



ELSEVIER

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com



Mathematical Analysis/Calculus of Variations

On optimality of c -cyclically monotone transference plans*Sur l'optimalité des plans de transport c -cycliques monotones*Stefano Bianchini^a, Laura Caravenna^b^a SISSA, via Beirut 2, 34014 Trieste, Italy^b CRM De Giorgi, Collegio Puteano, Scuola Normale Superiore, Piazza dei Cavalieri 3, 56100 Pisa, Italy

ARTICLE INFO

Article history:

Received 4 February 2010

Accepted after revision 9 March 2010

Available online 13 May 2010

Presented by Michel Talagrand

ABSTRACT

This Note deals with the equivalence between the optimality of a transport plan for the Monge–Kantorovich problem and the condition of c -cyclical monotonicity, as an outcome of the construction in Bianchini and Caravenna (2009) [7]. We emphasize the measurability assumption on the hidden structure of linear preorder, applied also to extremality and uniqueness problems among the family of transport plans.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Dans la présente note nous décrivons brièvement la construction introduite dans Bianchini and Caravenna (2009) [7] à propos de l'équivalence entre l'optimalité d'un plan de transport pour le problème de Monge–Kantorovich et la condition de monotonie c -cyclique—ainsi que d'autres sujets que cela nous amène à aborder. Nous souhaitons mettre en évidence l'hypothèse de mesurabilité sur la structure sous-jacente de pré-ordre linéaire.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Version française abrégée

Soient μ, ν deux mesures de probabilité boréliennes sur $[0, 1]$ et $c : [0, 1]^2 \rightarrow [0, +\infty]$ une fonction coût. On note \mathcal{B} la σ -algèbre borélienne. Le problème de Monge–Kantorovich traite de la minimisation de la *fonctionnelle coût*

$$\mathcal{I}(\pi) := \int c(x, y)\pi(dx dy)$$

sur la famille des *plans de transport* sur $\pi \in \Pi(\mu, \nu)$, qui sont des mesures de probabilité boréliennes sur $[0, 1]^2$ de marginales μ, ν :

$$\Pi(\mu, \nu) := \{\pi \in \mathcal{P}([0, 1]^2) : \pi(A \times [0, 1]) = \mu(A), \pi([0, 1] \times A) = \nu(A) \text{ for } A \in \mathcal{B}\}.$$

On suppose, tacitement, que c is π -mesurable pour tout $\pi \in \Pi(\mu, \nu)$ et que $\mathcal{I}(\pi) < +\infty$ pour un $\pi \in \Pi(\mu, \nu)$.

Si un plan de plan de transport π minimise \mathcal{I} pour un coût co-analytique c , alors il doit être concentré sur un ensemble Γ *c -cycliquement monotone* (en bref *c -monotone*), c'est-à-dire que Γ satisfait la condition ponctuelle :

$$\forall M \in \mathbb{N}, (x_i, y_i) \in \Gamma \quad c(x_0, y_0) + \dots + c(x_M, y_M) \leq c(x_1, y_0) + \dots + c(x_M, y_{M-1}) + c(x_1, y_M).$$

E-mail addresses: bianchin@sissa.it (S. Bianchini), laura.caravenna@sns.it (L. Caravenna).

Toutefois cette condition est simplement nécessaire [2]. Partant d'une idée de [17], en supposant des conditions supplémentaires, une investigation sur l'équivalence de la c -monotonie et de l'optimalité est mise en évidence ; les résultats sont donnés dans [2–7,13–15].

Le problème fondamental résolu dans cette Note porte sur la discussion d'une condition de préordre aboutissant à une réponse positive qui recouvre les autres cas rencontrés dans la littérature. Les démonstrations détaillées sont données dans [7], où la construction ébauchée ici est appliquée au problème de l'unicité d'un plan concentré sur un ensemble donné et extrémal sur $\Pi(\mu, \nu)$.

