RECENT RESULTS ON THE ESSENTIAL SELF-ADJOINTNESS OF SUB-LAPLACIANS, WITH SOME REMARKS ON THE PRESENCE OF CHARACTERISTIC POINTS

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Abstract. In this proceeding, we present some recent results obtained in [5] on the essential self-adjointness of sub-Laplacians on non-complete sub-Riemannian manifolds. A notable application is the proof of the essential self-adjointness of the Popp sub-Laplacian on the equiregular connected components of a sub-Riemannian manifold, when the singular region does not contain characteristic points. In their presence, the self-adjointness properties of (sub-)Laplacians are still unknown. We conclude the paper discussing the difficulties arising in this case.

Contents

1. A criterion for essential self-adjointness of sub-Laplacians

Let N be a complete sub-Riemannian manifold, with distribution $\mathcal{D}$. Let $\mathcal{Z} \subset N$ be a smooth embedded hypersurface with no characteristic points, i.e. $\mathcal{D} \pitchfork \mathcal{Z}$. Let ω be a measure on N, smooth on $M = N \setminus \mathcal{Z}$. In the following, we denote with $L^2(M)$ the complex Hilbert space of functions $u : M \to \mathbb{C}$, with scalar product

$$\langle u, v\rangle = \int_M u\bar{v}\, d\omega, \qquad u, v \in L^2(M), \tag{1}$$

where the bar denotes complex conjugation. The corresponding norm is $\|u\|^2 = \langle u, u\rangle$. Similarly, given a coordinate neighborhood $U \subseteq M$ and denoting by dx the Lebesgue measure on it, we denote by $L^2(U, dx)$ the complex Hilbert space of square-integrable functions $u : U \to \mathbb{C}$ satisfying (1) with $d\omega$ replaced by dx and M by U.

The sub-Laplacian Δ_ω is the operator

$$\Delta_\omega u := \operatorname{div}_\omega(\nabla u), \qquad \forall u \in C_c^\infty(M), \tag{2}$$

where the divergence $\operatorname{div}_\omega$ is computed with respect to the measure ω, and ∇ is the sub-Riemannian gradient. Equivalently, $-\Delta_\omega$ can be defined as the non-negative operator associated with the quadratic form

$$\mathcal{E}(u, v) := \int_M g(\nabla u, \nabla \bar{v})\, d\omega, \qquad \forall v, w \in C_c^\infty(M). \tag{3}$$

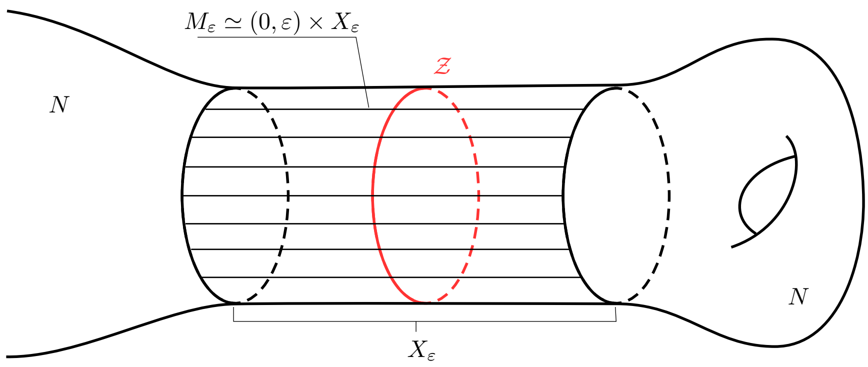

FIGURE 1. Tubular neighborhood of the singular region.

When $\mathcal{Z} = \emptyset$ and the sub-Riemannian structure on N is complete, the sub-Laplacian is essentially self-adjoint [11]. This problem, which is related with the quantum confinement phenomenon, is treated in [5]. There, the authors prove the following self-adjointness criterion.

Theorem 1.1 (Main quantum completeness criterion). *Let N be a complete sub-Riemannian manifold endowed with a measure ω. Assume ω to be smooth on $N \setminus \mathcal{Z}$, where the singular set $\mathcal{Z}$ is a smooth, embedded, compact hypersurface with no characteristic points. Assume also that, for some $\varepsilon > 0$, there exists a constant $\kappa \geq 0$ such that, letting $\delta = d(\mathcal{Z}, \cdot)$, we have*

$$V_{\text{eff}} = \left(\frac{\Delta_\omega \delta}{2}\right)^2 + \left(\frac{\Delta_\omega \delta}{2}\right)' \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta}, \qquad \text{for } 0 < \delta \leq \varepsilon, \tag{4}$$

where the prime denotes the derivative in the direction of $\nabla\delta$. Then Δ_ω with domain $C_c^\infty(M)$ is essentially self-adjoint in $L^2(M)$, where $M = N \setminus \mathcal{Z}$, or any of its connected components.

Moreover, if M is relatively compact, the unique self-adjoint extension of Δ_ω has compact resolvent. Therefore, its spectrum is discrete and consists of eigenvalues with finite multiplicity.

