$ as follows.

We define $\mathcal{F}_1(\epsilon,\iota,\theta^i,\theta^o,\psi^i,\psi^o)$ to be equal to the difference between the left hand side and the right hand side of equation (7.1), once we have replaced the term $(\delta_{2,n}\epsilon\log\epsilon)$ which appears in the right hand side by ι, for all $(\epsilon,\iota,\theta^i,\theta^o,\psi^i,\psi^o)\in]-\epsilon_1,\epsilon_1[\times\mathbb{R}\times X_{io}^2$.

We define $\mathcal{F}_2(\epsilon,\iota,\theta^i,\theta^o,\psi^i,\psi^o)$ to be equal to the difference between the left hand side and the right hand side of equation (7.2), once we have replaced the term $(\delta_{2,n}\epsilon\log\epsilon)$ which appears in the right hand side by ι, for all $(\epsilon,\iota,\theta^i,\theta^o,\psi^i,\psi^o)\in]-\epsilon_1,\epsilon_1[\times X_{io}^2$.

We define $\mathcal{F}_3(\epsilon,\iota,\theta^i,\theta^o,\psi^i,\psi^o)$ to be equal to the left hand side of equation (7.3), for all $(\epsilon,\iota,\theta^i,\theta^o,\psi^i,\psi^o)\in]-\epsilon_1,\epsilon_1[\times X_{io}^2$.

By definition of $\mathcal{F}$, if $\epsilon\in]0,\epsilon_1[$, then we can rewrite the system of equations (7.1)–(7.3) as

$$\mathcal{F}\big(\epsilon,\delta_{2,n}\epsilon\log\epsilon,\theta^i,\theta^o,\psi^i,\psi^o\big) = 0, \tag{8.2}$$

and $(\psi^i,\psi^o) \equiv \tilde{A}_\epsilon[\theta^i,\theta^o]$ is the unique solution of such equation whenever $(\theta^i,\theta^o) \in X_{io}$. So equation (8.2) characterizes $\tilde{A}_\epsilon[\theta^i,\theta^o]$ for each $(\theta^i,\theta^o) \in X_{io}$. We now wish to introduce an operator equation which characterizes the operator $\tilde{A}_\epsilon[\cdot,\cdot]$. Thus we introduce the operator valued map $\tilde{\mathcal{F}}$ from the set $]-\epsilon_1,\epsilon_1] \times \mathbb{R} \times \mathcal{L}(X_{io})$ to $\mathcal{L}(X_{io}, X_{io}\times\mathbb{R})$ which takes (ϵ,ι,Z) with $Z \equiv (Z_1,Z_2)$ to the operator $\tilde{\mathcal{F}}[\epsilon,\iota,Z](\cdot,\cdot)$ defined by the following formula

$$\tilde{\mathcal{F}}[\epsilon,\iota,Z]\big(\theta^i,\theta^o\big) \equiv \mathcal{F}\big(\epsilon,\iota,\theta^i,\theta^o,Z_1\big(\theta^i,\theta^o\big),Z_2\big(\theta^i,\theta^o\big)\big) \quad \forall\big(\theta^i,\theta^o\big) \in X_{io}.$$

Since the integral is a linear and continuous functional on $L^2(\partial\Omega^i)$ and on $L^2(\partial\Omega^o)$, we conclude that $m^i[\epsilon,\psi^i,\psi^o]$, $m^o[\epsilon,\psi^i,\psi^o]$ depend analytically on (ϵ,ψ^i,ψ^o) (cf. (6.1)). Similarly, $m^i[\epsilon,\theta^i,\theta^o]$ and $m^o[\epsilon,\theta^i,\theta^o]$ depend analytically on $(\epsilon,\theta^i,\theta^o)$. Then we know that the linear operators $\frac{1}{2}I + W^t_{\Omega^i}$, V_{Ω^i} and $-\frac{1}{2}I+W^t_{\Omega^o}$, V_{Ω^o} are linear and continuous in $L^2(\partial\Omega^i)$ and in $L^2(\partial\Omega^o)$, respectively. Moreover, an analyticity result for integral operators with real analytic kernel implies that the maps from $]-\epsilon_1,\epsilon_1[\,\times L^2(\partial\Omega^o)$ to $L^2(\partial\Omega^i)$ which take (ϵ,f^o) to the function $S_{1,\epsilon}[f^o](\cdot) \equiv \int_{\partial\Omega^o} \nu^i(\cdot) DS_n(\epsilon\cdot - y) f^o(y)\, d\sigma_y$ and (ϵ,f^o) to the function $S_{2,\epsilon}[f^o](\cdot) \equiv \int_{\partial\Omega^o} S_n(\epsilon\cdot - y) f^o(y)\, d\sigma_y$, and the maps from $]-\epsilon_1,\epsilon_1[\,\times L^2(\partial\Omega^i)$ to $L^2(\partial\Omega^o)$ which take (ϵ,f^i) to the function $S_{3,\epsilon}[f^i](\cdot) \equiv \int_{\partial\Omega^i} \nu^o(\cdot) DS_n(\cdot - \epsilon\eta) f^i(\eta)\, d\sigma_\eta$, and (ϵ,f^i) to the function $S_{4,\epsilon}[f^i](\cdot) \equiv \int_{\partial\Omega^i} S_n(\cdot - \epsilon\eta) f^i(\eta)\, d\sigma_\eta$, are analytic (cf. [22, Prop. 4.1(ii)]). As a consequence, the corresponding Fréchet derivatives with respect to the second variable are analytic and thus the maps from $]-\epsilon_1,\epsilon_1[$ to $\mathcal{L}(L^2(\partial\Omega^o),L^2(\partial\Omega^i))$ which take ϵ to $S_{1,\epsilon}[\cdot]$, $S_{2,\epsilon}[\cdot]$ are analytic, and the maps from $]-\epsilon_1,\epsilon_1[$ to $\mathcal{L}(L^2(\partial\Omega^i),L^2(\partial\Omega^o))$ which take ϵ to $S_{3,\epsilon}[\cdot]$, $S_{4,\epsilon}[\cdot]$ are analytic.

Hence, the map $\tilde{\mathcal{F}}$ is real analytic from $]-\epsilon_1,\epsilon_1[\,\times\mathbb{R}\times\mathcal{L}(X_{io})$ to the space $\mathcal{L}(X_{io},X_{io}\times\mathbb{R})$. Next we note that equation $\mathcal{F}(0,0,\theta^i,\theta^o,\psi^i,\psi^o) = 0$ is just another way of writing the limiting system (7.5), which implies that formulas (7.6), (7.7) hold and that accordingly $\mathcal{F}(0,0,\theta^i,\theta^o,\tilde{A}^i_0[\theta^i,\theta^o],\tilde{A}^o_0[\theta^i,\theta^o)] = 0$ (cf. Definition 7.2). Then the definition of $\tilde{\mathcal{F}}$ implies that $\tilde{\mathcal{F}}(0,0,\tilde{A}_0) = 0$. We now wish to compute the Fréchet differential of $\tilde{\mathcal{F}}$ at the point $(0,0,\tilde{A}_0)$ with respect to its last (operator) argument. Since $\tilde{\mathcal{F}}$ is linear with respect to its last argument, we have $d_Z\tilde{\mathcal{F}}(0,0,\tilde{A}_0)[\overline{Z}] = \tilde{\mathcal{F}}(0,0,\overline{Z})$ for all $\overline{Z} \in \mathcal{L}(X_{io})$. We now show that the linear map from $\mathcal{L}(X_{io})$ to $\mathcal{L}(X_{io},X_{io}\times\mathbb{R})$ which takes $\overline{Z}$ to $\tilde{\mathcal{F}}(0,0,\overline{Z})$ is a bijection.

