

## Research Article

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# Extremals for fractional Moser-Trudinger inequalities in dimension 1 via harmonic extensions and commutator estimates

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**Abstract:** We prove the existence of extremals for fractional Moser-Trudinger inequalities in an interval and on the whole real line. In both cases we use blow-up analysis for the corresponding Euler-Lagrange equation, which requires new sharp estimates obtained via commutator techniques.

**Keywords:** Moser-Trudinger, fractional Laplacian, extremals, blow-up, commutator estimates.

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## 1 Introduction

The celebrated Moser-Trudinger inequality [28] states that for  $\Omega \subset \mathbb{R}^n$  with finite measure  $|\Omega|$  we have

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} e^{\alpha_n |u|^{\frac{n}{n-1}}} dx \leq C|\Omega|, \quad \alpha_n := n\omega_{n-1}^{-1}, \quad (1)$$

where  $\omega_{n-1}$  is the volume of the unit sphere in  $\mathbb{R}^n$ . The constant  $\alpha_n$  is sharp in the sense that the supremum in (1) becomes infinite if  $\alpha_n$  is replaced by any  $\alpha > \alpha_n$ . In the case  $\Omega = \mathbb{R}^2$ , B. Ruf [34] proved a similar inequality, using the

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full  $W^{1,2}$ -norm instead of the  $L^2$ -norm of the gradient, then generalized to  $\mathbb{R}^n$ ,  $n \geq 2$  by Li-Ruf [20] as

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{L^n(\mathbb{R}^n)}^n + \|\nabla u\|_{L^n(\mathbb{R}^n)}^n \leq 1} \int_{\mathbb{R}^n} \left( e^{\alpha_n |u|^{\frac{n}{n-1}}} - 1 \right) dx < \infty. \quad (2)$$

Higher-order versions of (1) were proven by Adams [2] on the space  $W^{k, \frac{n}{k}}(\Omega)$  for  $n > k \in \mathbb{N}$ .

In [17] the authors proved the following 1-dimensional fractional extension of the previous results (for the definition of  $H^{\frac{1}{2},2}(\mathbb{R})$  and  $(-\Delta)^{\frac{1}{4}}$  see (65) in the Appendix).

**Theorem A.** *Set  $I := (-1, 1) \subset \mathbb{R}$  and  $\tilde{H}^{\frac{1}{2},2}(I) := \{u \in H^{\frac{1}{2},2}(\mathbb{R}) : u \equiv 0 \text{ on } \mathbb{R} \setminus I\}$ . Then we have*

$$\sup_{u \in \tilde{H}^{\frac{1}{2},2}(I), \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(I)} \leq 1} \int_I \left( e^{\alpha u^2} - 1 \right) dx = C_\alpha < \infty, \quad \text{for } \alpha \leq \pi, \quad (3)$$

and

$$\sup_{u \in H^{\frac{1}{2},2}(\mathbb{R}), \|u\|_{H^{\frac{1}{2},2}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} \left( e^{\alpha u^2} - 1 \right) dx = D_\alpha < \infty, \quad \text{for } \alpha \leq \pi, \quad (4)$$

where  $\|u\|_{H^{\frac{1}{2},2}(\mathbb{R})}^2 := \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2$ . The constant  $\pi$  is sharp in (3) and (4).

More general results have recently appeared, see e.g. [1; 12; 18; 27; 35; 38], in which both the dimension and the (fractional) order of differentiability have been generalized. For instance, (3) and (4) can be seen as 1-dimensional cases of the more general results of [18; 27; 12] that hold in arbitrary dimension  $n$ .

The existence of extremals for this kind of inequalities is a challenging question. Existence of extremals for (1) was originally proven by L. Carleson and A. Chang [5] in the case of the unit ball, a fundamental result later extended by Struwe [37] and Flucher [11] to the case of general bounded domains in  $\mathbb{R}^2$  and by K. Lin [22] to the case of bounded domains in  $\mathbb{R}^n$ . In the case of the Li-Ruf inequality (2), the existence of extremals appears in [20] when  $n \geq 3$  and was proven by Ishiwata [16] when  $n = 2$ . For the higher-order Adams inequality the existence of extremals has been proven in various cases, e.g. by Li-Ndiaye [21] on a 4 dimensional closed manifold, by Lu-Yang [23] for a 4 dimensional bounded domain and by DelaTorre-Mancini [7] for a bounded domain in  $\mathbb{R}^{2m}$ ,  $m \geq 1$  arbitrary.

On the other hand, the existence of extremals for the fractional Moser-Trudinger inequality has remained open until now, with the exception of Takahashi [38] considering a subcritical version of (4) of Adachi-Tanaka type [1], and Li-Liu [19] treating the case of a fractional Moser-Trudinger on  $H^{\frac{1}{2},2}(\partial M)$  with  $M$  a compact Riemann surface with boundary. The idea of Li and Liu is that working on the boundary of a compact manifold, one can localize the  $H^{\frac{1}{2},2}$ -norm.

Applying the same method for an interval  $I \subset \mathbb{R}$  creates problems near  $\partial I$ , which require additional care in the estimate, and the problem becomes even more challenging when working on the whole  $\mathbb{R}$ . The main purpose of this paper is to handle these two cases and prove that the suprema in (3) and (4) are attained.

**Theorem 1.1.** *For any  $0 < \alpha \leq \pi$ , the inequality (3) has an extremal i.e. there exists  $u_\alpha \in \tilde{H}^{\frac{1}{2},2}(I)$  such that  $\|(-\Delta)^{\frac{1}{4}} u_\alpha\|_{L^2(\mathbb{R})} \leq 1$  and*

$$\int_I \left( e^{\alpha u_\alpha^2} - 1 \right) dx = C_\alpha.$$

Theorem 1.1 is rather simple to prove for  $\alpha \in (0, \pi)$ , while the case  $\alpha = \pi$  relies on a delicate blow-up analysis for subcritical extremals.

A similar analysis can be carried out for the Ruf-type inequality (4). However, working on the whole real line we need to face additional difficulties due to the lack of compactness of the embedding of  $H = H^{\frac{1}{2},2}(\mathbb{R})$  into  $L^2(\mathbb{R})$ : vanishing at infinity might occur for maximizing sequences, even in the sub-critical case  $\alpha \in (0, \pi)$ . This issue is not merely technical indeed Takahashi [38] proved that (4) has no extremal when  $\alpha$  is small enough. Here, in analogy with the results in dimension  $n \geq 2$ , we prove that the supremum in (4) is attained if  $\alpha$  sufficiently close to  $\pi$ .

**Theorem 1.2.** *There exists  $\alpha^* \in (0, \pi)$  such that for  $\alpha^* \leq \alpha \leq \pi$  the inequality (4) has an extremal, namely, there exists  $\bar{u}_\alpha \in H^{\frac{1}{2},2}(\mathbb{R})$  such that  $\|\bar{u}_\alpha\|_{H^{\frac{1}{2},2}(\mathbb{R})} \leq 1$  and*

$$\int_{\mathbb{R}} \left( e^{\alpha \bar{u}_\alpha^2} - 1 \right) dx = D_\alpha.$$

As for Theorem 1.1, the proof of Theorem 1.2 for  $\alpha = \pi$  is based on blow-up analysis. In fact we need to study the blow-up of a non-local equation on the whole real line (no boundary conditions), as done in the following theorem.

**Theorem 1.3.** *Let  $(u_k) \subset H = H^{\frac{1}{2},2}(\mathbb{R})$  be a sequence of non-negative solutions to*

$$(-\Delta)^{\frac{1}{2}}u_k + u_k = \lambda_k u_k e^{\alpha_k u_k^2} \quad \text{in } \mathbb{R}, \quad (5)$$

where  $\alpha_k \rightarrow \pi$  and  $\lambda_k \rightarrow \lambda_\infty \geq 0$ . Assume  $u_k$  even and decreasing ( $u_k(-x) = u_k(x) \leq u_k(y)$  for  $x \geq y \geq 0$ ) for every  $k$  and set  $\mu_k := \sup_{\mathbb{R}} u_k = u_k(0)$ . Assume also that

$$\Lambda := \limsup_{k \rightarrow \infty} \|u_k\|_H^2 < \infty. \quad (6)$$

Then up to extracting a subsequence we have that either

(i)  $\mu_k \leq C$ ,  $u_k \rightarrow u_\infty$  in  $C_{\text{loc}}^\ell(\mathbb{R})$  for every  $\ell \geq 0$ , where  $u_\infty \in C_{\text{loc}}^\ell(\mathbb{R}) \cap H$  solves

$$(-\Delta)^{\frac{1}{2}}u_\infty + u_\infty = \lambda_\infty u_\infty e^{\pi u_\infty^2} \quad \text{in } \mathbb{R}, \quad (7)$$

or

(ii)  $\mu_k \rightarrow \infty$ ,  $u_k \rightarrow u_\infty$  weakly in  $H$  and strongly in  $C_{\text{loc}}^0(\bar{\mathbb{R}} \setminus \{0\})$  where  $u_\infty$  is a solution to (7). Moreover, setting  $r_k$  such that

$$\lambda_k r_k \mu_k^2 e^{\alpha_k \mu_k^2} = \frac{1}{\alpha_k}, \quad (8)$$

and

$$\eta_k(x) := 2\alpha_k \mu_k (u_k(r_k x) - \mu_k), \quad \eta_\infty(x) := -\log(1 + |x|^2), \quad (9)$$

one has  $\eta_k \rightarrow \eta_\infty$  in  $C_{\text{loc}}^\ell(\mathbb{R})$  for every  $\ell \geq 0$ ,  $\sup_k \|\eta_k\|_{L_s(\mathbb{R})} < \infty$  for any  $s > 0$  (cfr. (63)), and  $\Lambda \geq \|u_\infty\|_H^2 + 1$ .

The proof of Theorem 1.3 is quite delicate because local elliptic estimates of a nonlocal equation depend on global bounds as we shall prove in Lemma 3.6. This will be based on sharp commutator estimates (Lemma 3.3), as developed in [24] for the case of a bounded domain in  $\mathbb{R}^n$ , extending to the fractional case the approach of [26].

We expect similar existence results to hold for a perturbed version of inequalities (3)-(4), as in [25] and [39] (see also the recent results in [15]), but we will not investigate this issue here.

## 2 Proof of Theorem 1.1

### 2.1 Strategy of the proof

We will focus on the case  $\alpha = \pi$ , since the existence of extremals for (3) with  $\alpha \in (0, \pi)$  follows easily by Vitali's convergence theorem, see e.g. the argument in [25, Proposition 6].

Let  $u_k$  be an extremal of (3) for  $\alpha = \alpha_k = \pi - \frac{1}{k}$ . By replacing  $u_k$  with  $|u_k|$  we can assume that  $u_k \geq 0$ . Moreover  $\|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2(\mathbb{R})} = 1$ , and  $u_k$  satisfies the Euler-Lagrange equation

$$(-\Delta)^{\frac{1}{2}} u_k = \lambda_k u_k e^{\alpha_k u_k^2}, \tag{10}$$

with bounds on the Lagrange multipliers  $\lambda_k$  (see (13)).

Using the monotone convergence theorem we also get

$$\lim_{k \rightarrow \infty} \int_I \left( e^{\alpha_k u_k^2} - 1 \right) dx = \lim_{k \rightarrow \infty} C_{\alpha_k} = C_\pi, \tag{11}$$

where  $C_{\alpha_k}$  and  $C_\pi$  are as in (3).

If  $\mu_k := \max_I u_k = O(1)$  as  $k \rightarrow \infty$ , then up to a subsequence  $u_k \rightarrow u_\infty$  locally uniformly, where by (11)  $u_\infty$  maximizes (3) with  $\alpha = \pi$ . Therefore we will work by contradiction, assuming

$$\lim_{k \rightarrow \infty} \mu_k = \infty. \tag{12}$$

By studying the blow-up behavior of  $u_k$ , see in particular Propositions 2.2 and 2.8, we will show that (12) implies  $C_\pi \leq 4\pi$  (Proposition 2.9), but with suitable test functions we will also prove that  $C_\pi > 4\pi$  (Proposition 2.10), hence contradicting (12) and completing the proof of Theorem 1.1.

### 2.2 The blow-up analysis

The following proposition is well known in the local case, and its proof in the present setting is similar to the local one. We give it for completeness.

**Proposition 2.1.** *We have  $u_k \in C^\infty(I) \cap C^{0, \frac{1}{2}}(\bar{I})$ ,  $u_k > 0$  in  $I$ , and  $u_k$  is symmetric with respect to 0 and decreasing with respect to  $|x|$ . Moreover,*

$$0 < \lambda_k < \lambda_1(I). \tag{13}$$

Up to a subsequence we have  $\lambda_k \rightarrow \lambda_\infty$  and  $u_k \rightarrow u_\infty$  weakly in  $\tilde{H}^{\frac{1}{2},2}(I)$  and strongly in  $L^2(I)$ , where  $u_\infty$  solves

$$(-\Delta)^{\frac{1}{2}} u_\infty = \lambda_\infty u_\infty e^{\pi u_\infty^2}. \quad (14)$$

*Proof.* For the first claim see Remark 1.4 in [24]. The positivity follows from the maximum principle, and symmetry and monotonicity follow from the moving point technique, see e.g. [8, Theorem 11].

Now testing (10) with  $\varphi_1$ , the first eigenfunction of  $(-\Delta)^{\frac{1}{2}}$  in  $\tilde{H}^{\frac{1}{2},2}(I)$ , positive and with eigenvalue  $\lambda_1(I) > 0$ , we obtain

$$\lambda_1(I) \int_I u_k \varphi_1 dx = \lambda_k \int_I u_k e^{\alpha_k u_k^2} \varphi_1 dx > \lambda_k \int_I u_k \varphi_1 dx,$$

hence proving (13). By the theorem of Banach-Alaoglu and the compactness of the Sobolev embedding of  $\tilde{H}^{\frac{1}{2},2}(I) \hookrightarrow L^2(I)$ , we obtain the claimed convergence of  $u_k$  to  $u_\infty$ . Finally, to show that  $u_\infty$  solves (14), test with  $\varphi \in C_c^\infty(I)$ :

$$\begin{aligned} \int_I u_\infty (-\Delta)^{\frac{1}{2}} \varphi dx &= \lim_{k \rightarrow \infty} \int_I u_k (-\Delta)^{\frac{1}{2}} \varphi dx \\ &= \lim_{k \rightarrow \infty} \int_I \lambda_k u_k e^{\alpha_k u_k^2} \varphi dx \\ &= \int_I \lambda_\infty u_\infty e^{\pi u_\infty^2} \varphi dx, \end{aligned}$$

where the convergence of the last integral is justified by splitting  $I$  into  $I_1 := \{x \in I : u_k(x) \leq L\}$  and  $I_2 := \{x \in I : u_k(x) > L\}$ , applying the dominated convergence on  $I_1$  and bounding

$$\begin{aligned} \int_{I_2} \lambda_k u_k e^{\alpha_k u_k^2} \varphi dx &\leq \frac{\sup_I |\varphi|}{L} \int_I \lambda_k u_k^2 e^{\alpha_k u_k^2} dx \\ &= \frac{\sup_I |\varphi|}{L} \int_I u_k (-\Delta)^{\frac{1}{2}} u_k dx \\ &= \frac{\sup_I |\varphi|}{L} \|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

and letting  $L \rightarrow \infty$ . □

Let  $\tilde{u}_k$  be the harmonic extension of  $u_k$  to  $\mathbb{R}_+^2$  given by the Poisson integral, see (66) in the appendix. Notice that

$$\int_I \lambda_k u_k^2 e^{\alpha_k u_k^2} dx = \|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2(\mathbb{R})}^2 = \|\nabla \tilde{u}_k\|_{L^2(\mathbb{R}_+^2)}^2 = 1. \quad (15)$$

Let  $r_k = \frac{1}{\alpha_k \lambda_k \mu_k^2 e^{\alpha_k \mu_k^2}}$  and  $\eta_k(x) := 2\alpha_k \mu_k (u_k(r_k x) - \mu_k)$  be as in (8) and (9), and set

$$\tilde{\eta}_k(x, y) := 2\alpha_k \mu_k (\tilde{u}_k(r_k x, r_k y) - \mu_k).$$

Note that  $\tilde{\eta}_k$  is the Poisson integral of  $\eta_k$ .