Remark 1.1. The compactness of $\mathcal{Z}$ in Theorem 1.1 can be replaced by the weaker assumption that the (normal) injectivity radius from $\mathcal{Z}$ is strictly positive.

Following the strategy developed in [8, 9], the first part of Theorem 1.1 is proved by a two-step approach based on *Hardy inequality* and *Agmon-type estimates*, see Section 1.1. This strategy can be successfully implemented thanks to the existence of a normal tubular neighborhood close the singular region $\mathcal{Z}$, described in the following Proposition 1.2 (see also Figure 1). The distance from the singular region is $\delta : N \to [0,\infty)$,

$$\delta(p) = \inf\{d(q,p) \mid q \in \mathcal{Z}\}, \qquad \forall p \in N. \tag{5}$$

Proposition 1.2 (Tubular neighborhood for smooth hypersurfaces without characteristic points). *Let N be a smooth sub-Riemannian manifold and $\mathcal{Z} \subset N$ be a smooth, embedded, compact hypersurface with no characteristic points. Then:*

i) *$\delta : N \to [0,\infty)$ is Lipschitz w.r.t. the sub-Riemannian distance and $|\nabla\delta| \leq 1$ a.e.;*
ii) *there exists $\varepsilon > 0$ such that $\delta : M_\varepsilon \to [0,\infty)$ is smooth, where $M_\varepsilon = \{0 < \delta(p) < \varepsilon\}$;*
iii) *letting $X_\varepsilon = \{\delta(p) = \varepsilon\}$, there exists a smooth diffeomorphism $F : (0,\varepsilon) \times X_\varepsilon \to M_\varepsilon$, such that*

$$\delta(F(t,q)) = t \quad \text{and} \quad F_*\partial_t = \nabla\delta, \qquad \text{for } (t,q) \in (0,\varepsilon) \times X_\varepsilon. \tag{6}$$

Moreover, $|\nabla\delta| \equiv 1$ on M_ε.

Remark 1.2. Proposition 1.2 can be simplified if $\mathcal{Z}$ is two-sided (e.g. when N and $\mathcal{Z}$ are orientable). In this special case, $M_\varepsilon = (-\varepsilon, 0) \times \mathcal{Z} \sqcup (0, \varepsilon) \times \mathcal{Z}$ and there is no need to introduce X_ε. Moreover, the compactness assumption here can be replace by the weaker assumption that the (normal) injectivity radius from $\mathcal{Z}$ is strictly positive.

The main criterion presented in Teorem 1.1 is the sub-Riemannian generalization of Theorem 1 in [9]. The new aspects of the proof are the *exploitation of subellipticity* to obtain regularity properties of weak solutions, and the sub-Riemannian version of the *Rellich-Kondrachov theorem*. We present them in the next Lemmas 1.3 and 1.4. We introduce some notations. Given a sub-Riemannian manifold M equipped with a smooth measure ω, we denote by $W^1(M)$ the Sobolev space of functions in $L^2(M)$ with distributional sub-Riemannian gradient $\nabla u \in L^2(\mathcal{D})$, where the latter is the complex Hilbert space of sections of the complexified distribution $X : M \to \mathcal{D}^{\mathbb{C}} \subseteq TM^{\mathbb{C}}$, with scalar product

$$\langle X, Y\rangle = \int_M g(X, Y)\, d\omega, \qquad X, Y \in L^2(\mathcal{D}). \tag{7}$$

The Sobolev space $W^1(M)$ is a Hilbert space when endowed with the scalar product

$$\langle u, v\rangle_{W^1} = \langle \nabla u, \nabla v\rangle + \langle u, v\rangle. \tag{8}$$

Similarly, given a coordinate neighborhood $U \subseteq M$ and denoting by dx the Lebesgue measure on it, we denote by $W^1(U, dx)$ the Sobolev space of functions in $L^2(U, dx)$, with distributional (sub-Riemannian) gradient in $L^2(\mathcal{D}|_U, dx)$, that is the complex Hilbert space of sections of the of the complexified distribution $X : U \to \mathcal{D}^{\mathbb{C}} \subseteq TM^{\mathbb{C}}$, with the scalar product defined in (7) where $d\omega$ is replaced by dx. Moreover, we denote by $L^2_{\mathrm{loc}}(M)$ and $W^1_{\mathrm{loc}}(M)$ the space of functions $u : M \to \mathbb{C}$ such that, for any relatively compact domain $\Omega \subseteq M$, their restriction to Ω belongs to $L^2(\Omega)$ and $W^1(\Omega)$, respectively.

To adhere to the standard notation in quantum physics, let $H = -\Delta_\omega$, with domain $\mathrm{Dom}(H) = C_c^\infty(M)$. The associated symmetric bilinear form is

$$\mathcal{E}(u, v) = \int_M g(\nabla u, \nabla v)\, d\omega, \qquad u, v \in C_c^\infty(M). \tag{9}$$

We use the same symbol to denote the above integral, eventually equal to $+\infty$, for all functions $u, v \in W^1_{\mathrm{loc}}(M)$. We also let, for brevity, $\mathcal{E}(u) = \mathcal{E}(u, u)$.

