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ANTI-DE SITTER BLACK HOLES IN SUPERGRAVITY, BPS ENTROPY AND SQUASHED BOUNDARIES

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"Mamma! Nella vita a volte è necessario saper lottare,
non solo senza paura, ma anche senza speranza."
(Sandro Pertini)

A Babbo e Mamma,
per aver sempre creduto in me

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In every thesis, this kind of section begins with academic acknowledgements and I'm not going to break this paradigm.

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DECLARATION

I, *Lorenzo Papini*, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.
I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university.

DETAILS OF COLLABORATION AND PUBLICATIONS:

This work of thesis is mainly based on the following published works, which have been started and completed during my Ph.D. :

- D. Cassani and L. Papini, "Squashing the boundary of supersymmetric AdS5 black holes", JHEP 1812, 037 (2018), arXiv:1809.02149
- A. Bombini and L. Papini, "General supersymmetric AdS5 black holes with squashed boundary", Eur. Phys. J. C79, 515 (2019), arXiv:1903.00021.
- D. Cassani and L. Papini, "The BPS limit of rotating AdS black hole thermodynamics", JHEP 1909, 079 (2019), arXiv:1906.10148.

During my M.Sc. in Physics in the University of Perugia, I have also started the following work, which has been published during my Ph.D. :

- G. Grignani, A. Marini, L. Papini and A.-C. Pigna, "AC conductivities of a holographic Dirac semimetal", JHEP 1812, 109 (2018), arXiv:1807.10717.

ABSTRACT

Among the four fundamental interactions present in nature, gravity is the only one that has not been quantized. A fundamental task in theoretical physics is thus to formulate a theory of quantum gravity. In this context, the understanding of black hole physics plays a central role, since a quantum gravity theory must be able to fully describe these objects, solving the issues that emerge in general relativity. In particular, a quantum gravity theory should account for their macroscopical entropy in terms of some microstates describing the black hole.

String theory is one of the leading candidates for a quantum gravity theory. There are some striking achievements this theory has reached so far and we are interested mainly in two of them. The first one is that it manages to reproduce the Bekenstein-Hawking entropy of large classes of asymptotically flat black holes by counting microstates in a Boltzmann way. The second one is the formulation of the AdS/CFT correspondence, which establishes a duality between gravitational theories in Anti-de Sitter and conformal quantum field theory without gravity. It is tempting to combine these two achievements to study Anti-de Sitter black hole entropy via AdS/CFT correspondence, using a conformal field theory. However, this has been proven to be hard to do and there are aspects of the puzzle which have not been solved yet.

The main focus of this thesis is on adding some key ingredients to solve the task of reproducing rotating AdS black hole entropy using AdS/CFT. The first two chapters contain a detailed introduction on string theory and AdS/CFT together to the presentation of the problem we study. We review all the main related known results in the literature, and an extensive overview of known rotating AdS black hole solutions in all the dimensions between four and seven is provided. The third and four chapters contain our main results. There, we shed light on the physical interpretation of the so called *entropy functions*, which have proven to be crucial to reproduce the entropy of rotating AdS black holes. We show that they coincide with the on-shell action obtained in a particular limit of black hole thermodynamics that we carefully define. We also construct two families of new asymptotically locally AdS_5 rotating black hole solutions. As opposed to standard AdS_5 solutions, they present a conformal boundary which comprises a squashed three-sphere. However their near-horizon geometry turns out to be the same as standard AdS_5 black holes; thus the paradigm of holographic uniformization, stating that the IR behavior of the solution is independent of the UV one, is confirmed. Nevertheless, there are new important aspects linked to the form the Bekentein-Hawking entropy of these new black holes can be written in, which we underline and discuss.

CONTENTS

1

INTRODUCTION AND MOTIVATIONS

This thesis focuses on the studying of the black holes in asymptotically Anti-de Sitter (AdS) spaces in various dimensions and on how to reproduce their entropy using one of the main tools string theory and supergravity grants at our disposal: the Anti-De Sitter/Conformal Field Theory (AdS/CFT) correspondence. There are some questions that might arise in the mind of the readers, especially for the ones which do not work with black holes everyday, just by reading the title of this thesis or the last statement above. Some of them might be:

- why should we study black holes in Anti-de Sitter spacetime, which is so different from the one we live in? Why should we work with extra dimensions?
- why should we study Anti-de Sitter black holes in string theory or supergravity if we have not reached yet a complete knowledge of asymptotically flat ones?
- how can string theory and supergravity tools help us to shed light on black hole information paradox formulated by Hawking in the context of general relativity?

All these questions, and many other similar ones, present an answer that indeed motivates the study of AdS black holes in string theory/-supergravity and the study of their entropy using AdS/CFT correspondence; therefore the analysis of such black holes is in fact very much relevant for many important topics in theoretical physics. In this first chapter of the thesis, we will try to make these answers explicit and to fully motivate the study of AdS black holes in various dimensions showing how they may shed light on many important aspects of theoretical physics. To do this, we shall start from the very beginning by the notion of a black hole in general relativity, going forward to explore how these fascinating objects posed various challenges to the physicists who aimed to find the fundamental laws governing them and finally discuss if and how these challenges have been overcome today. Due to the vast amount of research topics and the complexity of them, this excursus does not pretend to be comprehensive and exhaustive, but rather to focus on the aspects which can help us to explain the importance of AdS black holes in today's theoretical physics research.

1.1 BLACK HOLES IN GENERAL RELATIVITY AND THEIR THERMODYNAMICS

1.1.1 *General relativity*

Among all the ideas in 20th-century physics, General Relativity (GR) has been one of the two hugely successful and groundbreaking ones [1]. It has been developed by Einstein [1] as a four-dimensional classical theory of gravity which extends his special theory of relativity so to include also the notions of gravitational acceleration and gravitational fields. The main revolutionary idea of GR is that space and time are a dynamic part of physics and not just a background for it to happen. Therefore, matter and spacetime influence each other: the first one can change the curvature of the second one which in turn govern the motion of the matter. This concept is totally conveyed in the form of the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 8\,\pi\, G_N\, T_{\mu\nu}\,, \tag{1.1}$$

where the left-hand side involves the form of the spacetime, represented by the metric $g_{\mu\nu}$, the Ricci scalar and the Ricci tensor, while the right-hand side puts matter into account via the energy-momentum tensor $T_{\mu\nu}$. After many tests and checks in lots of experiments, nowadays GR can be regarded as one of the best established and least controversial theories in physics.

Finding a solution of the Einstein equations means finding a spacetime with some matter configuration which is compatible with general relativity. Among all the solutions of the Einstein equations there are some that exhibit singularities. When the singularity is shielded by an event horizon we have a black hole, which will be the focus of this thesis.

Two conclusive remarks are in order. The first one is that GR has proven to be a non-renormalizable field theory [2]; therefore it may be regarded as an incomplete theory which is the classical limit at low energies (i.e. large distances) of a more fundamental theory of quantum gravity. The second remark is that, although it is usually quite simple to write the full Einstein equations for a given matter configuration, we are still very far from having a full classification of all the possible solutions. This is because there are infinite possibilities for matter to couple with gravity. It is at this point that supersymmetry and supersymmetric theories play a crucial role: supersymmetric solutions are governed by first order differential equations[2], while Einstein ones are of second

1 The other one is obviously quantum mechanics which we will introduce later.

2 Even though this is true in most of the cases, it is not true in general. Indeed, in the case of a null vector field Killing spinor bilinear, typically not all of the components of the Einstein equations hold as a consequence of supersymmetry. When this happens, there is one component of the Einstein equations which must be imposed by hand.

order. This makes the task of finding new solutions and classifying them much more easy and feasible in a supersymmetric theory. String theory and supergravity are both supersymmetric theory, so this may already be a hint to understand why these theories are indeed important and useful.

1.1.2 *Black holes in general relativity*

Black holes are among the most interesting solutions of general relativity. In the past, much experimental evidence pointed towards the existence of black holes in the universe and the presence even in our galaxy; finally in 2019 the first image of a supermassive black hole, situated at the centre of the galaxy M87, has been obtained [3].

Black holes are regions of spacetime where gravity becomes so strong to stop even light from escaping. They are characterized by the presence of an event horizon hiding a space-time singularity; particles that pass the horizon can classically never come back.

The first, simplest and best known black hole solution is the Schwarzschild one [4]. It is a spherical symmetric solution of pure gravity without any matter, i.e. the energy-momentum tensor vanishes. In spherical coordinates, its metric has the following form:

$$\mathrm{d}s^2 = -\left(1 - \frac{2\,M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{2\,M}{r}\right)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \quad (1.2)$$

with M being a parameter which may be interpreted as the mass of the black hole. Asymptotically, in the limit $r \to \infty$, flat spacetime is recovered, while the point $r = 0$ is a singularity of the solution. However the singularity is shielded by the coordinate singularity at $r_h = 2\,M$, which is a spherical surface corresponding to the event horizon. This latter thus prevents the outer region of spacetime from casually interacting with the inner part, in contact with the singularity, because no particles can classically come out from the horizon. A naked singularity would instead cause the breakdown of the theory; this also makes manifest the fact that it must be $M \geq 0$, otherwise no event horizon would exist to shield the singularity. The metric (1.2) is symmetric under time translation and space rotations; the full Lorentz symmetry $SO(1,3)$ is restored asymptotically. Therefore this black hole is static, spherically symmetric and asymptotically flat. The only arbitrary parameter of the Schwarzschild black hole is the mass M. More general black holes can be found in general relativity, which depend on more than one parameter.

The simplest of these more general black holes is the Reissner-Nordström solution [5,6]. This is a charged black hole solution in general relativity plus a Maxwell field. Its metric and non-zero gauge field components can be written as:

$$\mathrm{d}s^2 = -U^2(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{U^2(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,, \tag{1.3}$$

$$A_t = \frac{2\,Q}{r}\,, \qquad A_\varphi = -2\,P\,\cos\theta\,, \tag{1.4}$$

where

$$U^2(r) = 1 - \frac{2\,M}{r} + \frac{Z^2}{r^2}\,, \qquad Z^2 = Q^2 + P^2 \tag{1.5}$$

with M being the mass of the black hole, Q being its electric charge and P the magnetic one. The black hole is again asymptotically flat since the spacetime becomes just Minkowski when $r \to \infty$ and the electric and magnetic fields vanish. As for the Schwarzschild black hole, the point $r = 0$ is a true singularity; there are one or two more singularities in the points where $U(r) = 0$, corresponding to event horizons. They are given by the equation:

$$r_\pm = M \pm \sqrt{M^2 - Z^2}\,, \tag{1.6}$$

so that we have two event horizons if the mass is bigger than the charges and only one event horizon if the charge balances the mass. In the latter case the black hole is said *extremal*. We will return to these special black holes later. The Reissner-Nordström solution depend on three parameters: the mass M and the electric and magnetic charges Q and P.

There is finally the Kerr-Newman solution, which is the most general asymptotically flat black hole solution of the Einstein-Maxwell theory. We will not present the explicit form of the solution since it is quite involved, but we must underline that this black hole has also a constant non vanishing angular momentum J in addition to a mass and electric and magnetic charges. This most general solution is therefore governed by four parameters: M, J, Q and P. When $J \to 0$ the Reissner-Nordström black hole is recovered.

1.1.2.1 *Black hole thermodynamics*

Classically there seems to be nothing that does not work with the notion of a black hole since the naked singularity at the center of spacetime is always shielded by one or more event horizons. However as soon as one starts taking into account thermodynamics of black holes and their semiclassical behaviour, things change.

The story about black hole thermodynamics starts with the papers of Bekenstein [7–9], where it has been shown the need of introducing an entropy for black holes proportional to the area of their event horizon, so as not to violate the second principle of thermodynamics. Later Bardeen, Carter and Hawking proved the four laws of black hole mechanics in [10]; these can be put in close analogy with the ones of thermodynamics. The four laws can be stated as follows:

0. The surface gravity κ of a stationary black hole is constant over the event horizon. This is analogous to the zeroth law of thermodynamics, which states that the temperature is constant throughout a body in thermal equilibrium. It suggests that the surface gravity is analogous to temperature;

1. For perturbations of stationary black holes, the change of energy is related to change of area, angular momentum, and electric charge by:

$$\mathrm{d}E = \frac{\kappa}{8\pi}\,\mathrm{d}A + \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q\,, \tag{1.7}$$

   where E is the energy, A is the horizon area, Ω is the angular velocity, J is the angular momentum, Φ is the electrostatic potential and Q is the electric charge. It is quite immediate to put in analogy the equation above with the first law of thermodynamics [3];

2. If the energy-momentum tensor satisfies the weak energy condition[4] and assuming the cosmic censorship hypothesis[5] to be true, then in any physical process, the area A of the event horizon does not decrease, i.e.

$$\delta A \geq 0\,. \tag{1.8}$$

   This is the famous *area theorem* formulated by Hawking [13]. Analogously, the second law of thermodynamics states that the change in entropy in an isolated system will be greater than or equal to zero for a spontaneous process, suggesting a link between entropy and the area of a black hole horizon;

3. It is not possible to reach a vanishing surface gravity, i.e. $\kappa = 0$, via a physical process. This is analogous to the third law of thermodynamics, which states that a thermal system cannot reach zero temperature in a finite number of physical processes.

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3 Although the first law is usually stated in terms of perturbations of stationary black holes, there are also other formulations showing that it is even more fundamental than it is generally taken to be. An example of these alternative formulations is in the context of the isolated formalism, discussed in [11]. There, the authors review and redefine the notion of isolated horizons and show that requiring the time evolution to be Hamiltonian implies that the first law must hold and viceversa. This approach has been generalized to rotating black holes in [12].

4 The weak-energy condition states that for every timelike vector field, the matter density which is observed by the corresponding observer is always non-negative.

5 The cosmic censorship hypothesis states that no naked singularities (i.e. singularities not shielded by a horizon) exist in the universe.

1.1.3 *The problem of microstates counting and the information paradox*

It should be now clear the analogy between black hole mechanics and the laws of thermodynamics. But the story is very far from its ending. It has been established that black holes have a temperature and an entropy, which are given by

$$T = \frac{\kappa}{2\pi}\,, \qquad S = \frac{A}{4\,G_N}\,, \tag{1.9}$$

for example, by reinstating all the constants which have been suppressed in natural units, for a Schwarzschild black hole in four dimensions they are

$$T = \frac{\hbar\, c^3}{8\,\pi\, G_N\, k_B\, M}\,, \qquad S = \frac{c^3\, A}{4\,\hbar\, G_N}\,.$$

The temperature of a black hole is known as *the Hawking temperature*, while the entropy of a black hole is usually called *the Bekenstein-Hawking entropy.*

The precise physical meaning of these quantities was however still unclear and there were some problems, that in some fashion are still open nowadays, which demanded for an answer. The first one is about how to compute the entropy of the black hole by counting its microstates via the Boltzmaan law. Indeed, from classical thermodynamics we know that the entropy of a system is a macroscopic quantity which has a microscopic interpretation in terms of the microscopic configurations allowed for a given macroscopic state. Due to all the analogies we traced between black hole mechanics and thermodynamic quantities it is natural to seek for a microscopical interpretation of black hole entropy in this direction. Unfortunately, as far as we know, there is no way to describe and enumerating all the possible black hole microstates in GR; this is made sharper due to the existence of uniqueness theorems, also called no-hair theorems [14,15], showing that a classical black hole can only be described by three charges: the mass, the angular momentum and the electric or magnetic charge. However, the above mentioned no-hair theorems do not hold for AdS black holes[6], which are the main focus of this thesis; therefore for such solutions the situation is even more intricate. The task of counting and characterizing the black hole microstates, providing thus an interpretation for the entropy, seemed therefore to be possible only in a theory of "quantum gravity" which extends GR.

However the question of microstates counting was not the only one to puzzle theoretical physicists in understanding black hole physics. In the very famous paper [16] Hawking used a semiclassical approach to

6 This will be evident in Chapter 4, where we will explicitly construct hairy black hole solutions in AdS_5

study a quantum field on a classical curved background described by a black-hole geometry, proving that the first forces the black hole to emit a thermal radiation, the temperature of which coincides with the Hawking temperature. This discover basically shows that black holes can be regarded as black-body emitters of particles and has some very important consequences. Consider a black hole which is formed and then starts evaporating away, leaving at the end thermal radiation only. When the black hole is formed, the collapsing matter is in a definite, pure quantum state. We can associate to this state a density matrix which will be the one of a pure state. However a black hole presents an horizon, therefore the total set of states can be divided in states referred to the inside of the horizon and states referred to the outside. Outside observers can only have access to the outside states, so the description they can provide will be necessarily incomplete. From their point of view, the only state they can describe, i.e. the outside state, is mixed, in agreement with the fact that it contains thermal radiation. Accordingly, this mixed state will be represented by a reduced density matrix. At this point, there are no problems yet, since the interior state is not lost, but is entangled with the external one. However when the black hole has completely evaporated nothing is left in the interior and the system will continue to be described by a reduced density matrix which is referred to a mixed state. Therefore, to sum up, the black hole has transformed the pure state of the matter which constitutes it into a thermal state of pure radiation, once it is completely evaporated. This feature is in contrast with quantum mechanics, the other groundbreaking paradigm shift of 20th century physics explaining how things at the very small scales behave. Indeed, time evolution in quantum mechanics is described by unitary operators, but the black hole matter/radiation transition we have depicted in the paragraph above is a transition from a pure state to a mixed state and cannot be described by a unitary operator. Therefore it violates one of the principles of quantum mechanics. This is known as *black hole information paradox* [17, 18]. Once again, it seems it is not possible to solve the paradox inside GR; this problem demands for a "quantum gravity" framework in order to be solved.

To sum up, we have found two important related questions which seem impossible to solve within GR and ask for a "quantum gravity" theory:

- how to count microstates for a given black hole in order to reproduce its entropy via a Boltzmaan counting?
- how do we solve the black hole information paradox?

One of the biggest achievements of string theory, the leading candidate for a theory of quantum gravity, was the computation of the black hole entropy by microstate counting for some particular black holes [19]. We shall gradually proceed to examine this fundamental result by first exploring the current state of the art in the search for a quantum gravity theory and then introducing string theory and supergravity. We will then converge on one of the main goals of this thesis, which is to present recent developments on microstate counting for AdS black holes. However it is useful to provide before a brief illustration of the information paradox, in order to have a more rigorous understanding of it. The task of solving the paradox has been at the core of many developments in theoretical physics, so it is worth presenting it in some details.

1.1.3.1 *The Page formulation of the information paradox*

In the famous paper [20], Page provided an illuminating formulation of the black hole information paradox highlighting the contrast between a statistical-mechanical description of black holes and the thermal nature of Hawking radiation. We now briefly report this formulation here below since it is quite useful in better understanding the nature of the paradox.

Consider the Hilbert space $\mathcal{H}$ of a standard thermodynamic system which we would like to describe by using a microcanonical ensemble. In order to do this, we fix an energy E and denote as ΔE a small variation of it which is large enough to assure that in the interval $(E, E + \Delta E)$ there are a large number of microstates. The energy eigenstates whose eigenvalues are in the interval above span the Hilbert space $\mathcal{H}\,(E)$. At the beginning, the system under consideration is at microcanonical equilibrium with energy E_0; then it will cool down emitting thermal radiation. In its initial state, contained in $\mathcal{H}\,(E_0)$, the system is pure. In the following since the evolution is unitary, as prescribed by quantum mechanics, the total state system+radiation is again pure; however each emitted quanta forming the thermal radiation is in a mixed state, so to keep consistency each quanta has to be entangled with some other system. In other words, to allow the total state to be pure, each emitted quanta has to be entangled with the system. Let us explore what all this means in more concrete terms. Suppose the system cools down from E_0 to $E < E_0$ and call S_{RAD} the Von Neumann entropy of the radiation; then the Von Neumann entropy of the system $S\,(E_0|E)$ must be equal to S_{RAD}. At a given time t, the system has an energy which is in the interval $(E(t), E(t) + \Delta E)$ and the corresponding state is comprised into $\mathcal{H}\,(E(t))$; the system must have a Von Neumann entropy which is less than the microcanonical entropy $S_{\mathrm{MICRO}} = \log \dim \mathcal{H}\,(E\,(t))$. Therefore:

$$S\,(E_0|E) \leq S_{\mathrm{MICRO}}\,(E\,(t)) = \log \dim \mathcal{H}\,(E\,(t))\ . \tag{1.10}$$

Eventually, there will be a time t when the equal sign holds; this is due to the fact that the microcanonical entropy decreases with the energy as the system cools down. The time when the equal sign holds is called *Page time*. After that, the radiation cannot be exactly thermal anymore, therefore it should be entangled with the early time radiation. The time dependence of the entropy described now is represented in fig.1.1 and the corresponding curve is usually called *Page curve*.

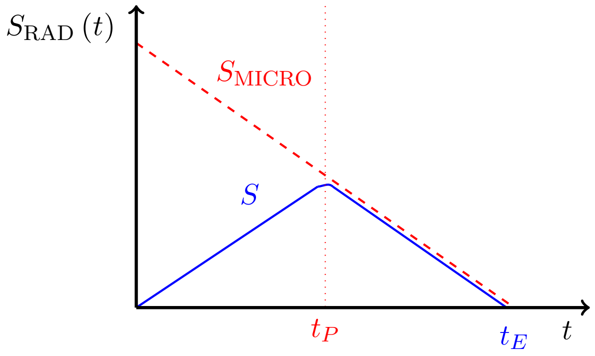

Figure 1.1: Entropy of radiation as function of time; the time where the microcanonical entropy equates the Von Neumann one is the Page time t_P, while t_E is the evaporation time. This curve is usually called Page curve.

For a Schwarzschild black hole described by (1.2), the Page time is around the half of the total evaporation time, at this point the half of the black hole mass has been radiated away. For the arguments we have introduced above, after the Page time the radiation should be entangled with the early time radiation, since it cannot be thermal anymore. Here it lays exactly the core of the black hole information paradox in this formulation: the Hawking radiation is exactly thermal according to quantum field theory computations so that there seem not to be any entanglement between early time radiation and late time one. It clearly emerges a serious contrast and disagreement between the description provided by QFT and the one developed using black hole statistical mechanics. What is more, the discrepancy between the two descriptions starts way long before complete evaporation, so Plack-size scale effect may not be advocated to solve the paradox since the horizon scale is still macroscopic at that time.

1.2 STRING THEORY AND SUPERGRAVITY

1.2.1 *Quantum gravity*

As we have already mentioned, GR cannot be a complete theory, since it fails at small scales. At these scales, phenomena start to manifest a "quantum behavior" and can therefore be described by the quantum theory.

Quantum theory was founded by Planck's work on black body radiation, giving birth to quantum mechanics; it was then corroborated by the work of Einstein on the photo-electric effect [21], by the atomic model of Bohr [22] and many others. All these revolutionaries and groundbreaking studies established the quantum theory of particle physics. In the quantum world deterministic concepts like trajectory or equations of kinematics exactly determining position and velocity no longer exist; they are replaced by the notions of uncertainty and probabilistic interpretation introduced by Heisenberg and Schroedinger among others. These new concepts are surely non-intuitive and groundbreaking and their introduction have been enormously important for humanity, both for a philosophical and technical point of view. Quantum mechanics subsequently evolved into the formulation of quantum field theory (QFT), which is the framework describing all particle interactions.

The desired theory of quantum gravity should be able to reconcile and put together quantum field theory and general relativity, i.e. it should be a "theory of everything" valid at all the scales where gravity can be quantized. It is generally desired and believed that such a theory can be formulated; in principle any consistent quantum theory which reduces to GR in its classical limit would be a candidate for a quantum gravity theory.

The task of formulating such a theory proved to be very hard. Even though there are notable candidates, such as loop quantum gravity and string theory, which present the desired features to be quantum gravity theories, there is no experimental evidence strongly pointing towards one of them.

Even though this is the present state of the art, it is very important to remind that we can investigate some relevant aspects of quantum gravity without using a complete theory of everything. Indeed, there are some quantum features which survive at low energy and can therefore be studied in a low energy limit theory of a quantum gravity theory. Black holes are particularly interesting in this sense, because they possess a non-vanishing entropy even in the classical regime. As we have already mentioned above, GR is unable to explain the microscopic origin of the entropy in terms of fundamental degrees of freedom, i.e. by counting black hole microstates, but this must be achieved by a theory of quantum gravity. Most of the entropy aspects of black holes can be investigated using a low energy limit theory; this is exactly what we will do in this thesis where we use supergravity, the low energy limit of string theory, to study black holes and their entropy.

1.2.2 *String Theory*

String theory is at present one of the most accredited candidates for a quantum gravity theory. The basic idea behind string theory is to consider one-dimensional extended strings as fundamental objects rather than point particles. Strings present an infinite number of possible excitations; each of them gives rise to either particles we observe in nature or to one of the fundamental interactions. Gravitational interaction in this context is just one of these possibilities. String theory was first studied in the late 1960's as a theory of the strong nuclear force before being abandoned in favor of the QCD theory due to the fact that it is not possible to describe hadronic physics with it because of the presence of an excitation with zero mass and spin two (today we know that this excitation is indeed the graviton). The first version of string theory, called bosonic string theory, described only bosonic particles and it was shown to be consistent only in a 26-dimensional spacetime. Apart from the very high number of dimensions, which may easily appear unphysical, it was soon realised that bosonic string theory had a quite relevant problem: there is a tachyon state in its spectrum, i.e. a state associated to a particle with $m^2 < 0$.

Both these problems can be overcome with the introduction of supersymmetry (see below). It is possible to build up five different types of supersymmetric string theories which manage to describe both fermions and bosons. All these different kind of string theories are related via a complicated web of dualities; in the 1990s it was conjectured that they are also all different limiting cases of a single theory in eleven dimensions known as M-theory.

In the low-energy limit, string theory can still be described by effective particle theories, like supergravities (see below); it is desirable that eventually the low-energy limit of supergravity will lead to the quantum field theory best describing all physical particle phenomena except gravity, the Standard Model of particle physics. One might be confused by the fact that supersymmetric string theories are consistent in a 10-dimensional spacetime, since it may appear not clear how to recover low-energy theories in dimension 4, which is the one of the universe we live in. In order to do this, one splits the original 10 dimensions into 6 small and compact directions, which we cannot perceive, and the 4 remaining ones, which are the ones we can experience. This topic is usually called *compactification*.

1.2.2.1 *Supersymmetry and supergravity*

In a quantum theory fundamental particles are either bosons or fermions. They are not described by the classical Boltzmann statistics, but their quantum nature demands for a different statistics accordingly to which of the above mentioned classes they belong to. The main difference between these two kind of particles is that only a single fermion can occupy a particular quantum state at a given time, while no such restriction applies for bosons. Accordingly, fermions are described by the so called Fermi-Dirac statistics, while bosons follow the Bose-Einstein one. Ordinary matter is made entirely of fermions; electrons, protons and neutrons are all fermions and so are the quarks constituting them. On the other hand, all the fundamental interactions are mediated by bosons; photon, the mediator of electromagnetic interaction, is a boson and so are the gluons which are responsible of the nuclear strong interaction.

Supersymmetry is a new type of symmetry between fields and spacetime which is able to relate bosons and fermions stating that each ordinary boson particle should have a fermion supersymmetric dual and vice versa. In some theories, supersymmetry can relate more than two particles; in this case it is said that the amount of supersymmetry is higher. This latter is "measured" by the amount of the supercharges $\mathcal{N}$ the theory presents, the higher is $\mathcal{N}$ the higher the theory is supersymmetric. The Standard Model is a non supersymmetric theory, i.e. it has $\mathcal{N} = 0$; this is because this symmetry does not seem to be manifest in the regime of validity of the Standard Model. However the supersymmetry breaking process makes it possible for a $\mathcal{N} > 0$ theory to reduce and be consistent with a theory, like the Standard Model, where supersymmetry is absent. This enters directly in the flux compactification topic in the context of string theory: there, one typically tries to build low-energy effective theories with the lowest amount of supersymmetry possible, in order to be consistent with the real world.

It should be clear how it can be tempting to try to combine supersymmetry and gravity. Supergravity theories are indeed the results of this attempt. In order to introduce gravity we need to describe the graviton, which is the boson carrying the gravitational interaction. Due to the presence of supersymmetry, there are supersymmetric fermionic partners to the graviton which are called gravitini; there will be exactly $\mathcal{N}$ gravitini in any $\mathcal{N}$ supergravity theory. There are no restrictions on the matter content that can be included into a supergravity theory; the only request is of course that it is consistent with supersymmetry. It is then clear how there are lots of possibilities for $\mathcal{N}$ matter-coupled supergravities to exist and this indeed produced a large number of different theories in different dimensions with different amounts of supersymmetry.

The first supergravity theory was formulated in the 1970s and discovered independently on string theory [23], other supergravity theories were rapidly found in the next years. They were regarded as candidates for theories of quantum gravity, but it was soon realized that they are non-renormalizable so that they do not behave well at high energies; however they arise as the low energy limit of string theory, which therefore provides their ultraviolet completion. As it is for string theories, also supergravity theories are related via a web of dualities and also by compactifications and reductions when it is needed to change the dimensionality of one theory to land to another one.

In a supersymmetric theory there exist supersymmetric states which are also called BPS states. They are background solutions that remain invariant under some of the supersymmetry transformations of the theory they belong to. All the BPS states saturate certain BPS bounds and they are the building blocks of the vacuum structure of any supergravity theory. BPS states have a supergroup of symmetries which is described by a superalgebra; they provide together with the latter a bridge between classical and quantum gravity since supersymmetry protects low energy solutions from any high energy corrections that can modify them. Therefore a BPS state in an effective theory will continue to exist retaining its properties also in the full quantum theory.

In this thesis, we will construct and work with black hole solutions in (mainly five-dimensional) supergravity theories. The majority of these black holes will be supersymmetric, i.e. their charges will satisfy a particular BPS relation and their entropy should be reproduced by counting BPS microstates. Right now, recalling also what we have stated in the above paragraph, it should be clear how the study of these solutions, although constructed in a low energy limit theory, can help to shed light on quantum entropy aspects of black holes.

1.3 THE STROMINGER-VAFA BLACK HOLE MICROSTATE COUNTING

The first black hole microstate counting has been performed by Strominger and Vafa in the seminal paper [19]. There, they constructed an extremal black hole which is a solution of a low energy limit of type II string theory and then managed to reproduce its Bekenstein-Hawking entropy by counting its stringy microstates; they also predicted the quantum corrections to the classical entropy.

In this section we aim at examining the Strominger-Vafa microstate counting in order to understand how and why it reproduces the Bekenstein-Hawking entropy of the black hole into account. This computation, which is one of the major achievements of string theory, lays on many fundamental results which characterized the second superstring revolution around 1995; among them it is mandatory to mention the seminal work of Polchinski on the Dp-branes [24] since in order to count microstates it was crucial to engineering a well-defined brane scenario. Due to the nature of the computation, in this section things start to be more technical and require a much more advanced mathematical and theoretical physics background. Nevertheless, we will try to underline the main concepts which should emerge and be clear also to the non expert readers.

1.3.1 *Strings, dualities and branes*

As we have mentioned in the last section, the main idea on which string theory is based on is to consider one-dimensional tiny strings as fundamental objects rather then point particles. By quantizing the fluctuations of these strings one reproduces the different particles of particle physics from their fluctuation modes. Among these fluctuations modes in the spectrum, there is one of them which corresponds to a spin-2 particle: this can be interpreted as the graviton so string theory immediately becomes a candidate theory of quantum gravity.

In order to obtain consistent theories we need to put supersymmetry into the game. However there are different possibilities to implement it: these correspond to five types of consistent superstring theories. They are type I, type IIA, type IIB and the two heterotic string theories with gauge group $SO(32)$ and $E_8 \times E_8$. Type I string theory has one spacetime supersymmetry, since it is a theory of open superstrings only. Type IIA and IIB are instead theories of closed superstrings and, as the label II suggests, they posses two independent supersymmetries. The label A and B refer instead to whether the two ten-dimensional supercharges have the opposite or the same chirality respectively. To prevent the theories to be inconsistent for the presence of a conformal anomaly, they are usually formulated in a 10-dimensional spacetime[7]. All these different theories are characterized by fundamental strings with different features and, as a consequences, the particles arising in the low-energy spectrum present also different features and symmetries.

There is one fundamental dimensionful parameter in string theories which is the string length ℓ_s; when we consider energy scales small with respect to ℓ_s^{-1} all the massive modes might be neglected since their mass is proportional to ℓ_s^{-3}. The low-energy limit of string theory is therefore characterized by only massless modes and it coincides with ten-dimensional supergravity. This is not a unique theory, but there are many variants of it which are related to the different types of superstring theories.

The black hole solutions we will consider in this thesis and we are interested in are solutions of either type IIA or type IIB superstring theories. These are the two theories we are mainly interested in. As we have already stated above, both the theories contain two supercharges, in the first one they have opposite chiralities, in the second one the chiralities are the same. The low-energy bosonic spectrum of the two theories is composed by:

- the graviton $g_{\mu\nu}$;
- the 2-form Kalb-Ramond gauge field $B_{\mu\nu}$;
- the dilaton ϕ;
- p-form gauge fields C_p, with p odd in type IIA and p even in type IIB (therefore type IIA contains as independent forms C_1, C_3 and C_5 while type IIB presents C_0, C_2 and C_4).

The spectrum is completed with the fermionic sector which contains all the $\mathcal{N} = 2$ superpartners.

From the analysis of the spectrum above it is evident that antisymmetric fields are present in both type IIA and type IIB theory.

7 There are actually other possibilities to get rid of the conformal anomaly without necessarily considering a 10-dimensional spacetime; the theories so obtained are called *non-critical string theories*. We will not examine neither discuss further these theories since we will not need them for this thesis.

There should be objects that carry charge under them since they are basically a generalization of electromagnetic gauge fields. Now there is a problem: if string theory contains strings only, such objects could not exist because strings can couple only with a two-rank field, that is the Kalb-Ramond field $B_{\mu\nu}$. The conclusion is that in string theory there must be also other objects that can couple to the C_p fields.: these objects prove to be the so called Dp-branes. Basically a Dp-brane is an hypersurface extended in p directions with a $p+1$-dimensional worldvolume. Dp-branes are so called since open strings can end on them: therefore they provide Dirichlet boundary conditions for the open strings. An U(1) gauge theory is obtained when a single massless open string moves on a single Dp-brane; when we have N coincident branes this gauge symmetry is enhanced to U(N).

A key realization in the development of string theory framework was the fact that all the different superstring theories are related by a complicated web of dualities; we report a pictorial representation of that in fig. 1.2. All the theories can be obtained as the limiting case of an even more fundamental theory: M-theory. In particular type IIA and type IIB theories are linked by the so called *T-duality*.

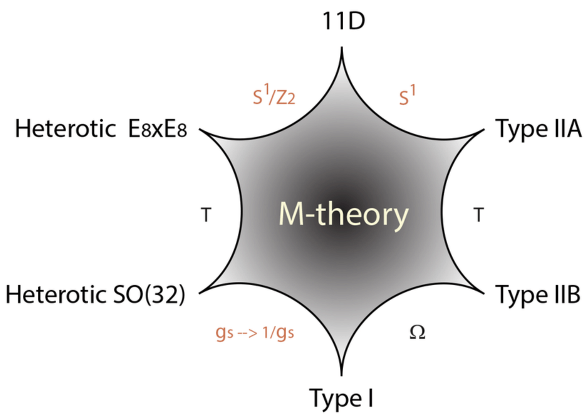

Figure 1.2: Pictorial representation of the web of dualities connecting every ten-dimensional superstring theory and eleven-dimensional supergravity. Picture taken from [25].

1.3.2 *The Strominger-Vafa black hole*

In [19], Strominger and Vafa consider a family of $\frac{1}{4}$-BPS black hole solutions in type IIA compactified on $S^1 \times K3$. We will now briefly introduce this family of black hole solutions and then proceed to review the original argument to reproduce the Bekenstein-Hawking entropy by microstate counting.

The bosonic part of the type IIA action compactified on $S^1 \times K3$ is

$$S_{IIA} = \frac{1}{16\,\pi} \int \mathrm{d}^5 x \sqrt{-g_5} \left[e^{-2\phi} \left(R_5 + 4\,(\nabla\phi)^2 - \frac{1}{4}\tilde{H}_2^2 \right) - \frac{1}{4}F_2^2 \right],$$

$$(1.11)$$

with ϕ being the dilation, F_2 being a Ramond-Ramond 2-form field strength and $\tilde{H}_2$ being a 2-form axion field strength arising from the NS-NS 3-form with one component tangent to the S^1. In this framework, black holes can be charged both with respect F_2 or $\tilde{H}_2$. Their charges are given by:

$$Q_H = \frac{1}{4}\pi^2 \int_{S^3} \star\, e^{-\frac{4}{3}\phi}\, \tilde{H}_2\,, \tag{1.12}$$

$$Q_F = \frac{1}{16}\pi \int_{S^3} \star\, e^{\frac{2}{3}\phi}\, F_2\,. \tag{1.13}$$

We want to construct an extremal black hole with a non-vanishing area; this is possible if its near-horizon geometry is $\mathrm{AdS}_2 \times S^3$ charged Robinson-Berttoti universe with constant dilaton $\phi = \phi_h$. Indeed, it can be shown that taking the dilation to be constant the metric ansatz

$$\begin{aligned} \mathrm{d}s_5^2 &= -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega_3^2\,, \\ f(r) &= 1 - \frac{r_0^2}{r^2}\,, \quad r_0 \equiv \left(\frac{8\,Q_H\,Q_F^2}{\pi^2}\right)^{1/6}\,, \end{aligned} \tag{1.14}$$

does solve the equations of motion coming from the action (1.11). This black hole solution presents the desired $\mathrm{AdS}_2 \times S^3$ near-horizon geometry; indeed, it is easy to show, by performing the near-horizon limit to the metric (1.14), that the near-horizon metric can be written as[8]

$$\mathrm{d}s_5^2 = \frac{r_0^2}{4}\left(-\bar{r}^2\,\mathrm{d}t^2 + \frac{\mathrm{d}\bar{r}^2}{\bar{r}^2}\right) + r_0^2\,\mathrm{d}\Omega_3^2\,, \qquad \text{with } \bar{r} = \frac{4}{r_0^2}\,r\ . \tag{1.15}$$

It is now quite trivial to compute the entropy via the Bekenstein-Hawking formula as

$$S_{\mathrm{BH}} = 2\,\pi\sqrt{\frac{Q_H\,Q_F^2}{2}}\,. \tag{1.16}$$

This is the entropy we aim to reproduce via microstate counting.

1.3.3 *The microstate counting*

From a brane point of view, the original Strominger-Vafa black hole family can be regarded as a type IIA system of the D0D4 kind compactified on $S^1 \times K3$. This system proves to be dual to a D1D5 system

8 In order to establish regularity at the horizon, another possible approach is to find the appropriate Gaussian null coordinates for the metric (1.14). This would be a preferable approach with respect to the one we followed here; however we will not present it in order to not burden the dissertation with technical details. Nevertheless, we will follow and present the Gaussian null coordinates approach for the AlAdS_5 black holes we will construct in Chapter 4.

compactified on the same space; indeed under T-duality we have that D1-D3-D5-P in type IIB maps into D0-D2-D4-F1 in type IIA. The authors consider D-brane BPS states which carry the charges Q_F and Q_H, which correspond to the ones of the extremal black hole family. What it is needed to do now is to count the degeneracy of the black hole system by counting the number of all BPS bound states. The main idea which we can exploit to perform the counting is that:

> *The number of microstates of the Strominger-Vafa black hole coincides with the the number of independent ways in which the D1 branes can move inside the $K3$ if the limit where $K3$ is small with respect to S^1 is taken.*

In the limit above we get a (1+1)-dimensional supersymmetric sigma model whose target space is the symmetric product of $\frac{1}{2}Q_F^2+1$ copies of $K3$. We need to compute the number of $\frac{1}{4}$-BPS states, i.e. those states which vanish under the action of the right-moving supercharges ($\bar{L}_0 = 0$). Remarkably, for $2d$ conformal field theories there is a tool which can greatly help us: the Cardy formula [26, 27]. This gives the asymptotic density of states in a two-dimensional conformal field theory that is determined by only a few features of the symmetry algebra, independent of any details of the dynamics[9]. The Cardy formula presents a great advantage: it is able to count states without requiring detailed knowledge of them, i.e. without having at disposal a full quantum theory of gravity[10]. For the case under consideration, the Cardy formula may be used to obtain the desired counting of the black hole microstates and it is given by:

$$d(n,c) \sim \exp\left[2\pi\sqrt{\frac{nc}{6}}\right] ; \tag{1.17}$$

this can be applied only when $n \gg c$ which is the central charge for the D1D5 system. It is a crucial point for the Strominger and Vafa computation to realise that c can be determined solely by the dimension of the moduli space; indeed it results:

$$n = Q_H\,, \quad c = 6\left(\frac{1}{2}\,Q_F^2+1\right) . \tag{1.18}$$

Evaluating the entropy from the Cardy formula we find:

$$S_{\text{micro}} = \log d(n,c) \sim 2\pi\sqrt{Q_H\left(\frac{1}{2}\,Q_F^2+1\right)}\,, \tag{1.19}$$

which in the correct D-brane limit $Q_H \gg Q_F^2 \gg 1$, coinciding with the one where the Cardy formula applies, reproduces the Bekenstein-Hawking entropy (1.16) at leading order. The macroscopic entropy of the black hole (1.14) is thus reproduced via microstate counting.

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9 Strictly speaking, the Cardy formula only exists for $2d$ CFTs due to the peculiar and unique modular properties conformal field theories present in two dimensions.

10 Note however that this is also the main weakness of the formula as the actual states being counted remain disguised.

1.4 MICROSTATE COUNTING FOR ANTI-DE SITTER BLACK HOLES

The Strominger-Vafa microstates counting described in the last section has been the first example of a computation of the entropy of a black hole by its microstates. The black hole considered by the authors is asymptotically flat; after this seminal paper an immense literature, which would be too long to refer to, of important similar results follow for this kind of black holes. One natural question that could arise was if and how it is possible to perform similar computations in order to account for the entropy of black holes with different asymptotics.

One year after the Strominger and Vafa paper, the most fruitful conjecture in modern contemporary physics, the Anti-De Sitter/Conformal Field Theory (AdS/CFT) correspondence, has been formulated by Maldacena in [28] and sharpened a few months later in [29,30]. The AdS/CFT correspondence relates quantum gravity on a d-dimensional AdS spacetime and a $(d-1)$-dimensional conformal field theory (CFT) in absence of gravity. The CFT is thought to live at the boundary of the AdS space. Applied to black holes, the correspondence implies that the microstates of black holes should correspond to states in the dual CFT and therefore this appears the natural setting to exploit in order to provide an interpretation of the black hole entropy in terms of a microscopical theory.

Thus, what we can imagine to do is to count the microstates of asymptotically AdS black holes using the dual CFT and compute the Bekenstein-Hawking entropy by them. These kind of computations would constitute for asymptotically AdS black holes a result analogous to the one by Strominger and Vafa for asymptotically flat ones. In this section we will discuss whether or not similar results have been reached and we will recap on the state of art of microstate counting for AdS black holes. Before doing this, we introduce more formally and rigorously the AdS/CFT correspondence by recalling the process and the arguments which led Maldacena to its formulation in [28]. Due to the nature and the complexity of the physical concepts involved, this part and some others in this subsection might be quite technical, but nevertheless the main physical concepts should emerge aside the technicalities.

1.4.1 *The original Maldacena's correspondence*

Sometimes, in theoretical physics, the key to find new and important results has proven to be to realise the fact that two seemingly different concepts are indeed related to each other at a deep and fundamental level. Examples of this kind are dualities which relate two appeareantly different theories to each other by stating that they are in fact equivalent. In particular, the Hilbert spaces and the dynamics of the two theories agree. Even though from a mathematical point of view the theories are identical, from a physical point of view their descriptions may differ, for example there may be different Lagrangians for the two theories.

AdS/CFT correspondence is a duality between a superstring theory and a certain conformal quantum field theory; it was originally conjectured in [28] as a duality between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$, SU(N) super Yang-Mills theory. We will start by briefly review how the original conjecture arose.

The bosonic spectrum of type IIB string theory is the following:

$$\begin{aligned} &g_{ij}\,, \quad B_{ij}\,, \quad e^{\phi} \qquad \text{in the NS-NS sector}\,, \\ &C_0\,, \quad C_2\,, \quad C_4 \qquad \text{in the R-R sector} \end{aligned}$$

the R-R fields are sourced by Dp-branes; in particular each C_p field is coupled to a D$(p-1)$-brane. Furthermore, each R-R field strength has a magnetic dual via

$$F_p = \pm F_{10-p}\,, \tag{1.20}$$

so that every Dp-brane has a magnetic D$(6-p)$-brane dual. Therefore we see that D3-branes are special objects since they are dual to themselves and so is the field strength F_5 they couple to.

There are brane solutions both in type IIA and type IIB string theory. A generic Dp-brane solution is provided by the following ansatz

$$\begin{aligned} \mathrm{d}s^2 = H_p^{-1/2}(r)\left(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_p^2\right) \\ + H_p^{1/2}(r)\left(\mathrm{d}y_{p+1}^2 + \cdots + \mathrm{d}y_{9-p}^2\right)\,, \end{aligned} \tag{1.21}$$

$$C_{p+1}^{t,x_1,\ldots,x_p} = -\frac{1}{2} H_p^{-1}(r)\,, \tag{1.22}$$

$$e^{\phi} = g_s \left[H_p(r)\right]^{\frac{3-p}{4}}\,, \tag{1.23}$$

here we have labelled as g_s the string coupling constant and we have divided the spacetime coordinates into x_{p+1} ones longitudinal to the brane and y_{9-p} transverse to it; furthermore we have defined as r the transverse distance between the branes

$$r^2 = y_{p+1}^2 + \cdots + y_{9-p}^2\,. \tag{1.24}$$

The type II equations of motion fix the function H_p to be

$$H_p(r) = 1 + \frac{L^{7-p}}{r^{7-p}}\,, \tag{1.25}$$

where L is a constant which must obey to the quantization condition

$$\frac{1}{(2\,\pi\,\ell_s)^{p+1}\, g_s} \int_{\Sigma_{8-p}} \star F_{p+2} = N \in \mathbb{N}\,, \tag{1.26}$$

with N being the number of p-branes and the integration contour Σ_{8-p} being a cycle which surrounds the brane at infinity. Recalling the form of the gauge fields (1.22) and of the H_p functions (1.25), we can integrate (1.26) in order to obtain a condition for L. Doing so, we find

$$L^{7-p} = 2\,(2\,\pi)^{p-2}\,\ell_s^{7-p}\,g_s\,N\,. \tag{1.27}$$

The brane solutions we have constructed are singular in $r \to 0$ for every p but $p = 3$. The case of D3-branes seems therefore to be very special and we shall examine it in detail. From (1.23) and (1.25) we see that for D3-branes we have:

$$e^{\phi} = g_s\,, \qquad H_3\,(r) = 1 + \frac{L^4}{r^4}\,, \tag{1.28}$$

note in particular that the dilaton is constant. From equation (1.27), we obtain that in the $p = 3$ case the constant L is fixed by the quantization condition to be $L^4 = 4\,\pi\,g_s\,\alpha'^2\,N$ with $\alpha' = \ell_s^2$. We can now study the near-horizon limit $r \to 0$ of the brane solution we have constructed[11]. This limit must be taken carefully; in particular we require that [28]

$$r \to 0\,, \quad \alpha' \to 0\,, \quad \text{so that } u = \frac{r}{\alpha'} \text{ is fixed} \tag{1.29}$$

and we also keep fixed the string coupling g_s and the number of branes N. This limit is equivalent to the situation in which the branes decouple; the α' correction to the supergravity approximation are suppressed in this limit. Performing the limit (1.29), the metric (1.21) becomes

$$\mathrm{d}s^2 = \alpha' \left[(4\,\pi\,g_s\,N)^{1/2} \left(\frac{\mathrm{d}u^2}{u^2} + \mathrm{d}s^2\left(S^5\right) \right) + \frac{u^2}{(4\,\pi\,g_s\,N)^{1/2}} \mathrm{d}x^{\mu}\mathrm{d}x_{\mu} \right]\,, \tag{1.30}$$

which is the geometry of $\mathrm{AdS}_5 \times S^5$. Therefore we have found that

> *the near-horizon geometry, i.e. the geometry near the D3-branes, is the one of $AdS_5 \times S^5$.*

Note that the radius of S^5 and the one of AdS_5 are exactly the same and they are given by $L^4 = 4\,\pi\,g_s\,\ell_s^4\,N$. We want the curvature of the geometry to be small and the string theory corrections to be negligible in order to preserve the supergravity approximation; this can be achieved by imposing $L^4 \gg \ell_s^4$, from which it follows $N \gg 1$.

For now we have provided a description of the D3-branes based on a closed string perspective, i.e. we have treated them merely as supergravity solitons. From the point of view of open strings, branes are extended objects on which open strings can attach and they may be connected to Super-Yang Mills theory. Let us consider a stack of N D3-branes in $\mathbb{R}^{1,3}$; we may ask which is the low-energy effective theory of N coincident Dp-branes in $p+1$ dimensions. The answer turns out to be

11 Strictly speaking, this is a near-brane limit, rather than a near-horizon one.

> $p+1$-*dimensional Super Yang-Mills theory with gauge group SU(N).*

There are several ways to show this; here we just recall that the Super Yang-Mills action can be easily obtained starting from the effective action for a Dp-brane. Indeed, the latter is just the Dirac-Born-Infeld action

$$S_{\text{DBI}} = -T_p \int \mathrm{d}x^{p+1} e^{-\phi} \sqrt{-\det\left(g + 2\,\pi\,\alpha'\,F\right)} \quad \text{with} \quad T_p = \frac{1}{(2\,\pi)^p\,(\alpha')^{\frac{p+1}{2}}\,g_s}\,, \tag{1.31}$$

performing the limit $\alpha' \to 0$ and keeping fixed $g_s\,(\alpha')^{\frac{p-3}{2}}$ we obtain from the above action

$$S_{\text{SYM}} = \frac{1}{g_{YM}^2} \int \mathrm{d}x^{p+1} e^{-\phi} F_{\mu\nu} F^{\mu\nu}\,, \tag{1.32}$$

which is the action of a Super Yang-Mills theory[12].

We have thus obtained two different descriptions of the same objects from two different perspectives and thus we are led to the conclusion that the physics behind both of them should be the same. Therefore we can state the following conjecture:

> *type IIB string theory in an* $AdS_5 \times S^5$ *background with* N *units of R-R fluxes and equal radius* L *for* AdS_5 *and* S^5 *is dual to* $\mathcal{N}=4$ *Super Yang-Mills conformal field theory in* $d = 1+3$ *dimensions with gauge group SU(N) and* $g_{YM}^2 = 2\,\pi\,g_s$.

This is the famous Maldacena's conjecture, originally stated in [28], which has overcome a very large amount of non trivial checks and tests. However, from a mathematical point of view, this is still a conjecture and one may even be tempted to invert the statement and use it to define a quantum gravity theory when the dual CFT is totally known and under control.

In this original form, AdS/CFT is an $\text{AdS}_5/\text{CFT}_4$ correspondence; since in this thesis we will mainly focus on AdS_5 black holes this is the form we are interested in the most. Nevertheless, this has been generalised over the years to different context and dimensions (see [31–34] for a review) such as $\text{AdS}_3/\text{CFT}_2$, which has many important applications and which naturally arises when studying the Strominger-Vafa black hole using the correspondence, or $\text{AdS}_4/\text{CFT}_3$, using which the Bekenstein-Hawking entropy for an AdS black hole has been reproduced for the first time via microstate counting.

It is worth mentioning that the AdS/CFT correspondence has proven to be suitable for various applications in experimentally reachable areas of physics. Indeed, even using only simple classical solutions such as

12 The Yang-Mills constant g_{YM} is given by $g_{YM}^2 = 2\,(2\pi)^{p-2}\left(\alpha'\right)^{\frac{p-3}{2}}\,g_s$.

AdS black holes, it is possible to simulate to a good approximation a certain amount of physically relevant field theories at strong coupling and finite temperature. Among the system which one may attempt to describe in such a way there is the quark-gluon plasma [35, 36], tested at particle accelerators such as LHC, and relevant condensed matter systems [37, 38].

1.4.2 *Microstate counting for AdS black holes: state of the art*

1.4.2.1 *The general idea*

Having AdS/CFT correspondence at disposal, it seemed very natural to use it to provide a description of asymptotically AdS black holes in terms of microscopical states in a dual quantum field theory, thus extending the results obtained for the asymptotically flat ones. In this thesis, we are interested in AdS_d black holes with $4 \leq d \leq 7$; we will not consider the AdS_3 case since it is somewhat special due to the additional properties the dual CFT_2 presents.