**Proposition 2.2.** *We have  $r_k \rightarrow 0$  and  $\tilde{\eta}_k \rightarrow \tilde{\eta}_\infty$  in  $C_{\text{loc}}^\ell(\overline{\mathbb{R}_+^2})$  for every  $\ell \geq 0$ , where*

$$\tilde{\eta}_\infty(x, y) = -\log((1+y)^2 + x^2)$$

is the Poisson integral (compare to (66)) of  $\eta_\infty := -\log(1+x^2)$ , and

$$(-\Delta)^{\frac{1}{2}} \eta_\infty = 2e^{\eta_\infty}, \quad \int_{\mathbb{R}} e^{\eta_\infty} dx = \pi. \quad (16)$$

*Proof.* According to Lemma 2.2, Theorem 1.5 and Proposition 2.7 in [24], we have  $r_k \rightarrow 0$ ,  $\eta_k \rightarrow \eta_\infty$  in  $C_{\text{loc}}^\ell(\mathbb{R})$  for every  $\ell \geq 0$  and  $(\eta_k)$  is uniformly bounded in  $L_{\frac{1}{2}}(\mathbb{R})$  (see (63)).

To obtain the local convergence of  $\tilde{\eta}_k$ , fix  $R > 0$  and split the integral in the Poisson integral (66) of  $\tilde{\eta}_k$  into an integral over  $(-R, R)$  and an integral over  $\mathbb{R} \setminus (-R, R)$ , for  $R$  large. The former is bounded by the convergence of  $\eta_k$  locally, the latter by the boundedness of  $\eta_k$  in  $L_{\frac{1}{2}}(\mathbb{R})$ , provided  $(x, y) \in B_{\frac{R}{2}} \cap \mathbb{R}_+^2$ . As a consequence we get that  $\tilde{\eta}_k$  is locally uniformly bounded in  $\overline{\mathbb{R}_+^2}$ . Since  $\tilde{\eta}_k$  is harmonic, we conclude by elliptic estimates.  $\square$

**Corollary 2.3.** *For  $R > 0$  and  $i = 0, 1, 2$ , we have*

$$\lim_{k \rightarrow \infty} \int_{-Rr_k}^{Rr_k} \lambda_k \mu_k^i u_k^{2-i} e^{\alpha_k u_k^2} dx = \frac{1}{\pi} \int_{-R}^R e^{\eta_\infty} dx. \quad (17)$$

Moreover,  $u_\infty \equiv 0$ , i.e. up to a subsequence  $u_k \rightarrow 0$  in  $L^2(I)$ , weakly in  $\tilde{H}^{\frac{1}{2}, 2}(I)$ , and a.e in  $I$ .

*Proof.* With the change of variables  $\xi = \frac{x}{r_k}$ , writing  $u_k(r_k \cdot) = \mu_k + \frac{\eta_k}{2\alpha_k \mu_k}$  and using (8) and Proposition 2.2, we see that

$$\begin{aligned} \int_{-Rr_k}^{Rr_k} \lambda_k \mu_k^i u_k^{2-i} e^{\alpha_k u_k^2} dx &= \underbrace{r_k \lambda_k \mu_k^2 e^{\alpha_k \mu_k^2}}_{=\frac{1}{\alpha_k}} \int_{-R}^R \left(1 + \frac{\eta_k}{2\alpha_k \mu_k^2}\right)^{2-i} e^{\eta_k + \frac{\eta_k}{4\alpha_k \mu_k^2}} d\xi \\ &\rightarrow \frac{1}{\pi} \int_{-R}^R e^{\eta_\infty} d\xi, \end{aligned}$$

as  $k \rightarrow \infty$ , as claimed in (17).

In order to prove the last statement, recalling that  $\|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2} = 1$ , we write

$$1 = \int_{-Rr_k}^{Rr_k} \lambda_k u_k^2 e^{\alpha_k u_k^2} dx + \int_{I \setminus (-Rr_k, Rr_k)} \lambda_k u_k^2 e^{\alpha_k u_k^2} dx =: (I)_k + (II)_k.$$

By (17) and (16) we get

$$\lim_{k \rightarrow \infty} (I)_k = \frac{1}{\pi} \int_{-R}^R e^{\eta_\infty} dx = 1 + o(1),$$

with  $o(1) \rightarrow 0$  as  $R \rightarrow \infty$ . This in turn implies that

$$\lim_{R \rightarrow \infty} \lim_{k \rightarrow \infty} (II)_k = 0,$$

which is possible only if  $u_\infty \equiv 0$ , or  $\lambda_\infty = 0$  (by Fatou's lemma). But on account of (14), also in the latter case we have  $u_\infty \equiv 0$ .  $\square$

**Lemma 2.4.** *For  $A > 1$ , set  $u_k^A := \min \{u_k, \frac{\mu_k}{A}\}$ . Then we have*

$$\limsup_{k \rightarrow \infty} \|(-\Delta)^{\frac{1}{4}} u_k^A\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{A}. \quad (18)$$

*Proof.* We set  $\bar{u}_k^A := \min \{\tilde{u}_k, \frac{\mu_k}{A}\}$ . Since  $\bar{u}_k^A$  is an extension (in general not harmonic) of  $u_k^A$ , we have

$$\|(-\Delta)^{\frac{1}{4}} u_k^A\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}_+^2} |\nabla \bar{u}_k^A|^2 dx dy. \quad (19)$$

Using integration by parts and the harmonicity of  $\tilde{u}_k$  we get

$$\begin{aligned} \int_{\mathbb{R}_+^2} |\nabla \bar{u}_k^A|^2 dx dy &= \int_{\mathbb{R}_+^2} \nabla \bar{u}_k^A \cdot \nabla \tilde{u}_k dx dy \\ &= - \int_{\mathbb{R}} u_k^A(x) \frac{\partial \tilde{u}_k(x, 0)}{\partial y} dx \\ &= \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A dx. \end{aligned} \quad (20)$$



Proposition 2.2 implies that  $u_k^A(r_k x) = \frac{\mu_k}{A}$  for  $|x| \leq R$  and  $k \geq k_0(R)$ . Then, with (16) and (17) we obtain

$$\begin{aligned} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A dx &\geq \int_{-Rr_k}^{Rr_k} \lambda_k u_k e^{\alpha_k u_k^2} u_k^A dx \\ &\xrightarrow[k \rightarrow \infty]{} \frac{1}{\pi A} \int_{-R}^R e^{\eta_\infty} d\xi \\ &\xrightarrow[R \rightarrow \infty]{} \frac{1}{A}. \end{aligned}$$

Set now  $v_k^A := (u_k - \frac{\mu_k}{A})^+ = u_k - u_k^A$ . With similar computations we get

$$\begin{aligned} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k v_k^A dx &\geq \int_{-Rr_k}^{Rr_k} \lambda_k u_k v_k^A e^{\alpha_k u_k^2} dx \\ &\xrightarrow[k \rightarrow \infty]{} \frac{1}{\pi} \left(1 - \frac{1}{A}\right) \int_{-R}^R e^{\eta_\infty} d\xi \\ &\xrightarrow[R \rightarrow \infty]{} \frac{A-1}{A}. \end{aligned}$$

Since

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A dx + \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k v_k^A dx = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k dx = 1,$$

we get that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A dx = \frac{1}{A}.$$

Then, we conclude using (19), and (20). □

**Proposition 2.5.** *We have*

$$C_\pi = \lim_{k \rightarrow \infty} \frac{1}{\lambda_k \mu_k^2}. \tag{21}$$

Moreover

$$\lim_{k \rightarrow \infty} \mu_k \lambda_k = 0. \tag{22}$$

*Proof.* Fix  $A > 1$  and let  $u_k^A$  be defined as in Lemma 2.4. We split

$$\begin{aligned} & \int_I \left( e^{\alpha_k u_k^2} - 1 \right) dx \\ &= \int_{I \cap \{u_k \leq \frac{\mu_k}{A}\}} \left( e^{\alpha_k (u_k^A)^2} - 1 \right) dx + \int_{I \cap \{u_k > \frac{\mu_k}{A}\}} \left( e^{\alpha_k u_k^2} - 1 \right) dx =: (I) + (II). \end{aligned}$$

Using Corollary 2.3 and Vitali's theorem, we see that

$$(I) \leq \int_I \left( e^{\alpha_k (u_k^A)^2} - 1 \right) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

since  $e^{\alpha_k (u_k^A)^2}$  is uniformly bounded in  $L^A(I)$  by Lemma 2.4 together with Theorem A.

By (15) and Corollary 2.3, we now estimate

$$(II) \leq \frac{A^2}{\lambda_k \mu_k^2} \int_{I \cap \{u_k > \frac{\mu_k}{A}\}} \lambda_k u_k^2 \left( e^{\alpha_k u_k^2} - 1 \right) dx \leq \frac{A^2}{\lambda_k \mu_k^2} (1 + o(1)),$$

with  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ . Together with (11), and letting  $A \downarrow 1$ , this gives

$$C_\pi \leq \lim_{k \rightarrow \infty} \frac{1}{\lambda_k \mu_k^2}.$$

The converse inequality follows from (17) as follows:

$$\begin{aligned} \int_I \left( e^{\alpha_k u_k^2} - 1 \right) dx &\geq \int_{-Rr_k}^{Rr_k} e^{\alpha_k u_k^2} dx + o(1) \\ &= \frac{1}{\lambda_k \mu_k^2} \left( \frac{1}{\pi} \int_{-R}^R e^{\eta^\infty} dx + o(1) \right) + o(1), \end{aligned}$$

with  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ . Letting  $R \rightarrow \infty$  and recalling (16) we obtain (21).

Finally, (22) follows at once from (21), because otherwise we would have  $C_\pi = 0$ , which is clearly impossible.  $\square$

**Proposition 2.6.** *Let us set  $f_k := \lambda_k \mu_k u_k e^{\alpha_k u_k^2}$ . Then we have*

$$\int_I f_k \varphi dx \rightarrow \varphi(0),$$

as  $k \rightarrow \infty$ , for any  $\varphi \in C(\bar{I})$ . In particular,  $f_k \rightarrow \delta_0$  in the sense of Radon measures in  $I$ .

*Proof.* Take  $\varphi \in C(\bar{I})$ . For given  $R > 0$ ,  $A > 1$ , we split

$$\begin{aligned} \int_I \varphi f_k dx &= \int_{-Rr_k}^{Rr_k} \varphi f_k dx + \int_{\{u_k > \frac{\mu_k}{A}\} \setminus (-Rr_k, Rr_k)} \varphi f_k dx + \int_{\{u_k \leq \frac{\mu_k}{A}\}} \varphi f_k dx \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

On  $\{u_k \leq \frac{\mu_k}{A}\}$  we have  $u_k = u_k^A$  and Lemma 2.4 and Theorem A imply that  $u_k e^{\alpha_k u_k^2}$  is uniformly bounded in  $L^1$  (depending on  $A$ ). Thus using (22) we get  $I_3 \rightarrow 0$ .

With (15) and (17) we also get

$$\begin{aligned} I_2 &\leq A \|\varphi\|_{L^\infty(I)} \int_{\{u_k > \frac{\mu_k}{A}\} \setminus (-Rr_k, Rr_k)} \lambda_k u_k^2 e^{\alpha_k u_k^2} dx \\ &\leq A \|\varphi\|_{L^\infty(I)} \left( 1 - \int_{-Rr_k}^{Rr_k} \lambda_k u_k^2 e^{\alpha_k u_k^2} dx \right) \\ &= A \|\varphi\|_{L^\infty(I)} \left( 1 - \frac{1}{\pi} \int_{-R}^R e^{\eta_\infty} dx + o(1) \right) \end{aligned}$$

with  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ . Thanks to (16), we conclude that  $I_2 \rightarrow 0$  as  $k \rightarrow \infty$  and  $R \rightarrow \infty$ .

As for  $I_1$ , again with (17) we compute

$$I_1 = (\varphi(0) + o(1)) \left( \frac{1}{\pi} \int_{-R}^R e^{\eta_\infty} dx + o(1) \right),$$

so that  $I_1 \rightarrow \varphi(0)$  as  $k \rightarrow \infty$  and  $R \rightarrow \infty$ . □

Given  $x \in I$ , let  $G_x : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be the Green's function of  $(-\Delta)^{\frac{1}{2}}$  on  $I$  with singularity at  $x$ . We recall that we have the explicit formula (see e.g. [3])

$$G_x(y) := \begin{cases} \frac{1}{\pi} \log \left( \frac{1-xy + \sqrt{(1-x^2)(1-y^2)}}{|x-y|} \right), & y \in I, \\ 0 & y \in \mathbb{R} \setminus I. \end{cases} \quad (23)$$

In the following we further denote

$$S(x, y) := G_x(y) - \frac{1}{\pi} \log \frac{1}{|x-y|}. \quad (24)$$

**Lemma 2.7.** *We have  $\mu_k u_k \rightarrow G := G_0$  in  $L^\infty_{\text{loc}}(\bar{I} \setminus \{0\}) \cap L^1(I)$  as  $k \rightarrow +\infty$ .*

*Proof.* Let us set  $v_k := \mu_k u_k - G$  and  $f_k = \mu_k \lambda_k u_k e^{\alpha_k u_k^2}$ . Arguing as in Proposition 2.6, we show that  $\|f_k\|_{L^1(\mathbb{R})} \rightarrow 1$  as  $k \rightarrow \infty$ . Moreover, since  $u_k$  is decreasing with respect to  $|x|$ , we get that  $u_k \rightarrow 0$  and  $f_k \rightarrow 0$  locally uniformly in  $\bar{I} \setminus \{0\}$  as  $k \rightarrow \infty$ . By Green's representation formula, we have

$$\begin{aligned} |v_k(x)| &= \left| \int_I G_x(y) f_k(y) dy - G(x) \right| \\ &\leq \int_I |G_x(y) - G(x)| f_k(y) dy + \|f_k\|_{L^1(I)} - 1 |G(x)|, \quad x \in I. \end{aligned} \quad (25)$$

Fix  $\sigma \in (0, 1)$ . If we assume  $|x| \geq \sigma$ ,  $|y| \leq \frac{\sigma}{2}$ , then we have

$$\begin{aligned} |G_x(y) - G(x)| &\leq \frac{1}{\pi} \left| \log \frac{|x|}{|x-y|} \right| + |S(x, y) - S(x, 0)| \\ &\leq \frac{1}{\pi} \left| \log \left| \frac{x}{|x|} - \frac{y}{|x|} \right| \right| + \sup_{|x| \geq \sigma, |y| \leq \frac{\sigma}{2}} |\nabla_y S(x, y)| |y| \\ &\leq C|y|, \end{aligned} \quad (26)$$

where  $C$  is a constant depending only on  $\sigma$ . Then, for any  $\varepsilon \in (0, \frac{\sigma}{2})$ , we can write

$$\begin{aligned} |v_k(x)| &\leq \int_I |G_x(y) - G(x)| f_k(y) dy + o(1) \\ &= \int_{-\varepsilon}^{\varepsilon} |G_x(y) - G(x)| f_k(y) dy + \int_{I \setminus (-\varepsilon, \varepsilon)} |G_x(y) - G(x)| f_k(y) dy + o(1) \\ &\leq C\varepsilon \|f_k\|_{L^1(-\varepsilon, \varepsilon)} + \left( \sup_{z \in I} \|G_z\|_{L^1(I)} + |G(x)| \right) \|f_k\|_{L^\infty(I \setminus (-\varepsilon, \varepsilon))} + o(1) \\ &\leq C\varepsilon + o(1), \end{aligned} \quad (27)$$

where  $o(1) \rightarrow 0$  uniformly in  $I \setminus (-\sigma, \sigma)$  as  $k \rightarrow \infty$ . Clearly, (27) implies

$$\limsup_{k \rightarrow \infty} \|v_k\|_{L^\infty(I \setminus (-\sigma, \sigma))} \leq C\varepsilon.$$

Since  $\varepsilon$  and  $\sigma$  can be arbitrarily small, this shows that  $v_k \rightarrow 0$  in  $L^\infty_{\text{loc}}(\bar{I} \setminus \{0\})$ . With a similar argument, we prove the  $L^1$  convergence. Indeed, integrating (25),

for  $\varepsilon \in (0, 1)$  we get

$$\begin{aligned}
 \|v_k\|_{L^1(I)} &\leq \int_I \int_I |G_x(y) - G(x)| f_k(y) \, dy \, dx + \|f_k\|_{L^1(I)} - 1 \|G\|_{L^1(I)} \\
 &\leq \int_I f_k(y) \int_I |G_x(y) - G(x)| \, dx \, dy + o(1) \\
 &\leq \int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I |G_x(y) - G(x)| \, dx \, dy \\
 &\quad + 2 \sup_{z \in I} \|G_z\|_{L^1(I)} \|f_k\|_{L^\infty(I \setminus (-\varepsilon, \varepsilon))} + o(1) \\
 &= \int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I |G_x(y) - G(x)| \, dx \, dy + o(1). \tag{28}
 \end{aligned}$$

Since

$$\sup_{y \in (-\varepsilon, \varepsilon)} \sup_{x \in I} |S(x, y) - S(x, 0)| = O(\varepsilon),$$

we get

$$\int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I |G_y(x) - G(x)| \, dx \, dy = \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I \left| \log \frac{|x|}{|x-y|} \right| \, dx \, dy + O(\varepsilon).$$