Lemma 1.3. *Let M be a sub-Riemannian manifold equipped with a smooth measure ω. Then* $\mathrm{Dom}(H^*) \subseteq W^1_{\mathrm{loc}}(M)$.

Lemma 1.3 implies that for any $\psi \in \mathrm{Dom}(H^*)$, its energy $\mathcal{E}(\psi)$ is well defined. This is crucial in the proof of the Agmon estimate of Proposition 1.6 (see inequality (17)).

Lemma 1.4 (Sub-Riemannian Rellich-Kondrachov theorem)**.** *Let M be a sub-Riemannian manifold equipped with a smooth measure ω. Let $\Omega \subseteq M$ be a relatively compact domain with Lipschitz boundary. Then $W^1(\Omega)$ is compactly embedded into $L^2(\Omega)$.*

Lemma 1.4 will be used to prove compactness of the resolvent of Δ (under the assumption that M is relatively compact), see Section 1.2

1.1. **Weak Hardy inequality and Agmon-type estimates.** By using the the diffeomorphism of Proposition 1.2 to identify $M_\varepsilon \simeq (0, \varepsilon) \times X_\varepsilon$, the measure ω reads

$$d\omega(t, q) = e^{2\theta(t,q)} dt\, d\mu(q), \qquad (t, q) \in M_\varepsilon, \tag{10}$$

where $d\mu$ is a fixed smooth measure on X_ε, and θ is a smooth function. This leads to the following expression for V_{eff}:

$$V_{\mathrm{eff}} = (\partial_t \theta)^2 + \partial_t^2 \theta. \tag{11}$$

Hence, condition (4) reads locally

$$V_{\text{eff}} \geq \frac{3}{4t^2} - \frac{\kappa}{t} \qquad \text{for } 0 < t \leq \varepsilon. \tag{12}$$

A combination of (12) together with the 1-dimensional Hardy inequality leads to the weak Hardy inequality (14) presented in the next Proposition 1.5.

Proposition 1.5 (Weak Hardy Inequality)**.** *Let N be a complete sub-Riemannian manifold endowed with a measure ω. Assume ω to be smooth on $M = N \setminus \mathcal{Z}$, where the singular set $\mathcal{Z}$ is a smooth, embedded, compact hypersurface with no characteristic points. Assume also that there exist $\kappa \geq 0$ and $\varepsilon > 0$ such that*

$$V_{\text{eff}} \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta}, \qquad \text{for } \delta \leq \varepsilon. \tag{13}$$

Then, there exist $\eta \leq 1/\kappa$ and $c \in \mathbb{R}$ such that

$$\int_M |\nabla u|^2 \, d\omega \geq \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |u|^2 \, d\omega + c\|u\|^2, \qquad \forall u \in W^1_{\text{comp}}(M), \tag{14}$$

where $M_\eta = \{0 < \delta < \eta\}$. In particular, the operator $H = -\Delta_\omega$ is semibounded on $C_c^\infty(M)$.

The proof of Proposition 1.5 in the case $u \in W^1_{\text{comp}}(M_\varepsilon)$ follows by (12) and the 1-dimensional Hardy inequality. To extend it to $u \in W^1_{\text{comp}}(M)$ one needs a localization argument, exploiting the boundlessness of $|\nabla \delta|$ (see Proposition 1.2).

We now state the Agmon-type estimate that, combined with Proposition 1.5, allows to prove the self-adjointness statement in Theorem 1.1.

Proposition 1.6 (Agmon-type estimate)**.** *Let N be a complete sub-Riemannian manifold endowed with a measure ω. Assume ω to be smooth on $M = N \setminus \mathcal{Z}$, where the singular set $\mathcal{Z}$ is a smooth embedded hypersurface with no characteristic points. Assume also that there exist $\kappa \geq 0$, $\eta \leq 1/\kappa$ and $c \in \mathbb{R}$ such that,*

$$\int_M |\nabla u|^2 \, d\omega \geq \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |u|^2 d\omega + c\|u\|^2, \qquad \forall u \in W^1_{\text{comp}}(M). \tag{15}$$

Then, for all $E < c$, the only solution of $H^*\psi = E\psi$ is $\psi \equiv 0$.