Let $B \equiv (B^i,B^o,b) \in \mathcal{L}(X_{io},X_{io}\times\mathbb{R})$. We must show that there exists a unique $\overline{Z} \in \mathcal{L}(X_{io})$ such that

$$\tilde{\mathcal{F}}(0,0,\overline{Z}) = B. \tag{8.3}$$

By the definition of $\tilde{\mathcal{F}}$, we rewrite such an equality in the form

$$\mathcal{F}\big(0,0,\theta^i,\theta^o,\overline{Z}^i\big(\theta^i,\theta^o\big),\overline{Z}^o\big(\theta^i,\theta^o\big)\big) = B\big(\theta^i,\theta^o\big) \quad \forall\big(\theta^i,\theta^o\big) \in X_{io}. \tag{8.4}$$

In order to shorten our notation, we set $\psi^i \equiv \overline{Z}^i(\theta^i,\theta^o)$, $\psi^o \equiv \overline{Z}^o(\theta^i,\theta^o)$. Then we can rewrite equation (8.4) in the following form.

$$\begin{aligned}
&-\frac{1}{2}\psi^i(\xi) - W^t_{\Omega^i}\big[\psi^i\big](\xi) + \nu^i(\xi)\cdot\int_{\partial\Omega^o} DS_n(y)\psi^o(y)\, d\sigma_y \\
&= \int_{\partial\Omega^o} S_n(-y)\left[\theta^o(y) - \fint_{\partial\Omega^o}\theta^o\, d\sigma\chi_{\partial\Omega^o}(y)\right] d\sigma_y \\
&\quad - \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o}\theta^o\, d\sigma\chi_{\partial\Omega^o}\right] d\sigma + B^i\big(\theta^i,\theta^o\big)(\xi) \quad \forall\xi \in \partial\Omega^i,
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\psi^o(x) + W^t_{\Omega^o}\big[\psi^o\big](x) \\
&\quad = V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma \chi_{\partial\Omega^o}\right](x) \\
&\qquad - \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma \chi_{\partial\Omega^o}\right] d\sigma + B^o\big(\theta^i,\theta^o\big)(x) \quad \forall x \in \partial\Omega^o, \\
&\int_{\partial\Omega^o} \psi^o\, d\sigma = b\big(\theta^i,\theta^o\big).
\end{aligned}
\tag{8.5}
$$

By Lemma 4.1, the map Ξ_{Ω^o} from $L^2(\partial\Omega^o)$ onto $L^2(\partial\Omega^o)_0 \times \mathbb{R}$ which takes f^o to the pair $((-\frac{1}{2}I + W^t_{\Omega^o})[f^o], \int_{\partial\Omega^o} f^o\, d\sigma)$ is an isomorphism. Then the last two equations can be rewritten as

$$
\begin{aligned}
\psi^o &\equiv \overline{S}^o\big(\theta^i,\theta^o\big) \\
&= \Xi^{(-1)}_{\Omega^o}\Bigg[V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma \chi_{\partial\Omega^o}\right] \\
&\quad - \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma \chi_{\partial\Omega^o}\right] d\sigma + B^o\big(\theta^i,\theta^o\big), b\big(\theta^i,\theta^o\big)\Bigg] \quad \text{on } \partial\Omega^o,
\end{aligned}
$$

and the right hand side is linear and continous in $(\theta^i,\theta^o) \in X_{io}$. Since Ω^{i-} is connected, Lemma 3.3 implies that $\frac{1}{2}I + W^t_{\Omega^i}$ is an isomorphism in $L^2(\partial\Omega^i)$, and thus the first equation of (8.5) can be rewritten as

$$
\begin{aligned}
\psi^i &\equiv \overline{Z}^i\big(\theta^i,\theta^o\big) \\
&= -\left(\frac{1}{2}I + W^t_{\Omega^i}\right)^{(-1)} \Bigg[-\nu^i(\cdot)\cdot \int_{\partial\Omega^o} DS_n(y)\overline{Z}^o\big(\theta^i,\theta^o\big)(y)\, d\sigma_y \\
&\quad + \int_{\partial\Omega^o} S_n(-y)\left[\theta^o(y) - \fint_{\partial\Omega^o} \theta^o\, d\sigma \chi_{\partial\Omega^o}(y)\right] d\sigma_y \\
&\quad - \fint_{\partial\Omega^o} V_{\Omega^o}\left[\theta^o - \fint_{\partial\Omega^o} \theta^o\, d\sigma \chi_{\partial\Omega^o}\right] d\sigma + B^i\big(\theta^i,\theta^o\big)\Bigg] \quad \text{on } \partial\Omega^i,
\end{aligned}
$$

and the right hand side is linear and continous in $(\theta^i,\theta^o) \in X_{io}$. In particular, equation (8.3) has one and only one solution $\overline{Z}$ for each B as above and $d_Z\tilde{\mathcal{F}}(0,0,\tilde{A}_0)$ is a linear isomorphism. Since $d_Z\tilde{\mathcal{F}}(0,0,\tilde{A}_0)$ is continuous, the Open Mapping Theorem implies that it is a homeomorphism. Then the Implicit Function Theorem in Banach space implies the existence of $\epsilon_2 \in]0,\epsilon_1[$ and of $\iota_2 \in]0,+\infty[$, and of an open neighborhood U of $\tilde{A}_0$ in $\mathcal{L}(X_{io})$ and of a real analytic family as in (8.1) in U such that

$$
\begin{aligned}
&\big\{(\epsilon,\iota,\mathcal{A}_{(\epsilon,\iota)}) : (\epsilon,\iota) \in]-\epsilon_2,\epsilon_2[\times]-\iota_2,\iota_2[\big\} \\
&\quad = \big\{(\epsilon,\iota,Z) \in]-\epsilon_2,\epsilon_2[\times]-\iota_2,\iota_2[\times U : \tilde{\mathcal{F}}[\epsilon,\iota,Z] = 0\big\}.
\end{aligned}
$$

In particular, $\mathcal{A}_{(0,0)} = \tilde{A}_0$ and thus (ii) holds true. Possibly shrinking ϵ_2, we can assume that $\delta_{2,n}\epsilon\log\epsilon$ belongs to $]-\iota_2,\iota_2[$ for all $\epsilon \in]0,\epsilon_2[$. By the definition of $\tilde{\mathcal{F}}$, we know that $\tilde{\mathcal{F}}(\epsilon,\delta_{2,n}\epsilon\log\epsilon,\tilde{A}_\epsilon) = 0$

for all ϵ in $]0,\epsilon_2[$, and we know that $\tilde{A}_\epsilon$ is the only solution in $\mathcal{L}(X_{io})$ of such an equation. Hence, $\mathcal{A}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)} = \tilde{A}_\epsilon$ for all $\epsilon \in]0,\epsilon_2[$. $\square$

9. Symmetric functions of multiple eigenvalues of the operators of the family $\{\tilde{A}_\epsilon\}_{\epsilon\in[0,\epsilon_0[}$

Propositions 4.2, 4.5, 7.4 imply that if we fix $\lambda \in \mathbb{R}\setminus\{0\}$, then λ is an eigenvalue of multiplicity $r \in \mathbb{N}\setminus\{0\}$ for the Steklov problem (1.1) in Ω^o if and only if $\gamma = 1/\lambda$ is an eigenvalue of multplicity r for the operator A_{Ω^o}, if and only if $\gamma = 1/\lambda$ is an eigenvalue of multplicity r for $\tilde{A}_0$.

Similarly, Propositions 4.2, 4.5, 5.1 imply that if $\epsilon \in]0,\epsilon_1[$, then a number $\lambda \in \mathbb{R}\setminus\{0\}$ is an eigenvalue of multiplicity $r \in \mathbb{N}\setminus\{0\}$ for the Steklov problem (1.2) in $\Omega(\epsilon)$ if and only if $\gamma = 1/\lambda$ is an eigenvalue of multplicity r for $\tilde{A}_\epsilon$.

Since the operators A_{Ω^o} and $\tilde{A}_\epsilon$ are compact and self adjoint with respect to suitable scalar products, their nonzero eigenvalues are real isolated and positive.

We now assume that $\tilde{\lambda} \in]0,+\infty[$ is an eigenvalue of mutiplicity $r \in \mathbb{N}\setminus\{0\}$ of the Steklov problem (1.1) in Ω^o. Then $\tilde{\gamma} \equiv 1/\tilde{\lambda}$ is a nonzero isolated eigenvalue of multiplicity r for $\tilde{A}_0$.