Moreover, using the change of variables  $x = yz$ , we obtain

$$\int_I \left| \log \frac{|x|}{|x-y|} \right| \, dx = |y| \int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \left| \log \frac{|z|}{|z-1|} \right| \, dz = O\left(|y| \log \frac{1}{|y|}\right).$$

Then, we have

$$\begin{aligned}
 \int_{-\varepsilon}^{\varepsilon} f_k(y) \int_I |G_y(x) - G_0(x)| \, dx \, dy &= \int_{-\varepsilon}^{\varepsilon} f_k(y) O\left(|y| \log \frac{1}{|y|}\right) \, dy + O(\varepsilon) \\
 &= O\left(\varepsilon \log \frac{1}{\varepsilon}\right). \tag{29}
 \end{aligned}$$

Clearly (28) and (29) yield  $\limsup_{k \rightarrow +\infty} \|v_k - G\|_{L^1(I)} = O(\varepsilon \log \frac{1}{\varepsilon})$ . Since  $\varepsilon$  can be arbitrarily small we get the conclusion.  $\square$

**Proposition 2.8.** *We have  $\mu_k \tilde{u}_k \rightarrow \tilde{G}$  in  $C_{\text{loc}}^0(\overline{\mathbb{R}_+^2} \setminus \{(0, 0)\}) \cap C_{\text{loc}}^1(\mathbb{R}_+^2)$ , where  $\tilde{G}$  is the Poisson extension of  $G$ .*

*Proof.* As in the proof of Lemma 2.7 we denote  $v_k := \mu_k u_k - G$ . Let us consider the Poisson extension  $\tilde{v}_k = \mu_k \tilde{u}_k - \tilde{G}$ . For any fixed  $\varepsilon > 0$ , we can split

$$\tilde{v}_k(x, y) = \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{y v_k(\xi)}{(x - \xi)^2 + y^2} d\xi + \frac{1}{\pi} \int_{I \setminus (-\varepsilon, \varepsilon)} \frac{y v_k(\xi)}{(x - \xi)^2 + y^2} d\xi,$$

By Lemma 2.7, we have

$$\begin{aligned} \left| \frac{1}{\pi} \int_{I \setminus (-\varepsilon, \varepsilon)} \frac{y v_k(\xi)}{(x - \xi)^2 + y^2} d\xi \right| &\leq \frac{1}{\pi} \|v_k\|_{L^\infty(I \setminus (-\varepsilon, \varepsilon))} \int_{\mathbb{R}} \frac{y v_k(\xi)}{(x - \xi)^2 + y^2} d\xi \\ &= \|v_k\|_{L^\infty(I \setminus (-\varepsilon, \varepsilon))} \rightarrow 0, \end{aligned}$$

as  $k \rightarrow \infty$ . Moreover, assuming  $(x, y) \in \mathbb{R}_+^2 \setminus B_{2\varepsilon}(0, 0)$ , we get

$$\left| \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{y v_k(\xi)}{(x - \xi)^2 + y^2} d\xi \right| \leq \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{y |v_k(\xi)|}{|(x, y) - (\xi, 0)|^2} d\xi \leq \frac{y}{\pi \varepsilon^2} \|v_k\|_{L^1(I)} \rightarrow 0.$$

Hence  $\tilde{v}_k \rightarrow 0$  in  $C_{\text{loc}}^0(\mathbb{R}_+^2 \setminus B_{2\varepsilon}(0, 0))$ . Finally, since can  $\varepsilon$  be arbitrarily small and  $\tilde{v}_k$  is harmonic in  $\mathbb{R}_+^2$ , we get  $\tilde{v}_k \rightarrow 0$  in  $C_{\text{loc}}^0(\overline{\mathbb{R}_+^2} \setminus \{(0, 0)\}) \cap C_{\text{loc}}^1(\mathbb{R}_+^2)$ .  $\square$

### 2.3 The two main estimates and completion of the proof

We shall now conclude our contradiction argument by showing the incompatibility of (12) with (11) and the the definition of  $C_\pi$ . In this final part of the proof, we will use the precise asymptotic of  $\tilde{G}$  near  $(0, 0)$ . Since  $\log |(x, y)|$  is the Poisson integral of  $\log |x|$  (see Proposition A.3), and since  $S \in C(\mathbb{R})$ , (24) guarantees the existence of the limit

$$S_0 := \lim_{(x, y) \rightarrow (0, 0)} \tilde{G}(x, y) + \frac{1}{\pi} \log |(x, y)| = \lim_{x \rightarrow 0} G(x) + \frac{1}{\pi} \log |x|.$$

In fact, using (23) we get  $S_0 = \frac{\log 2}{\pi}$ . More precisely, noting that  $S \in C^\infty(I)$ , we can write

$$\tilde{G}(x, y) = \frac{1}{\pi} \log \frac{1}{|(x, y)|} + S_0 + h(x, y), \quad (30)$$

with  $h \in C^\infty(\overline{\mathbb{R}_+^2} \cap B_1(0, 0)) \cap C(\overline{\mathbb{R}_+^2})$  and  $h(0, 0) = 0$ .

**Proposition 2.9.** *If (12) holds, then  $C_\pi \leq 2\pi e^{\pi S_0} = 4\pi$ .*

*Proof.* For a fixed large  $L > 0$  and a fixed and small  $\delta > 0$  set

$$a_k := \inf_{B_{Lr_k} \cap \mathbb{R}_+^2} \tilde{u}_k, \quad b_k := \sup_{B_\delta \cap \mathbb{R}_+^2} \tilde{u}_k, \quad \tilde{v}_k := (\tilde{u}_k \wedge a_k) \vee b_k.$$

Recalling that  $\|\nabla \tilde{u}_k\|_{L^2}^2 = 1$ , we have

$$\int_{(B_\delta \setminus B_{Lr_k}) \cap \mathbb{R}_+^2} |\nabla \tilde{v}_k|^2 dx dy \leq 1 - \int_{\mathbb{R}_+^2 \setminus B_\delta} |\nabla \tilde{u}_k|^2 dx dy - \int_{\mathbb{R}_+^2 \cap B_{Lr_k}} |\nabla \tilde{u}_k|^2 dx dy \quad (31)$$

Clearly the left-hand side bounds

$$\begin{aligned} \inf_{\substack{\tilde{u}|_{\mathbb{R}_+^2 \cap \partial B_{Lr_k}} = a_k \\ \tilde{u}|_{\mathbb{R}_+^2 \cap \partial B_\delta} = b_k}} \int_{(B_\delta \setminus B_{Lr_k}) \cap \mathbb{R}_+^2} |\nabla \tilde{u}|^2 dx dy &= \int_{(B_\delta \setminus B_{Lr_k}) \cap \mathbb{R}_+^2} |\nabla \tilde{\Phi}_k|^2 dx dy \\ &= \pi \frac{(a_k - b_k)^2}{\log \delta - \log(Lr_k)}, \end{aligned}$$

where the function  $\tilde{\Phi}_k$  is the unique solution to

$$\begin{cases} \Delta \tilde{\Phi}_k = 0 & \text{in } \mathbb{R}_+^2 \cap (B_\delta \setminus B_{Lr_k}), \\ \tilde{\Phi}_k = a_k & \text{on } \mathbb{R}_+^2 \cap \partial B_{Lr_k}, \\ \tilde{\Phi}_k = b_k & \text{on } \mathbb{R}_+^2 \cap \partial B_\delta, \\ \frac{\partial \tilde{\Phi}_k}{\partial y} = 0 & \text{on } \partial \mathbb{R}_+^2 \cap (B_\delta \setminus B_{Lr_k}), \end{cases}$$

given explicitly by

$$\tilde{\Phi}_k = \frac{b_k - a_k}{\log \delta - \log(Lr_k)} \log |(x, y)| + \frac{a_k \log \delta - b_k \log Lr_k}{\log \delta - \log(Lr_k)}.$$

Using Proposition 2.2 we obtain

$$a_k = \mu_k + \frac{-\frac{1}{\pi} \log L + O(L^{-1}) + o(1)}{\mu_k},$$

where for fixed  $L > 0$  we have  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ , and  $|O(L^{-1})| \leq \frac{C}{L}$  uniformly for  $L$  and  $k$  large. Moreover, using Proposition 2.8 and (30), we obtain

$$b_k = \frac{-\frac{1}{\pi} \log \delta + S_0 + O(\delta) + o(1)}{\mu_k},$$

where for fixed  $\delta > 0$  we have  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ , and  $|O(\delta)| \leq C\delta$  uniformly for  $\delta$  small and  $k$  large.

Still with Proposition 2.2 we get

$$\begin{aligned} \lim_{k \rightarrow \infty} \mu_k^2 \int_{\mathbb{R}_+^2 \cap B_{Lr_k}} |\nabla \tilde{u}_k|^2 dx dy &= \frac{1}{4\pi^2} \int_{\mathbb{R}_+^2 \cap B_L} |\nabla \tilde{\eta}_\infty|^2 dx dy \\ &= \frac{1}{\pi} \log \frac{L}{2} + O\left(\frac{\log L}{L}\right). \end{aligned}$$

Similarly with Proposition 2.8 we get

$$\begin{aligned} \liminf_{k \rightarrow \infty} \mu_k^2 \int_{\mathbb{R}_+^2 \setminus B_\delta} |\nabla \tilde{u}_k|^2 dx dy &\geq \int_{\mathbb{R}_+^2 \setminus B_\delta} |\nabla \tilde{G}|^2 dx dy \\ &= \int_{\mathbb{R}_+^2 \cap \partial B_\delta} -\frac{\partial \tilde{G}}{\partial r} \tilde{G} d\sigma + \int_{(\mathbb{R} \times \{0\}) \setminus B_\delta} -\frac{\partial \tilde{G}(x, 0)}{\partial y} G(x) dx \\ &= \int_{\mathbb{R}_+^2 \cap \partial B_\delta} \left(\frac{1}{\pi\delta} + O(1)\right) \left(-\frac{1}{\pi} \log \delta + S_0 + O(\delta)\right) d\sigma \\ &= -\frac{1}{\pi} \log \delta + S_0 + O(\delta \log \delta), \end{aligned}$$

where we used the expansion in (30) and the boundary conditions

$$\begin{cases} \tilde{G}(x, 0) = G(x) = 0, & \text{for } x \in \mathbb{R} \setminus I, \\ -\frac{\partial \tilde{G}(x, 0)}{\partial y} = (-\Delta)^{\frac{1}{2}} G(x) = 0, & \text{for } x \in I \setminus \{0\}. \end{cases}$$

We then get

$$\frac{\pi(a_k - b_k)^2}{\log \delta - \log(Lr_k)} \leq 1 - \frac{-\frac{1}{\pi} \log \delta + S_0 + O(\delta \log \delta) + \frac{1}{\pi} \log \frac{L}{2} + O\left(\frac{\log L}{L}\right)}{\mu_k^2},$$

or

$$\begin{aligned} \pi(a_k - b_k)^2 &= \pi\mu_k^2 - 2 \log L + O(L^{-1}) + 2 \log \delta - 2\pi S_0 + O(\delta) + o(1) \\ &\quad + \frac{O(\log^2 L + \log^2 \delta)}{\mu_k^2} \\ &\leq (\log \delta - \log L + \log(\lambda_k \mu_k^2) + \alpha_k \mu_k^2 + \log \alpha_k) \\ &\quad \times \left(1 - \frac{-\frac{1}{\pi} \log \delta + S_0 + O(\delta \log \delta) + \frac{1}{\pi} \log \frac{L}{2} + O\left(\frac{\log L}{L}\right)}{\mu_k^2}\right) \\ &= \log \frac{\delta}{L} + \log(\lambda_k \mu_k^2) + \alpha_k \mu_k^2 + \log \alpha_k + \alpha_k \left(\frac{1}{\pi} \log \frac{2\delta}{L} - S_0\right) \\ &\quad + O(\delta \log \delta) + O\left(\frac{\log L}{L}\right) + \frac{O(\log^2 \delta) + O(\log^2 L) + O(1)}{\mu_k^2}. \end{aligned}$$



Rearranging gives

$$\begin{aligned} \log \frac{1}{\lambda_k \mu_k^2} &\leq \left(1 - \frac{\alpha_k}{\pi}\right) \log \frac{L}{\delta} + (\alpha_k - \pi) \mu_k^2 + (2\pi - \alpha_k) S_0 + \frac{\alpha_k}{\pi} \log 2 + \log \alpha_k \\ &\quad + O(\delta \log \delta) + O\left(\frac{\log L}{L}\right) + o(1), \end{aligned}$$

with  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ . Then, recalling that  $\alpha_k \uparrow \pi$ , letting  $k \rightarrow \infty$  first and then  $L \rightarrow \infty$ ,  $\delta \rightarrow 0$ , we obtain

$$\limsup_{k \rightarrow \infty} \log \frac{1}{\lambda_k \mu_k^2} \leq \pi S_0 + \log(2\pi) = \log(4\pi),$$

and using Proposition 2.5 we conclude.  $\square$

**Proposition 2.10.** *There exists a function  $u \in \tilde{H}^{\frac{1}{2},2}(I)$  with  $\|(-\Delta)^{\frac{1}{4}}u\|_{L^2(\mathbb{R})} \leq 1$  such that*

$$\int_I \left( e^{\pi u^2} - 1 \right) dx > 2\pi e^{\pi S_0} = 4\pi.$$

*Proof.* For  $\varepsilon > 0$  choose  $L = L(\varepsilon) > 0$  such that as  $\varepsilon \rightarrow 0$  we have  $L \rightarrow \infty$  and  $L\varepsilon \rightarrow 0$ . Fix

$$\Gamma_{L\varepsilon} := \left\{ (x, y) \in \mathbb{R}_+^2 : \tilde{G}(x, y) = \gamma_{L\varepsilon} := \min_{\mathbb{R}_+^2 \cap \partial B_{L\varepsilon}} \tilde{G} \right\},$$

and

$$\Omega_{L\varepsilon} := \{ (x, y) \in \mathbb{R}_+^2 : \tilde{G}(x, y) > \gamma_{L\varepsilon} \}.$$

By the maximum principle we have  $\mathbb{R}_+^2 \cap B_{L\varepsilon} \subset \Omega_{L\varepsilon}$ . Indeed,  $\tilde{G}$  is harmonic in  $\mathbb{R}_+^2$ ,  $\tilde{G} \geq \gamma_{L\varepsilon}$  on  $\partial(\mathbb{R}_+^2 \cap B_{L\varepsilon}) \setminus \{(0, 0)\}$ , and  $\tilde{G} \rightarrow +\infty$  as  $(x, y) \rightarrow (0, 0)$ . Notice also that (30) gives

$$\gamma_{L\varepsilon} = -\frac{1}{\pi} \log(L\varepsilon) + S_0 + O(L\varepsilon). \tag{32}$$

For some constants  $B$  and  $c$  to be fixed we set

$$U_\varepsilon(x, y) := \begin{cases} c - \frac{\log\left(\frac{x^2}{\varepsilon^2} + \left(1 + \frac{y}{\varepsilon}\right)^2\right) + 2B}{2\pi c} & \text{for } (x, y) \in \mathbb{R}_+^2 \cap B_{L\varepsilon}(0, -\varepsilon) \\ \frac{\gamma_{L\varepsilon}}{c} & \text{for } (x, y) \in \Omega_{L\varepsilon} \setminus B_{L\varepsilon}(0, -\varepsilon) \\ \frac{\tilde{G}(x, y)}{c} & \text{for } (x, y) \in \mathbb{R}_+^2 \setminus \Omega_{L\varepsilon}. \end{cases}$$

Observe that  $\mathbb{R}_+^2 \cap B_{L\varepsilon}(0, -\varepsilon) \subseteq \mathbb{R}_+^2 \cap B_{L\varepsilon} \subseteq \Omega_{L\varepsilon}$ . To have continuity on  $\mathbb{R}_+^2 \cap \partial B_{L\varepsilon}(0, -\varepsilon)$  we impose

$$-\frac{\log L^2 - 2B}{2\pi c} + c = \frac{\gamma_{L\varepsilon}}{c}$$

which, together with (32), gives the relation

$$B = \pi c^2 + \log \varepsilon - \pi S_0 + O(L\varepsilon). \quad (33)$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}_+^2 \cap B_{L\varepsilon}(0, -\varepsilon)} |\nabla U_\varepsilon|^2 dx dy &= \frac{1}{4\pi^2 c^2} \int_{\mathbb{R}_+^2 \cap B_L(0, -1)} |\nabla \log(x^2 + (1+y)^2)|^2 dx dy \\ &= \frac{\frac{1}{\pi} \log\left(\frac{L}{2}\right) + O\left(\frac{\log L}{L}\right)}{c^2}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}_+^2 \setminus \Omega_{L\varepsilon}} |\nabla U_\varepsilon|^2 dx dy &= \frac{1}{c^2} \int_{\mathbb{R}_+^2 \setminus \Omega_{L\varepsilon}} |\nabla \tilde{G}|^2 dx dy \\ &= \frac{1}{c^2} \int_{\mathbb{R}_+^2 \cap \partial \Omega_{L\varepsilon}} \frac{\partial \tilde{G}}{\partial \nu} \tilde{G} d\sigma - \underbrace{\frac{1}{c^2} \int_{(\mathbb{R} \times \{0\}) \setminus \bar{\Omega}_{L\varepsilon}} \frac{\partial \tilde{G}}{\partial y} \tilde{G} dx}_{=0} \\ &= \frac{\frac{1}{\pi} \log\left(\frac{1}{L\varepsilon}\right) + S_0 + O(L\varepsilon \log(L\varepsilon))}{c^2}, \end{aligned}$$

where the last equality follows from (30). We now impose  $\|\nabla U_\varepsilon\|_{L^2(\mathbb{R}_+^2)} = 1$ , obtaining

$$-\log \varepsilon - \log 2 + \pi S_0 + O(L\varepsilon \log(L\varepsilon)) + O\left(\frac{\log L}{L}\right) = \pi c^2, \quad (34)$$

which, together with (33), implies

$$B = -\log 2 + O(L\varepsilon \log(L\varepsilon)) + O\left(\frac{\log L}{L}\right). \quad (35)$$

Let now  $I_{L,\varepsilon}^1 = (-\varepsilon\sqrt{L^2 - 1}, \varepsilon\sqrt{L^2 - 1})$  and  $I_{L,\varepsilon}^2$  be the disjoint sub-intervals of  $I$  obtained by intersecting  $I \times \{0\}$  respectively with  $B_{L\varepsilon}(0, -\varepsilon)$  and  $\overline{\mathbb{R}_+^2 \setminus \Omega_{L\varepsilon}}$ .