Sketch of the proof. The proof follows the ideas of [8, 4] and is divided into two steps:

Step 1: Let ψ be a solution of $(H^* - E)\psi = 0$ for some $E < c$. For any bounded function $f : M \to \mathbb{R}$ which is Lipschitz w.r.t. the sub-Riemannian distance and satisfies $\operatorname{supp} f \subseteq \overline{M \setminus M_\zeta}$, for some $\zeta > 0$, we have:

$$(c - E)\|f\psi\|^2 \leq \langle \psi, |\nabla f|^2 \psi \rangle - \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |f\psi|^2 d\omega. \tag{16}$$

This is due to the Hardy inequality (15) and to an easy computation that leads to

$$\mathcal{E}(f\psi) = E\|f\psi\|^2 + \langle \psi, |\nabla f|^2 \psi \rangle. \tag{17}$$

Step 2. A particular f is now plugged into (16), by setting

$$f(p) := \begin{cases} F(\delta(p)) & 0 < \delta(p) \leq \eta, \\ 1 & \delta(p) > \eta, \end{cases} \tag{18}$$

for a Lipschitz function F to be chosen in order to satisfy the assumptions of Step 1. Since $|\nabla \delta| \leq 1$ a.e. on M, we have, on M_η, $|\nabla f| = |F'(\delta)||\nabla \delta| \leq |F'(\delta)|$. Thus, (16) implies

$$(c - E)\|f\psi\|^2 \leq \int_{M_\eta} \left[F'(\delta)^2 - \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) F(\delta)^2 \right] |\psi|^2 d\omega. \tag{19}$$

We continue the proof for the case $\kappa = 0$. The following arguments can be adapted also to the cases $\kappa \neq 0$. For $0 < \zeta < 2\zeta < \eta$, we choose F for $\tau \in [2\zeta, \eta]$ to be the solution of

$$F'(\tau) = \frac{1}{\tau} F(\tau), \qquad \text{with } F(\eta) = 1, \tag{20}$$

to be zero on $[0, \zeta]$, and linear on $[\zeta, 2\zeta]$. Namely:

$$F(t) = \begin{cases} 0 & t \in [0,\zeta], \\ \frac{2}{\eta}(t-\zeta) & t \in [\zeta, 2\zeta], \\ \frac{1}{\eta} t & t \in [2\zeta, \eta), \end{cases} \qquad F'(t) = \begin{cases} 0 & t \in [0,\zeta], \\ \frac{2}{\eta} & t \in [\zeta, 2\zeta], \\ \frac{1}{\eta} & t \in [2\zeta, \eta). \end{cases} \tag{21}$$

From (19), this leads to

$$(c-E)\|f\psi\|^2 \leq \int_{M_{2\zeta}\setminus M_\zeta} \left[F'(\delta)^2 - \frac{1}{\delta^2} F(\delta)^2 \right] |\psi|^2 d\omega \leq K^2 \int_{M_{2\zeta}\setminus M_\zeta} |\psi|^2 d\omega \tag{22}$$

for a constant $K > 0$. If we let $\zeta \to 0$, then f tends to an almost everywhere strictly positive function. Recalling that $E < c$, and taking the limit, (22) implies $\psi \equiv 0$. □

Remark 1.3. If (15) is replaced with the weaker assumption

$$\int_M |\nabla u|^2 \, d\omega \geq a \int_{M_\eta} \left(\frac{1}{\delta^2} - \frac{\kappa}{\delta} \right) |u|^2 d\omega + c\|u\|^2, \qquad \forall u \in W^1_{\text{comp}}(M) \tag{23}$$

for $\frac{3}{4} < a < 1$, then the arguments in the previous proof cannot be applied. In fact, defining F as in (20) it is impossible to find a constant $K > 0$ such that (22) is satisfied.

1.2. **Sketch of the proof of Theorem 1.1.** To prove that Δ_ω with domain $C_c^\infty(M)$ is essentially self-adjoint in $L^2(M)$ we apply the classical criterion of [10, Thm. X.I and Corollary]: since H is semibounded (by Proposition 1.5), H is essentially self-adjoint if and only if there exists $E < 0$ such that the only solution of $H^*\psi = E\psi$ is $\psi \equiv 0$. This is guaranteed by the Agmon-type estimate of Proposition 1.6, whose hypotheses are satisfied again by the conclusion of Proposition 1.5.

To prov compactness of the resolvent it is sufficient to show existence of a value $z < c$ such that the resolvent $(H^* - z)^{-1}$ is compact on $L^2(M)$. To this purpose one must prove that for any bounded sequence $\psi_n \in L^2(M)$, say $\|\psi_n\| \leq (c - z)$, the image $u_n = (H^* - z)^{-1}\psi_n \in \text{Dom}(H^*)$ has a subsequence converging in $L^2(M)$. To prove it we decompose $u_n = u_{n,1} + u_{n,2}$ where $u_{n,1}$ is supported in a neighborhood of $\mathcal{Z}$ and $u_{n,2}$ is compactly supported in a neighborhood of $M \setminus \mathcal{Z}$. By using the the sub-Riemannian Rellich-Kondrachov theorem of Lemma 1.4 it is possible to show that $u_{n,2}$ converges up to subsequences in $L^2(M)$. Moreover, by the weak Hardy inequality (14), it is possible to show that for all $k \in \mathbb{N}$, there is a subsequence $n \mapsto \gamma_k(n)$ such that $u_{\gamma_k(n)} = \sum_{i=1}^2 u_{\gamma_k(n),i}$ with $\|u_{\gamma_k(n),1}\| \leq C/k$ and $u_{\gamma_k(n),2}$ is convergent in $L^2(M)$. Exploiting these facts, a Cauchy subsequence of u_n can be extracted, yielding the compactness of $(H^* - z)^{-1}$, and concluding the proof. □

2. Applications to the Popp sub-Laplacian

Theorem 1.1 can be applied to study essential self-adjointness of the sub-Laplacian $\Delta = \Delta_{\mathcal{P}}$, where $\mathcal{P}$ is the intrinsic Popp's measure.