Soient π et Γ comme ci-dessus. Le résultat fondamental est le suivant :

Définition 0.1. On définit le préordre suivant : $x \preceq x'$ s'il existe un *chemin axial* de coût fini les reliant, c'est-à-dire :

$$\exists(x_i, y_i) \in \Gamma, \quad i \leq I \in \mathbb{N}, \quad x_0 = x, \quad x_{I+1} = x' : (x_{i+1}, y_i) \in \{c < +\infty\} \quad \forall i = 0, \dots, I.$$

Il est μ -linéarisable s'il existe un préordre linéaire ρ tel que $\{x\rho x'\}$ est π' -mesurable pour tout $\pi' \in \Pi(\mu, \mu)$ et $x \preceq x'$ implique $x\rho x'$, mais si de plus $x'\rho x$ alors ou $x = x'$ ou bien $x' \preceq x$.

Théorème 0.2. Si le préordre $\{x \preceq x'\}$ est μ -linéarisable, alors $\bar{\pi}$ minimise \mathcal{I} .

Corollaire 0.3. S'il existe une famille dénombrable d'ensembles boréliens $A_i, B_i \subset [0, 1], i \in \mathbb{N}$ tel que

$$\pi\left(\bigcup_i A_i \times B_i\right) = 1, \quad \mu \otimes \nu\left(\bigcup_i (A_i \times B_i) \cap \{c = +\infty\}\right) = 0,$$

alors le préordre $\{x \preceq x'\}$ est μ -linéarisable et donc $\bar{\pi}$ minimise \mathcal{I} .

1. Introduction to the problem

Optimal mass transportation has been an exceptionally prolific field in the very last decades, both in theory and applications. What we reconsider is though a basic question in the foundations.

Let μ, ν be two Borel probability measures on $[0, 1]$ and $c : [0, 1]^2 \rightarrow [0, +\infty]$ a cost function. Denote by \mathcal{B} the Borel σ -algebra. The Monge–Kantorovich problem deals with the minimization of the *cost functional*

$$\mathcal{I}(\pi) := \int c(x, y) \pi(dx dy)$$

among the family of *transport plans* $\pi \in \Pi(\mu, \nu)$, which are Borel probability measures on $[0, 1]^2$ having marginals μ, ν :

$$\Pi(\mu, \nu) := \{\pi \in \mathcal{P}([0, 1]^2) : \pi(A \times [0, 1]) = \mu(A), \pi([0, 1] \times A) = \nu(A) \text{ for } A \in \mathcal{B}\}.$$

We tacitly assume that c is π -measurable for all $\pi \in \Pi(\mu, \nu)$ and that $\mathcal{I}(\pi) < +\infty$ for some $\pi \in \Pi(\mu, \nu)$.

When the cost c is l.s.c., then by [2] any optimal transport plan π must be concentrated on a c -cyclically monotone (briefly c -monotone) set Γ , meaning [16] that Γ satisfies the pointwise condition

$$\forall M \in \mathbb{N}, (x_i, y_i) \in \Gamma \quad c(x_0, y_0) + \dots + c(x_M, y_M) \leq c(x_1, y_0) + \dots + c(x_M, y_{M-1}) + c(x_1, y_M). \tag{1}$$

This expresses that one cannot lower the cost of π by cyclic perturbations of the transport plan (see also below). In [2] it is also provided the following counterexample, showing that the condition is not sufficient in general.

Example 1.1. Consider $\mu = \nu = \mathcal{L}^1_{\lfloor [0,1]}$ and define the following cost function (Fig. 1):

$$c(x, y) = \begin{cases} 1 & \text{if } y = x, \\ 2 & \text{if } y = x + \alpha \pmod 1, \quad \alpha \in [0, 1] \setminus \mathbb{Q}. \\ +\infty & \text{otherwise.} \end{cases}$$

Being α irrational, the plan $(x, x + \alpha \pmod 1)_{\#} \mathcal{L}^1$ is trivially c -cyclically monotone: the verification with $\Gamma := \{(x, x + \alpha \pmod 1)\}_{x \in [0,1]}$ leads to $2M < +\infty, M \in \mathbb{N}$. However it is not optimal, since the plan $(x, x)_{\#} \mathcal{L}^1$ has lower cost.

Since c -cyclical monotonicity is more handily verifiable, [17] rose the question of its equivalence with optimality for $c(x, y) = \|y - x\|^2$. Improvements of [2] were soon given, independently, in the case both of atomic marginals or continuous cost [13] and in the case of real valued, l.s.c. cost functions c [14], answering Villani's question. Since then other cases have been covered [15,3,6,4,5]. We briefly outline here the approach we pursued in [7].