We begin by considering, for simplicity, a supersymmetric and extremal black hole. In this thesis we will call black holes presenting both these two properties as *BPS black holes*[13]. The general idea is that the entropy should be a function of the charges of the AdS black hole

$$S_{\mathrm{BH}} = S_{\mathrm{BH}}\left(Q_I, J_i\right) \,, \tag{1.33}$$

where Q_I are electric or magnetic charges and J_i are angular momenta. Here we have not considered the energy E since for a BPS black hole the BPS relation among its charges does hold. This is a linear relation involving the conserved charges and the chemical potentials which is a consequence of supersymmetry algebra. We may think of having exploited the BPS relation to replace the energy with the other charges; this is why the energy does not appear in the formulae related to BPS black holes. The boundary metric associated to the black hole solution would be of the type

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2} + r^2\,\mathrm{d}s^2_{\mathcal{M}_{d-2}\times\mathbb{R}} + \dots \qquad\qquad r \gg 1\,, \tag{1.34}$$

and the entropy should be recovered by counting states in the dual CFT_{d-1} on $\mathcal{M}_{d-2}\times\mathbb{R}$

$$S_{\mathrm{BH}} = S_{\mathrm{BH}}\left(Q_I, J_i\right) = \log n\left(Q_I, J_i\right) \,. \tag{1.35}$$

13 We shall see in the following that the two properties of supersymmetry and extremality correspond to two different conditions and that the first one does not necessarily imply the second or viceversa. Indeed, there are black hole solutions which are extremal but not supersymmetric [39, 40]. Nevertheless, it is worth mentioning that for large classes of solutions there are arguments showing that *regular* supersymmetric black holes must be extremal. For instance, this is the case of the five-dimensional supergravity theory that we will consider in Chapter 3, 4. However, it is fundamental to remark that these arguments only hold for *regular* black hole solutions and not for the complexified ones we will consider in Chapter 3. Furthermore, these are theories for which such arguments are not available: this is the case of $D = 11$ supergravity.

In order to try to perform the computation above some ingredients are needed: a full BPS AdS_d black hole solution for which the Bekenstein-Hawking entropy is computed, knowledge of the CFT dual to the black hole solution and some tool to count black hole microstates with given charges in the dual field theory.

1.4.2.2 *The first unsuccessful attempts*

These requirements seemed to be met for AdS_5 black holes when the first rotating AdS_5 BPS black hole solution was constructed in 2004, in the famous paper [41], and many generalizations quickly followed [42–45]. Indeed, being them AdS_5 black holes, we are in the domain of the original, emblematic, $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, so that the dual field theory is just the well known $\mathcal{N} = 4$ Super Yang-Mills theory and one could have hoped that the counting of the microstates could have been realised by evaluating the large N limit of the superconformal index proposed in [46, 47][14]. It is important to remark that the index is defined in Euclidean signature, while (1.34) is still written in Lorentzian one. Therefore, in order to compare the field theory result form the index with the supergravity black hole entropy, one needs to compactify the time direction. This latter becomes an S^1 with radius β. From a supersymmetric localization point of view, changing this radius is a Q-exact deformation. The computation was indeed tried in [47], but the result did not match the expectations. Indeed, the Bekenstein-Hawking entropy of five-dimensional black holes goes like N^2 in the large N limit, while the index, also if generalized as proposed in [49–51], behaves as $\mathcal{O}(1)$ in the same large N limit; thus it fails to account for the correct $\mathcal{O}(N^2)$ leading behaviour of the Bekenstein-Hawking entropy. A possible explanation to this result that has been proposed is that the index retains different signs when counting bosonic and fermionic states, therefore there could be large cancellations which reduce the index drastically. To solve this problem one should be able to identify the exact contribution in the index of the supersymmetric black hole microstates, separating it from the gravitational ensemble, but this has proven to be very difficult and the task of reproducing the entropy of AdS black holes via microstate counting has remained in a somewhat inconclusive state for a long time.

1.4.2.3 *A solution for AdS_4 static black holes*

The same kind of problem has been solved before in one dimension lower, for asymptotically AdS_4 static supersymmetric black holes. A key ingredient that allowed the counting to be performed has been the advent of localization techniques for supersymmetric quantum field

14 It is worth mentioning that the superconformal indices for 3d, 5d, 6d SCFTs have been introduced and studied in [48]. These are relevant for the AdS_4, AdS_6 and AdS_7 black holes that we will analyze in this thesis in the next chapters.

theories [52, 53]. The computation was performed in 2015 in the very interesting paper [54]. The black holes there involved are quite different from the rotating ones considered in five-dimensions, since they have magnetic charges that correspond to a topological twist in the dual field theory; this latter is the ABJM theory [55] in $d = 3$. For four-dimensional AdS black holes, the entropy scales as $N^{3/2}$ in the large N limit. This behaviour and the Bekenstein-Hawking entropy itself are indeed reproduced in the large N limit in [54] by the three-dimensional supersymmetric partition functions of ABJM theory on $\Sigma_g \times S^1$ with a topological twist along the Riemann surface Σ_g.

The main difference between the above-mentioned case, which regards magnetic charged AdS_4 black holes, and the AdS_5 rotating one is that in this latter there is no topological twist. Therefore, some other technology has to be found in order to perform the black hole microstate computation for such black holes from a CFT_4.

However it is still possible to seek for an inspiration from the AdS_4 case. In particular we can note that the topologically twisted index is a function of magnetic charges p_I and fugacities Δ_I for the global symmetries of the theory; in order to obtain the entropy for a black hole with electric charges q_I one has to Legendre transform $\log Z_{\text{twisted}}$ in the following fashion

$$S_{\mathrm{BH}}\left(q_I, p_I\right) = \mathcal{I}\left(\Delta_I\right)\Big|_{\bar{\Delta}_I} = \log Z_{\text{twisted}}\left(p_I, \Delta_I\right) - i \sum_I q_I\, \Delta_I \Big|_{\bar{\Delta}_I} , \quad (1.36)$$

with $\bar{\Delta}_I$ being the extremum of $\mathcal{I}\left(\Delta_I\right)$ and because of this the procedure is called *$\mathcal{I}$-extremization* [54, 56–58][15]. This has been proven to correspond with the attractor mechanism in gauged supergravity [61–64]. One possible inspiration which one may follow is to try to find a function of some fugacities associated to the charges of an AdS_5 rotating black hole that reproduces the entropy once Legendre transformed and extremized. This is exactly what is done in [65].

1.4.2.4 *Inspiration from the gravity side*

Although attractor mechanism in five-dimensional gauged supergravity is not known in details, nevertheless the authors of [65] managed to find an extremization principle for the entropy of rotating AdS_5 black holes.

In particular, they consider the class of black holes found in [41–45], which are asymptotic to $\mathrm{AdS}_5 \times S^5$, that presents three electric

15 It is worth mentioning that c-extremization principle [59] is capable of reproducing the entropy of AdS_5 black strings. This has been investigated in [60]. There, the authors constructed black string solutions which can be embedded in $\mathrm{AdS}_5 \times S^5$ and correspond to a twisted compactification of $\mathcal{N} = 4$ SYM on Σ_g. Holographic arguments suggest that this theory flows to an IR two-dimensional SCFT; therefore the entropy of such black strings can be reproduced from field theory by computing the value of the central charge via c-extremization [60].

charges Q_I with $I = 1, 2, 3$ and two angular momenta J_i with $i = 1, 2$[16]. Due to supersymmetry the five charges are connected by a non trivial constraint and therefore only four of them are independent. In [65] it has been shown that the Bekenstein-Hawking entropy of the black holes can be obtained as the Legendre transform of a quantity proportional to[17]

$$I = \frac{\pi}{4} \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2} , \tag{1.37}$$

where Δ_I are chemical potentials conjugated to the electric charges Q_I and ω_i chemical potentials conjugated to the angular momenta J_i. Note that the function (1.37) is independent of β. This is in agreement with the fact that, as we have mentioned in the last subsection, in the supersymmetric localization computation one obtains a Q-exact deformation by changing the radius β of the S^1 factor of the superconformal index signature. The function I, which reproduces the entropy once Legendre transformed and extremized, is called *entropy function*[18]. There is the following constraint that the chemical potentials have to satisfy as a consequence of the one for the charges:

$$\sum_i \omega_i - \sum_I \Delta_I = \pm 2 \pi i , \tag{1.38}$$

this makes clear that the conjugated chemical potentials are complex[19]. However, the entropy obtained by a Legendre transformation must be real; imposing this, one obtains a non-linear and non-trivial constraint among the charges which remarkably is verified by the BPS black hole solution. Later, it has been shown that similar entropy functions exist also for AdS_4, AdS_6 and AdS_7 black holes [68–70]. For every dimension, the entropy function is an homogeneous function of conjugated chemical potentials where one has the product of all the Δ_I (as many as the electric charges Q_I) appearing in the numerator, and the product of the ω_i (as many as the independent angular momenta J_i) appearing in the denominator. For each black hole, the conjugated chemical potentials appearing in the entropy functions must satisfy a linear con-

16 These are the black holes with whom we will work the most in this thesis; we will introduce them in much more details in the following chapters.

17 Here we change conventions with respect to [65] in order to match with the one of [66, 67], which are the one we will use in the whole thesis. As consequence, our potentials and the one used in [65] differ for some overall constants.

18 Note that sometimes in the literature the term *entropy function* is used to denote the right-hand side of (1.39) before evaluating it at the extremal values of the chemical potentials.

19 Again, we have changed conventions for the chemical potentials with respect to [65] in order to match with [66, 67]; these conventions make clear that the chemical potentials are complex since at the right-hand side of the constraint (1.38) there is an imaginary number.

straint which is analogous to (1.38). The entropy is always obtained by Legendre transforming and extremizing the entropy function

$$S_{\mathrm{BH}}\left(Q_I, J_i\right) = I\left(\Delta_I,\, \omega_i\right) - i\left(\Delta_I\, Q_I + \omega_i\, j_i\right)\bigg|_{\bar{\Delta}_I,\, \bar{\omega}_i} ; \tag{1.39}$$

for this reason, this principle is known in the literature as the *extremization principle for rotating AdS black holes*. Note that the fact that the chemical potentials satisfy a linear constraint similar to (1.38) is crucial to recover the entropy: had the potentials been completely free, the Legendre transform would have vanished. However we should underline that none of the above papers tell us how to extract the complex conjugated chemical potentials from the supergravity black hole solutions nor what is the physical interpretation of the entropy function from a gravitational point of view nor what is the origin and the interpretation of the linear constraint satisfied by the chemical potentials. We will return on these fundamental points in the following.

1.4.2.5 *Results on the CFT side*

The results found on the gravity side have been greatly inspiring for quantum field theory computations. For AdS_5 black holes, we have already mentioned that the computation performed in [47], using the supersymmetric index, fails in reproducing the Bekenstein-Hawking entropy; however that same computation is valid only for *real fugacities* while the constraint (1.38) seems to point in the direction of *complex fugacities*. Therefore, the gravity side results suggest that the saddle point at large N that one should consider to reproduce the black hole entropy might be complex. In more concrete words, we can describe the dual $\mathcal{N}=4$ Super Yang-Mills theory from a $\mathcal{N}=1$ point of view; then we may say that it contains a vector multiplet W_α and three chiral multiplets ϕ_a with the superpotential $\mathcal{W} = \mathrm{Tr}\left(\phi_3, [\phi_1, \phi_2]\right)$. There are three R-symmetries $\mathcal{R}_I$ which are associated to the Cartan $\mathrm{U}(1)^3 \subset \mathrm{SO}(6)$; the exact R-symmetry is $R = \left(\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3\right)/3$. Furthermore, there are three global symmetries $q_I = \left(\mathcal{R}_I - R\right)/2$, but only two of them are indepedent, since it must hold the constraint $q_1 + q_2 + q_3 = 0$. The superconformal index, which is the most natural place where to look for the entropy, can be written as

$$\mathcal{I}\left(\Delta_I, \omega_i\right) = \mathrm{Tr}\bigg|_{Q=0} \left(-1\right)^F e^{i\left(\Delta_I\, Q_I + \omega_i\, J_i\right)}\,, \tag{1.40}$$

with Δ_I and ω_i being the chemical potentials and $Q_I = \mathcal{R}_I/2$. the chemical potentials must fulfil a constraint so that the index depends only on four independent parameters, as the family of BPS black holes. We know that there are cancellations between bosonic and fermionic supersymmetric states in the index, therefore the result we can obtain should be only a lower bound on the number of BPS states; nevertheless we may expect that for large N it saturates the entropy exactly as it happens for magnetically charged black holes in $d = 4$. The first computations in the literature have been performed for *real fugacities*; we know that in this case they do not account for the entropy, giving a result of order $\mathcal{O}(1)$ for large N; however the gravity side results suggest to look at what happens for *complex fugacities*. Computations with complex fugacities have been performed in the literature in the last couple of years, starting with [66, 69, 71] and they have been successful in the two following partial overlapping limits

1. Large N limit and $J_1 = J_2$ [71, 72]: the index presents a Stokes behavior at large N as a function of the chemical potentials and it may provide a contribution of order $\mathcal{O}\left(N^2\right)$ if one picks up the right direction in the complex plane. To perform the computation, a successful approach has been proven to be writing the superconformal index as a sum over Bethe vacua [73–75]. An other useful approach leading to the same result is presented in [76, 77] while the extension to the unequal angular momenta case is discussed in [78].

2. The Cardy limit [69, 79]: this limit corresponds to $\omega_i \ll 1$ with Δ_I being fixed and it describes large black holes with

$$q_I \sim \frac{1}{\omega^2}\,, \quad J_i = \frac{1}{\omega^3}\,, \qquad \omega_1 \sim \omega_2 \sim \omega \to 0\,. \tag{1.41}$$

   In this approach, chemical potentials must be complex so that their imaginary parts at the saddle point present phases that optimally obstruct cancellations between bosonic and fermionic states. Using such complex potentials, the index in the Cardy limit is able to reproduce the entropy of large AdS_5 black holes. Generalization to this approach have been provided in [80–85].

Saying that the computations performed with both approaches we have presented here agree with the gravity side means that the index is found to match with the entropy function (1.37), i.e.

$$\log \mathcal{I}\left(\omega_i, \Delta_I\right) = \frac{\pi}{4}\,\frac{\Delta_1\,\Delta_2\,\Delta_3}{\omega_1\,\omega_2}\,, \tag{1.42}$$

with the constraint $\omega_1 + \omega_2 - \Delta_1 - \Delta_2 - \Delta_3 = \pm\, 2\,\pi\, i$ obtained analytically.

It is worth to mention that the expression (1.42) can be recovered from quantum field theory also using a third different approach. In [65] it was pointed out that the expression for the supersymmetric Casimir energy E_c matches exactly (1.42)[20], with the precise coefficient for $\mathcal{N} = 4$ Super Yang-Mills theory[21]. Building on this observation, in [66]

20 The supersymmetric Casimir energies formulae for the related maximally supersymmetric theories have been first written down in [86].

21 In [68] the same fact has been observed for AdS_7 black holes, i.e. the supersymmetric Casimir energy of $(0, 2)$ CFT matches the AdS_7 black hole entropy function.

it has been proved that, by considering a modified supersymmetric partition function on $S^3 \times S^1$ and imposing the constraint (1.38), the resulting supersymmetric Casimir energy still matches with (1.42). This is quite intriguing, since it is not clear why this quantity, which is the energy of the CFT vacuum [87], should have something to do with the entropy of the black hole. One proposal is that this is the consequence of some modular properties of the partition function, which has still to be understood.

The $\mathcal{O}(N^2)$ behavior of the index in the complex chemical potential case has been confirmed by numerical analysis performed in [88, 89]; however there seems to be instabilities when decreasing the charges in all the approaches above mentioned. This might due to the contribution of other types of black holes, which might be for example the supersymmetric hairy black holes found in [90, 91].

The results for AdS_5 have been generalized to other dimensions. The entropy of AdS_4 rotating black holes has been reproduced in the Cardy limit in [92–94], while computations for AdS_6 and AdS_7 black holes have been perfomed in [95–98].

1.5 THESIS CONTRIBUTIONS AND CONTENT

1.5.1 *Thesis contribution*

From everything we have reported in the last section it should be evident that the problem of reproducing the entropy for AdS black holes via microstate counting is complex, involved and interesting and there are still pieces which must be added to the puzzle in order to have a full comprehension of this topic.

The main result that has been obtained is that by using holography the entropy of BPS AdS black holes with large charges can be evaluated via a superconformal index in the dual conformal field theory. Once the computation is done correctly, the logarithm of the index results to be an homegeneous function of chemical potentials which exactly matches the entropy function on the gravity side. By Legendre transforming and extremizing such function, we obtain the correct Bekenstein-Hawking entropy.

Although this general picture is quite clear and it has been proven to work for all the dimensions $4 \leq d \leq 7$, there are still some open questions to address on the gravity side. In this thesis we aim to answer some of these questions, mostly by summarizing and reporting the results of the papers [67, 99, 100] on which this work is based, but also introducing and presenting some original unpublished material. We may now briefly present the main problems we try to face in the next chapters and the most important results we find; instead in the next subsection we present how the thesis is organized in a more detailed manner.

The first topic we would like to investigate is the physical interpretation of the entropy function, both on the gravity and the field theory side of the holographic correspondence. For rotating black holes this question is rather subtle because, due to the constraint (1.38), the saddle point values of both the rotational and electric chemical potentials turn out to be complex and it is not obvious how to read the chemical potentials in (1.38) from the black hole solution. These aspects have been firstly investigated in [66], there it has been found that the entropy function for a particular class of rotating BPS black holes in AdS_5 is the supergravity on-shell action after taking a specific BPS limit. This latter goes along a supersymmetric trajectory in the space of complexified solutions.

We provide an extension of the gravity side results of the above mentioned work in [67]. There, we analyze other classes of rotating, asymptotically AdS black holes in four, five, six and seven dimensions, discussing also the cases with more than one electric charge; while the authors of [66] only consider AdS_5 black holes with two angular momenta and only one electric charge. For any dimension, we start from a non-supersymmetric and non extremal family of black holes and we reach extremality by performing the limit we propose in the paper, which follows a supersymmetric trajectory in the space of complexified solutions. In this way, we are able to define the appropriated chemical potentials for the family of BPS black holes under consideration and we verify that they satisfy a constraint of the type (1.38). Moreover, the supersymmetric on-shell action I takes the form of a simple function of these variables, that precisely matches the entropy functions proposed in [65, 68, 70]. The Legendre transform of I (subject to the constraint (1.38)) is in general a complex quantity, so it cannot be immediately identified with the entropy of the Lorentzian solution. However, demanding reality of the Legendre transform, which amounts to a specific condition on the charges, one finds precisely the Bekenstein-Hawking entropy of the BPS black hole [66, 69]. The saddle point values of the chemical potentials remain complex and match the ones that we obtain from the solution. Therefore we have provided a physical derivation of the proposed entropy functions and the related extremization principles via the BPS limit of black hole thermodynamics described throughout the paper.

In each of the field theoretic computations aiming to reproduce the entropy of AdS black holes that we have described in the last section, it is crucial to understand which are the field theory states that contribute to the entropy. Further information on them might come from studying whether the black hole solutions continue to exist when one tries to deform the geometry of the conformal boundary, and if so how this affects their thermodynamics. This problem has been studied in [101]. There, the authors work in five-dimensional minimal gauged supergravity and construct both supersymmetric and non-supersymmetric black holes which are not globally asymptotically AdS, but just locally, since the boundary is non conformally-flat. Such solutions are therefore AlAdS_5 (asymptotically locally Anti-de Sitter) black holes. While the squashing at the boundary is arbitrary, in the supersymmetric case the event horizon geometry turns out to be completely frozen and therefore the entropy takes a fixed value. This behaviour is qualitatively different from the one of asymptotically AdS_5 black hole solutions to minimal gauged supergravity with the same symmetry [41], where the entropy depends on one parameter controlling the horizon geometry.

Motivated by the above mentioned paper, in [99] we construct more general supersymmetric AlAdS_5 black hole solutions in the context of five-dimensional Fayet-Iliopoulos supergravity; this theory, to be formally introduced in the next chapter, is five-dimensional supergravity coupled to an arbitrary number n_V of vector multiplets and with a $\mathrm{U}(1)$ gauging of the R-symmetry. This is a more general theory with respect to minimal gauged supergravity. In the paper, we keep n_V general and we look for black hole solutions having the energy, one angular momentum and $n_V + 1$ electric charges as conserved charges. The system of equations for a supersymmetric solution to Fayet-Iliopoulos gauged supergravity has been given in [42]. In [99] we partially solve these conditions and we impose an ansatz on the scalar fields in order to reduce the system to two coupled ordinary differential equations; however these are very hard to solve and we could not find new analytic solutions. Therefore we resort to the numerical approach: we construct the near-horizon and near-boundary solutions perturbatively and then interpolate numerically. By using this approach, we find a two-parameter family of supersymmetric black holes with both running gauge fields and scalar fields; this family of solutions generalizes the one presented in [101] which depends only on one parameter. We find two main results in this paper: the first one is that the horizon does not depend on the squashing at the boundary since of the two parameters the one that controls the squashing is washed away in the near-horizon region. In other words, whatever is the squashing at the boundary, the radial flow towards the horizon acts as an attractor that brings the transverse geometry into a form which only depends on the other parameter. This result is in agreement with other ones already present in the literature suggesting a general principle of holographic uniformization [102], where the UV freedom of the solution is fixed in the IR region[22]. Nevertheless, we have still found an interesting near-horizon geometry since this latter depends non trivially on the remaining parameter and is not frozen as in [101]. The second main result is that the black hole entropy can be expressed as a simple function of the Page electric charges rather than the holographic electric charges. These are two different types of

22 In particular, in [102] black string solutions of the same five-dimensional gauged supergravity exhibiting the same uniformization behaviour have been studied. See also [103] for more recent results on the same topic.

charges which however are numerically different only when there is a non vanishing Chern-Simons term at the boundary, as it is for the family of AlAdS_5 black holes.

The solution of [99] is quite general and valid for an arbitrary number of vector multiplets n_V; however it is still obtained by imposing a particular ansatz on the scalar fields which constrains all their components orthogonal to the scalar vacuum expectation values in the supersymmetric AdS_5 vacuum to be the same. In [100] we look for a more general solution, obtained without imposing any ansatz, for the theory with $n_V = 2$. Therefore we let the scalar fields unconstrained by abandoning the restrictive conditions imposed in [99]. The choice of $n_V = 2$ is motivated by the fact that the supergravity theory so obtained can be uplifted to type IIB supergravity in ten dimensions; the solutions obtained in this theory are thus particularly relevant from a string-theoretical perspective. In order to find such solutions, we consider the system of $n_V + 1$ coupled differential equations obtained in [99], which is the result of a rearrangement process applied to the conditions given in [42], to be imposed in order to have a supersymmetric solution, and we specialize it to the case $n_V = 2$; in this way we obtain three coupled differential equations to be solved. These are very cumbersome and complicated and again we have to resort to the numerical approach. We construct the near-horizon family of candidate black hole solutions, a near-boundary family of candidate AlAdS_5 solutions and we find numerically that the two perturbative solutions thus obtained match in the bulk; we therefore obtained a new three-parameter family of supersymmetric black holes. This family of solution generalizes the one of [99] for the case $n_V = 2$ since the scalar fields are now left unconstrained. The main results obtained by analyzing the properties of this new family of solutions are quite similar to the one of [99]. The horizon geometry and the horizon properties are controlled by two of the three total parameters; the last parameter regulates the squashing at the boundary geometry, but does not influence the horizon. Therefore once again the near-horizon geometry is independent of the squashing. By evaluating the entropy of the solutions we find that it is remarkably reproduced by a simple formula containing the Page charges, rather than the holographic charges.

1.5.2 *Thesis content*

This thesis is composed of five chapters and two appendices.

In Chapter 2 we provide a review of various AdS black hole solutions of different supergravity theories in all the dimensions $4 \leq d \leq 7$. For each dimension, we introduce a finite-temperature solution and we present the corresponding BPS solution, illustrating how it is possible to land on the latter starting from the former. We will see that, by taking the BPS limit as it is usually performed, one cannot obtain the

BPS chemical potentials ω^i and Δ^I, which play a central role in the extremization principles. In this Chapter, in particular, we extensively analyze the black holes constructed by Gutowski and Reall in [41, 42], since they would be particularly relevant for the topics of this thesis.

In Chapter 3 we introduce our BPS limit of rotating AdS black hole thermodynamics, which goes along a supersymmetric trajectory in the space of complexified solutions. We then apply it to the AdS black holes we have introduced in Chapter 2. For each dimension, we start from the finite-temperature solution and we arrive at the BPS locus by taking our BPS limit; we will see that in the limiting procedure complex quantities naturally emerges and it is possible to obtain the BPS chemical potentials ω^i and Δ^I from the supergravity black hole solution. Furthermore, we will show that, in this limit, the Euclidean on-shell action coincides with the entropy function. Thus, we will provide a physical derivation of the extremization principles for AdS black holes in all the dimensions $4 \leq d \leq 7$.

In Chapter 4 we turn our attention on black hole solutions which are just asymptotically *locally* AdS_5, rather than asymptotically AdS_5 like the ones examined in Chapters 2, 3. We present two different families of solutions: the first one is composed by solutions of $\mathcal{N} = 2$ Fayet-Iliopoulos supergravity with an arbitrary number n_V of vector multiplets, while the second one contains solutions of the theory with $n_V = 2$. The latter case is particularly interesting since the solutions can be uplifted to $d = 10$ supergravity on $\mathrm{AdS}_5 \times S^5$, as explained in [104]. All the solutions present a boundary geometry containing a squashed S^3. We begin the chapter by writing the supersymmetry equations one has to solve in order to find supersymmetric black hole solutions in the Fayet-Iliopoulos supergravity theories under consideration. Then, we proceed to solve these supersymmetry conditions. They will turn out to be very hard to be solved analytically, therefore we will attack them with a perturbative and a numerical approach. For both the cases of arbitrary n_V and $n_V = 2$, we will construct near-boundary and near-horizon perturbative solutions and then we will show numerically that the two smoothly interpolate in the bulk, giving rise to fully regular black hole solutions. Although the solutions thus obtained are numerical, we will still able to evaluate analytically many of their relevant physical properties. In particular, we will evaluate all the conserved charges of the solutions and we will discuss how the entropy can be written as a function of them.

In Chapter 5 we draw the conclusions of this thesis and we discuss some possible outlooks. This chapter concludes the relevant physical discussion in this thesis. In order to make the main text less technical, some more mathematical aspects are left to the appendices.

In Appendix A we briefly review the main features of holographic renormalization in $d = 5$ and how this is used to compute the physical properties described in Chapters 3, 4 for both AdS_5 and AlAdS_5 black holes. In particular, we focus on the computation of the renormalized on-shell action and of the one-point functions. Appendix B contains an analogous review on holographic renormalization in $d = 4$. Here, we also review how this can be used to compute some relevant physical properties of the AdS_4 black holes discussed in Chapters 3, 4.

2

ADS BLACK HOLES IN SUPERGRAVITY

For a long time, Anti-de Sitter black holes have been somewhat neglected in the literature with respect to Minkowski ones; this is mainly because such solutions seemed of little relevance for describing observable objects in our universe. As already depicted in the last chapter, the situation changed with the introduction of the AdS/CFT correspondence, for various reasons. One main reason is the importance AdS black holes retain in the microstate counting problem, as we have already stated. Another important reason is that thermal black objects in AdS are relevant for the holographic description of various condensed matter phenomena at strong coupling, which cannot be attacked using the usual perturbative QFT approaches. Among these phenomena, there are high temperature superconductivity and quantum Hall effect. Furthermore the same thermal objects are also involved in the description of the quark-gluon plasma.

It is worth mentioning the fact that the zoo of AdS black holes is much larger compared to the asymptotically flat case; the term "black hole" in AdS has a much broader meaning than in Minkowski. Consider for example the four-dimensional case: the topology of the horizon of AdS_4 black holes is not unique as for asymptotically flat black holes; in particular the horizon geometry can be a Riemann surface of any genus [105]. Therefore, black holes in AdS_4 fall in three separate classes: spherical, toroidal and higher genus. However, AdS black holes exhibit thermodynamic properties analogous to the asymptotically flat ones, in particular the entropy is given by $1/4$ of the area of the event horizon.

We devote this section to introduce all the black hole solutions we will analyze throughout this thesis. We start by looking at AdS_5 black holes, since they are the ones which we will study the most, introducing the $\mathcal{N} = 2$ Fayet-Iliopoulos supergravity theory of which they are solutions. Then we move to introduce AdS_4, AdS_6 and AdS_7 black holes.

The large majority of what we will introduce in this chapter is background material, in preparation for the original results that will be presented in the next chapters. However, we will also present original results that provide a better understanding of the thermodynamics of the various black hole solutions. Among these, there are the computations, performed using the framework of holographic renormalization, of the on-shell action for AdS_5 and AdS_4 black holes, as well as the evaluation of the conserved charges of the same black holes using the same framework.

2.1 AdS$_5$ BLACK HOLES

2.1.1 *Fayet-Iliopoulos gauged supergravity*

The AdS$_5$ black hole solutions we are interested in belong to five-dimensional $\mathcal{N} = 2$ supergravity with a Fayet-Iliopoulos gauging of the R-symmetry [106]. This theory is known for an arbitrary number n_V of vector multiplets, although it can be uplifted to ten-dimensional type IIB supergravity on AdS$_5 \times S^5$ only for $n_V = 2$ [104].

2.1.1.1 *The theory with arbitrary n_V*

We start by presenting the theory for an arbitrary number of vector multiplets [106]. The bosonic content of the theory is constituted by the metric $g_{\mu\nu}$, $n_V + 1$ Abelian gauge fields A^I_μ, $I = 1, \ldots, n_V + 1$, of which one is the graviphoton in the gravity multiplet, and by n_V real scalar fields. It is convenient to parametrize the latter in terms of $n_V + 1$ real variables X^I, subject to the following constraint

$$\frac{1}{6} C_{IJK} X^I X^J X^K = 1 \,, \tag{2.1}$$

with C_{IJK} being a constant, symmetric tensor. In the mostly plus signature $(-, +, +, +, +)$, the bosonic action can be written as:

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \bigg[(R - 2\mathcal{V}) \star 1 - Q_{IJ} F^I \wedge \star F^J \\ - Q_{IJ} \mathrm{d}X^I \wedge \star\, \mathrm{d}X^J - \tfrac{1}{6} C_{IJK} A^I \wedge F^J \wedge F^K \bigg], \tag{2.2}$$

where $F^I = \mathrm{d}A^I$ and κ^2 is the five-dimensional gravitational coupling constant.

All known black holes in this theory have been found by assuming the additional property that the scalar target space is symmetric, which is equivalent to require that the C_{IJK} tensor satisfies the following identity [107]

$$C_{IJK} C_{J'(LM} C_{PQ)K'}\, \delta^{JJ'} \delta^{KK'} = \frac{4}{3}\, \delta_{I(L} C_{MPQ)} \,. \tag{2.3}$$

It is important to underline that in principle there is no reason to believe that this is a necessary condition to be imposed to have black hole solutions, i.e. it should be totally possible to find solutions that do not respect this property; however they have not been found yet at the time this thesis is written.

It is convenient to introduce the lower-index scalars

$$X_I = \frac{1}{6} C_{IJK} X^J X^K \,, \tag{2.4}$$

so that (2.1) reads

$$X_I X^I = 1\,. \tag{2.5}$$

Using the property (2.3), we can express the upper-index variables with respect to the lower-index ones

$$X^I = \frac{9}{2} C^{IJK} X_J X_K\;, \tag{2.6}$$

with the upper-index tensor C^{IJK} being defined as

$$C^{IJK} = \delta^{II'}\delta^{JJ'}\delta^{KK'} C_{I'J'K'}\;. \tag{2.7}$$

From (2.6) and (2.5) it also follows

$$C^{IJK} X_I X_J X_K = \frac{2}{9}\;. \tag{2.8}$$

The kinetic matrix Q_{IJ} appearing in the action and its inverse Q^{IJ} read[23]:

$$Q_{IJ} = \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K\;, \tag{2.9}$$

$$Q^{IJ} = 2 X^I X^J - 6\, C^{IJK} X_K\;, \tag{2.10}$$

it is also useful to note that

$$Q_{IJ} X^J = \frac{3}{2} X_I\;. \tag{2.11}$$

The theory is gauged by choosing $n_V + 1$ Fayet-Iliopoulos parameters V_I so as to gauge a U(1) subgroup of the SU(2) R-symmetry that the ungauged supergravity theory presents. The vector field gauging this U(1) is specified by the parameters V_I through the linear combination $V_I A^I$. Doing so, nothing changes in the bosonic sector apart from the introduction of the scalar potential

$$\mathcal{V} = -27\, C^{IJK} V_I V_J X_K\;, \tag{2.12}$$

as required by supersymmetry.

From the action (2.2), one obtains the following Einstein, Maxwell and scalar equations

$$R_{\mu\nu} - Q_{IJ} F^I_{\mu\kappa} F^J{}_\nu{}^\kappa - Q_{IJ}\partial_\mu X^I \partial_\nu X^J + \frac{1}{6}\, g_{\mu\nu}\left(Q_{IJ} F^I_{\kappa\lambda} F^{J\,\kappa\lambda} - 4\mathcal{V}\right) = 0\,, \tag{2.13}$$

$$\mathrm{d}\left(Q_{IJ} \star F^J\right) + \frac{1}{4} C_{IJK} F^J \wedge F^K = 0\;, \tag{2.14}$$

$$\begin{aligned}
&\mathrm{d}(\star \mathrm{d}X_I) - \left(\frac{1}{6}\, C_{MNI} - \frac{1}{2}\, X_I C_{MNJ} X^J\right) \mathrm{d}X^M \wedge \star \mathrm{d}X^N \\
&+ \left(X_M X^P C_{NPI} - \frac{1}{6}\, C_{MNI} - 6 X_I X_M X_N + \frac{1}{6}\, X_I C_{MNJ} X^J\right) F^M \wedge \star F^N \\
&+ 6\left(6 X_I C^{MPQ} V_M V_P X_Q - C^{MPQ} V_M V_P C_{QIJ} X^J\right) \star 1 = 0\,.
\end{aligned} \tag{2.15}$$

23 Note that the inverse matrix Q^{IJ} can be expressed in this form only if the property (2.3) does hold.

Assuming the condition $C^{IJK}V_IV_JV_K > 0$, the theory admits a supersymmetric AdS_5 vacuum of radius ℓ where the scalars are constant; their constant value is determined by the Fayet-Iliopoulous parameters as

$$\bar{X}_I = \ell\, V_I \ . \tag{2.16}$$

In the following, since we find it convenient, we exploit equation (2.16) to get rid of every parameter V_I and use instead the constant values of the scalars in the vacuum $\bar{X}_I$. Then, the scalar potential can be written as

$$\mathcal{V} = -6\ell^{-2}\, \bar{X}^I X_I \ . \tag{2.17}$$

The most general set of black hole conserved charges, in the theory we have presented in this subsection, is given by the energy, $n_V + 1$ electric charges and two angular momenta. To impose supersymmetry and extremality means to require that the BPS black hole satisfies two more constraints, hence the BPS solution carries $n_V + 2$ independent conserved charges [45].

2.1.1.2 *The theory with $n_V = 2$ and the uplift to type IIB supergravity*

A supergravity theory like the one we have described is particularly interesting when it can be uplifted to string theory or M-theory. This is also the case in which the holographic interpretation is well under control. A consistent uplift of the theory we are considering is provided by type IIB supergravity on $\mathrm{AdS}_5 \times S^5$ [104], whose SCFT dual is the $\mathcal{N} = 4$, $\mathrm{SU}(N)$ Super Yang-Mills at large N. Indeed, starting from type IIB theory one can truncate to Fayet-Iliopoulos gauged supergravity with $n_V = 2$ and with the C_{IJK} tensor given by

$$C_{IJK} = C^{IJK} = |\epsilon_{IJK}| \begin{cases} 1 & \text{if } (IJK) \text{ is a permutation of } (123)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.18}$$

In this theory[24], the constraint on the scalars (2.1) becomes

$$X^1\, X^2\, X^3 = 1\,, \tag{2.19}$$

while the scalar kinetic matrix (2.9) is given by

$$Q_{IJ} = \frac{9}{2}\,\mathrm{diag}\left((X_1)^2\,,\, (X_2)^2\,,\, (X_3)^2 \right). \tag{2.20}$$

24 Note that the supergravity theory with $n_V = 2$ vector multiplets has gauge group $\mathrm{U}(1)^3$; for this reason this particular case is sometimes dubbed as $\mathrm{U}(1)^3$ theory [42, 45]. We will sometimes use this denomination to refer to this particualr theory throughout the thesis. We underline the fact that it remains true that the fermion fields are only charged under one linear combination of the vector fields

The scalars assume the following values in the AdS_5 vacuum:

$$\bar{X}^I = 1\,, \qquad \Rightarrow \qquad \bar{X}_I = \frac{1}{3}\,, \tag{2.21}$$

accordingly, the corresponding kinetic matrix in the same vacuum becomes

$$\bar{Q}_{IJ} = \frac{1}{2}\,\mathcal{I}_{3\times 3}\,. \tag{2.22}$$

Since this theory is obtained by setting $n_V = 2$, the most general set of conserved charges a black hole can present is given by the energy, three electric charges and two angular momenta. Imposing supersymmetry and extremality there are two more relations to be satisfied and therefore the maximum number of independent charges a BPS solution can present is equal to four.

We conclude this section by briefly reviewing how we can embed $\mathcal{N} = 2$, $D = 5$ supergravity with $\mathrm{U}(1)^3$ gauge group in ten-dimensional type IIB supergravity following [104]. Starting from type IIB on $\mathrm{AdS}_5 \times S^5$, we can have a consistent truncation turning on the $\tau = C_0 + ie^{-\Phi}$, C_4 and g_{MN} fields, where $x^M = (x^\mu, y^a)$ with x^μ being the AdS_5 coordinates and $y^a = (\tilde{\theta}, \tilde{\psi}, \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3)$ being the S^5 coordinates. In this set of coordinates the round S^5 is

$$\mathrm{d}\Omega_5^2 = \sum_i \left(\mathrm{d}\mu_i^2 + \mu_i\,\mathrm{d}\tilde{\phi}_i^2\right). \tag{2.23}$$

Then we have:

$$\mathrm{d}s_{10}^2 = \sqrt{\widetilde{\Delta}}\,\mathrm{d}s_5^2 + \frac{\ell^2}{\sqrt{\widetilde{\Delta}}}\,\mathrm{d}\tilde{s}_5^2\,, \qquad \widetilde{\Delta} = \sum_{i=1}^3 X_i \mu_i\,, \tag{2.24a}$$

$$\mathrm{d}\tilde{s}_5^2 = G_{ab}\mathrm{d}y^a\mathrm{d}y^b = \sum_i X_i^{-1}\left[\mathrm{d}\mu_i^2 + \mu_i^2\left(\mathrm{d}\tilde{\phi}_i^2 + \ell^{-1}A_i^{(1)}\right)^2\right], \tag{2.24b}$$

$$\mu_1 = \sin\tilde{\theta}\,, \quad \mu_2 = \cos\tilde{\theta}\sin\tilde{\psi}\,, \quad \mu_3 = \sin\tilde{\theta}\cos\tilde{\psi}\,, \tag{2.24c}$$

$$X_i = e^{-\frac{1}{2}\boldsymbol{a}_i\cdot\boldsymbol{\varphi}}\,, \quad X_1X_2X_3 = 1\,, \quad F^i_{(2)} = \mathrm{d}A^i_{(1)}\,, \tag{2.24d}$$

$$a_1 = \left(\frac{2}{\sqrt{6}}\,, +\sqrt{2}\right), \quad a_2 = \left(\frac{2}{\sqrt{6}}\,, -\sqrt{2}\right), \quad a_3 = \left(-\frac{4}{\sqrt{6}}\,, 0\right), \tag{2.24e}$$

$$C_2 = 0 = B_2\,, \quad F_5 = \mathrm{d}C_4 = G_5 + \star_{10}G_5\,, \tag{2.24f}$$

$$\begin{aligned} G_5 = {} & 2\ell^{-1}\sum_i\left(X_i\mu_i^2 - \widetilde{\Delta}\,X_i\right)\mathrm{vol}_5 - \frac{\ell}{2}\sum_i \star_5\mathrm{d}\log X_i \wedge \mathrm{d}\mu_i^2 \\ & + \frac{\ell^2}{2}\sum_i \mathrm{d}\mu_i^2 \wedge \left(\mathrm{d}\tilde{\phi}_i^2 + \ell A_i^{(1)}\right)\wedge\star_5 F^i_{(2)}\,, \end{aligned} \tag{2.24g}$$

where $\star_5 \equiv \star$ and $\mathrm{v}ol_5$ are referred to the AdS_5 metric. In the formulae above X_i are variables which are parametrized with respect to the real scalar fields φ. Considering the only relevant part of the usual type IIB lagrangian

$$\mathcal{L}_{\text{IIB}} \supseteq \sqrt{-g_E}\left[R_E - \frac{1}{2}\frac{\partial\tau\cdot\partial\bar{\tau}}{\operatorname{Im}\tau} - \frac{1}{2}F_5\wedge\star_{10}F_5\right], \tag{2.25}$$

and inserting the ansatz (2.24), with an appropriate field-redefinition, we land to eq. (2.2).

2.1.2 *Finite-temperature AdS_5 black holes*

Here we present the finite-temperature AdS_5 black hole solutions which are relevant for this work of thesis and we briefly discuss their thermodynamic properties. For finite-temperature black holes, we directly assume $n_V = 2$ and work in the $\text{U}(1)^3$ theory; while for BPS black holes we will also consider the general case of n_V arbitrary as we shall see in the following.

Since we assume $n_V = 2$, the most general black hole solution can present a total of six independent conserved charges: the energy, two angular momenta and three electric charges. Such a solution has been constructed in [108]; however the expressions of the physical fields and properties of this general solution are quite involved and cumbersome, thus we choose to work in a slightly less general case. In particular we assume the angular momenta to be equal and the three charges to be independent; a solution with these properties was originally constructed in [109] and further discussed in [110]. This solution contains the three-parameter BPS black hole of [42], as we will see in detail later[25]. Another interesting solution is the one obtained by setting the three charges equal and leaving the two angular momenta unconstrained; this solution belongs to minimal gauged supergravity and has been constructed in [44] and extensively examined in [66]. This solution becomes the one of [41] in the BPS limit.

We now present the non-supersymmetric, finite temperature solution of [109], following the notation of [110]. The five-dimensional action (2.2) can be rewritten for the $\text{U}(1)^3$ theory in the following fashion

$$\mathcal{S} = \frac{1}{16\pi}\int\Bigg[\left(R + 4g^2\sum_{I=1}^{3}\left(X^I\right)^{-1} - \frac{1}{2}\partial\vec{\phi}^{\,2}\right)\star 1 \\ - \frac{1}{2}\sum_{I=1}^{3}\left(X^I\right)^{-2}F^I\wedge\star F^I - \frac{1}{6}\left|\epsilon_{IJK}\right| A^I\wedge F^J\wedge F^K\Bigg], \tag{2.26}$$

where A^I, $I = 1,2,3$, are the three Abelian gauge fields, with field strength $F^I = \mathrm{d}A^I$, while $\vec{\phi} = (\phi_1,\phi_2)$ are real scalar fields and we have parametrized the three variables X^I as

$$X^1 = \mathrm{e}^{-\frac{1}{\sqrt{6}}\phi_1 - \frac{1}{\sqrt{2}}\phi_2}, \qquad X^2 = \mathrm{e}^{-\frac{1}{\sqrt{6}}\phi_1 + \frac{1}{\sqrt{2}}\phi_2}, \qquad X^3 = \mathrm{e}^{\frac{2}{\sqrt{6}}\phi_1}. \tag{2.27}$$

25 Solutions with restricted set of independent charges were also found in [43,111,112].

Furthermore, we have defined $g = \frac{1}{\ell}$ and we have made use of (2.18). We choose the set of coordinates $(t, r, \theta, \phi, \psi)$ to express the solution and, in order to make the SU(2)× U(1) symmetry manifest, we introduce the following left-invariant 1-forms on a three-sphere S^3 parameterized by (θ, ϕ, ψ):

$$\begin{aligned} \sigma_1 + i\,\sigma_2 &= \mathrm{e}^{-i\psi}(\mathrm{d}\theta + i\sin\theta\,\mathrm{d}\phi)\,, \\ \sigma_3 &= \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\,. \end{aligned} \tag{2.28}$$

The metric, scalar fields and gauge fields of the solution of [109] are given by

$$\begin{aligned} \mathrm{d}s_5^2 = (H_1 H_2 H_3)^{1/3}\bigg[&-\frac{r^2\,Y}{f_1}\mathrm{d}t^2 + \frac{r^4}{Y}\mathrm{d}r^2 \\ &+ \frac{r^2}{4}\Big(\sigma_1^2+\sigma_2^2\Big) + \frac{f_1}{4r^4 H_1 H_2 H_3}\Big(\sigma_3 - \frac{2f_2}{f_1}\mathrm{d}t\Big)^2\bigg], \end{aligned} \tag{2.29}$$

$$X^I = \frac{(H_1 H_2 H_3)^{1/3}}{H_I}\,, \tag{2.30}$$

$$A^I = A^I_t\,\mathrm{d}t + A^I_\psi\,\sigma_3\,, \tag{2.31}$$

where

$$A^I_t = \frac{2\,m}{r^2\,H_I}\,s_I\,c_I + \alpha^I\,, \qquad A^I_\psi = \frac{m\,\mathfrak{a}}{r^2\,H_I}\,\left(c_I\,s_J\,s_K - s_I\,c_J\,c_K\right)\,, \tag{2.32}$$

and the indices I, J, K in A^I_ψ are never equal. In the temporal component of the gauge fields we also introduced a constant gauge choice α^I, that will be fixed in the following. The solution depends on the following functions of the radial coordinate r:

$$\begin{aligned} H_I &= 1 + \frac{2\,m\,s_I^2}{r^2}\,, \\ f_1 &= r^6 H_1 H_2 H_3 + 2\,m\,\mathfrak{a}^2 r^2 + 4\,m^2\mathfrak{a}^2\bigg[2\left(c_1c_2c_3 - s_1s_2s_3\right)s_1s_2s_3 \\ &\qquad\qquad - s_1^2s_2^2 - s_2^2s_3^2 - s_3^2s_1^2\bigg], \\ f_2 &= 2\,m\,\mathfrak{a}\left(c_1c_2c_3 - s_1s_2s_3\right)\,r^2 + 4\,m^2\mathfrak{a}\,s_1s_2s_3\,, \\ f_3 &= 2\,m\,\mathfrak{a}^2(1+g^2r^2) + 4\,g^2m^2\mathfrak{a}^2\bigg[2(c_1c_2c_3 - s_1s_2s_3)s_1s_2s_3 \\ &\qquad\qquad - s_1^2s_2^2 - s_2^2s_3^2 - s_3^2s_1^2\bigg], \\ Y &= f_3 + g^2r^6H_1H_2H_3 + r^4 - 2\,m\,r^2\,, \end{aligned} \tag{2.33}$$

with s_I, c_I being shorthand notations for:

$$s_I = \sinh\delta_I\,, \qquad c_I = \cosh\delta_I\,, \qquad I = 1,2,3\,. \tag{2.34}$$

Looking at the relevant fields above, it is easy to see that the solution depends on the five parameters m, δ_1, δ_2, δ_3, $\mathfrak{a}$; this number of parameters is expected since it is equal to the number of independent conserved charges. The horizons are given by the roots of $Y(r)$; we demand that the spatial components of the metric are positive for $r > r_+$, where r_+ is the largest positive root of this function and thus denotes the position of the outer horizon. This implies that the five parameters should satisfy suitable inequalities. Provided this, the outer horizon is actually a Killing horizon since the Killing vector

$$V = \frac{\partial}{\partial t} + 2\,\frac{f_2(r_+)}{f_1(r_+)}\,\frac{\partial}{\partial \psi} \tag{2.35}$$

is null at $r = r_+$. The following physical properties can be associated to the outer horizon

$$\begin{aligned}
S &= \frac{\pi^2}{2}\sqrt{f_1(r_+)}\,, \qquad \beta = 4\,\pi\,r_+\sqrt{f_1(r_+)}\left(\frac{\mathrm{d}Y}{\mathrm{d}r}(r_+)\right)^{-1}\,,\\
\Omega &= 2\,\frac{f_2(r_+)}{f_1(r_+)}\,,\\
\Phi^I &= \frac{2\,m}{r_+^2\,H_I(r_+)}\Big(s_I\,c_I + \tfrac{1}{2}\,\mathfrak{a}\,\Omega\,(c_I\,s_J\,s_K - s_I\,c_J\,c_K)\Big)\,,
\end{aligned} \tag{2.36}$$

with S being the Bekenstein-Hawking entropy computed as $\frac{1}{4}$ the area of the horizon, $\beta = T^{-1} = \frac{2\pi}{\kappa}$ being the inverse Hawking temperature obtained from the surface gravity κ, Ω being the angular velocity relative to a non-rotating frame at infinity as read off from the Killing vector V, and Φ^I being the electrostatic potentials,[26] defined as

$$\Phi^I = \iota_V A^I|_{r_+} - \iota_V A^I|_{\infty}\,. \tag{2.37}$$

The five independent conserved charges are given by the energy E for translations along $\frac{\partial}{\partial t}$, the angular momentum J for rotations along $-\frac{\partial}{\partial \psi}$, and three electric charges Q_I. They result to be:

$$\begin{aligned}
E &= E_0 + \frac{1}{4}\,m\,\pi\left(3 + \mathfrak{a}^2 g^2 + 2\,s_1^2 + 2\,s_2^2 + 2\,s_3^2\right)\,,\\
J &= \frac{1}{2}\,m\,\mathfrak{a}\,\pi\,\left(c_1\,c_2\,c_3 - s_1\,s_2\,s_3\right)\,,\\
Q_I &= \frac{1}{2}\,m\,\pi\,s_I\,c_I\,.
\end{aligned} \tag{2.38}$$

The electric charges and the angular momentum have been computed in [110] using the boundary integrals

$$\begin{aligned}
Q_I &= -\frac{1}{16\pi}\int_{S^3_{\mathrm{bdry}}}\left(X^I\right)^{-2}\star F^I\,,\\
J &= \frac{1}{16\pi}\int_{S^3_{\mathrm{bdry}}}\star\,\mathrm{d}\left(g_{\psi\mu}\mathrm{d}x^\mu\right)\,,
\end{aligned} \tag{2.39}$$

26 In eq. (3.10) of [110] there was a minus sign typo in the expression for Φ^I; here we have corrected this, see also [67, 69].

while the energy E was obtained by integrating the first law of thermodynamics,

$$\mathrm{d}E = T\,\mathrm{d}S + \Omega\,\mathrm{d}J + \Phi^I\,\mathrm{d}Q_I\,. \tag{2.40}$$

The integration constant E_0 was fixed to zero in [110] by requiring that E vanishes in the limiting case $m = 0$ where the solution becomes empty AdS$_5$, which is regarded as the vacuum solution (see [113] for more details on this approach to compute the energy).