Then, for  $u_\varepsilon(x) := U_\varepsilon(x, 0)$ , using a change of variables and (34)-(35) we get

$$\begin{aligned} \int_{I_{L,\varepsilon}^1} e^{\pi u_\varepsilon^2} dx &= \varepsilon \int_{-\sqrt{L^2-1}}^{\sqrt{L^2-1}} \exp\left(\pi\left(c - \frac{\log(1+x^2)+2B}{2\pi c}\right)^2\right) dx \\ &> \varepsilon e^{\pi c^2-2B} \int_{-\sqrt{L^2-1}}^{\sqrt{L^2-1}} \frac{1}{1+x^2} dx \\ &= 2e^{\pi S_0+O(L\varepsilon \log(L\varepsilon))+O(\frac{\log L}{L})} \pi \left(1 + O\left(\frac{1}{L}\right)\right) \\ &= 2\pi e^{\pi S_0} + O(L\varepsilon \log(L\varepsilon)) + O\left(\frac{\log L}{L}\right). \end{aligned}$$

Moreover

$$\int_{I_{L\varepsilon}^2} \left(e^{\pi u_\varepsilon^2} - 1\right) dx \geq \int_{I_{L\varepsilon}^2} \pi u_\varepsilon^2 dx = \frac{1}{c^2} \int_{I_{L\varepsilon}^2} \pi G^2 dx =: \frac{\nu_{L\varepsilon}}{c^2},$$

with

$$\nu_{L\varepsilon} > \nu_{\frac{1}{2}} > 0, \quad \text{for } L\varepsilon < \frac{1}{2}.$$

Now observe that  $c^2 = -\frac{\log \varepsilon}{\pi} + O(1)$  by (34), and choose  $L = \log^2 \varepsilon$  to obtain

$$O(L\varepsilon \log(L\varepsilon)) + O\left(\frac{\log L}{L}\right) = O\left(\frac{\log \log \varepsilon}{\log^2 \varepsilon}\right) = o\left(\frac{1}{c^2}\right),$$

so that

$$\int_I \left(e^{\pi u_\varepsilon^2} - 1\right) dx \geq 2\pi e^{\pi S_0} + \frac{\nu_{\frac{1}{2}}}{c^2} + o\left(\frac{1}{c^2}\right) > 2\pi e^{\pi S_0}$$

for  $\varepsilon$  small enough.

Finally notice that

$$\|(-\Delta)^{\frac{1}{4}} u_\varepsilon\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}_+^2} |\nabla \tilde{u}_\varepsilon|^2 dx dy \leq \int_{\mathbb{R}_+^2} |\nabla U_\varepsilon|^2 dx dy \leq 1,$$

since the Poisson extension  $\tilde{u}_\varepsilon$  minimizes the Dirichlet energy among extensions with finite energy. □

### 3 Proof of Theorem 1.3

Let  $u_k \in H \cap C^\infty(\mathbb{R})$  be a sequence of positive even and decreasing solutions to (5) satisfying the energy bound (6) and with  $\lambda_k \rightarrow \lambda_\infty \geq 0$  as  $k \rightarrow \infty$ .

First we show that case (i) holds when  $\mu_k \leq C$ .

**Lemma 3.1.** *If  $\mu_k \leq C$  then (i) holds.*

*Proof.* By assumption we know that  $u_k$  and  $f_k := (-\Delta)^{\frac{1}{2}} u_k = \lambda_k u_k e^{\alpha_k u_k^2} - u_k$  are uniformly bounded in  $L^\infty(\mathbb{R})$ . Then, by elliptic estimates and a bootstrap argument, we can find  $u_\infty \in C^\infty(\mathbb{R})$  such that up to a subsequence  $u_k \rightarrow u_\infty$  in  $C_{\text{loc}}^\ell(\mathbb{R})$  for every  $\ell \geq 0$ . To prove that  $u_\infty$  satisfies (7), note that  $f_k \rightarrow f_\infty := \lambda_\infty u_\infty e^{\pi u_\infty^2} - u_\infty$  locally uniformly on  $\mathbb{R}$  and set  $M = \sup_k (\|f_k\|_{L^\infty(\mathbb{R})} + \mu_k)$ . For any  $\varphi \in \mathcal{S}(\mathbb{R})$  (the Schwarz space of rapidly decreasing functions) and any  $R > 0$ , we have that

$$\begin{aligned} \int_{\mathbb{R}} |f_k - f_\infty| |\varphi| dx &\leq \|f_k - f_\infty\|_{L^\infty((-R, R))} \int_{-R}^R |\varphi| + 2M \|\varphi\|_{L^1((-R, R)^c)} \\ &\stackrel{k \rightarrow +\infty}{\rightarrow} M \|\varphi\|_{L^1((-R, R)^c)} \\ &\stackrel{R \rightarrow +\infty}{\rightarrow} 0. \end{aligned}$$

Similarly, recalling that  $(-\Delta)^{\frac{1}{2}} \varphi$  has quadratic decay at infinity (see e.g. [14, Prop. 2.1]), we get

$$\begin{aligned} \int_{\mathbb{R}} |u_k - u_\infty| |(-\Delta)^{\frac{1}{2}} \varphi| dx &\leq \|(-\Delta)^{\frac{1}{2}} \varphi\|_{L^\infty((-R, R))} \|u_k - u_\infty\|_{L^1((-R, R))} \\ &\quad + C \int_{(-R, R)^c} \frac{|u_k(x) - u_\infty(x)|}{|x|^2} dx \\ &\leq \|(-\Delta)^{\frac{1}{2}} \varphi\|_{L^\infty((-R, R))} \|u_k - u_\infty\|_{L^1((-R, R))} \\ &\quad + 2CM \int_{(-R, R)^c} \frac{dx}{x^2} \\ &\stackrel{k, R \rightarrow +\infty}{\rightarrow} 0. \end{aligned}$$

Hence  $u$  is a weak solution of (7). □

From now on we will assume that  $\mu_k \rightarrow +\infty$  and prove that (ii) of Theorem 1.3 holds.

**Lemma 3.2.** *Let  $\eta_k$  be defined as in Theorem 1.3. Then  $\eta_k$  is bounded in  $C_{\text{loc}}^{0,\alpha}(\mathbb{R})$  for  $\alpha \in (0, 1)$ .*

*Proof.* Note that

$$\begin{aligned} r_k \mu_k^2 &= \frac{1}{\alpha_k \lambda_k e^{\alpha_k \mu_k^2}} = \frac{1}{\alpha_k \|u_k\|_H^2 e^{\alpha_k \mu_k^2}} \int_{\mathbb{R}} u_k^2 e^{\alpha_k u_k^2} dx \\ &\leq C \frac{1}{\alpha_k \|u_k\|_H^2 e^{\frac{\alpha_k}{2} \mu_k^2}} \int_{\mathbb{R}} u_k^2 e^{\frac{\alpha_k}{2} u_k^2} dx \\ &\leq C \frac{\|u_k\|_{L^4}^2 \sqrt{D_{\alpha_k}}}{\alpha_k \|u_k\|_H^2 e^{\frac{\alpha_k}{2} \mu_k^2}} \\ &\leq C \frac{\sqrt{D_\pi}}{\alpha_k e^{\frac{\alpha_k}{2} \mu_k^2}} \rightarrow 0. \end{aligned}$$

Moreover we have that

$$(-\Delta)^{\frac{1}{2}} \eta_k = 2 \frac{u_k(r_k \cdot)}{\mu_k} e^{\alpha_k u_k^2(r_k \cdot) - \alpha_k \mu_k^2} - 2 \alpha_k r_k \mu_k^2 \frac{u_k(r_k \cdot)}{\mu_k}$$

is bounded in  $L^\infty$ . Since  $\eta_k \leq 0$ , and  $\eta_k(0) = 0$  this implies that  $\eta_k$  is bounded in  $L_{\text{loc}}^\infty(\mathbb{R})$  and then in  $C_{\text{loc}}^\alpha(\mathbb{R})$  for any  $\alpha \in (0, 1)$ .  $\square$

The bound of Lemma 3.2 implies that, up to a subsequence  $\eta_k \rightarrow \eta_\infty$  in  $C_{\text{loc}}^{0,\alpha}(\mathbb{R})$  for some function  $\eta_\infty$ . However, it does not provide a limit equation for  $\eta_\infty$ . In order to prove that  $\eta_\infty$  solves

$$(-\Delta)^{\frac{1}{2}} \eta_\infty = 2e^{\eta_\infty}$$

we will prove that that  $\eta_k$  is bounded in  $L_s(\mathbb{R})$  for any  $s > 0$ . This bound can be obtained thanks to the commutator estimates proved in [24]. Part of the argument must be modified since the  $u_k$ 's are not compactly supported. We start by recalling the following technical lemma, which is a consequence of the estimates in [24].

**Lemma 3.3.** *For any  $s \in (0, 1)$ , there exists a constant  $C = C(s)$  such that, for any  $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ ,  $\rho \in \mathbb{R}^+$ , we have*

$$\|\varphi(-\Delta)^{\frac{s}{2}} \psi\|_{L^{(\frac{1}{s}, \infty)}((-\rho, \rho))} \leq C (E_1(\varphi, \psi) + E_{2,2\rho}(\varphi, \psi)),$$

where

$$\begin{aligned} E_1(\varphi, \psi) &= \|(-\Delta)^{\frac{1}{4}} \varphi\|_{L^2(\mathbb{R})} \|(-\Delta)^{\frac{1}{4}} \psi\|_{L^2(\mathbb{R})} \\ E_{2,\rho}(\varphi, \psi) &= \|(-\Delta)^{\frac{1}{4}} \varphi\|_{L^2(\mathbb{R})} \|(-\Delta)^{\frac{1}{2}} \psi\|_{L^{\log \frac{1}{2}} L(-\rho, \rho)}, \end{aligned}$$

*Proof.* Let  $\theta \in C_c^\infty((-2, 2))$  be a cut-off function such that  $\theta \equiv 1$  on  $(-1, 1)$  and  $0 \leq \theta \leq 1$ . Let us denote  $\theta_\rho = \theta(\frac{\cdot}{\rho})$ . Let us also introduce the Riesz operators

$$I_{1-s}u := \kappa_s |\cdot|^{-s} * u \quad \text{for } s \in (0, 1),$$

where the constant  $\kappa_s$  is defined by the identity  $\widehat{\kappa_s |\cdot|^{-s}} = |\cdot|^{s-1}$ . With this definition  $I_{1-s}$  is the inverse of  $(-\Delta)^{\frac{1-s}{2}}$ . Then we can split

$$\begin{aligned} \varphi(-\Delta)^{\frac{s}{2}}\psi &= \varphi I_{1-s}(-\Delta)^{\frac{1}{2}}\psi \\ &= \varphi I_{1-s} \left( \theta_{2\rho}(-\Delta)^{\frac{1}{2}}\psi \right) + \varphi I_{1-s} \left( (1 - \theta_{2\rho})(-\Delta)^{\frac{1}{2}}\psi \right) \\ &= \varphi I_{1-s} \left( \theta_{2\rho}(-\Delta)^{\frac{1}{2}}\psi \right) + [\varphi, I_{1-s}] \left( (1 - \theta_{2\rho})(-\Delta)^{\frac{1}{2}}\psi \right) \\ &\quad + I_{1-s} \left( (1 - \theta_{2\rho})\varphi(-\Delta)^{\frac{1}{2}}\psi \right) \\ &=: J_1 + J_2 + J_3, \end{aligned}$$

where we use the commutator notation  $[u, I_{1-s}](v) = uI_{1-s}v - I_{1-s}(uv)$  for any  $u, v \in C_c^\infty(\mathbb{R})$ . Applying respectively Proposition 3.2, Proposition 3.4 and Proposition A.3. in [24], we get that

$$\begin{aligned} \|J_1\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)} &= \|I_{\frac{1}{2}} \left( (-\Delta)^{\frac{1}{4}}\varphi \right) I_{1-s} \left( \theta_{2\rho}(-\Delta)^{\frac{1}{2}}\psi \right)\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)} \\ &\leq C \|(-\Delta)^{\frac{1}{4}}\varphi\|_{L^2(\mathbb{R})} \|(-\Delta)^{\frac{1}{2}}\psi\|_{L \log^{\frac{1}{2}} L(-2\rho, 2\rho)} \\ &= CE_{2,2\rho}(\varphi, \psi), \end{aligned}$$

that

$$\begin{aligned} \|J_2\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)} &= \|[\varphi, I_{1-s}] \left( (1 - \theta_{2\rho})(-\Delta)^{\frac{1}{4}}(-\Delta)^{\frac{1}{4}}\psi \right)\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)} \\ &\leq C \|(-\Delta)^{\frac{1}{4}}\varphi\|_{L^2(\mathbb{R})} \|(-\Delta)^{\frac{1}{s}}\psi\|_{L^2(\mathbb{R})} \\ &= CE_1(\varphi, \psi), \end{aligned}$$

and that

$$\begin{aligned} \|J_3\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)} &\leq \|I_{1-s} \left( \varphi(-\Delta)^{\frac{1}{2}}\psi \right)\|_{L^{(\frac{1}{s}, \infty)}(\mathbb{R})} \\ &\leq C \|\varphi(-\Delta)^{\frac{1}{2}}\psi\|_{L^1(\mathbb{R})} \\ &= C \|(-\Delta)^{\frac{1}{4}}\varphi(-\Delta)^{\frac{1}{4}}\psi\|_{L^1(\mathbb{R})} \\ &\leq CE_1(\varphi, \psi). \end{aligned}$$

□

As a consequence of Lemma 3.3 we obtain the following crucial estimate.

**Lemma 3.4.** *For any  $s \in (0, 1)$  there exists a constant  $C = C(s)$  such that*

$$\int_{(-\rho, \rho)} |u(-\Delta)^{\frac{s}{2}} u| dx \leq C \rho^{1-s} (E_1(u, u) + E_{2,2\rho}(u, u))$$

for any  $\rho > 0$ , and  $u \in H \cap C^\infty(\mathbb{R})$ . Here  $E_1$  and  $E_{2,2\rho}$  are defined as in Lemma 3.3.