In order to study the behavior of the eigenvalues of $\tilde{A}_\epsilon$ as ϵ approaches zero, we plan to complexify the operators $\tilde{A}_0, \tilde{A}_\epsilon$ in X_{io}. The complexification of X_{io} coincides with the space $\hat{X}_{io} \equiv \{x+iy : x,y \in X_{io}\}$. Since $X_{io} = L^2(\partial\Omega^i,\mathbb{R}) \times L^2(\partial\Omega^o,\mathbb{R})$, we have $\hat{X}_{io} \equiv L^2(\partial\Omega^i,\mathbb{C}) \times L^2(\partial\Omega^o,\mathbb{C})$. If Q is a real bilinear form on X_{io}^2, then its complexification $\hat{Q}$ coincides with the sesquilinear form on $\hat{X}_{io}^2$ defined by

$$
\begin{aligned}
&\hat{Q}[x_1+iy_1, x_2+iy_2] \equiv \big(Q[x_1,x_2] + Q[y_1,y_2]\big) + i\big(Q[y_1,x_2] - Q[x_1,y_2]\big) \\
&\qquad \forall (x_1+iy_1),(x_2+iy_2) \in \hat{X}_{io}.
\end{aligned}
$$

If Q is symmetric, then $\hat{Q}$ is conjugate symmetric, *i.e.*, $\hat{Q}[x_1+iy_1,x_2+iy_2] = \overline{\hat{Q}[x_2+iy_2,x_1+iy_1]}$ for all $(x_1+iy_1),(x_2+iy_2) \in \hat{X}_{io}$. If Q is a scalar product on the real space X, then $\hat{Q}$ is a scalar product on the complex space $\hat{X}_{io}$. So for example, the complexification of the usual scalar product

$$
\big((\theta_1^i,\theta_1^o),(\theta_2^i,\theta_2^o)\big)_{X_{io}} \equiv \int_{\partial\Omega^i} \theta_1^i\theta_2^i\,d\sigma + \int_{\partial\Omega^o} \theta_1^o\theta_2^o\,d\sigma \quad \forall (\theta_1^i,\theta_1^o),(\theta_2^i,\theta_2^o) \in X_{io},
$$

in X_{io}, coincides with the usual scalar product

$$
\big((\theta_1^i,\theta_1^o),(\theta_2^i,\theta_2^o)\big)_{\hat{X}_{io}} \equiv \int_{\partial\Omega^i} \theta_1^i\overline{\theta_2^i}\,d\sigma + \int_{\partial\Omega^o} \theta_1^o\overline{\theta_2^o}\,d\sigma \quad \forall (\theta_1^i,\theta_1^o),(\theta_2^i,\theta_2^o) \in \hat{X}_{io}
$$

in the complexified space $\hat{X}_{io}$. Then if T is a linear operator from X_{io} to X_{io}, the complexified operator $\hat{T}$ of T is defined by $\hat{T}[x+iy] \equiv T[x] + iT[y]$ for all $x+iy \in \hat{X}_{io}$. If T is selfadjoint in the real Hilbert space (X_{io},Q), then $\hat{T}$ is easily verified to be selfadjoint in the complex Hilbert space $(\hat{X}_{io},\hat{Q})$.

We also note that if λ is a real eigenvalue of finite geometric multiplicity r of the operator T in X_{io}, *i.e.*, the real dimension of the eigenspace corresponding to λ equals r, then λ is also an eigenvalue for the operator $\hat{T}$ in $\hat{X}_{io}$ of geometric multiplicity r, *i.e.*, the complex dimension of the eigenspace corresponding to λ equals r. By the Spectral Stability Theorem, we immediately deduce the validity of the following statement.

Theorem 9.1. *Let $\alpha \in]0,1[$, $r \in \mathbb{N} \setminus \{0\}$. Let Ω^i, Ω^o be as in (5.1). Let $\tilde{\gamma} \in]0,+\infty[$ be an eigenvalue of multiplicity r of $\tilde{A}_0$. Then $\tilde{\gamma}$ is an eigenvalue of geometric multiplicity r of the compact complexified operator $\hat{\tilde{A}}_0$ and is an isolated point of the spectrum of $\hat{\tilde{A}}_0$. Let ϵ_2, ι_2 be as in Theorem 8.1.*

Let $\delta \in]0,+\infty[$ such that $\mathrm{cl}\,\mathbb{B}_{\mathbb{C}}(\tilde{\gamma},2\delta) \setminus \{\tilde{\gamma}\}$ does not contain 0 and does not contain any point of the spectrum of $\hat{\tilde{A}}_0$. Then there exist $(\epsilon_3,\iota_3) \in]0,\epsilon_2[\times]0,\iota_2[$ such that

$$\delta_{2,n}\epsilon \log \epsilon \in]-\iota_3,\iota_3[\quad \forall \epsilon \in]0,\epsilon_3[, \tag{9.1}$$

and such that the spectrum of the complexified operator $\hat{\mathcal{A}}_{(\epsilon,\iota)}$ does not contain elements of the set $\mathrm{cl}\,\mathbb{B}_2(\tilde{\gamma},2\delta) \setminus \mathbb{B}_2(\tilde{\gamma},\delta)$ for all $(\epsilon,\iota) \in]-\epsilon_3,\epsilon_3[\times]-\iota_3,\iota_3[$.

Proof. Since $\tilde{\gamma}$ is a real eigenvalue of multiplicity r for $\tilde{A}_0$, then $\tilde{\gamma}$ is also an eigenvalue of geometric multiplicity r for its complexified operator $\hat{\tilde{A}}_0$. Since $\hat{\tilde{A}}_0$ is compact and $\tilde{\gamma} \neq 0$, $\tilde{\gamma}$ is an isolated eigenvalue of $\hat{\tilde{A}}_0$ and there exists δ as in the statement. Then $U \equiv \mathbb{C} \setminus (\mathrm{cl}\,\mathbb{B}_2(\tilde{\gamma},2\delta) \setminus \mathbb{B}_2(\tilde{\gamma},\delta))$ is an open set which contains the spectrum of $\hat{\tilde{A}}_0$. Since $\{\mathcal{A}_{(\epsilon,\iota)}\}_{(\epsilon,\iota)\in]-\epsilon_2,\epsilon_2[\times]-\iota_2,\iota_2[}$ is a real analytic family in $\mathcal{L}(X_{io})$, then the family of complexified operators $\{\hat{\mathcal{A}}_{(\epsilon,\iota)}\}_{(\epsilon,\iota)\in]-\epsilon_2,\epsilon_2[\times]-\iota_2,\iota_2[}$ is real analytic in $\mathcal{L}(\hat{X}_{io})$. In particular, such a family is continuous at $(0,0)$ and $\hat{\mathcal{A}}_{(0,0)} = \hat{\tilde{A}}_0$. Then the Spectral Stability Theorem ensures that there exist $(\epsilon_3,\iota_3) \in]0,\epsilon_2[\times]0,\iota_2[$ such that the spectrum of $\hat{\mathcal{A}}_{(\epsilon,\iota)}$ is contained in U for all $(\epsilon,\iota) \in]-\epsilon_3,\epsilon_3[\times]-\iota_3,\iota_3[$ (cf. *e.g.*, Rudin [34, Thm. 10.20, p. 257]). Possibly shrinking ϵ_3, we can ensure the validity of (9.1). □

By the exploiting Kato Projection, we can prove the following.

Theorem 9.2. *Let $\alpha \in]0,1[$, $r \in \mathbb{N} \setminus \{0\}$. Let Ω^i, Ω^o be as in (5.1). Let $\tilde{\gamma} \in]0,+\infty[$ be an eigenvalue of multiplicity r of $\tilde{A}_0$. Let δ, ϵ_3, ι_3 be as in Theorem 9.1. Let ς be the curve defined by $\varsigma(\varphi) \equiv \tilde{\gamma} + \delta e^{i\varphi}$ for all $\varphi \in [0,2\pi]$. Then the following statements hold.*

(i) *If $(\epsilon,\iota) \in]-\epsilon_3,\epsilon_3[\times]-\iota_3,\iota_3[$, then the operator*

$$P^{\sharp}_{(\epsilon,\iota)} \equiv \frac{1}{2\pi i}\int_{\varsigma}(\zeta I_{\hat{X}_{io}} - \hat{\mathcal{A}}_{(\epsilon,\iota)})^{(-1)} d\zeta, \tag{9.2}$$

is a projection of the complexified space $\hat{X}_{io}$ onto an $\hat{\mathcal{A}}_{(\epsilon,\iota)}$-invariant subspace of $\hat{X}_{io}$.