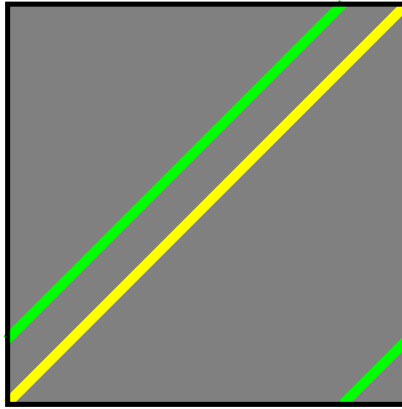


Fig. 1. Level sets of the cost c .
 Fig. 1. Ensemble de niveau de c .

2. Main statement

Let $\bar{\pi} \in \Pi(\mu, \nu)$ a transference plan, with finite cost $\mathcal{I}(\bar{\pi})$, concentrated on a c -monotone subset Γ of $\{c < +\infty\}$. The aim is to give a concrete construction in order to test whether $\bar{\pi}$ is optimal by exploiting the c -monotonicity of Γ .

The answer we propose relies on the following intrinsic preorder on $[0, 1]$. We recall that a *preorder* is a transitive relation $R \subset [0, 1]^2$: whenever $xRx', x'Rx''$ then xRx'' .¹ It reduces to a *partial order* when it is antisymmetric: if $xRx', x'Rx$ then $x = x'$. A relation is *linear* if every two elements are comparable: for $x, x' \in [0, 1]$ either xRx' or $x'Rx$. If R is a linear preorder, notice also that $R \cap R^{-1}$ is a natural equivalence relation associated to it.

Definition 2.1. Define $x \preceq x'$ if there exists an *axial path* with finite cost connecting them:

$$\exists(x_i, y_i) \in \Gamma, i \leq I \in \mathbb{N}, \quad x_0 = x, x_{I+1} = x': (x_{i+1}, y_i) \in \{c < +\infty\} \quad \forall i = 0, \dots, I.$$

Define $x \sim x'$ if there exists a *closed cycle* with finite cost connecting them:

$$\exists(x_i, y_i) \in \Gamma, i \leq I \in \mathbb{N}, \exists j \in \{0, \dots, I\}: \quad x_0 = x_{I+1} = x, x_j = x', (x_{i+1}, y_i) \in \{c < +\infty\} \quad \forall i = 0, \dots, I.$$

Lemma 2.2. The relation \preceq is a preorder. Moreover $x \sim x'$ iff $x \preceq x'$ and $x' \preceq x$.

As a corollary, the relation $\{(x, x'): x \sim x'\}$ is an equivalence relation on a subset of $[0, 1]$, easily extendable as $\{x \sim x'\} \cup \{x = x'\}$. The preorder \preceq induces a partial order on the quotient space $[0, 1]/\sim$, but its extension is more subtle. Let m be a Borel probability measure.

Definition 2.3. A preorder P on $[0, 1]$ is *Borel linearizable* if there exists a Borel linear order $B \supset P$ such that $B \cap B^{-1} = P \cap P^{-1} \cup \{x = y\}$. It is *m-linearizable* if instead B is π' -measurable for all $\pi' \in \Pi(m, m)$.

The following result does not rely on the Axiom of Choice. However, under it any preorder is a subset of a linear preorder B , which in general fails to be $m \otimes m$ -measurable [7, C.12]: μ -linearizability is then a measurability condition.

Theorem 2.4. If the preorder $\{x \preceq x'\}$ is μ -linearizable, then $\bar{\pi}$ minimizes \mathcal{I} .

An analogous statement holds with ν instead of μ if one applies Definition 2.1 after inverting the coordinates; let \approx and \approx be the corresponding partial order and equivalence relation.

Corollary 2.5. If there exists a countable family of Borel sets $A_i, B_i \subset [0, 1], i \in \mathbb{N}$, such that

$$\bar{\pi} \left(\bigcup_i A_i \times B_i \right) = 1, \quad \mu \otimes \nu \left(\bigcup_i (A_i \times B_i) \cap \{c = +\infty\} \right) = 0,$$

then the preorder $\{x \preceq x'\}$ is μ -linearizable and therefore $\bar{\pi}$ minimizes \mathcal{I} .