There are other approaches one can adopt to compute the same charges. One possibility is to use the framework of holographic renormalization [114–116]. We perform the computation of the charges using this framework in app. A, adopting a minimal subtraction scheme. As expected from the analysis of [117], we find agreement with the expressions above for the angular momentum J and the electric charges Q_I. The energy E also agrees, except that the AdS mass E_0 now takes the non-vanishing value

$$E_0 = \frac{3\,\pi}{32\,g^2}\,. \tag{2.41}$$

The thermodynamical quantities introduced above must satisfy the *quantum statistical relation*

$$I = \beta E - S - \beta\,\Omega\,J - \beta\,\Phi^I\,Q_I\,, \tag{2.42}$$

with I being the on-shell action of the solution under consideration. In order to provide an interpretation for the extremization principle for AdS$_5$ black holes, we will need to evaluate this quantity. This has been computed for the first time in [67], again using holographic renormalization. The action must be evaluated on a regular Euclidean section of the solution. The Euclideanization is obtained by the Wick rotation $t = -i\tau$, together with the continuation of the parameter $\mathfrak{a}$ to purely imaginary values. After the action is computed one can take $\mathfrak{a}$ back to the original real domain, or choose to analytically continue the solution to more general complex values of the parameters [118]. As usual, regularity of the Euclidean section leads to identify the length of the circle parameterized by the Euclidean time τ with the inverse Hawking temperature, that is $\int \mathrm{d}\tau = \beta$. A further regularity condition is that the contraction of the Killing vector (2.35) with the gauge fields vanishes at the horizon,

$$\iota_V\,A^I|_{r=r_+} = 0\,. \tag{2.43}$$

This leads us to fix the constant gauge choice α^I introduced in (2.32) as

$$\alpha^I = -\Phi^I\;, \tag{2.44}$$

where Φ^I is the electrostatic potential (2.36). As for the charges, we describe this computation in detail in app. A while here we just provide the final result

$$\begin{aligned} I = I_0 - \frac{\pi\beta}{12}\Big[& 2m\left(c_1 s_1 \Phi^1 + c_2 s_2 \Phi^2 + c_3 s_3 \Phi^3\right) \\ & + 4m^2 g^2\left(s_1^2 s_2^2 + s_1^2 s_3^2 + s_2^2 s_3^2\right) + 3m(g^2 a^2 - 1) \\ & + 3g^2 r_+^4 + 2m\left(2g^2 r_+^2 - 1\right)\left(s_1^2 + s_2^2 + s_3^2\right)\Big] , \end{aligned} \tag{2.45}$$

where

$$I_0 = \beta E_0 \tag{2.46}$$

is the on-shell action of empty AdS$_5$ at temperature β. Using the above result for the on-shell action I and the expressions for the thermodynamical quantities provided in (2.36), (2.38), it is easy to check that the quantum statistical relation (2.42) is indeed satisfied. This equation has also a microscopic interpretation: it can be seen as the relation between a grand-canonical partition function $I = -\log Z_{\text{grand}}$, seen as a function of the chemical potentials, $I = I(\beta, \Omega, \Phi^I)$, and the microcanonical partition function $S = \log Z_{\text{micro}}$, seen as a function of the charges $S = S(E, J, Q_I)$. The latter can be obtained by varying I with respect to the chemical potentials as

$$E = \frac{\partial I}{\partial \beta} , \qquad J = -\frac{1}{\beta}\frac{\partial I}{\partial \Omega} , \qquad Q_I = -\frac{1}{\beta}\frac{\partial I}{\partial \Phi^I} . \tag{2.47}$$

Two conclusive remarks are in order.

The first one is about the contribution E_0 to the energy and the corresponding I_0 in the on-shell action. These two are not fixed, but their specific values depend on the regularization adopted. Indeed, if we had performed the computation by using the background subtraction method as done for similar solutions in e.g. [113, 119, 120], we would have found the same result (2.45), but with $I_0 = 0$. This is because this method regularizes the divergences due to the infinite spacetime volume by subtracting the action of empty AdS space, with a boundary at large distance $\bar{r}$ matched to the boundary of the black hole solution, and then sends $\bar{r} \to \infty$. In this way the action I is measured relative to the action of the AdS vacuum which results from taking $m = 0$. Therefore in this approach $I_0 = 0$ by construction. However the quantum statistical relation is still satisfied, provided one sets $E_0 = 0$ for consistency. This regularization is different from the one associated to holographic renormalization; in this framework one can shift E_0 (and I_0) to any desidered value by adding a finite, local counterterm to the action. In particular, we can add to the Lorentzian action the finite counterterm $\frac{\varsigma}{8\pi}\int \mathrm{d}^4 x \sqrt{h} R^2$, where ς is a parameter, h_{ij} is the boundary metric and R its Ricci curvature, so to obtain the shift one obtains the shift $I_0 \to I_0 - 9\pi g\beta\varsigma$ and $E_0 \to E_0 - 9\pi g\varsigma$.[27] Because of the fact that E_0 and I_0 can be shifted to any desired arbitrary constant via a local counterterm, they may be considered ambiguous quantities, without an intrinsic physical meaning. This is confirmed by the AdS/CFT correspondence, which identifies E_0 with the Casimir energy of the dual conformal field theory on $S^3 \times \mathbb{R}$, that does not have intrinsic value [87]. However we are mainly interested in supersymmetric solutions and the presence of supersymmetry changes the above situation. Indeed, the R^2 counterterm we have introduced is not allowed in a supersymmetric setup and thus E_0 acquires physical meaning [87, 121]. In [122, 123] E_0 has been interpreted as the consequence of a supercurrent anomaly, which is physical in nature. It may be possible to see this as a mixed anomaly and thus shift it away by adding local counterterms that restore supersymmetry at the expense of breaking part of the diffeomorphisms, along the lines of [124–126].

The second remark concerns the fact that in principle, in the integral for the electric charges (2.39) there should also be a contribution given by the Chern-Simons term in the action (2.26); however the latter vanishes in the solution of interest because $F^I \to 0$ as $r \to \infty$, so we omitted this term. Nevertheless, it is worth remarking that it implies a priori different definitions of the electric charge such as the Maxwell charge, the Page charge, and the charge that arises from integrating the holographic currents; these charges are indeed different when there is a non vanishing Chern-Simons term. This is the case, for example, of the AlAdS$_5$ black holes constructed in [99–101]; we will return to this point extensively in Chapter 4 when we will turn to examine such black holes.

2.1.3 *BPS AdS$_5$ black holes*

In this subsection we provide a detailed presentation of the first and best known BPS AdS$_5$ black holes: the ones found by Gutowski and Reall in [41,42]. The first one is a solution to the $\mathcal{N} = 2$ Fayet-Iliopoulos gauged supergravity under consideration, while the second one is a solution to $\mathcal{N} = 2$ minimal gauged supergravity. The Gutowski-Reall black holes are fundamental for this work of thesis, since the BPS limit of the family of solutions of [110] introduced in the last subsection coincides with the solution of [42]; furthermore the AlAdS$_5$ solutions constructed in [99–101] can be regarded as deformations of the solutions of [41] and [42].

Before starting with the review, we remind one more time that we denote as *BPS black holes* the ones which are *both supersymmetric and*

27 This counterterm also yields a trivial $-\frac{3\varsigma}{2\pi}\nabla^2 R$ contribution to the trace of the energy-momentum tensor, while the "minimal subtraction" scheme that we used to reach (2.41) is characterized by the fact that the trace of the holographic energy-momentum tensor does not contain trivial $\nabla^2 R$ terms.

extremal and that neither of these two properties automatically implies the other. This will be made clearer in the following.

2.1.3.1 *The Gutoswki-Reall black holes*

Historically, the first rotating AdS_5 BPS black hole which has been found is the minimal gauged supergravity one of [41]; some days later the more general solution in Fayet-Iliopoulos gauged supergravity has been presented in [42]. Here we will start by the more general solution of [42], then we discuss how it is possible to recover the minimal gauged supergravity solution.

The black hole of [42] has been constructed for an arbitrary number of vector multiplets n_V and it is a solution of the action (2.2). It is obtained by assuming the property (2.3), i.e. that the scalar target space is symmetric. We now briefly review how this solution has been obtained.

The authors of [42] are interested in bosonic, supersymmetric solutions with a local SU(2)× U(1)× U(1) symmetry. As a consequence of supersymmetry, the existence of a Killing vector is guaranteed and the form of the solutions depends on whether it is timelike or null; here we will just consider the timelike case. Furthermore, as a consequence of the additional SU(2)× U(1) symmetry, the supersymmetry conditions reduce to ODE's. The necessary and sufficient conditions for solutions of this type have been provided in [42]; we shall now summarize them and review how they can be solved to find the black hole solution.

The starting point to obtain these conditions is to analyse the differential forms that can be constructed from a commuting Killing spinor; in the spirit of [127] the authors search for algebraic and differential properties of these forms. Then, they specialize the conditions so obtained to the timelike case we are interested in. For this latter, we consider the set of coordinates $(y, \rho, \theta, \phi, \hat{\psi})$ and we introduce the SU(2) left-invariant one-forms

$$\begin{aligned}
\hat{\sigma}_1 &= \cos\hat{\psi}\,\mathrm{d}\theta + \sin\hat{\psi}\sin\theta\,\mathrm{d}\phi\ , \\
\hat{\sigma}_2 &= -\sin\hat{\psi}\,\mathrm{d}\theta + \cos\hat{\psi}\sin\theta\,\mathrm{d}\phi\ , \\
\hat{\sigma}_3 &= \mathrm{d}\hat{\psi} + \cos\theta\,\mathrm{d}\phi\ ,
\end{aligned} \tag{2.48}$$

satisfying $\mathrm{d}\hat{\sigma}_1 = -\hat{\sigma}_2 \wedge \hat{\sigma}_3$, $\mathrm{d}\hat{\sigma}_2 = -\hat{\sigma}_3 \wedge \hat{\sigma}_1$, $\mathrm{d}\hat{\sigma}_3 = -\hat{\sigma}_1 \wedge \hat{\sigma}_2$. The hat symbol on $\hat{\psi}$ (and thus on the σ's) distinguishes this coordinate from a different coordinate ψ which is relevant for $AlAdS_5$ solutions and will be introduced later in Chapter 4.

Supersymmetry equations

Supersymmetry fixes the timelike Killing vector as $V = \frac{\partial}{\partial y}$, while the left-invariant vector $\frac{\partial}{\partial \hat{\psi}}$ generates the other Abelian symmetry. The following form for the five-dimensional metric is assumed[28]

$$\mathrm{d}s^2 = -f^2(\mathrm{d}y + w\,\hat{\sigma}_3)^2 + f^{-1}\left[\mathrm{d}\rho^2 + a^2(\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + (2aa')^2\,\hat{\sigma}_3^2\right], \tag{2.49}$$

where a, w, f are functions of the radial coordinate ρ, and throughout this subsection a prime denotes differentiation with respect to ρ. The square parenthesis contains a Kahler metric on a four-dimensional base space $\mathcal{B}$; this is requied by supersymmetry[29]. The scalar fields depend on the ρ coordinate only

$$X^I = X^I(\rho)\,, \tag{2.50}$$

while the gauge fields are given by

$$A^I = X^I f\,(\mathrm{d}y + w\,\hat{\sigma}_3) + U^I \hat{\sigma}_3\,, \tag{2.51}$$

with $U^I(\rho)$ being additional functions that depend only on the radial coordinate ρ.

To find a solution, one has to determine $a(\rho), w(\rho), f(\rho), X^I(\rho), U^I(\rho)$. The analysis of the supersymmetry conditions, developed by combining also these latter with the Maxwell equation, performed in [42] results in the following set of equations

$$f = f_{\min}\,\bar{X}^I X_I\,, \tag{2.52}$$

$$\left(a^2 U^I\right)' = 36\frac{\epsilon}{\ell}\,a^3 a' f^{-1}\,C^{IJK}\bar{X}_J X_K\,, \tag{2.53}$$

$$f^{-1}X_I\left(a^{-2}U^I\right)' = -\frac{2}{3}\left(a^{-2}w\right)'\,, \tag{2.54}$$

$$\bar{X}_I U^I = \frac{\epsilon\ell}{3}\,p\,, \tag{2.55}$$

$$\left[a^3 a'\left(f^{-1}X_I\right)' + \frac{\epsilon}{\ell}\bar{X}_I a^2 w + \frac{1}{12}C_{IJK}U^J U^K\right]' = 0\,, \tag{2.56}$$

in the expressions above $\epsilon = \pm 1$ is just an arbitrary sign choice related to the versus of rotation of the solution along $\frac{\partial}{\partial \hat{\psi}}$. We have furthermore denoted as $f_{\min}$ the function[30]

$$f_{\min} = \frac{12\,a^2 a'}{\ell^2(a^2 a''' - a' + 7aa'a'' + 4(a')^3)} \tag{2.57}$$

28 In [42] the authors work with a negative signature metric while here we present their solution using a positive signature one; this is because the mostly plus signature is the one we will always use in the thesis. As a consequence, there are some quantities we introduce below that differ for an overall minus sign when compared to [42].

29 Even though there are in general obstructions that prevent a Kahler metric to provide supersymmetric solutions [128, 129], these are automatically solved by the symmetric ansatz (2.49).

30 This function has a geometrical meaning: it is proportional to the inverse scalar curvature $R_{\mathcal{B}}$ of the four-dimensional Kahler base, i.e. it results $f_{\min} = -\frac{24}{\ell^2 R_{\mathcal{B}}}$.

which is the expression for f that is obtained when working in minimal gauged supergravity [41]. The last function we have introduced is in eq. (2.55) and it results

$$p = -1 + 2aa'' + 4(a')^2 \,. \tag{2.58}$$

This function fulfils the following identity[31]

$$a^3 a' f_{\rm min}^{-1} = \frac{\ell^2}{24}\left(a^2 p\right)' \,. \tag{2.59}$$

Derivation of the black hole solution

We now review how Gutowski and Reall solved the system of equations provided above and constructed the BPS black hole. The key ingredient to solve the system is to make the following guess

$$f^{-1} X_I = \bar{X}_I + \frac{q_I^{\rm GR}}{4\,a^2} \,, \tag{2.60}$$

with $q_I^{\rm GR}$ being constants. As a consequence of (2.4) and (2.1), it must be

$$C^{IJK} X_I X_J X_K = \frac{2}{9} \,, \tag{2.61}$$

combining this with eq. (2.60) we easily get

$$f^{-3} = \frac{9}{2} C^{IJK}\left(\bar{X}_I + \frac{q_I^{\rm GR}}{4\,a^2}\right)\left(\bar{X}_J + \frac{q_J^{\rm GR}}{4\,a^2}\right)\left(\bar{X}_K + \frac{q_K^{\rm GR}}{4\,a^2}\right) \,, \tag{2.62}$$

from which we obtain the expression for f

$$f = \left(1 + \frac{\alpha_1^{\rm GR}}{4\,a^2} + \frac{\alpha_2^{\rm GR}}{16\,a^4} + \frac{\alpha_3^{\rm GR}}{64\,a^6}\right)^{-1/3} \,, \tag{2.63}$$

where we have defined the $\alpha_i^{\rm GR}$ constants as

$$\begin{aligned}
\alpha_1^{\rm GR} &= \frac{27}{2} C^{IJK} \bar{X}_I \bar{X}_J q_K^{\rm GR} \,, \\
\alpha_2^{\rm GR} &= \frac{27}{2} C^{IJK} \bar{X}_I q_J^{\rm GR} q_K^{\rm GR} \,, \\
\alpha_3^{\rm GR} &= \frac{9}{2} C^{IJK} q_I^{\rm GR} q_J^{\rm GR} q_K^{\rm GR} \,.
\end{aligned} \tag{2.64}$$

Plugging (2.60) in the equation for U^I given by (2.53), the latter becomes a total derivative and it is possible to solve for those functions as

$$U^I = \frac{9\,\epsilon}{\ell} C^{IJK} \bar{X}_J \left(a^2 \bar{X}_K + \frac{q_K^{\rm GR}}{2}\right) \,, \tag{2.65}$$

31 The function p determines the Ricci form on the Kähler base as $\mathcal{R} = \epsilon\, \mathrm{d}(p\,\hat{\sigma}_3)$. The identity (2.59) expresses the fact that in Kähler geometry the trace of the Ricci form is proportional to the Ricci scalar, $J^{mn}\mathcal{R}_{mn} = R$. Here $J = -\epsilon\, \mathrm{d}(a^2 \hat{\sigma}_3)$ is the Kähler form on the Kähler base $\mathcal{B}$ [42].

where we have set to zero a possible term of the form constant of integration times a^{-2} for compatibility with the black hole solution of the minimal theory constructed in [41]. Having the explicit expressions of the U^I functions, we can determine w from (2.54):

$$w = \frac{\epsilon}{\ell}\left(w_0\, a^2 - \frac{\alpha_1^{\rm GR}}{2} - \frac{\alpha_2^{\rm GR}}{16\, a^2}\right) , \tag{2.66}$$

with w_0 being a constant of integration. Now eq. (2.55) can be integrated twice to obtain

$$a = \frac{\ell}{2}\sqrt{1 + \frac{\alpha_1^{\rm GR}}{\ell^2}}\,\sinh\left(\frac{\rho}{\ell}\right) . \tag{2.67}$$

It is remarkable that the geometry of the base space, determined by the a function found above, is constituted by the same singular deformation of the Bergmann manifold one can found in the minimal theory [41]. We have one last equation to satisfy, which is (2.56). Plugging in all the results we have obtained for the various functions and using the property (2.3), it can be shown that this equation is equivalent to the condition

$$w_0 = -2\,, \tag{2.68}$$

which fixes the integration constant and therefore the w function as

$$w = -\frac{\epsilon}{\ell}\left(2\, a^2 + \frac{\alpha_1^{\rm GR}}{2} + \frac{\alpha_2^{\rm GR}}{16\, a^2}\right) . \tag{2.69}$$

Properties of the solution

The solution is controlled by the $n_V + 1$ constants $q_I^{\rm GR}$, with the $\alpha_i^{\rm GR}$ determined by these using eq. (2.64). To better describe the horizon geometry and to better investigate which solutions have regular horizons, it is convenient to switch to gaussian null coordinates adapted to the supersymmetric Killing vector field V. The appropriate coordinate transformations are given by:

$$\begin{aligned}
{\rm d}y &= {\rm d}u + \left(\frac{f w^2}{(2aa')^2} - \frac{1}{f^2}\right){\rm d}\tilde\rho\ ,\\
{\rm d}\hat\psi &= {\rm d}\tilde\psi - \frac{f\, w}{(2aa')^2}\,{\rm d}\tilde\rho\ ,\\
{\rm d}\rho &= \sqrt{\frac{1}{f} - \frac{f^2 w^2}{(2aa')^2}}\,{\rm d}\tilde\rho\,,
\end{aligned} \tag{2.70}$$

using them, the original five-dimensional metric (2.49) becomes

$$\begin{aligned}
{\rm d}s^2 = &-f^2{\rm d}u^2 + 2\,{\rm d}u\,{\rm d}\tilde\rho - 2f^2 w\,{\rm d}u\,\tilde\sigma_3\\
&+ f^{-1}a^2(\sigma_1^2+\sigma_2^2) + \left(f^{-1}(2aa')^2 - f^2w^2\right)\tilde\sigma_3^2\,.
\end{aligned} \tag{2.71}$$

To guarantee a regular horizon at $\tilde{\rho} = 0$, we must require that $f^{-1}a^2$ approaches a positive constant and that $f^{-1}(2aa')^2 - f^2 w^2$ does the same. These two conditions imply respectively

$$\alpha_3^{\rm GR} > 0\,, \tag{2.72}$$

$$\alpha_3^{\rm GR}\left(1+\frac{\alpha_1^{\rm GR}}{\ell^2}\right) - \frac{\left(\alpha_2^{\rm GR}\right)^2}{4\,\ell^2} > 0\,. \tag{2.73}$$

Having the $\alpha_i^{\rm GR}$ fulfilling the above inequalities guarantees that the solution has a regular horizon; in order to avoid problems outward the horizon, for $\tilde{\rho} > 0$, one must impose further restrictions on the $\alpha_i^{\rm GR}$ which we do not report here[32].

The Bekenstein-Hawking entropy can be computed as $\frac{1}{4}$ the area of the horizon[33] and results to be

$$S = \frac{\pi^2}{2}\sqrt{\alpha_3^{\rm GR}\left(1+\frac{\alpha_1^{\rm GR}}{\ell^2}\right) - \frac{(\alpha_2^{\rm GR})^2}{4\,\ell^2}}\,, \tag{2.74}$$

note that, due to the second inequality of (2.72), the entropy is always real and well defined. The angular velocity of the event horizon calculated with respect to a stationary frame at infinity is[34]

$$\Omega_H = \frac{2\,\epsilon}{\ell}\,. \tag{2.75}$$

The energy and the angular momentum of the solutions are evaluated in [42] using the Ashtekar and Das (AD) approach [131] and they assume the following values[35]

$$E = \frac{\pi}{4}\left(\alpha_1^{\rm GR} + \frac{3\,\alpha_2^{\rm GR}}{2\,\ell^2} + \frac{2\,\alpha_3^{\rm GR}}{\ell^4}\right)\,, \tag{2.76}$$

$$J = \frac{\epsilon\,\pi}{8\,\ell}\left(\alpha_2^{\rm GR} + \frac{2\,\alpha_3^{\rm GR}}{\ell^2}\right)\,. \tag{2.77}$$

The conserved electric charges are defined as[36]

$$Q_I = -\frac{1}{8\,\pi}\int_{S^3_{\rm bdry}} Q_{IJ} \star F^J\,, \tag{2.78}$$

32 Note that they are neither reported in the original paper [42].
33 Here we set the five-dimensional gravitational constant to 1.
34 Note that taking the limit $\ell \to \infty$ with $q_I^{\rm GR}$ held fixed, one obtains the static BPS black holes of the ungauged supergravity theory constructed in [130].
35 As for the finite temperature case, these charges can be evaluated also using other approaches, such as holographic renormalization. Using this latter, the expectation is that the energy so evaluated differs from the AD result for an additive term which is the Casimir energy; while the angular momentum remains the same. We will discuss this topic in more details as we proceed further in the thesis.
36 Note that this definition is in agreement with (2.39), since in the $\mathrm{U}(1)^3$ theory the Q_{IJ} matrix is given by (2.20) and due to the parametrization (2.27) for the finite-temperature solution it results $X_I = \frac{1}{3}\left(X^I\right)^{-1}$.

and they evaluate to

$$Q_I = \pi \left(\frac{3}{4} q_I^{\rm GR} - \frac{3\,\alpha_2^{\rm GR}}{8\,\ell^2} \bar{X}_I + \frac{9}{8\,\ell^2} C_{IJK} \bar{X}^J C^{KLM} q_L^{\rm GR} q_M^{\rm GR} \right) . \quad (2.79)$$

By looking at (2.76), (2.77), (2.79) one can establish that the following BPS equality does hold

$$E - \frac{2}{\ell}|J| - |\bar{X}^I Q_I| = 0\,. \quad (2.80)$$

The non-extremal generalization of the black hole solution presented here is known only in the $n_V = 2$ case and coincides with the solution of [109] we presented in the last subsection. Below, we will explicitly see how starting from the latter solution one can recover the BPS black hole under consideration. The solution of [108] is a further generalization with respect to the one of [109], since it presents different angular momenta. The BPS generalization of the black hole of [42], presenting two different angular momenta and retaining three independent conserved charges, is instead known for any n_V and it coincides with the solution of [45].

The $n_V = 2$ black hole of the $U(1)^3$ theory

When $n_V = 2$, the black hole solution presented above is a solution of the $\mathrm{U}(1)^3$ theory; in this case the index I runs from 1 to 3. In this paragraph we briefly describe this particular solution. To ease the notation, we assume $\ell = 1$ and $\epsilon = 1$.

For convenience, we define rescaled parameters for the $n_V = 2$ black hole solution as:

$$q_I^{\rm GR} = \frac{\mu_I}{3} \quad \text{with } I = 1, 2, 3\,, \quad (2.81)$$

recalling the form of the C_{IJK} tensor in this theory (2.18) and the definitions of the $\alpha_i^{\rm GR}$ (2.64), it is easy to see that

$$\alpha_1^{\rm GR} = \mu_1 + \mu_2 + \mu_3\,, \quad \alpha_2^{\rm GR} = \mu_1\,\mu_2 + \mu_2\,\mu_3 + \mu_3\,\mu_1\,, \quad \alpha_3^{\rm GR} = \mu_1\,\mu_2\,\mu_3\,. \quad (2.82)$$

We then define the functions:

$$H_I^{\rm GR} = 1 + \frac{\mu_I}{4\,a^2}\,, \quad (2.83)$$

these allow to write f in (2.63) as

$$f = \left(H_1^{\rm GR}\, H_2^{\rm GR}\, H_3^{\rm GR} \right)^{-1/3}, \quad (2.84)$$

and the lower-index scalars as

$$X_I = \frac{1}{3} H_I^{\rm GR} \left(H_1^{\rm GR}\, H_2^{\rm GR}\, H_3^{\rm GR} \right)^{-1/3}. \quad (2.85)$$

The regularity conditions (2.72) to be imposed in order to have a regular horizon translate in the $\mathrm{U}(1)^3$ theory into the following restrictions on the μ_I:

$$\begin{aligned}&\mu_I > 0 \quad \text{for any } I = 1,2,3\,,\\ &4\,\mu_1\,\mu_2\,\mu_3\,(\mu_1+\mu_2+\mu_3+1) > (\mu_1\,\mu_2+\mu_2\,\mu_3+\mu_3\,\mu_1)^2\,,\end{aligned} \tag{2.86}$$

it is worth noting that this last constraint is highly non-trivial; for example it is not satisfied when $\mu_1 = \mu_2 \gg \mu_3$. In the following, we always assume that the μ_I are chosen so that the conditions (2.86) are satisfied.

The Bekenstein-Hawking entropy of the solution is

$$S = \frac{\pi^2}{4}\sqrt{4\mu_1\mu_2\mu_3\,(\mu_1+\mu_2+\mu_3+1) - (\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)^2}\,, \tag{2.87}$$

while the angular velocity and the electrostatic potentials take the fixed values

$$\Omega = 2, \qquad \Phi^I = 1\,, \tag{2.88}$$

finally the BPS charges are given by

$$\begin{aligned}E &= \frac{\pi}{4}\left(2\,\mu_1\,\mu_2\,\mu_3 + \frac{3}{2}\,(\mu_1\,\mu_2+\mu_2\,\mu_3+\mu_3\,\mu_1) + \mu_1+\mu_2+\mu_3\right),\\ J &= \frac{\pi}{8}\,(2\,\mu_1\,\mu_2\,\mu_3 + \mu_1\,\mu_2+\mu_2\,\mu_3+\mu_3\,\mu_1)\,,\\ Q_1 &= \frac{\pi}{8}\,(2\,\mu_1 + \mu_1\,\mu_2 + \mu_1\,\mu_3 - \mu_2\,\mu_3)\,,\end{aligned} \tag{2.89}$$

with Q_2, Q_3 being obtained from Q_1 by a cyclic permutation of the indices $1,2,3$. The charges satisfy the linear relation

$$E - \Omega\,J - \Phi^I\,Q_I = 0\,, \tag{2.90}$$

which is a consequence of supersymmetry; furthermore, the electric charges and angular momentum satisfy the non-linear relation

$$Q_1\,Q_2\,Q_3 + \frac{\pi}{4}\,J^2 = \left(Q_1\,Q_2+Q_2\,Q_3+Q_3\,Q_1 - \frac{\pi}{2}\,J\right)\left(Q_1+Q_2+Q_3+\frac{\pi}{4}\right), \tag{2.91}$$

which is related to well-definiteness of the horizon area, that is of the entropy. The BPS entropy can be written as a function of the charges in the suggestive form [132]

$$S = 2\pi\sqrt{Q_1\,Q_2+Q_2\,Q_3+Q_3\,Q_1 - \frac{\pi}{2}J}\,. \tag{2.92}$$

From the solution of [109] to the one of [42]

Here we show that the finite-temperature black hole solution of [109], that we have introduced above, admits a BPS limit which coincides with the solution of [42] with $n_V = 2$.

Let us begin by showing that the solution introduced in sec. 2.1.2 admits an extremal limit. To do this, we consider the function $Y(r)$ in the metric (2.29), which, being a cubic polynomial, can be written as

$$Y(r) = g^2(r^2 - r_+^2)(r^2 - r_0^2)(r^2 - r_-^2)\,, \tag{2.93}$$

where the roots $r_+^2 \geq r_0^2 \geq r_-^2$ are related to the parameters of the solution as:

$$\begin{aligned} r_+^2 + r_0^2 + r_-^2 &= -2m(s_1^2 + s_2^2 + s_3^2) - g^{-2}\ , \\ r_+^2 r_0^2 + r_0^2 r_-^2 + r_-^2 r_+^2 &= 4m^2(s_1^2 s_2^2 + s_2^2 s_3^2 + s_3^2 s_1^2) + 2m(\mathfrak{a}^2 - g^{-2})\ , \\ r_+^2 r_0^2 r_-^2 &= -8m^3 s_1^2 s_2^2 s_3^2 - g^{-2} f_3(r=0)\ . \end{aligned} \tag{2.94}$$

Recalling the expressions of the physical properties (2.36), we can note that the product of the temperature and the entropy is proportional to

$$T\,S\ =\ \frac{\pi}{8}\,\frac{Y'(r_+)}{r_+}\ =\ \frac{\pi g^2}{4}(r_+^2 - r_0^2)(r_+^2 - r_-^2)\,, \tag{2.95}$$

hence the limit in which the roots r_+^2 and r_0^2 coalesce corresponds to the extremality condition $T = 0$ (as long as the horizon area remains finite). It is important to notice that this condition does not imply supersymmetry.

However, in order to obtain a BPS black hole, we have to impose both supersymmetry and extremality. The authors of [110] found that one solution to the supergravity Killing spinor equations is guaranteed if the parameters satisfy:

$$\mathfrak{a}\,g\ =\ \frac{1}{\mathrm{e}^{\delta_1+\delta_2+\delta_3}}\,, \tag{2.96}$$

so that two supercharges are preserved. To ease the notation, from now on we set $g = 1$ in this subsection. We also trade the parameters δ_I for new parameters μ_I defined as

$$\mathrm{e}^{4\delta_I}\ =\ \frac{\mu_I\,(\mu_J+2)\,(\mu_K+2)}{(\mu_I+2)\,\mu_J\,\mu_K}\,, \tag{2.97}$$

where the indices I, J, K are never equal. We will see that these parameters coincide with the μ_I defined in the previous subsections. We can express the supersymmetry condition (2.96) in terms of the new parameters μ_I as

$$\mathfrak{a} = \left(\frac{\mu_1\,\mu_2\,\mu_3}{(\mu_1+2)\,(\mu_2+2)\,(\mu_3+2)}\right)^{1/4}\ . \tag{2.98}$$

Once this condition is imposed, closed timelike curves in the solution are avoided by taking

$$m = m_\star \equiv \frac{1}{2}\sqrt{\mu_1\mu_2\mu_3(\mu_1+2)(\mu_2+2)(\mu_3+2)}\,, \tag{2.99}$$

which implies that the outer horizon r_+ merges with the inner horizon r_0

$$r_0 \to r_\star \leftarrow r_+ \ , \tag{2.100}$$

with their common location being the BPS horizon, given by

$$r_\star^2 \equiv \frac{1}{2}\left(\sqrt{\mu_1\mu_2\mu_3(\mu_1+2)(\mu_2+2)(\mu_3+2)} - \mu_1\mu_2 - \mu_2\mu_3 - \mu_3\mu_1 - \mu_1\mu_2\mu_3\right) . \tag{2.101}$$

We have denoted the BPS horizon with a $\star$ symbol: in this thesis we will always use this subscript to denote the physical quantities of a BPS black hole obtained after a BPS limit is performed on a finite-temperature one; in this way there should be no confusion between labels referred to quantities of the BPS solutions and to the finite-temperature ones.

The supersymmetry condition (2.96) together with the requirement (2.99) of no casual pathologies implies extremality; therefore we indeed landed on a BPS black hole solution. It is then immediate to show that, by imposing both these conditions at the same time, one obtains the $\mathrm{U}(1)^3$ solution of Gutowski and Reall, described in the previous subsection; in particular the BPS charges exactly match (2.89), the same does the BPS entropy with (2.87) and the chemical potentials $\Omega^\star$ and $\Delta^{\star I}$ with (2.88).

It is worth mentioning that in this BPS limit there is no emergence of the complex BPS chemical potentials ω and $\phi^{\star I}$. Therefore we have no clues about how to construct the entropy function (1.37) from the black hole solution. We will return on this very important point in Chapter 3.

Limit to the minimal black hole solution of [41]

It is possible to recover the minimal gauged supergravity black hole solution of Gutowski and Reall, originally constructed in [41], by taking the limit to this theory. In this subsection we show how this can be done.

In [41], the authors assume for the black hole the metric ansatz (2.49) and they show that the supersymmetry conditions obtained working in the $\mathcal{N}=2$ minimal gauged supergravity theory can be reduced to a single equation, which we dub *Gutowski-Reall equation*[37]. This latter reads

$$\left(\nabla^2 f_{\min}^{-1} + 8\ell^{-2} f_{\min}^{-2} - \frac{\ell^2\mathfrak{g}^2}{18} + f_{\min}^{-1}\mathfrak{g}\right)' + \frac{4a'\mathfrak{g}}{a f_{\min}} = 0\,, \tag{2.102}$$

37 See [99–101, 133] for a detailed study of the Gutowski-Reall equation.

where we have defined the function $\mathfrak{g}$ as

$$\mathfrak{g} = -\frac{a'''}{a'} - 3\frac{a''}{a} - \frac{1}{a^2} + 4\frac{a'^2}{a^2}\,, \tag{2.103}$$

and

$$\nabla^2 f_{\rm min}^{-1} = \frac{1}{a^3 a'}\partial_\rho\left[a^3 a'\partial_\rho(f_{\rm min}^{-1})\right] \tag{2.104}$$

is the Laplacian on the Kähler base. Since $f_{\rm min}$ and $\mathfrak{g}$ may be expressed in terms of a, equation (2.102) can be transformed into an equation for $a(\rho)$ only. We want to show that the system of equations (2.52)-(2.56) of the $\mathcal{N}=2$ Fayet-Iliopoulos supergravity theory reduces to (2.102) in a particular limit, which will define the minimal limit of our theory.

As asserted in [42], in order to obtain the minimal theory one must have:

$$dX_I = 0\,, \tag{2.105}$$

so the scalars are constant and equal to the value taken in the AdS$_5$ solution:

$$X^I_{\rm min} = \bar{X}^I\,. \tag{2.106}$$

Furthermore the gauge fields must obey:

$$A^I = \bar{X}^I A_{\rm min}\,. \tag{2.107}$$

Given the above, we can reduce (2.52) to:

$$f = f_{\rm min}\,. \tag{2.108}$$

The next step is to examine (2.53), which, since the scalars are constant, reduces to:

$$\left(a^2 U^I\right)' = \frac{\epsilon\ell}{3}\bar{X}^I\left(a^2\,p\right)'\,, \tag{2.109}$$

so this equation can be immediately integrated to obtain:

$$U^I = \frac{U_0^I}{a^2} + \frac{\epsilon\ell}{3}\,\bar{X}^I p\,, \tag{2.110}$$

where U_0^I are constants of integration. As prescribed in [42], in order to obtain the minimal solution the right choice is to require $U_0^I = 0$ for each I. Doing this, we obtain:

$$U^I = \frac{\epsilon\ell}{3}\,\bar{X}^I p\,. \tag{2.111}$$

Plugging this relation in (2.56) and noticing that $\bar{X}_I$ factorizes, we get:

$$\partial_\rho\left[a^3 a'\partial_\rho(f^{-1}) + \frac{\epsilon}{\ell}\,a^2 w + \frac{1}{18}\,\ell^2 p^2\right] = 0\,. \tag{2.112}$$

Finally plugging (2.55) in (2.54) we find:

$$2\partial_\rho(a^{-2}w)+\epsilon\,\ell\,f^{-1}\,\partial_\rho(a^{-2}p)=0\,. \tag{2.113}$$

Now equations (2.113) and (2.112) contain both w and w'. Eliminating w' we obtain the following expression for w:

$$w\;=\;-\frac{\epsilon\,\ell\,a^2}{4}\left[\nabla^2(f_{\rm min}^{-1})+\frac{8}{\ell^2}f_{\rm min}^{-2}-\frac{\ell^2\mathfrak{g}^2}{18}+f_{\rm min}^{-1}\,\mathfrak{g}\right]\,, \tag{2.114}$$

plugging this back either in (2.113) or (2.112) we land exactly on equation (2.102). We have therefore defined the correct limit to obtain the minimal solutions starting from the more general Fayet-Iliopoulos theory.

Among the solutions that equation (2.102) presents, there is the black hole of [41]. This corresponds to the following solution of the Gutowski-Reall equation

$$a=\alpha_{\rm min}\,\ell\,\sinh\frac{\rho}{\ell}\,, \tag{2.115}$$

the functions f and w assume the values

$$f_{\rm min}=\frac{12\,\alpha_{\rm min}^2\,\sinh^2(\rho/\ell)}{12\,\alpha_{\rm min}^2\,\sinh^2(\rho/\ell)+4\alpha_{\rm min}^2-1}\,, \tag{2.116}$$

$$\begin{aligned}w=-2\,\epsilon\,\alpha_{\rm min}^2\,\ell\,\sinh^2(\rho/\ell)\Bigg[1+&\frac{4\,\alpha_{\rm min}^2-1}{4\,\alpha_{\rm min}^2\sinh^2(\rho/\ell)}+\\&+\frac{(4\,\alpha_{\rm min}^2-1)^2}{96\,\alpha_{\rm min}^4\,\sinh^4(\rho/\ell)}\Bigg]\,.\end{aligned} \tag{2.117}$$

The gauge field is given by

$$A=\frac{\sqrt{3}}{2}\,f_{\rm min}\left[{\rm d}y+\frac{\epsilon\,\ell\,(4\,\alpha_{\rm min}^2-1)^2}{144\,\alpha_{\rm min}^2\,\sinh^2(\rho/\ell)}\,\hat\sigma_3\right]\,. \tag{2.118}$$

It is more convenient to express the physical properties of the minimal black hole solution with respect to the parameter

$$R_0=\ell\sqrt{\frac{4\,\alpha_{\rm min}^2-1}{3}}\,, \tag{2.119}$$

the conserved charges of the solution, which are the energy E, one angular momentum J and one electric charge Q, are then given by

$$E=\frac{3\,\pi\,R_0^2}{4}\left(1+\frac{3\,R_0^2}{2\,\ell^2}+\frac{2\,R_0^4}{3\,\ell^4}\right)\,, \tag{2.120}$$

$$J=\frac{3\,\epsilon\,\pi\,R_0^4}{8\,\ell}\left(1+\frac{2\,R_0^2}{3\,\ell^2}\right)\,, \tag{2.121}$$

$$Q=\frac{\sqrt{3}\,\pi\,R_0^2}{2}\left(1+\frac{R_0^2}{2\,\ell^2}\right)\,, \tag{2.122}$$

while the Bekenstein-Hawking entropy of the solution is

$$S = \frac{\pi^2}{4} R_0^3 \sqrt{1 + \frac{3 R_0^2}{4 \ell^2}} \,. \tag{2.123}$$

It is worth mentioning that the supersymmetric black hole of the minimal theory we have examined here can be recovered from the more general Fayet-Iliopoulos solution by taking the parameters α_i^{GR} to be $\alpha_1^{\mathrm{GR}} = 3 R_0^2$, $\alpha_2^{\mathrm{GR}} = 3 R_0^4$ and $\alpha_3^{\mathrm{GR}} = R_0^6$.

We conclude by noting that solutions to equation (2.102) different from the one of [41] have been searched and found in [101,133]; although the solutions found in these papers are only known numerically they are still interesting and remarkable. In particular, in [133] the authors look for AlAdS$_5$ solutions with a squashed boundary and extensively analyze equation (2.102) with a perturbative approach; they find a new numerical solution which turns out to be a soliton. Instead, in [101] the authors use a similar approach to construct a new AlAdS$_5$ black hole with a squashed boundary; this solution presents a frozen horizon, i.e. the horizon geometry is completely fixed. We will discuss in much more detail the solutions found in [101, 133] and the perturbative approach used in the two papers in Chapter 4, when we will devote ourselves to the construction of AlAdS$_5$ black holes in Fayet-Iliopoulos gauged supergravity.

2.2 ADS$_4$ BLACK HOLES

In this section we introduce some AdS$_4$ black hole supergravity solutions which will be relevant for our future discussions. Since the main focus of this thesis is on five-dimensional black holes, we will keep the presentation shorter.

2.2.1 *Finite-temperature AdS$_4$ black holes*

We consider a class of rotating, electrically charged asymptotically AdS$_4$ black hole solutions, originally constructed in [134] within a consistent truncation of four-dimensional $\mathcal{N} = 8$, SO(8) gauged supergravity. The truncation is obtained by restricting to the U(1)^4 Cartan subgroup of SO(8) and setting the corresponding four gauge fields pairwise equal. The action of the theory under consideration is:

$$\begin{aligned}\mathcal{S} = \frac{1}{16\pi} \int \Big[& (R - 2\,\mathcal{V}) \star 1 - \frac{1}{2}\, \mathrm{d}\xi \wedge \star\, \mathrm{d}\xi - \frac{1}{2}\, \mathrm{e}^{2\xi}\, \mathrm{d}\chi \wedge \star\, \mathrm{d}\chi \\ & - \frac{1}{2}\, \mathrm{e}^{-\xi}\, F_3 \wedge \star F_3 - \frac{1}{2}\, \chi\, F_3 \wedge F_3 \\ & - \frac{1}{2\,(1 + \chi^2 \mathrm{e}^{2\xi})} \Big(\mathrm{e}^{\xi}\, F_1 \wedge \star F_1 - \mathrm{e}^{2\xi}\, \chi\, F_1 \wedge F_1 \Big) \Big] \,,\end{aligned} \tag{2.124}$$

with F_1 and F_3 being field strengths of the Abelian gauge fields $A_1 = A_2$ and $A_3 = A_4$,[38] and with $\mathcal{V}$ being the scalar potential for the axion and dilaton scalar fields χ, ξ:

$$\mathcal{V} = -\frac{1}{2}\, g^2 \left(4 + 2\cosh\xi + \mathrm{e}^{\xi}\chi^2\right) . \tag{2.125}$$

To describe the solution, we take the set of coordinates (t, r, θ, ϕ), where $\theta \in [0, \pi]$, $\phi \sim \phi + 2\pi$ parameterize a two-sphere. The metric, in a frame that rotates at infinity, assumes the following form

$$\begin{aligned} \mathrm{d}s_4^2 = -\frac{\Delta_r}{W}\left(\mathrm{d}t - \frac{\mathfrak{a}}{\Xi}\sin^2\theta\,\mathrm{d}\phi\right)^2 + W\left(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}\theta^2}{\Delta_\theta}\right) + \\ + \frac{\Delta_\theta \sin^2\theta}{W}\left(\mathfrak{a}\,\mathrm{d}t - \frac{r_1 r_2 + \mathfrak{a}^2}{\Xi}\mathrm{d}\phi\right)^2 , \end{aligned} \tag{2.126}$$

where

$$\begin{aligned} & r_i = r + 2\,m\,s_i^2\,, \qquad \Xi = 1 - \mathfrak{a}^2 g^2\,, \\ & \Delta_r = r^2 + \mathfrak{a}^2 - 2\,m\,r + g^2\, r_1\, r_2\left(r_1\, r_2 + \mathfrak{a}^2\right) , \\ & \Delta_\theta = 1 - \mathfrak{a}^2 g^2\, \cos^2\theta\,, \qquad W = r_1\, r_2 + \mathfrak{a}^2\, \cos^2\theta\,, \end{aligned} \tag{2.127}$$

and we have introduced again the shorthand notations $s_i = \sinh\delta_i$, $c_i = \cosh\delta_i$, $i = 1, 2$. The scalar fields result to be

$$\mathrm{e}^{\xi} = 1 + \frac{r_1\,(r_1 - r_2)}{W}\,, \qquad \chi = \frac{\mathfrak{a}\,(r_2 - r_1)\cos\theta}{r_1^2 + \mathfrak{a}^2\cos^2\theta}\,, \tag{2.128}$$

finally the gauge fields read

$$\begin{aligned} A_1 &= \frac{2\sqrt{2}\,m\,s_1 c_1 r_2}{W}\left(\mathrm{d}t - \frac{\mathfrak{a}}{\Xi}\,\sin^2\theta\,\mathrm{d}\phi\right) , \\ A_3 &= \frac{2\sqrt{2}\,m\,s_2 c_2 r_1}{W}\left(\mathrm{d}t - \frac{\mathfrak{a}}{\Xi}\,\sin^2\theta\,\mathrm{d}\phi\right) . \end{aligned} \tag{2.129}$$

There are four parameters controlling the solution, which are $m, \mathfrak{a}, \delta_1, \delta_2$. Accordingly, there should be four independent conserved charges; these are the energy E, the angular momentum J and two independent electric charges Q_1 and Q_3. The most general set of conserved charges a black hole may present in this theory is composed by the energy, one angular momentum and four independent electric charges $Q_{1,\ldots 4}$; here we have only two independent charges since we have set the corresponding gauge fields pairwise equal, so we have $Q_1 = Q_2$ and $Q_3 = Q_4$. This most general finite-temperature black hole solution has not been constructed yet at the time this thesis is written, however the corresponding BPS black hole solution has recently been presented in [135]. Since it is contained in SO(8) gauged supergravity, the solution uplifts to eleven-dimensional supergravity on S^7 (see [134] and references

38 In this section we use lower indices on the vector fields and the respective chemical potentials Φ.

therein for the explicit uplift formulae). The dual SCFT$_3$ is then the ABJM theory.

The solution presents an outer horizon at $r = r_+$, which coincides with the largest root of Δ_r. This is a Killing horizon, which is generated by the vector

$$V = \frac{\partial}{\partial t'} + \Omega \, \frac{\partial}{\partial \phi'} \,, \tag{2.130}$$

with the coordinates

$$\phi' = \phi + \mathfrak{a} \, g^2 \, t \,, \qquad t' = t \tag{2.131}$$

defining a frame that is non-rotating at infinity and Ω is the angular velocity at the horizon, which is given by:

$$\Omega = \frac{\mathfrak{a} \, (1 + g^2 \, r_1 \, r_2)}{r_1 \, r_2 + \mathfrak{a}^2} \,. \tag{2.132}$$

The Bekenstein-Hawking entropy, the inverse temperature and the electrostatic potentials of the solution assume the following values:

$$\begin{aligned} S &= \frac{\pi \, (r_1 \, r_2 + \mathfrak{a}^2)}{\Xi} \,, \qquad \beta = 4\pi \left(r_1 \, r_2 + \mathfrak{a}^2 \right) \left(\frac{\mathrm{d}\Delta_r}{\mathrm{d}r} \right)^{-1} \,, \\ \Phi_1 = \Phi_2 &= \frac{2 \, m \, s_1 \, c_1 \, r_2}{r_1 \, r_2 + \mathfrak{a}^2} \,, \qquad \Phi_3 = \Phi_4 = \frac{2 \, m \, s_2 \, c_2 \, r_1}{r_1 \, r_2 + \mathfrak{a}^2} \,, \end{aligned} \tag{2.133}$$

where all the functions of the radial coordinate are evaluated in r_+. The electrostatic potentials Φ_I, $I = 1, \ldots, 4$, have been evaluated from the four vector fields gauging the Cartan subgroup of SO(8); since these are set pairwise equal in the action (2.124), necessarily we have $\Phi_1 = \Phi_2$ and $\Phi_3 = \Phi_4$. The conserved charges, which are the energy (that is the charge associated with translations generated by $\frac{\partial}{\partial t'}$), the angular momentum (that is the charge associated with rotations generated by $-\frac{\partial}{\partial \phi'}$) and the electric charges, are given by:

$$\begin{aligned} E &= \frac{m}{\Xi^2} \left(1 + s_1^2 + s_2^2 \right) \,, \qquad J = \frac{m \, \mathfrak{a}}{\Xi^2} \left(1 + s_1^2 + s_2^2 \right) \,, \\ Q_1 = Q_2 &= \frac{m \, s_1 \, c_1}{2 \, \Xi} \,, \qquad Q_3 = Q_4 = \frac{m \, s_2 \, c_2}{2 \, \Xi} \,. \end{aligned} \tag{2.134}$$

The electric charges and the angular momentum were obtained in [110] evaluating the standard Maxwell and Komar asymptotic integrals respectively, while the energy was computed by integrating the first law of thermodynamics,

$$\mathrm{d}E = T \, \mathrm{d}S + \Omega \, \mathrm{d}J + 2 \, \Phi_1 \, \mathrm{d}Q_1 + 2 \, \Phi_3 \, \mathrm{d}Q_3 \,. \tag{2.135}$$

All the conserved charges can be also computed by using holographic renormalization in order to check whether the obtained expressions agree with (2.134). We do this explicitly in B, finding that the holographic renormalization approach leads to the same results given in (2.134).

The thermodynamical quantities (2.132), (2.133), (2.134) must satisfy the quantum statistical relation

$$I = \beta E - S - \beta\,\Omega\,J - 2\,\beta\,\Phi_1\,Q_1 - 2\beta\,\Phi_3\,Q_3\,, \tag{2.136}$$

with I being the Euclidean on-shell action of the solution. As in the five-dimensional case, we will need the expression of this quantity in order to provide an interpretation for the extremization principle for this family of black holes; one possibility to obtain it is to assume the quantum statistical relation and use it to compute I. We choose another road: we we compute I by means of holographic renormalization, as we have done for the five-dimensional case. We report the details of this computation in app. B; here we just provide the final result which is

$$\begin{aligned} I = \frac{\beta}{2(a^2g^2-1)} \Big\{ & g^2 r_+^3 + 3\,m\,g^2 r_+^2 \left(s_1^2+s_2^2\right) \\ & + r_+ \left[a^2g^2 + 2\,m^2 g^2 \left(s_1^4 + 4 s_1^2 s_2^2 + s_2^4\right)\right] \\ & + m\left(a^2 g^2 + 4\,m^2 g^2 s_1^2 s_2^2 - 1\right)\left(s_1^2+s_2^2\right) - m \\ & + \frac{2\,m^2\left[c_1^2 s_1^2\left(2\,m\,s_2^2 + r_+\right) + c_2^2 s_2^2 \left(2\,m\,s_1^2+r_+\right)\right]}{a^2 + \left(2\,m\,s_1^2+r_+\right)\left(2\,m\,s_2^2+r_+\right)} \Big\}\,. \end{aligned} \tag{2.137}$$

We have explicitly verified that the on-shell action and the thermodynamical quantities (2.132), (2.133), (2.134) satisfy the quantum statistical relation (2.136).

2.2.2 *BPS AdS_4 black holes*

Here we present the four-dimensional BPS black hole solution corresponding to the finite-temperature black hole introduced in the last subsection. To ease the notation we will set $g = 1$ from now on.

As for the five-dimensional case, in order to obtain a BPS solution we have to require both supersymmetry and extremality, which correspond to two different conditions. Supersymmetry is imposed by requiring that[39]

$$\mathfrak{a} = \frac{2}{\mathrm{e}^{2\,(\delta_1+\delta_2)} - 1}\,, \tag{2.138}$$

we assume this condition and in the following we use it to eliminate $\mathfrak{a}$ from all the expressions. The remaining parameters m, δ_1, δ_2 describe then a supersymmetric family of solutions. For real values of these parameters, it was shown in [110] that the equation $\Delta_r(r) = 0$ determining the existence of a horizon only has a solution if

$$m^2 = m_\star^2 \equiv \frac{\cosh^2(\delta_1+\delta_2)}{4\,\mathrm{e}^{\delta_1+\delta_2}\sinh^3(\delta_1+\delta_2)\,c_1\,s_1\,c_2\,s_2}\,, \tag{2.139}$$

39 In (2.138) and (2.139), we are using the expressions given in [136], which correct typos in the corresponding expressions of [110].

in this case there is a regular horizon at the location

$$r = r_\star \equiv \frac{2\, m_\star\, s_1\, s_2}{\cosh(\delta_1 + \delta_2)} \,. \tag{2.140}$$

The value $r_\star$ is a double root of Δ_r, therefore the supersymmetric solution becomes extremal and the temperature vanishes. We thus obtain a BPS solution that is regular everywhere[40]. If we further impose $\delta_1 = \delta_2$, we obtain a solution to pure $\mathcal{N} = 2$ gauged supergravity in four dimensions, which has been originally discussed in [138].