(ii) *The map from $]-\epsilon_3,\epsilon_3[\times]-\iota_3,\iota_3[$ to $\mathcal{L}(\hat{X}_{io})$ which takes (ϵ,ι) to $P^{\sharp}_{(\epsilon,\iota)}$ is real analytic.*

(iii) *$P^{\sharp}_{(0,0)}$ coincides with the projection onto the eigenspace of $\hat{\tilde{A}}_0$ corresponding to the eigenvalue $\tilde{\gamma}$, which has complex dimension equal to r.*

(iv) *There exist $(\epsilon_4,\iota_4) \in]0,\epsilon_3[\times]0,\iota_3[$ such that $\delta_{2,n}\epsilon\log\epsilon \in]-\iota_4,\iota_4[$ for all $\epsilon \in]0,\epsilon_4[$ and such that if $(\epsilon,\iota) \in]-\epsilon_4,\epsilon_4[\times]-\iota_4,\iota_4[$, then the image of $P^{\sharp}_{(\epsilon,\iota)}$ has complex dimension equal to r.*

(v) *If $\epsilon \in]0,\epsilon_4[$, then $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ is selfadjoint in $(\hat{X}_{io}, \hat{\mathcal{Q}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)})$ and the restriction of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ to the $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$-invariant subspace $\mathrm{Im}\,P^{\sharp}_{(\epsilon,\iota)}$ has precisely r real eigenvalues counted with their multiplicity $\gamma_r(\epsilon) \leqslant \cdots \leqslant \gamma_1(\epsilon)$, in the interval $]\tilde{\gamma}-\delta,\tilde{\gamma}+\delta[$. Moreover, $]\tilde{\gamma}-\delta,\tilde{\gamma}+\delta[$ contains no other eigenvalue of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$.*

Proof. For a proof of statement (i), we refer to Kato [15, p. 178]. We now turn to the proof of (ii). The space $C^0_b(\varsigma[0,2\pi],\mathcal{L}(\hat{X}_{io}))$ of bounded and continuous functions from the compact set $\varsigma[0,2\pi]$ to $\mathcal{L}(\hat{X}_{io})$ with the $\sup_{\varsigma[0,2\pi]}\|\cdot\|_{\mathcal{L}(\hat{X}_{io})}$-norm is a complex Banach algebra with unity. Let $\mathcal{I}(\hat{X}_{io})$ denote the set of invertible elements of $\mathcal{L}(\hat{X}_{io})$. Then $C^0_b(\varsigma[0,2\pi],\mathcal{I}(\hat{X}_{io}))$ is the set of invertible elements of $C^0_b(\varsigma[0,2\pi],\mathcal{L}(\hat{X}_{io}))$ and is open in the space $C^0_b(\varsigma[0,2\pi],\mathcal{L}(\hat{X}_{io}))$ of bounded and continuous functions from the compact set $\varsigma[0,2\pi]$ to $\mathcal{L}(\hat{X}_{io})$, and the map from $C^0_b(\varsigma[0,2\pi],\mathcal{I}(\hat{X}_{io}))$ to itself which takes an element to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [14, Thms. 4.3.2 and 4.3.4]).

Since the map which takes $(\epsilon,\iota)\in]-\epsilon_3,\epsilon_3[\times]-\iota_3,\iota_3[$ to the map $(\zeta I_{\hat{X}_{io}}-\hat{\mathcal{A}}_{(\epsilon,\iota)})$ of the variable $\zeta\in\varsigma([0,2\pi])$ of $C^0_b(\varsigma[0,2\pi],\mathcal{I}(\hat{X}_{io}))$ is analytic, and the line integral $\int_\varsigma$ is a linear and continuous from $C^0_b(\varsigma[0,2\pi],\mathcal{L}(\hat{X}_{io}))$ to $\mathcal{L}(\hat{X}_{io})$, we conclude that $P^\sharp_{(\epsilon,\iota)}$ is real analytic from $]-\epsilon_3,\epsilon_3[\times]-\iota_3,\iota_3[$ to $\mathcal{L}(\hat{X}_{io})$.

For a proof of statement (iii), we refer to Kato [15, p. 178]. Indeed, $\tilde{\gamma}$ is the only element of the spectrum of $\tilde{A}_0$ which belongs to $\mathbb{B}_2(\tilde{\gamma},\delta)$.

We now consider statement (iv). Since $P^\sharp_{(0,0)}$ is a projection onto the eigenspace $\hat{E}(\tilde{\gamma})$ of $\tilde{A}_0$ corresponding to $\tilde{\gamma}$, which has finite dimension equal to r a known result of functional analysis implies that there exists a neighborhood of $P^\sharp_{(0,0)}$ in $\mathcal{L}(\hat{X}_{io})$ such that all projection operators which belong to such a neighborhood have an image of dimension r (*cf.* Dunford and Schwartz [9, Ch. VII, Lem. 7]). Since $\lim_{(\epsilon,\iota)\to(0,0)}P^\sharp_{(\epsilon,\iota)}=P^\sharp_{(0,0)}$ in $\mathcal{L}(\hat{X}_{io})$, there exists $(\epsilon_4,\iota_4)\in]0,\epsilon_3[\times]0,\iota_3[$ such that $\dim\operatorname{Im}P^\sharp_{(\epsilon,\iota)}=r$ for all $(\epsilon,\iota)\in]-\epsilon_4,\epsilon_4[\times]-\iota_4,\iota_4[$. By Theorem 8.1, $\mathcal{A}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}=\tilde{A}_\epsilon$. Then Proposition 5.1 implies that $\mathcal{A}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ is selfadjoint in $(X_{io},\mathcal{Q}_\epsilon)$ and accordingly that $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ is selfadjoint in $(\hat{X}_{io},\hat{\mathcal{Q}}_\epsilon)$. Then the restriction of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ to the r-dimensional invariant subspace $\operatorname{Im}P^\sharp_{(\epsilon,\iota)}$ has r real eigenvalues counted with their multiplicity.

By Theorem 9.1, the intersection of $\mathbb{B}_2(\tilde{\gamma},\delta)$ and the spectrum of the operator $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ is a spectral set and the image of $P^\sharp_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ is the corresponding projection (cf. (9.2)). Then the intersection of the spectrum of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ and of $\mathbb{B}_2(\tilde{\gamma},\delta)$ equals the spectrum of the restriction of the operator $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ to $\operatorname{Im}P^\sharp_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$, *i.e.*, the set of the eigenvalues $\gamma_r(\epsilon)$, …, $\gamma_1(\epsilon)$ (cf. Dunford and Schwartz [9, Ch. VII, Thm. 20]). □

Next we turn to show that possibly shrinking ϵ_4, ι_4, we can choose r vectors $u_l[\epsilon,\iota]$ for $l\in\{1,\dots,r\}$ which generate the space $\operatorname{Im}P^\sharp_{(\epsilon,\iota)}$ and which satisfy the orthonormality conditions

$$\hat{\mathcal{Q}}_\epsilon\big[u_l[\epsilon,\iota],u_j[\epsilon,\iota]\big]=\delta_{lj}, \tag{9.3}$$

for all $l,j\in\{1,\dots,r\}$ and which depend real analytically on (ϵ,ι) when $(\epsilon,\iota)\in]-\epsilon_4,\epsilon_4[\times]-\iota_4,\iota_4[$. Namely, we prove the following by a variant of an argument of the proof of a corresponding result of paper [18, Prop. 2.20] with Lamberti.

Proposition 9.3. *Let $\alpha\in]0,1[$, $r\in\mathbb{N}\setminus\{0\}$. Let Ω^i, Ω^o be as in (5.1). Let $\tilde{\gamma}\in]0,+\infty[$ be an eigenvalue of multiplicity r of $\tilde{A}_0$. Let δ, ϵ_4, ι_4 be as in Theorem 9.2.*

Then there exist $(\epsilon_5,\iota_5)\in]0,\epsilon_4[\times]0,\iota_4[$ such that $\delta_{2,n}\epsilon\log\epsilon\in]-\iota_5,\iota_5[$ for all $\epsilon\in]0,\epsilon_5[$ and r real analytic functions $u_l[\cdot,\cdot]$ for $l\in\{1,\dots,r\}$, from $]-\epsilon_5,\epsilon_5[\times]-\iota_5,\iota_5[$ to $\hat{X}_{io}$ which satisfy the orthonor-

mality conditions (9.3) and such that $\{u_l[\epsilon,\iota] : l \in \{1,\dots,r\}\}$ *generates the space* $\operatorname{Im} P^\sharp_{(\epsilon,\iota)}$ *for all* $(\epsilon,\iota) \in]-\epsilon_5,\epsilon_5[\times]-\iota_5,\iota_5[$.