*Proof.* By the Hölder inequality for Lorentz spaces (see e.g. [31, Theorem 3.5]), we have

$$\begin{aligned} \|u(-\Delta)^{\frac{s}{2}} u\|_{L^1(-\rho, \rho)} &\leq \|\chi_{(-\rho, \rho)}\|_{L^{(\frac{1}{1-s}, 1)}(\mathbb{R})} \|u(-\Delta)^{\frac{s}{2}} u\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)} \\ &\leq C \rho^{1-s} \|u(-\Delta)^{\frac{s}{2}} u\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)}. \end{aligned} \tag{36}$$

We shall bound the RHS of (36) by approximating  $u$  with compactly supported functions and applying Lemma 3.3. To this purpose, we take a sequence of cut-off function  $(\tau_j)_{j \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R})$  such that  $\tau_j(x) = 1$  for  $|x| \leq j$ ,  $\tau_j(x) = 0$  for  $|x| \geq j + 1$ ,  $0 \leq \tau_j \leq 1$  and  $|\tau_j'| \leq 2$ . We define  $u_j := \tau_j u$ . We claim that

$$u_j \rightarrow u \quad \text{in } H^{\frac{1}{2}, 2}(\mathbb{R}) \cap L^q(\mathbb{R}), \quad q \in (2, \infty) \tag{37}$$

and

$$(-\Delta)^{\frac{s}{2}} u_j \rightarrow (-\Delta)^{\frac{s}{2}} u \quad \text{in } L_{\text{loc}}^\infty(\mathbb{R}). \tag{38}$$

The first claim is proved in [10, Lemma 12]. We shall prove the second claim. Set  $v_j = u_j - u$ . Then, for any fixed  $R_0 > 0$  and  $x \in (-R_0, R_0)$ , if  $j > 2R_0$  we have

$$|(-\Delta)^{\frac{s}{2}} v_j| \leq K_s \int_{\mathbb{R} \setminus (-j, j)} \frac{|v_j(y)|}{|x - y|^{1+s}} dy \leq 2^{1+s} K_s \int_{\mathbb{R} \setminus (-j, j)} \frac{|u(y)|}{|y|^{1+s}} dy \leq \frac{C \|u\|_{L^2(\mathbb{R})}}{j^{1+2s}}.$$

with  $C$  depending only on  $s$ . As  $j \rightarrow \infty$ , we get (38).

Now, By Lemma 3.3, we know that, for any  $j$ ,

$$\|u_j(-\Delta)^{\frac{s}{2}} u_j\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)} \leq C (E_1(u_j, u_j) + E_{2,2\rho}(u_j, u_j)), \tag{39}$$

where  $C$  depends only on  $s$ . Clearly, (37) yields

$$E_1(u_j, u_j) \rightarrow E_1(u, u).$$

Moreover

$$E_{2,2\rho}(u_j, u_j) = E_{2,2\rho}(u, u), \quad \text{for } j \geq 2\rho.$$

Finally, (37) and (38) imply that  $u_j(-\Delta)^{\frac{s}{2}}u_j \rightarrow u(-\Delta)^{\frac{s}{2}}u$  in  $L^q_{\text{loc}}(\mathbb{R})$  for every  $q \in [1, \infty)$ , and therefore in  $L^{(\frac{1}{s}, \infty)}(-\rho, \rho)$ . Then, passing to the limit in (39) we get

$$\|u(-\Delta)^{\frac{s}{2}}u\|_{L^{(\frac{1}{s}, \infty)}(-\rho, \rho)} \leq C(E_1(u, u) + E_{2,2\rho}(u, u)),$$

and together with (36) we conclude.  $\square$

We can now apply Lemma 3.4 to  $u_k$ . After scaling, we get the following bound on  $\eta_k$ .

**Lemma 3.5.** *For any  $s \in (0, 1)$ , there exists a constant  $C = C(s) > 0$  such that*

$$\int_{-R}^R |(-\Delta)^{\frac{s}{2}}\eta_k| dx \leq CR^{1-s}, \quad \text{for any } R > 0 \text{ and } k \geq k_0(R).$$

*Proof.* First we observe that  $f_k := (-\Delta)^{\frac{1}{2}}u_k = \lambda_k u_k e^{\alpha_k u_k^2} - u_k$  is bounded in  $L \log^{\frac{1}{2}} L_{\text{loc}}(\mathbb{R})$ . Indeed, we have

$$\log^{\frac{1}{2}}(2 + |f_k|) \leq C(1 + u_k),$$

so that

$$|f_k| \log^{\frac{1}{2}}(2 + |f_k|) \leq C|f_k|(1 + u_k) = O(|f_k|u_k + 1).$$

Since  $|f_k|u_k$  is bounded in  $L^1(\mathbb{R})$  by (5) and (6), we get that  $f_k$  is bounded in  $L \log^{\frac{1}{2}} L_{\text{loc}}(\mathbb{R})$ .

Then Lemma 3.4 and (6) imply the existence of  $C = C(s)$  such that

$$\int_{-\rho}^{\rho} |u_k(-\Delta)^{\frac{s}{2}}u_k| dx \leq C\rho^{1-s}, \quad \rho \in (0, 1).$$

For any  $R > 0$ , we can apply this with  $\rho = Rr_k$  and rewrite it in terms of  $\eta_k$ . Then, we obtain

$$\int_{-R}^R \left(1 + \frac{\eta_k}{\mu_k^2}\right) |(-\Delta)^{\frac{s}{2}}\eta_k| \leq CR^{1-s}.$$

Since, by Lemma 3.2,  $\eta_k$  is locally bounded, if  $k$  is sufficiently large we get  $1 + \frac{\eta_k}{\mu_k^2} \geq \frac{1}{2}$  and the proof is complete.  $\square$

**Lemma 3.6.** *The sequence  $(\eta_k)$  is bounded in  $L_s(\mathbb{R})$  for any  $s > 0$ .*



*Proof.* It is sufficient to prove the statement for  $s \in (0, 1/2)$ . Since  $\eta_k \leq 0$ , Lemma 3.5 gives

$$\begin{aligned} C &\geq \frac{1}{K_s} \int_{-1}^1 |(-\Delta)^s \eta_k| dx \\ &\geq \left| \int_{-1}^1 \int_{\mathbb{R}} \frac{\eta_k(x) - \eta_k(y)}{|x-y|^{1+2s}} dy dx \right| \\ &\geq \underbrace{\int_{-1}^1 \int_{-2}^2 \frac{\eta_k(x) - \eta_k(y)}{|x-y|^{1+2s}} dy dx}_{=: I_1} + \underbrace{\int_{-1}^1 \int_{(-2,2)^c} \frac{\eta_k(x)}{|x-y|^{1+2s}} dy dx}_{=: I_2} + \underbrace{\int_{-1}^1 \int_{(-2,2)^c} \frac{-\eta_k(y)}{|x-y|^{1+2s}} dy dx}_{=: I_3}. \end{aligned}$$

Take  $2s < \alpha < 1$ . Since  $\eta_k$  is bounded in  $C_{\text{loc}}^\alpha(\mathbb{R})$  by Lemma 3.2, we have that

$$|I_1| \leq C \int_{-1}^1 \int_{-2}^2 \frac{dy dx}{|x-y|^{1+2s-\alpha}} \leq C \int_{-3}^3 \frac{dz}{|z|^{1+2s-\alpha}} = C.$$

Similarly

$$|I_2| \leq \int_{-1}^1 |\eta_k(x)| \int_{(x-1, x+1)^c} \frac{1}{|x-y|^{1+2s}} dy dx \leq C.$$

Therefore, we obtain that

$$I_3 = \int_{-1}^1 \int_{(-2,2)^c} \frac{|\eta_k(y)|}{|x-y|^{1+2s}} dy dx \leq C.$$

But for  $x \in (-1, 1)$  and  $y \notin (-2, 2)$  we have  $|x-y| \leq |y| + |x| \leq 2|y| \leq 2(1+|y|^{1+2s})^{\frac{1}{1+2s}}$ . Hence

$$I_3 = \int_{-1}^1 \int_{(-2,2)^c} \frac{|\eta_k(y)|}{|x-y|^{1+2s}} dy dx \geq \frac{1}{2^{2s}} \int_{(-2,2)^c} \frac{|\eta_k(y)|}{1+|y|^{1+2s}} dy.$$

This and Lemma 3.2 imply that  $\eta_k$  is bounded in  $L_s(\mathbb{R})$ . □

*Proof of Theorem 1.3 (completed).* By Lemma 3.2, up to a subsequence we can assume that  $\eta_k \rightarrow \eta_\infty$  in  $C_{\text{loc}}^\alpha(\mathbb{R})$  for any  $\alpha \in (0, 1)$ , with  $\eta_\infty \in C_{\text{loc}}^\alpha(\mathbb{R})$ . Let us denote

$$f_k := (-\Delta)^{\frac{1}{2}} \eta_k = 2 \left( 1 + \frac{\eta_k}{2\alpha_k \mu_k^2} \right) e^{\eta_k + \frac{\eta_k^2}{4\alpha_k \mu_k^2}} - 2r_k \alpha_k \mu_k^2 \left( 1 + \frac{\eta_k}{2\alpha_k \mu_k^2} \right).$$

As observed in the proof of Lemma 3.2, we have  $r_k \mu_k^2 \rightarrow 0$  as  $k \rightarrow \infty$  and thus  $f_k \rightarrow 2e^{\eta_\infty}$  locally uniformly on  $\mathbb{R}$ . Moreover  $f_k$  is bounded in  $L^\infty(\mathbb{R})$ . Then, for any Schwarz function  $\varphi \in \mathcal{S}(\mathbb{R})$  we have

$$\int_{\mathbb{R}} |f_k - 2e^{\eta_\infty}| |\varphi| dx \leq o(1) \int_{(-R, R)} |\varphi| dx + (\|f_k\|_{L^\infty(\mathbb{R})} + \|2e^{\eta_\infty}\|_{L^\infty(\mathbb{R})}) \int_{(-R, R)^c} |\varphi| dx \rightarrow 0$$

as  $k, R \rightarrow +\infty$ . On the other hand, we know by Lemma 3.6 that  $\eta_k$  is bounded in  $L_s(\mathbb{R})$  and, consequently,  $\eta_\infty \in L_s(\mathbb{R})$ ,  $s > 0$ . In particular, for  $s \in (0, \frac{1}{2})$ , letting  $k \rightarrow \infty$  first, and then  $R \rightarrow \infty$  we get

$$\begin{aligned} & \int_{\mathbb{R}} |\eta_k - \eta_\infty| |(-\Delta)^{\frac{1}{2}} \varphi| dx \\ & \leq \|(-\Delta)^{\frac{1}{2}} \varphi\|_{L^\infty(-R, R)} \|\eta_k - \eta_\infty\|_{L^1(-R, R)} + C \int_{(-R, R)^c} \frac{|\eta_k(x) - \eta_\infty(x)|}{|x|^2} dx \\ & \leq C \|\eta_k - \eta_\infty\|_{L^1(-R, R)} + CR^{2s-1} (\|\eta_k\|_{L_s(\mathbb{R})} + \|\eta_\infty\|_{L_s(\mathbb{R})}) \rightarrow 0. \end{aligned}$$

Then  $\eta_\infty$  is a weak solution  $(-\Delta)^{\frac{1}{2}} \eta_\infty = 2e^{\eta_\infty}$  and  $\eta_\infty \in L_s(\mathbb{R})$  for any  $s$ . Moreover, repeating the argument of Corollary 2.3 and using (6), we get

$$\frac{1}{\pi} \int_{-R}^R e^{\eta_\infty} d\xi = \lim_{k \rightarrow \infty} \int_{-Rr_k}^{Rr_k} \lambda_k u_k^2 e^{\alpha_k u_k^2} dx \leq \limsup_{k \rightarrow \infty} \|u_k\|_H^2 = \Lambda, \quad (40)$$

which implies  $e^{\eta_\infty} \in L^1(\mathbb{R})$ . Then  $\eta_\infty(x) = -\log(1 + x^2)$ , see e.g. [6, Theorem 1.8].

To complete the proof, we shall study the properties of the weak limit  $u_\infty$  of  $u_k$  in  $H$ . First, we show that  $u_\infty$  is a weak solution of (7). Let us denote

$$g_k := \lambda_k u_k e^{\alpha_k u_k^2}, \quad g_\infty := \lambda_\infty u_\infty e^{\pi u_\infty^2}.$$

Take any function  $\varphi \in \mathcal{S}(\mathbb{R})$ . On the one hand, since  $(-\Delta)^{\frac{1}{2}} \varphi \in L^2(\mathbb{R})$  and  $u_k \rightharpoonup u_\infty$  weakly in  $L^2(\mathbb{R})$ , we have

$$\int_{\mathbb{R}} (u_k - u_\infty) (-\Delta)^{\frac{1}{2}} \varphi dx + \int_{\mathbb{R}} (u_k - u_\infty) \varphi dx \rightarrow 0,$$

as  $k \rightarrow \infty$ . On the other hand, for any large  $t > 0$  we get

$$\begin{aligned} \int_{\mathbb{R}} |g_k - g_\infty| |\varphi| dx & \leq \int_{\{u_k \leq t\}} |g_k - g_\infty| |\varphi| dx + \frac{\|\varphi\|_{L^\infty(\mathbb{R})}}{t} \int_{\mathbb{R}} u_k (g_k + g_\infty) dx \\ & = o(1) + O(t^{-1}) \rightarrow 0 \end{aligned}$$

as  $k, t \rightarrow \infty$ , where we used that  $g_\infty \in L^2(\mathbb{R})$  by Theorem A (see e.g. Lemma 2.3 of [17]) together with the dominated convergence theorem and the bounds  $\|u_k g_k\|_{L^1(\mathbb{R})} \leq \Lambda$  and  $\|u_k\|_{L^2(\mathbb{R})} \leq \Lambda$ . Then,  $u_\infty$  is a weak solution of (7).

Now, observe that

$$\|u_k\|_H^2 = \int_{\mathbb{R}} g_k u_k dx = \int_{-Rr_k}^{Rr_k} g_k u_k dx + \int_{\mathbb{R} \setminus (-Rr_k, Rr_k)} g_k u_k dx$$

with

$$\lim_{k \rightarrow \infty} \int_{-Rr_k}^{Rr_k} u_k g_k dx = \frac{1}{\pi} \int_{-R}^R e^{\eta_\infty} dx \rightarrow 1$$

as  $R \rightarrow \infty$ , and

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{R} \setminus (-Rr_k, Rr_k)} g_k u_k dx = \int_{\mathbb{R}} g_\infty u_\infty dx = \|u_\infty\|_H^2,$$

for any  $R > 1$ , by Fatou's lemma. Thus we conclude that

$$\|u_k\|_H^2 \geq \|u_\infty\|_H^2 + 1.$$

Finally, to prove that  $u_k \rightarrow u_\infty$  in  $C_{loc}^\ell(\mathbb{R} \setminus \{0\})$  for every  $\ell \geq 0$ , we use the monotonicity of  $u_k$ , which implies that  $u_k$  is locally bounded away from 0, hence we can conclude by elliptic estimates, as in Lemma 3.1.

## 4 Proof of Theorem 1.2

Let us denote

$$E_\alpha(u) = \int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx, \quad D_\alpha := \sup_{u \in H: \|u\|_H \leq 1} E_\alpha(u).$$

The proof of Theorem 1.2 is organized as follows. First, we prove that  $D_\alpha$  is attained for  $\alpha \in (0, \pi)$  sufficiently close to  $\pi$ . Then, we fix a sequence  $(\alpha_k)_{k \in \mathbb{N}}$  such that  $\alpha_k \nearrow \pi$  as  $k \rightarrow +\infty$ , and for any large  $k$  we take a positive extremal  $u_k \in H$  for  $D_{\alpha_k}$ . With a contradiction argument similar to the one of Section 2, we show that  $\mu_k := \sup_{\mathbb{R}} u_k \leq C$ . Finally, we show that  $u_k \rightarrow u_\infty$  in  $L_{loc}^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ , where  $u_\infty$  is a maximizer for  $D_\pi$ .

### 4.1 Subcritical extremals: Ruling out vanishing

The following lemma describes the effect of the lack of compactness of the embedding  $H \subseteq L^2(\mathbb{R})$  on  $E_\alpha$ , and holds uniformly for  $\alpha \in [0, \pi]$ .

**Lemma 4.1.** *Let  $(\alpha_k) \subseteq [0, \pi]$  and  $(u_k) \subseteq H$  be two sequences such that:*

1.  $\alpha_k \rightarrow \alpha_\infty \in [0, \pi]$  as  $k \rightarrow \infty$ .
2.  $\|u_k\|_H \leq 1$ ,  $u_k \rightharpoonup u_\infty$  weakly in  $H$ ,  $u_k \rightarrow u_\infty$  a.e. in  $\mathbb{R}$ , and  $e^{\alpha_k u_k^2} \rightarrow e^{\alpha_\infty u_\infty^2}$  in  $L^1_{\text{loc}}(\mathbb{R})$  as  $k \rightarrow \infty$ .
3. The  $u_k$ 's are even and monotone decreasing i.e.  $u_k(-x) = u_k(x) \geq u_k(y)$  for  $0 \leq x \leq y$ .

Then we have

$$E_{\alpha_k}(u_k) = E_{\alpha_\infty}(u_\infty) + \alpha_\infty \left( \|u_k\|_{L^2(\mathbb{R})}^2 - \|u_\infty\|_{L^2(\mathbb{R})}^2 \right) + o(1),$$

as  $k \rightarrow \infty$ .