We now turn to examine the thermodynamical properties of the BPS solution. The chemical potentials are fixed to the BPS values

$$\Omega^\star = 1\,, \qquad \Phi_1^\star = \Phi_3^\star = 1\,, \qquad \beta \to \infty\,, \tag{2.141}$$

the BPS charges are given by the following expressions

$$\begin{aligned}
E^\star &= \frac{(c_1\, c_2 - s_1\, s_2)\, \sqrt{\mathrm{e}^{-(\delta_1+\delta_2)}\, (c_1\, s_2 + c_2\, s_1)}}{2\, (\coth(\delta_1 + \delta_2) - 2)^2\, \sqrt{c_1\, c_2\, s_1\, s_2}}\,, \\
J^\star &= \frac{c_1\, c_2 - s_1\, s_2}{2\, (\coth(\delta_1 + \delta_2) - 2)^2\, \sqrt{\mathrm{e}^{3(\delta_1+\delta_2)}\, c_1\, c_2\, s_1\, s_2\, (c_1\, s_2 + c_2\, s_1)}}\,, \\
Q_1^\star &= \frac{\sqrt{c_1\, c_2\, s_1\, s_2\, \left(\mathrm{e}^{2(\delta_1+\delta_2)} - 1\right)}}{2\, \sqrt{2}\, c_2\, s_2\, \left(\mathrm{e}^{2(\delta_1+\delta_2)} - 3\right)}\,, \\
Q_3^\star &= \frac{\sqrt{c_1\, c_2\, s_1\, s_2\, \left(\mathrm{e}^{2(\delta_1+\delta_2)} - 1\right)}}{2\, \sqrt{2}\, c_1\, s_1\, \left(\mathrm{e}^{2(\delta_1+\delta_2)} - 3\right)}\,.
\end{aligned} \tag{2.142}$$

The above introduced thermodynamical quantities satisfy the relation

$$E^\star - \Omega^\star J^\star - 2\, \Phi_1^\star\, Q_1^\star - 2\, \Phi_3^\star\, Q_3^\star = 0\,, \tag{2.143}$$

that is a consequence of supersymmetry algebra. This is a linear relation between the charges which allows to fix one of them with respect to the other three. There is also another non trivial and non linear relation that the BPS angular momentum and electric charges also satisfy, which is [70]

$$J^\star = (Q_1^\star + Q_3^\star)\left(\sqrt{1 + 64 Q_1^\star Q_3^\star} - 1\right). \tag{2.144}$$

The validity of this relation is related to the fact that we have imposed (2.139) on top of the supersymmetry condition (2.138); indeed, having fixed two of the four free parameters of the solution, there cannot be more than two independent charges. Finally we have the BPS entropy, which is

$$S^\star = \frac{2\, \pi}{\mathrm{e}^{2\delta_1 + 2\delta_2} - 3}\,, \tag{2.145}$$

40 We underline that the black hole solutions we are discussing here are different from the rotating solutions with magnetic charges recently found in [137].

and can be expressed in terms of the charges as [70]:

$$S^\star = \frac{\pi\, J^\star}{2(Q_1^\star + Q_3^\star)}\,. \tag{2.146}$$

The BPS entropy must be positive: this condition restricts the allowed range of $\delta_1 + \delta_2$.

2.3 ADS$_6$ BLACK HOLES

We now turn to six-dimensional AdS black holes. As we have done in the two previous sections, we introduce the finite-temperature solution of interest for this work of thesis and we review its corresponding BPS solution which is obtained by imposing both supersymmetry and extremality.

2.3.1 *Finite-temperature AdS$_6$ black holes*

We consider the asymptotically AdS$_6$ black hole of [139], which is a solution to the six-dimensional $\mathcal{N} = (1, 0)$, SU(2) gauged supergravity of [140]. This uplifts to massive type IIA supergravity on $S^4/\mathbb{Z}_2$ [141].

The set of conserved charges the black hole presents is composed by the energy E, two angular momenta J_a, J_b and one U(1) $\subset$ SU(2) electric charge Q. This is the first solution we look at which has two independent angular momenta: because of this it looks slightly different in form from the other solutions introduced in the previous sections.

The metric of the solution is given by:

$$\begin{aligned}
\mathrm{d}s^2 = H^{1/2}\Bigg[&\frac{(r^2+y^2)(r^2+z^2)}{\mathcal{R}}\mathrm{d}r^2 + \frac{(r^2+y^2)(y^2-z^2)}{Y}\mathrm{d}y^2 \\
&+ \frac{(r^2+z^2)(z^2-y^2)}{Z}\mathrm{d}z^2 - \frac{\mathcal{R}}{H^2(r^2+y^2)(r^2+z^2)}\mathcal{A}^2 \\
&+ \frac{Y}{(r^2+y^2)(y^2-z^2)}\bigg(\mathrm{d}t' + (z^2-r^2)\,\mathrm{d}\psi_1 \\
&\qquad - r^2z^2\,\mathrm{d}\psi_2 - \frac{qr\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\bigg)^2 \\
&+ \frac{Z}{(r^2+z^2)(z^2-y^2)}\bigg(\mathrm{d}t' + (y^2-r^2)\,\mathrm{d}\psi_1 \\
&\qquad - r^2y^2\,\mathrm{d}\psi_2 - \frac{qr\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\bigg)^2\Bigg],
\end{aligned} \tag{2.147}$$

while the gauge and scalars fields assume the following expressions

$$
\begin{aligned}
& X = H^{-1/4}, \quad A_{(1)} = \frac{2mscr}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}, \\
& A_{(2)} = \frac{q}{H(r^2+y^2)^2(r^2+z^2)^2}\Bigg[-\frac{yz[2r(2r^2+y^2+z^2)+q]}{H}\mathrm{d}r\wedge\mathcal{A} \\
& + z[(r^2+z^2)(r^2-y^2)+qr]\,\mathrm{d}y\wedge \\
& \wedge\left(\mathrm{d}t'+(z^2-r^2)\,\mathrm{d}\psi_1 - r^2z^2\,\mathrm{d}\psi_2 - \frac{qr\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\right) \\
& + y[(r^2+y^2)(r^2-z^2)+qr]\,\mathrm{d}z\wedge \\
& \wedge\left(\mathrm{d}t'+(y^2-r^2)\,\mathrm{d}\psi_1 - r^2y^2\,\mathrm{d}\psi_2 - \frac{qr\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\right)\Bigg],
\end{aligned}
\tag{2.148}
$$

where:

$$
\begin{aligned}
& Y = -(1-g^2y^2)(\mathfrak{a}^2-y^2)(\mathfrak{b}^2-y^2), \\
& Z = -(1-g^2z^2)(\mathfrak{a}^2-z^2)(\mathfrak{b}^2-z^2), \\
& H = 1+\frac{qr}{(r^2+y^2)(r^2+z^2)}, \quad q = 2ms^2, \quad s=\sinh\delta, \quad c=\cosh\delta, \\
& \mathcal{A} = \mathrm{d}t' + (y^2+z^2)\,\mathrm{d}\psi_1 + y^2z^2\,\mathrm{d}\psi_2\,.
\end{aligned}
\tag{2.149}
$$

and $\mathcal{R}(r)$ is the blackening function

$$
\begin{aligned}
\mathcal{R}(r) = g^2\left[r\left(\mathfrak{a}^2+r^2\right)+2\,m\,s^2\right]\left[r\left(\mathfrak{b}^2+r^2\right)+2\,m\,s^2\right]+ \\
+\left(\mathfrak{a}^2+r^2\right)\left(\mathfrak{b}^2+r^2\right)-2\,m\,r\,.
\end{aligned}
\tag{2.150}
$$

As it is evident from the expressions of the functions above reported, the solution is controlled by the four parameters m, $\mathfrak{a}$, $\mathfrak{b}$, δ and presents an outer horizon at $r=r_+$, which is defined as the largest root of $\mathcal{R}(r)$. The entropy and the chemical potentials of the solution are given by

$$
\begin{aligned}
& S = \frac{2\,\pi^2\left[(r_+^2+\mathfrak{a}^2)\,(r_+^2+\mathfrak{b}^2)+2\,m\,r_+\,s^2\right]}{3\,\Xi_a\,\Xi_b}\,, \\
& \Omega_a = \mathfrak{a}\,\frac{(1+g^2\,r_+^2)\,(r_+^2+\mathfrak{b}^2)+2\,m\,g^2\,r_+\,s^2}{(r_+^2+\mathfrak{a}^2)\,(r_+^2+\mathfrak{b}^2)+2\,m\,r_+\,s^2}\,, \\
& \Omega_b = \mathfrak{b}\,\frac{(1+g^2\,r_+^2)\,(r_+^2+\mathfrak{a}^2)+2\,m\,g^2\,r_+\,s^2}{(r_+^2+\mathfrak{a}^2)\,(r_+^2+\mathfrak{b}^2)+2\,m\,r_+\,s^2}\,, \\
& \Phi = \frac{2\,m\,r_+\,s\,c}{(r_+^2+\mathfrak{a}^2)\,(r_+^2+\mathfrak{b}^2)+2\,m\,r_+\,s^2}\,, \\
& \frac{1}{\beta} = \frac{1}{\mathcal{D}}\Bigg(2r_+^2(1+g^2r_+^2)(2r_+^2+\mathfrak{a}^2+\mathfrak{b}^2)+ \\
& \qquad -(1-g^2r_+^2)(r_+^2+\mathfrak{a}^2)(r_+^2+\mathfrak{b}^2)+8mg^2r_+^3s^2-4m^2g^2s^4\Bigg)\,, \\
& \mathcal{D} = 4\,\pi\,r_+\left[\left(r_+^2+\mathfrak{a}^2\right)\left(r_+^2+\mathfrak{b}^2\right)+2\,m\,r_+\,s^2\right]\,,
\end{aligned}
\tag{2.151}
$$

where $\Xi_a = 1 - \mathfrak{a}^2 g^2$ and $\Xi_b = 1 - \mathfrak{b}^2 g^2$. The conserved charges of the black hole solution, which are the energy, the angular momenta and the electric charge, are given by

$$\begin{aligned} E &= \frac{2\pi m}{3\,\Xi_a\,\Xi_b}\left[\frac{1}{\Xi_a} + \frac{1}{\Xi_b} + s^2\left(1 + \frac{\Xi_a}{\Xi_b} + \frac{\Xi_b}{\Xi_a}\right)\right], \qquad Q = \frac{2\pi m\, s\, c}{\Xi_a\,\Xi_b}\,, \\ J_a &= \frac{2\pi m\,\mathfrak{a}}{3\,\Xi_a^2\,\Xi_b}\left(1 + \Xi_b\, s^2\right), \qquad J_b = \frac{2\pi m\,\mathfrak{b}}{3\,\Xi_a\,\Xi_b^2}\left(1 + \Xi_a\, s^2\right)\,, \end{aligned} \tag{2.152}$$

as expected, they satisfy the first law of black hole thermodynamics:

$$\mathrm{d}E = T\,\mathrm{d}S + \Omega_a\,\mathrm{d}J_a + \Omega_b\,\mathrm{d}J_b + \Phi\,\mathrm{d}Q\,. \tag{2.153}$$

For the black hole under consideration, the quantum statistical relation reads

$$I = \beta\,E - S - \beta\,\Omega_a\,J_a - \beta\,\Omega_b\,J_b - \beta\,\Phi\,Q\,, \tag{2.154}$$

this relation must be verified by the thermodynamical quantities of the solution. Once again, as for the five- and four-dimensional cases, we will need the Euclidean on-shell action I in order to provide a physical interpretation to the extremization principle applied to this six-dimensional black hole. One possibility to obtain I is to assume the quantum statistical relation to hold and use it to compute the Euclidean on-shell action. This is exactly what we do in sec. 3.3 where we analyze the extremization principle for six-dimensional black holes. Therefore, while in the four- and five-dimensional cases we explicitly verified the quantum statistical relation by computing the on-shell action via holographic renormalization, in the present case we will just assume this to hold. We will demonstrate that chemical potentials satisfying the correct complex constraint arise from a suitably complexified family of supersymmetric solutions, and that the expression of I on these solutions is precisely the entropy function given in [70].

2.3.2 *BPS AdS$_6$ black holes*

Here we introduce the AdS$_6$ BPS black hole solution we are interested in; this can be obtained by imposing supersymmetry and extremality to the finite-temperature solution we presented in the last subsection. For ease of notation we set $g = 1$ from now on.

The finite-temperature black hole is supersymmetric if

$$\mathrm{e}^{2\delta} = 1 + \frac{2}{\mathfrak{a} + \mathfrak{b}}\,, \tag{2.155}$$

in the following we shall always assume this condition to eliminate δ in all the expressions. The solution depends now on the remaining three free parameters m, $\mathfrak{a}$, $\mathfrak{b}$. In order to guarantee that it is free from closed timelike curves, one has to impose that

$$m = m_\star = \frac{(\mathfrak{a}+\mathfrak{b})^2\,(1+\mathfrak{a})\,(1+\mathfrak{b})\,(2+\mathfrak{a}+\mathfrak{b})}{2\,(1+\mathfrak{a}+\mathfrak{b})}\sqrt{\frac{\mathfrak{a}\,\mathfrak{b}}{1+\mathfrak{a}+\mathfrak{b}}}\,. \tag{2.156}$$

Imposing both this condition and (2.155), the temperature vanishes and we obtain a BPS solution whose horizon coordinate is given by

$$r_\star = \sqrt{\frac{\mathfrak{a}\,\mathfrak{b}}{1+\mathfrak{a}+\mathfrak{b}}}\,. \tag{2.157}$$

For the BPS solution, the chemical potentials are fixed to the BPS values

$$\Omega_a^\star = \Omega_b^\star = 1\,, \qquad \Phi^\star = 1\,, \qquad \beta \to \infty\,, \tag{2.158}$$

the BPS charges are given instead by

$$\begin{aligned}
E^\star &= -\frac{\pi\, r_\star\,(\mathfrak{a}+\mathfrak{b})\,\left[2\,\mathfrak{a}^2+\mathfrak{a}\,(\mathfrak{b}-1)+(\mathfrak{b}+1)\,(2\mathfrak{b}-3)\right]}{3\,(\mathfrak{a}-1)^2\,(\mathfrak{b}-1)^2\,(\mathfrak{a}+\mathfrak{b}+1)}\,,\\
J_a^\star &= -\frac{\pi\, r_\star^3\,(\mathfrak{a}+\mathfrak{b})\,(\mathfrak{a}+2\,\mathfrak{b}+1)}{3\,\mathfrak{b}\,(\mathfrak{a}-1)^2\,(\mathfrak{b}-1)}\,,\\
J_b^\star &= -\frac{\pi\, r_\star^3\,(\mathfrak{a}+\mathfrak{b})\,(2\,\mathfrak{a}+\mathfrak{b}+1)}{3\,\mathfrak{a}\,(\mathfrak{a}-1)\,(\mathfrak{b}-1)^2}\,,\\
Q^\star &= \frac{\pi\, r_\star\,(\mathfrak{a}+\mathfrak{b})}{(\mathfrak{a}-1)\,(\mathfrak{b}-1)}\,.
\end{aligned}$$

There is the following linear relation satisfied by the thermodynamical quantities above

$$E^\star - \Omega_a^\star\,J_a^\star - \Omega_b^\star\,J_b^\star - \Phi^\star\,Q^\star = 0\,, \tag{2.159}$$

which is a consequence of supersymmetry algebra. The BPS entropy of the BPS black hole solution reads

$$S^\star = \frac{2\,\pi^2\,r_\star^2\,(\mathfrak{a}+\mathfrak{b})}{3\,(1-\mathfrak{a})\,(1-\mathfrak{b})}\,; \tag{2.160}$$

in the following we assume $0 < a < 1$, $0 < b < 1$, which guarantee $r_\star$ to be real and the BPS entropy to be real and positive. The BPS entropy satisfies the following two non-linear relations, which involve also the BPS charges [70]

$$\begin{aligned}
&S^{\star\,3} - \frac{2\,\pi^2}{3}S^{\star\,2} - 12\,\pi^2\left(\frac{Q^\star}{3}\right)^2 S^\star + \frac{8\,\pi^4}{3}\,J_a^\star\,J_b^\star = 0\,,\\
&\frac{Q^\star}{3}S^{\star\,2} + \frac{2\,\pi^2}{9}\,(J_a^\star + J_b^\star)\,S^\star - \frac{4\,\pi^2}{3}\left(\frac{Q^\star}{3}\right)^3 = 0\,.
\end{aligned} \tag{2.161}$$

The two relations above may be combined to express the BPS entropy in terms of the charges and to obtain a relation between $J_a^\star$, $J_b^\star$, $Q^\star$, analogously to what happens in the other spacetime dimensions.

2.4 AdS$_7$ black holes

The final stage of our journey through AdS black holes is given by the seven-dimensional ones. As usual, we introduce the finite-temperature solution of interest for this thesis in the next subsection and later we look at its BPS version, which we construct by imposing supersymmetry and extremality.

2.4.1 *Finite-temperature AdS$_7$ black holes*

We consider the seven-dimensional black hole originally found in [142] and further discussed in [110]. This is a solution to maximal SO(5) gauged supergravity and uplifts to eleven-dimensional supergravity on S^4 [104]. We begin with a brief summary of its relevant physical properties[41].

The metric of the solution under consideration is given by:

$$\mathrm{d}s_7^2 = (H_1 H_2)^{1/5} \Big[-\frac{Y\,\mathrm{d}t^2}{f_1\,\Xi_-^2} + \frac{r^2\,\rho^4\,\mathrm{d}r^2}{Y} + \frac{f_1}{\rho^4\,H_1 H_2\,\Xi^2}\left(\sigma - \frac{2f_2}{f_1}\,\mathrm{d}t\right)^2 + \frac{r^2+\mathfrak{a}^2}{\Xi}\,\mathrm{d}\Sigma_2^2 \Big]\,, \tag{2.162}$$

while the gauge and scalar fields assume the following form:

$$\begin{aligned} A_1^i &= \frac{2m\,s_i}{\rho^4\,\Xi\,H_i}\,(\alpha_i\,\mathrm{d}t + \beta_i\,\sigma)\,, \\ A_2 &= \frac{m\,\mathfrak{a}\,s_1\,s_2}{\rho^4\,\Xi_-^2}\left(\frac{1}{H_1}+\frac{1}{H_2}\right)\mathrm{d}t\wedge\sigma\,, \qquad A_3 = \frac{2m\,\mathfrak{a}\,s_1\,s_2}{\rho^2\,\Xi\,\Xi_-}\,\sigma\wedge J\,, \\ X_i &= (H_1 H_2)^{2/5}\,H_i^{-1}\,, \end{aligned} \tag{2.163}$$

with the various functions being given by:

$$\begin{aligned} &\Xi_\pm = 1 \pm \mathfrak{a}\,g\,, \quad \Xi = 1-\mathfrak{a}^2 g^2\,, \quad \rho = \sqrt{\Xi}\,r\,, \quad H_I = 1 + \frac{2\,m\,s_I^2}{\rho^4}\,, \\ &\alpha_1 = c_1 - \frac{1}{2}\left(1-\Xi_+^2\right)(c_1-c_2)\,, \quad \alpha_2 = c_2 + \frac{1}{2}\left(1-\Xi_+^2\right)(c_1-c_2)\,, \\ &\beta_1 = -\mathfrak{a}\,\alpha_2\,, \quad \beta_2 = -\mathfrak{a}\,\alpha_1\,, \quad s_I = \sinh\delta_I\,, \quad c_I = \cosh\delta_I\,, \end{aligned} \tag{2.164}$$

41 We correct a few misprints in [110] following [68, 69].

$$
\begin{aligned}
f_1(r) &= \Xi\rho^6 H_1 H_2 - \frac{4\Xi_+^2 m^2 \mathfrak{a}^2 s_1^2 s_2^2}{\rho^4} \\
&\quad + \frac{m\mathfrak{a}^2}{2}\Big[4\Xi_+^2 + 2c_1c_2(1-\Xi_+^4) + (1-\Xi_+^2)^2(c_1^2+c_2^2)\Big], \\
f_2(r) &= -\frac{1}{2}\, g\, \Xi_+\, \rho^6 H_1 H_2 \\
&\qquad + \frac{1}{4} m\mathfrak{a}\left[2\left(1+\Xi_+^4\right)c_1c_2 + \left(1-\Xi_+^4\right)\left(c_1^2+c_2^2\right)\right], \\
Y(r) &= g^2\rho^8 H_1 H_2 + \Xi\rho^6 \\
&\quad + \frac{1}{2} m\mathfrak{a}^2\left[4\Xi_+^2 + 2\left(1-\Xi_+^4\right)c_1c_2 + \left(1-\Xi_+^2\right)^2\left(c_1^2+c_2^2\right)\right] \\
&\quad - \frac{1}{2} m\rho^2\Big[4\Xi + 2\mathfrak{a}^2 g^2\left(6+8\mathfrak{a}g+3\mathfrak{a}^2g^2\right)c_1c_2 \\
&\qquad\qquad - \mathfrak{a}^2g^2\left(2+\mathfrak{a}g\right)\left(2+3\mathfrak{a}g\right)\left(c_1^2+c_2^2\right)\Big].
\end{aligned} \tag{2.165}
$$

Looking at the expressions above, we clearly see that there are four parameters controlling the solution, which are m, $\mathfrak{a}$, δ_1, δ_2, while r is the radial coordinate. The largest root of the equation $Y(r) = 0$ defines the outer horizon, which we dub as r_+.

The entropy, inverse temperature, angular velocity and electrostatic potentials on the horizon result to be:

$$
\begin{aligned}
S &= \frac{\pi^3\,\rho^2\sqrt{f_1}}{4\,\Xi^3}, \qquad \beta = T^{-1} = 4\,\pi\, g\,\rho^3\sqrt{\Xi\, f_1}\left(\frac{\mathrm{d}Y}{\mathrm{d}r}\right)^{-1}, \\
\Omega &= -\frac{1}{g}\left(g + \frac{2\,f_2}{f_1}\,\Xi_-\right), \qquad \Phi_I = \frac{4ms_I}{\rho^4\Xi H_I}\left(\alpha_I\Xi_- + \beta_I\frac{2f_2\Xi_-}{f_1}\right),
\end{aligned} \tag{2.166}
$$

these are measured in a non-rotating frame at infinity. The conserved charges of the black hole solution, which are the energy, the angular momentum and the electric charges, are given by

$$
\begin{aligned}
E &= \frac{m\,\pi^2}{32\,g\,\Xi^4}\Big[12\Xi_+^2\left(\Xi_+^2-2\right) \\
&\qquad - 2c_1c_2\mathfrak{a}^2g^2\left(21\Xi_+^4 - 20\Xi_+^3 - 15\Xi_+^2 - 10\Xi_+ - 6\right) \\
&\qquad + \left(c_1^2+c_2^2\right)\left(21\Xi_+^6 - 62\Xi_+^5 + 40\Xi_+^4 + 13\Xi_+^2 - 2\Xi_+ + 6\right)\Big], \\
J &= -\frac{m\mathfrak{a}\pi^2}{16\,\Xi^4}\Big[4\mathfrak{a}g\Xi_+^2 - 2c_1c_2(2\Xi_+^5 - 3\Xi_+^4 - 1) + \\
&\qquad + \mathfrak{a}g(c_1^2+c_2^2)(\Xi_+ + 1)(2\Xi_+^3 - 3\Xi_+^2 - 1)\Big], \\
Q_1 &= \frac{m\,s_1\,\pi^2}{8\,g\,\Xi^3}\Big[\mathfrak{a}^2g^2c_2\left(2\Xi_+ + 1\right) - c_1\left(2\Xi_+^3 - 3\Xi_+^2 - 1\right)\Big], \\
Q_2 &= \frac{m\,s_2\,\pi^2}{8\,g\,\Xi^3}\Big[\mathfrak{a}^2g^2c_1\left(2\Xi_+ + 1\right) - c_2\left(2\Xi_+^3 - 3\Xi_+^2 - 1\right)\Big].
\end{aligned} \tag{2.167}
$$

The conserved charges are computed in a scheme in which the energy of the vacuum AdS_7 solution vanishes, i.e. it is $E_0 = 0$. In a different scheme, the expression above should be regarded as $E - E_0$; considerations similar to the ones discussed in the five-dimensional case apply also here. The thermodynamical quantities introduced above satisfy the first law of black hole thermodynamics

$$\mathrm{d}E = T\,\mathrm{d}S + 3\,\Omega\,\mathrm{d}J + \Phi_1\,\mathrm{d}Q_1 + \Phi_2\,\mathrm{d}Q_2\,, \tag{2.168}$$

while the quantum statistical relation reads

$$I = \beta E - S - 3\beta\,\Omega\,J - \beta\,\Phi_1\,Q_1 - \beta\,\Phi_2\,Q_2\,; \tag{2.169}$$

as for the six-dimensional case, we will assume this is satisfied and use it to evaluate I. Therefore we will not evaluate the on-shell action independently and check the validity of the quantum statistical relation, as instead we have done for the five- and four-dimensional cases. To ensure consistency with the assumption made for the vacuum energy, we assume we are working in a scheme where $I_0 = \beta E_0 = 0$.

The most general finite-temperature black hole solution of this theory should carry two independent electric charges and three independent angular momenta; however this solution has not been found yet, even though it is likely to exist. The black hole we are considering here presents two independent electric charges and three equal angular momenta; it is worth mentioning that there is a solution, constructed in [143], which carries three independent angular momenta with the two electric charges being equal.

2.4.2 *BPS AdS$_7$ black holes*

Now we turn to AdS$_7$ BPS black hole solutions, introducing the BPS version of the finite-temperature solution presented in the last subsection. As we have already stated for the previous cases, in order to land to the BPS black hole we need to impose both supersymmetry and extremality, which are two separate conditions. For ease of notation, we will set $g = 1$ from now on.

The solution is supersymmetric (preserving two supercharges) if [110]

$$\mathfrak{a} = \frac{2}{3(1-\mathrm{e}^{\delta_1+\delta_2})}\,. \tag{2.170}$$

We will always use this relation to eliminate $\mathfrak{a}$ in the expressions below. The solution depends now on three parameters, which are m, δ_1, δ_2. For simplicity, we restrict to the case of two equal electric charges by imposing $\delta_1 = \delta_2 \equiv \delta$ (and similarly $c_1 = c_2 \equiv c$, $s_1 = s_2 \equiv s$); the extension $\delta_1 \neq \delta_2$ is in principle straightforward although the computations are much more involved. Now we therefore have $\Phi_1 = \Phi_2 \equiv \Phi$ and $Q_1 = Q_2 \equiv Q$.

To avoid closed timelike curves from our family of solutions, we take

$$m = m_\star = \frac{4\,\mathrm{e}^{-3\delta}\,(c+2\,s)^3}{729\,c^2\,s^6}\,, \tag{2.171}$$

imposing this in addition to (2.170) implies vanishing of the temperature and thus leads to the BPS solution, whose horizon is located at

$$r_\star^2 = -\frac{16}{3\,(2\,\mathrm{e}^{2\delta} - 3\,\mathrm{e}^{4\delta} + 5)}\,. \tag{2.172}$$

The square of the horizon coordinate must be positive, so the equation above implies

$$\mathrm{e}^{2\delta} > \frac{5}{3}\,; \tag{2.173}$$

this is a physical condition on the parameter δ that we shall always assume in the following.

The chemical potentials are fixed to the BPS values

$$\Omega^\star = 1\,, \qquad \Phi^\star = 2\,, \qquad \beta \to \infty\,, \tag{2.174}$$

while the BPS charges are given by

$$\begin{aligned}
E^\star &= \frac{16\,\pi^2\left(-21\,\mathrm{e}^{4\delta} + 18\,\mathrm{e}^{6\delta} + 7\right)}{3\,(5 - 3\,\mathrm{e}^{2\delta})^4\,(\mathrm{e}^{2\delta}+1)^2}\,,\\
J^\star &= \frac{16\,\pi^2\left[9\,\mathrm{e}^{2\delta}\left(\mathrm{e}^{2\delta}+2\right) - 23\right]}{9\,(5 - 3\,\mathrm{e}^{2\delta})^4\,(\mathrm{e}^{2\delta}+1)^2}\,,\\
Q^\star &= -\,\frac{\pi^2\,\tanh\delta\,\mathrm{e}^{-3\delta}}{(c-4\,s)^3}\,,
\end{aligned} \tag{2.175}$$

these satisfy the supersymmetry linear relation

$$E^\star - 3\,\Omega^\star J^\star - 2\,\Phi^\star\,Q^\star = 0\,. \tag{2.176}$$

The BPS entropy can be immediately computed and it is given by

$$S^\star = \frac{2\,\pi^3\sqrt{c+8\,s}}{3\,\mathrm{e}^{4\delta}\,\sqrt{3\,c^3}\,(4\,s-c)^3}\,, \tag{2.177}$$

this can be written in terms of the charges as [69]:

$$S^\star = 2\pi\sqrt{\frac{32\,(Q^\star)^3 - 3\,\pi^2\,(J^\star)^2}{32\,Q^\star - \pi^2}}\,. \tag{2.178}$$

3 THE BPS LIMIT OF ROTATING ADS BLACK HOLE THERMODYNAMICS

In Chapter 2 we have introduced some families of finite-temperature AdS black holes for each dimension $4 \leq d \leq 7$. Furthermore, we have shown how, by imposing supersymmetry and extremality to the finite-temperature solution, one can obtain BPS black holes. In Chapter 1, sec. 1.4.2, we have reviewed a general feature that has been identified in the last few years, namely the fact that the Bekenstein-Hawking entropy of BPS black holes in AdS arises from an *extremization principle* [54,56, 65–68, 70]: the entropy is reproduced as the Legendre transform of an *entropy function* of rotational and electrical chemical potentials ω^i, Δ^I. These are subject to a linear, complex constraint of the following kind

$$\sum_i \omega^i - \sum_I \Delta^I = 2\,\pi\,i \ . \tag{3.1}$$

For none of the BPS black hole solutions we looked at in the previous chapter it is obvious how to retrieve the BPS chemical potentials ω^i and Δ^I. These cannot be the usual chemical potentials Ω^i and Φ^I, since, being fixed to a specific value, they are always trivial in the BPS solutions. Furthermore, in our analysis there is no emergence of complex quantities, since everything seems to stay real. Therefore it is not obvious at all how to read the chemical potentials in (3.1) from the black hole solution. As a consequence, being the entropy function a function of the BPS chemical potentials, it is not clear how we can obtain it from the black hole solution, neither what is its physical interpretation, both on the gravity and on the field theory side of the holographic correspondence.

Another natural question that may arise in the mind of the reader is what the physical interpretation of the constraint (3.1) could be. This was understood in [66] as a regularity condition for the Killing spinor of the supersymmetric solution, ensuring that this is *antiperiodic* around the Euclidean time circle of finite length β corresponding to the orbit of the Killing generator of the horizon. In fact the only spin structure allowed in the topology of the cigar formed by the radial direction and the orbit of the Killing generator is the one of an antiperiodic spinor.

In this chapter, we solve the problems stated above by showing that, for each family of rotating AdS_d black holes we have introduced in Chapter 2, the entropy function is the supergravity on-shell action after taking a specific BPS limit, that goes along a supersymmetric trajectory in the space of complexified solutions[42]. By performing the mentioned limit, the BPS complex chemical potentials ω^i and Δ^I can be retrieved from the black hole solutions. These results have been obtained for the first time in [66], where the authors applied the BPS limit to the family of AdS_5 black holes constructed in [142] which presents two independent angular momenta and one electric charge. Later, important generalizations have been provided in [67], where the analysis of [66] has been extended to other classes of rotating AdS black holes in different dimensions, thus showing the universality of the approach of [66]. Here we will review the results of [67] and we will show that, for every family of AdS black holes we have introduced in Chapter 2, by performing the proposed BPS limit, the BPS chemical potentials can be read from the black hole solution and that the entropy function can be always identified with the supergravity on-shell action.

Before proceeding to analyze every specific case in each dimension, we briefly summarize the main results we will get. Starting from the finite-temperature solution, we want to reach the BPS locus in parameter space, namely the BPS solution. Motivated by the fact that in the dual field theory one is mostly interested in studying a supersymmetric ensemble of states, we adopt the strategy of [66] and first impose supersymmetry, namely that the supergravity Killing spinor equations are solved. As we have stressed various times in the previous chapters, imposing supersymmetry amounts to precisely one condition that the parameters of the original solution have to satisfy, and it does not imply extremality, i.e. vanishing of the temperature[43]. However we have also seen that, in Lorentzian signature, the supersymmetric solution has causal pathologies unless one also imposes another condition on the parameters which is equivalent to send the temperature to zero [110]. This condition is usually imposed together with supersymmetry, since one wants to get rid immediately of these pathologies. Here, following [66, 67], we choose to do something different: we only impose super-

42 The entropy function of *non-rotating* BPS black holes has been related to the supergravity on-shell action in [144–146] for AdS_4 black holes and in [147] for AdS_6 black holes. Using the approaches proposed in those papers, everything stays real and there is no complexification to be considered. However complexification do emerge in the recent paper [148]. There, the authors construct supersymmetric black saddle solutions in the STU model of four-dimensional gauged supergravity which are holographically dual to partially twisted ABJM theory on $S^1 \times \Sigma_g$ for arbitrary values of the deformations parameters of the theory. They explicitly show that the regularized on-shell action of the black saddles agrees with the topologically twisted index. Some of these black saddle solutions can be Wick-rotated to the well known supersymmetric dyonic AdS_4 black holes of the supergravity theory under consideration; therefore the entropy of this black holes may be reproduce starting from the black saddle solutions, which are complex.

43 For convenience, we remind what we already stated in the previous chapters, i.e. that a quantity evaluated after imposing both supersymmetry and extremality will be called "BPS" and denoted by the symbol $\star$ in the formulae. For instance, $S^\star$ is the BPS entropy.

symmetry and carefully study the so obtained family of solutions. We are interested in semi-classical saddle points of the Euclidean path integral, and thus allow for more general solutions by complexifying (one of) the remaining parameters. Thus, we obtain a complexified family of supersymmetric solutions; this is interesting since it gives in principle the possibility to build complex quantities and so are the BPS chemical potentials and the entropy function we are looking for. To move on this direction, building on ideas of [149], we introduce the variables

$$\omega^i = \beta\left(\Omega^i - \Omega^{i\star}\right), \qquad \Delta^I = \beta\left(\Phi^I - \Phi^{I\star}\right), \tag{3.2}$$

where $\Omega^{i\star}$ and $\Phi^{I\star}$ are the (frozen) values taken by Ω^i and Φ^I in the BPS solution. The variables (3.2) are the chemical potentials conjugate to the angular momenta and electric charges when one identifies the generator of "time" translations with the conserved quantity $\{\mathcal{Q}, \overline{\mathcal{Q}}\}$, where $\mathcal{Q}$ is the supercharge.

The dual superconformal field theory partition function, Z, is defined by the asymptotic behavior of the supergravity solution; starting from the considerations above, the following Hamiltonian representation can be inferred:

$$Z = \mathrm{Tr}\left[\mathrm{e}^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+\omega^i J_i+\Delta^I Q_I}\right], \tag{3.3}$$

where there is no $(-1)^F$ due to anti-periodicity of the supercharge. The authors of [66] have shown that, upon using (3.1), the partition function Z is proportional to the superconformal index [46, 47]; in particular there is an identification between the SCFT chemical potentials appearing in (3.3) and the black hole variables (3.2).

Following [67], here we will show, for each black hole we introduced in Chapter 2, that after imposing supersymmetry the variables (3.2) satisfy a linear constraint of the type (2.5) and are otherwise free. We also show that the supersymmetric on-shell action I can indeed be written as a simple function of the variables (3.2), so that it precisely matches the entropy functions proposed in [65, 68, 70]. It is important to underline the fact that these results are referred to solutions which are supersymmetric only: we have not imposed extremality yet. The supersymmetric thermodynamical variables satisfy the following supersymmetric form of the quantum statistical relation

$$I = -S - \omega^i J_i - \Delta^I Q_I, \tag{3.4}$$

while the first law of thermodynamics in the supersymmetric ensemble reads

$$\mathrm{d}S + \omega^i \,\mathrm{d}J_i + \Delta^I \,\mathrm{d}Q_I = 0. \tag{3.5}$$

In the expressions above the energy does not appear since we have replaced it with the other charges by using the linear supersymmetric relation between the charges which emerges as a consequence of supersymmetry algebra. It is not immediately obvious to identify the Legendre transform of I, subject to the constraint (3.1), with the entropy of the Lorentzian solution, since the former turns out to be in general a complex quantity while the latter is naturally real. However, demanding reality of the Legendre transform, which amounts to a non-linear condition on the charges, one finds precisely the Bekenstein-Hawking entropy of the *supersymmetric and extremal* black hole [66,69]. The saddle point values of the chemical potentials remain complex and match the ones that we obtain from the solution by taking the zero-temperature limit of (3.2). We have thus reached the two following important results:

- taking the proposed BPS limit, the variables (3.2) become indeed the BPS chemical potentials we were looking for. Now we are thus able to read them from the supergravity black hole solution;

- the BPS limit of black hole thermodynamics we have described gives a derivation of the proposed entropy functions and the related extremization principles.

Some remarks are in order. The first one is that it is important to underline that the on-shell action I, which enters in the quantum statistical relation, is first defined in a regular Euclidean solution where the Wick-rotated time has been compactified and the metric is positive-definite. Then, after the on-shell action has been computed in this way, it is possible to extend its value to a complexified solution by analytic continuation [118]. As it should be clear from the discussion above, we find that this complexification is crucial for the on-shell action to eventually match the proposed entropy functions. A further remark regards the fact that the same family of black hole solutions we have introduced in Chapter 2, on which we will proceed to apply our BPS limit, have been considered also in [69, 70]. In these papers, the BPS limit of quantum statistical relation has been discussed; however our limit is different from the one considered there precisely because it reaches the physical BPS black hole through a complexified family of solutions, specified by supersymmetry. Instead, the limit taken in [69, 70] appears similar to the one originally discussed in [149], in that it yields real chemical potentials that satisfy just the real part of (3.1). Therefore, the on-shell action in this other limit does not match the entropy functions proposed in [65, 68, 70].

A final comment regards the relation between our BPS limit and Sen's entropy function formalism. This latter is based on a near-horizon analysis of extremal black holes, which are not necessarily supersymmetric [150,151]. Using Sen's approach, one obtains real chemical potentials and real entropy function, therefore it appears quite different from ours. It was shown in [152] that Sen's formalism is matched by an extremal limit of black hole thermodynamics where all quantities remain real. In [66] it has been shown how there is in fact a continuous family of extremal limits of black hole thermodynamics, all leading to a meaningful entropy function and an associated constraint between chemical potentials. However, it is fundamental to underline the fact that, of all these possible limits, only the manifestly supersymmetric one discussed above leads to the entropy functions proposed in [65, 68, 70]. One task which would be surely interesting to complete in the future is to clarify further the relation between our limit and Sen's near-horizon approach.

In the next four sections, we proceed to analyze the BPS limit for AdS_5, AdS_4, AdS_6 and AdS_7 black holes respectively[44]. In each case, we will introduce and discuss our BPS limit, we will get the explicit values of the BPS chemical potentials from the supergravity black hole solution and we will show that the on-shell action matches the proposed entropy function. For the AdS_5 and AdS_4 black hole solutions we have evaluated the on-shell action explicitly by using holographic renormalization; appendix A and appendix B contain the details on these computations in five and four dimensions, respectively.

3.1 BPS LIMIT FOR ADS$_5$ BLACK HOLES

We start describing our BPS limit for the AdS_5 finite temperature solution presented in sec. 2.1.2.

As we have already noted in sec.. 2.1.3 when we have looked at the BPS version of this black hole, in the BPS solution the chemical potentials take the fixed values $\Omega = \Omega^\star$, $\Phi^I = \Phi^{I\,\star}$, $\beta^{-1} = 0$. Since these are just trivial, they cannot be the BPS chemical potentials ω and Δ^I. Therefore, one can ask if the BPS black hole satisfies non-trivial thermodynamic relations and, in particular, what is the BPS version of the quantum statistical relation (2.42). There are many possible limits towards the BPS solution [149]; among these there is the one proposed in [66] that reaches the BPS point along a supersymmetric trajectory in parameter space, thus fully respecting supersymmetry. This limit was applied in [66] to the AdS_5 black hole of [43], with two independent angular momenta and only one electric charge; this provided a result that agrees with dual supersymmetric field theory computations. The proposed BPS limit has then been generalized in [67], where it has been applied to other AdS black holes, including the finite-temperature one we have presented in sec. 2.1.3.

We start by eliminating the parameter $\mathfrak{a}$ imposing the supersymmetry condition (2.96) and for now we do not require (2.99) to clear the solution from closed timelike curves; recall that imposing this last condition would imply extremality. Although for $m \neq m^\star$ the Lorentzian solution has closed timelike curves, we are interested in saddle points of the quantum gravity path integral, and thus allow ourselves to work

44 As we have done in the previous chapter, here we choose to present first the AdS_5 black holes rather than the AdS_4 ones, since the former will be the main focus of this thesis and the ones we will analyze the most.

with a complex section of the solution, to be specified momentarily. The supersymmetric family of solutions we are considering now depends on the four parameters m, μ_1, μ_2, μ_3. One natural manner to impose extremality would be to send the outer horizon coordinate r_+ to the BPS horizon $r_\star$, given in (2.101); this would be easy to do if we trade one of the four parameters of the supersymmetric solution for the outer horizon position r_+. This can be achieved by solving the equation $Y(r_+) = 0$ for m so as to trade m for r_+. Since the equation $Y(r_+) = 0$ is of third order in m, its solution is quite complicated. To circumvent this complication, we first change the radial coordinate r into a new coordinate R, such that[45]:

$$r^2 = R^2 + \frac{m}{m_\star}\left(r_\star^2 - \mu_1\right) . \tag{3.6}$$

Performing this change of coordinates, we break the symmetry in the μ_i, but this will be restored in the final results regarding the BPS solution. In the new radial coordinate, the outer horizon position is given by the largest root R_+ of the equation $Y(R) = 0$. From (3.6) we see that in the new coordinate the BPS horizon is found at

$$R_\star^2 = \mu_1 . \tag{3.7}$$

Now the equation $Y(R_+) = 0$ is only quadratic in m, and its solution can be written as:

$$m = \frac{2m_\star R_+^4(R_+^2+1)}{R_+^4(2\mu_1-\mu_2-\mu_3)+R_+^2(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1+2\mu_1)-\mu_1\mu_2\mu_3 \mp (R_+^2-\mu_1)\mathcal{R}}, \tag{3.8}$$

where we introduced the quantity:

$$\mathcal{R} = \sqrt{R_+^4(\mu_2-\mu_3)^2 - 2R_+^2\mu_2\mu_3(\mu_2+\mu_3+2)+\mu_2^2\mu_3^2} . \tag{3.9}$$

It is now crucial to study the sign of the argument of the above square root. It is easy to see that this is undefined: indeed, for very large R_+^2 we have that $\mathcal{R}$ is real, thus m is real; on the other hand, for R_+^2 sufficiently close to $R_\star^2 = \mu_1$, that is sufficiently close to the extremal value, the square root $\mathcal{R}$ is purely imaginary as a consequence of (2.86). Therefore, the expression for m in (3.8) may be complex and in particular it is complex close to extremality. In the strict extremal limit $R_+^2 = R_\star^2$ we have that the factor multiplying $\mathcal{R}$ in (3.8) goes to zero, so although $\mathcal{R}$ is purely imaginary, m becomes real and reaches its BPS value (2.99).

Thus, we have discovered that fixing $\mathfrak{a}$ as in (2.96) and trading m for R_+ as in (3.8), we reach a family of complexified, supersymmetric solutions. Obtaining complex quantities is promising since we know

45 In terms of the old parameters δ_I, the change of coordinate is expressed as $r^2 = R^2 - 2m\sinh^2\delta_1$. This implies $r^2 H_1 = R^2$. The new coordinate R should not be confused with the Ricci scalar.

that the BPS chemical potentials and the BPS entropy function we are looking for must be complex. Evaluating the quantities (2.36) for this family of solutions, we obtain quite cumbersome expressions, that we will not display here. Remarkably, we find that the chemical potentials obtained in this way satisfy the constraint:

$$\beta\,(1+\Omega-\Phi^1-\Phi^2-\Phi^3) = \mp 2\,\pi\,i\,, \tag{3.10}$$

where the sign choice follows from the one in (3.8). Although we have obtained the full expressions of the complex chemical potentials, deriving the constraint above has proven to be not a trivial task due to the complexity of these expressions and we did not manage to do it in full generality. However we verified the validity of (3.10) with many numerical checks over a wide range of the parameters as well as in a perturbative expansion near the BPS point that we will show below. Before showing this, it is worth mentioning that while performing this analysis over the range of the parameters we noted that eq. (3.10) is satisfied in two slightly different ways, depending on the value of R_+^2. As we described above, for sufficiently large R_+^2, m is real; however in this case β is purely imaginary, so that (3.10) holds true[46]. On the other hand, when R_+^2 is close to the extremal value, m is complex, and so are the chemical potentials β, Ω, Φ^I; still (3.10) is satisfied.

We now display some perturbative expansions near the BPS point of some relevant physical quantities, in order to make their near-extremal behavior more explicit and to show that the constraint (3.10) is satisfied. To obtain these expansions, we set $R_+ = R_\star + \epsilon$ and study the limit $\epsilon \to 0$. We perform the computations choosing the upper sign in (3.8) for simplicity. In this limit, the "inverse temperature" β diverges as:

$$\beta = \frac{4\,S^\star + i\,\pi^2\,\left[2\,\mu_1^2 - \mu_2\,\mu_3 + \mu_1\,(2+\mu_2+\mu_3)\right]}{4\,\pi\,\epsilon\,\sqrt{\mu_1}\,(1+\mu_1+\mu_2+\mu_3)} + \mathcal{O}(\epsilon^0)\,, \tag{3.11}$$

where $S^\star$ is the BPS entropy given in (2.87). Hence β is complex at leading order near the BPS point. The same holds for the other chemical potentials, which at first order in ϵ read

$$\begin{aligned}
\Omega &= \Omega^\star - \frac{4\,S^\star - i\,\pi^2\,(\mu_1\mu_2+\mu_3\mu_1-\mu_2\mu_3)}{S^\star\,\sqrt{\mu_1}\;(1+\mu_1)}\,\epsilon\,,\\
\Phi^1 &= \Phi^{1\,\star} + \frac{4S^\star\,[\mu_1\mu_2+\mu_3\mu_1-\mu_2\mu_3] - 2i\pi^2\left[(1+\mu_1)\mu_1\mu_2\mu_3 - 8(S^\star/\pi^2)^2\right]}{2\,S^\star\,\mu_1^{5/2}\,(1+\mu_1)\,\mu_2\,\mu_3}\,\epsilon
\end{aligned} \tag{3.12}$$

with Φ^2, Φ^3 being obtained from Φ^1 by a cyclic permutation of the μ_I. It follows that

$$1+\Omega-\sum_I \Phi^I = \frac{-4\,\mu_1^2 + 2\,\mu_2\,\mu_3 - 2\,\mu_1\,(2+\mu_2+\mu_3) - 8\,i\,S^\star/\pi^2}{\mu_1^{3/2}\;(1+\mu_1)}\,\epsilon + \mathcal{O}(\epsilon^2)\,.$$

46 We recall that β is given by the expression in (2.36), so it is purely imaginary when f_1 is negative. We find that this happens precisely in the regime where m is real. In this discussion we are assuming that R_+^2 is real.

(3.13)

It is now easy to check that the constraint (3.10) holds in this near-extremal limit: indeed multiplying the complex quantities (3.11) and (3.13), the factors of ϵ cancel out and the finite result (3.10) is obtained.

Our next step is to introduce the variables

$$\omega = \beta\,(\Omega - \Omega^\star)\,, \qquad \Delta^I = \beta\left(\Phi^I - \Phi^{I\,\star}\right)\,; \tag{3.14}$$

these will prove to be the right chemical potentials for the supersymmetric and BPS family of solutions. Furthermore, we introduce the supersymmetric Hamiltonian, which reads

$$\begin{aligned} H &= E - \Omega^\star J - \Phi^{I\,\star}\,Q_I \\ &= E - 2J - Q_1 - Q_2 - Q_3\;. \end{aligned} \tag{3.15}$$

There are crucial differences between E and H: the former is the charge for translations generated by $\frac{\partial}{\partial t}$, while the latter is the charge for translations generated by the Killing vector $K = \frac{\partial}{\partial t} + \Omega^\star \frac{\partial}{\partial \psi}$ that arises as a bilinear of the Killing spinor, covariantized by the term $\iota_K A^I|_\infty Q_I = -\Phi^{I\,\star}\,Q_I$. There is a relation between the supersymmetric Hamiltonian and the anticommutator of the supercharges: this is given by $\{\mathcal{Q}, \overline{\mathcal{Q}}\} = H - E_0$, where E_0 is the anomalous term induced by the supercurrent anomaly [122, 123]. With respect to the new variables, the quantum statistical relation (2.42) can be expressed as:

$$I = \beta H - S - \omega\,J - \Delta^I\,Q_I\,, \tag{3.16}$$

from the relation above, one can see that ω and Δ^I are the chemical potentials conjugate to J and Q_I, respectively, when the time translations are generated by the supersymmetric Hamiltonian H. We are considering a supersymmetric family of solutions, therefore the anticommutator $\{\mathcal{Q}, \overline{\mathcal{Q}}\}$ evaluates to zero and thus $H = E_0$. Consequently, the quantum statistical relation becomes the *supersymmetric quantum statistical relation*

$$I - I_0 = -S - \omega\,J - \Delta^I\,Q_I\,, \tag{3.17}$$

where we used $I_0 = \beta E_0$. Using the new variables, the constraint (3.10) can be written as

$$\omega - \Delta^1 - \Delta^2 - \Delta^3 = \mp\,2\,\pi\,i\,. \tag{3.18}$$

This is promising since it presents the same form of (3.1); although we must recall that here we are still considering a supersymmetric (and not BPS) family of solutions. Varying the supersymmetry relation between the charges and subtracting this from the first law (2.40), we obtain a supersymmetric form of the first law:

$$\mathrm{d}S + \omega\,\mathrm{d}J + \Delta^I\,\mathrm{d}Q_I = 0\,. \tag{3.19}$$

Moreover, plugging the supersymmetric condition (2.96) and the expression (3.8) for m into the on-shell action (2.45), we find that the latter takes the simple form:

$$I - I_0 = \pi \, \frac{\Delta^1 \Delta^2 \Delta^3}{(\omega)^2} \,. \tag{3.20}$$

Notice that the right hand side of (3.20) is independent of β. Once again, this is promising since it has exactly the same form of the BPS entropy function, even though we are looking at a family of only supersymmetric solutions.

Before taking the limit to extremality, it is worth noting that the supersymmetric charges E, J and Q_I can be evaluated by substituting the supersymmetry condition (2.96) and the formula (3.8) for m in (2.38). We do not display the expressions so obtained, since they are quite cumbersome; however we checked that, although they are generically complex, they satisfy the supersymmetry relation $H = E_0$. In the same fashion we obtain a complex expression for the entropy. The fact that the entropy (that is the area of the horizon) is complex is related to the fact that when continued back to Lorentzian signature, the supersymmetric but non-extremal solution presents a pseudo-horizon rather than a horizon [110].

Now we proceed to take the limit to extremality by sending $R_+ \to R_\star$. Doing this, our complexified family of supersymmetric solutions reaches the real, BPS solution of [42]. All the main physical quantities become real; in particular the entropy and the charges take the values (2.87), (2.89). In the extremal limit $R_+ \to R_\star$, the chemical potentials ω and Δ^I stay finite; this is possible because even if the temperature vanishes, and therefore β diverges, at the same time $\Omega \to \Omega^\star$, $\Phi^I \to \Phi^{I\,\star}$. We denote the BPS values of the redefined chemical potentials as

$$\omega^\star = \lim_{R_+ \to R_\star} \omega \,, \qquad \Delta^{I\,\star} = \lim_{R_+ \to R_\star} \Delta^I \,. \tag{3.21}$$

By evaluating these limits we obtain

$$\begin{aligned}
\omega^\star &= \frac{-2\pi}{\sum_I \mu_I + 1} \left[\frac{\mu_1\,\mu_2 + \mu_2\,\mu_3 + \mu_3\,\mu_1}{\sqrt{4\mu_1\mu_2\mu_3 \left(\sum_I \mu_I + 1\right) - (\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)^2}} \pm i \right], \\
\Delta^{1\,\star} &= \frac{\pi}{\sum_I \mu_I + 1} \left[\frac{\mu_1(\mu_2^2 + \mu_3^2) - \mu_2\,\mu_3(\mu_2 + \mu_3 + 2)}{\sqrt{4\mu_1\mu_2\mu_3 \left(\sum_I \mu_I + 1\right) - (\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)^2}} \right. \\
&\qquad\qquad \left. \pm\, i\,(\mu_2 + \mu_3) \right],
\end{aligned} \tag{3.22}$$

with the expressions for $\Delta^{2\,\star}$ and $\Delta^{3\,\star}$ being obtained from the one for $\Delta^{1\,\star}$ through straightforward permutations of the indices $1, 2, 3$. As it is evident from the expressions above, the supersymmetirc chemical potentials remain complex even after the BPS limit is taken[47]. Therefore,

47 Note that the argument of the square roots in (3.22) is positive due to assumption (2.86), and proportional to the BPS entropy (2.87).

the non-trivial BPS chemical potentials providing interesting thermodynamical relations are not the leading order terms (2.88) in the expansion of the chemical potentials Ω and Φ^I around their BPS value, but rather the next-to-leading-order terms:

$$\Omega = \Omega^\star + \frac{1}{\beta}\,\omega^\star + \ldots\,, \qquad\qquad \Phi^I = \Phi^{I\,\star} + \frac{1}{\beta}\,\Delta^{I\,\star} + \ldots\,. \tag{3.23}$$

The BPS limit we have presented is totally smooth, so the BPS chemical potentials still satisfy the constraint

$$\omega^\star - \Delta^{1\,\star} - \Delta^{2\,\star} - \Delta^{3\,\star} = \mp 2\,\pi\,i\,, \tag{3.24}$$

and the on-shell action can still be written as

$$(I - I_0)^\star = \pi\,\frac{\Delta^{1\,\star}\Delta^{2\,\star}\Delta^{3\,\star}}{(\omega^\star)^{\,2}}\,. \tag{3.25}$$

As we have mentioned in Chapter 2, the supersymmetric on-shell action $I - I_0$, seen as a function of the chemical potentials (3.14), should be regarded as minus the logarithm of the supersymmetric grand-canonical partition function in the semi-classical approximation to the quantum gravity path integral, in the spirit of [118]. Therefore, its Legendre transform must be the logarithm of the microcanonical partition function, that is the entropy. This gives a physical derivation of the extremization principle proposed in [65].