Proof. Since $\tilde\gamma$ is an eigenvalue of multiplicity r of $\tilde A_0$, Proposition 7.4 ensures that $\tilde\gamma$ is an eigenvalue of multiplicity r of the compact self adjoint operator A_{Ω^o} in $(L^2(\partial\Omega^o,\mathbb{R}), \langle\langle\cdot,\cdot\rangle\rangle_{\Omega^o})$. Then the eigenspace of A_{Ω^o} corresponding to $\tilde\gamma$ has an orthonormal basis $\{\theta^o_l : l \in \{1,\dots,r\}\}$, and we have $\langle\langle\theta^o_l,\theta^o_j\rangle\rangle_{\Omega^o} = \delta_{l,j}$ for all $l,j \in \{1,\dots,r\}$. Now let θ^i_l be the function defined by the right hand side of formula (7.8) when $\theta^o = \theta^o_l$, $\gamma = \tilde\gamma$. By Proposition 7.4, $\{(\theta^i_l,\theta^o_l) : l \in \{1,\dots,r\}\}$, is a basis of the eigenspace of $\tilde A_0$ corresponding to $\tilde\gamma$ and Theorem 6.1(ii) ensures the validity of the equalities $\mathcal{Q}_0[(\theta^i_l,\theta^o_l),(\theta^i_j,\theta^o_j)] = \langle\langle\theta^o_l,\theta^o_j\rangle\rangle_{\Omega^o} = \delta_{l,j}$ for all $l,j \in \{1,\dots,r\}$. Then $\{(\theta^i_l,\theta^o_l) : l \in \{1,\dots,r\}\}$ is a basis of the (complex) eigenspace of the complexification $\hat{\tilde A}_0$ of $\tilde A_0$ corresponding to the real eigenvalue $\tilde\gamma$. By the definition of the complexified Hermitian form $\hat{\mathcal{Q}}_0$, we have $\hat{\mathcal{Q}}_0[(\theta^i_l,\theta^o_l),(\theta^i_j,\theta^o_j)] = \mathcal{Q}_0[(\theta^i_l,\theta^o_l),(\theta^i_j,\theta^o_j)] = \delta_{l,j}$ for all $l,j \in \{1,\dots,r\}$. By Lemma A.1 of the Appendix, and by the continuity of $P^\sharp_{(\epsilon,\iota)}$ in (ϵ,ι) at $(\epsilon,\iota) = (0,0)$, there exist $(\epsilon_5,\iota_5) \in]0,\epsilon_4[\times]0,\iota_4[$ such that the restriction $P^\sharp_{(\epsilon,\iota)|\hat E(\tilde\gamma)}$ is injective when $(\epsilon,\iota) \in]-\epsilon_5,\epsilon_5[\times]-\iota_5,\iota_5[$ and accordingly, $\{v_l[\epsilon,\iota] \equiv P^\sharp_{(\epsilon,\iota)}[(\theta^i_l,\theta^o_l)] : l \in \{1,\dots,r\}\}$ is a system of r linearly independent vectors which generates $\operatorname{Im} P^\sharp_{(\epsilon,\iota)}$. If $\epsilon \in]-\epsilon_5,\epsilon_5[$, we know that $\hat{\mathcal{Q}}_\epsilon$ is a sesquilinear conjugate symmetric form. If $\epsilon \in]0,\epsilon_5[$, then we also know that $\mathcal{Q}_\epsilon = Q_\epsilon$ is a scalar product in X_{io} and that accordingly $\hat{\mathcal{Q}}_\epsilon$ is a scalar product on $\hat X_{io}$. By Proposition 7.3, Q_0 is positive definite on the subspace $\operatorname{Im}\tilde A_0$ of X_{io} and accordingly on the eigenspace of $\tilde A_0$ corresponding to $\tilde\gamma$. Hence, $\hat{\mathcal{Q}}_0$ is a scalar product on the eigenspace $\hat E(\tilde\gamma)$ of $\hat{\tilde A}_0$ corresponding to the real eigenvalue $\tilde\gamma$. Now the equalities

$$\hat{\mathcal{Q}}_0\big[v_l[0,0],v_j[0,0]\big] = \hat{\mathcal{Q}}_0\big[P^\sharp_{(0,0)}\big[(\theta^i_l,\theta^o_l)\big],P^\sharp_{(0,0)}\big[(\theta^i_j,\theta^o_j)\big]\big] = \hat{\mathcal{Q}}_0\big[(\theta^i_l,\theta^o_l),(\theta^i_l,\theta^o_l)\big] = \delta_{j,l}$$

for all $l,j \in \{1,\dots,r\}$, and the continuity of $\hat{\mathcal{Q}}_\epsilon$ at $\epsilon = 0$ and of $P^\sharp_{(\epsilon,\iota)}$ at $(\epsilon,\iota) = (0,0)$ ensure that possibly shrinking ϵ_5, ι_5 $\hat{\mathcal{Q}}_\epsilon[v_l[\epsilon,\iota],v_l[\epsilon,\iota]] > 0$ for all $l \in \{1,\dots,r\}$, for all $(\epsilon,\iota) \in]-\epsilon_5,\epsilon_5[\times]-\iota_5,\iota_5[$. Then by following the Gram–Schmidt procedure, we set $u_1[\epsilon,\iota] \equiv \frac{v_1[\epsilon,\iota]}{\hat{\mathcal{Q}}_\epsilon[v_1[\epsilon,\iota],v_1[\epsilon,\iota]]^{1/2}}$ for all pairs (ϵ,ι) in $]-\epsilon_5,\epsilon_5[\times]-\iota_5,\iota_5[$. Next we argue by induction and we assume that possibly shrinking ϵ_5, ι_5, the real analytic functions $u_l[\cdot,\cdot]$ have been defined for $l = 1,\dots,h$ with $h < r$ and that $\{u_l[\epsilon,\iota] : l \in \{1,\dots,h\}\}$ generates the space generated by $\{v_l[\epsilon,\iota] : l \in \{1,\dots,h\}\}$ for all $(\epsilon,\iota) \in]-\epsilon_5,\epsilon_5[\times]-\iota_5,\iota_5[$ and we define $u_{h+1}[\cdot,\cdot]$. To do so, we consider the vector $v_{h+1}[\epsilon,\iota] - \sum_{l=1}^h \hat{\mathcal{Q}}_\epsilon[v_{h+1}[\epsilon,\iota],u_l[\epsilon,\iota]]u_l[\epsilon,\iota]$. For $(\epsilon,\iota) = (0,0)$, we have

$$\hat{\mathcal{Q}}_0\Bigg[v_{h+1}[0,0] - \sum_{l=1}^h \hat{\mathcal{Q}}_0\big[v_{h+1}[0,0],u_l[0,0]\big]u_l[0,0],$$
$$v_{h+1}[0,0] - \sum_{l=1}^h \hat{\mathcal{Q}}_0\big[v_{h+1}[0,0],u_l[0,0]\big]u_l[0,0]\Bigg] > 0. \tag{9.4}$$

Indeed, $\hat{\mathcal{Q}}_0$ is positive semidefinite, and it is positive definite on the eigenspace $\hat E(\tilde\gamma)$ of $\hat{\mathcal{A}}$ corresponding to $\tilde\gamma$, and if the number in (9.4) were to be equal to zero, then we would have the equality $v_{h+1}[0,0] -$

$\sum_{l=1}^{h} \hat{\mathcal{Q}}_0[v_{h+1}[0,0], u_l[0,0]]u_l[0,0] = 0$, and $v_{h+1}[0,0]$ would belong to the space generated by the set $\{u_l[0,0] : l \in \{1,\dots,h\}\}$, which by inductive assumption equals the space generated by the set $\{v_l[0,0] : l \in \{1,\dots,h\}\}$, a contradiction. By inequality (9.4) and possibly shrinking ϵ_5, ι_5, we can assume that

$$\begin{aligned}
&\left\| v_{h+1}[\epsilon,\iota] - \sum_{l=1}^{h} \hat{\mathcal{Q}}_\epsilon\big[v_{h+1}[\epsilon,\iota], u_l[\epsilon,\iota]\big]\, u_l[\epsilon,\iota] \right\|_{(\operatorname{Im} P^\sharp_{(\epsilon,\iota)}, \hat{\mathcal{Q}}_\epsilon)} \\
&\quad \equiv \hat{\mathcal{Q}}_\epsilon \Bigg[v_{h+1}[\epsilon,\iota] - \sum_{l=1}^{h} \hat{\mathcal{Q}}_\epsilon\big[v_{h+1}[\epsilon,\iota], u_l[\epsilon,\iota]\big]\, u_l[\epsilon,\iota], v_{h+1}[\epsilon,\iota] \\
&\qquad - \sum_{l=1}^{h} \hat{\mathcal{Q}}_\epsilon\big[v_{h+1}[\epsilon,\iota], u_l[\epsilon,\iota]\big]\, u_l[\epsilon,\iota] \Bigg]
\end{aligned}$$

is strictly positive for all $(\epsilon,\iota) \in \,]-\epsilon_5, \epsilon_5[\, \times \,]-\iota_5, \iota_5[$. Then we can set