¹ We denote xRx' if $(x, x') \in R$.

The corollary covers the presently known cases where c -monotonicity implies optimality: there are countably many classes of \sim and consequently the linearization of the preorder, on a discrete set, is easily solvable by induction.

Notice moreover what happens in Example 1.1 choosing the diagonal as $\Gamma: \sim$ is the trivial equivalence relation $\{y = x\}$ and \preceq ends to the standard example of nonlinearizable Borel preorder—the Vitali one [9]; with $\Gamma = \{c < +\infty\}$ instead the quotient is the Vitali set and there is no disintegration of L^1 strongly consistent with \sim .

If the preorder \preceq is Borel, then either Theorem 2.4 holds or the preorder embeds a copy of the Vitali preorder [11].

3. Sketch of the proof

The result of Theorem 2.4 is based on a reduction argument. More precisely, we split the optimal transport problem within the classes of \sim and to a problem of uniqueness in the quotient space $[0, 1]/\sim$. This is formalized by means of the Disintegration Theorem, a very useful tool which decomposes a measure in a superposition of *conditional measures* concentrated on given subsets, thus ‘localizing’ it. We recall the Disintegration Theorem in the last section.

The solution of the reduced transport problems will follow from the next property of the equivalence classes.

Lemma 3.1. *Let μ', ν' be Borel probability measures on $[0, 1]$. If μ' is concentrated on a class of \sim , then each $\pi' \in \Pi(\mu', \nu')$ concentrated on Γ is a minimizer of the cost functional \mathcal{I} in $\Pi(\mu', \nu')$.*

Sketch. Optimality is obtained constructing by Rüschemdorf’s formula optimal Kantorovich potentials in each equivalence class C , for any transport plan concentrated on² $\Gamma \cap (C \times \Gamma(C))$. A different proof is also in [15]. \square

The reduction argument, performed just below, consists in disintegrating μ in probability measures $\{\mu_\alpha\}_{\alpha \in [0, 1]}$ on the equivalence classes of \sim , ν in probabilities $\{\nu_\alpha\}_{\alpha \in [0, 1]}$ on the classes of \approx and every transport plan $\pi \in \Pi(\mu, \nu)$ with finite cost in transport plans $\pi_\alpha \in \Pi(\mu_\alpha, \nu_\alpha)$. From $\tilde{\pi}(\Gamma) = 1$ one would obtain $\tilde{\pi}_\alpha \in \Pi(\mu_\alpha, \nu_\alpha)$ concentrated on Γ : the perturbations occur within the equivalence classes. Lemma 3.1 would ensure that $\mathcal{I}(\tilde{\pi}_\alpha) \leq \mathcal{I}(\pi_\alpha)$ for m -a.e. α . Therefore, in this case, by the disintegration formula and the optimality within the classes we would get

$$\begin{aligned} \mathcal{I}(\tilde{\pi}) &= \int c(x, y) \tilde{\pi}(dx dy) \stackrel{(3)}{=} \int \left\{ \int c(x, y) \tilde{\pi}_\alpha(dx dy) \right\} m(d\alpha) \\ &\stackrel{L.3.1}{\leq} \int \left\{ \int c(x, y) \pi_\alpha(dx dy) \right\} m(d\alpha) \stackrel{(3)}{=} \int c(x, y) \pi(dx dy) = \mathcal{I}(\pi). \end{aligned}$$

If \preceq is μ -linearizable, the next theorem provides a μ -measurable quotient map $q_1 : [0, 1] \rightarrow [0, 1]$ for \sim .

Theorem 3.2. *Let $\tilde{\pi} \in \Pi(\check{\mu}, \check{\nu})$ be concentrated on a linear preorder L which is π' -measurable for all $\pi' \in \Pi(\check{\mu}, \check{\nu})$. Let E be the equivalence relation $L \cap L^{-1}$. Then $\tilde{\pi}(E) = 1$ and the disintegration of $\check{\mu}$ is strongly consistent.*