However, because of the constraint (3.18), the Legendre transformation is not completely straightforward and it is worth recalling here the main steps one has to follow in order to obtain it[48]. We consider the supersymmetric quantum statistical relation (3.17) and we enforce the constraint (3.18) through a Lagrange multiplier Λ, as follows

$$I - I_0 = -S - \omega\,J - \Delta^I\,Q_I - \Lambda\left(\omega - \Delta^1 - \Delta^2 - \Delta^3 \pm 2\,\pi\,i\right)\,. \tag{3.26}$$

One may be confused by the fact that the constraint is identically satisfied in the supersymmetric solution, but at this stage we are not assuming any explicit expression for ω, Δ^I, since we want to treat them as the basic variables to be varied. The next step is to extremize (3.26) with respect to $\Lambda, \omega, \Delta^I$; doing so we retrieve the constraint (3.18) and the equations

$$-\frac{\partial(I - I_0)}{\partial\omega} = J + \Lambda\,, \qquad -\frac{\partial(I - I_0)}{\partial\Delta^I} = Q_I + \Lambda\,, \quad I = 1,2,3\,. \tag{3.27}$$

These state the conjugacy relations between supersymmetric charges and chemical potentials which we have already mentioned above. Now

48 The Legendre transformation for black holes in the $\mathrm{U}(1)^3$ theory has been described in detail in [66]. In [67] a further generalization has been provided by illustrating the Legendre transformation for black holes in $\mathcal{N} = 2$ Fayet Iliopoulos gauged supergravity with n_V arbitrary. We explicitly discuss this Legendre transformation in sec. 3.1.1.

we have to solve the five equations we have obtained for ω, Δ^I and Λ in terms of the charges J, Q_I. The explicit solutions are reported in [66]. Once these equations are solved, the solution must be plugged into (3.26); doing so, one obtains a formula for the entropy S that is a complex function of the charges. Further demanding reality of S, as well as of J, Q_I, one obtains precisely the non-linear relation (2.91) between the BPS charges, together with the expression (2.92) for the Bekenstein-Hawking entropy of the BPS black hole. The reality condition on S may be understood as a well-definiteness condition for the horizon area, and this is what leads to (2.91) in the extremization procedure.

As a check, we verified in [99] that this extremization is indeed realized in the black hole solutions, verifying in particular that the BPS chemical potentials (3.22) match the saddle point values of ω, Δ^I obtained by solving the extremization equations (3.27) in terms of the charges J, Q_I, demanding reality of the entropy, and substituting the parameterization (2.89) of the BPS charges. We also checked that this match still holds true when one compares the supersymmetric but non-extremal values of ω, Δ^I, even if in this case the entropy and the charges are generically complex.

3.1.1 *Entropy function for general AdS$_5$ black holes with arbitrary n_V*

We would like to go beyond S^5 compactification of type IIB supergravity, analyzing more general AdS$_5$ black holes. As we have already mentioned in sec. 2.1.1, the imprint of S^5 in the five-dimensional supergravity considered in this section is found in the specific number of vector multiplets (three, gauging the $\mathrm{U}(1)^3 \subset \mathrm{SO}(6)$ isometry group of S^5) and in the form of the C_{IJK} tensor controlling the matter couplings, which is given by (2.18). However, multi-charge, supersymmetric AdS$_5$ black holes are known more generally in five-dimensional Fayet-Iliopoulos gauged supergravity with an arbitrary number n_V of vector multiplets and an arbitrary choice of the tensor C_{IJK}, $I, J, K = 1, \ldots, n_V + 1$; some examples are provided by the family of black holes constructed in [42], which we have introduced in sec. 2.1.3.1, and by the more general solutions with two independent angular momenta found in [45]. The most general set of independent conserved charges carried by these solutions is composed by two angular momenta $J_1^\star, J_2^\star$ and $n_V + 1$ electric charges $Q_I^\star$.[49] An entropy function whose Legendre transform should reproduce the Bekenstein-Hawking entropy of these black holes has been conjectured in [68] and reads

$$I = \frac{\pi}{24} \frac{C_{IJK} \Delta^I \Delta^J \Delta^K}{\omega_1\, \omega_2}\,; \tag{3.28}$$

49 Here we do not mention the energy E since we eliminated it by using the linear relation between the charges that holds as a consequence of supersymmetry algebra.

following [67], here we prove that this is the correct entropy function and that the entropy is indeed reproduced, provided the chemical potentials satisfy the constraint

$$\omega_1 + \omega_2 - 3\,\bar{X}_I\,\Delta^I = \mp 2\,\pi\,i\,, \tag{3.29}$$

and in addition one demands reality of the Legendre transform.

We start from the function (3.28) of the rotational and electric chemical potentials ω_i, $i = 1, 2$, and Δ^K, $K = 1, \dots, n_V + 1$; we want to compute the Legendre transform, subject to the more general constraint

$$\omega_1 + \omega_2 - 3\bar{X}_K\Delta^K = 2\pi i n\,, \tag{3.30}$$

where n is a real number. We set up the extremization problem by writing the entropy using the quantum statistical relation and enforcing the constraint

$$\begin{aligned} S(Q_K, J_i) = \text{ext}_{\{\Delta^K,\omega^i,\Lambda\}}\Big[&- I - \Delta^K Q_K - \omega^i J_i \\ &- \Lambda(\omega_1 + \omega_2 - 3\bar{X}_K\Delta^K - 2\pi i n)\Big]\,, \end{aligned} \tag{3.31}$$

with Λ being a Lagrange multiplier and $\omega^i = \omega_i$. As we have done for the $\mathrm{U}(1)^3$ case, we vary (3.31) with respect to the chemical potentials and the Lagrange multiplier Λ, thus obtaining the extremization equations

$$-\frac{\partial I}{\partial \Delta^K} = Q_K - 3\bar{X}_K\,\Lambda\;, \qquad -\frac{\partial I}{\partial \omega^i} = J_i + \Lambda\;, \tag{3.32}$$

together with the constraint (3.30). The supergravity black hole solutions with arbitrary n_V constructed in [42, 45] present a symmetric scalar manifold, namely the C_{IJK} tensor satisfy the property (2.3); therefore we are allowed to use the same property in this derivation. Using (2.3), it is not hard to see that the equations above imply

$$0 = \frac{1}{6}C^{IJK}(-Q_I + 3\bar{X}_I\,\Lambda)(-Q_J + 3\bar{X}_J\,\Lambda)(-Q_K + 3\bar{X}_K\,\Lambda) - \frac{\pi}{4}(J_1 + \Lambda)(J_2 + \Lambda)\,, \tag{3.33}$$

the above condition can be written as a cubic equation for Λ in the following fashion

$$0 = p_0 + p_1\Lambda + p_2\Lambda^2 + \Lambda^3\;, \tag{3.34}$$

with

$$\begin{aligned} p_0 &= -\frac{1}{6}C^{IJK}Q_IQ_JQ_K - \frac{\pi}{4}J_1J_2\;, \\ p_1 &= \frac{3}{2}C^{IJK}\bar{X}_IQ_JQ_K - \frac{\pi}{4}(J_1 + J_2)\;, \\ p_2 &= -\frac{9}{2}C^{IJK}\bar{X}_I\bar{X}_JQ_K - \frac{\pi}{4}\;. \end{aligned} \tag{3.35}$$

Being a cubic equation, (3.34) can be straightforwardly solved for Λ; we denote by "Roots" the set of three solutions of the equation above. The next step is to solve the rest of the equations (3.32), together with the constraint (3.30); in this way the saddle point values of the chemical potentials ω^i, Δ^I will be determined. Solving the above mentioned equations and the constraint, we find

$$\omega^i = \frac{1}{6}\,\Xi\, C^{IJK}\tilde{Q}_I\tilde{Q}_J\tilde{Q}_K|\epsilon^{ij}|\tilde{J}_j\,, \qquad \Delta^I = -\frac{1}{2}\,\Xi\, C^{IJK}\tilde{Q}_J\tilde{Q}_K\tilde{J}_1\tilde{J}_2\,, \tag{3.36}$$

where we introduced $\tilde{Q}_I = Q_I - 3\bar{X}_I\Lambda$, $\tilde{J}_i = J_i + \Lambda$, along with

$$\Xi = \frac{4\pi i n}{3\tilde{J}_1\tilde{J}_2\, C^{IJK}\bar{X}_I\tilde{Q}_J\tilde{Q}_K + \frac{1}{3}(\tilde{J}_1+\tilde{J}_2)C^{IJK}\tilde{Q}_I\tilde{Q}_J\tilde{Q}_K}\,, \tag{3.37}$$

and it is understood that $\Lambda \in$ Roots.

It is possible to exploit the same argument used in [66] to show that the Legendre transform reads

$$S = \mathrm{ext}_{\Lambda\,\in\,\mathrm{Roots}}\left(2\pi i n\,\Lambda\right)\,, \tag{3.38}$$

and that S is real and positive if and only if one imposes

$$p_0 = p_1\,p_2\,, \tag{3.39}$$

and picks the purely imaginary root $\Lambda = i\sqrt{p_1}$ if $n<0$, or $\Lambda = -i\sqrt{p_1}$ if $n>0$. From the explicit expressions of the p_i, given in (3.35), we see that (3.39) is a constraint on the charges. Assuming the condition (3.39) to be satisfied, the Legendre transform (3.38) becomes

$$\begin{aligned} S &= 2\pi|n|\,\sqrt{p_1} \\ &= \pi|n|\sqrt{6\,C^{IJK}\bar{X}_I Q_J Q_K - \pi\left(J_1+J_2\right)}\,. \end{aligned} \tag{3.40}$$

For $n = \mp 1$, this is the Bekenstein-Hawking entropy of the black holes of [42, 45], in the form first given in [132]. Thus we have indeed proved that the Legendre transform of the entropy function (3.28) leads to the entropy of the asymptotically AdS_5 black holes of [42, 45]. However, in order to carry out the computation we presented so far for the explicit black hole solution, we would need to start from the non-supersymmetric finite temperature black hole solution which corresponds to the BPS one of [45] in the BPS limit. Unfortunately, this solution has not been found yet. The same problem holds in principle for the multi-charge solution of [42]; however in this latter case we will be able to infer the BPS chemical potentials from the ones we have found for the $U(1)^3$ solution, which have been given in eq. (3.22).

We can specialize the expressions above to the $U(1)^3$ model, so as to perform a consistency check with the results obtained in [66] for this theory. It is easy to see that indeed all the expressions we have presented reduce to those given in the above mentioned paper, upon identifying $Q_I^{\text{here}} = -Q_I^{\text{there}}$, $\Delta^I_{\text{here}} = -\Delta_{I\,\text{there}}$ and $\mu_{\text{there}} = -\frac{\pi}{4}$. Moreover, for $n = \mp 1$ the constraint (3.30) is perfectly consistent with (3.18).

Consider now the multi-charge black hole solution of [42], that we have introduced in sec. 2.1.3.1, where the two angular momenta are set equal. We recall that this solution is controlled by the real parameters q_I^{GR}, α_1^{GR}, α_2^{GR}, α_3^{GR}, with the α_i^{GR} given in eq. (2.64). Generalizing our formulae (3.22), we infer that the BPS chemical potentials for this general solution read

$$\begin{aligned}\omega_1^\star = \omega_2^\star \equiv \tfrac{1}{2}\,\omega^\star &= -\frac{\pi}{1+\alpha_1^{\text{GR}}}\left[\frac{\alpha_2^{\text{GR}}}{\sqrt{4\,(1+\alpha_1^{\text{GR}})\,\alpha_3^{\text{GR}} - (\alpha_2^{\text{GR}})^2}} \pm i\right],\\ \Delta^{I\,\star} &= \frac{9\pi}{1+\alpha_1^{\text{GR}}} C^{IJK} q_J^{\text{GR}}\left[\frac{\alpha_2^{\text{GR}}\,\bar{X}_K - (1+\alpha_1^{\text{GR}})\,q_K^{\text{GR}}}{\sqrt{4\,(1+\alpha_1^{\text{GR}})\,\alpha_3^{\text{GR}} - (\alpha_2^{\text{GR}})^2}} \pm i\,\bar{X}_K\right].\end{aligned} \tag{3.41}$$

We now show that the explicit expressions for the chemical potentials above satisfy the saddle point expressions (3.36). The entropy and charges for the BPS black hole of [42] are given by (2.74), (2.77) and (2.79), respectively. The charges satisfy the relation (3.39); correspondingly, the BPS solution has just $n_V + 1$ independent parameters, though there are $n_V + 2$ charges $J^\star, Q_I^\star$. In order to compare (3.36) with (3.41), we need to evaluate $C^{IJK}\tilde{Q}_J\tilde{Q}_K$, $C^{IJK}\tilde{Q}_I\tilde{Q}_J\tilde{Q}_K$ and $C^{IJK}\bar{X}_I\tilde{Q}_J\tilde{Q}_K$ in terms of the parameters. Straightforward though tedious computations making repeated use of the identity (2.3) lead us to:[50]

$$
\begin{aligned}
C^{IJK}\tilde{Q}^\star_J\tilde{Q}^\star_K &= \frac{1}{8}\left[\pi^2\left((1+\alpha_1)\alpha_3-\tfrac{1}{4}\alpha_2^2\right)+16\,\Lambda^2\right]\bar{X}^I \\
&\quad+\frac{9}{16}\left[\pi^2\left(1+\alpha_1+\tfrac{1}{2}\alpha_2\right)-4\pi\Lambda\right]C^{IJK}q^{\rm GR}_J q^{\rm GR}_K \\
&\quad-\frac{9}{16}\left[\pi^2\left(\alpha_2+2\alpha_3\right)+8\pi\Lambda\right]C^{IJK}\bar{X}_J q^{\rm GR}_K\,, \\
C^{IJK}\tilde{Q}^\star_I\tilde{Q}^\star_J\tilde{Q}^\star_K &= \frac{3\pi^3}{32}\bigg[\left(1+2\alpha_1+\alpha_1^2-\alpha_3\right)\alpha_3 \\
&\qquad+\frac{1}{2}(\alpha_1-1)\alpha_2\alpha_3-\frac{1}{4}(2+\alpha_1)\alpha_2^2-\frac{1}{8}\alpha_2^3\bigg] \\
&\qquad-\frac{3\pi^2}{8}\left(\alpha_1\alpha_3+\alpha_2+3\alpha_3-\frac{1}{4}\alpha_2^2\right)\Lambda \\
&\qquad+\frac{3\pi}{4}(2\alpha_1+\alpha_2)\Lambda^2-6\Lambda^3\,, \\
C^{IJK}\bar{X}_I\tilde{Q}^\star_J\tilde{Q}^\star_K &= \frac{\pi^2}{24}\left(\alpha_1\alpha_3+\alpha_2+3\alpha_3-\tfrac{1}{4}\alpha_2^2\right) \\
&\qquad-\frac{\pi}{6}\left(2\alpha_1+\alpha_2\right)\Lambda+2\Lambda^2\,.
\end{aligned}
\tag{3.42}
$$

For the purposes of this section, we are interested in the case where the Legendre transform is real and positive, and $n=\mp1$. Accordingly, we substitute $\Lambda=\pm i\sqrt{p_1}=\pm\frac{i}{2\pi}S^\star$ in the formulae above. Note that then the term proportional to $\bar{X}^I$ in the first line of (3.42) vanishes. Plugging (2.77), (2.79), (3.42) into (3.36), (3.37), we obtain precisely the expressions for the chemical potentials given in (3.41). As a final consistency check, we also verified that the thermodynamical quantities given in (3.28), (3.41), (2.77), (2.79) satisfy the supersymmetric quantum statistical relation $I^\star=-S^\star-\omega^\star J^\star-\Delta^{I\star}Q^\star_I$.

It is important to mention that the entropy function (3.28) has been reproduced from a dual SCFT$_4$ viewpoint by taking the Cardy-like limit of the superconformal index in [85]. There is the possibility that some of the black hole solutions of [42, 45] uplit to to type IIB supergravity on Sasaki-Einstein manifolds and thus have an SCFT$_4$ dual; however, at the time this thesis is written, the uplift is only known for the case of S^5, or in a single-charge limit where the black holes are solutions to minimal gauged supergravity. Furthermore, there is no known consistent truncation of type IIB supergravity on five-dimensional Sasaki-Einstein manifolds including all Kaluza-Klein vector fields gauging the relevant internal symmetries; therefore it is not completely clear how to match with the dual SCFT.

Another important remark is that the starting point of the BPS limit we have proposed is constituted by a finite-temperature solu-

50 These expressions also hold for the solution of [45], where the two angular momenta are different.

tion and these are only known within the $U(1)^3$ theory discussed here; this means that there is no known finite-temperature black solution in $\mathcal{N} = 2$ Fayet-Iliopoulos gauged supergravity with $n_V \neq 2$. Clearly, it would be interesting to construct new asymptotically AdS black holes in five-dimensional supergravity (for instance relaxing the assumption that the scalar manifold is symmetric), study their uplift to type IIB supergravity on different Sasaki-Einstein manifolds, and extend the results, originally presented in [67], we have review in this thesis by investigating their BPS limit. At this regard, it is worth mentioning that the near-horizon geometry of a black hole in type IIB on $T^{1,1}$ has been recently constructed in [78].

3.2 BPS LIMIT FOR ADS$_4$ BLACK HOLES

Here we describe our BPS limit for the AdS$_4$ finite-temperature family of black holes presented in sec. 2.2.1. This proceeds similarly to the five-dimensional case.

Since, as a first step, we want to reach the supersymmetric but not extremal family of solutions, we impose the supersymmetry condition (2.138) without requiring (2.139) for the moment. This last condition would imply extremality and $\Delta_r(r) = 0$, with Δ_r being given in (2.127). However, the equation $\Delta_r(r) = 0$ can be solved in a more general way than (2.139), (2.140) if we allow for complex values of the parameter m. In fact, $\Delta_r(r) = 0$ can be seen as an equation for m, where the solution depends on δ_1, δ_2 and on the position of the outer horizon, r_+. It turns out that $\Delta_r(r)$ is a quartic polynomial in m, therefore the solutions would be rather cumbersome; as we have already done in the five-dimensional case it is better to change the radial coordinate r into a new coordinate R so that the analysis is simplified. A good change of radial coordinate turns out to be

$$r = R - 2\, m\, s_1^2\,, \tag{3.43}$$

indeed, the equation for the horizon now becomes $\Delta_r(R) = 0$ and it is only quadratic in m. The solution of this equation is given by:

$$m = \frac{R_+^2 + 1 - (1 \pm i\, R_+)\coth(\delta_1 + \delta_2)}{R_+\,(c_1^2 + s_1^2 - c_2^2 - s_2^2) \mp 2\, i\, s_1\, c_1}\,, \tag{3.44}$$

where we have denoted as R_+ the position of the outer horizon. Now, R_+ is treated as a parameter, on the same footing as δ_1, δ_2.

We now plug the expression (3.44) for m, together with the supersymmetry condition (2.138), into the different quantities summarized in sec. 2.2.1. The conditions we have imposed imply supersymmetry but not extremality, so we land on a complexified supersymmetric but not extremal family of solutions. After these manipulations, we find that the chemical potentials in (2.133) satisfy the relation

$$\beta(1 + \Omega - \Phi_1 - \Phi_3) = \mp 2\,\pi\, i\,, \tag{3.45}$$

which is completely analogous to the constraint we have found in five dimensions. Therefore, we can argue that it has the same interpretation as an anti-periodicity condition for the Killing spinor when this is translated around the Euclidean time circle.

We proceed in analogy with the five-dimensional case by introducing the variables

$$\omega = \beta\,(\Omega - \Omega^\star)\,, \qquad \Delta_I = \beta\left(\Phi_I - \Phi_I^\star\right)\,, \tag{3.46}$$

where the BPS values $\Omega^\star, \Phi_I^\star$ were given in (2.141). Using these variables, the constraint (3.45) can be written as

$$\omega - \Delta_1 - \Delta_3 = \mp 2\,\pi\, i\,. \tag{3.47}$$

These variables are the right supersymmetric chemical potentials providing non trivial thermodynamical relations. Their explicit expressions are:

$$\begin{aligned}
\omega &= \frac{4\pi}{\Upsilon}\left[c_1\,(c_2 - 2\,s_2) + s_1(s_2 - 2\,c_2)\right]\left[R_+(c_1^2 - c_2^2 + s_1^2 - s_2^2) \mp 2\,i\,c_1\,s_1\right]\,,\\
\Delta_1 = \Delta_2 &= \frac{4\pi}{\Upsilon}\left(-c_1^2 + 2\,c_1\,s_1 + c_2^2 - s_1^2 + s_2^2\right)\left[R_+(c_1\,s_2 + c_2\,s_1) \mp i\,\mathrm{e}^{-\delta_1-\delta_2}\right]\,,\\
\Delta_3 = \Delta_4 &= \frac{4\pi}{\Upsilon}\left[R_+(c_1^2 - c_2^2 + s_1^2 - s_2^2) \mp 2i\,c_1 s_1\right]\\
&\qquad\qquad\left[(c_1c_2 + s_1s_2) - (1 \mp i\,R_+)(c_1s_2 + c_2s_1)\right]\,,
\end{aligned} \tag{3.48}$$

where we introduced:

$$\begin{aligned}
\Upsilon = {} & 2\,R_+\,(c_1\,s_2 + c_2\,s_1)\left[R_+\left(c_1^2 - c_2^2 + s_1^2 - s_2^2\right) \mp 4\,i\,c_1\,s_1\right]\\
& - c_1\,s_2 + c_2\,s_1 - 2\,\sinh\left(3\delta_1 + \delta_2\right) + \sinh\left(\delta_1 + 3\delta_2\right)\\
& + \cosh\left(3\delta_1 + \delta_2\right) - \cosh\left(\delta_1 + 3\delta_2\right)\,.
\end{aligned} \tag{3.49}$$

The conserved charges in (2.134) satisfy the linear relation

$$E - \Omega^\star J - 2\,\Phi_1^\star\,Q_1 - 2\,\Phi_3^\star\,Q_3 = 0\,, \tag{3.50}$$

which as already remarked is purely a consequence of supersymmetry. For this supersymmetric family of complexified solutions, the expressions for the charges, as well as the one for the entropy, are complex. We need them to become real when we perform the limit to the BPS solution.

After using (2.138), (3.43), the on-shell action (2.137) can be written in terms of the chemical potentials $\omega, \Delta_1, \Delta_3$ as:

$$I = \frac{1}{2\,i}\frac{\Delta_1\,\Delta_3}{\omega}\,. \tag{3.51}$$

Notice that this is independent of β, as also found in the five-dimensional analysis. The expression (3.51) is promising since the form of the right-hand side matches with the one of the entropy function for these black holes proposed in [70]; however it is important to remark that we are not yet considering BPS black holes, but just supersymmetric ones. The natural expectation that we have to verify is that eq. (3.51) will still hold for the BPS black hole solution. Using (3.50), the quantum statistical relation (2.136) takes the supersymmetric form:

$$I = -S - \omega\, J - 2\,\Delta_1\, Q_1 - 2\,\Delta_3\, Q_3\,. \tag{3.52}$$

Having fully analyzed the supersymmetric complexified solutions, we are ready to take the BPS limit by sending $R_+ \to R_\star$, where $R_\star$ is the map of the BPS horizon position $r_\star$ in (2.140) under the change of coordinate (3.43), that is:

$$R_\star = 2\, s_1\, c_1\, m_\star \tanh\left(\delta_1 + \delta_2\right)\,. \tag{3.53}$$

As it appears clear by looking at (3.44), when $R_+ \to R_\star$ the complex expression for m becomes real and gives $m \to m_\star$. Accordingly, the original chemical potentials are trivially fixed to the BPS values (2.141). Instead, the BPS limit of the redefined chemical potentials ω, Δ_1, Δ_3 appears to be more interesting, since β is going to infinity but at the same time the parenthesis in (3.46) vanish. We define the BPS chemical potentials as

$$\omega^\star = \lim_{R_+\to R_\star} \omega\,, \qquad \Delta_I^\star = \lim_{R_+\to R_\star} \Delta_I\,. \tag{3.54}$$

By evaluating them explicitly from (3.48), we find that they stay finite and that their explicit expressions are given by

$$\begin{aligned}
\omega^\star =& -\frac{16\,\pi}{\Theta}\left(\mathrm{e}^{2(\delta_1+\delta_2)}-3\right)\Big[4\,\mathfrak{R}\left(s_1\,c_1+s_2\,c_2\right)\\
&\qquad\qquad \pm 4\,i\,s_1\,s_2\,c_1\,c_2\,\left(c_1\,c_2+s_1\,s_2\right)\mathrm{e}^{\delta_1+\delta_2}\Big]\,,\\
\Delta_1^\star =& -\frac{16\,\pi}{\Theta}\Big\{\mathfrak{R}\left[\left(\mathrm{e}^{4\delta_2}-3\right)\mathrm{e}^{2\delta_1}-4s_2\,c_2+2\,\mathrm{e}^{-2\delta_1}\right]\\
&\mp 2\,i\,s_2\,c_2\left[\mathrm{e}^{2(\delta_1+\delta_2)}\left(c_1^2+s_1^2+2s_1\,c_1+2s_2\,c_2\right)-c_2^2-s_2^2-4s_2\,c_2\right]\Big\}\,,
\end{aligned} \tag{3.55}$$

where $\Delta_3^\star$ is obtained from $\Delta_1^\star$ by switching δ_1 and δ_2, and

$$\begin{aligned}
\Theta =& \,\mathrm{e}^{2(\delta_1+\delta_2)}\left(\mathrm{e}^{4\delta_1}+\mathrm{e}^{4\delta_2}-10\right)+6\,\mathrm{e}^{4(\delta_1+\delta_2)}+\mathrm{e}^{6(\delta_1+\delta_2)}\\
&-2\left(\mathrm{e}^{-4\delta_1}+\mathrm{e}^{-4\delta_2}\right)-2\,\left[4\left(\mathrm{e}^{4\delta_1}+\mathrm{e}^{4\delta_2}\right)-7\right]\\
&+\mathrm{e}^{-2(\delta_1+\delta_2)}\left[5\left(\mathrm{e}^{4\delta_1}+\mathrm{e}^{4\delta_2}\right)-3\right]\,,\\
\mathfrak{R} =& \sqrt{s_1\,s_2\,c_1\,c_2\,(s_1\,c_2+s_2\,c_1)\,\mathrm{e}^{\delta_1+\delta_2}}\,.
\end{aligned} \tag{3.56}$$

Therefore, the chemical potentials $\omega, \Delta_1, \Delta_3$ remain complex even after the BPS limit is taken; moreover, since the BPS limit is entirely smooth, they still satisfy the constraint (3.47)

$$\omega^\star - \Delta_1^\star - \Delta_3^\star = \mp\, 2\, i\, \pi\,. \tag{3.57}$$

The on-shell action at the BPS point retains the form of (3.51) and it is thus given by

$$I^\star = \frac{1}{2\,i}\frac{\Delta_1^\star\,\Delta_3^\star}{\omega^\star}\,, \tag{3.58}$$

moreover it satisfies

$$I^\star = -S^\star - \omega^\star J^\star - 2\,\Delta_1^\star\,Q_1^\star - 2\,\Delta_3^\star\,Q_3^\star\,. \tag{3.59}$$

The supersymmetric on-shell action (3.51) matches the entropy function proposed in [70]. There, it was shown that the BPS entropy follows from Legendre transforming this entropy function and demanding reality of the Legendre transform. Here we have shown a derivation of the entropy function from imposing supersymmetry in the black hole thermodynamics. This derivation has been provided for the first time in [67]. The relation with the entropy is clear from (3.58). The expressions (3.55) for the BPS chemical potentials match the saddle point values obtained from Legendre transforming the entropy function, as it can be checked by plugging the formulae (2.142) for the BPS charges in the saddles given in [70], and comparing with (3.55).

3.3 BPS LIMIT FOR ADS$_6$ BLACK HOLES

We now turn to show how our BPS limit can be applied to the six-dimensional black holes we have introduced in sec. 2.3.1. The procedure is very similar to the one illustrated for the four- and five-dimensional cases.

The complexified family of supersymmetric solutions is obtained by imposing the supersymmetry condition (2.155) and by solving the equation $\mathcal{R}(r_+) = 0$, with $\mathcal{R}$ given by eq. (2.150), for the parameter m, so that this is traded for the position of the outer horizon r_+. Here there is no need to change the radial coordinate as in the previous cases, since the equation is already of quadratic order in m. The solutions of $\mathcal{R}(r_+) = 0$ are given by

$$m = \frac{1}{2}\,(r_+ \mp i)\,(\mathfrak{a} \pm i\,r_+)\,(\mathfrak{b} \pm i\,r_+)\,(\mathfrak{a} + \mathfrak{b})\,(\mathfrak{a} + \mathfrak{b} + 2)\;, \tag{3.60}$$

therefore m is complex for real values of $\mathfrak{a}, \mathfrak{b}, r_+$. Plugging this expression for m in the chemical potentials (2.151), we find that these are complex and non trivially fixed as in the BPS case; furthermore they satisfy the constraint

$$\beta\,(1 + \Omega_a + \Omega_b - 3\,\Phi) = \mp 2\,\pi\,i\,. \tag{3.61}$$

As for the previous cases, we introduce the new chemical potentials

$$\omega_a = \beta\,(\Omega_a - \Omega_a^\star)\;, \qquad \omega_b = \beta\,(\Omega_b - \Omega_b^\star)\;, \qquad \Delta = \beta\,(\Phi - \Phi^\star)\;, \tag{3.62}$$

which satisfy

$$\omega_a + \omega_b - 3\,\Delta = \mp 2\,\pi\, i\,. \tag{3.63}$$

As a consequence of the supersymmetry algebra, the complex black hole charges satisfy the supersymmetry condition

$$E - \Omega_a^\star\, J_a - \Omega_b^\star\, J_b - \Phi^\star\, Q = 0\,. \tag{3.64}$$

Using (3.62), (3.64) in (2.154), we obtain the supersymmetric quantum statistical relation for these black holes

$$I = -S - \omega_a\, J_a - \omega_b\, J_b - \Delta\, Q\,. \tag{3.65}$$

For the AdS_6 case we did not evaluate the on-shell action explicitly, therefore we have to compute it from the expression above. Doing so, we find

$$I = \frac{\pi\, i}{3}\,\frac{\Delta^3}{\omega_a\,\omega_b}\,, \tag{3.66}$$

the right-hand side has the same form of the entropy function proposed in [70]. However, this result is referred to the complexified family of supersymmetric but non extremal solutions; we want to show that this also holds for the BPS black hole.

To do so, we take the extremal limit by sending $r_+ \to r_\star$. As in the previous cases, the BPS chemical potentials stay finite and take the limiting values

$$\begin{aligned}
\omega_a^\star &= \frac{2\, i\, \pi\, (\mathfrak{a}-1)\,(\mathfrak{b}+i\, r_\star)}{2\, i\, \mathfrak{a}\, \mathfrak{b}\, r_\star^{-1} + \mathfrak{a}\, \mathfrak{b} + \mathfrak{a} + \mathfrak{b} - 3\, r_\star^2}\,,\\
\omega_b^\star &= \frac{2\, i\, \pi\, (\mathfrak{b}-1)\,(\mathfrak{a}+i\, r_\star)}{2\, i\, \mathfrak{a}\, \mathfrak{b}\, r_\star^{-1} + \mathfrak{a}\, \mathfrak{b} + \mathfrak{a} + \mathfrak{b} - 3\, r_\star^2}\,,\\
\Delta^\star &= \frac{2\, i\, \pi\, (\mathfrak{a}+i\, r_\star)\,(\mathfrak{b}+i\, r_\star)}{2\, i\, \mathfrak{a}\, \mathfrak{b}\, r_\star^{-1} + \mathfrak{a}\, \mathfrak{b} + \mathfrak{a} + \mathfrak{b} - 3\, r_\star^2}\,.
\end{aligned} \tag{3.67}$$

Since the limit is smooth, the constraint

$$\omega_a^\star + \omega_b^\star - 3\,\Delta^\star = \mp 2\,\pi\, i \tag{3.68}$$

is still satisfied by the BPS chemical potentials (3.67); moreover, the BPS on-shell action results to be

$$I^\star = \frac{\pi\, i}{3}\,\frac{(\Delta^\star)^3}{\omega_a^\star\,\omega_b^\star}\,, \tag{3.69}$$

so that it coincides with the entropy function proposed in [70]. We obtained in this way a derivation of the BPS AdS_6 black hole entropy function.

3.4 BPS LIMIT FOR ADS$_7$ BLACK HOLES

The final case we discuss is the one of AdS$_7$ black holes. In this section we illustrate the BPS limit for the family of finite-temperature solutions we have introduced in sec. 2.4.2. The BPS limit for the AdS$_7$ black hole we examine has also been discussed in [96].

The procedure is totally analogous to the previous cases, therefore we will keep the presentation a bit shorter with respect to the previous sections. The first step is to obtain the the complexified family of supersymmetric solutions; to land on them, we impose the supersymmetry condition (2.170) and we solve the equation $Y(r_+) = 0$ for the parameter m, thus trading it for the position of the outer horizon r_+. This equation is of quadratic order in m, therefore it is immediately solved as:

$$m = -\frac{c+2\,s}{648\,s^6}\bigg\{\Big[3\,r_+^2\left(c^2-8\,c\,s+s^2+1\right)+8\,\mathrm{e}^{-2\delta}\Big]\,\mathcal{R}+16\,\mathrm{e}^{-3\delta} \\ +r_+^2\,(4\,s-c)\Big[2\,r_+^2\left(7\left(c^2+s^2\right)+4\,c\,s-9\right)-2\mathrm{e}^{-2\,\delta}-18\Big]\bigg\}\,, \tag{3.70}$$

where

$$\mathcal{R} = \sqrt{4\,\mathrm{e}^{-2\delta}-2\,r_+^2\left[7\,\left(c^2+s^2\right)+4\,c\,s-9\right]}\,. \tag{3.71}$$

We would like to establish whether the argument of this square root is positive or negative. Using the expression for $r_\star$ given in (2.172), we can write

$$\mathcal{R} = \sqrt{4-\frac{16\,r_+^2\left(r_\star^2-r_\star\sqrt{r_\star^2+1}+1\right)}{r_\star^2}}\,, \tag{3.72}$$

and using the physical condition $r_+ > r_\star$ it is easy to see that the argument satisfies the inequality

$$4-\frac{16\,r_+^2\left(r_\star^2-r_\star\sqrt{r_\star^2+1}+1\right)}{r_\star^2} < 4-\frac{8\,r_+^2}{r_\star^2} < 0\,, \tag{3.73}$$

so that it is negative for all the permitted values of r_+. Consequently, the square root is always imaginary and the expression for m given in (3.70) is therefore complex. This identifies our complexified family of solutions.

Plugging the above expression for m in the chemical potentials, we find that these are complex as expected and that they satisfy the constraint

$$\beta\,(1+3\,\Omega-2\,\Phi) = \mp 2\,\pi\,i\,. \tag{3.74}$$

As for the previous cases, we introduce the redefined chemical potentials

$$\omega = \beta\,(\Omega-\Omega^\star)\,, \qquad \Delta = \beta\,(\Phi-\Phi^\star)\,; \tag{3.75}$$

with respect to which the constraint (3.74) is written as:

$$3\,\omega - 2\,\Delta = \mp\, 2\,\pi\, i\,. \tag{3.76}$$

The on-shell action can be obtained by the quantum statistical relation (2.169), as for the six-dimensional case. We checked that it is given in terms of the supersymmetric chemical potentials by

$$I = -\frac{\pi^3}{128}\,\frac{\Delta^4}{\omega^3}\,. \tag{3.77}$$

Although we are considering a supersymmetric but not extremal family of solutions, this is very promising since the form of the right-hand side matches with the one of the entropy function proposed in [68]. Furthermore, the supersymmetric on-shell action and the chemical potentials satisfy the supersymmetric quantum statistical relation

$$I = -S - 3\,\omega\, J - 2\,\Delta\, Q\,. \tag{3.78}$$

We are ready to take the extremal limit by sending $r_+ \to r_\star$. The limiting values of the chemical potentials are:

$$\begin{aligned}
\omega^\star &= -\frac{6\,\pi\sqrt{2\,(16\,c\,s + c^2 + s^2 + 1)}\,(c - 4\,s)}{(c + 8\,s)\;\Theta}\times\\
&\qquad \times\left[c\left(\sqrt{3} - \sqrt{-8\,\tanh\delta - 1}\right) + 4\,s\left(\sqrt{-8\tanh\delta - 1} + 2\,\sqrt{3}\right)\right],\\
\Delta^\star &= -\frac{64\,\pi\,\left(\sqrt{3}\,\mathrm{e}^{-\delta}\,(c + 8\,s) + 9\,c\,s\sqrt{-8\tanh\delta - 1}\right)}{\Theta\,\mathrm{e}^{\delta}\sqrt{2\,(c^2 + 16\,c\,s + s^2 + 1)}}\,,
\end{aligned} \tag{3.79}$$

where we have defined

$$\begin{aligned}
\Theta &= 8\,c\,s\left(\sqrt{-3\,(8\,\tanh\delta + 1)} - 18\right) - 9\left(\sqrt{-3\,(8\,\tanh\delta + 1)} + 1\right)\\
&\quad + \left(23\sqrt{-3\,(8\,\tanh\delta + 1)} - 9\right)\left(c^2 + s^2\right)\,.
\end{aligned} \tag{3.80}$$

Since the limit towards extremality is smooth, the BPS chemical potentials satisfy the constraint

$$3\,\omega^\star - 2\,\Delta^\star = \mp\, 2\,\pi\, i\,, \tag{3.81}$$

which is completely analogous to what we have found in lower dimensions. In terms of these chemical potentials, the BPS n-shell action reads

$$I^\star = -\frac{\pi^3}{128}\,\frac{(\Delta^\star)^4}{(\omega^\star)^3}\,, \tag{3.82}$$

therefore it matches exactly with the entropy function proposed in [68]. This completes our derivation of the BPS entropy function from black hole thermodynamics.

d	Charges	BPS entropy	Entropy function
4	$J,\ Q_1 = Q_2,\, Q_3 = Q_4$	$\frac{\pi\, J^\star}{2(Q_1^\star + Q_3^\star)}$	$I^\star = \frac{1}{2\,i}\, \frac{\Delta_1^\star\, \Delta_3^\star}{\omega^\star}$
5	$J_1 = J_2,\ Q_1,\, Q_2,\, Q_3$	$2\pi \sqrt{\sum_{I<J} Q_I^\star\, Q_J^\star - \frac{\pi}{2}\, J^\star}$	$I^\star = \pi\, \frac{\Delta^{1\,\star} \Delta^{2\,\star} \Delta^{3\,\star}}{(\omega^\star)^2}$
6	$J_1,\ J_2,\ Q$	$\frac{-3\,\pi^2 \left(\sum_i J_i\right)^2 + \pi \sqrt{9\,\pi^2 \left(\sum_i J_i\right)^2 + 12\, Q^4}}{9\, Q}$	$I^\star = \frac{\pi\, i}{3}\, \frac{(\Delta^\star)^3}{\omega_1^\star\, \omega_2^\star}$
7	$J_1 = J_2 = J_3,\ Q_1 = Q_2$	$2\pi \sqrt{\frac{32\,(Q^\star)^3 - 3\,\pi^2\,(J^\star)^2}{32\, Q^\star - \pi^2}}$	$I^\star = -\frac{\pi^3}{128}\, \frac{(\Delta^\star)^4}{(\omega^\star)^3}$

Table 3.1: For each spacetime dimension d, we have reported the set of conserved charges, the BPS entropy and the on-shell action matching the entropy function of the AdS_d black hole solutions we have analyzed.

3.5 RECAP AND DISCUSSION

The main result of this section has been to provide a derivation of the extremization principle leading to the Bekenstein-Hawking entropy of AdS_d black holes, by showing that the entropy functions of [65, 68, 70] are the supergravity on-shell action $I = I(\omega_i, \Delta_I)$ evaluated on a complexified family of supersymmetric solutions. The BPS chemical potentials ω_i, Δ_I indeed satisfy the corresponding extremization equations and they fulfil a linear complex constraint analogous to (3.1). The correct BPS chemical potentials and the on-shell actions exactly reproducing the entropy functions are obtained by taking the particular BPS limit of AdS rotating black hole thermodynamics we have presented in this section. The analysis of several examples across different spacetime dimensions demonstrates that this approach is general and should play a role towards understanding the thermodynamics of BPS black holes in AdS.

We report in tab. 3.1 a summary of all the different cases we have studied throughout this section: the set of conserved charges, the expression for the entropy and the one for the on-shell action matching the entropy functions are reported for each black hole. For each dimension, the solutions we have analyzed are not the most general ones, as it is evident from the table since there are conserved charges which are assumed to be equal; therefore there are some generalizations that it would be interesting to consider. The analysis of the solutions of [110, 139] we have provided strongly indicates that the same BPS limit will work when the most general set of electric charges and angular momenta is turned on in each spacetime dimension, although in many cases the corresponding asymptotically AdS black hole solutions

d	Known uplift	Most general set of black hole conserved charges
4	$\mathrm{AdS}_4 \times S^7$	$J \in SU(2) \subset SO(2,3) \quad , \quad Q_1, Q_2, Q_3, Q_4 \in U(4) \subset SO(8)$
5	$\mathrm{AdS}_5 \times S^5$	$J_1, J_2 \in U(2) \subset SO(2,4) \quad , \quad Q_1, Q_2, Q_3 \in U(3) \subset SO(6)$
6	$\mathrm{AdS}_6 \times S^4/\mathbb{Z}_2$	$J_1, J_2 \in SO(4) \subset SO(2,5) \quad , \quad Q$
7	$\mathrm{AdS}_7 \times S^4$	$J_1, J_2, J_3 \in SO(6) \subset SO(2,6) \quad , \quad Q_1, Q_2 \in U(2) \subset SO(4)$

Table 3.2: For each spacetime dimension d, we have reported the known uplifts and the most general sets of conserved charges a black hole solution can present.

are still to be constructed, and the explicit check may be technically hard to perform. We display in tab. 3.2 the known uplifts and the most general sets of conserved charges a black hole solution can present. Here below we briefly discuss each case in few details.

Starting from the context of eleven-dimensional supergravity on S^7, one could relax the condition of pairwise equal electric charges within the $\mathrm{U}(1)^4$ consistent truncation of $\mathrm{SO}(8)$ maximal supergravity, though the finite-temperature asymptotically AdS_4 solution with four independent electric charges in addition to the angular momentum has not been found yet[51]. In four dimensions, one could also switch on magnetic charges. For type IIB supergravity on S^5, one could analyze the solution carrying two independent angular momenta and three independent electric charges given in [108]. For massive type IIA supergravity on $S^4/\mathbb{Z}_2$, the solution of [139] discussed here already carries all possible independent electric charges and angular momenta available within known consistent truncations, although one may still search for asymptotically AdS_6 black holes carrying a non-vanishing electric charge for the additional $\mathrm{U}(1) \subset \mathrm{SU}(2)$ isometry of S^4 working directly in ten dimensions (this charge would be dual to a flavor charge of the D4-D8-O8 SCFT_5). For eleven-dimensional supergravity on S^4, a solution carrying two independent electric charges and three independent angular momenta is likely to exist within the $\mathrm{U}(1)^2$ truncation of seven-dimensional $\mathrm{SO}(5)$ maximal supergravity, but has not been found yet. Nevertheless, the known solutions allow to partially relax the condition of equal electric charges and equal angular momenta we imposed: one could take the two electric charges in the solution of [110] to be independent, or consider the solution with equal electric charges but independent angular momenta given in [143].

For all the cases where the most general solution has still to be constructed, although constructing the full asymptotically AdS_d black hole would be desirable, for the purpose of studying the extremization principle it may be sufficient to focus on the simpler near-horizon geometry, upon identifying the near-horizon counterpart of our BPS limit. This approach, once promoted to the full ten- or eleven-dimensional

51 Very recently, the corresponding BPS solution has been constructed in [135].

supergravity theory, may also lead to a generalization of the extremization principle of [57, 58, 153] to the case of rotating horizons with no magnetic charge. However, an important issue that must be remarked is that if we only have the near-horizon at our disposal, we cannot evaluate the various thermodynamical quantities by using holographic renormalization, since there would no longer be an asymptotically AdS region to perform this technique in. Therefore, we must seek for another approach in order to obtain them. There are several possibilities to be explored: one is using Sen's formalism, in the spirit of [154], another one could be exploiting the isolated horizon formalism defined in [11] and extended to the rotating case in [12].

There is a large class of new black hole solutions in five-dimensional minimal ungauged supergravity which has been recently found in [155]. The black holes of the family are supersymmetric, stationary and asymptotically flat. They contain non-trivial topology outside the event horizon and have the same conserved charges at infinity as the famous BMPV solution found in [156]. The most striking feature of this family of solutions is that there are some black holes which have greater entropy than the BMPV solution. This is a surprising fact since it poses a puzzle about why the the original counting of states [156] reproduces the the BMPV entropy. This seemed like a perfect agreement at the time, since the BMPV black hole was the only known solution with such charges. It would be interesting to search for asymptotically AdS_5 solutions of similar kind to the ones found in [155]. If these solutions turn out to exist, then it would be worth to study them applying an appropriate version of the BPS limit studied in this chapter.

In the next chapter, we are going to focus on black holes that are just *asymptotically locally* AdS_5, the S^3 spatial part of the conformal boundary being squashed. These black holes have been constructed in [99–101]; in particular in the first paper the minimal gauged supergravity black hole was presented, while in the others black hole solutions to $\mathcal{N} = 2$ Fayet-Iliopoulos supergravity are introduced. It has been shown in [99, 100] that the expression of the Bekenstein-Hawking entropy of these black holes in terms of the charges is the same as in the round S^3 case, provided one uses the Page electric charges of the solution. Hence the entropy function should also be the same, provided the electric potentials Δ^I are those conjugate to the Page charges. It would be interesting to show this from the on-shell action by implementing the BPS limit discussed here. Likewise, it would be worth performing the same analysis for asymptotically locally AdS black hole solutions in dimensions different than five.

4

ALAdS$_5$ BLACK HOLES WITH SQUASHED BOUNDARY

In Chapter 1 we have discussed how the entropy of supersymmetric asymptotically AdS$_d$ black holes can be reproduced via a CFT$_{d-1}$ computation. Furthermore, we have introduced the extremization principle for rotating AdS$_d$ black holes, which states that the entropy of these solutions can be reproduced by Legendre transforming an entropy function which is function of some BPS chemical potentials satisfying a linear, complex constraint. In Chapter 3 we have described the BPS limit originally proposed in [66] and generalized in [67] that, when applied to a finite-temperature AdS$_d$ black hole solution, is able to provide the BPS chemical potentials above mentioned and the entropy function, which coincides with the on-shell action. By introducing this BPS limit, we provided a physical interpretation to the extremization principle, by showing that the entropy function can be identified with the on-shell action of the solution.

One question which might be interesting to answer is whether the picture above changes when one considers solutions which are not asymptotically AdS$_d$, but only asymptotically *locally* AdS$_d$. In order to avoid confusion, it is worth spending some words to define AlAdS spacetimes. To this aim, we recall that AdS$_d$ is the maximally symmetric solution of Einstein's equations with negative cosmological constant. Its curvature tensor is given by

$$R_{\mu\nu\rho\sigma} = g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} \,, \tag{4.1}$$

and it has a conformal boundary with topology $\mathbb{R} \times S^{d-2}$. AAdS spacetimes are solutions of Einstein's equations which asymptotically become exactly AdS spacetimes. AlAdS spacetimes are characterized instead by the fact that the Riemann tensor approaches (4.1) asymptotically. For more details on this definitions we refer the reader to [116, 117].

For instance, such solutions might be obtained by trying to deform the geometry of an AdS$_d$ black hole at the conformal boundary; if the solution continues to exist it would be interesting to study how the deformation affects the thermodynamics of the black hole. Furthermore, constructing this solution might also shed some light on which field theory states contribute to the black hole entropy.

In five-dimensions, this question has been firstly investigated in [101, 157][52].

Working in minimal five-dimensional gauged supergravity and using a cohomogeneity-one ansatz with local SU(2)×U(1)×U(1) symmetry, the authors constructed both supersymmetric and non-supersymmetric black holes where the three-sphere sitting at the conformal boundary of global AdS$_5$ is squashed. Since the boundary is non conformally-flat, the solutions are indeed AlAdS$_5$. The supersymmetric black hole presents an arbitrary squashing at the boundary; however the event horizon geometry turns out to be completely frozen and therefore the entropy takes a fixed value. This behaviour is different from the minimal gauged supergravity black hole of Gutowski-Reall, originally constructed in [41] and presented in detail in sec. 2.1.3.1, which presents the same symmetry: indeed for this solution the entropy depends on one parameter controlling the horizon geometry.

It is natural to look for more general solutions in a more general theory which do not present a frozen horizon geometry and would be thus more interesting. This is what we have done in [99,100]: we constructed more general supersymmetric black holes having a local SU(2)× U(1)× U(1) symmetry and displaying a squashed three-sphere at the boundary in the context of the five-dimensional Fayet-Iliopoulos gauged supergravity theory we have introduced in sec. 2.1.1. In the solutions we will look for, one of the Abelian Killing vectors is timelike while the remaining SU(2)× U(1) symmetry acts on a three-sphere. The a priori non-vanishing, conserved charges carried by the solutions thus are the energy, one angular momentum and $n_V + 1$ electric charges. Previously known supersymmetric solutions with the same symmetry include the black holes of [41,42], the black hole with a squashed boundary of [101,157] and a solitonic deformation of AdS found in [133]. Apart for the solution of [42], these were obtained by restricting to minimal gauged supergravity and thus have just one electric charge.

The difference between the approaches taken in [99] and in [100] is the following. In the former paper we insisted on constructing a solution in the general theory with arbitrary n_V, this was possible by imposing an ansatz on the scalar fields. Instead, in the latter paper we searched for AlAdS black holes in the U$(1)^3$ theory, i.e. in the theory with $n_V = 2$; as we have already mentioned, these solutions are particularly interesting since they can be uplifted to type IIB ten-dimensional string theory [104] in the fashion of sec. 2.1.1.2. The black holes found in [100] are totally general, i.e. no arbitrary ansatz has been imposed to solve the supersymmetry equations. For both general theory with n_V arbitrary and the U$(1)^3$ one, we found that the equations to be solved in order to find a solution are very complicated and we could not find new analytic solutions in both cases. Therefore, we rather construct

52 See also [158] for a non-supersymmetric study.

the near-horizon and near-boundary solutions perturbatively and then interpolate numerically.

In [99] a two-parameter family of supersymmetric black holes displaying both running gauge fields and scalar fields is obtained, while the solutions of [100] present three arbitrary parameters; both the families of solutions generalize the one-parameter solution of [101,157]. For both cases, we will show that for a certain range of the parameters the solution is regular on and outside the horizon.

For both families of solutions, we find that of the total set of parameters one controls the event horizon geometry as well as the angular momentum and the Page electric charges of the solution, while the remaining ones are responsible for the squashing at the boundary and do not affect the horizon. This means that whatever is the squashing at the boundary, the radial flow towards the horizon acts as an attractor that brings the transverse geometry into a form which only depends on the remaining parameter(s). Still, the horizon is not frozen and the entropy is a non-trivial function of the parameters.