$$u_{h+1}[\epsilon,\iota] \equiv \frac{v_{h+1}[\epsilon,\iota] - \sum_{l=1}^{h} \hat{\mathcal{Q}}_\epsilon[v_{h+1}[\epsilon,\iota], u_l[\epsilon,\iota]]u_l[\epsilon,\iota]}{\| v_{k+1}[\epsilon,\iota] - \sum_{l=1}^{h} \hat{\mathcal{Q}}_\epsilon[v_{h+1}[\epsilon,\iota], u_l[\epsilon,\iota]]u_l[\epsilon,\iota] \|_{(\operatorname{Im} P^\sharp_{(\epsilon,\iota)}, \hat{\mathcal{Q}}_\epsilon)}}$$

for all $(\epsilon,\iota) \in \,]-\epsilon_5, \epsilon_5[\, \times \,]-\iota_5, \iota_5[$. Then conditions (9.3) hold true for all $l, j \in \{1,\dots,h+1\}$ and $\{u_l[\epsilon,\iota] : l \in \{1,\dots,h+1\}\}$ generates the space generated by $\{v_l[\epsilon,\iota] : l \in \{1,\dots,h+1\}\}$ for all $(\epsilon,\iota) \in \,]-\epsilon_5, \epsilon_5[\, \times \,]-\iota_5, \iota_5[$.

So by finite induction, the same is true for $h = r-1$ and $\{u_l[\epsilon,\iota] : l \in \{1,\dots,r\}\}$ satisfies conditions (9.3) and generates the space $\operatorname{Im} P^\sharp_{(\epsilon,\iota)}$ for all $(\epsilon,\iota) \in \,]-\epsilon_5, \epsilon_5[\, \times \,]-\iota_5, \iota_5[$. Possibly shrinking ϵ_5, we can assume that $\delta_{2,n}\epsilon \log \epsilon \in \,]-\iota_5, \iota_5[$ for all $\epsilon \in \,]0, \epsilon_5[$. □

Next we introduce an (ϵ,ι)-dependent family of (complex) $r \times r$ matrices, which represents the matrix of the restriction of $\hat{\mathcal{A}}_{(\epsilon,\iota)}$ to the invariant subspace $P^\sharp_{(\epsilon,\iota)}$. We do so by means of the following.

Proposition 9.4. *Let $\alpha \in \,]0,1[$. Let Ω^i, Ω^o be as in (5.1). Let $\tilde{\gamma} \in \,]0,+\infty[$ be an eigenvalue of multiplicity r of $\tilde{A}_0$. Let δ, ϵ_5, ι_5, $\{u_l[\epsilon,\iota] : l \in \{1,\dots,r\}\}$ be as in Proposition 9.3. Let $\mathcal{S}$ be the map from the set $]-\epsilon_5, \epsilon_5[\, \times \,]-\iota_5, \iota_5[$ to the space $M_r(\mathbb{C})$ of $r \times r$ matrices with complex entries defined by*

$$\mathcal{S}(\epsilon,\iota) \equiv \big(\mathcal{S}_{h,k}(\epsilon,\iota)\big)_{h,k\in\{1,\dots,r\}} \equiv \Big(\hat{\mathcal{Q}}_\epsilon\big[\hat{\mathcal{A}}_{(\epsilon,\iota)}\big[u_h[\epsilon,\iota]\big], u_k[\epsilon,\iota]\big]\Big)_{h,k\in\{1,\dots,r\}}$$

for all $(\epsilon,\iota) \in \,]-\epsilon_5, \epsilon_5[\, \times \,]-\iota_5, \iota_5[$. Then the following statements hold.

(i) *$\mathcal{S}$ is real analytic.*

(ii) *If $\epsilon \in \,]0, \epsilon_5[$, then $\mathcal{S}(\epsilon, \delta_{2,n}\epsilon \log \epsilon)$ is the Hermitian $r \times r$ matrix associated to the restriction of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ to the invariant space $\operatorname{Im} P^\sharp_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ and has precisely r real eigenvalues counted with their multiplicity $\gamma_r[\epsilon] \leqslant \cdots \leqslant \gamma_1[\epsilon]$, in the interval $]\tilde{\gamma}-\delta, \tilde{\gamma}+\delta[$. Moreover, $]\tilde{\gamma}-\delta, \tilde{\gamma}+\delta[$ contains no other eigenvalue of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$.*

(iii) *$\mathcal{S}(0,0)$ is the Hermitian matrix of the restriction of $\hat{\hat{A}}_0$ to the eigenspace $\hat{E}(\tilde{\gamma})$ of $\hat{\hat{A}}_0$ corresponding to $\tilde{\gamma}$, which has the real eigenvalue $\tilde{\gamma}$ with multiplicity r, and we set $\gamma_l[0] \equiv \tilde{\gamma}\ \forall l \in \{1,\dots,r\}$.*

Proof. Since $\hat{\mathcal{Q}}_\epsilon$ depends real analytically on $\epsilon \in]-\epsilon_5, \epsilon_5[$ and $\hat{\mathcal{A}}_{(\epsilon,\iota)}$, $u_l[\epsilon,\iota]$ depend real analytically on $(\epsilon,\iota) \in]-\epsilon_5,\epsilon_5[\times]-\iota_5,\iota_5[$, then $\mathcal{S}_{h,k}(\epsilon,\iota)$ depends real analytically on $(\epsilon,\iota) \in]-\epsilon_5,\epsilon_5[\times]-\iota_5,\iota_5[$ for all $h,k \in \{1,\ldots,r\}$.

(ii) By Proposition 5.1 and Theorem 8.1(i), $\mathcal{A}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)} = \tilde{A}_\epsilon$ is selfadjoint in $(X_{io},\mathcal{Q}_\epsilon)$. Then $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ is selfadjoint in $(\hat{X}_{io},\hat{\mathcal{Q}}_\epsilon)$. By Proposition 9.3, $\{u_l[\epsilon,\delta_{2,n}\epsilon\log\epsilon] : l \in \{1,\ldots,r\}\}$ is an orthonormal basis of the invariant space $\mathrm{Im}\, P^\sharp_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$, which has dimension r (cf. Theorem 9.2(iv)). Hence, $\mathcal{S}(\epsilon,\delta_{2,n}\epsilon\log\epsilon)$ is the matrix of the restriction of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ to $\mathrm{Im}\, P^\sharp_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$. Hence, $\mathcal{S}(\epsilon,\delta_{2,n}\epsilon\log\epsilon)$ is Hermitian and has r real eigenvalues counted with their multiplicity. Also, Theorem 9.2 implies that $]\tilde{\gamma}-\delta,\tilde{\gamma}+\delta[$ can contain no other eigenvalue of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$.

(iii). Since $\mathcal{S}$ is analytic, then it is continuous at $(\epsilon,\iota) = (0,0)$, and accordingly $\mathcal{S}(0,0) = \lim_{\epsilon\to 0^+}\mathcal{S}(\epsilon,\delta_{2,n}\epsilon\log\epsilon)$ must be Hermitian. By Proposition 9.3, $\{u_l[0,0] : l \in \{1,\ldots,r\}\}$ is an orthonormal basis of the $\hat{\mathcal{A}}_{(0,0)}$-invariant space $\mathrm{Im}\, P^\sharp_{(0,0)}$, which equals the eigenspace $\hat{E}(\tilde{\gamma})$ of $\hat{\tilde{A}}_0$ corresponding to the eigenvalue $\tilde{\gamma}$ (see Theorem 9.2(iii)). Then the $r\times r$ Hermitian matrix $\mathcal{S}(0,0)$ is the matrix of the restriction of $\hat{\mathcal{A}}_{(0,0)}$ to $\hat{E}(\tilde{\gamma})$, which has only $\tilde{\gamma}$ as eigenvalue and accordingly has r real eigenvalues counted with their multplicity, all of them equal to $\tilde{\gamma}$. $\square$

We are now ready to analyze the behavior of the eigenvalues $\gamma_r[\epsilon]$, …, $\gamma_1[\epsilon]$ of Proposition 9.4(ii), which ‘split’ from the multiple eigenvalue $\tilde{\gamma}$ of $\tilde{A}_0$ (or of A_{Ω^o}). We do so by means of the following.