Sketch. Let \trianglelefteq be the lexicographic ordering in $[0, 1]^\alpha$, with $\alpha \in \omega_1$ countable ordinal. We exhibit a Borel map $h_\alpha : [0, 1] \rightarrow [0, 1]^\alpha$ s.t., up to an $\check{\mu}$ -negligible set, $x \triangleleft x'$ if and only if $h_\alpha(x) \trianglelefteq h_\alpha(x')$. Since $([0, 1]^\alpha, \mathcal{B})$ can be measurably injected in $([0, 1], \mathcal{B})$, the existence of h_α ensures on one hand that the disintegration of $\check{\mu}$ w.r.t. E is strongly consistent (Theorem A.2). On the other hand, $\tilde{\pi}(L) = 1$ and $\tilde{\pi}(E) = 1$ become equivalent to $(h_\alpha \otimes h_\alpha)_\# \tilde{\pi}(\{\alpha = \beta\}) = 1$: indeed, one proves by transfinite induction on α that there is a unique transport plan from a measure to itself concentrated on $\{\alpha \trianglelefteq \beta\} \subset [0, 1]^\alpha \times [0, 1]^\alpha$, induced by the identity map.

The definition of h_α , by transfinite induction, is based on the Disintegration Theorem, introduced just below. The first component is $h_1(x) := \check{\mu}(\{x' : x' \preceq x\})$; then we disintegrate $\check{\mu}$ w.r.t. h_1 and define $h_2(x) = (h_1(x), \check{\mu}_{h_1(x)}(\{x' : x' \preceq x\})), \dots$. The sequence becomes constant in $|\alpha|$ steps. For L Borel, in [9] one finds a different Borel order preserving immersion in $([0, 1]^\alpha, \trianglelefteq)$ without reference measures. \square

Now it is worth noticing the *crosswise structure* of the relations: the nontrivial classes of \approx are of the form

$$\Gamma(A) = \{y : (x, y) \in \Gamma \text{ for some } x \in A\} \quad \text{with } A = \{x' : x' \sim x\}, \quad x \in [0, 1]. \quad (2)$$

As a consequence, one can define the quotient projection w.r.t. \approx by setting $q_2(\Gamma(x)) := q_1(x)$ and also ν has a disintegration strongly consistent with \approx , by $\tilde{\pi}$ ’s marginal conditions. By (2) then the quotient probability spaces $([0, 1], \mu)/\sim$ and $([0, 1], \nu)/\approx$ can be identified with a Borel probability space $([0, 1], m)$.

For any plan $\pi \in \Pi(\mu, \nu)$ its quotient measure n w.r.t. the product equivalence relation $q_1 \otimes q_2$ belongs consequently to $\Pi(m, m)$. Since π has finite cost, n is clearly concentrated on $q_1 \otimes q_2(\{c < +\infty\})$.

² We use the notation of multivalued function $\Gamma(A) := \{y : \exists x \in A : (x, y) \in \Gamma\}$, as well as $\Gamma^{-1} := \{(y, x) : (x, y) \in \Gamma\}$.

Lemma 3.3. *The set $q_1 \otimes q_2(\{c < +\infty\})$ is the partial order $q_1 \otimes q_1(\{x \preceq x'\})$ in the quotient space.*

The assumption of Theorem 2.4 grants that this partial order can be extended to a linear order which is π' -measurable for every $\pi' \in \Pi(m, m)$. Applying Theorem 3.2 one obtains then that $n = (\mathbb{I}, \mathbb{I})_{\#} m$ for every plan of finite cost π . As a consequence, any $\pi \in \Pi(\mu, \nu)$ admits the strongly consistent disintegration $\pi = \int \pi_{\alpha} m(d\alpha)$ w.r.t. the partition $\{(q_1 \otimes q_2)^{-1}(\alpha)\}_{\alpha \in [0, 1]}$. The crosswise structure yields then the following statement.

Lemma 3.4. *By the marginal conditions, $\pi_{\alpha} \in \Pi(\mu_{\alpha}, \nu_{\alpha})$ for m -a.e. α .*

4. Mention of side studies and remarks

Our basic tool has been the Disintegration Theorem. The main references for our review on that has been [1,8]. We also applied it to a family of equivalence relations closed under countable intersection, establishing that there is an element of the family which is the finest partition, in a measure theoretic sense. In particular, in the construction of the immersion h_{α} of Theorem 3.2 we ended up with such a family, which was not closed under uncountable intersection because uncountable intersections of sets generally are not measurable. Having a finest element, we could value the projection in the Polish space $[0, 1]^{\omega}$, $\alpha \in \omega_1$, instead of $[0, 1]^{\omega_1}$.