2.1. **Popp's measure.** Popp's measure was introduced in [7]. It was used in [1] to define an intrinsic sub-Laplacian in the sub-Riemannian setting. In the following, we will use the explicit formula for Popp's measure given in [2] in terms of adapted frames, in order to define Popp's measure. For an intrinsic definition, we refer to [7, 2].

Let $r(q) = \dim(\mathcal{D}_q)$ be the *rank* of the distribution at $q \in N$. Moreover, for $k \in \mathbb{N}$, let

$$\mathcal{D}^k_q := \text{span}\{[X_1, \ldots, [X_{j-1}, X_j]]_q : X_i \in \Gamma(\mathcal{D}),\ j \leq k\}. \tag{24}$$

We call the *step* of the sub-Riemannian structure at q the minimal integer $s = s(q) \in \mathbb{N}$ such that $\mathcal{D}^s_q = T_qN$.

Definition 2.1. Let $A \subseteq N$. We say that a sub-Riemannian structure on N is *equiregular* on A if $\dim(\mathcal{D}^k_q)$ is constant for $q \in A$ and for any $k \in \mathbb{N}$.

Remark 2.1. Already $r(q) = \dim(\mathcal{D}^1_q)$ can be non-constant. For instance, this is the case of almost-Riemannian manifolds, where there exists a closed set $\mathcal{Z} \subset N$ such that $\dim(\mathcal{D}^1_q) = \dim N$ for every $q \in N \setminus \mathcal{Z}$.

Let $\mathcal{O} \subseteq N$ be an equiregular neighborhood of an n-dimensional sub-Riemannian manifold N. A local frame $X_1, \dots, X_n$ on $\mathcal{O}$ is said to be *adapted* to the sub-Riemannian structure if $X_1, \dots, X_{k_i}$ is a local frame for $\mathcal{D}^i$, where $k_i = \dim(\mathcal{D}^i)$ is constant on $\mathcal{O}$. In particular $r(q) \equiv r$ is constant on $\mathcal{O}$. Notice that, the equiregularity assumption means that, on $\mathcal{O}$, $\mathcal{D}^i$ are "true" distributions, and hence that there always exists a local adapted frame. Define the smooth functions $b^\ell_{i_1\dots i_j} \in C^\infty(N)$ as

$$[X_{i_1}, [X_{i_2}, \dots, [X_{i_{j-1}}, X_j]]] = \sum_{\ell=k_{j-1}+1}^{k_j} b^\ell_{i_1 i_2 \dots i_j} X_\ell \qquad \mathrm{mod} \;\; \mathcal{D}^{j-1}, \tag{25}$$

where $1 \le i_1, \dots, i_j \le m = \dim(\mathcal{D}^1)$. Consider the $k_j - k_{j-1}$ dimensional square matrices

$$(B_j)^{h\ell} = \sum_{i_1,\dots,i_j=1}^{r} b^h_{i_1,\dots,i_j} b^\ell_{i_1,\dots,i_j}, \qquad \forall j = 1, \dots, s, \tag{26}$$

where s is the step of the structure. Then, denoting by $\nu^1, \dots, \nu^n$ the dual frame to $X_1, \dots, X_n$, the Popp's measure reads

$$\mathcal{P} = \frac{1}{\sqrt{\prod_{j=1}^s \det B_j}} |\nu^1 \wedge \dots \wedge \nu^n|. \tag{27}$$

One can check that the measure defined by (27) does not depend on the choice of the local adapted frame, and can be taken as the definition of Popp's measure. It is not hard to see, using the very definition, that if $q \in \bar{\mathcal{O}}$ is a non-equiregular point, then $\lim \sqrt{\prod \det B_j} = 0$ hence the Radon-Nikodym derivative of Popp's measure computed with respect to any globally smooth measure on N diverges to $+\infty$ on the singular region $\mathcal{Z}$. Uniform estimates of this divergence can be found in [6].