*Proof.* Since  $u_k$  is even and decreasing, we know that

$$u_k(x)^2 \leq \frac{\|u_k\|_{L^2(\mathbb{R})}^2}{2|x|} \leq \frac{1}{2|x|}, \tag{41}$$

for any  $x \in \mathbb{R} \setminus \{0\}$ . In particular, there exists a constant  $C > 0$ , such that

$$e^{\alpha_k u_k^2(x)} - 1 - \alpha_k u_k^2(x) \leq C|x|^{-4},$$

for  $|x| \geq 1$ . Applying the dominated convergence theorem for  $|x| \geq 1$ , using the assumption that  $e^{\alpha_k u_k^2} \rightarrow e^{\alpha_\infty u_\infty^2}$  in  $L^1_{\text{loc}}(\mathbb{R})$ , and recalling that  $(u_k)$  is precompact in  $L^1_{\text{loc}}(\mathbb{R})$ , we find that

$$\int_{\mathbb{R}} (e^{\alpha_k u_k^2} - 1 - \alpha_k u_k^2) dx \rightarrow \int_{\mathbb{R}} (e^{\alpha_\infty u_\infty^2} - 1 - \alpha_\infty u_\infty^2) dx,$$

and the Lemma follows. □

**Lemma 4.2.** *Take  $\alpha \in (0, \pi)$ . If  $D_\alpha > \alpha$ , then  $D_\alpha$  is attained by an even and decreasing function, i.e. there exists  $u_\alpha \in H$  even and decreasing s.t.  $\|u_\alpha\|_H = 1$  and  $E_\alpha(u_\alpha) = D_\alpha$ .*

*Proof.* Let  $(u_k) \subset H$  be a maximizing sequence for  $E_\alpha$ . W.l.o.g. we can assume  $u_k \rightarrow u_\infty \in H$  weakly in  $H$  and a.e. on  $\mathbb{R}$ . Moreover, up to replacing  $u_k$  with its symmetric decreasing rearrangement, we can assume that  $u_k$  is even and decreasing (see [30]). Since  $\alpha \in (0, \pi)$  the sequence  $e^{\alpha u_k^2} - 1$  is bounded in

$L^{\frac{\pi}{\alpha}}(\mathbb{R})$ , with  $\frac{\pi}{\alpha} > 1$ . Then, by Vitali's theorem, we get  $e^{\alpha u_k^2} \rightarrow e^{\alpha u_\infty^2}$  in  $L^1_{\text{loc}}(\mathbb{R})$ , and Lemma 4.1 yields

$$E_\alpha(u_k) = E_\alpha(u_\infty) + \alpha \left( \|u_k\|_{L^2(\mathbb{R})}^2 - \|u_\infty\|_{L^2(\mathbb{R})}^2 \right) + o(1). \quad (42)$$

This implies that  $u_\infty \not\equiv 0$ , since otherwise we have  $E_\alpha(u_k) = \alpha \|u_k\|_{L^2(\mathbb{R})}^2 + o(1) \leq \alpha + o(1)$ , which contradicts the assumption  $D_\alpha > \alpha$ . Let us denote

$$L := \limsup_{k \rightarrow \infty} \|u_k\|_{L^2(\mathbb{R})}^2, \quad \tau := \frac{\|u_\infty\|_{L^2(\mathbb{R})}^2}{L}.$$

Observe that  $L, \tau \in (0, 1]$ . Let us consider the sequence  $v_k(x) = u_k(\tau x)$ . Clearly, we have  $v_k \rightharpoonup v_\infty$  weakly in  $H$ , where  $v_\infty(x) := u_\infty(\tau x)$ . Moreover, since  $\|u_\infty\|_{L^2}^2 = L$  and

$$\|(-\Delta)^{\frac{1}{4}} v_\infty\|_{L^2}^2 \leq \liminf_{k \rightarrow \infty} \|(-\Delta)^{\frac{1}{4}} v_k\|_{L^2}^2 = \liminf_{k \rightarrow \infty} \|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2}^2 \leq 1 - L,$$

we get  $\|v_\infty\|_H \leq 1$ . By (42) we have

$$D_\alpha \leq E_\alpha(u_\infty) + \alpha L(1 - \tau) = \tau E_\alpha(v_\infty) + \alpha L(1 - \tau) \leq \tau D_\alpha + \alpha L(1 - \tau). \quad (43)$$

If  $\tau < 1$ , this implies  $D_\alpha \leq \alpha L \leq \alpha$ , contradicting the assumptions. Hence  $\tau = 1$  and (43) gives  $D_\alpha = E_\alpha(u_\infty)$ . Finally we have  $\|u_0\|_H = 1$ , otherwise  $E_\alpha\left(\frac{u_\infty}{\|u_\infty\|_H}\right) > E_\alpha(u_\infty) = D_\alpha$ .  $\square$

**Lemma 4.3.** *There exists  $\alpha^* \in (0, \pi)$  such that  $D_\alpha > \alpha$  for any  $\alpha \in (\alpha^*, \pi]$ . In particular  $D_\alpha$  is attained by an even and decreasing function  $u_\alpha$  for any  $\alpha \in (\alpha^*, \pi)$  by Lemma 4.2.*

*Proof.* This follows from Proposition 4.14 by continuity. Indeed Proposition 4.14 gives  $D_\pi > 2\pi e^{-\gamma} > \pi$ .  $\square$

## 4.2 The critical case

Next, we take a sequence  $\alpha_k$  such that  $\alpha_k \nearrow \pi$  as  $k \rightarrow \infty$ . For any large  $k$ , Lemma 4.3 yields the existence  $u_k \in H$  even and decreasing such that  $D_{\alpha_k} = E_{\alpha_k}(u_k)$ . Each  $u_k$  satisfies

$$(-\Delta)^{\frac{1}{2}} u_k + u_k = \lambda_k u_k e^{\alpha_k u_k^2},$$

and  $\|u_k\|_H = 1$ . Note that  $u_k \in C^\infty(\mathbb{R})$  by elliptic estimates. Multiplying the equation by  $u_k$  and using the basic inequality  $te^t \geq e^t - 1$ , for  $t \geq 0$ , we infer

$$\frac{1}{\lambda_k} = \int_{\mathbb{R}} u_k^2 e^{\alpha_k u_k^2} dx \geq \frac{1}{\alpha_k} E_{\alpha_k}(u_k) = \frac{1}{\alpha_k} D_{\alpha_k}.$$

Since  $D_{\alpha_k} \rightarrow D_\pi > 0$ , we get that  $\lambda_k$  is uniformly bounded.

Then the sequence  $u_k$  satisfies the alternative of Theorem 1.3. If case (i) holds, then we can argue as in Lemma 4.2 and Lemma 4.3 and prove that  $D_\pi$  is attained. Therefore, we shall assume by contradiction that case (ii) occurs.

Let  $r_k$  and  $\eta_k$  be as in Theorem 1.3. Let  $\tilde{\eta}_k$  denote the Poisson integral of  $\eta_k$ .

**Proposition 4.4.** *We have  $\tilde{\eta}_k \rightarrow \tilde{\eta}_\infty$  in  $C_{\text{loc}}^\ell(\overline{\mathbb{R}_+^2})$  for every  $\ell \geq 0$ , where*

$$\tilde{\eta}_\infty(x, y) = -\log((1+y)^2 + x^2)$$

*is the Poisson integral (compare to (66)) of  $\eta_\infty := -\log(1+x^2)$ .*

*Proof.* By Theorem 1.3 we know that  $\eta_k \rightarrow \eta_\infty$  in  $C_{\text{loc}}^\ell(\mathbb{R})$  and that  $\eta_k$  is bounded in  $L_{\frac{1}{2}}$ . Then, we can repeat the argument of the proof of Proposition 2.2. □

**Remark 4.5.** *As in (17), the convergence  $\eta_k \rightarrow \eta_\infty$  in  $L_{\text{loc}}^\infty(\mathbb{R})$  implies*

$$\lim_{k \rightarrow \infty} \int_{-r_k R}^{r_k R} \lambda_k \mu_k^i u_k^{2-i} e^{\alpha_k u_k^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{\eta_\infty} dx,$$

*for  $i = 0, 1, 2$  and for any  $R > 0$ .*

**Lemma 4.6.** *We have  $u_k \rightarrow 0$  in  $L^2(\mathbb{R})$ .*

*Proof.* Indeed, otherwise up to a subsequence we would have  $\|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2(\mathbb{R})} \leq \frac{1}{A}$  for some  $A > 1$ . Consider, the function  $v_k = (u_k - u_k(1))^+$ . Then,  $v_k \in \dot{H}^{\frac{1}{2}, 2}(I)$  and  $\|(-\Delta)^{\frac{1}{4}} v_k\|_{L^2(\mathbb{R})} \leq \frac{1}{A}$ . The Moser-Trudinger inequality (3) gives that  $e^{\alpha_k v_k^2}$  is bounded in  $L^A(\mathbb{R})$ . Since

$$u_k^2 \leq (1 + \varepsilon)v_k^2 + \frac{1}{\varepsilon}(u_k - v_k)^2$$

and  $|v_k - u_k| \leq u_k(1) \rightarrow 0$  as  $k \rightarrow \infty$ , we get that  $e^{\alpha_k u_k^2}$  is uniformly bounded in  $L^p(\mathbb{R})$  for every  $1 < p < A$ . Therefore, we have

$$\int_{(-1,1)} (e^{\alpha_k u_k^2} - 1) dx \rightarrow 0$$

as  $k \rightarrow \infty$ . But then, by Lemma 4.1 we find  $D_g^\pi \leq \pi$ , which contradicts Lemma 4.3. □

**Lemma 4.7.** For  $A > 1$ , set  $u_k^A := \min \left\{ u_k, \frac{\mu_k}{A} \right\}$ . Then we have

$$\limsup_{k \rightarrow \infty} \|(-\Delta)^{\frac{1}{4}} u_k^A\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{A}. \tag{44}$$

*Proof.* The proof is similar to the one of Lemma 2.4. We set  $\tilde{u}_k^A := \min \left\{ \tilde{u}_k, \frac{\mu_k}{A} \right\}$ . Since  $\tilde{u}_k^A$  is an extension of  $u_k^A$ , using integration by parts and the harmonicity of  $\tilde{u}_k$  we get

$$\begin{aligned} \|(-\Delta)^{\frac{1}{4}} u_k^A\|_{L^2(\mathbb{R})}^2 &\leq \int_{\mathbb{R}^2_+} |\nabla \tilde{u}_k^A|^2 dx dy = \int_{\mathbb{R}^2_+} \nabla \tilde{u}_k^A \cdot \nabla \tilde{u}_k dx dy \\ &= - \int_{\mathbb{R}} u_k^A(x) \frac{\partial \tilde{u}_k(x, 0)}{\partial y} dx \\ &= \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A dx. \end{aligned} \tag{45}$$

Proposition 4.4 implies that  $u_k^A(r_k x) = \frac{\mu_k}{A}$  for  $|x| \leq R$  and  $k \geq k_0(R)$ . Noting that  $u_k^A \leq u_k$  and using Lemma 4.6, and Remark 4.5, we get

$$\begin{aligned} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A dx &\geq \int_{-Rr_k}^{Rr_k} \lambda_k u_k e^{\alpha_k u_k^2} u_k^A dx - \int_{\mathbb{R}} u_k u_k^A dx \\ &= \frac{1}{A} \int_{-Rr_k}^{Rr_k} \lambda_k \mu_k u_k e^{\alpha_k u_k^2} u_k^A dx + O(\|u_k\|_{L^2(\mathbb{R})}^2) \\ &\xrightarrow{k \rightarrow \infty} \frac{1}{\pi A} \int_{-R}^R e^{\eta_\infty} d\xi \\ &\xrightarrow{R \rightarrow \infty} \frac{1}{A}. \end{aligned}$$

Set now  $v_k^A := (u_k - \frac{\mu_k}{A})^+$ . With similar computations we get

$$\begin{aligned} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k v_k^A dx &\geq \int_{-Rr_k}^{Rr_k} \lambda_k u_k v_k^A e^{\alpha_k u_k^2} dx + O(\|u_k\|_{L^2(\mathbb{R})}^2) \\ &\xrightarrow{k \rightarrow \infty} \frac{1}{\pi} \left( 1 - \frac{1}{A} \right) \int_{-R}^R e^{\eta_\infty} d\xi \\ &\xrightarrow{R \rightarrow \infty} \frac{A-1}{A}. \end{aligned}$$

Since

$$\int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A dx + \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k v_k^A dx = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k dx = 1 - \|u_k\|_{L^2(\mathbb{R})}^2 \rightarrow 1$$

as  $k \rightarrow \infty$ , we get that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u_k u_k^A dx = \frac{1}{A}.$$

Then, we conclude using (45). □

**Proposition 4.8.** *We have*

$$D_\pi = \lim_{k \rightarrow \infty} \frac{1}{\lambda_k \mu_k^2}. \quad (46)$$

Moreover

$$\lim_{k \rightarrow \infty} \mu_k \lambda_k = 0. \quad (47)$$

*Proof.* Fix  $A > 1$  and write

$$\int_{\mathbb{R}} \left( e^{\alpha_k u_k^2} - 1 \right) dx = (I) + (II) + (III),$$

where (I), (II) and (III) denote respectively the integrals over the regions  $\{u_k \leq \frac{\mu_k}{A}\} \cap (-1, 1)$ ,  $\{u_k \leq \frac{\mu_k}{A}\} \cap (-1, 1)^c$  and  $\{u_k > \frac{\mu_k}{A}\}$ . Using Lemmas 4.11 and 4.7 together with Theorem A we see that

$$(I) \leq \int_{-1}^1 \left( e^{\alpha_k (u_k^A)^2} - 1 \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

since  $e^{\alpha_k (u_k^A)^2} - 1$  is uniformly bounded in  $L^p$ , for any  $1 \leq p < A$ . By (41) and Lemma 4.6, we find

$$(II) \leq \int_{(-1,1)^c} \left( e^{\alpha_k u_k^2} - 1 \right) dx \leq C \int_{\mathbb{R}} u_k^2 dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We now estimate

$$(III) \leq \frac{A^2}{\lambda_k \mu_k^2} \int_{\{u_k > \frac{\mu_k}{A}\}} \lambda_k u_k^2 e^{\alpha_k u_k^2} dx \leq \frac{A^2}{\lambda_k \mu_k^2} (1 + o(1)),$$



with  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ , where we used that

$$\int_{I \cap \{u_k > \frac{\mu_k}{A}\}} \lambda_k u_k^2 e^{\alpha_k u_k^2} dx \leq \|u_k\|_H^2 = 1.$$

Letting  $A \downarrow 1$ , this gives

$$\sup_H E_\pi \leq \lim_{k \rightarrow \infty} \frac{1}{\lambda_k \mu_k^2}.$$

The converse inequality follows from Remark 4.5:

$$\int_{\mathbb{R}} \left( e^{\alpha_k u_k^2} - 1 \right) dx \geq \int_{-Rr_k}^{Rr_k} e^{\alpha_k u_k^2} dx + o(1) = \frac{1}{\lambda_k \mu_k^2} \left( \int_{-R}^R e^{\eta_\infty} dx + o(1) \right) + o(1).$$

with  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ . Letting  $R \rightarrow \infty$  we obtain (46).

Finally, (47) follows at once from (46), because otherwise we would have  $D_\pi = 0$ , which is clearly impossible.  $\square$

**Lemma 4.9.** *We have*

$$f_k := \lambda_k \mu_k u_k e^{\alpha_k u_k^2} \rightharpoonup \delta_0$$

as  $k \rightarrow \infty$ , in the sense of Radon measures in  $\mathbb{R}$ .

*Proof.* The proof follows step by step the one Proposition 2.6, with (4), Proposition 4.4, Remark 4.5 Lemma 4.6 and Lemma 4.7 used in place of (3), Proposition 2.2, (17), Lemma 2.3 and Lemma 2.4. We omit the details.  $\square$

For  $x \in \mathbb{R}$ , let  $G_x$  be the Green function of  $(-\Delta)^{\frac{1}{2}} + Id$  on  $\mathbb{R}$  with singularity at  $x$ . In the following we denote  $G := G_0$ . By translation invariance, we get  $G_x(y) = G(y - x)$  for any  $x, y \in \mathbb{R}$ ,  $x \neq y$ . Moreover, the inversion formula for the Fourier-transform implies that

$$G(x) = \frac{1}{2} \sin |x| - \frac{1}{\pi} \sin(|x|) \text{Si}(|x|) - \frac{1}{\pi} \cos(|x|) \text{Ci}(|x|), \tag{48}$$

where

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt \quad \text{and} \quad \text{Ci}(x) = - \int_x^{+\infty} \frac{\cos t}{t} dt.$$

We recall that the identity

$$\text{Ci}(x) = \log x + \gamma + \int_0^x \frac{\cos t - 1}{t} dt \tag{49}$$

holds for any  $x \in \mathbb{R} \setminus \{0\}$ , where  $\gamma$  denotes the Euler-Mascheroni constant see e.g. [13, Chapter 12.2].