As briefly sketched, optimality holds in the equivalence classes basically by Kantorovich duality, the equivalence relation is indeed chosen for having real valued optimal Kantorovich potentials by Rüschemdorf's formula.

Moreover, the necessity of c -monotonicity with co-analytic cost functions—clearly assuming that the optimal cost is finite—is a corollary of the general duality in [12].³ It follows just by the fact that there is no cyclic perturbation λ of the optimal plan π such that $\mathcal{I}(\pi + \lambda) < \mathcal{I}(\pi)$, where cyclic perturbations of π are defined as nonzero measures λ with Jordan decomposition $\lambda = \lambda^+ - \lambda^-$ satisfying $\lambda^- \preceq \pi$ and which can be written, for some $m_I \in \mathcal{M}^+([0, 1]^{2I})$, $I \in \mathbb{N}$, as

$$\lambda^+ = \sum_I \frac{1}{I} \int_{[0, 1]^{2I}} \sum_{i=1}^I \delta_{(w_{2i-1}, w_{2i})} m(dw), \quad \lambda^- = \sum_I \frac{1}{I} \int_{[0, 1]^{2I}} \sum_{i=1}^I \delta_{(w_{2i+1 \bmod 2n}, w_{2i})} m(dw).$$

Observe that if \preceq is μ -linearizable, each transport plan of finite cost is concentrated on $(q_1 \otimes q_2)^{-1}(\{\alpha = \beta\})$; as a separate observation based on Von Neumann's Selection Theorem, one can construct optimal potentials for the cost which is $+\infty$ out of that set, gluing the ones in the classes.

As a final remark on the topic, we observe the following asymmetry: for universally measurable cost functions, c -cyclically monotone transference plans are optimal under the universally measurable linear preorder condition; however, in this case the necessity of c -cyclical monotonicity is not proven, since duality is provided in [12] for Souslin functions, corresponding to co-analytic costs.

In general, \preceq can be μ -linearizable for some c -cyclically monotone set Γ , $\pi(\Gamma) = 1$, and not for others, and we do not see how to choose a best one—which in the case of continuous cost would be the support. This problem is intrinsic in the 'pointwise' definition of c -monotonicity. Another question is what happens when there is no such set Γ such that \preceq is μ -linearizable. Examples show a crazy behavior. As already mentioned, for Borel sets [11] states that in this case there is a situation analogous to Example 1.1 and the quotient projection w.r.t. $\bigcup_{M \in \mathbb{N}} (\{x \preceq y\} \cup \{x \preceq y\}^{-1})^M$ is not universally measurable—but, however, since we have fixed measures optimality could still hold.

We conclude noticing that in [7] we study with the same approach the problems of establishing if a plan $\pi \in \Pi(\mu, \nu)$ is extremal and if it is the unique plan in $\Pi(\mu, \nu)$ concentrated on a given set A , say universally measurable. A necessary condition is respectively *acyclicity/A-acyclicity*: $\pi(\Gamma) = 1$ for a Γ , say universally measurable, such that the r.h.s. of (1) is always $+\infty$ when $c = \chi_{\Gamma} / c = \chi_A$. In the first case we precisely recover the Borel Countable Limb Condition in [10]. The second case comes from the problem of uniqueness in the quotient space described above. Considering the same axial preorder \preceq defined above, but with Γ/A in place of $\{c < +\infty\}$, the statement becomes

Theorem 4.1. *If Γ is acyclic/A-acyclic and \preceq is μ -linearizable, then π is extremal/the unique transport on A .*

Appendix A. The Disintegration Theorem

Consider an equivalence relation \approx on $[0, 1]$ and its quotient map $q : [0, 1] \rightarrow [0, 1]/\approx$. Given a Borel probability measure ξ on $[0, 1]$, let Θ_{ξ} be the σ -algebra of ξ -measurable sets. The *push forward* of a σ -algebra of $\mathcal{A} \subset \Theta_{\xi}$ and the push forward probability measure $\eta = q_{\#} \xi$ are defined on $[0, 1]/\approx$ as

$$S \in q_{\#} \mathcal{A} \iff q^{-1}(S) \in \mathcal{A}, \quad \eta(S) := \xi(q^{-1}(S)) \text{ for } S \in q_{\#} \Theta_{\xi}.$$

³ Necessity was proven in [2] by Kantorovich duality for l.s.c. cost functions, and extended to the Borel case in [3] with Kellerer results. We provide in [7] a different proof, still based on Keller duality, for co-analytic cost functions.