Theorem 9.5. *Let $\alpha \in]0,1[$, $r \in \mathbb{N}\setminus\{0\}$. Let Ω^i, Ω^o be as in (5.1). Let $\tilde{\gamma} \in]0,+\infty[$ be an eigenvalue of multiplicity r of $\tilde{A}_0$. Let δ be as in Theorem 9.1. Then there exist $\epsilon^* \in]0,\epsilon_0[$ and $\iota^* \in]0,+\infty[$ and r real analytic functions $\Gamma_1,\ldots,\Gamma_r$ from $]-\epsilon^*,\epsilon^*[\times]-\iota^*,\iota^*[$ to $\mathbb{C}$ such that $\delta_{2,n}\epsilon\log\epsilon \in]-\iota^*,\iota^*[$ for all $\epsilon \in]0,\epsilon^*[$, and such that $\tilde{A}_\epsilon$ has precisely r real eigenvalues $\gamma_r[\epsilon] \leqslant \cdots \leqslant \gamma_1[\epsilon]$ counted with their multiplicity in the interval $]\tilde{\gamma}-\delta,\tilde{\gamma}+\delta[$ and*

$$\begin{aligned}
&\Gamma_1(\epsilon,\delta_{2,n}\epsilon\log\epsilon) \equiv \sum_{j_1=1}^{r}\gamma_{j_1}[\epsilon],\\
&\Gamma_2(\epsilon,\delta_{2,n}\epsilon\log\epsilon) \equiv \sum_{\substack{j_1,j_2=1,\ldots,r\\ j_1<j_2}}\gamma_{j_1}[\epsilon]\cdot\gamma_{j_2}[\epsilon],\\
&\cdots\cdots\cdots\\
&\Gamma_{r-1}(\epsilon,\delta_{2,n}\epsilon\log\epsilon) \equiv \sum_{\substack{j_1,\ldots,j_r=1,\ldots,r\\ j_1<\cdots<j_{r-1}}}\gamma_{j_1}[\epsilon]\cdot\cdots\cdot\gamma_{j_{r-1}}[\epsilon],\\
&\Gamma_r(\epsilon,\delta_{2,n}\epsilon\log\epsilon) \equiv \gamma_1[\epsilon]\cdot\cdots\cdot\gamma_r[\epsilon],
\end{aligned} \tag{9.5}$$

for all $\epsilon \in]0,\epsilon^*[$. Moreover,

$$\Gamma_s(0,0) = \binom{r}{s}\tilde{\gamma}^s \quad \forall s \in \{1,\ldots,r\}, \tag{9.6}$$

and

$$\lim_{\epsilon\to 0^+} \gamma_l[\epsilon] = \tilde{\gamma} \quad \forall l \in \{1,\dots,r\}. \tag{9.7}$$

If we further assume that $n \geqslant 3$, then there exists $\epsilon^\sharp \in]0,\epsilon^*[$ such that for each $l \in \{1,\dots,r\}$ there exists an analytic function ζ_l from $]-\epsilon^\sharp,\epsilon^\sharp[$ to $\mathbb{R}$ such that $\gamma_l[\epsilon] = \zeta_l(\epsilon)$ for all $\epsilon \in]0,\epsilon^\sharp[$, and $\zeta_l(0) = \tilde{\gamma}$.

Proof. Let ϵ_5, ι_5 be as in Proposition 9.3. Then we take $\epsilon^* = \epsilon_5$, $\iota^* = \iota_5$. If $\epsilon \in]0,\epsilon^*[$, Proposition 9.4(ii) ensures that the operator $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)} = \hat{\tilde{A}}_\epsilon$ has precisely r real eigenvalues $\gamma_r[\epsilon] \leqslant \cdots \leqslant \gamma_1[\epsilon]$ counted with their multiplicity in the interval $]\tilde{\gamma}-\delta,\tilde{\gamma}+\delta[$. Since $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ is the complexification of $\mathcal{A}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$, the above real eigenvalues are also eigenvalues of $\mathcal{A}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ and have the same multiplicity. Then we note that if I is the identity matrix in $M_r(\mathbb{C})$, then we must have

$$\det\bigl(\gamma I - \mathcal{S}(\epsilon,\delta_{2,n}\epsilon\log\epsilon)\bigr) = \prod_{l=1}^{r}\bigl(\gamma - \gamma_l[\epsilon]\bigr) \quad \forall \gamma \in \mathbb{R}, \tag{9.8}$$

for all $\epsilon \in]0,\epsilon^*[$. Indeed, $\mathcal{S}(\epsilon,\delta_{2,n}\epsilon\log\epsilon)$ is the Hermitian matrix of the restriction of $\hat{\mathcal{A}}_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ to $\operatorname{Im} P^\sharp_{(\epsilon,\delta_{2,n}\epsilon\log\epsilon)}$ (see Proposition 9.4). We now define $\Gamma_s(\epsilon,\iota)$ to be the coefficient of γ^{r-s} multiplied by $(-1)^s$ of the polynomial $\det(\gamma I - \mathcal{S}(\epsilon,\iota))$ for all $(\epsilon,\iota) \in]-\epsilon^*,\epsilon^*[\times]-\iota^*,\iota^*[$ and $s \in \{1,\dots,r\}$. Since the $\Gamma_s(\epsilon,\iota)$ are sums of products of the entries of the matrix $\mathcal{S}(\epsilon,\iota)$, Proposition 9.4 ensures that the functions Γ_s are analytic in $]-\epsilon^*,\epsilon^*[\times]-\iota^*,\iota^*[$. Moreover, equality (9.8) ensures the validity of the equalities in (9.5) and that equality (9.6) holds true.

Since $\lim_{\epsilon\to 0^+} \mathcal{S}(\epsilon,\delta_{2,n}\epsilon\log\epsilon) = \mathcal{S}(0,0)$ and the spectrum of $\mathcal{S}(0,0)$ equals $\{\tilde{\lambda}\}$, the Spectral Stability Theorem implies that equality (9.7) holds true (cf. *e.g.*, Rudin [34, Thm. 10.20, p. 257]).

If we further assume that $n \geqslant 3$, then by applying the Rellich Theorem to the analytic family $\{\mathcal{S}(\epsilon,0)\}_{\epsilon\in]-\epsilon^*,\epsilon^*[}$ of Hermitian matrixes, we deduce the existence of $\epsilon^\sharp$ and of the analytic functions ζ_l (cf. *e.g.*, Rellich [33, Thm. 1, p. 57]). □

10. Continuity of the eigenvalues of the operators of the family $\{\tilde{A}_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$

If $\epsilon \in]0,\epsilon_0[$, then the operator $\tilde{A}_\epsilon$ is self adjoint in the Hilbert space (X_{io}, Q_ϵ) and thus all of its eigenvalues ar real and positive, and we can write the nonzero ones as a decreasing sequence $\{\gamma_j[\tilde{A}_\epsilon]\}_{j\in\mathbb{N}\setminus\{0\}}$, which has 0 as limiting point and we know that $\gamma_j[\tilde{A}_\epsilon] = 1/\lambda_j[\Omega(\epsilon)]$ for all $j \in \mathbb{N}\setminus\{0\}$ (cf. Proposition 5.1).

Then we know that the nonzero eigenvalues of $\tilde{A}_0$ coincide with those of the compact self adjoint operator A_{Ω^o}, which in turn equal the reciprocals of the nonzero Steklov eigenvalues in Ω^o. Thus the eigenvalues of $\tilde{A}_0$ are real and positive and we can write the nonzero ones as a decreasing sequence $\{\gamma_j[\tilde{A}_0]\}_{j\in\mathbb{N}\setminus\{0\}}$, which has 0 as limiting point. Moreover, we know that $\gamma_j[\tilde{A}_0] = 1/\lambda_j[\Omega^o]$ for all $j \in \mathbb{N}\setminus\{0\}$ (cf. Proposition 7.4).

By Theorems 6.1 and 8.1, the family $\{\tilde{A}_\epsilon\}_{\epsilon\in[0,\epsilon_0[}$ of operators of $\mathcal{L}(X_{io})$ and the family $\{Q_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ of bilinear and continuous maps on X_{io} are continuous in ϵ and a continuity result for parameter dependent self adjoint and compact operators in a Hilbert space with a parameter dependent scalar product implies that for each $j \in \mathbb{N} \setminus \{0\}$, the eigenvalue $\gamma_j[\tilde{A}_\epsilon]$ depends continuously on $\epsilon \in]0, \epsilon_0[$ (cf. *e.g.*, paper [19, Thm. 5.5] with Lamberti). Then by the continuity result for $\lambda_j[\Omega(\epsilon)]$ at $\epsilon = 0$ of Nazarov [27, Thm. 2.1, p. 288], we deduce the validity of the following.