In both cases, we set up holographic renormalization for Fayet-Iliopoulos gauged supergravity, providing the needed counterterms. This allows us to compute the holographic one-point function for the SCFT energy-momentum tensor, R-current, flavour currents and the scalar superpartners of the latter. These in turn provide the holographic energy, the angular momentum and the R- and flavour charges. While these conserved quantities are naturally interpreted as expectation values of the corresponding SCFT operators in the state dual to the black holes, they also make sense in the gravitational solution, independently of holography. In addition we compute the renormalized on-shell actions for both families and verify that they satisfy the quantum statistical relation. Finally, we find in both cases that the black hole entropy can be expressed as simple functions of the angular momentum and the Page electric charges, but apparently not the holographic electric charges.

This chapter is organized as follows. In section 4.1 we partially solve the supersymmetry conditions given in sec. 2.1.3.1, originally derived in [42], so to reduce the problem to a few coupled ordinary differential equations in both cases. In section 4.2 we present our families of solutions. In section 4.3 we discuss holographic renormalization in Fayet-Iliopoulos supergravity and apply it to the evaluation of the holographic charges as well as the on-shell action. We also discuss the entropy of the solutions. We devote section 4.4 to recap and discuss the results we have shown in the chapter.

4.1 SUPERSYMMETRY CONDITIONS FOR ALADS5 SOLUTIONS

4.1.1 *Equations for the theory with n_V arbitrary*

We are working in the same setting of [42], so we are still assuming for our solution the form (2.49) for the metric and the one given in (2.51) for the gauge fields. Therefore, supersymmetric solutions can be still obtained by solving the set of supersymmetry equations (2.52)- (2.56). In [42], the above system was solved by making the guess (2.60), while here we would like to proceed with no loss of generality as much as possible. We can express the lower-index scalars $X_I(\rho)$ by separating the component along the constant vector $\bar{X}_I$ and the orthogonal ones:

$$f^{-1}X_I = f_{\min}^{-1}\bar{X}_I + h_I \ , \tag{4.2}$$

where h_I are functions of ρ satisfying

$$\bar{X}^I h_I = 0 \ , \tag{4.3}$$

while the component along $\bar{X}_I$ has already been fixed using (2.52). Plugging (4.55) in the constraint (2.8), we find that f is expressed as:

$$f = \left(f_{\min}^{-3} + \frac{27}{2} f_{\min}^{-1} C^{IJK}\bar{X}_I h_J h_K + \frac{9}{2} C^{IJK} h_I h_J h_K\right)^{-1/3} . \tag{4.4}$$

Using the identity (2.59), equation (2.53) for U^I becomes

$$\left(a^2 U^I\right)' = \frac{\epsilon\ell}{3}\bar{X}^I\left(a^2 p\right)' + \frac{36\epsilon}{\ell} a^3 a'\, C^{IJK}\bar{X}_J h_K \, . \tag{4.5}$$

The right-hand side of the above equation becomes a total derivative if we trade h_I for some new functions, $H_I(\rho)$, defined as

$$h_I = \frac{H_I'}{a^3 a'} \, . \tag{4.6}$$

Exploiting the replacement above, we can solve the equation for U^I as

$$U^I = \frac{\epsilon\ell}{3}\bar{X}^I p + \frac{36\epsilon}{\ell a^2} C^{IJK}\bar{X}_J H_K + \frac{U_0^I}{a^2} \ , \tag{4.7}$$

where U_0^I are integration constants. In the following we will choose $U_0^I = 0$[53] and thus require that

$$\bar{X}^I H_I = 0 \ . \tag{4.8}$$

We have started solving the system of supersymmetry conditions by expressing X_I, f and U^I in terms of a and H_I. Our next step is to use

53 A preliminary analysis with $U_0^I \neq 0$ was performed in [99] and no regular solutions has been found due to a divergence appearing in the perturbative expansion at small ρ.

the above results to manipulate eq. (2.54) containing w and the Maxwell equation (2.56), as we have already done for the Gutowski-Reall minimal black hole solution in sec. 2.1.3.1. Recalling the expression of the $\mathfrak{g}$ function given in (2.103) and observing that the following identity holds

$$\left(a^{-2}p\right)' = -\frac{2a'\mathfrak{g}}{a}\,, \tag{4.9}$$

we are able to write eq. (2.54) as

$$\frac{a}{2a'}(a^{-2}w)' \equiv \frac{w'}{2aa'} - \frac{w}{a^2} = \epsilon\left[\frac{\ell}{2}f_{\rm min}^{-1}\,\mathfrak{g} - \frac{27a}{\ell\,a'}\,\bar{X}_I C^{IJK}\frac{H_J'}{a^3a'}\left(\frac{H_K}{a^4}\right)'\right]. \tag{4.10}$$

It remains to massage the Maxwell equation (2.56). After some computations involving the identity (2.3), we find that

$$C_{IJK}U^JU^K = \frac{2\ell^2}{3}\bar{X}_I\,p^2 + \frac{8p}{a^2}H_I + \frac{288}{\ell^2a^4}\bar{Q}_{IJ}(CHH)^J\,, \tag{4.11}$$

where we used the shorthand notation $(CHH)^J = C^{JKL}H_KH_L$, while by $\bar{Q}^{IJ}$ we denote the kinetic matrix (2.10) evaluated on $X=\bar{X}$. Then, eq. (2.56) becomes

$$\begin{aligned}\Bigg[a^3a'\left(f_{\rm min}^{-1}\bar{X}_I + \frac{H_I'}{a^3a'}\right)' + \bar{X}_I\left(\frac{\epsilon}{\ell}a^2w + \frac{\ell^2p^2}{18}\right)&\\ + \frac{2p}{3a^2}H_I + \frac{24}{\ell^2a^4}\bar{Q}_{IJ}(CHH)^J\Bigg]' &= 0\,.\end{aligned} \tag{4.12}$$

The component along $\bar{X}_I$, which is obtained by contracting with $\bar{X}^I$, reads

$$\left[a^3a'\left(f_{\rm min}^{-1}\right)' + \frac{\epsilon}{\ell}a^2w + \frac{\ell^2p^2}{18} + \frac{36}{\ell^2a^4}C^{IJK}\bar{X}_IH_JH_K\right]' = 0\,. \tag{4.13}$$

The components having vanishing contraction with $\bar{X}^I$, which are given by $\mathrm{Maxw}_I - \bar{X}_I\bar{X}^J\mathrm{Maxw}_J$, where Maxw_I is eq. (4.12), read instead

$$\begin{aligned}\Bigg[H_I'' - \left(\frac{3a'}{a} + \frac{a''}{a'}\right)H_I' + \frac{2p}{3a^2}H_I&\\ + \frac{24}{\ell^2a^4}\left(\bar{Q}_{IJ} - \frac{3}{2}\bar{X}_I\bar{X}_J\right)(CHH)^J\Bigg]' &= 0\,.\end{aligned} \tag{4.14}$$

We can rearrange eq. (4.13) as follows

$$\begin{aligned}\frac{w'}{2aa'} + \frac{w}{a^2} = -\frac{\epsilon\ell}{2}\Bigg[\nabla^2(f_{\rm min}^{-1}) + 8\ell^{-2}f_{\rm min}^{-2} - \frac{\ell^2\mathfrak{g}^2}{18}&\\ + \frac{36}{\ell^2a^3a'}\bar{X}_IC^{IJK}\left(\frac{H_JH_K}{a^4}\right)'\Bigg]&,\end{aligned} \tag{4.15}$$

where

$$\nabla^2 f_{\rm min}^{-1} = \frac{1}{a^3 a'}\left(a^3 a'\left(f_{\rm min}^{-1}\right)'\right)' \tag{4.16}$$

is the Laplacian of $f_{\rm min}^{-1}$ on the Kähler base $\mathcal{B}$.

Now we can follow the strategy we have announced above, i.e. we combine (4.15) with (4.10) in order to eliminate w' and solve for w as

$$\begin{aligned} w \;=\; -\frac{\epsilon\ell a^2}{4}\Bigg\{&\nabla^2(f_{\rm min}^{-1}) + \frac{8}{\ell^2}f_{\rm min}^{-2} - \frac{\ell^2\mathfrak{g}^2}{18} + f_{\rm min}^{-1}\,\mathfrak{g} \\ &+ \frac{36}{\ell^2 a^3 a'}\bar{X}_I C^{IJK}\left[\left(\frac{H_J H_K}{a^4}\right)' - \frac{3a}{2a'}H_J'\left(\frac{H_K}{a^4}\right)'\right]\Bigg\}\,, \end{aligned} \tag{4.17}$$

the final step is to plug this back into either (4.10) or (4.13), doing this we finally arrive at

$$\begin{aligned} &\left(\nabla^2 f_{\rm min}^{-1} + \frac{8}{\ell^2}f_{\rm min}^{-2} - \frac{\ell^2\mathfrak{g}^2}{18} + f_{\rm min}^{-1}\,\mathfrak{g}\right)' + \frac{4a'\mathfrak{g}}{a f_{\rm min}} \\ &+\bar{X}_I C^{IJK}\left\{\frac{36}{\ell^2 a^3 a'}\left[\left(\frac{H_J H_K}{a^4}\right)' - \frac{3a}{2a'}H_J'\left(\frac{H_K}{a^4}\right)'\right]\right\}' \\ &-\frac{216}{\ell^2}\bar{X}_I C^{IJK}\frac{H_J'}{a^3 a'}\left(\frac{H_K}{a^4}\right)' = 0\,. \end{aligned} \tag{4.18}$$

The main result we have obtained for now is that we have partially solved the system of equations (2.52)– (2.56) for a, f, w, U^I, X_I and we have reduced the problem of finding new solutions into solving the equations (4.14), (4.18) involving just the unknown functions H_I and a. Eq. (4.14) is third order in the variable ρ, while eq. (4.18) contains up to six derivatives.

The equations above reduce to the supersymmetry conditions for the minimal gauged supergravity theory [41] when $H_I = 0$. Indeed, in this case (4.4) yields $f = f_{\rm min}$ while from (4.55) we see that the scalars are set to the constant value taken in the AdS$_5$ solution, $X^I = \bar{X}^I$. The expression (2.51) for the gauge fields becomes

$$A^I = \bar{X}^I A\,, \quad \text{with} \quad A = f\,({\rm d}y + w\,\hat{\sigma}_3) + \frac{\epsilon\ell}{3}p\,\hat{\sigma}_3 \tag{4.19}$$

being the graviphoton of minimal gauged supergravity. Moreover, (4.14) trivializes while eqs. (4.10), (4.13), (4.17), (4.18) reduce to those of the minimal case given in [41].

4.1.1.1 *Imposing an ansatz*

So far we have manipulated the original supersymmetry equations of [42] without any restriction,[54] arriving at eqs. (4.14), (4.18). Although studying the solutions of the two equations above in the general case of arbitrary n_V without imposing any ansatz would be very interesting and

54 Apart for fixing the integration constants $U_0^I = 0$ when solving for U^I.

desirable, this appears to be a very complicated task. There are two possibilities to simplify the analysis: the first one is to choose a particular value for n_V, the second one is to impose an ansatz which helps us to solve the equations. Here we choose the second road and search for solutions with arbitrary n_V; in the next section we will instead follow the first strategy and fix $n_V = 2$, looking for solutions of the $\mathrm{U}(1)^3$ theory.

We find it convenient to impose the ansatz

$$H_I = q_I\, H\,, \qquad I = 0, \ldots, n_V\ , \tag{4.20}$$

where $H(\rho)$ is a real function and q_I is a constant vector, which for consistency with (4.8) must be orthogonal to $\bar{X}_I$,

$$\bar{X}^I q_I = 0\,. \tag{4.21}$$

Although this ansatz will not be enough for solving the equations analytically, it will reduce the number of independent equations we have to study and will be helpful while performing the perturbative and numerical analysis in the next sections.

Plugging our ansatz in, eq. (4.14) becomes

$$q_I \left[H'' - \left(\frac{3a'}{a} + \frac{a''}{a'}\right) H' + \frac{2p}{3a^2} H\right]' - \frac{4}{\ell^2}\, W_I \left(\frac{H^2}{a^4}\right)' = 0\ , \tag{4.22}$$

where the constant vector W_I is defined as

$$W_I = (-6\bar{Q}_{IJ} + 9\bar{X}_I\bar{X}_J)\, C^{JKL} q_K q_L\ . \tag{4.23}$$

There are now some different cases we have to analyze:

- The case when $W_I = 0$, then one can see that necessarily $q_I = 0$. Indeed multiplying $(-6\bar{Q}_{IJ} + 9\bar{X}_I\bar{X}_J)\,(Cqq)^J = 0$ by $\bar{Q}^{-1}$ and using (2.11) we obtain $(Cqq)^I = (C\bar{X}qq)\bar{X}^I$. Contracting (2.3) with four q's one finds that this implies $(C\bar{X}qq) = 0$. This in turn means that $q_I = 0$;

- The case when the constant vectors q_I and W_I are linearly independent. Then, their coefficients in (4.22) have to vanish separately. In this case, from the term proportional to W_I we obtain

  $$H = \mathrm{const}\, a^2\ , \tag{4.24}$$

  while the rest of (4.22) has, up to trivial symmetries involving shifts and rescalings of the coordinate ρ, the general solution:

  $$a \;=\; \alpha\ell \sinh(\rho/\ell)\,, \tag{4.25}$$

  where α is a parameter. This automatically satisfies (4.18) without imposing further conditions; it is not a new solution, but just the AdS black hole found in [42] and described in sec. 2.1.3.1 which presents a conformally flat boundary;

- The case when the vectors W_I and q_I are parallel to each other. Since the overall scale of q_I is immaterial (as it can always be reabsorbed in the function H), there is no loss of generality in assuming $W_I = q_I$. Recalling the definition of W_I, we conclude that we have to impose the condition

$$(-6\bar{Q}_{IJ} + 9\bar{X}_I\bar{X}_J)\, C^{JKL} q_K q_L \;=\; q_I\,. \tag{4.26}$$

Note that this implies (4.21). Thus we have a system of $n_V + 1$ equations for $n_V + 1$ unknowns q_I, which in general determines the q_I.

From the above analysis, we conclude that new solutions within the ansatz (4.20) may only be found if we assume that the vectors W_I and q_I are parallel to each other. It can be shown that (4.26) also implies (see [99] for a detailed proof of the identities below)

$$C^{IJK} q_J q_K = -\frac{1}{18}\bar{X}^I + \bar{Y}^I\;, \quad \text{where} \quad \bar{Y}^I = C^{IJK}\bar{X}_J q_K\;, \tag{4.27}$$

$$C^{IJK}\bar{X}_I q_J q_K = C^{IJK} q_I q_J q_K = -\frac{1}{18}\;. \tag{4.28}$$

$$\tag{4.29}$$

With these relations at disposal, we can simplify the supersymmetry conditions we have introduced in this section in such a way that one can look for solutions independently of the specific values taken by the q_I. Indeed, we find that a solution is obtained by solving the following coupled ODE's for the functions $a(\rho)$, $H(\rho)$:

$$\left[H'' - \left(\frac{3a'}{a} + \frac{a''}{a'}\right)H' + \frac{2p}{3a^2}H - \frac{4}{\ell^2}\frac{H^2}{a^4}\right]' = 0\;, \tag{4.30}$$

$$\begin{aligned}&\left(\nabla^2 f_{\min}^{-1} + \frac{8}{\ell^2} f_{\min}^{-2} - \frac{\ell^2\mathfrak{g}^2}{18} + f_{\min}^{-1}\,\mathfrak{g}\right)' + \frac{4a'\mathfrak{g}}{a f_{\min}}\\ &- \left\{\frac{2}{\ell^2 a^3 a'}\left[\left(\frac{H^2}{a^4}\right)' - \frac{3a}{2a'}H'\left(\frac{H}{a^4}\right)'\right]\right\}' + \frac{12}{\ell^2}\frac{H'}{a^3 a'}\left(\frac{H}{a^4}\right)' = 0\;,\end{aligned} \tag{4.31}$$

where we recall that $f_{\min}$, p and $\mathfrak{g}$ are the functions of a and its derivatives given in (2.57), (2.58), and (2.103), respectively. Once a solution for a and H is obtained, the five-dimensional supergravity fields are fully determined. The metric and the gauge fields take the form (2.49), (2.51), where the functions f, w and U^I read:

$$f = \left[f_{\text{min}}^{-1/3} - \frac{3}{4} f_{\text{min}}^{-1} \left(\frac{H'}{a^3 a'} \right)^2 - \frac{1}{4} \left(\frac{H'}{a^3 a'} \right)^3 \right]^{-3} , \tag{4.32}$$

$$\begin{aligned} w = -\frac{\epsilon \ell a^2}{4} \bigg\{ & \nabla^2 (f_{\text{min}}^{-1}) + \frac{8}{\ell^2} f_{\text{min}}^{-2} - \frac{\ell^2 \mathfrak{g}^2}{18} + f_{\text{min}}^{-1} \mathfrak{g} \\ & - \frac{1}{2\ell^2 a^3 a'} \left[\left(\frac{H^2}{a^4} \right)' - \frac{3a}{2a'} H \left(\frac{H}{a^4} \right)' \right] \bigg\} , \end{aligned} \tag{4.33}$$

$$U^I = \frac{\epsilon \ell}{3} \bar{X}^I p + \frac{36\epsilon}{\ell} \bar{Y}^I \frac{H}{a^2} . \tag{4.34}$$

From (4.20) we obtain the expression of the lower-index scalars X_I

$$X_I = \bar{X}_I f f_{\text{min}}^{-1} + q_I f \frac{H'}{a^3 a'} , \tag{4.35}$$

while the scalars X^I can be obtained from these by using (2.6), and read

$$X^I = \bar{X}^I f^2 \left[f_{\text{min}}^{-2} - \frac{1}{4} \left(\frac{H'}{a^3 \, a'} \right)^2 \right] + 9 \, \bar{Y}^I f^2 \left[f_{\text{min}}^{-1} + \frac{H'}{2a^3 \, a'} \right] \frac{H'}{a^3 \, a'} . \tag{4.36}$$

Note that all the quantities that carry an upper index I split into a part aligned to $\bar{X}^I$ and one aligned to $\bar{Y}^I$. We recall that $\bar{Y}^I = C^{IJK} \bar{X}_J q_K$, with the constants q_K being in general determined by condition (4.26). In the U$(1)^3$ theory, the only allowed choices for the q_I are either $q_1 = q_2 = \frac{1}{6}$, $q_3 = -\frac{1}{3}$, or the similar expressions obtained by cyclically permuting the indices $1, 2, 3$. This implies that the $\bar{Y}^I$ take the values $\bar{Y}^1 = \bar{Y}^2 = -\frac{1}{18}$, $\bar{Y}^3 = \frac{1}{9}$ (or their cyclic permutations).

4.1.1.2 *First integrals and conserved charges*

Here we discuss some conserved charges and we introduce some first integrals of the equations of motion. Both of them will play an important role in the following. The analysis we report has been performed in [99, 100], generalizing to Fayet-Iliopoulos gauged supergravity similar considerations made in [101, 133] for minimal gauged supergravity.

We begin by considering a Cauchy surface (that is, a hypersurface of constant time). This is foliated by three-dimensional spacelike, compact hypersurfaces of constant ρ, that we denote by Σ_ρ. By considering the hypersurface Σ_∞ at $\rho = \infty$, we introduce the Page electric charges [159]:

$$P_I = \frac{1}{\kappa^2} \int_{\Sigma_\infty} \left(Q_{IJ} \star F^J + \frac{1}{4} C_{IJK} A^J \wedge F^K \right) . \tag{4.37}$$

Since by the Maxwell equation (2.14) the integrand is a closed three-form, it follows from the Stokes theorem that P_I is a constant of the flow along the radial direction and can equally well be evaluated on any other hypersurface Σ_ρ (moreover it should be quantized in appropriate units). In particular, it can be measured at the horizon, that is on $\Sigma_{\rho=0}$.

In a similar fashion, we can associate a conserved angular momentum to the symmetry generated by the vector $K = \frac{\partial}{\partial\psi}$ by considering the following generalization of the Komar integral:

$$J = \frac{1}{2\kappa^2}\int_{\Sigma_\infty}\left[\star\,\mathrm{d}K + 2\,\iota_K A^I\left(Q_{IJ}\star F^J + \frac{1}{6}C_{IJL}A^J\wedge F^L\right)\right]\,. \tag{4.38}$$

Using both the Einstein and the Maxwell equation, one can show that the integrand is closed on the Cauchy surface and thus J can also be evaluated on any Σ_ρ. We emphasize that in general the standard Komar integral $\int_{\Sigma_\infty}\star\mathrm{d}K$ would not satisfy this property, because of the gauge field energy-momentum tensor in the Einstein equation. These conserved charges are strictly connected to the first integrals we introduce below, as we shall see in the following.

The analysis performed in [100] is able to find a total number of n_V+3 first integrals for the general case of arbitrary n_V. One of them come from the component of the Maxwell equation parallel to $\bar{X}_I$, n_V+1 comes from the orthogonal components and finally the last one can be derived by manipulating the ${}^t{}_\psi$ component of the Einstein equations using the Maxwell equation and the supersymmetry conditions. These first integrals are given by

$$\mathcal{K}_1 = a^3a'\left(f_{\min}^{-1}\right)' + \frac{1}{\ell}a^2w + \frac{\ell^2p^2}{18} + \frac{36}{\ell^2a^4}C^{IJK}\bar{X}_IH_JH_K\,, \tag{4.39a}$$

$$\begin{aligned}\mathcal{K}_2^{(I)} &= H_I'' - \left(\frac{3a'}{a}+\frac{a''}{a'}\right)H_I' + \frac{2p}{3a^2}H_I\\ &\qquad + \frac{24}{\ell^2a^4}\left(\bar{Q}_{IJ} - \frac{3}{2}\bar{X}_I\bar{X}_J\right)(CHH)^J\,,\end{aligned} \tag{4.39b}$$

$$\begin{aligned}\mathcal{K}_3 &= \frac{a}{a'f^3}\left(f^3w^2-4a^2(a')^2\right)^2\left[\frac{f^3w}{f^3w^2-4a^2(a')^2}\right]'\\ &\qquad - 12A^I_\psi\left(\mathcal{K}_1\bar{X}_I + \mathcal{K}_2^{(I)}q_I\right) + \frac{1}{3}\,C_{IJK}A^I_\psi A^J_\psi A^K_\psi\,,\end{aligned} \tag{4.39c}$$

where the q_I are defined such that $\bar{X}^Iq_I = 0$ and they have to satisfy eq. (4.26). Note that in the general n_V case, the H_I functions satisfy the constraint $\bar{X}^I H_I = 0$, therefore one of the above first integrals is dependent from the others and thus there are n_V+2 independent first integrals.

By manipulating the definitions of the conserved charges (4.37) and (4.38), it is possible to show that the Page charges P_I decompose in a term proportional to $\bar{X}_I$ and a term proportional to q_I, and that it results

$$P_I = -\frac{48\pi^2\ell^2}{\kappa^2}\left(\mathcal{K}_1\bar{X}_I + \mathcal{K}_2^{(I)}q_I\right)\,, \tag{4.40}$$

with the overall factor introduced for convenience. The Page charges are therefore described by all the first integrals which derive from the Maxwell equation. Furthermore, evaluating the angular momentum J, given in (4.38), on the supergravity background we are considering, we find

$$J = \frac{4\pi^2\ell^3}{\kappa^2}\,\mathcal{K}_3\,. \tag{4.41}$$

so that J is proportional to the last first integral.

Imposing the special ansatz (4.20), all the H_I functions are equal (up to a constant) and therefore all the components of the Maxwell equation orthogonal to $\bar{X}^I$ are also equal (up to a constant), so they globally provide only one non-trivial first integral. Thus a total number of three first integrals is obtained: two coming from the Maxwell equation and one from the Einstein equations. These first integrals read

$$\mathcal{K}_1 \;=\; a^3 a' \left(f_{\min}^{-1}\right)' + \frac{1}{\ell} a^2 w + \frac{\ell^2 p^2}{18} - \frac{2}{\ell^2 a^4}\, H^2 \;, \tag{4.42}$$

$$\mathcal{K}_2 \;=\; H'' - \left(\frac{3a'}{a} + \frac{a''}{a'}\right) H' + \frac{2p}{3a^2} H - \frac{4}{\ell^2 a^4} H^2 \;, \tag{4.43}$$

$$\begin{aligned}\mathcal{K}_3 \;=\; & \frac{a}{a' f^3}\left(f^3 w^2 - 4a^2 (a')^2\right)^2 \left(\frac{f^3 w}{f^3 w^2 - 4a^2(a')^2}\right)' \\ & - 12\, A^I_\psi \left(\mathcal{K}_1 \bar{X}_I + \mathcal{K}_2\, q_I\right) + \frac{1}{3} C_{IJK} A^I_\psi A^J_\psi A^K_\psi \;.\end{aligned} \tag{4.44}$$

Using the special ansatz, the conserved charges P_I and J can be written as

$$P_I = -\frac{48\pi^2\ell^2}{\kappa^2}\left(\mathcal{K}_1 \bar{X}_I + \mathcal{K}_2\, q_I\right) \;, \tag{4.45}$$

$$J = \frac{4\pi^2\ell^3}{\kappa^2} \mathcal{K}_3 \;. \tag{4.46}$$

The Page charges P_I and the quantity J defined by a Komar integral can be regarded as the electric charges and the angular momentum of the solution. However the procedure of holographic renormalization, which we will employ later in this thesis, gives the possibility to define in a different manner analogous conserved quantities playing the same role; we will therefore compare them with the conserved Page charges and angular momentum we have defined in the present section. In particular, since the contribution provided by the Chern-Simons term to the holographic charges is different to the one for the Page charges, we should expect that those conserved quantities are not equal.

One might be confused by the term *holographic charges*: this may seem ambiguous since also the Page charges can be obtained by holographic renormalization adding suitable finite counterterms. In this thesis, we denote as holographic charges the ones which are obtained by using holographic renormalization in a minimal subtraction scheme, i.e.

without adding any finite counterterm except those coming from the usual divergent ones. We refer to sec. 4.3.1 and to app. A for more details.

4.1.2 *Equations for the $U(1)^3$ theory*

In the previous section, we have obtained the $n_V + 1$ independent supersymmetry equations one has to solve in order to find a supersymmetric solution to the $\mathcal{N} = 2$ Fayet-Iliopoulos gauged supergravity theory we are considering. These equations are given by (4.14), (4.18). Given the fact that it is pretty difficult to find a solution of these equations in the case of arbitrary n_V, we have imposed the ansatz (4.20) so as to simplify the analysis. Here, we choose another strategy: we focus on the theory with $n_V = 2$; as we already know, this is the $\mathrm{U}(1)^3$ theory introduced in sec. 2.1.1.2, which is particularly interesting since it can be uplifted to type IIB supergravity on $\mathrm{AdS}_5 \times S^5$ as we have seen in the same section.

Our aim is thus to explicitly rewrite the supersymmetry equations (4.14), (4.18) and all the various objects defined in the section above for the case $n_V = 2$. Since we have the index I running from 1 to 3, we will have three functions $H_1(\rho)$, $H_2(\rho)$, $H_3(\rho)$ which control the scalars. The equations will depend on these three functions and on $a(\rho)$. However from eq. (4.8) we have the constraint

$$H_1 + H_2 + H_3 = 0\,. \tag{4.47}$$

This implies that we can eliminate one of the H_I functions. For example we choose to use this constraint to replace H_3 with

$$H_3 = -H_1 - H_2\,, \tag{4.48}$$

so that H_3 will never appear anymore throughout the thesis. It is convenient to define two particular combinations of H_1 and H_2 which will appear in the supersymmetry equations:

$$\Sigma(H_1, H_2) = -\left(H_1^2 + H_2^2 + H_1\, H_2\right), \tag{4.49a}$$

$$\begin{aligned}\Lambda(H_1, H_2) = -\Bigg[&2\, H_1' \left(\frac{H_1}{a^4}\right)' + 2\, H_2' \left(\frac{H_2}{a^4}\right)' \\ &+ H_1' \left(\frac{H_2}{a^4}\right)' + H_2' \left(\frac{H_1}{a^4}\right)'\Bigg].\end{aligned} \tag{4.49b}$$

In order to obtain these equations, we start from (4.14), (4.18), we let the index I run from 1 to 3, we use (4.48) to eliminate H_3 whenever it appears and we perform all the necessary contractions recalling (2.18). Doing so we obtain the following three equations

$$\Bigg[H_1'' - \left(\frac{3a'}{a} + \frac{a''}{a'}\right) H_1' + \frac{2p}{3a^2} H_1 + \frac{8}{\ell^2 a^4}\left(H_1^2 - 2H_2^2 - 2H_1 H_2\right)\Bigg]' = 0\,, \tag{4.50}$$

$$\Bigg[H_2'' - \left(\frac{3a'}{a} + \frac{a''}{a'}\right) H_2' + \frac{2p}{3a^2} H_2 + \frac{8}{\ell^2 a^4}\left(-2H_1^2 + H_2^2 - 2H_1 H_2\right)\Bigg]' = 0\,, \tag{4.51}$$

$$\left(\nabla^2 f_{\rm min}^{-1} + \frac{8}{\ell^2} f_{\rm min}^{-2} - \frac{\ell^2 \mathfrak{g}^2}{18} + f_{\rm min}^{-1}\mathfrak{g}\right)' + \frac{4a'\mathfrak{g}}{a f_{\rm min}} + \left\{\frac{12}{\ell^2\, a^3\, a'}\left[2\left(\frac{\Sigma}{a^4}\right)' - \frac{3\,a}{2\,a'}\,\Lambda\right]\right\}' - \frac{72\,\Lambda}{\ell^2\, a^3\, a'} = 0\,. \tag{4.52}$$

The problem of finding new solutions in the $\mathrm{U}(1)^3$ theory is thus reduced into solving these three equations (4.50), (4.51) and (4.52)

Once a, H_1 and H_2 are determined, all the other functions are fixed in terms of these. The explicit expressions of them are straightforwardly obtained by the general relations reported in sec. 4.1 by setting $n_V = 2$ and performing the necessary contractions. Starting with the function f, it is easy to show that it is given by

$$f = \left[f_{\rm min}^{-3} - 9\, f_{\rm min}^{-1}\left(h_1^2 + h_2^2 + h_1\, h_2\right) - 27\left(h_1^2\, h_2 + h_1\, h_2^2\right)\right]^{-1/3}\,, \tag{4.53}$$

with the h_I being given in terms of H_I as

$$h_I = \frac{H_I'}{a^3 a'}\,, \tag{4.54}$$

so that

$$f^{-1} X_I = f_{\rm min}^{-1}\bar{X}_I + h_I\,. \tag{4.55}$$

The three h_I functions obviously satisfy the constraint $h_1 + h_2 + h_3 = 0$ as the H_I do. The function w can be written as

$$w = -\frac{\ell\, a^2}{4}\Bigg\{\nabla^2 (f_{\rm min}^{-1}) + \frac{8}{\ell^2} f_{\rm min}^{-2} - \frac{\ell^2\mathfrak{g}^2}{18} + f_{\rm min}^{-1}\,\mathfrak{g} + \frac{12}{\ell\, a^3\, a'}\left[2\left(\frac{\Sigma\,(H_1, H_2)}{a^4}\right)' - \frac{3\,a}{2\,a'}\,\Lambda\,(H_1, H_2)\right]\Bigg\}.$$

(4.56)

The functions U^1, U^2 and U^3 are given in terms of H_I by

$$U^1 = \frac{\ell}{3}\,p - \frac{12}{\ell\,a^2}\,H_1\,, \tag{4.57}$$

$$U^2 = \frac{\ell}{3}\,p - \frac{12}{\ell\,a^2}\,H_2\,, \tag{4.58}$$

$$U^3 = \frac{\ell}{3}\,p + \frac{12}{\ell\,a^2}\,(H_1 + H_2)\,, \tag{4.59}$$

and it is immediate to see that they satisfy

$$U^1 + U^2 + U^3 = \ell\,p\,, \tag{4.60}$$

so that only U^1 and U^2 are independent. Finally let us consider the scalars; the lower-index scalars X_I we have can be computed from equation (4.55) and they result to be

$$X_1 = \frac{f\,f_{\min}^{-1}}{3} + h_1\,f\,, \tag{4.61}$$

$$X_2 = \frac{f\,f_{\min}^{-1}}{3} + h_2\,f\,, \tag{4.62}$$

$$X_3 = \frac{f\,f_{\min}^{-1}}{3} - (h_1 + h_2)\,\,f\,. \tag{4.63}$$

Note that only X_1 and X_2 are independent since it holds the following constraint

$$X_1 + X_2 + X_3 = f\,f_{\min}^{-1}\,. \tag{4.64}$$

For the upper-index scalars X^I it is easy to show that the following relations hold

$$X^1 = \left(f\,f_{\min}^{-1}\right)^2 - 3\,f^2\,f_{\min}^{-1}\,h_1 - 9\,f^2\,\,(h_1 + h_2)\,\,h_2\,, \tag{4.65}$$

$$X^2 = \left(f\,f_{\min}^{-1}\right)^2 - 3\,f^2\,f_{\min}^{-1}\,h_2 - 9\,f^2\,\,(h_1 + h_2)\,\,h_1\,, \tag{4.66}$$

$$X^3 = \left(f\,f_{\min}^{-1}\right)^2 + 3\,f^2\,f_{\min}^{-1}\,\,(h_1 + h_2) + 9\,f^2\,h_1\,h_2\,, \tag{4.67}$$

and we have that only X^1 and X^2 are independent, since it results

$$X^1 + X^2 + X^3 = 3\left(f\,f_{\min}^{-1}\right)^2 - 9\,f^2\,\left(h_1^2 + h_2^2 + h_1\,h_2\right). \tag{4.68}$$

We conclude this subsection by briefly discussing which conditions one should impose to reduce to previously known solutions of the $\mathcal{N} = 2$, $D = 5$ Fayet-Iliopoulos gauged supergravity and to minimal gauged supergravity. To obtain the latter it is sufficient to take $H_1 = H_2 = 0$. Indeed, in this case the equations for H_1 and H_2 (4.30), (4.51) are trivially satisfied, while the equation for a (4.52) becomes the same given in [41]. All the physical relevant functions become the ones of minimal gauged supergravity. Indeed equation (4.53) gives $f = f_{\min}$, therefore the scalars (4.65) become just constants, the $U^{1,2}$ functions are equal and provide the same gauge field of [41] and equations (4.56), (4.61) reduce to the same form they take in minimal gauged supergravity. We can also easily obtain the $\mathrm{U}(1)^3$ version of the general family of solutions given in [99], which we will introduce and analyze below. Indeed, as we have already seen, the solutions of that paper are obtained by taking the simplifying ansatz $H_I(\rho) = q_I\, H(\rho)$ and fixing the charges q_I to assume in the $\mathrm{U}(1)^3$ theory the values $q_1 = q_2 = \frac{1}{6}$, $q_3 = -\frac{1}{3}$ (or cyclic permutations). Therefore to reduce to this class of solutions we have to take the limit $H_1 = H_2 = \frac{1}{6}\,H$, $H_3 = -\frac{1}{3}\,H$ (or cyclic permutations). Doing so, equations (4.50) and (4.51) become equal and become the same as equation (2.58) of [99], which is nothing but (4.30), while equation (4.52) reduces to (2.59) of the same paper, which coincides with (4.31). The Gutowski-Reall solution [42] is also recovered, since it is just a particular limit of the more general solutions of [99].

We conclude this section by noting that the supersymmetry equations (4.50), (4.51) and (4.52) possess a scaling symmetry [99]: indeed rescaling the coordinates such that $\rho = \lambda^{-1}\tilde{\rho}$, $y = \lambda^2\tilde{y}$, the functions $a(\rho)$, $H_1(\rho)$ and $H_2(\rho)$ become $\tilde{a}(\tilde{\rho}) = \lambda a(\lambda^{-1}\tilde{\rho})$, $\tilde{H}_1(\tilde{\rho}) = \lambda^2 H_1(\lambda^{-1}\tilde{\rho})$, $\tilde{H}_2(\tilde{\rho}) = \lambda^2 H_2(\lambda^{-1}\tilde{\rho})$ which still provide a solution for the supersymmetry equations. We shall use this scaling symmetry later to eliminate unphysical parameters and to help us interpolating the near-boundary and near-horizon perturbative solutions we will construct.

4.1.2.1 *First integrals and conserverd charges for the $n_V = 2$ case*

The first integrals (4.39) we have introduced in sec. 4.1.1.2 are defined for an arbitrary number of vector multiplets n_V, therefore we can use these general definitions to obtain their expressions for the $\mathrm{U}(1)^3$ theory case with $n_V = 2$.

By setting $n_V = 2$ in (4.39) and performing the needed contractions, we find

$$\mathbb{K}_1 = a^3 a' \left(f_{\text{min}}^{-1}\right)' + \frac{1}{\ell} a^2 w + \frac{\ell^2 p^2}{18} + \frac{24}{\ell^2 a^4}\,\Sigma\,, \tag{4.69a}$$

$$\mathbb{K}_2^{(1)} = H_1'' - \left(\frac{3a'}{a} + \frac{a''}{a'}\right) H_1' + \frac{2p}{3a^2} H_1 + \frac{8}{\ell^2 a^4}\left(H_1^2 - 2H_1H_2 - 2H_2^2\right), \tag{4.69b}$$

$$\mathbb{K}_2^{(2)} = H_2'' - \left(\frac{3a'}{a} + \frac{a''}{a'}\right) H_2' + \frac{2p}{3a^2} H_2 + \frac{8}{\ell^2 a^4}\left(-2H_1^2 - 2H_1H_2 + H_2^2\right), \tag{4.69c}$$

$$\mathbb{K}_2^{(3)} = -(H_1'' + H_2)'' + \left(\frac{3a'}{a} + \frac{a''}{a'}\right)(H_1' + H_2') - \frac{2p}{3a^2}(H_1 + H_2) + \frac{8}{\ell^2 a^4}\left(H_1^2 + 4H_1H_2 + H_2^2\right), \tag{4.69d}$$

$$\mathbb{K}_3 = \frac{a}{a' f^3}\left(f^3 w^2 - 4a^2 (a')^2\right)^2 \left[\frac{f^3 w}{f^3 w^2 - 4a^2 (a')^2}\right]' - 12\left(\mathbb{K}_1 A_\psi^I \bar{X}_I + \frac{1}{6}(\mathbb{K}_2^{(1)} A_\psi^1 + \mathbb{K}_2^{(2)} A_\psi^2) - \frac{1}{3}\,\mathbb{K}_2^{(3)} A_\psi^3\right) + 2A_\psi^1 A_\psi^2 A_\psi^3\,, \tag{4.69e}$$

where we have renamed $\mathcal{K}_I$ as $\mathbb{K}_I$ so as to distinguish between the first integrals for the general theory with arbitrary n_V and the ones for the U$(1)^3$ theory. For this theory, the q_I are fixed to be $q_1 = q_2 = \frac{1}{6}, q_3 = -\frac{1}{3}$, as it is for the U$(1)^3$ version of the solution of [99]. Moreover, equations (4.40) and (4.41) are still valid with the I index which runs from 1 to 3. In (4.69) we already used the constraint (4.47) to eliminate H_3 in favour of H_1 and H_2. As a consequence, we immediately see that $\mathbb{K}_2^{(1)}$, $\mathbb{K}_2^{(2)}$ and $\mathbb{K}_2^{(3)}$ are not independent, but they satisfy the relation

$$\mathbb{K}_2^{(1)} + \mathbb{K}_2^{(2)} + \mathbb{K}_2^{(3)} = 0\,, \tag{4.70}$$

so that we have 4 independent first integrals in total. We will always use (4.70) to trade $\mathbb{K}_2^{(3)}$ with $\mathbb{K}_2^{(1)}$ and $\mathbb{K}_2^{(2)}$, so the set of independent first integrals we choose is $(\mathbb{K}_1, \mathbb{K}_2^{(1)}, \mathbb{K}_2^{(2)}, \mathbb{K}_3)$. As we shall see later in the thesis, and as it is shown in [100], these first integrals will also help us to connect the parameters of the perturbative near-boundary solution we will construct with the parameters of the near-horizon one.

4.2 PERTURBATIVE AND NUMERICAL SOLUTIONS

The ODE's we have obtained, both for the arbitrary n_V case and the U$(1)^3$ theory one, are difficult to solve analytically, therefore we resort to a numerical method to find new solutions. The strategy we adopt is the following. The first step is to series-expand the fields both in the near-boundary region $\rho \to \infty$ and in the near-horizon one $\rho \to 0$, and to fix the series coefficients by solving the ODE's order by order. The second one is to build the numerical solution by matching the two expansions in the bulk. For both the general theory and the $\mathrm{U}(1)^3$ one, in the near-boundary region we find expansions that are compatible with AlAdS solutions, in the near-horizon one we note that there are solutions which possess the characteristics of black holes and, using a numerical procedure, we establish that there are well-behaved solutions interpolating between these two regimes. For ease of notations, we will set $\ell = 1$ in the whole section. We divide this section in two main parts: in the first one we construct the solution for the case of arbitrary n_V, while in the second one we turn to the $\mathrm{U}(1)^3$ theory case.

4.2.1 *The solution with arbitrary n_V*

4.2.1.1 *Near-boundary solution*

We study our equations (4.30) and (4.31) perturbatively around $\rho \to \infty$, which as we will see corresponds to a limit where a conformal boundary is approached. We assume the following asymptotic expansions for the unknown functions a and H:

$$\begin{aligned} a(\rho) &= a_0 e^{\rho}\left[1 + \sum_{k\geq 1}\sum_{0\leq n\leq k} a_{2k,n}\,\rho^n\,\left(a_0\,e^{\rho}\right)^{-2k}\right] \\ &= a_0 e^{\rho}\left[1 + (a_{2,0} + a_{2,1}\rho)\,\frac{e^{-2\rho}}{a_0^2}\right. \\ &\qquad\left. + \left(a_{4,0} + a_{4,1}\,\rho + a_{4,2}\,\rho^2\right)\frac{e^{-4\rho}}{a_0^4} + \dots\right]\,, \end{aligned} \tag{4.71}$$

$$\begin{aligned} H(\rho) &= a_0^4 e^{4\rho}\sum_{k\geq 0}\sum_{0\leq n\leq k} H_{2k,n}\,\rho^n\,\left(a_0\,e^{\rho}\right)^{-2k} \\ &= a_0^4 e^{4\rho}\left[H_{0,0} + (H_{2,0} + H_{2,1}\rho)\,\frac{e^{-2\rho}}{a_0^2}\right. \\ &\qquad\left. + \left(H_{4,0} + H_{4,1}\,\rho + H_{4,2}\,\rho^2\right)\frac{e^{-4\rho}}{a_0^4} + \dots\right]\,, \end{aligned} \tag{4.72}$$

with $a_0 \neq 0$. The reason why the expansion of a only involves odd powers of e^{ρ} is that terms involving even powers would have been set to zero by the equations. For the same reason the expansion of H only involves even powers of e^{ρ}. We have obtained a perturbative solution for the two equations (4.30) and (4.31) which is valid up to order $\mathcal{O}(e^{-10\rho})$ and is controlled by the following eight parameters

$$\begin{aligned} &a_0\,, \quad a_2 \equiv a_{2,0}\,, \quad c \equiv a_{2,1}\,, \quad a_4 \equiv a_{4,0}\,, \quad a_6 \equiv a_{6,0}\,, \\ &H_2 \equiv H_{2,0}\,, \quad H_4 \equiv H_{4,0}\,, \quad \tilde{H} \equiv H_{2,1}\,. \end{aligned} \tag{4.73}$$

We report here the first terms in the expansion of H and a:

$$\begin{aligned} a(\rho) = {} & a_0\, e^{\rho} + (a_2 + c\rho)\,\frac{e^{-\rho}}{a_0} \\ + & \left[a_4 + \frac{2 - 16a_2 - 5c}{12} c\rho + \frac{3}{8}\Big(2H_2 + 3\tilde{H}\Big)\tilde{H}\rho \right. \\ & \left. - \frac{2}{3}c^2\rho^2 + \frac{3}{8}\tilde{H}^2\rho^2\right]\frac{e^{-3\rho}}{a_0^3} + \mathcal{O}(e^{-4\rho})\,, \end{aligned} \tag{4.74}$$

$$\begin{aligned} H(\rho) = {} & \Big(H_2 + \tilde{H}\rho\Big)a_0^2 e^{2\rho} + H_4 + 2\Big(H_2 + \tilde{H}\Big)\tilde{H}\rho \\ & + \left(\frac{2}{3}c\tilde{H} + \tilde{H}^2\right)\rho^2 + \frac{1}{6}\Big(4a_2\tilde{H} + 4cH_2 - 2c\tilde{H} + \tilde{H}\Big)\rho + \mathcal{O}(e^{-2\rho})\ . \end{aligned} \tag{4.75}$$

Notice that the backreaction of the fields in the supergravity vector multiplets introduces a dependence on H_4, H_2, $\tilde{H}$ in the metric functions.[55]

Using the near-boundary solution we found, we can perturbatively evaluate all the other relevant functions. However before computing them, we introduce the following parameter

$$v^2 = 1 - 4c\ , \tag{4.76}$$

which will be related to the squashing of the three-sphere at the boundary. We will trade the parameter c for v^2 when writing the main results of the present section, since the latter has a clearer physical interpretation. We also change the coordinates y, $\hat{\psi}$ into new coordinates t, ψ, defined as:

$$y = t\ , \qquad \hat{\psi} = \psi - \frac{2}{v^2}\,t\ . \tag{4.77}$$

These lead to a static (rather than stationary) metric on the conformal boundary. In these coordinates, the supersymmetric Killing vector V reads

$$V = \frac{\partial}{\partial y} = \frac{\partial}{\partial t} + \frac{2}{v^2}\frac{\partial}{\partial \psi}\ . \tag{4.78}$$

In the new set of coordinates, the metric and the gauge fields turn to:

$$\mathrm{d}s^2 = g_{\rho\rho}\mathrm{d}\rho^2 + g_{\theta\theta}(\sigma_1^2 + \sigma_2^2) + g_{\psi\psi}\sigma_3^2 + g_{tt}\mathrm{d}t^2 + 2g_{t\psi}\,\sigma_3\,\mathrm{d}t\ , \tag{4.79}$$

$$A^I = A^I_t\,\mathrm{d}t + A^I_{\psi}\,\sigma_3\ , \tag{4.80}$$

55 We also found a different solution for $H(\rho)$, having $H_{0,0} = 1$ (while $H_{0,0} = 0$ in (4.75)) and governed by the free parameter $H_{4,0}$. However the leading term of the corresponding metric turns out to be of order $\mathcal{O}(e^{4\rho})$, indicating that the latter is not AlAdS. For this reason we will not discuss this other solution in the following.

where the one-form σ's are defined as the $\hat{\sigma}$'s in (2.48), but using ψ instead of $\hat{\psi}$.

The functions f and w, which are independent on the change of coordinates (4.77), can be easily evaluated using (4.32) and (4.33). From their explicit expansions, we see that f goes to 1 in the near-boundary limit, while the w function has a $e^{2\rho}$ leading term[56]. Both of these two near-boundary behaviours are fully consistent with an AlAdS solution.

We find that at leading order the five-dimensional metric reads:

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{2\rho}\,\mathrm{d}s^2_{\mathrm{bdry}} + \ldots\ , \tag{4.81}$$

where the metric on the conformal boundary is:

$$\mathrm{d}s^2_{\mathrm{bdry}} \;=\; (2a_0)^2\left[-\frac{1}{v^2}\mathrm{d}t^2 + \frac{1}{4}\left(\sigma_1^2+\sigma_2^2+v^2\sigma_3^2\right)\right]\ . \tag{4.82}$$

As anticipated this is static in the chosen coordinates. The three-dimensional part of the metric involving the σ's is locally the metric on a Berger three-sphere, with v controlling the SU(2)$\times$ U(1) invariant squashing of the Hopf fiber.

The gauge fields A^I have a part along $\bar{X}^I$ and a part along $\bar{Y}^I$. At leading order, these are given by:

$$\bar{X}_I A^I = \frac{v^2+2}{3\,v^2}\,\mathrm{d}t + \frac{1}{3}(v^2-1)\,\sigma_3 + \mathcal{O}(e^{-\rho}) \tag{4.83}$$

and

$$\bar{Y}_I A^I = 36\,\frac{\tilde{H}}{v^2}\,\mathrm{d}t - 18\,\tilde{H}\,\sigma_3 + \mathcal{O}(e^{-\rho})\ . \tag{4.84}$$

Note that both $\bar{X}_I A^I$ and $\bar{Y}_I A^I$ have a non-trivial boundary field-strength proportional to $\sigma_1 \wedge \sigma_2$.

For the scalar fields X^I we find the following expressions

$$X^I = \bar{X}^I + 9\,\bar{Y}^I\,(2H_2 + \tilde{H} + 2\tilde{H}\,\rho)\,\frac{e^{-2\rho}}{a_0^2} + \mathcal{O}(e^{-3\rho})\ . \tag{4.85}$$

We recall that in the AdS$_5$ solution, our scalar fields have mass $m^2\ell^2 = -4$, hence the conformal dimension of the dual operator, following from the well-known formula $m^2\ell^2 = \Delta(\Delta-4)$, is $\Delta = 2$. This is also reflected in the asymptotic behavior displayed above.

For AlAdS solutions, there are natural coordinates which are preferable to use; these are the Fefferman-Graham ones. The analysis of the near-boundary solution we displayed above in these coordinates has been performed in [99]. There, it has been shown that the free parameters a_0, c and $\tilde{H}$ specify the boundary conditions of the bulk fields and are thus associated to sources in the dual field theory. As already apparent from the expressions above, a_0 and c determine both the metric

56 Since they are quite long and cumbersome, here we do not report the explicit expressions of the expansions for f and w. They can be found in [99].

and the value of $\bar{X}_I A^I$ at the conformal boundary, while $\tilde{H}$ fixes the asymptotic mode of the scalar fields. The three parameters a_0, c and $\tilde{H}$ together also determine $\bar{Y}_I A^I$. The remaining parameters a_2, a_4, a_6, H_2, H_4 instead control dual field theory one-point functions. In particular, H_2 controls the normalizable mode of the scalar fields. We will return on the holographic interpretation of our solution in sec. 4.3.1

We recall that in sec. 4.1.1.2 we have introduced the three independent first integrals for this theory with n_V arbitrary but the ansatz (4.20) imposed. These first integrals are given by $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ and are reported in eq. (4.39). Since they are constants, we can evaluate them in the near-boundary region using the expansions (4.71), (4.72) we have found. Doing so, we find that the two first integrals coming from the Maxwell equation, $\mathcal{K}_1$, $\mathcal{K}_2$ depend on the various near-boundary free parameters, among which the most subleading are a_4 and H_4. We might use these relations to express the latter free parameters with respect to the others and the first integrals. Furthermore, we also find that the first integral coming from the Einstein equations, $\mathcal{K}_3$, depends on a_6; therefore we also get an expression for a_6 with respect the other near-boundary free parameters and the first integrals. These relations for a_4, H_4 and a_6 are given by

$$\begin{aligned} a_4 = \tfrac{5}{384} + \tfrac{1}{6}a_2 - \tfrac{2}{3}a_2^2 + (1-5a_2)\tfrac{c}{12} - \tfrac{13}{48}c^2 \\ + \tfrac{3}{8}H_2^2 + \tfrac{9}{8}H_2\tilde{H} + \tfrac{51}{64}\tilde{H}^2 - \tfrac{3}{8}\mathcal{K}_1 \ , \end{aligned} \tag{4.86}$$

$$\begin{aligned} H_4 = \tfrac{1}{6}(4a_2H_2 + H_2 - 2\tilde{H}a_2 - 4\tilde{H}c + \tilde{H}) \\ + H_2^2 + 2H_2\tilde{H} + \tfrac{3}{2}\tilde{H}^2 + \tfrac{1}{4}\mathcal{K}_2 \ , \end{aligned} \tag{4.87}$$

$$\begin{aligned} a_6 = {} & \tfrac{1}{1296} - \tfrac{5}{18}a_2^2 + \tfrac{70}{81}a_2^3 + \left(\tfrac{1913}{3888}a_2 - \tfrac{125}{1944}\right)c^2 + \tfrac{1105}{11664}c^3 \\ & + \tfrac{1}{16}H_2^2 + \tfrac{1}{6}H_2^3 + \left(\tfrac{169}{144}H_2 + \tfrac{557}{3456}\right)\tilde{H}^2 + \tfrac{1229}{1728}\tilde{H}^3 \\ & + c\left(\tfrac{25}{3456} + \tfrac{197a_2-61}{324}a_2 - \tfrac{13}{72}H_2^2 - \tfrac{137}{216}H_2\tilde{H} - \tfrac{971}{1728}\tilde{H}^2 + \tfrac{19}{216}\mathcal{K}_1\right) \\ & + a_2\left(-\tfrac{29}{3456} - \tfrac{17}{24}H_2^2 - \tfrac{137}{72}H_2\tilde{H} - \tfrac{2129}{1728}\tilde{H}^2 + \tfrac{43}{72}\mathcal{K}_1\right) \\ & + \tilde{H}\left(\tfrac{7}{36}H_2 + \tfrac{17}{24}H_2^2 + \tfrac{29}{288}\mathcal{K}_2\right) - \tfrac{5}{288}\mathcal{K}_1 + \tfrac{1}{12}H_2\mathcal{K}_2 - \tfrac{1}{384}\mathcal{K}_3 \ . \end{aligned} \tag{4.88}$$

The advantage we get from these equations is the following: we are able to evaluate all the first integrals both in the near-boundary region as well as in the near-horizon region, obtaining them as functions of, respectively, the near-boundary and the near-horizon parameters; combining the relations obtained in the near-horizon with the ones obtained in the near-boundary we will be able to express a_4, H_4, a_6 as functions of only the remaining near-boundary parameters and the near-horizon ones. This will allow us to replace the most subleading parameters a_4, H_4, a_6 with the others.