Definition A.1. The disintegration of a Borel probability measure ξ on $[0, 1]$ strongly consistent with a map $q : [0, 1] \rightarrow [0, 1]/\approx$ is a family of Borel probability measures $\{\xi_\alpha\}_{\alpha \in [0, 1]}$, the *conditional probabilities*, such that $\alpha \mapsto \int_S \xi_\alpha(O)$ is η -measurable for all $S \in q_\# \mathcal{B}$, where $\eta := q_\# \xi$, and

$$\xi(O \cap h^{-1}(S)) = \int_S \xi_\alpha(O) \eta(d\alpha) \quad \text{for all } O \in \mathcal{B}, S \in q_\# \mathcal{B}, \quad (3a)$$

$$\xi_\alpha(X_\alpha) = 1 \quad \text{for } \eta\text{-a.e. } \alpha \in [0, 1]. \quad (3b)$$

It is *unique* if for any other family $\{\xi'_\alpha\}_{\alpha \in [0, 1]}$ satisfying (3a) then $\xi_\alpha = \xi'_\alpha$ for η -a.e. α .

Theorem A.2 (Disintegration Theorem). *If there exists a measurable injection $([0, 1]/\approx, q_\# \Theta_\xi) \hookrightarrow ([0, 1], \mathcal{B})$, then there exists a unique disintegration of ξ strongly consistent with q .*

References

- [1] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press, Oxford, 2000.
- [2] L. Ambrosio, A. Pratelli, *Existence and Stability Results in the L^1 Theory of Optimal Transportation*, Lecture Notes in Mathematics, vol. 1813, Optimal Transportation and Applications, Springer, Berlin, 2001.
- [3] M. Beiglböck, M. Goldstern, G. Maresch, W. Schachermayer, Optimal and better transport plans, *J. Funct. Anal.* 256 (6) (2009) 1907–1927.
- [4] M. Beiglböck, C. Leonard, W. Schachermayer, A general duality theorem for the Monge–Kantorovich transport problem.
- [5] M. Beiglböck, C. Leonard, W. Schachermayer, On the duality theory for the Monge–Kantorovich transport problem.
- [6] M. Beiglböck, W. Schachermayer, Duality for Borel measurable cost function, preprint.
- [7] S. Bianchini, L. Caravenna, On the extremality, uniqueness and optimality of transference plans, *Bull. Inst. Math. Acad. Sin. (N.S.)* 4 (4) (2009) 353–455.
- [8] D.H. Fremlin, *Measure Theory*, vol. 1–4, Torres Fremlin, Colchester, 2001–2003.
- [9] L. Harrington, D. Marker, S. Shelah, Borel orderings, *Trans. Amer. Math. Soc.* 310 (1) (1988) 293–302.
- [10] K. Hestir, S.C. Williams, Supports of doubly stochastic measures, *Bernoulli* 1 (3) (1995) 217–243.
- [11] V. Kanovei, When a partial Borel order is linearizable, *Fund. Math.* 155 (3) (1998) 301–309.
- [12] H.G. Kellerer, Duality theorems for marginals problems, *Z. Wahrsch. Verw. Gebiete* 67 (4) (1984) 399–432.
- [13] A. Pratelli, On the sufficiency of c -cyclical monotonicity for optimality of transport plans, *Math. Z.* (2007).
- [14] W. Schachermayer, J. Teichmann, Solution of a problem in Villani's book, preprint, 2005.
- [15] W. Schachermayer, J. Teichmann, Characterization of optimal transport plans for the Monge–Kantorovich problem, *Proc. Amer. Math. Soc.* 137 (2) (2009) 519–529.
- [16] C. Smith, M. Knott, On Hoeffding–Fréchet bounds and cyclic monotone relations, *J. Multivariate Anal.* 40 (2) (1992) 328–334.
- [17] C. Villani, *Topics in Optimal Transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, AMS, Providence, RI, 2003.