2.2. **Popp-regular structures.** The study of condition (4) is a difficult task, because it requires the explicit knowledge of the distance from the singular set. In the following we define a class of sub-Riemannian structures, to which Theorem 1.1 applies, without knowing an explicit expression for δ. Let ϖ be a reference measure, smooth and positive on the whole N and let $\mathcal{P}$ denote Popp's measure, smooth on $M = N \setminus \mathcal{Z}$. We define the function $\rho : N \to \mathbb{R}$ by setting

$$\rho(p) = \begin{cases} \left(\frac{d\mathcal{P}}{d\varpi}\right)^{-1}(p) & \text{if } p \in N \setminus \mathcal{Z}, \\ 0 & \text{if } p \in \mathcal{Z}. \end{cases} \tag{28}$$

This is the unique continuous extension to $\mathcal{Z}$ of the reciprocal of the Radon-Nikodym derivative of $\mathcal{P}$ with respect to ϖ. Notice that ρ is smooth on $N \setminus \mathcal{Z}$.

Definition 2.2. We say that a sub-Riemannian manifold N is *Popp-regular* if it is equiregular outside a smooth embedded hypersurface $\mathcal{Z}$ containing no characteristic points, and there exists $k \in \mathbb{N}$ such that, for all $q \in \mathcal{Z}$ there exists a neighborhood $\mathcal{O}$ of q and a smooth submersion $\psi : \mathcal{O} \to \mathbb{R}$ such that the function ρ defined in (28) satisfies $\rho|_{\mathcal{O}} = \psi^k$.

Definition 2.2 generalizes the notion of *regular* almost-Riemannian structure given in [9, Def. 7.10]. Notice that the sub-Riemannian structure in Example 2.1 is Popp-regular.

Proposition 2.3. *Let N be a complete and Popp-regular sub-Riemannian manifold, with compact singular set $\mathcal{Z}$. Then, the sub-Laplacian $\Delta_{\mathcal{P}}$ with domain $C_c^\infty(M)$ is essentially self-adjoint in $L^2(M)$, where $M = N \setminus \mathcal{Z}$ or one of its connected components. Moreover, if M is relatively compact, the unique self-adjoint extension of $\Delta_{\mathcal{P}}$ has compact resolvent.*

This settles, at least in the Popp-regular case, the conjecture proposed in [3] on the essential self-adjointness of the intrinsic sub-Laplacian.

2.3. **Examples.** We start by considering a family of structures generalizing the Martinet structure. These are complete sub-Riemannian structures on $\mathbb{R}^3$, equiregular outside a hypersurface $\mathcal{Z} \subset \mathbb{R}^3$, on which the distance from $\mathcal{Z}$ is explicit. Using Theorem 1.1 we deduce essential self-adjointness of $\Delta = \Delta_{\mathcal{P}}$ defined on $C_c^\infty(N \setminus \mathcal{Z})$.

Example 2.1 (k-Martinet distribution). Let $k \in \mathbb{N}$. We consider the sub-Riemannian structure on $\mathbb{R}^3$ defined by the following global generating family of vector fields:

$$X_1 = \partial_x, \qquad X_2 = \partial_y + x^{2k}\partial_z. \tag{29}$$

The singular region is $\mathcal{Z} = \{x = 0\}$ and the distance from $\mathcal{Z}$ is $\delta(x, y, z) = |x|$. Using formula (27), the associated Popp's measure turns out to be

$$\mathcal{P} = \frac{1}{2\sqrt{2}k|x|^{2k-1}}\, dx \wedge dy \wedge dz. \tag{30}$$

The case $k = 1$ is the standard Martinet structure considered in the introduction. Notice that the injectivity radius from $\mathcal{Z}$ is infinite, hence even if $\mathcal{Z}$ is not compact we can apply Theorem 1.1. We compute the effective potential V_{eff} using (11). Indeed we have

$$\theta = \theta(x) = \frac{1}{2}\log\frac{1}{2\sqrt{2}kx^{2k-1}}, \tag{31}$$

and thus, using (11), we have

$$V_{\text{eff}}(x) = \frac{4k^2 - 1}{4x^2} \geq \frac{3}{4x^2}, \qquad \forall k \geq 1. \tag{32}$$

Hence (4) is satisfied, and $\Delta_{\mathcal{P}}$ with domain $C_c^\infty(\mathbb{R}^3 \setminus \mathcal{Z})$ is essentially self-adjoint.

We generalize Example 7.2 in [9], showing an example of non-Popp-regular sub-Riemannian structure to which Theorem 1.1 does not apply.

Example 2.2 (non-Popp-regular sub-Riemannian structure). Consider the sub-Riemannian structure on $\mathbb{R}^4$ given by the following generating family of vector fields:

$$X_1 = \partial_1 + x_3\partial_4, \qquad X_2 = x_1(x_1^{2\ell} + x_2^2)\partial_2, \qquad X_3 = \partial_3. \tag{33}$$

The singular region is $\mathcal{Z} = \{x_1 = 0\}$. The following set of vector fields is an adapted frame on $\mathbb{R}^4 \setminus \mathcal{Z}$.

$$\underbrace{X_1,\ X_2,\ X_3}_{\mathcal{D}^1}, \qquad \underbrace{X_4 = [X_3, X_1] = \partial_4}_{\mathcal{D}^2/\mathcal{D}^1}. \tag{34}$$