Theorem 10.1. *Let $\alpha \in]0,1[$, $r \in \mathbb{N} \setminus \{0\}$. Let Ω^i, Ω^o be as in (5.1). Then the map from $[0, \epsilon_0[$ to $]0, +\infty[$ which takes ϵ to $\gamma_j[\tilde{A}_\epsilon]$ is continuous for all $j \in \mathbb{N} \setminus \{0\}$ and the map from $]0, \epsilon_0[$ to $]0, +\infty[$ which takes ϵ to $\lambda_j[\Omega(\epsilon)]$ is continuous and $\lim_{\epsilon \to 0^+} \lambda_j[\Omega(\epsilon)] = \lambda_j[\Omega^o]$ for all $j \in \mathbb{N} \setminus \{0\}$.*

One could also prove Theorem 10.1 by exploiting Theorem 9.5, but for brevity we omit such a proof.

11. Symmetric functions of multiple Steklov eigenvalues

Theorem 11.1. *Let $\alpha \in]0,1[$. Let Ω^i, Ω^o be as in (5.1). Let $t \in \mathbb{N} \setminus \{0\}$ be such that $\tilde{\lambda} \equiv \lambda_t[\Omega^o]$ is an eigenvalue of multiplicity $r \in \mathbb{N} \setminus \{0\}$ of the Steklov problem (1.1) in Ω^o and that*

$$\tilde{\lambda} = \lambda_t\left[\Omega^o\right] = \cdots = \lambda_{t+r-1}\left[\Omega^o\right].$$

Then there exist $\epsilon^* \in]0, \epsilon_0[$ and $\iota^* \in]0, +\infty[$ and r real analytic functions $\Lambda_{t,1}, \ldots, \Lambda_{t,r}$ from the set $]-\epsilon^*, \epsilon^*[\times]-\iota^*, \iota^*[$ to $\mathbb{R}$ such that $\delta_{2,n}\epsilon \log \epsilon \in]-\iota^*, \iota^*[$ for all $\epsilon \in]0, \epsilon^*[$, and such that the equalities in (1.3) hold true for all $\epsilon \in]0, \epsilon^*[$. Moreover,

$$\Lambda_{t,s}(0,0) = \binom{r}{s} \tilde{\lambda}^s \quad \forall s \in \{1, \ldots, r\}. \tag{11.1}$$

If we further assume that $n \geqslant 3$, then there exists $\epsilon^\sharp \in]0, \epsilon^*[$ such that for each $l \in \{1, \ldots, r\}$ there exists an analytic function ξ_l from $]-\epsilon^\sharp, \epsilon^\sharp[$ to $\mathbb{R}$ such that $\lambda_{t+l-1}[\Omega(\epsilon)] = \xi_l(\epsilon)$ for all $\epsilon \in]0, \epsilon^\sharp[$ and $\xi_l(0) = \tilde{\lambda}$.

Proof. Since $\tilde{\lambda}$ is a non zero eigenvalue of multiplicity r of the Steklov eigenvalue problem (1.1) in Ω^o, then Proposition 4.2 implies that $\tilde{\gamma} \equiv 1/\tilde{\lambda}$ is an eigenvalue of multiplicity r of A_{Ω^o}. Then Proposition 7.4 implies that $\tilde{\gamma} \equiv 1/\tilde{\lambda}$ is an eigenvalue of multiplicity r of $\tilde{A}_0$. Then Theorem 9.1 implies that there exists $\delta > 0$ such that $]\tilde{\gamma} - 2\delta, \tilde{\gamma} + 2\delta[\setminus \{\tilde{\gamma}\}$ does not contain 0 and does not contain any point of the spectrum of $\tilde{A}_0$ and Theorem 9.5 implies that there exist $\epsilon^* \in]0, \epsilon_0[$ and $\iota^* \in]0, +\infty[$ and r real analytic functions $\Gamma_1, \ldots, \Gamma_r$ from $]-\epsilon^*, \epsilon^*[\times]-\iota^*, \iota^*[$ to $\mathbb{C}$ such that $\delta_{2,n}\epsilon \log \epsilon \in]-\iota^*, \iota^*[$ for all $\epsilon \in]0, \epsilon^*[$, and such that $\tilde{A}_\epsilon$ has precisely r real eigenvalues counted with their multiplicity $\gamma_r[\epsilon] \leqslant \cdots \leqslant \gamma_1[\epsilon]$ in the interval $]\tilde{\gamma} - \delta, \tilde{\gamma} + \delta[$ and that equalities (9.5) and (9.6) hold true.

By the continuity Theorem 10.1, $\lim_{\epsilon \to 0^+} \gamma_{t+l-1}[\Omega(\epsilon)] = \tilde{\gamma}$ for all $l \in \{1, \ldots, r\}$. Then possibly shrinking ϵ^* we can assume that $\gamma_{t+l-1}[\Omega(\epsilon)] \in]\tilde{\gamma} - \delta, \tilde{\gamma} + \delta[$ and thus we must necessarily have $\gamma_l[\epsilon] = \gamma_{t+l-1}[\Omega(\epsilon)]$ for all $\epsilon \in]0, \epsilon^*[$, and thus $\lambda_{t+l-1}[\Omega(\epsilon)] = 1/\gamma_l[\epsilon]$ for all $\epsilon \in]0, \epsilon^*[$ (see Proposition 5.1). Then we set $\Gamma_0(\epsilon, \iota) \equiv 1$ for all $(\epsilon, \iota) \in]-\epsilon^*, \epsilon^*[\times]-\iota^*, \iota^*[$, and $\Lambda_{t,s}(\epsilon, \iota) \equiv \Re \frac{\Gamma_{r-s}(\epsilon,\iota)}{\Gamma_r(\epsilon,\iota)}$ for all $(\epsilon, \iota) \in]-\epsilon^*, \epsilon^*[\times]-\iota^*, \iota^*[$ and for all $s \in \{1, \ldots, r\}$, then the equalities in (1.3) and (11.1) hold true. Here $\Re$ denotes the real part.

Then the last part of the statement follows by the equalities $\lambda_{t+l-1}[\Omega(\epsilon)] = 1/\gamma_{t+l-1}[\Omega(\epsilon)] = 1/\gamma_l[\epsilon]$ for all $\epsilon \in]0, \epsilon^*[$ and for all $l \in \{1, \ldots, r\}$ and by the last part of the statement of Theorem 9.5. □

Acknowledgements

The author acknowledges the support of ‘Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni’, of ‘INdAM’ and of the project ‘BIRD168373/16: Singular perturbation problems for the heat equation in a perforated domain’ of the University of Padova.

Appendix. An elementary lemma of operator theory

Lemma A.1. *Let Y, Z be (real or complex) normed spaces. Let Y_1 be a finite dimensional subspace of Y. Let $\tilde{T} \in \mathcal{L}(Y,Z)$. If the restriction of $\tilde{T}$ to Y_1 is injective, then there exists an open neighborhood W of $\tilde{T}$ in $\mathcal{L}(Y,Z)$ such that the restriction of T to Y_1 is injective for all $T \in W$.*

Proof. Assume by contradiction that W does not exist. Then there exists a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{L}(Y,Z)$ such that $T_{j|Y_1}$ is not injective for each j and $\lim_{j\to\infty} T_j = \tilde{T}$ in $\mathcal{L}(Y,Z)$.

Let $y_j \in Y_1$ be such that $\|y_j\|_Y = 1$ and $T_j[y_j] = 0$ for all $j \in \mathbb{N}$. Since the sequence $\{y_j\}_{j\in\mathbb{N}}$ is bounded in the finite dimensional normed space Y_1, then there exists a subsequence $\{y_{j_k}\}_{k\in\mathbb{N}}$ which converges to an element $\tilde{y} \in Y_1$. By continuity of the norm, we have $\|\tilde{y}\|_Y = 1$. Next we note that

$$\big\|\tilde{T}[\tilde{y}] - T_{j_k}[y_{j_k}]\big\|_Z \leqslant \|\tilde{T}\|_{\mathcal{L}(Y,Z)}\|\tilde{y} - y_{j_k}\|_Y + \|\tilde{T} - T_{j_k}\|_{\mathcal{L}(Y,Z)}\|y_{j_k}\|_Y \quad \forall k \in \mathbb{N}.$$

Hence, $\lim_{k\to\infty} \|\tilde{T}[\tilde{y}] - T_{j_k}[y_{j_k}]\|_Z = 0$. Since $T_{j_k}[y_{j_k}] = 0 \ \forall k \in \mathbb{N}$, we have $\tilde{T}[\tilde{y}] = 0$, a contradiction. □

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