4.2.1.2 *Near-horizon solution*

In the previous section we have shown that solutions compatible with an AlAdS$_5$ behaviour exist in the near-boundary, now we proceed to solve the ODEs (4.30), (4.31) near to $\rho = 0$, which we identify with the interior region of our solution. It is natural to assume that both the a and H functions can be Taylor expanded as:

$$\begin{aligned} a(\rho) &= \alpha_0 + \alpha_1\,\rho + \alpha_2\,\rho^2 + \dots\ , \\ H(\rho) &= \eta_0 + \eta_1\,\rho + \eta_2\,\rho^2 + \dots\ . \end{aligned} \tag{4.89}$$

We are interested a priori in two different types of solutions, either closing off regularly or meeting an event horizon when $\rho \to 0$. Solutions of first kind have been constructed for example in [133]. They are globally equivalent to AdS$_5$, with topology $\mathbb{R} \times \mathbb{R}^4$, with the first factor being a time direction and the second one arising from the three-sphere of the boundary which shrinks smoothly to zero size in the interior. This solution does not present any horizon, therefore around $\rho = 0$ the solution is non-singular. Solutions of second kind are quite different and have been constructed for example in [101]. In a neighbourhood of $\rho = 0$, the radial component blows up, revealing the presence of a horizon; therefore these are indeed black hole solutions, as we have already stated before in this thesis.

In both cases, given the form (2.49) of the metric we should take $\alpha_0 = 0$ in the expansion above. Moreover, due to the symmetries of the ODE's, we can take $\alpha_1 > 0$ with no loss of generality (we are not interested in solutions with $\alpha_1 = 0$).

In order to construct the perturbative solution in the interior region, we solved equations (4.30), (4.31) order by order in powers of ρ, up to $\mathcal{O}(\rho^{18})$. We found different branches of solutions, most of them corresponding to the small-ρ expansion of (4.24), (4.25), that is the well-known solution of [42] we have presented in sec. 2.1.3.1. However we also obtain one interesting branch of solutions to (4.30), yielding the following expression for H:

$$\begin{aligned} H(\rho) \;=&\; \eta\,\alpha^2\rho^2 + \frac{2\,\alpha\,\alpha_2\,\eta\,(2 - 3\,\alpha^2 + 24\,\eta)}{2 + \alpha^2 + 24\eta}\,\rho^3 \\ &+ \frac{\eta}{81}\Bigg[81(\alpha_2^2 + 2\,\alpha\,\alpha_3) - \frac{16(-2 + 17\,\alpha^2)\alpha_2^2}{1 - 4\alpha^2 + 12\,\eta} \\ &\qquad - \frac{288\alpha^2\,\alpha_2^2\,(2 + \alpha^2)}{(2 + \alpha^2 + 24\eta)^2} - \frac{8(8 + 175\alpha^2)\alpha_2^2}{2 + \alpha^2 + 24\eta}\Bigg]\rho^4 + \mathcal{O}(\rho^5) \end{aligned} \tag{4.90}$$

where we defined

$$\alpha \equiv \alpha_1\ , \qquad \eta \equiv \frac{\eta_2}{\alpha_1^2}\ , \tag{4.91}$$

Therefore, the function $H(\rho)$ is entirely determined by η and the coefficients of $a(\rho)$; these latter in turn are controlled by eq. (4.31). The next step is thus to solve this equation perturbatively order by order. The first non-trivial order of yields:

$$\alpha_2\left(8+13\alpha^2+\frac{576\,\eta^2}{2+\alpha^2+24\,\eta}\right) = 0\ , \tag{4.92}$$

so we have two possibility, namely to set either α_2 or the parenthesis to zero. In this thesis we will choose $\alpha_2 = 0$ and will not discuss the other option. In minimal gauged supergravity, the analogous analysis for equation (2.102) has been performed in [133]. This can be recovered by setting $H = 0$, which, due to (4.90), is equivalent to set $\eta = 0$. In [133], the above condition corresponded to $\alpha_2\left(8+13\alpha^2\right) = 0$, hence the choice $\alpha_2 = 0$ was the only possible. We conclude that only solutions with $\alpha_2 = 0$ admit a minimal gauged supergravity limit; this is one more reason to make this choice, since we want our new solutions to have a minimal limit. At the next order we get:

$$\alpha_4\left(-8+11\alpha^2-\frac{576\,\eta^2}{2-23\alpha^2+24\eta}\right) = 0\,. \tag{4.93}$$

This equation can be satisfied if $\alpha_4 = 0$ or the parenthesis vanishes. When $\eta = 0$ this reduces to $\alpha_4\left(-8+11\alpha^2\right) = 0$, that is the equation found in [133] for the minimal theory. In [133], the choice $\alpha_4 = 0$ led to either the solution of [41] (given by (4.25) above) or to a regular soliton that was identified as the gravity dual of the vacuum state of four-dimensional superconformal field theories on a squashed $S^3\times\mathbb{R}$. The choice $\alpha^2 = \frac{8}{11}$ led to the near-horizon expansion of a new supersymmetric black hole, as later confirmed and studied in greater detail in [101, 157]. We can expect the situation in the more general theory we are looking at to be similar, however this is not the case. Indeed, setting $\alpha_4 = 0$ leads to either (again) the solution of [42], or to a new solution, but this latter is not the generalization of the regular soliton found in [133], but rather a singular solution. In order to establish this, we have integrated numerically this new solution and found that it develops a singularity in the bulk for all initial conditions we tried. So we could not find a counterpart of the regular soliton of [133] in the presence of running scalars, neither we could find new regular solution with the choice $\alpha_4 = 0$. Thus we choose the second option to solve (4.93), that is we fix η in terms of α as:

$$\eta = \frac{1}{48}\left(-8+11\alpha^2\pm 9\alpha\sqrt{8-11\alpha^2}\right)\,, \tag{4.94}$$

implying that we must take $0<\alpha\leq\sqrt{\frac{8}{11}}$. There are two possible values of η, depending on the sign we choose in the equation above; for now we keep this choice unspecified and continue further. Proceeding with the perturbative approach to solving the supersymmetry equation (4.31) near $\rho = 0$, we find that the coefficients α_3 and α_4 in the expansion of $a(\rho)$ remain free together with α, while all the others are determined in terms of these ones. The first terms in the expansion of a and H read:

$$\begin{aligned} a &= \alpha\,\rho + \alpha_3\,\rho^3 + \alpha_4\,\rho^4 + \frac{3\alpha_3}{10\alpha}\rho^5 + \frac{\alpha_3\alpha_4}{4\alpha}\rho^6 + \mathcal{O}(\rho^7)\;, \\ H &= \eta\,\alpha^2\rho^2 + 2\eta\,\alpha\,\alpha_3\rho^4 + \frac{2\alpha\alpha_4(-2+15\alpha^2-24\eta)\eta}{-2+23\alpha^2-24\eta}\rho^5 \\ &\qquad + \frac{8\alpha_3^2\eta}{5}\rho^6 + \mathcal{O}(\rho^7)\;. \end{aligned} \tag{4.95}$$

We would like to establish which of our three free parameters α, α_3 and α_4 are physical. We note that it is possible to rescale at will one of the parameters without changing the five-dimensional solution. This is because eqs. (4.14), (4.18) imply that under a rescaling of the coordinates $\rho = \lambda^{-1}\tilde{\rho}$, $y = \lambda^2\,\tilde{y}$, a solution $a(\rho)$, $H_I(\rho)$ is transformed into another solution $\tilde{a}(\tilde{\rho}) = \lambda\,a(\lambda^{-1}\tilde{\rho})$, $\tilde{H}_I(\tilde{\rho}) = \lambda^2 H_I(\lambda^{-1}\tilde{\rho})$. This leaves the parameters α and η invariant, while it rescales α_3, α_4. We choose to rescale α_3 and we therefore regard this as an unphysical parameter. In the large-ρ solution of the previous subsection, this freedom has been fixed by assuming that for $\rho \to \infty$ the function a goes like e^ρ. Even though for now we will keep this arbitrary, when later on we will construct an interpolation between the small-ρ and the large-ρ solution we will need to tune it so that the assumed large-ρ asymptotics are matched. Furthermore, we find convenient to trade α_4 for a new parameter ξ, which is invariant under such symmetry transformation and is thus physical

$$\alpha_4 = \xi\,\alpha_3^{3/2}. \tag{4.96}$$

We will always use the definition above to replace α_4 with ξ wherever the former appears.

Due to the form of η given in (4.94), imposing $\alpha = \sqrt{\frac{8}{11}}$ corresponds to $\eta = 0$, that is $H = 0$. In this case, it is possible to check that the near-horizon expansion of the superymmetric black hole studied in [101,157] is recovered. So we can expect that choosing η as in (4.94), but with $\alpha \neq \sqrt{\frac{8}{11}}$, will lead to a generalization of such black hole, where the scalars will be running. In the remainder of this section and the next ones we will show that this is indeed the case.

We now provide the first terms in the small-ρ expansion of the metric, the gauge fields and the scalar fields. Although these depend on the free parameters α, α_3, ξ only, for convenience in the expressions below we also employ η, being understood that this is fixed in terms of α as in (4.94). Our main purpose will be to show that our small-ρ solution has a regular horizon at $\rho = 0$. We begin with the functions f and w, which can be evaluated by using (4.32), (4.33) and result to be

$$f = \frac{12\,\alpha^2}{\Delta}\rho^2 + \frac{1}{\Delta^4}\Big\{24\alpha\alpha_3\left(4\alpha^2+12\eta-1\right)\\ \Big[128\alpha^4-(1-12\eta)(1+24\eta)-4\alpha^2(7+96\eta)\Big]\Big\}\rho^4$$

$$w = -\frac{\left(1-4\,\alpha^2\right)^2-144\,\eta^2}{48\,\alpha^2}\,\frac{1}{\rho^2}\\ +\frac{\alpha_3\left(-272\,\alpha^4+64\,\alpha^2-144\,\eta^2+1\right)}{24\,\alpha^3}+\mathcal{O}(\rho)\;, \tag{4.97}$$

where we have defined the quantity:

$$\Delta = \left(4\alpha^2-24\eta-1\right)^{1/3}\left(4\alpha^2+12\eta-1\right)^{2/3}\;. \tag{4.98}$$

Keeping only the leading order terms in a small ρ expansion, the five dimensional metric reads:

$$\mathrm{d}s^2 = -\frac{48\alpha^6}{\Delta^2\Theta}\rho^4\,\mathrm{d}t^2+\Delta\left[\frac{\mathrm{d}\rho^2}{12\,\alpha^2\rho^2}+\frac{1}{12}(\sigma_1^2+\sigma_2^2)+\Theta\left(\sigma_3-\frac{2}{v^2}\mathrm{d}t\right)^2\right]\;, \tag{4.99}$$

where

$$\Theta = \frac{16\,\alpha^4+\alpha^2(8-96\,\eta)-3(12\,\eta+1)^2}{48\,(4\,\alpha^2-24\,\eta-1)}\;. \tag{4.100}$$

The scalar fields can be computed by (4.36) and they result to be

$$X^I = \left[\frac{\left(4\alpha^2-1\right)^2-144\eta^2}{\Delta^2}-\frac{20736\,\alpha\,\alpha_3\,\eta^2\left(4\alpha^2+12\eta-1\right)^2}{\Delta^5}\rho^2\right]\bar{X}^I\\ +\left[\frac{216\eta\left(4\alpha^2+12\eta-1\right)}{\Delta^2}\right.\\ \left.-\frac{15552\alpha\alpha_3\eta\left(4\alpha^2-1\right)\left(4\alpha^2+12\eta-1\right)^2}{\Delta^5}\rho^2\right]\bar{Y}^I+\mathcal{O}(\rho^3). \tag{4.101}$$

Finally, the perturbative expansion for the gauge fields around $\rho = 0$ is

$$A^I_\psi = \frac{\left(4\alpha^2-36\eta-1\right)\left(4\alpha^2+12\eta-1\right)}{12\left(4\alpha^2-24\eta-1\right)}\bar{X}^I\\ -\frac{18\eta\left(4\alpha^2+12\eta-1\right)}{4\alpha^2-24\eta-1}\bar{Y}^I+\mathcal{O}(\rho^2)\;,$$

$$A^I_t = -\frac{2}{v^2}A^I_\psi(\rho=0)+\mathcal{O}(\rho^2)\;. \tag{4.102}$$

We can argue that the solution above describes the vicinity of an event horizon of finite size situated at $\rho = 0$. Indeed the elsewhere timelike supersymmetric Killing vector V, whose norm is $-f^2$, becomes null as $\rho \to 0$. Moreover the metric has a divergent term $\mathcal{O}(\rho^{-2})\mathrm{d}\rho^2$, while the remaining spatial part remains finite. In addition, both the scalar fields and the gauge fields have a regular behaviour as $\rho \to 0$. In particular, note that in the gauge we are using the gauge fields at the horizon are transverse to the supersymmetric Killing vector V,

$$V^\mu A^I_\mu = A^I_y = A^I_t + \tfrac{2}{v^2} A^I_\psi = 0 \quad \text{for} \quad \rho = 0\,. \tag{4.103}$$

In order to better describe the geometry of the horizon, we introduce gaussian null coordinates adapted to the supersymmetric Killing vector field V [41, 42]. This is done by the transformation:

$$\begin{aligned}
\mathrm{d}y &= \mathrm{d}u + \left(\frac{f w^2}{(2aa')^2} - \frac{1}{f^2}\right)\mathrm{d}\tilde{\rho}\,,\\
\mathrm{d}\hat{\psi} &= \mathrm{d}\tilde{\psi} - \frac{f\,w}{(2aa')^2}\,\mathrm{d}\tilde{\rho}\,,\\
\mathrm{d}\rho &= \sqrt{\frac{1}{f} - \frac{f^2 w^2}{(2aa')^2}}\,\mathrm{d}\tilde{\rho}\,.
\end{aligned} \tag{4.104}$$

Note that this is the same coordinate transformation of eq. (2.70), therefore the form assumed by the five-dimensional metric is also the same as (2.71). We report this latter explicitly here below for convenience

$$\begin{aligned}
\mathrm{d}s^2 = -f^2\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}\tilde{\rho} - 2f^2 w\,\mathrm{d}u\,\tilde{\sigma}_3 \\
+ f^{-1}a^2(\sigma_1^2+\sigma_2^2) + \left(f^{-1}(2aa')^2 - f^2w^2\right)\tilde{\sigma}_3^2\,.
\end{aligned} \tag{4.105}$$

Plugging our near-horizon solution in, we obtain that the metric at the horizon is

$$\mathrm{d}s^2_{\text{horizon}} \;=\; 2\,\mathrm{d}u\,\mathrm{d}\tilde{\rho} + \frac{\Delta}{12}(\sigma_1^2+\sigma_2^2) + \Delta\,\Theta\,\tilde{\sigma}_3^2\,, \tag{4.106}$$

which is manifestly well-definite and regular provided $\Delta > 0$ and $\Theta > 0$. We have plotted these quantities in Figure 4.1, choosing the minus sign in the determination (4.94) for η. We note that Δ is positive for every value of the parameter α except for $\alpha = \sqrt{2/3} \simeq 0.816$, while Θ is real and positive for $0.657 < \alpha < \sqrt{8/11} \simeq 0.853$. Regularity of the horizon however does not guarantee regularity outward the horizon. In the next section, we will see that that regularity in the bulk in fact further constrains the allowed range of α.[57]

57 Similarly, we find a narrow regularity range for the horizon geometry when the plus sign is chosen in the formula (4.94) for η. This is also further reduced when regularity away from the horizon is imposed.

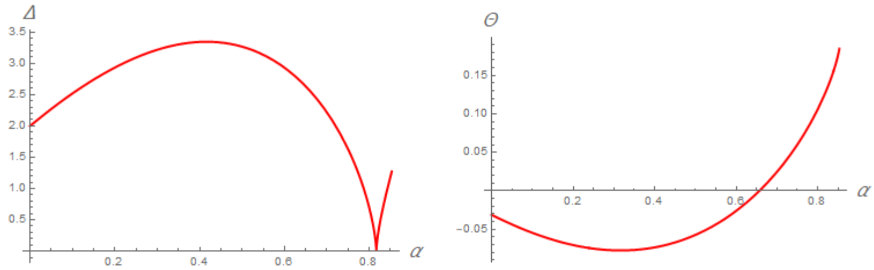


Figure 4.1: The two functions $\Delta(\alpha)$ and $\Theta(\alpha)$ whose positiveness is needed to have a regular horizon. We observe that Δ is always positive except in the cusp point at $\alpha = \sqrt{2/3}$, while Θ is positive only for $\alpha \gtrsim 0.657$.

From (4.106), we can compute the area of the horizon, which results to be

$$Area = \frac{\pi^2}{3\sqrt{3}}\left(4\alpha^2 + 12\eta - 1\right)\left[16\alpha^4 + \alpha^2(8 - 96\eta) - 3(12\eta + 1)^2\right]^{1/2} , \tag{4.107}$$

note that this is finite in the allowed range of the parameters.

By everything we have shown above, it is demonstrated that our small-ρ solution describes the vicinity of the horizon of a new two-parameter family of black holes with running scalars, controlled by the parameters α and ξ (recall that in general our q_I are not free parameters as they are fixed by condition (4.26)).

Three concluding remarks are in order.

The first is that ξ is sufficiently subleading in the small-ρ expansion of a not to appear in the leading terms of the supergravity fields as $\rho \to 0$. In other words, the horizon is not affected by ξ. We will confirm later that this parameter is anyway physical, as when it is non-zero it leads to a squashing of the conformal boundary, making the solution asymptotically *locally* AdS (as opposed to asymptotically AdS).

The second fact to point out is that when $\xi = 0$ we can resum the perturbative series and obtain the exact solution $H = \eta\, a^2$, $a = \alpha \sinh\rho$, where η is fixed in terms of α as discussed above. This solution matches the one of [42], with our parameter α being mapped into the three parameters $\alpha_1^{\rm GR}, \alpha_2^{\rm GR}, \alpha_3^{\rm GR}$ appearing in that paper. The precise relation between the parameters is worked out in [99] and results to be

$$\begin{aligned}
\alpha_1^{\rm GR} &= (4\alpha^2 - 1)\ell^2 , \\
\alpha_2^{\rm GR} &= \tfrac{1}{3}(4\alpha^2 - 1)^2\ell^4 - 48\,\eta^2 , \\
\alpha_3^{\rm GR} &= \tfrac{1}{27}(4\alpha^2 - 1)^3\ell^6 - 16\,\eta^2(4\alpha^2 - 1)\ell^2 - 128\,\eta^3 ,
\end{aligned} \tag{4.108}$$

where we have reinstated the AdS radius ℓ for completeness. Furthermore, the following relation between our q_I and the $q_I^{\rm GR}$ of [42] is found:

$$q_I^{\rm GR} = \tfrac{1}{3}(4\alpha^2 - 1)\ell^2 \bar{X}_I + 8\,\eta\, q_I , \tag{4.109}$$

Therefore, for $\xi = 0$ our solution is equivalent to a one-parameter subfamily of the black hole of [42]. Instead, when $\xi \neq 0$ we have a new branch of solutions. Nevertheless, since ξ does not affect the horizon geometry, the latter remains the same as in the black hole of [42], with the identification of the parameters above. In particular, using this dictionary the area of the horizon (4.107) matches the expression given in [42].

The third and final remark is about the minimal limit, obtained by taking $\alpha = \sqrt{\frac{8}{11}}$. In this case, the scalars are constant and the perturbative near-horizon solution reduces to the one of [101, 157], which is controlled by the parameter ξ only and presents a frozen horizon geometry. We have thus demonstrated that by allowing for running scalars one can introduce a new parameter so that the horizon geometry gets unfrozen.

We conclude this subsection by evaluating the first integrals (4.39) in the near-horizon region. We find that in the limit $\rho \to 0$ they evaluate to:

$$\mathcal{K}_1 = -\frac{1}{9}\left(\alpha^2+1\right)\alpha^2 + \eta^2 + \frac{5}{144}\,, \tag{4.110}$$

$$\mathcal{K}_2 = -\frac{2}{3}\eta\left(2\alpha^2+6\eta+1\right)\,, \tag{4.111}$$

$$\mathcal{K}_3 = -4\left(8\alpha^2+1\right)\eta^2 + \frac{1}{108}\left(8\alpha^2+7\right)\left(1-4\alpha^2\right)^2 - 64\eta^3\,. \tag{4.112}$$

Recalling that η is fixed as in (4.94), these are functions of the near-horizon parameter α only. Combining these relations with the one we have obtained by evaluating the first integrals in the near-boundary region, given by eq. (4.86), (4.87), (4.88), we can determine a_4, a_6, H_4 in terms of the other near-boundary parameters a_0, a_2, v, $\tilde{H}$, H_2 and the near-horizon parameter α. On the other hand, in order to determine the relation of the remaining near-boundary parameters with the only two physical near-horizon parameters α and ξ we will have to resort to numerics.

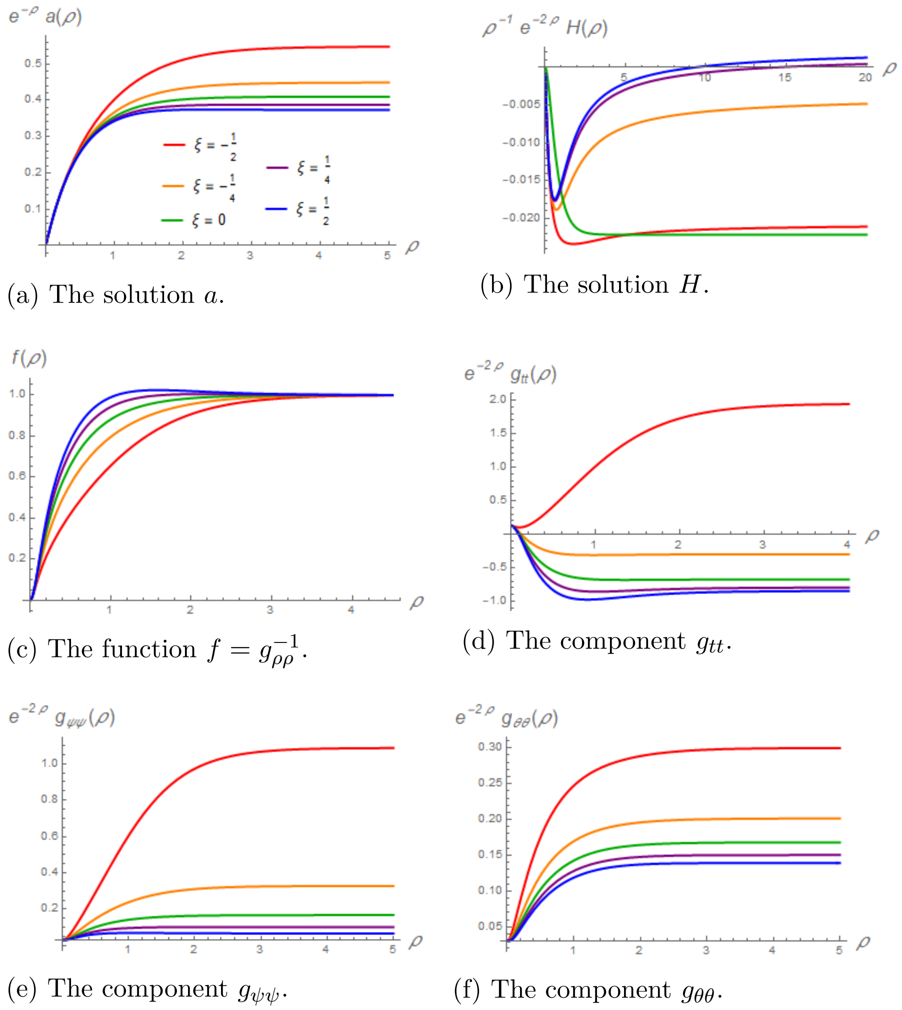
As a cross-check, we can evaluate the relations above in the limit leading to minimal gauged supergravity and compare with the expressions previously found within this theory [101, 133]. We thus take $H_2 = H_4 = \tilde{H} = 0$. Then (4.87) merely gives $\mathcal{K}_2 = 0$, while (4.86), (4.88) reduce to expressions that are in agreement with eqs. (4.21), (4.22) of [101].[58] The values of $\mathcal{K}_1$, $\mathcal{K}_3$ specific to the black hole solution of minimal gauged supergravity studied in [101] are correctly retrieved by sending $\alpha \to \sqrt{\frac{8}{11}}$ in (4.110), (4.112). We can also compare with the expressions for a_4 and a_6 given in eq. (B.1) of [133]: we find agreement upon setting $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}_3 = 0$, which are the appropriate values for a solution capping off smoothly such as the one presented in that paper.

58 Upon identifying the constants c_t, c_W appearing there as $c_t = -4\sqrt{3}\mathcal{K}_1$, $c_W = -\mathcal{K}_3$.

4.2.1.3 *The numerical solution*

Our aim is now to interpolate the near-horizon solution and the near-boundary one by a numerical approach, thus showing there is indeed a black hole solution which is regular for every $\rho > 0$. This happens only in a certain region of the parameter space, that we will determine.

We begin by describing our strategy to perform the numerical analysis.



(a) The solution a.

(b) The solution H.

(c) The function $f = g_{\rho\rho}^{-1}$.

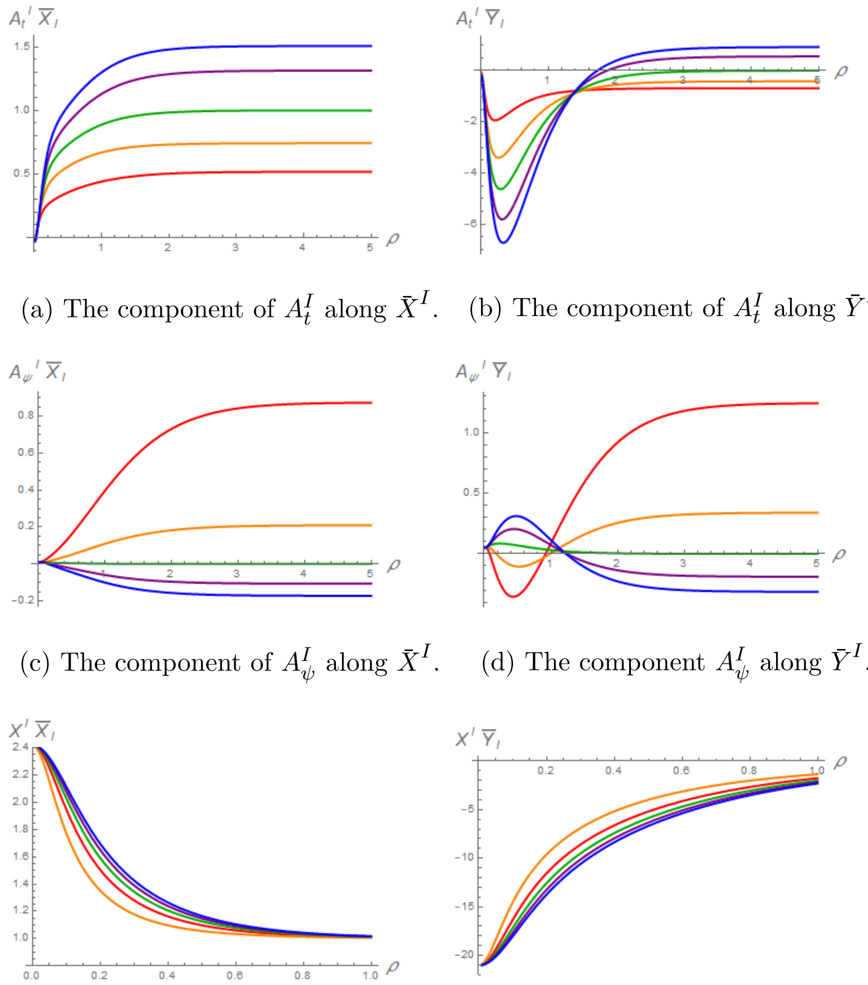
(d) The component g_{tt}.

(e) The component $g_{\psi\psi}$.

(f) The component $g_{\theta\theta}$.

Figure 4.2: Relevant functions and metric components of our solution, rescaled by their asymptotic behaviour at large ρ. The different values of the near-horizon parameter ξ are indicated in the label. We emphasize that although this is not immediately recognized from the plots, $g_{\theta\theta}$ and $g_{\psi\psi}$ go to a small but positive constant, leading to an even horizon of finite size. This is clear from (4.99).

We fix the initial conditions at $\rho \simeq 0$ using the expressions in sec. 4.2.1.2 and integrate equations (4.30), (4.31) numerically towards larger values of ρ. It is obvious that in order to do this we need to assign a numerical value to the two physical parameters ξ and α; in particular we will



(a) The component of A_t^I along $\bar{X}^I$. (b) The component of A_t^I along $\bar{Y}^I$.

(c) The component of A_ψ^I along $\bar{X}^I$. (d) The component A_ψ^I along $\bar{Y}^I$.

(e) Scalar fields X^I along $\bar{X}^I$. (f) Scalar fields X^I along $\bar{Y}^I$.

Figure 4.3: Components of the gauge fields A^I and of the scalar fields X^I along $\bar{X}^I$ and $\bar{Y}^I$.

choose for α numerical values that are in the range $0.657 \leq \alpha \leq \sqrt{\frac{8}{11}}$, with $\alpha \neq \sqrt{\frac{2}{3}}$, so as to meet the regularity conditions for the horizon we have found in sec. 4.2.1.2. Moreover we rescale the unphysical parameter α_3 in such a way that the assumed AlAdS behaviour of a for $\rho \to \infty$ holds.[59]

From the numerical analysis we have performed emerges that that the solution is regular only in the range:

$$\sqrt{\frac{2}{3}} < \alpha \leq \sqrt{\frac{8}{11}} \ , \tag{4.113}$$

59 In order to achieve this we exploit the rescaling properties we have described in sec. 4.2.1.2. We integrate a first time choosing $\alpha_3 = 1$, then we look at the large-ρ behaviour of the solution and determine the rescaling factor λ^2 by requiring that $f \to 1$ asymptotically. This is equivalent to impose $a \sim e^\rho$ as $\rho \to \infty$. Then we fix $\alpha_3 = 1/\lambda^2$ and repeat the integration.

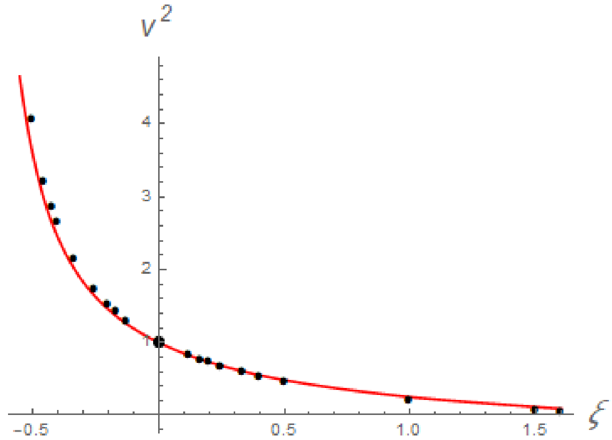
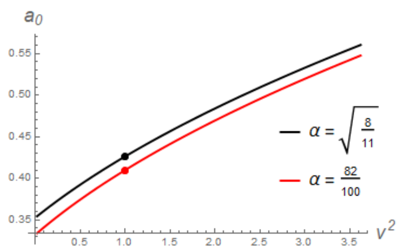
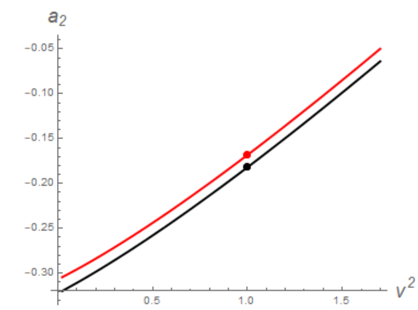


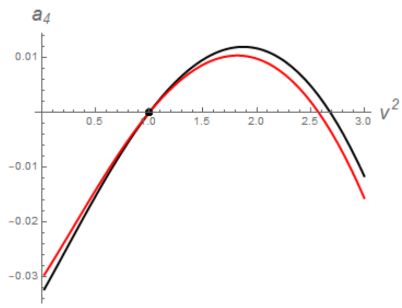
Figure 4.4: Relation between the near-horizon parameter ξ and the squashing v^2 of the boundary metric, for $\alpha = 0.82$. v^2 is positive and finite for $-0.7 \lesssim \xi \lesssim 1.6$. The black dots represent the values effectively calculated by means of the numerical analysis. The larger dot at $(\xi = 0, v = 1)$ represents the solution of [42].



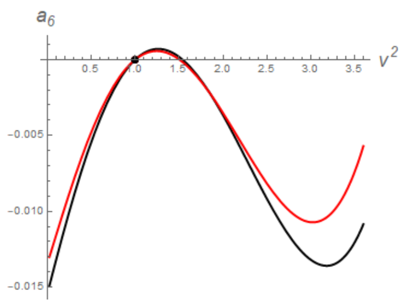
(a) The parameter a_0.



(b) The parameter a_2.

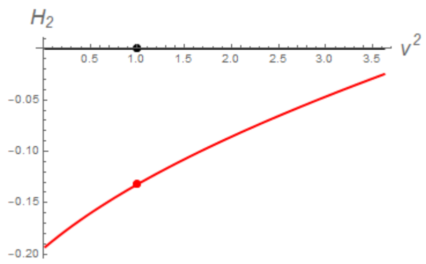


(c) The parameter a_4.

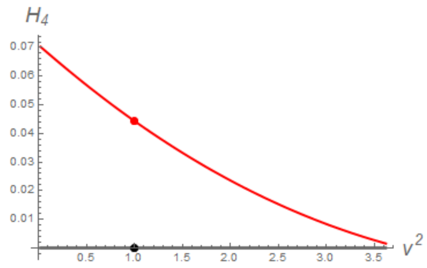


(d) The parameter a_6.

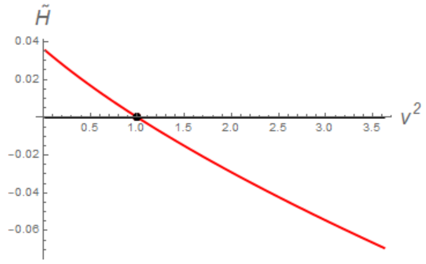
Figure 4.5: The near-boundary parameters a_0, a_2, a_4, a_6 in terms of the squashing v^2, for $\alpha = 0.82$ (red) and $\alpha = \sqrt{8/11}$ (black).



(a) The parameter H_2.



(b) The parameter H_4.



(c) The parameter $\tilde{H}$.

Figure 4.6: The parameters of $H_2, H_4, \tilde{H}$ in terms of the squashing v^2, for $\alpha = 0.82$ (red). For $\alpha = \sqrt{\frac{8}{11}}$ they vanish identically (black).

while outside of this the function f presents a divergence at finite ρ and the same do other components of the metric and the gauge fields. Instead, when α takes a values within this range, all the components of the metric and the gauge fields should be regular. We have checked for several values of α within this range that this indeed happens, provided ξ lies in a certain range that depends on α and is determined by regularity of the boundary geometry.

In order to provide an illustrative example, we report here all the relevant physical functions for the value $\alpha = 0.82$ and for different choices of ξ. In Figure 4.2 we display the functions a and H and the components of the metric (4.118), while Figure 4.9 shows the components of the gauge field (4.119) and of the scalar fields X^I. The plots demonstrate that the solution is smooth on and outside the event horizon.

In the previous subsections, we have underlined how the free parameters appearing in the general near-boundary solution (a_0, a_2, a_4, a_6, v, H_2, H_4, $\tilde{H}$) can be in principle expressed as functions of the only two near-horizon parameters α, ξ characterizing the black hole solution; however this has been proven impossible to do analytically. Instead, we can do this here by following a numerical approach. In order to do this we compare the numerical solution for the functions a and H with the near-boundary expansion discussed in sec. 4.2.1.1 at some reasonably large values of the radial coordinate ρ (we find it sufficient to use several points in the interval $3 < \rho < 6$), and evaluate the near-boundary parameters using a best-fit technique. In Figures 4.4, 4.5, 4.6 we present the results obtained using this method for the two values $\alpha = 0.82$ and

$\alpha_1 = \sqrt{\frac{8}{11}}$ and for about 20 values of ξ. Figure 4.4 shows the relation between the squashing parameter v^2 and the near-horizon parameter ξ, with $\alpha = 0.82$ (we are not presenting the plot for $\alpha = \sqrt{\frac{8}{11}}$ as it is not significantly different from the displayed one). Notice that for ξ running between $\xi \sim 1.6$ and $\xi \sim -0.7$ the squashing v^2 spans the whole positive line. From an AdS/CFT perspective, the squashing parameter v of the boundary geometry seems to play a more significant role than ξ, so once α has been fixed, we choose to regard the family of solutions as parametrized by v^2 rather than ξ. Consequently, in the Figures 4.5 and 4.6 we plot the near-boundary parameters as function of v^2. Recall that the solution with $\alpha = \sqrt{\frac{8}{11}}$ fits into minimal gauged supergravity and coincides with the black hole of [101], so with the plots of Figures 4.5 and 4.6 we are comparing our new family of solutions with that one.

Guided and helped by the figures, let now discuss some physical properties of our solution. From Figure 4.2 we can exclude the presence of closed timelike curves, which would appear whenever the $g_{\psi\psi}$ component of the metric becomes negative. Although the figure displays just the behavior for $\alpha = 0.82$, we have verified that closed timelike curves are also absent for different values of α in the range (4.113). Furthermore we should note from Figure 4.2 that in the near-horizon region g_{tt} becomes positive, implying that the vector $\frac{\partial}{\partial t}$ becomes spacelike. This means that if this vector is regarded as the generator of time translations, then our solution presents an ergoregion for all the values of ξ and α in the allowed range (4.113). However we may also take as generator of time translations the supersymmetric Killing vector field (4.78), which corresponds to working in a frame that is co-rotating with the event horizon. In this case there is no ergoregion as this vector is timelike everywhere outside the horizon. This feature is common in rotating, asymptotically AdS black holes and in the supersymmetric context it was noted in [41].

4.2.2 *The solution with $n_V = 2$*

4.2.2.1 *Near-boundary solution*

We now move to the $\mathrm{U}(1)^3$ theory case and we construct the near-boundary perturbative solution by solving the equations (4.50), (4.51) and (4.52) around $\rho \to \infty$. The strategy we follow is the same of the arbitrary n_V case presented in the previous section and the physical concept which will emerge are basically the same; therefore we will keep the presentation shorter. First of all, we change the labels of the functions H_1 and H_2 to:

$$H_1(\rho) \to Z(\rho), \qquad H_2(\rho) \to K(\rho). \tag{4.114}$$

Then, we assume for the unknown functions the following asymptotic expansion

$$\begin{aligned} a(\rho) &= a_0 e^{\rho}\left[1+\sum_{k\geq 1}\sum_{0\leq n\leq k} a_{2k,n}\,\rho^n\,\left(a_0\,e^{\rho}\right)^{-2k}\right] \\ &= a_0 e^{\rho}\left[1+\left(a_{2,0}+a_{2,1}\rho\right)\frac{e^{-2\rho}}{a_0^2}\right. \\ &\left.+\left(a_{4,0}+a_{4,1}\,\rho+a_{4,2}\,\rho^2\right)\frac{e^{-4\rho}}{a_0^4}+\dots\right], \end{aligned} \tag{4.115}$$

$$\begin{aligned} Z(\rho) &= a_0^4 e^{4\rho}\left[\sum_{k\geq 0}\sum_{0\leq n\leq k} Z_{2k,n}\,\rho^n\,\left(a_0\,e^{\rho}\right)^{-2k}\right] \\ &= a_0^4 e^{4\rho}\left[Z_{0,0}+\left(Z_{2,0}+Z_{2,1}\rho\right)\frac{e^{-2\rho}}{a_0^2}\right. \\ &\qquad\left.+\left(Z_{4,0}+Z_{4,1}\,\rho+Z_{4,2}\,\rho^2\right)\frac{e^{-4\rho}}{a_0^4}+\dots\right], \end{aligned} \tag{4.116}$$

$$\begin{aligned} K(\rho) &= a_0^4 e^{4\rho}\left[\sum_{k\geq 0}\sum_{0\leq n\leq k} K_{2k,n}\,\rho^n\,\left(a_0\,e^{\rho}\right)^{-2k}\right] \\ &= a_0^4 e^{4\rho}\left[K_{0,0}+\left(K_{2,0}+K_{2,1}\rho\right)\frac{e^{-2\rho}}{a_0^2}\right. \\ &\qquad\left.+\left(K_{4,0}+K_{4,1}\,\rho+K_{4,2}\,\rho^2\right)\frac{e^{-4\rho}}{a_0^4}+\dots\right]. \end{aligned} \tag{4.117}$$

As for the previous case, we assume $a_0 \neq 0$. Note that in the expansion for a there are only odd powers of e^{ρ}: that is exactly for the same reason as the arbitrary n_V case. The same argument applies to the other two expansions, in which only even powers of e^{ρ} appear.

We have obtained a perturbative solution for the three equations (4.30), (4.51) and (4.52) which is valid up to order $\mathcal{O}\left(e^{-10\,\rho}\right)$ and is controlled by the following eleven parameters[60]:

$$\begin{aligned} &a_0\,, \quad a_2 = a_{2,0}\,, \quad c = a_{2,1}\,, \quad a_4 = a_{4,0}\,, \quad a_6 = a_{6,0}\,, \\ &\quad Z_2 = Z_{2,0}\,, \quad Z_4 = Z_{4,0}\,, \quad \tilde{Z} = Z_{2,1}\,, \\ &\quad K_2 = K_{2,0}\,, \quad K_4 = K_{4,0}\,, \quad \tilde{K} = K_{2,1}\,. \end{aligned}$$

60 In principle, other solutions are possible. They have $Z_{0,0} \neq 0$ or $K_{0,0} \neq 0$, so the Z and K functions have a different leading behaviour. However these solutions present metrics which are not AlAdS, since their leading term is of order $\mathcal{O}\left(e^{4\,\rho}\right)$. We are interested only in AlAdS behaviours, therefore we will not discuss these solutions in the following.

Since they are quite long and cumbersome, we do not provide the explicit expressions of the perturbative solutions for a, Z and K; these can be found in [100].

The parameter v is defined exactly as for the previous case, i.e. is again given by eq. (4.76). Also for this solution it is convenient to always trade the parameter c for v^2 so as to eliminate the former. We also apply the same change of coordinates used in the previous section, given by eq. (4.77), to this solution; doing so we obtain the following form for the metric and the gauge fields:

$$\mathrm{d}s^2 = g_{\rho\rho}\mathrm{d}\rho^2 + g_{\theta\theta}(\sigma_1^2+\sigma_2^2) + g_{\psi\psi}\sigma_3^2 + g_{tt}\mathrm{d}t^2 + 2g_{t\psi}\,\sigma_3\,\mathrm{d}t \ , \tag{4.118}$$

$$A^I = A_t^I\,\mathrm{d}t + A_\psi^I\,\sigma_3 \ , \tag{4.119}$$

the σ_i being defined in the same way as the $\hat{\sigma}^i$ with ψ replacing $\hat{\psi}$.

The near-boundary behaviors of the functions f and w are consistent with an AlAdS$_5$ solution; indeed f goes to 1 as $\rho \to \infty$ while the function w presents a $e^{2\rho}$ leading term.

The metric (4.118) results to be static, as it was in the previous case. Indeed, we find that it can be rewritten as

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{2\rho}\,\mathrm{d}s^2_{\mathrm{bdry}} + \dots \ , \tag{4.120}$$

with $\mathrm{d}s^2_{\mathrm{bdry}}$ being the metric at the boundary, which reads

$$\mathrm{d}s^2_{\mathrm{bdry}} \;=\; (2a_0)^2\left[-\frac{1}{v^2}\,\mathrm{d}t^2 + \frac{1}{4}\left(\sigma_1^2+\sigma_2^2+v^2\sigma_3^2\right)\right]. \tag{4.121}$$

Here we do not display the explicit expressions for the metric and gauge fields components, neither the ones for the scalars X^I, since they are quite cumbersome and we do not want to burden the discussion. These expressions can be found in [100]. In the same paper, the analysis of the near-boundary solution in Fefferman-Graham coordinates is performed.

The possibility to write the solution in Fefferman-Graham form confirms once more that it is indeed AlAdS$_5$ and the analysis of [100] shows that four of the eleven free parameters, $a_0, c, \tilde{Z}$ and $\tilde{K}$ determine the bulk fields at the boundary and therefore play the role of a source in the dual quantum field theory. In particular, looking at (4.121) and recalling that v^2 is related to c as in (4.76), it is possible to predict that a_0 and c would control the metric at the conformal boundary, while $\tilde{Z}$ and $\tilde{K}$ should determine the scalar fields. The analysis in Fefferman-Graham coordinates of [100] reveals that this is indeed the case and we refer to this paper for further details.

As for the arbitrary n_V case, we can evaluate the first integrals for the $\mathrm{U}(1)^3$ theory, given in eq. (4.69), in the near-boundary and, doing so, we can obtain relations between them and the free parameters. In particular we find relations between the most subleading parameters

a_4, Z_4, K_4, a_6 and the first integrals, which involve also the other parameters of the solution. We use these relations to trade the subleading parameters above with the first integrals, so that they will never appear into the equations anymore. Later, we will evaluate all the first integrals also in the near-horizon region, obtaining them as functions of the near-horizon parameters; therefore we will be able to express the subleading parameters a_4, Z_4, K_4, a_6 with respect to the near-horizon ones and the remaining near-boundary ones.

4.2.2.2 *Near-horizon solution*

In this subsection we construct the solution perturbatively in the interior region, near $\rho \to 0$. The procedure is very similar to the one we followed in sec. 4.2.1.2 and the emerging physical concepts will be basically the same, therefore we will proceed faster keeping the presentation shorter.

In order to solve the three ODEs (4.30), (4.51) and (4.52) in the desired region, we assume that a, Z and K can be Taylor-expanded as

$$\begin{aligned} a(\rho) &= \alpha_0 + \alpha_1\,\rho + \alpha_2\,\rho^2 + \dots \;, \\ Z(\rho) &= \eta_0 + \eta_1\,\rho + \eta_2\,\rho^2 + \dots \;, \\ K(\rho) &= \iota_0 + \iota_1\,\rho + \iota_2\,\rho^2 + \dots \;. \end{aligned} \tag{4.122}$$

Here we call the coefficients of Z as η_i, while in sec. 4.2.1.2 we used the same notation for the coefficients of H; however this would be not confusing since we will never use the old coefficients anymore. Furthermore, keeping this notation allows to compare with [100] much more easily. We want to search for either new black hole solutions or new soliton solutions. Looking at the metric (2.49), we see that both the types of solutions require $\alpha_0 = 0$, which we therefore assume. Moreover, due to the symmetries of the ODE's we can take $\alpha_1 > 0$ with no loss of generality.

Solving (4.30), (4.51) and (4.52) perturbatively up to $\mathcal{O}(\rho^{13})$, we find that equations (4.30) and (4.51) fix uniquely the form of K and Z with the coefficients ι_0, η_0, ι_1, η_1 forced to vanish and all the others determined by the free parameters ι_2, η_2 and by the coefficients of a. As for the previous case of the solution with arbitrary n_V, we find it convenient to define the new parameters

$$\alpha \equiv \alpha_1 \;, \qquad \eta \equiv \frac{\eta_2}{\alpha_1^2} \;, \qquad \iota \equiv \frac{\iota_2}{\alpha_1^2} \tag{4.123}$$

with respect which the expansions of Z and K can be written as

$$\begin{aligned}Z(\rho) \simeq \eta\,\alpha^2\,\rho^2 - &\frac{2\,\alpha\,\alpha_2}{\alpha^4+4\,\alpha^2-6912\,(\eta^2+\eta\,\iota+\iota^2)+4}\times\\ &\times\Big[192\,\eta^2\left(\alpha^2+36\,\iota\right)-384\,\alpha^2\,\iota^2\\ &\quad+\eta\,\left(3\,\alpha^4+\alpha^2(4-384\,\iota)+6912\,\iota^2-4\right)+6912\,\eta^3\Big]\rho^3,\end{aligned} \tag{4.124}$$

$$\begin{aligned}K(\rho) \simeq \iota\,\alpha^2\,\rho^2 - &\frac{2\,\alpha\,\alpha_2}{\alpha^4+4\,\alpha^2-6912\,(\iota^2+\iota\,\eta+\eta^2)+4}\times\\ &\times\Big[192\,\iota^2\left(\alpha^2+36\,\eta\right)-384\,\alpha^2\,\eta^2\\ &\quad+\iota\,\left(3\,\alpha^4+\alpha^2(4-384\,\eta)+6912\,\eta^2-4\right)+6912\,\iota^3\Big]\rho^3.\end{aligned} \tag{4.125}$$

Note that switching the parameters $\eta \leftrightarrow \iota$ we have that $Z \leftrightarrow K$, as expected.

Plugging the expansions (4.124), (4.72) into the equations (4.50), (4.51), we find that these are solved without imposing any condition on a. This function will then be constrained by the remaining equation (4.52) on which we now focus. The solution process of the latter equation brings us to distinguish between different cases. To solve the first non-trivial order of (4.52) we must satisfy the condition:

$$\begin{aligned}\alpha_2\Big[&13\,\alpha^6+60\,\alpha^4-12\,\alpha^2\left(6912\left(\eta^2+\eta\,\iota+\iota^2\right)-7\right)\\ &-32(36\,\eta+1)(36\,\iota+1)(36\,\eta+36\,\iota-1)\Big]=0\,,\end{aligned} \tag{4.126}$$

which means that either $\alpha_2 = 0$ or the parenthesis vanishes. This condition is analogous to the one in (4.92) that we have found in the previous section. As before, we are interested in solutions with a minimal supergravity limit and therefore we have to choose $\alpha_2 = 0$. At the next non trivial order we then find the condition

$$\begin{aligned}\alpha_4\Big[&5819\,\alpha^6-5244\,\alpha^4+12\,\alpha^2\left(6912\left(\eta^2+\eta\,\iota+\iota^2\right)+65\right)\\ &+32(36\,\eta+1)(36\,\iota+1)(36\,\eta+36\,\iota-1)\Big]=0\,,\end{aligned} \tag{4.127}$$

this equation can be satisfied if $\alpha_4 = 0$ or the parenthesis vanishes. In the case of the previous section, which was originally discussed in [99], setting the corresponding parenthesis to zero led to the black hole solution studied there, while setting α_4 to zero the only regular solution obtained was the one of [42]. In the minimal theory it is possible to obtain the black hole of [101] by setting $\alpha = \sqrt{\frac{8}{11}}$ while the choice $\alpha_4 = 0$ leads either to the regular soliton of [133] or to the black hole of [41]. In [100] it has been shown that there is no possibility of finding a regular soliton solution which presents running scalars, i.e. with at least one of the two functions Z and K which does not vanish. We now briefly report this proof here. First of all, we consider the expression for f (4.53) and we rewrite it as

$$f(\rho) = \frac{12\, a^3\, a'}{\left[(36\, Z' + \mathcal{P})\,(36\, K' + \mathcal{P})\,(-36(Z+K)' + \mathcal{P})\right]^{1/3}}\,, \tag{4.128}$$

in order to describe a soliton, the f function must start with a constant term in a small ρ-expansion, so that the solution closes smoothly. However from (4.128) we can argue that this is possible only if the numerator and the denominator have the same leading behaviour at small ρ. Plugging the expansions (4.122) in (4.128), we can easily check whether this is possible or not; in particular we note that the numerator goes as ρ^3 at small ρ, so we have to impose the same behaviour to the denominator. In the minimal case $Z = K = 0$ this is easily achieved by taking $\alpha = \pm\frac{1}{2}$, since

$$\mathcal{P} = a'''\, a^3 + a\, a' \left(7\, a\, a'' + 4\,(a')^2 - 1\right)\,, \tag{4.129}$$

indeed starts with a ρ^3 term if and only if $\alpha = \frac{1}{2}$. This choice for α is the one taken in [133] and leads the author to find a soliton solution. In the general case we are considering in this paper, Z and K are non vanishing, therefore recalling (4.122) it is evident that the denominator of (4.128) goes always as ρ, while the numerator begins with ρ^3. We then conclude that there is no possibility to find a soliton solution in the $\mathrm{U}(1)^3$ theory with non trivial scalars. We have furthermore verified that, even in our general framework where we do not have imposed any ansatz on the scalar fields, the choice $\alpha_4 = 0$ leads only to the solution of [42] or to a singular solution[61].