Using formula (27), we have the following expression for Popp's measure

$$\mathcal{P} = \frac{1}{\sqrt{2}x_1(x_1^{2\ell} + x_2^2)} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4, \tag{35}$$

or, equivalently, $\mathcal{P} = x_1^{a(x)} e^{2\varphi(x)} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$, where

$$a(x) = \begin{cases} -(2\ell+1) & x_2 = 0, \\ -1 & x_2 \neq 0, \end{cases} \qquad \varphi(x) = \begin{cases} -\frac{1}{2}\log\sqrt{2} & x_2 = 0, \\ -\frac{1}{2}\log\left(\sqrt{2}(x_1^{2\ell} + x_2^2)\right) & x_2 \neq 0. \end{cases} \tag{36}$$

Noticing that $\delta(x_1,x_2,x_3,x_4)=x_1$, the effective potential reads

$$V_{\text{eff}} = \frac{a(x)(a(x)-2)}{4x_1^2} + R(x), \text{ with } R(x) = \frac{a(x)}{x_1}\partial_1\varphi(x) + (\partial_1\varphi(x))^2 + \partial_1^2\varphi(x). \tag{37}$$

We have

$$R(x) = \begin{cases} 0 & x_2 = 0, \\ \frac{\ell t^{2\ell-2}}{(t^{2\ell}+x_2^2)^2}\left[(\ell+2)t^{2\ell} + (2-2\ell)x_2^2\right] & x_2 \neq 0. \end{cases} \tag{38}$$

Combining (36)-(38) we deduce that $V_{\text{eff}} = 3/(4x_1^2) + R(x)$ if $x_2 \neq 0$, and it is easy to see that the behavior of $R(x)$ depends on the choice of the parameter ℓ. In particular, if $\ell = 1$, $R(x) \geq 0$ and we deduce essential self-adjointness of $\Delta = \Delta_{\mathcal{P}}$ by Theorem 1.1. On the other hand, if $\ell > 1$, along any sequence $x^i = (1/i, 1/i, 0, 0)$, we have $x_1^i R(x_i) \to -\infty$. Hence, we cannot apply Theorem 1.1.

3. Characteristic points

In this section, we discuss the case in which the singular region has characteristic points. This is a subtle and difficult technical issue, so we consider the easiest case which appears already in the setting of almost-Riemannian geometry. In this case, the metric structure in the regular region is actually Riemannian, and the Popp sub-Laplacian is simply the Laplace-Beltrami operator in the regular region. See [9, Section 7] for a self-contained and concise introduction to almost-Riemannian geometry.

Consider the almost-Riemannian structure on $\mathbb{R}^2$ defined by the global vector fields:

$$X_1 = \partial_x, \qquad X_2 = (y-x^2)\partial_y. \tag{39}$$

The singular region is the parabola $\mathcal{Z} = \{y = x^2\}$ and the origin is a characteristic (or tangency) point for $\mathcal{Z}$. We stress that the essential self-adjointness properties of the Laplace-Beltrami operator in the regular region remain unknown even in this simple case.

In presence of characteristic points, the distance from the singular region is not smooth, and in particular the normal tubular neighborhood of Proposition 1.2 does not exist. Therefore, the arguments of [9, 5] cannot be applied. To see that δ is non-smooth arbitrarily close to a characteristic point, notice that X_1, X_2 are invariant under the reflection $(x,y) \in \mathbb{R}^2 \mapsto (-x,y)$. Therefore, for any $p \in \{(x,y) \in \mathbb{R}^2 \mid x = 0, y \neq 0\}$ there exist at least two distinct minimizing geodesics joining p with the characteristic point. In this case, it is well known that the distance is not differentiable at p. (See Figure 2).

Let M be either connected component of $\mathbb{R}^2 \setminus \mathcal{Z}$. The Riemannian measure is

$$d\mathcal{P} = \tfrac{1}{|y-x^2|}dxdy, \tag{40}$$

and the Laplace-Beltrami operator $H = -\Delta$ is

$$H = -\partial_x^2 - (y-x^2)^2\partial_y^2 - \frac{2x}{y-x^2}\partial_x - (y-x^2)\partial_y, \qquad \text{Dom}(H) = C_c^\infty(M). \tag{41}$$

Consider the unitary transformation $T : L^2(\mathbb{R}^2, dxdy) \to L^2(\mathbb{R}^2, \mathcal{P})$ given by

$$Tu(x,y) = \sqrt{|y-x^2|}u(x,y). \tag{42}$$

The operator H is unitary equivalent to $\tilde{H} = T^{-1} \circ H \circ T$, with domain $C_c^\infty(M)$. A straightforward computation yields

$$\tilde{H} = -\partial_x^2 - (y-x^2)^2\partial_y^2 - 2(y-x^2)\partial_y + \left(\frac{1}{(y-x^2)} + \frac{3x^2}{(y-x^2)^2} - \frac{1}{4}\right), \tag{43}$$