**Proposition 4.10.** *The function  $G$  satisfies the following properties.*

1. We have  $G \in C^\infty(\mathbb{R} \setminus \{0\})$  and

$$G(x) = -\frac{1}{\pi} \log|x| - \frac{\gamma}{\pi} + O(|x|), \quad G'(x) = -\frac{1}{\pi x} + O(1), \quad \text{as } x \rightarrow 0. \quad (50)$$

2. We have  $G(x) = O(|x|^{-2})$  and  $G'(x) = O(|x|^{-3})$  as  $|x| \rightarrow \infty$ .

3. Let  $\tilde{G}$  be the Poisson extension of  $G$ . There exists a function  $f \in C^1(\overline{\mathbb{R}_+^2})$  such that  $f(0,0) = 0$  and

$$\tilde{G}(x,y) = -\frac{1}{\pi} \ln|(x,y)| - \frac{\gamma}{\pi} + \frac{x}{\pi} \arctan \frac{x}{y} - \frac{y}{2\pi} \log(x^2 + y^2) + f(x,y) \quad \text{in } \mathbb{R}_+^2. \quad (51)$$

*Proof.* Property 1. follows directly by formula (48) and the identity in (49). Similarly, since

$$Si(t) = \frac{\pi}{2} - \frac{\cos t}{t} - \frac{\sin t}{t^2} + O(t^{-3}), \quad Ci(t) = \frac{\sin t}{t} - \frac{\cos t}{t^2} + O(t^{-3}),$$

as  $t \rightarrow +\infty$ , we get 2.

Given  $R > 0$ , let  $\psi \in C_c^\infty(\mathbb{R})$  be a cut-off function with  $\psi \equiv 1$  on  $(-R, R)$ . Let us denote  $g_0 := -\frac{1}{\pi} \log|\cdot| - \frac{\gamma}{\pi}$ ,  $g_1 := \frac{1}{2}|\cdot|\psi$ ,  $g_2 := G - g_0 - g_1$ . By Proposition A.3, we have

$$\tilde{g}_0(x,y) = -\frac{1}{\pi} \log|(x,y)| - \frac{\gamma}{\pi}, \quad (x,y) \in \mathbb{R}^2.$$

Denoting  $\theta(x,y) := \arctan \frac{x}{y}$  the angle between the  $y$ -axis and the segment connecting the origin to  $(x,y)$ , the function

$$h(x,y) := \tilde{g}_1(x,y) - \frac{1}{\pi} x \theta(x,y) + \frac{1}{2\pi} y \log(x^2 + y^2)$$

is harmonic in  $\mathbb{R}_+^2$ , continuous on  $\overline{\mathbb{R}_+^2}$ , and identically 0 on  $(-R, R) \times \mathbb{R}$ . By [33, Theorem C], we get that  $h \in C^\infty(\overline{\mathbb{R}_+^2} \cap B_R(0,0))$ . Finally, note that formula (48) implies  $\underline{g}_2 \in C^2(\mathbb{R})$  and  $g_2(0) = 0$ . Hence, standard elliptic regularity yields  $\tilde{g}_2 \in C^{1,\alpha}(\overline{\mathbb{R}_+^2} \cap B_R(0,0))$ , for any  $\alpha \in (0,1)$ . In particular  $\tilde{g}_2(0,0) = g_2(0) = 0$ .  $\square$

**Lemma 4.11.** *We have  $\mu_k u_k \rightarrow G$  in  $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R} \setminus (-\varepsilon, \varepsilon))$ , for any  $\varepsilon > 0$ ,*

*Proof.* Let us set  $v_k := \mu_k u_k - G$  and  $f_k = \mu_k \lambda_k u_k e^{\alpha_k u_k^2}$ . By Lemma 4.9 we have  $\|f_k\|_{L^1(I)} \rightarrow 1$  as  $k \rightarrow +\infty$ ,  $I = (-1, 1)$ . Then, arguing as in Lemma 2.8,

we get

$$\begin{aligned}
 |v_k(x)| &= \left| \int_{\mathbb{R}} G(y-x)f_k(y) dy - G(x) \right| \\
 &\leq \int_I |G(x-y) - G(x)|f_k(y) dy + \underbrace{\|f_k\|_{L^1(I)} - 1}_{=o(1)} |G(x)| \\
 &\quad + \underbrace{\int_{\mathbb{R} \setminus I} G(x-y)f_k(y)dy}_{=:w_k(x)}. \tag{52}
 \end{aligned}$$

Using (41), Lemma 4.6 and (47), we get that  $f_k \rightarrow 0$  in  $L^2(\mathbb{R} \setminus I)$ . In particular

$$|w_k(x)| \leq \|f_k\|_{L^2(\mathbb{R} \setminus I)} \|G\|_{L^2(\mathbb{R})} \rightarrow 0.$$

Fix  $\sigma \in (0, 1)$  and assume  $|x| \geq \sigma$ . If we further take  $|y| \leq \frac{\sigma}{2}$ , then Proposition 4.10 implies

$$|G(x-y) - G(x)| \leq C|y|,$$

where  $C$  is a constant depending only on  $\sigma$ . Thus, for any  $\varepsilon \in (0, \frac{\sigma}{2})$ , setting  $I_\varepsilon = \mathbb{R} \setminus (-\varepsilon, \varepsilon)$  we can write

$$\begin{aligned}
 |v_k(x)| &\leq \int_I |G(x-y) - G(x)|f_k(y) dy + o(1)\|G\|_{L^\infty(\mathbb{R} \setminus (-\sigma, \sigma))} + o(1) \\
 &\leq C \int_{-\varepsilon}^{\varepsilon} |y|f_k(y) dy + \int_{I_\varepsilon} |G(x-y)|f_k(y) dy + |G(x)| \int_{I_\varepsilon} f_k(y) dy + o(1) \\
 &\leq C\varepsilon \|f_k\|_{L^1(I)} + \|f_k\|_{L^\infty(I_\varepsilon)} (\|G\|_{L^1(\mathbb{R})} + \|G\|_{L^\infty(\mathbb{R} \setminus (-\sigma, \sigma))}) + o(1) \\
 &\leq C\varepsilon + o(1), \tag{53}
 \end{aligned}$$

where  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$  (depending on  $\varepsilon$  and  $\sigma$ ). Here, we used that  $f_k \rightarrow 0$  in  $L^\infty(\mathbb{R} \setminus (-\varepsilon, \varepsilon))$  by (41) and (47). Since  $\varepsilon$  is arbitrarily small, (53) shows that  $v_k \rightarrow 0$  in  $L^\infty(\mathbb{R} \setminus (-\sigma, \sigma))$ .

Next, we prove the  $L^2$  convergence. First, Hölder's inequality and Fubini's theorem give

$$\|w_k\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left( \int_{\mathbb{R} \setminus I} G(x-y)f_k(y)dy \right)^2 dx \leq \|G\|_{L^1(\mathbb{R})}^2 \|f_k\|_{L^2(\mathbb{R} \setminus I)}^2 \rightarrow 0$$

as  $k \rightarrow \infty$ . With a similar argument, after integrating (52) and using the triangular inequality in  $L^2$ , we find

$$\begin{aligned} \|v_k\|_{L^2(\mathbb{R})} &\leq \left( \int_{\mathbb{R}} \left( \int_I |G(x-y) - G(x)| f_k(y) dy \right)^2 dx \right)^{\frac{1}{2}} \\ &\quad + \|f_k\|_{L^1(I)} - 1 \|G\|_{L^2(\mathbb{R})} + \|w_k\|_{L^2(\mathbb{R})} \\ &\leq \underbrace{\left( \int_I f_k(y) dy \right)^{\frac{1}{2}}}_{=1+o(1)} \left( \int_I f_k(y) \int_{\mathbb{R}} |G(x-y) - G(x)|^2 dx dy \right)^{\frac{1}{2}} + o(1). \end{aligned}$$

Since  $G \in L^2(\mathbb{R})$ , the function  $\psi(y) := \int_{\mathbb{R}} |G(x-y) - G(x)|^2 dx$  is continuous on  $\mathbb{R}$  and  $\psi(0) = 0$ . Let  $\varphi \in C(\mathbb{R})$  be a compactly supported function such that  $\varphi \equiv \psi$  on  $I$ . Then, Lemma 4.9 implies

$$\int_I f_k(y) \int_{\mathbb{R}} |G(x-y) - G(x)|^2 dx dy = \int_{\mathbb{R}} f_k(y) \varphi(y) dy + o(1) \rightarrow 0,$$

as  $k \rightarrow \infty$ , and the conclusion follows.  $\square$

Repeating the argument of Proposition 2.8, we get the following:

**Lemma 4.12.** *We have  $\mu_k \tilde{u}_k \rightarrow \tilde{G}$  in  $C_{\text{loc}}^0(\overline{\mathbb{R}_+^2} \setminus \{(0,0)\}) \cap C_{\text{loc}}^1(\mathbb{R}_+^2)$ , where  $\tilde{G}$  is the Poisson extension of  $G$ .*

With Proposition 4.4 and Lemma 4.12 we can give an upper bound on  $D_\pi$ .

**Proposition 4.13.** *Under the assumption that  $\mu_k \rightarrow \infty$  as  $k \rightarrow \infty$ , we have  $D_\pi \leq 2\pi e^{-\gamma}$ .*

*Proof.* For a fixed and small  $\delta > 0$  set

$$a_k := \inf_{B_{Lr_k} \cap \mathbb{R}_+^2} \tilde{u}_k, \quad b_k := \sup_{B_\delta \cap \mathbb{R}_+^2} \tilde{u}_k, \quad \tilde{v}_k := (\tilde{u}_k \wedge a_k) \vee b_k.$$

Recalling that  $\|\nabla \tilde{u}_k\|_{L^2(\mathbb{R})}^2 = \|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2(\mathbb{R})}^2 = 1 - \|u_k\|_{L^2(\mathbb{R})}^2$ , we have

$$\int_{(B_\delta \setminus B_{Lr_k}) \cap \mathbb{R}_+^2} |\nabla \tilde{v}_k|^2 dx dy \leq 1 - \|u_k\|_{L^2}^2 - \int_{\mathbb{R}_+^2 \setminus B_\delta} |\nabla \tilde{u}_k|^2 dx - \int_{\mathbb{R}_+^2 \cap B_{Lr_k}} |\nabla \tilde{u}_k|^2 dx \quad (54)$$

Clearly the left-hand side bounds

$$\inf_{\substack{\tilde{u}|_{\mathbb{R}_+^2 \cap \partial B_{Lr_k}} = a_k \\ \tilde{u}|_{\mathbb{R}_+^2 \cap \partial B_\delta} = b_k}} \int_{(B_\delta \setminus B_{Lr_k}) \cap \mathbb{R}_+^2} |\nabla \tilde{u}|^2 dx dy = \pi \frac{(a_k - b_k)^2}{\log \delta - \log(Lr_k)}.$$

Using Proposition 4.4, Proposition 4.10 and Lemma 4.12 we obtain

$$\begin{aligned} a_k &= \mu_k + \frac{-\frac{1}{\pi} \log L + O(L^{-1}) + o(1)}{\mu_k}, \\ b_k &= \frac{-\frac{1}{\pi} \log \delta - \frac{\gamma}{\pi} + O(\delta |\log \delta|) + o(1)}{\mu_k}, \end{aligned} \tag{55}$$

where  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$  for fixed  $L > 0$ ,  $\delta > 0$ , and  $|O(L^{-1})| \leq CL^{-1}$ ,  $|O(\delta |\log \delta|)| \leq C\delta |\log \delta|$ , uniformly for  $\delta$  small, and  $L, k$  large. Still with Proposition 4.4 we get

$$\begin{aligned} \lim_{k \rightarrow \infty} \mu_k^2 \int_{B_{Lr_k}^+} |\nabla \tilde{u}_k|^2 dx dy &= \frac{1}{4\pi^2} \int_{B_L^+} |\nabla \tilde{\eta}_\infty|^2 dx dy \\ &= \frac{1}{\pi} \log \frac{L}{2} + O\left(\frac{\log L}{L}\right). \end{aligned}$$

Similarly Lemma 4.12 and Proposition 4.10 yield

$$\liminf_{k \rightarrow \infty} \mu_k^2 \int_{\mathbb{R}_+^2 \setminus B_\delta} |\nabla \tilde{u}_k|^2 dx dy \geq \int_{\mathbb{R}_+^2 \setminus B_\delta} |\nabla \tilde{G}|^2 dx dy$$

with

$$\begin{aligned} \int_{\mathbb{R}_+^2 \setminus B_\delta} |\nabla \tilde{G}|^2 dx dy &= \int_{\mathbb{R}_+^2 \cap \partial B_\delta} -\frac{\partial \tilde{G}}{\partial r} \tilde{G} d\sigma + \int_{(\mathbb{R} \times \{0\}) \setminus B_\delta} -\frac{\partial \tilde{G}(x, 0)}{\partial y} G(x) dx \\ &= \int_{\mathbb{R}_+^2 \cap \partial B_\delta} \left(\frac{1}{\pi\delta} + O(|\log \delta|)\right) \left(-\frac{1}{\pi} \log \delta - \frac{\gamma}{\pi} + O(\delta |\log \delta|)\right) d\sigma \\ &\quad - \int_{\mathbb{R} \setminus (-\delta, \delta)} G(x)^2 dx \\ &= -\frac{1}{\pi} \log \delta - \frac{\gamma}{\pi} - \|G\|_{L^2(\mathbb{R})}^2 + O(\delta \log^2 \delta), \end{aligned}$$

where we used that

$$-\frac{\partial \tilde{G}(x, 0)}{\partial y} = (-\Delta)^{\frac{1}{2}} G(x) = -G(x), \quad \text{for } x \in \mathbb{R} \setminus \{0\}.$$

From Lemma 4.11 we get that  $\mu_k u_k \rightarrow G$  in  $L^2(\mathbb{R})$ , hence

$$\|u_k\|_{L^2(\mathbb{R})}^2 = \frac{\|G\|_{L^2(\mathbb{R})}^2 + o(1)}{\mu_k^2}$$

as  $k \rightarrow +\infty$ . We then get

$$\frac{\pi(a_k - b_k)^2}{\log \delta - \log(Lr_k)} \leq 1 - \frac{\frac{1}{\pi} \log \frac{L}{2\delta} - \frac{\gamma}{\pi} + O(\delta \log^2 \delta) + O\left(\frac{\log L}{L}\right) + o(1)}{\mu_k^2}.$$

Using (55) and rearranging as in the proof of Proposition 2.9, we find

$$\begin{aligned} \log \frac{1}{\lambda_k \mu_k^2} &\leq \left(1 - \frac{\alpha_k}{\pi}\right) \log \frac{L}{\delta} + (\alpha_k - \pi) \mu_k^2 + \left(\frac{\alpha_k}{\pi} - 2\right) \gamma + \frac{\alpha_k}{\pi} \log 2 + \log \alpha_k \\ &\quad + O(\delta \log^2 \delta) + O\left(\frac{\log L}{L}\right) + o(1), \end{aligned}$$

with  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ . Then, recalling that  $\alpha_k \uparrow \pi$ , letting  $k \rightarrow \infty$  first and then  $L \rightarrow \infty$ ,  $\delta \rightarrow 0$ , we obtain

$$\limsup_{k \rightarrow \infty} \log \frac{1}{\lambda_k \mu_k^2} \leq -\gamma + \log(2\pi),$$

and using Proposition 4.8 we conclude.  $\square$

**Proposition 4.14.** *There exists a function  $u \in H^{\frac{1}{2},2}(\mathbb{R})$  such that  $\|u\|_H \leq 1$  and  $E_\pi(u) > 2\pi e^{-\gamma}$ .*

*Proof.* For  $\varepsilon > 0$  choose  $L = L(\varepsilon) > 0$  such that as  $\varepsilon \rightarrow 0$  we have  $L \rightarrow \infty$  and  $L\varepsilon \rightarrow 0$ . Fix

$$\Gamma_{L\varepsilon} := \left\{ (x, y) \in \mathbb{R}_+^2 : \tilde{G}(x, y) = \gamma_{L\varepsilon} := \min_{\mathbb{R}_+^2 \cap \partial B_{L\varepsilon}} \tilde{G} \right\},$$

and

$$\Omega_{L\varepsilon} := \{(x, y) \in \mathbb{R}_+^2 : \tilde{G}(x, y) > \gamma_{L\varepsilon}\}.$$

By the maximum principle we have  $\mathbb{R}_+^2 \cap B_{L\varepsilon} \subset \Omega_{L\varepsilon}$ . Notice also that Proposition 4.10 gives

$$\gamma_{L\varepsilon} = -\frac{1}{\pi} \log(L\varepsilon) - \frac{\gamma}{\pi} + O(L\varepsilon |\log(L\varepsilon)|).$$

and  $\Omega_{L\varepsilon} \subseteq \mathbb{R}_+^2 \cap B_{2L\varepsilon}$ . For suitable constants  $B, c \in \mathbb{R}$  to be fixed we set

$$U_\varepsilon(x, y) := \begin{cases} c - \frac{\log\left(\frac{x^2}{\varepsilon^2} + \left(1 + \frac{y}{\varepsilon}\right)^2\right) + 2B}{2\pi c} & \text{for } (x, y) \in B_{L\varepsilon}(0, -\varepsilon) \cap \mathbb{R}_+^2 \\ \frac{\gamma L\varepsilon}{c} & \text{for } (x, y) \in \Omega_{L\varepsilon} \setminus B_{L\varepsilon}(0, -\varepsilon) \\ \frac{\tilde{G}(x, y)}{c} & \text{for } (x, y) \in \mathbb{R}_+^2 \setminus \Omega_{L\varepsilon}. \end{cases}$$

Observe that  $\mathbb{R}_+^2 \cap B_{L\varepsilon}(0, -\varepsilon) \subseteq \mathbb{R}^2 \cap B_{L\varepsilon} \subseteq \Omega_{L\varepsilon}$ . We choose  $B$  in order to have continuity on  $\mathbb{R}_+^2 \cap \partial B_{L\varepsilon}(0, -\varepsilon)$ , i.e. we impose

$$\frac{-\log L^2 - 2B}{2\pi c} + c = \frac{\gamma L\varepsilon}{c},$$

which gives the relation

$$B = \pi c^2 + \log \varepsilon + \gamma + O(L\varepsilon |\log(L\varepsilon)|). \tag{56}$$

This choice of  $B$  also implies that the function  $cU_\varepsilon$  does not depend on the value of  $c$ . Then we can choose  $c$  by imposing

$$\|\nabla U_\varepsilon\|_{L^2(\mathbb{R}_+^2)}^2 + \|u_\varepsilon\|_{L^2(\mathbb{R})}^2 = 1, \tag{57}$$

where we set  $u_\varepsilon(x) = U_\varepsilon(x, 0)$ . Since the harmonic extension  $\tilde{u}_\varepsilon$  minimizes the Dirichlet energy among extensions with finite energy, we have

$$\|(-\Delta)^{\frac{1}{4}} u_\varepsilon\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}_+^2} |\nabla \tilde{u}_\varepsilon|^2 dx dy \leq \int_{\mathbb{R}_+^2} |\nabla U_\varepsilon|^2 dx dy,$$

and (57) implies  $\|u_\varepsilon\|_{H^{\frac{1}{2}, 2}(\mathbb{R})}^2 \leq 1$ .