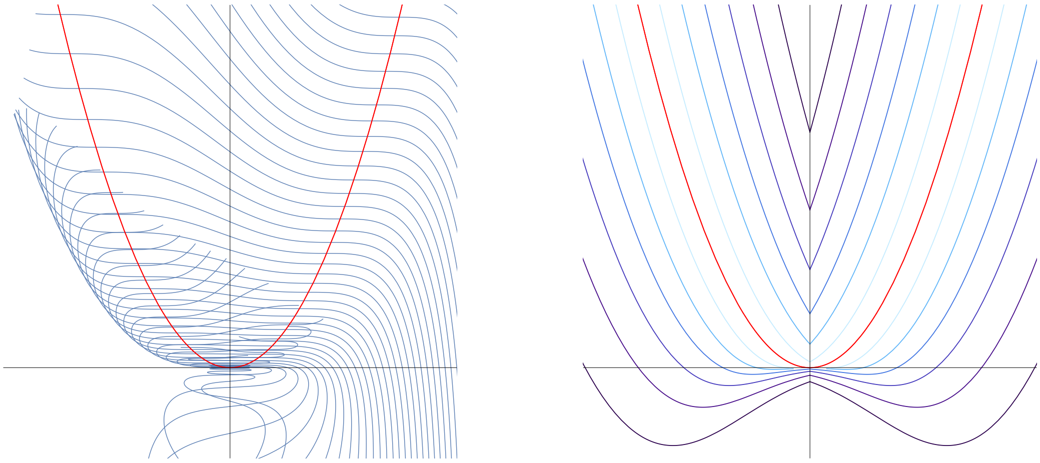

FIGURE 2. Some metric properties of the almost-Riemannian structure (39). *Left*: geodesics satisfying the necessary condition for minimality from $\mathcal{Z}$ (red parabola) and starting at points of $\mathcal{Z}$ with $x > 0$. Numerical evidence suggests that they are minimizing until they cross the vertical axis. *Right*: Level sets of the distance from $\mathcal{Z}$.

The above operator is of the form $\tilde{H} = -\tilde{\Delta} + V$, where $\tilde{\Delta}$ is the Laplace operator of the non-complete Riemannian metric defined on M by (39), but associated with the Lebesgue measure, and

$$V(x,y) = \frac{1}{(y-x^2)} + \frac{3x^2}{(y-x^2)^2} - \frac{1}{4}. \tag{44}$$

This class of Schrödinger-type operators has been studied in [9]. There, at least when the almost-Riemannian distance from $\mathcal{Z}$ is smooth, it was proved that a sufficient condition for essential-self adjointness is

$$V(x,y) \geq \frac{3}{4}\frac{1}{\delta(x,y)^2} - \frac{\kappa}{\delta(x,y)}, \tag{45}$$

for some $\kappa \geq 0$. As we remarked, the almost-Riemannian distance from $\mathcal{Z}$ is not smooth, but condition (45) still make sense. One could hope that, at least in this case, (45) is still sufficient for essential self-adjointess.

Although in principle it is possible to compute δ explicitly, and hence verify (45), such a computation seems to be very hard to obtain due to the apparent non-integrability of the associated Hamilton equations. In order to obtain some insights on the validity of (45) we approximate δ by assuming that the minimizing geodesic from (x,y) to $\mathcal{Z}$ to be given by the integral curves of $\pm X_1$, that is

$$\delta(x,y) \simeq |\sqrt{y} - |x||. \tag{46}$$

Numerical experiments suggest that this approximation is reasonable at least for (x,y) sufficiently near the singularity (i.e. $|y-x^2|$ small). Unfortunately, a simple computation shows that, assuming the validity of this approximation, (45) is not satisfied near the origin on either side of the singularity.

This suggests that a direct extension of the techniques of [9, 5], with some technical workaround to deal with the non-smoothness of δ, is not the right approach.

We stress that Proposition 1.2 is merely a technical tool to prove the Agmon-type estimate 1.6. The latter is the fundamental result which prevents weak solutions of $H^*\psi = E\psi$ to be supported arbitrarily close to $\mathcal{Z}$. We observe that for any $\varepsilon > 0$, the set $\mathcal{Z} \cap \{|x| > \varepsilon\}$ possess no characteristic points, and thus an Hardy-type inequality as (14) can be deduced outside a small ball B_ε centered in 0 (with constants that are not uniform, and actually explode, for $\varepsilon \to 0$). Therefore, our current line of investigation aims to pair the aforementioned inequality with a second one, valid on B_ε, and where δ is replaced by the almost-Riemannian distance from the origin.

The obstacle to this line of proof is the helplessness of the current state-of-the-art techniques to derive Hardy-type inequalities close to the characteristic point. This requires a precise and deep investigation of the properties of geodesics, which we are not able to carry out. However, we remark that numerical experiments suggest that such an Hardy inequality should hold.

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