In order to obtain a more precise expansion of  $B$  and  $c$  we compute

$$\begin{aligned} \int_{B_{L\varepsilon}(0, -\varepsilon) \cap \mathbb{R}_+^2} |\nabla U_\varepsilon|^2 dx dy &= \frac{1}{4\pi^2 c^2} \int_{B_L(0, -1) \cap \mathbb{R}_+^2} |\nabla \log(x^2 + (1+y)^2)|^2 dx dy \\ &= \frac{\frac{1}{\pi} \log\left(\frac{L}{2}\right) + O\left(\frac{\log L}{L}\right)}{c^2}, \end{aligned} \tag{58}$$

and

$$\begin{aligned}
 \int_{\mathbb{R}_+^2 \setminus \Omega_{L\varepsilon}} |\nabla U_\varepsilon|^2 dx dy &= \frac{1}{c^2} \int_{\mathbb{R}_+^2 \setminus \Omega_{L\varepsilon}} |\nabla \tilde{G}|^2 dx dy \\
 &= -\frac{1}{c^2} \int_{\mathbb{R}_+^2 \cap \partial \Omega_{L\varepsilon}} \frac{\partial \tilde{G}}{\partial \nu} \tilde{G} d\sigma - \frac{1}{c^2} \int_{(\mathbb{R} \times \{0\}) \setminus \bar{\Omega}_{L\varepsilon}} \frac{\partial \tilde{G}}{\partial y} \tilde{G} d\sigma \\
 &= (I) + (II).
 \end{aligned}$$

By the divergence theorem we have for  $\tau < L\varepsilon$  and letting  $\tau \rightarrow 0$ ,

$$\begin{aligned}
 (I) &= -\frac{\gamma L\varepsilon}{c^2} \int_{(\mathbb{R} \times \{0\}) \cap (\bar{\Omega}_{L\varepsilon} \setminus B_\tau)} \frac{\partial \tilde{G}}{\partial \nu} d\sigma - \frac{\gamma L\varepsilon}{c^2} \int_{\mathbb{R}_+^2 \cap \partial B_\tau} \frac{\partial \tilde{G}}{\partial \nu} d\sigma \\
 &= \frac{\gamma L\varepsilon}{c^2} \left( \int_{(\mathbb{R} \times \{0\}) \cap \bar{\Omega}_{L\varepsilon}} G d\sigma + 1 \right) \\
 &= \frac{\gamma L\varepsilon}{c^2} (1 + O(L\varepsilon \log(L\varepsilon))) \\
 &= \frac{\frac{1}{\pi} \log\left(\frac{1}{L\varepsilon}\right) - \frac{\gamma}{\pi} + O(L\varepsilon \log^2(L\varepsilon))}{c^2}, \tag{59}
 \end{aligned}$$

where in the third identity we used that  $\Omega_{L\varepsilon} \subset B_{2L\varepsilon}$  for  $L\varepsilon$  small enough. Observe also that

$$\|u_\varepsilon\|_{L^2(\mathbb{R})}^2 = \frac{1}{c^2} \int_{(\mathbb{R} \times \{0\}) \setminus \bar{\Omega}_{L\varepsilon}} G^2 dx + \frac{O(L\varepsilon \log^2(L\varepsilon))}{c^2} = -(II) + \frac{O(L\varepsilon \log^2(L\varepsilon))}{c^2}.$$

Together with (57)-(59) this gives

$$-\log \varepsilon - \log 2 - \gamma + O(L\varepsilon \log^2(L\varepsilon)) + O\left(\frac{\log L}{L}\right) = \pi c^2,$$

which, together with (56), implies

$$B = -\log 2 + O(L\varepsilon \log^2(L\varepsilon)) + O\left(\frac{\log L}{L}\right).$$



Now, observe that  $B_{L\varepsilon}(0, -\varepsilon) \cap (\mathbb{R} \times \{0\}) = (-\varepsilon\sqrt{L^2-1}, \varepsilon\sqrt{L^2-1})$  and that

$$\begin{aligned} \int_{-\varepsilon\sqrt{L^2-1}}^{\varepsilon\sqrt{L^2-1}} e^{\pi u_\varepsilon^2} dx &= \varepsilon \int_{-\sqrt{L^2-1}}^{\sqrt{L^2-1}} \exp\left(\pi\left(c - \frac{\log(1+x^2) + 2B}{2\pi c}\right)^2\right) dx \\ &> \varepsilon e^{\pi c^2 - 2B} \int_{-\sqrt{L^2-1}}^{\sqrt{L^2-1}} \frac{1}{1+x^2} dx \\ &= 2e^{-\gamma + O(L\varepsilon \log^2(L\varepsilon)) + O\left(\frac{\log L}{L}\right)} \pi \left(1 + O\left(\frac{1}{L}\right)\right) \\ &= 2\pi e^{-\gamma} + O(L\varepsilon \log^2(L\varepsilon)) + O\left(\frac{\log L}{L}\right). \end{aligned}$$

Moreover

$$\int_{(\mathbb{R} \times \{0\}) \setminus \bar{\Omega}_{L\varepsilon}} \left(e^{\pi u_\varepsilon^2} - 1\right) dx \geq \int_{(\mathbb{R} \times \{0\}) \setminus \bar{\Omega}_{L\varepsilon}} \pi u_\varepsilon^2 dx = \frac{1}{c^2} \int_{(\mathbb{R} \times \{0\}) \setminus \bar{\Omega}_{L\varepsilon}} \pi G^2 dx =: \frac{\nu_{L\varepsilon}}{c^2},$$

with

$$\nu_{L\varepsilon} > \nu_{\frac{1}{2}} > 0, \quad \text{for } L\varepsilon < \frac{1}{2}.$$

Now choose  $L = \log^2 \varepsilon$  to obtain

$$O(L\varepsilon \log^2(L\varepsilon)) + O\left(\frac{\log L}{L}\right) = O\left(\frac{\log \log \varepsilon}{\log^2 \varepsilon}\right) = o\left(\frac{1}{c^2}\right),$$

so that

$$E_\pi(u_\varepsilon) = \int_{\mathbb{R}} \left(e^{\pi u_\varepsilon^2} - 1\right) dx \geq 2\pi e^{-\gamma} + \frac{\nu_{\frac{1}{2}}}{c^2} + o\left(\frac{1}{c^2}\right) > 2\pi e^{-\gamma}$$

for  $\varepsilon$  small enough. □

*Proof of Theorem 1.2 (completed).* By Propositions 2.10 and 4.14, we know that  $\mu_k \leq C$ . Then, by dominated convergence theorem we have  $e^{\alpha_k u_k^2} \rightarrow e^{\pi u_\infty^2}$  in  $L^1_{\text{loc}}(\mathbb{R})$ . Then, by Lemma 4.2, we infer

$$E_{\alpha_k}(u_k) = E_\pi(u_\infty) + \pi(\|u_k\|_{L^2(\mathbb{R})}^2 - \|u_\infty\|_{L^2(\mathbb{R})}^2) + o(1). \tag{60}$$

This implies that  $u_\infty \neq 0$ , otherwise we would have  $E_{\alpha_k}(u_k) \leq \pi\|u_k\|_{L^2(\mathbb{R})}^2 + o(1) \leq \pi + o(1)$ , which contradicts the strict inequality  $D_\pi > 2\pi e^{-\gamma} > \pi$ , since  $E_{\alpha_k}(u_k) = D_{\alpha_k} \rightarrow D_\pi$  as  $k \rightarrow \infty$ .

Let us denote  $L := \limsup_{k \rightarrow \infty} \|u_k\|_2^2$ ,  $\tau = \frac{\|u_\infty\|_{L^2(\mathbb{R})}^2}{L}$  and observe that  $L, \tau \in (0, 1]$ . Let us consider the sequence  $v_k(x) = u_k(\tau x)$ . Clearly, we have  $v_k \rightharpoonup v_\infty$  in  $H$ , where  $v_\infty(x) := u_\infty(\tau x)$ . Since  $\|v_\infty\|_{L^2}^2 = L$ , and

$$\|(-\Delta)^{\frac{1}{4}} v_\infty\|_{L^2}^2 \leq \liminf_{k \rightarrow \infty} \|(-\Delta)^{\frac{1}{4}} v_k\|_{L^2}^2 = \liminf_{k \rightarrow \infty} \|(-\Delta)^{\frac{1}{4}} u_k\|_{L^2}^2 \leq 1 - L,$$

we get  $\|v_\infty\|_{H^{\frac{1}{2}, 2}} \leq 1$ . By (60) we have

$$D_\pi = E_\pi(u_\infty) + \pi L(1 - \tau) = \tau E_\pi(v_\infty) + \pi L(1 - \tau) \leq \tau D_\pi + \pi L(1 - \tau).$$

If  $\tau < 1$ , this implies  $D_\pi \leq \pi L$ , which is not possible. Hence, we must have  $\tau = 1$  and  $E_\pi(u_\infty) = D_\pi$ .  $\square$

## A Appendix: The half-Laplacian on $\mathbb{R}$

For  $u \in \mathcal{S}$  (the Schwarz space of rapidly decaying functions) we set

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi), \quad \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} dx. \quad (61)$$

One can prove that it holds (see e.g.)

$$(-\Delta)^s u(x) = K_s P.V. \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy := K_s \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy, \quad (62)$$

from which it follows that

$$\sup_{x \in \mathbb{R}} |(1 + x^{1+2s})(-\Delta)^s \varphi(x)| < \infty, \quad \text{for every } \varphi \in \mathcal{S}.$$

Then one can set

$$L_s(\mathbb{R}) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) : \|u\|_{L_s} := \int_{\mathbb{R}} \frac{|u(x)|}{1 + |x|^{1+2s}} dx < \infty \right\}, \quad (63)$$

and for every  $u \in L_s(\mathbb{R})$  one defines the tempered distribution  $(-\Delta)^s u$  as

$$\langle (-\Delta)^s u, \varphi \rangle := \int_{\mathbb{R}} u (-\Delta)^s \varphi dx = \int_{\mathbb{R}} u \mathcal{F}^{-1}(|\xi| \hat{\varphi}(\xi)) dx, \quad \text{for every } \varphi \in \mathcal{S}. \quad (64)$$

Moreover we will define for  $p \geq 1$  and  $s \in (0, 1)$

$$H^{s,p}(\mathbb{R}) := \{u \in L^p(\mathbb{R}) : (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R})\}. \quad (65)$$

In the case  $s = \frac{1}{2}$  we have  $K_{\frac{1}{2}} = \frac{1}{\pi}$  in (62) and a simple alternative definition of  $(-\Delta)^{\frac{1}{2}}$  can be given via the Poisson integral. For  $u \in L_{\frac{1}{2}}(\mathbb{R})$  define the Poisson integral

$$\tilde{u}(x, y) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{yu(\xi)}{(y^2 + (x - \xi)^2)} d\xi, \quad y > 0, \tag{66}$$

which is harmonic in  $\mathbb{R}_+^2 = \mathbb{R} \times (0, \infty)$  and satisfies the boundary condition  $\tilde{u}|_{\mathbb{R} \times \{0\}} = u$  in the following sense:

**Proposition A.1.** *If  $u \in L^{\frac{1}{2}}(\mathbb{R})$ , then  $\tilde{u}(\cdot, y) \in L^1_{\text{loc}}(\mathbb{R})$  for  $y \in (0, \infty)$  and  $\tilde{u}(\cdot, y) \rightarrow u$  in the sense of distributions as  $y \rightarrow 0^+$ . If  $u \in L^{\frac{1}{2}}(\mathbb{R}) \cap C((a, b))$  for some interval  $(a, b) \subseteq \mathbb{R}$ , then  $\tilde{u}$  extends continuously to  $(a, b) \times \{0\}$  and  $\tilde{u}(x, 0) = u(x)$  for any  $x \in (a, b)$ . If  $u \in H^{\frac{1}{2}}(\mathbb{R})$ , then  $\tilde{u} \in H^1(\mathbb{R}_+^2)$ , the identity  $\|\nabla \tilde{u}\|_{L^2(\mathbb{R}_+^2)} = \|(-\Delta)^{\frac{1}{4}}u\|_{L^2(\mathbb{R})}$  holds, and  $\tilde{u}|_{\mathbb{R} \times \{0\}} = u$  in the sense of traces.*

Then we have (see e.g [4])

$$(-\Delta)^{\frac{1}{2}}u = -\left. \frac{\partial \tilde{u}}{\partial y} \right|_{y=0}, \tag{67}$$

where the identity is pointwise if  $u$  is regular enough (for instance  $C^{1,\alpha}_{\text{loc}}(\mathbb{R})$ ), and has to be read in the sense of tempered distributions in general, with

$$\left\langle -\left. \frac{\partial \tilde{u}}{\partial y} \right|_{y=0}, \varphi \right\rangle := \left\langle u, -\left. \frac{\partial \tilde{\varphi}}{\partial y} \right|_{y=0} \right\rangle, \quad \varphi \in \mathcal{S}, \quad \tilde{\varphi} \text{ as in (66)}. \tag{68}$$

More precisely:

**Proposition A.2.** *If  $u \in L_{\frac{1}{2}}(\mathbb{R}) \cap C^{1,\alpha}_{\text{loc}}((a, b))$  for some interval  $(a, b) \subset \mathbb{R}$  and some  $\alpha \in (0, 1)$ , then the tempered distribution  $(-\Delta)^{\frac{1}{2}}u$  defined in (64) coincides on the interval  $(a, b)$  with the functions given by (62) and (67). For general  $u \in L_{\frac{1}{2}}(\mathbb{R})$  the definitions (64) and (67) are equivalent, where the right-hand side of (67) is defined by (68).*

It is known that the Poisson integral of a function  $u \in L^{\frac{1}{2}}(\mathbb{R})$  is the unique harmonic extension of  $u$  under some growth constraints at infinity. In fact, combining [36, Theorem 2.1 and Corollary 3.1] and [33, Theorem C] we get:

**Proposition A.3.** *For any  $u \in L_{\frac{1}{2}}(\mathbb{R})$ , the Poisson extension  $\tilde{u}$  satisfies  $\tilde{u}(x, y) = o(y^{-1}(x^2 + y^2))$  as  $|(x, y)| \rightarrow \infty$ . Moreover, if  $U$  is a harmonic function in  $\mathbb{R}_+^2$  which satisfies  $U(x, y) = o(y^{-1}(x^2 + y^2))$  as  $|(x, y)| \rightarrow \infty$  and  $U(\cdot, y) \rightarrow u$  as  $y \rightarrow 0^+$  in the sense of distributions, then  $U = \tilde{u}$  in  $\mathbb{R}_+^2$ .*

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