The only new solution we find is thus obtained by setting the parenthesis in (4.127) to zero. This can be achieved by imposing that the parameter ι assumes the following value

$$\iota = -\frac{\eta}{2} \pm \frac{1}{144\sqrt{2}\,(2\,\alpha^2 + 36\,\eta + 1)} \Big\{ \left(72\,\eta - 23\,\alpha^2 + 2\right)\left(2\,\alpha^2 + 36\,\eta + 1\right) \left[253\,\alpha^4 + \alpha^2(792\,\eta - 206) + 16(1 - 18\,\eta)^2\right] \Big\}^{1/2}\,. \tag{4.130}$$

Here one has the possibility to choose either the plus or the minus sign; we leave this choice unspecified for now and proceed further. Setting ι as in (4.130), we continue to perturbatively solve the equation (4.52)

61 Indeed, setting $\alpha_4 = 0$ we find a near-horizon expansion which is compatible with a new black hole, but when integrated numerically towards $\rho \to \infty$ this solution presents divergences in the interior region for all the different initial integration conditions we tried.

finding that the solution is uniquely determined in terms of the free parameters α, η, α_3 and α_4. We do not report here the expansions of the functions a, Z and K; they can be found in [100] together with further details on the near-horizon solution. One important remark is that, due to the fact that we have fixed ι as in (4.130), the symmetry between Z and K is now broken since ι is not a free parameter anymore.

It is important to discuss how we can reduce our general solution to the We now briefly show how to reduce our general solution to the $\mathrm{U}(1)^3$ version of the one constructed in [99] and analyzed in sec. 4.2.1.2. As discussed at the end of sec. 4.1.2, we need to impose $Z = K$, which means $\eta = \iota$. The condition (4.130) then becomes an equation for η which gives:

$$\eta_{\text{ limit}} = \frac{1}{288}\bigg(-8 + 11\alpha^2 \pm 9\alpha\sqrt{8 - 11\alpha^2}\bigg) = \frac{\eta_{\text{ there}}}{6} \,, \qquad (4.131)$$

which is consistent with the fact that it must be $Z_{\text{limit}} = \frac{1}{6}\,H$ as already stated above. As consequence, all the expansions reproduce the one of [99], as it has been shown in [100].

As it was for the solution of the general theory with arbitrary n_V, also here we have unphysical parameters and we can use the scaling symmetry discussed at the end of sec. 4.1.2. In particular we note that α and η are left invariant under the action of these symmetries, while α_3 and α_4 can be rescaled. We can therefore argue that only three of the free parameters we found are physical and we choose to consider α_3 as an unphysical parameter. We will explicitly use the possibility to rescale α_3 to numerically match our small-ρ behaviour with the near-boundary one discussed in the previous section, showing that an interpolating solution indeed exists for different values of the remaining physical free parameters. Furthermore, we also introduce the parameter ξ, which is defined as in (4.96); we will trade α_4 with this parameter wherever the former appears. From now on, our set of independent near-horizon free parameters will then be $(\alpha,\, \eta,\, \xi)$.

We report the explicit near-horizon form of the metric, of the gauge feilds and the scalars, so that it will be manifest that they are compatible with a black hole which is a generalization of the $\mathrm{U}(1)^3$ version of the solution presented in [99]. At leading order, in the small ρ expansion, the metric assumes the following form

$$\mathrm{d}s^2 = -\frac{48\,\alpha^6}{\Delta^2\,\Theta}\,\rho^4\,\mathrm{d}t^2 + \Delta\left[\frac{\mathrm{d}\rho^2}{12\,\alpha^2\,\rho^2} + \frac{1}{12}\left(\sigma_1^2 + \sigma_2^2\right) + \Theta\left(\sigma_3 - \frac{2}{v^2}\,\mathrm{d}t\right)^2\right] \,, \qquad (4.132)$$

where we have defined as Δ and Θ the two following quantities

$$\Delta = \left[\left(4\alpha^2 + 72\eta - 1\right)\left(4\alpha^2 + 72\iota - 1\right)\left(4\alpha^2 - 72(\eta+\iota) - 1\right)\right]^{1/3}, \tag{4.133}$$

$$\begin{aligned}\Theta = \frac{1}{48\,\Delta^3}\Big\{&256\alpha^8 - 96\alpha^4\left[1728\left(\eta^2+\eta\,\iota+\iota^2\right)+1\right]\\ &- 32\alpha^2\left[186624\,\eta\,\iota\,(\eta+\iota)-1\right]\\ &- 3\left[1-1728\left(\eta^2+\eta\,\iota+\iota^2\right)\right]^2\Big\}.\end{aligned} \tag{4.134}$$

The scalar fields are given by

$$\begin{aligned}X^1 = {}&\frac{\Delta}{4\alpha^2+72\,\eta-1} + \frac{1}{\Delta^5}\Big\{5184\,\alpha\,\alpha_3\left(4\alpha^2+72\iota-1\right)\\ &\left[4\alpha^2-72\,(\eta+\iota)-1\right]\left[\eta\left(4\alpha^2-24\eta-1\right)+48\,\eta\,\iota+48\iota^2\right]\Big\}\,\rho^2+\mathcal{O}(\rho^4),\end{aligned} \tag{4.135}$$

$$\begin{aligned}X^2 = {}&\frac{\Delta}{4\alpha^2+72\,\iota-1} + \frac{1}{\Delta^5}\Big\{5184\,\alpha\,\alpha_3\left(4\alpha^2+72\eta-1\right)\\ &\left[\iota\left(4\alpha^2-24\iota-1\right)+48\,\eta^2+48\,\eta\,\iota\right]\left(4\alpha^2-72\,[\eta+\iota)-1\right]\Big\}\,\rho^2+\mathcal{O}(\rho^4)\end{aligned} \tag{4.136}$$

with the last scalar field X^3 that can be easily determined using the constraint between them, given by eq (4.68). Finally the gauge fields in the near-horizon result to be

$$\begin{aligned}A^1 = {}&-\frac{2}{v^2}\,A^1_\psi(\rho=0)\,\mathrm{d}t\\ &+\frac{16\alpha^4+8\alpha^2(72\eta-1)+5184\left(\iota^2+\iota\,\eta-\eta^2\right)-144\eta+1}{12\left(4\alpha^2+72\eta-1\right)}\,\sigma^3+\mathcal{O}(\rho^2),\end{aligned} \tag{4.137}$$

$$\begin{aligned}A^2 = {}&-\frac{2}{v^2}\,A^2_\psi(\rho=0)\,\mathrm{d}t\\ &+\frac{16\alpha^4+8\alpha^2(72\iota-1)+5184\left(\eta^2+\eta\,\iota-\iota^2\right)-144\iota+1}{12\left(4\alpha^2+72\iota-1\right)}\,\sigma^3+\mathcal{O}(\rho^2),\end{aligned} \tag{4.138}$$

and, again, the third gauge field A^3 can be easily determined by the other two and will not be presented here. The perturbative solution we have found can indeed be regarded as the near-horizon expansion of a black hole whose horizon is located at $\rho = 0$. Indeed, the metric has a divergent radial component which is $\mathcal{O}(\rho^{-2})$ while its spatial part stays finite as the limit $\rho \to 0$ is approached. Furthermore, the supersymmetric Killing vector V, given by eq. (4.78), is everywhere timelike but on the horizon, where its norm $-f^2$ vanishes[62]. All the

62 In order to explicitly see this, one can consider the explicit form of f in the near-horizon, which is reported in [100].

scalar fields stay regular as the limit $\rho \to 0$ towards the horizon is approached and the same do the gauge fields, which are furthermore transverse to V in the gauge we have chosen[63].

Our next step is to identify for which choice of parameters the solution has a well defined horizon at $\rho = 0$. In order to ensure regularity of the horizon we need that all the spatial diagonal metric components in eq. (4.132) retain their sign for every value of the radial coordinate ρ. This means we must assure that $g_{ii} > 0$ for $i = \rho, \theta, \phi, \psi$[64]. From (4.132) it is easy to see that this translates into imposing the conditions $\Delta > 0$ and $\Theta > 0$. Indeed, this ensures the positivity of g_{ii} for every value of ρ. We are still left with the possible sign choice in eq. (4.130); both choices give a well defined black hole solution and we will analyze the parameter space for both of them, even if in the following we will report the numerical results only for the minus sign choice, which is the choice that leads to the largest space of regular solutions. In fig. 4.7 we report the parameter space in terms of α and ϵ, with the latter defined via

$$\eta = \frac{1}{288}\left(-8 + 11\alpha^2 \pm 9\alpha\sqrt{8 - 11\alpha^2}\right) + \epsilon = \eta_{\,\text{limit}} + \epsilon\,, \quad (4.139)$$

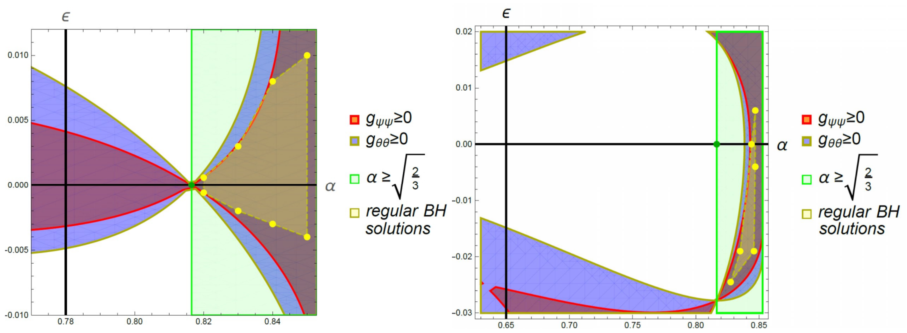
so that the limit to $Z = K$ case of [99] is simply reproduced by the choice $\epsilon = 0$. Note that we can trade η with ϵ using (4.139) only if $\alpha \leq \sqrt{\frac{8}{11}}$, that is the maximum value considered for α in fig. 4.7; we have analyzed the parameter space for $\alpha > \sqrt{\frac{8}{11}}$ and for generic values of η finding that no regular black hole horizon with real coefficients appears in this region[65].

In fig. 4.7 we have reported the parameter space for both the sign choices. The region colored in red is the region where $g_{\psi\psi} > 0$, while the region in blue is where $g_{\theta\theta} > 0$. We have colored in yellow the regions where we managed to find numerically a regular black hole solution with real coefficients, by interpolating the near-horizon solution of the present section with the near-boundary one of the previous section. In particular, the yellow dots are the point characterized by the most extreme values for the parameters $\alpha\,, \epsilon$ for which we found a regular numerical solution. The points in the purple region that are not in the yellow one represent values of $\alpha\,, \epsilon$ for which the horizon is well behaved but a full solutions seems not to exist. This is because we find divergences in the bulk when we try to numerically interpolate

63 Indeed, it results $V^\mu A^I_\mu = 0$ as it is easy to verify using eq. (4.137).

64 Actually to ensure regularity of the horizon we should also require that $g_{yy} \leq 0$, where the equal sign holds only at the horizon $\rho = 0$. However this is already guaranteed by the fact that $g_{yy} = -f^2$ with f being, as already stated, a real function which vanishes at the horizon.

65 Note that the regularity conditions $\Delta > 0$ and $\Theta > 0$ must be combined with the existence condition of the square root in (4.130). We found that these three requests are never simultaneously verified when $\alpha > \sqrt{\frac{8}{11}}$.



(a) Parameter space for − sign. (b) Parameter space for + sign.

Figure 4.7: On the left we show the parameter space with the minus sign choice in eq. (4.130), while on the right we show the parameter space for the plus sign choice. We shaded in yellow the region where regular black hole solutions are found.

the near-horizon region with the near-boundary one. Notice that we have reproduced the results of [99] on the axis $\epsilon = 0$. We have a nice explanation for the peculiar behaviour appearing at the point $(\alpha\,, \epsilon) = (\sqrt{\frac{2}{3}}\,, 0)$, reported in fig. 4.7 with a green dot, where in [99] it emerges a non-analytic behaviour of Δ: this is due to the peculiar structure of the Δ function in the $(\alpha\,, \epsilon)$ plane.

Notice that, as it can be easily seen in fig. 4.7, the region of existence of regular black hole solutions with the plus choice in eq. (4.130) is smaller than the one obtained with the minus choice; this is clearly visible from the form of the “yellow triangle” of solutions in the two cases. This is what is also found in the case $\epsilon = 0$ of [99]. We also stress the fact that both $g_{\theta\theta}$ and $g_{\psi\psi}$ quickly drop to be negative outside the region of the parameter space we have shown in the figure, so no regular horizon can be found there. There is a possible exception only for $\alpha \in (0, \frac{1}{2})$, where instead we have found a region of regular positive $g_{\theta\theta}$ and $g_{\psi\psi}$, but there H and K becomes complex.

As we have done for the solution of the general theory with arbitrary n_V, we conclude this section by observing that we can use the near-horizon solution to obtain the dependence of the first integrals (4.69) on the near-horizon parameters α and η [66]. In order to do this, we have just to plug our near-horizon expansions for the supergravity functions into (4.69) and perform the computations. For conciseness we do not report the expressions such obtained here, but they are given in [100], together with the relations between the near-boundary parameters and the $\mathbb{K}_I$. As we mentioned at the end of sec. 4.2.2, and as we have already done for the previous solution in sec. 4.2.1.1, 4.2.1.2, confronting the near-boundary and the near-horizon expressions for the first integrals,

66 In principle also the parameter ξ could appear in such relations, but it turns out that, since it is quite subleading, it is instead absent.

we are able to write the most subleading near-boundary parameters a_4, Z_4, K_4 and a_6 in terms of the remaining near-boundary ones and the near horizon ones. This allows us to eliminate these four parameters in all the expressions and we will proceed by doing it throughout the paper, as it simplifies many expressions.

4.2.2.3 *The numerical solution*

In this section we construct the full numerical solution by interpolating the near-horizon and the near-boundary expansions, as we have already done for the general solution with arbitrary n_V in sec. 4.2.1.3. The numerical approach we take and the strategy we follow are basically the same and are extensively described in [100], therefore here we present only the essential details and we refer to this paper for further details.

We recall that we found in the near-horizon an unphysical parameter, α_3, which may be rescaled at will; we use this possibility to set the appropriate rescaling such that the AlAdS behaviour of a holds in the near-boundary region. Obviously, in order to integrate the equations, we need to give numerical values to the near-horizon parameters α, ϵ and ξ. We tried many different values for the parameters α and ϵ in the whole possible region of regularity of the solution (which coincides with the region colored in purple in fig. 4.7) finding regularity in the interior only in the points (α, ϵ) inside the yellow region. This means that for every point in the yellow region there is an interval of allowed values of ξ for which all the components of the metric, the scalars and the gauge fields are regular. The allowed interval of ξ depends on α and ϵ and is determined by regularity of the boundary geometry. All the points outside the yellow region lead to solutions which present fields that are not regular in the bulk; in particular for such solutions the function f turns out to have always a divergence at finite ρ. We shall therefore discard such solutions.

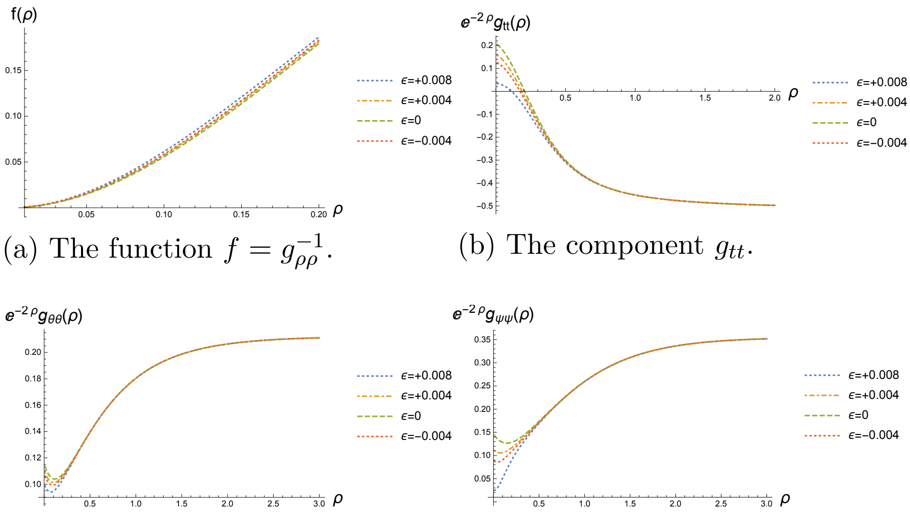
In particular, the region of regularity of the solution corresponding to the minus sign choice in (4.130) is inside

$$\sqrt{\frac{2}{3}} \leq \alpha \leq \sqrt{\frac{8}{11}} \qquad \text{and} \qquad -0.005 \leq \epsilon \leq 0.008\,, \tag{4.140}$$

while a similar, but smaller, range is found for the plus sign solution. From now on we will specialize on the minus sign choice, but all the characteristics of the solutions we will discuss are present also in the ones obtained choosing the plus sign.

We constructed the full interpolating solution for many values of the near-horizon parameters inside the bounds reported in (4.140). As illustrative examples, we discuss in the following two different analyses performed on the solution: the first is made by fixing α and ξ and studying solutions with different ϵ, the second one is made by fixing α and ϵ and studying solutions for various values of ξ. The first analysis gives us the possibility to compare the characteristics of the new solutions we have found with the ones of [99], which are obtained by setting $\epsilon = 0$. These are the solutions that we have presented in sec. 4.2.1.3. Since, as we know from [99, 101, 133], the parameter ξ is related to the squashing at the boundary, the second analysis allows us to study the new solutions (which present $\epsilon \neq 0$) with different squashed boundary geometries.

We begin by presenting the solutions with different ϵ. We choose $\alpha = 0.84$ and $\xi = -\frac{1}{4}$.



(a) The function $f = g_{\rho\rho}^{-1}$.

(b) The component g_{tt}.

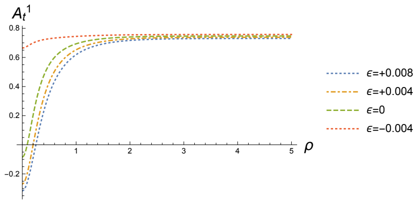
(c) The component $g_{\theta\theta}$.

(d) The component $g_{\psi\psi}$.

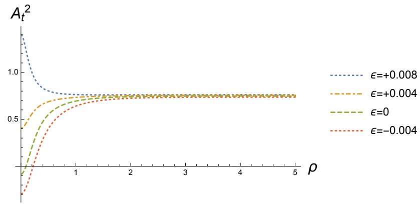
Figure 4.8: Relevant functions and metric components of our solution for $\alpha = 0.84$, $\xi = -\frac{1}{4}$ and different values of ϵ, reported in the label. Each function is rescaled by its asymptotic behaviour at large ρ. We emphasize that both $g_{\theta\theta}$ and $g_{\psi\psi}$ are positive in all the $\rho \geq 0$ region, so our solution does not have any CTCs. Since instead g_{tt} assumes positive values near the horizon, our solution does have an ergoregion.

In fig. 4.8 we show the numerical behaviour of the metric components. It is easy to see that in the near-horizon region their behaviour is in general different for the various choices of ϵ; an exception is the f function for which the differences are very small. We should notice that $g_{\theta\theta}$ and $g_{\psi\psi}$ tend to a positive non-zero value for $\rho = 0$ and are always positive; this means that our solution has no Closed Timelike Curves in the whole region $\rho \geq 0$. We have verified that the same happens for many different values of the parameters in the yellow region of regularity of fig. 4.7. Also, since our solutions are rotating solutions, it is clear from the plot of g_{tt} that an ergoregion emerges.

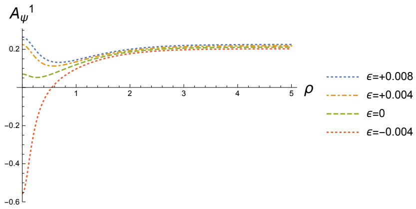
In fig. 4.9 we show the numerical solutions for the scalar and gauge fields. From this picture is quite evident how the change in ϵ affects the global structure of the solution, since in the near-horizon region the fields get attracted to different asymptotic values, while, as for the metric components, they are all attracted to the same large ρ value.



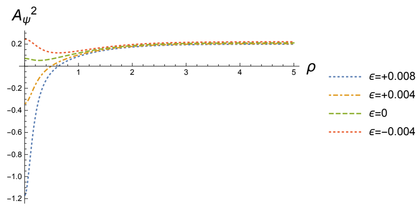
(a) The component of A_t^1.



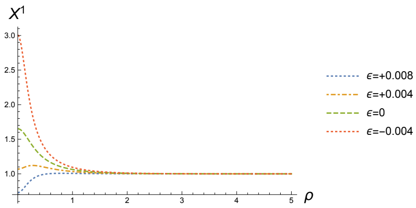
(b) The component of A_t^2.



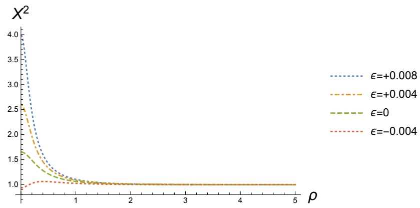
(c) The component of A_ψ^1.



(d) The component A_ψ^2.



(e) Scalar fields X^1.



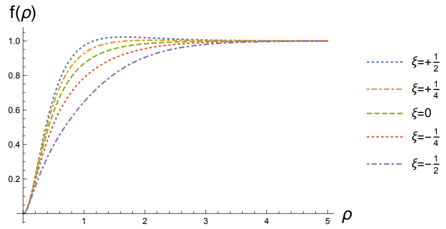
(f) Scalar fields X^2.

Figure 4.9: Components of the gauge fields A^I and scalar fields X^I at $\alpha = 0.84$, $\xi = -\frac{1}{4}$ for various ε. It is evident that in the near-horizon region the differences among the fields for the various ϵ are quite large.

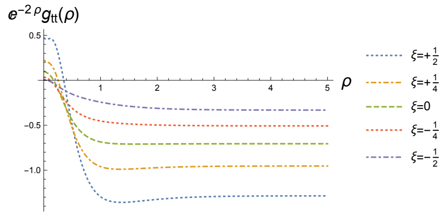
Next, we perform the other study we have announced before: we fix α and ϵ to be certain values and we study the particular solution so obtained for some different values of ξ. We choose the values $\alpha = 0.84$ and $\epsilon = 0.008$.

We reported in fig. 4.10 the metric components of the solutions. As opposed to the fixed ξ case, here the components go for $\rho \to \infty$ to different values. Furthermore, their behaviour is very similar in the near-horizon region. This is because the effect of having a different ξ is almost negligible in the near-horizon, since the horizon geometry is controlled by α and ϵ, while in the near-boundary region the same effect is relevant, being the ξ parameter related to the squashing at the boundary. Again, we have an ergoregion, where g_{tt} becomes positive, and no CTCs, since both $g_{\theta\theta}$ and $g_{\psi\psi}$ are positive everywhere.

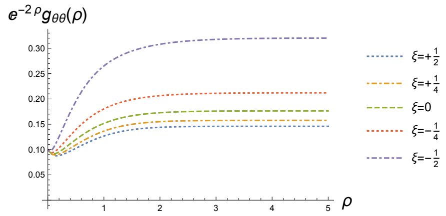
We then show fig. 4.11 where we reported the gauge and scalar fields for various ξ; again, we see that, in contrast with the fixed ξ case, their behaviour in the near-horizon region is similar for all the ξ, while they go to different values in the near-boundary region. The only exception



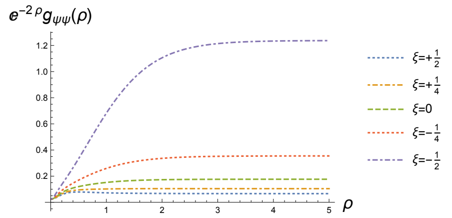
(a) The function $f = g_{\rho\rho}^{-1}$.



(b) The component g_{tt}.



(c) The component $g_{\theta\theta}$.

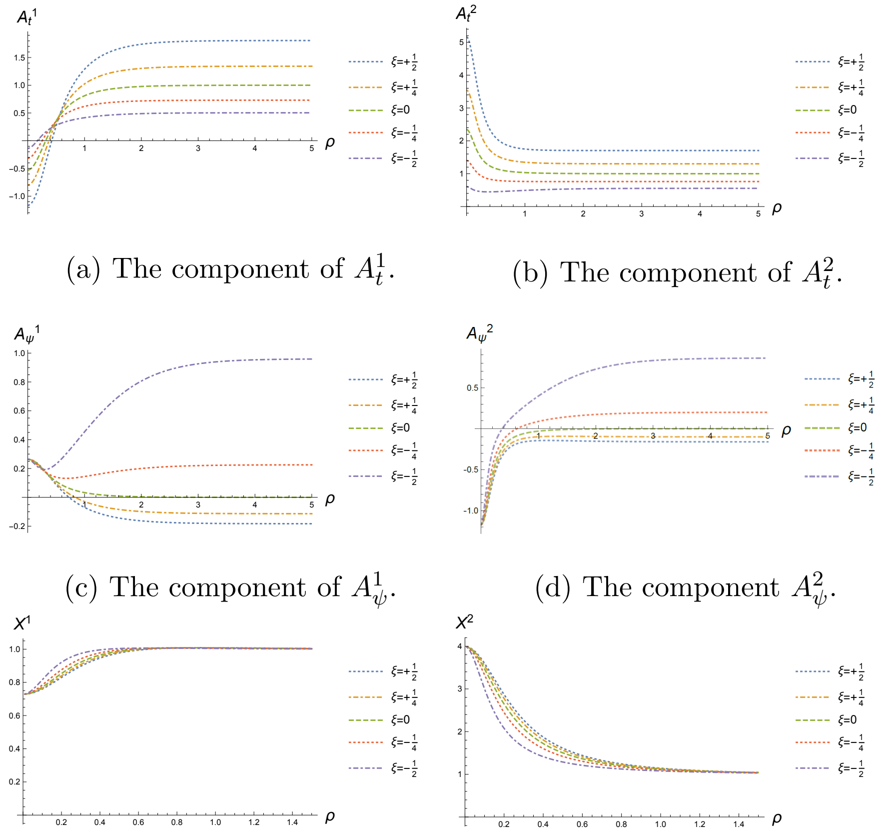


(d) The component $g_{\psi\psi}$.

Figure 4.10: Relevant functions and metric components of our solution for $\alpha = 0.84$, $\varepsilon = +0.008$ and different values of ξ, reported in the label. Each function is rescaled by its asymptotic behaviour at large ρ. We emphasize that both $g_{\theta\theta}$ and $g_{\psi\psi}$ are positive in all the $\rho \geq 0$ region, so our solution does not have any CTCs. Since instead g_{tt} assumes positive values near the horizon, our solution does have an ergoregion.

is the value of A_t^I, that also differs in the near-horizon region. This is due to the fact that in the coordinates (t, ψ) we are using the time component of the gauge fields explicitly depends on the squashing v, as it is clearly visible by (4.137). If we had used instead the coordinates $(y, \hat{\psi})$, the time component A_t^I would vanish at the horizon and would not be influenced by ξ.

We end this section by summarizing the main characteristics of the family of solutions we have constructed. Both the near-horizon analysis and the numerical one prove that our solutions are black hole solutions whose horizon geometry is controlled by two of the three near-horizon parameters, α and η. Therefore this family of solutions presents an horizon geometry which is described by one parameter more compared with the $\mathrm{U}(1)^3$ version of the solution presented in the previous section. The last near-horizon parameter, ξ, is related to the squashing at the boundary, and is therefore related with the parameter v^2 controlling the squashing of the boundary three-sphere. Both the near-boundary analysis and the numerical one show that our solutions are AlAdS, a conformally flat boundary being obtained only when the S^3 is round ($v^2 = 1$). We have numerically shown that the near-boundary and near-horizon behaviours we have found interpolate smoothly in the bulk, giving rise to regular solutions which are free from CTCs.



(a) The component of A_t^1.

(b) The component of A_t^2.

(c) The component of A_ψ^1.

(d) The component A_ψ^2.

(e) Scalar fields X^1.

(f) Scalar fields X^2.

Figure 4.11: Components of the gauge fields A^I and scalar fields X^I at $\alpha = 0.84$, $\varepsilon = +0.008$ for various ξ.

4.3 PHYSICAL PROPERTIES AND HOLOGRAPHIC RENORMALIZATION

We now compute the relevant physical quantities that characterize the family of solutions built in sec. 4.2. These are the energy, the angular momentum, the holographic and Page charges, which can be computed using the near-boundary perturbative solution, the chemical potentials and the entropy, which instead can be derived by means of the near-horizon expansions. Once these quantities are known, we can perform some consistency checks, for example by verifying the quantum statistical relation.

In order to compute some of the above physical properties, we will use the technology of holographic renormalization [30,114,115,160–162]. We perform such computations using the Fefferman-Graham radial coordiante r, introduced in app. A[67], instead of the usual one ρ. This

67 For further details on how to express the solutions we have built in Fefferman-Graham coordinates we refer the reader to [99, 100, 133].

is because the use of the Fefferman-Graham coordinate is standard in holography and may help to compare our results with other references. The general form that the metric assumes in Fefferman-Graham coordinats is

$$\mathrm{d}s^2 = \ell^2 \frac{\mathrm{d}r^2}{r^2} + h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \tag{4.141}$$

where the five-dimensional coordinates split as $x^\mu = (r, x^i)$ with $x^i = \{t, \theta, \phi, \psi\}$ and where h_{ij} is the induced metric at the boundary of the spacetime. The boundary gauge fields A^I_i and the boundary field strengths F^I_{ij} are similarly defined.

In the two following sections we report the results we got using holographic renormalization, for the general solutions with arbitrary n_V and for the one with $n_V = 2$, respectively. We refer to app. A and to [99, 100] for a more detailed discussions about how these results are obtained. We remark that we performed holographic renormalization using a minimal subtraction scheme; all the physical quantities evaluated by means of this formalism refer to this renormalization scheme.

4.3.1 *The solution with arbitrary n_V*

We start by computing via holographic renormalization the stress-energy tensor of the family of solutions under consideration. This is given by the following expression[68]

$$\begin{aligned}\langle T_{ij}\rangle = -\frac{1}{\kappa^2}\lim_{r_0\to\infty}\frac{r_0^2}{\ell^2}\Bigg[&K_{ij} - K\,h_{ij} + \mathcal{W}\,h_{ij} - \frac{\mathcal{W}-3\ell^{-1}}{\log\frac{r_0^2}{\ell^2}}\,h_{ij}\\ &- 2\,\Xi\left(R_{ij} - \frac{1}{2}\,R\,h_{ij}\right) - \frac{\ell^3}{4}\log\frac{r_0^2}{\ell^2}\Bigg(-\frac{1}{2}\,B_{ij}\\ &-\frac{2}{\ell^2}\,Q_{IJ}F^I_{ik}F^J{}_j{}^k + \frac{1}{2\ell^2}\,h_{ij}\,Q_{IJ}F^I_{kl}F^{J\,kl}\Bigg)\Bigg],\end{aligned} \tag{4.142}$$

where the Ricci tensor R_{ij}, the Ricci scalar R and the Bach tensor[69] B_{ij} are those of the induced metric h_{ij}. The other ingredients appearing in the expression (4.142) for the stress-energy tensor are the extrinsic curvature K_{ij} of the induced metric h_{ij}, its trace K, the superpotential $\mathcal{W}$ and the function Ξ[70]. The superpotential can be read from the supersymmetry variation of the gravitino field and satisfies

$$\mathcal{V} = \frac{1}{2}\left(Q^{IJ} - \frac{2}{3}X^I X^J\right)\partial_I\mathcal{W}\,\partial_J\mathcal{W} - \frac{2}{3}\mathcal{W}^2\,, \tag{4.143}$$

68 In the stress-energy tensor formula, as well as in all the formulae below, the quantity r_0 is the cutoff we used to regulate the large-distance divergences which appear. At the end of the computation it is removed by sending it to infinity.
69 See e.g. [133] for more details on the Bach tensor and how it arises here.
70 This must not be confused with the function introduced in (3.37): there is no relation between the two, although we are using the same notation. Confusion will be avoided anyway since we will not consider the function defined in (3.37) anymore.

where $\mathcal{V}$ is the scalar potential. For our Fayet-Iliopoulos gauging with scalar potential (2.17), the superpotential reads:

$$\mathcal{W} = 3\,\ell^{-1}\bar{X}_I X^I \ . \tag{4.144}$$

The function Ξ instead appears in the counterterms necessary to renormalize the bulk action; we will introduce them below and review them in much details in app. A. We find that Ξ is given by the following expression

$$\Xi = \frac{\ell}{4}\,\bar{X}^I X_I \ . \tag{4.145}$$

We find that the energy momentum tensor (4.142) can be expressed in Fefferman-Graham coordinates as:

$$\langle T_{ij}\rangle\,\mathrm{d}x^i\,\mathrm{d}x^j = \langle T_{tt}\rangle\,\mathrm{d}t^2 + \langle T_{\theta\theta}\rangle\,\left(\sigma_1^2+\sigma_2^2\right) + \langle T_{\psi\psi}\rangle\,\sigma_3^2 + 2\langle T_{t\psi}\rangle\,\mathrm{d}t\,\sigma_3\ , \tag{4.146}$$

where the components read:

$$\begin{aligned}
\langle T_{tt}\rangle &= \frac{1}{\kappa^2 a_0^2\,v^4\ell}\bigg(\Big(\tfrac{1}{9}-\tilde{H}^2-2\,\mathcal{K}_1\Big)v^2-\tfrac{7}{36}v^4+\tfrac{89}{864}v^6 \\
&\qquad + 2\tilde{H}\Big(2\tilde{H}^2-\tilde{H}+6\,\mathcal{K}_2\Big)+\tfrac{1}{27}\Big(2-108\mathcal{K}_1+27\mathcal{K}_3\Big)\bigg)\ , \\
\langle T_{\theta\theta}\rangle &= \frac{\ell}{384\,\kappa^2\,a_0^2}\bigg(16(16a_2-5)v^2+67\,v^4 \\
&\qquad + 288\,\tilde{H}\,(4\,H_2+\tilde{H})+32-576\,\mathcal{K}_1\bigg)\ , \\
\langle T_{\psi\psi}\rangle &= \frac{\ell}{3456\kappa^2 a_0^2}\bigg(4320\,v^2\tilde{H}^2-480\,(1-18\,\mathcal{K}_1)\,v^2 \\
&\qquad - 24\,(192\,a_2-53)\,v^4-1117v^6 \\
&\qquad + 1728\,\tilde{H}\,(2\tilde{H}^2-\tilde{H}+6\,\mathcal{K}_2) \\
&\qquad + 32\,(2-108\,\mathcal{K}_1+27\,\mathcal{K}_3)\bigg)\ , \\
\langle T_{t\psi}\rangle &= \frac{1}{\kappa^2 a_0^2\,v^2}\bigg(\tfrac{1}{27}(v^2-1)^3-(v^2-1)\tilde{H}^2-2\tilde{H}^3 \\
&\qquad 2\mathcal{K}_1(v^2-1)-6\tilde{H}\mathcal{K}_2-\tfrac{1}{2}\mathcal{K}_3\bigg)\ .
\end{aligned} \tag{4.147}$$

Note that in the expressions above the first integrals $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ do appear; this is because we have traded a_4, a_6, H_4 for them so as to considerably simplify the expressions. The trace of the stress-energy tensor is

$$\langle T_i{}^i\rangle = \frac{3}{\kappa^2 a_0^4}\tilde{H}\,(2H_2+\tilde{H})\ \ . \tag{4.148}$$

Next, we evaluate the conserved electric current. This arises from holographic renormalization and it is given by [99, 100]

$$\langle j^i_I \rangle = -\frac{1}{\kappa^2} \lim_{r_0 \to \infty} \frac{r_0^4}{\ell^4} \Bigg[\tfrac{1}{6} \epsilon^{ijkl} \left(Q_{IJ} \star F^J + \tfrac{1}{6} C_{IJK} A^J \wedge F^K \right)_{jkl} \\ + \ell \, \nabla_j \Big(Q_{IJ} F^{J\, ji} \Big) \log \frac{r_0}{\ell} \Bigg] . \tag{4.149}$$

Its non-vanishing components read

$$\langle j^t_I \rangle = \frac{-1}{36 \kappa^2 \ell^2 a_0^4} \Bigg[\left(54 \, \mathcal{K}_1 - \Big(v^2 - 1\Big)^2 + 9 \tilde{H}^2 \right) \bar{X}_I \\ + 6 \left(9 \mathcal{K}_2 + \Big(v^2 - 1\Big) \tilde{H} + 3 \tilde{H}^2 \right) q_I \Bigg],$$

$$\langle j^\psi_I \rangle = \frac{1}{72 \kappa^2 \ell^2 a_0^4 v^2} \Bigg\{ \Bigg[4 \Big(36 a_2 - 5\Big) v^2 - 36 \, \tilde{H}^2 \\ - 216 \, \mathcal{K}_1 + 25 \, v^4 + 4 \Bigg] \bar{X}_I \\ - 12 \Bigg[18 \left(H_2 \, v^2 + \mathcal{K}_2 \right) \\ + \tilde{H} \Big(6 \, \tilde{H} + 5 \, v^2 - 2 \Big) \Bigg] q_I \Bigg\} . \tag{4.150}$$

In the limit $\tilde{H} = H_2 = \mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}_3 = 0$, (4.147) and (4.150) are consistent with the energy-momentum tensor and current for minimal gauged supergravity solutions presented in [133].

Once both the stress-energy tensor and the electric current are evaluated, we are in the position to compute the energy and the angular momentum, which are visible as the charges associated to the two Killing vectors of the metric $\frac{\partial}{\partial t}$ and $-\frac{\partial}{\partial \psi}$

$$E = Q_{\frac{\partial}{\partial t}} = + \int_{\Sigma_\infty} \mathrm{vol}_\Sigma \, u_i \left(\langle T^i{}_t \rangle + A^I_t \langle j^i_I \rangle \right) , \tag{4.151}$$

$$J = Q_{-\frac{\partial}{\partial \psi}} = - \int_{\Sigma_\infty} \mathrm{vol}_\Sigma \, u_i \left(\langle T^i{}_\psi \rangle + A^I_\psi \langle j^i_I \rangle \right) , \tag{4.152}$$

where $u^i \, \partial_i = \frac{v}{2 \, a_0} \, \partial_t$ is a unit timelike vector for the metric on the conformal boundary. By using our expressions for the energy-momentum tensor and the electric currents, we find:

$$E = \frac{\pi^2 \ell^2}{\kappa^2} \left(\frac{16}{9} - \frac{14}{9} v^2 + \frac{19}{36} v^4 - 16 \, \tilde{H}^2 + \frac{8}{v^2} \mathcal{K}_3 \right) \tag{4.153}$$

$$J = \frac{4 \pi^2 \ell^3}{\kappa^2} \mathcal{K}_3 \tag{4.154}$$

where for the last equality we used (4.46). This shows that the holographic angular momentum coincides with the generalized Komar integral (4.38). Since the holographic electric currents $\langle j^i_I\rangle$ are conserved, we can introduce holographic electric charges Q_I as:

$$Q_I = \int_{\Sigma_\infty} \mathrm{vol}_\Sigma\, u_i \langle j^i_I\rangle \ . \tag{4.155}$$

Using (4.149) it is not hard to show that this is the same as:[71]

$$Q_I = -\frac{1}{\kappa^2}\int_{\Sigma_\infty}\left(Q_{IJ}\star F^J + \frac{1}{6}C_{IJK}\,A^J\wedge F^K\right) . \tag{4.156}$$

It is fundamental for our discussion to remark that these differ on our solutions from the Page charges

$$P_I = \frac{1}{\kappa^2}\int_{\Sigma_\infty}\left(Q_{IJ}\star F^J + \frac{1}{4}\,C_{IJK}A^J\wedge F^K\right). \tag{4.157}$$

In fact, on our solutions, we have

$$Q_I = \frac{16\pi^2\ell^2}{\kappa^2}\Bigg[\left(3\,\mathcal{K}_1 - \tfrac{1}{18}\left(v^2-1\right)^2 + \tfrac{1}{2}\tilde{H}^2\right)\bar{X}_I + \left(3\,\mathcal{K}_2 + \tfrac{1}{3}(v^2-1)\,\tilde{H} + \tilde{H}^2\right) q_I\Bigg] , \tag{4.158}$$

while

$$P_I = -\frac{48\pi^2\ell^2}{\kappa^2}\left(\mathcal{K}_1\bar{X}_I + \mathcal{K}_2\,q_I\right) \ , \tag{4.159}$$

so that the following relation does hold

$$Q_I = -P_I + \frac{1}{12\kappa^2}\int_{\Sigma_\infty} C_{IJK}A^J\wedge F^K \ . \tag{4.160}$$

We want to stress that this fact is a consequence of the squashing of the boundary: it is trivial to see from eqs. (4.156), (4.157) that the difference between holographic and Page charges is related to the Chern-Simons term which gives a different contribution to the two quantities. In usual non-squashed solutions like [41, 42] the same term gives no contribution, since the field strength F^I vanishes asymptotically, and the two different types of charges are therefore equal. This implies that we may have some relevant departure from the equation of [132] that relates the entropy and the charges, since we have no unique way to choose which charge is the correct one for reproducing the entropy; in fact, it will turn out later that the entropy is indeed reproduced in terms of the Page charges, instead of the holographic charges obtained in (4.184).

71 The overall minus sign can be traced back to the fact that our choice of orientation for the bulk and the boundary is such that $\mathrm{vol}(M) = -\frac{\mathrm{d}r}{r}\wedge \mathrm{vol}(\partial M)$.

These results for the electric charges Q_I, the energy E and the angular momentum J hold for any AlAdS solution to Fayet-Iliopoulos gauged supergravity satisfying the supersymmetry equations (4.30), (4.31). The expressions depend only on the squashing at the boundary v, on the scalar source $\tilde{H}$ and on the constants $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$. As explained in sec. 4.2.1.2, the latter can be fixed by studying how the solution caps off in the interior.

Our next focus is on the near-horizon properties of the solution. The entropy can be computed by looking at the horizon metric (4.106):

$$S = \frac{2\pi}{\kappa^2}\text{Area} = \frac{8\pi^3\ell^3}{\kappa^2}\sqrt{48\,\mathcal{K}_1^2 - 12\,\mathcal{K}_2^2 - \mathcal{K}_3}$$
$$= 2\pi\ell\sqrt{\frac{3}{2}C^{IJK}\bar{X}_I P_J P_K - \frac{4\pi^2\ell}{\kappa^2}J}\ . \tag{4.161}$$

This is the anticipated result: the entropy of the black hole solutions with squashed boundary can be reproduced by a simple formula containing only the conserved charges; however the formula does *not* involve the usual holographic electric charges but rather the Page charges, signaling a relevance for the entropy counting of the Chern-Simons term that was previously unnoticed by the non-squashed solutions of [41,42]. Therefore, the entropy formula found for example in [132] for asymptotically AdS_5 black holes retains its validity for these $AlAdS_5$ solutions if we identify the charges appearing therein with the Page ones instead of the holographic ones. This result has been obtained for the first time in [99] and, as we shall see in the following section, it does hold also for the more general solutions of [100]. This result is somewhat anticipated by the explicit form the different types of charges take: indeed the Q_I depend on both the squashing v and the scalar source $\tilde{H}$ on which the horizon geometry is independent and therefore appear to be inadequate to describe an horizon quantity like the entropy. The Page charges P_I, instead, depend only on the first integrals which are immediately related to the horizon geometry only and seem thus the right charges to describe the entropy.

From the supersymmetric Killing vector (4.78), it is easy to evaluate the angular velocity Ω of our family of solutions. This results (4.78) and is:

$$\Omega = \frac{2}{\ell v^2}\ , \tag{4.162}$$

while the electrostatic potential is given by:[72]

$$\Phi^I \equiv V^\mu A^I_\mu\,|_{\text{hor}} = 0\ , \tag{4.163}$$

72 Here we are using the definitions of [117], where the electric potential is measured just at the horizon, $\Phi^I = V^\mu A^I_\mu\,|_{\text{hor}}$ and E, J are those introduced in (4.151), (4.152). In another possible definition, the electric potential also receives a contribution from the gauge field at infinity, $\Phi^I = V^\mu A^I_\mu\,|_{\text{hor}} - V^\mu A^I_\mu\,|_\infty$, while E and J are computed just from the energy-momentum tensor (if conserved), without the term involving the gauge field. In any case the combination $E - \Omega J - \Phi^I Q_I$ remains the same.

Now we want to use all the quantities we have evaluated in this section to verify the extremal limit of the quantum statistical relation which we have already introduced in Chapter 2 and discussed in Chapter 3. We recall that the quantum statistical relation is given by

$$\frac{I}{\beta} = E - T\,S - \Omega\,J - \Phi^I\,Q_I\,, \tag{4.164}$$

where, as we have already explained in the previous chapters, I is the Euclidean on-shell action, T is the temperature of the black hole and $\beta = \frac{1}{T}$. We can take the extremal limit of this relation by recalling that for extremal black holes we have $T = 0$. Sending the temperature to zero, we obtain at the leading order the relation:

$$\frac{I}{\beta} = E - \Omega\,J - \Phi^I\,Q_I\,, \tag{4.165}$$

which is valid for extremal black holes and should therefore be satisfied by our family of solutions.

To verify eq. (4.165) we are missing the Euclidean on-shell action. This can be computed using again holographic renormalization. Within this framework, renormalized Lorentzian on-shell action can be defined as

$$\mathcal{S}_{\text{ren}} = \lim_{r_0 \to \infty} \mathcal{S}_{\text{reg}}\,, \tag{4.166}$$

where the regularized action is

$$\mathcal{S}_{\text{reg}} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{GH}} + \mathcal{S}_{\text{ct}}\,. \tag{4.167}$$

There are various terms in the equation above: $\mathcal{S}_{\text{bulk}}$ is the bulk action (2.2), while $\mathcal{S}_{\text{GH}}$ and $\mathcal{S}_{\text{ct}}$ are the Gibbons-Hawking term and the counterterms piece respectively. We show them explicitly in app. A and we refer the reader to [99, 100, 133] for further details. In app. A we also show how the whole computation of the renormalized action is performed; here we report only the final result, which is

$$\mathcal{S}_{\text{ren}} = -\frac{\pi^2\ell^2}{\kappa^2}\left[\frac{16}{9} - \frac{14}{9}\,v^2 + \frac{19}{36}\,v^4 - 16\,\tilde{H}^2\right]\int \mathrm{d}t\,. \tag{4.168}$$

This depends only on the squashing at the boundary v^2 and on the scalar source term $\tilde{H}$. Moreover, we should remark that the regularized action is gauge-dependent due to the Chern-Simons term in the bulk action (2.2); the result we reported above is valid when the gauge condition $V^\mu\,A^I_\mu = 0$ is imposed at the horizon. This particular gauge is justified since it ensures regularity of the solution by avoiding divergences in the square norm of the gauge fields. It should be noted that when taking the minimal limit $\tilde{H} \to 0$, this gauge choice leads to an expression for the on-shell action that is different from the one given in [133, eq. (4.13)]. Indeed in [133] a different gauge choice was made,[73] ensuring that the Killing spinor is instead preserved by the vector $\frac{\partial}{\partial t}$.

The Euclidean continuation of the Lorentzian regularized action can now be obtained by performing a Wick rotation on the time and by making the latter periodic of period β, so that we have

$$\frac{I}{\beta} = -\frac{\mathcal{S}_{\text{ren}}}{\int \mathrm{d}t}\,. \tag{4.169}$$

Thus, we have obtained I, i.e. the last quantity involved in the quantum statistical relation. Plugging all these quantities into eq. (4.165), we see that it is indeed verified. Note also that by recalling that $\Phi^I = 0$ for our family of solutions and by defining the holographic charge associated to the Killing vector (4.78) as:

$$Q_V = E - \frac{2}{\ell\, v^2}\, J\,, \tag{4.170}$$

the quantum statistical relation assumes the form:

$$\frac{I}{\beta} = Q_V\,, \tag{4.171}$$

which can also be seen as the BPS relation between the holographic charges, the anomalous contribution of [122,123] being already included.

Now we finally computed the last quantity involved in the quantum statistical relation. Plugging all the ingredients into eq. (4.165), we see that it is indeed verified. Note also that by recalling that $\Phi^I = 0$ for our family of solutions and by defining the holographic charge associated to the Killing vector (4.78) as:

$$Q_V = E - \frac{2}{\ell\, v^2}\, J\,, \tag{4.172}$$

the quantum statistical relation assumes the form:

$$\frac{I}{\beta} = Q_V\,, \tag{4.173}$$

which can also be seen as the BPS relation between the holographic charges, the anomalous contribution of [122,123] being already included.

4.3.2 *The solution with $n_V = 2$*

Here we compute and collect the main physical properties of the solutions with $n_V = 2$ constructed in 4.2.2.3. The approach and the logic we follow is the same as the previous case, as well as most of the formulae we use to compute the various quantities; for these reasons the presentation will be shorter.

73 From (4.83) we see that the present gauge satisfies $\lim_{r\to\infty} V^\mu A^I_\mu \bar{X}_I = 1$, while the gauge chosen in [133] corresponds to $\lim_{r\to\infty} V^\mu A^I_\mu \bar{X}_I = 1 - \frac{2}{3v^2}$.

The stress-energy tensor for this family of solutions is still given by (4.142) and, when evaluated in Fefferman-Graham coordinates, it assumes the form (4.146). The explicit components are given by We find that the stress-energy tensor can be written as

$$\langle T_{ij}\rangle \, \mathrm{d}x^i \, \mathrm{d}x^j = \langle T_{tt}\rangle \, \mathrm{d}t^2 + \langle T_{\theta\theta}\rangle \left(\sigma_1^2 + \sigma_2^2\right) + \langle T_{\psi\psi}\rangle \, \sigma_3^2 + 2 \, \langle T_{t\psi}\rangle \, \mathrm{d}t \, \sigma_3 \,, \tag{4.174}$$

where the components explicitly read

$$\begin{aligned}\langle T_{tt}\rangle = \frac{1}{\kappa^2 a_0^2 v^4 \ell}\Bigg[&\frac{2}{27} + \frac{v^2}{9} - \frac{7v^4}{36} + \frac{89v^6}{864} - 4\mathbb{K}_1 + \mathbb{K}_3 \\ &+ 24\Big(\tilde{Z}^2(18\tilde{K}-1) + \tilde{Z}(3\mathbb{K}_2^{(2)} + 6\mathbb{K}_2^{(1)} + \tilde{K}(18\tilde{K}-1)) + \\ &\tilde{K}(6\mathbb{K}_2^{(2)} + 3\mathbb{K}_2^{(1)} - \tilde{K})\Big) - 2v^2\left(6\left(\tilde{Z}^2 + \tilde{Z}\tilde{K} + \tilde{K}^2\right) + \mathbb{K}_1\right)\Bigg],\end{aligned} \tag{4.175}$$

$$\begin{aligned}\langle T_{t\psi}\rangle = \frac{1}{\kappa^2 a_0^2 v^2}\Bigg[&\frac{1}{54}\left(-108\mathbb{K}_1\left(v^2-1\right) - 27\mathbb{K}_3 + 2\left(v^2-1\right)^3\right) \\ &- 12\Bigg(\tilde{Z}^2\left(18\tilde{K} + v^2 - 1\right) + \tilde{Z}\left(3\mathbb{K}_2^{(2)} + 6\mathbb{K}_2^{(1)} + \tilde{K}\left(18\tilde{K} + v^2 - 1\right)\right) \\ &+ \tilde{K}\left(6\mathbb{K}_2^{(2)} + 3\mathbb{K}_2^{(1)} + \tilde{K}\left(v^2-1\right)\right)\Bigg)\Bigg],\end{aligned} \tag{4.176}$$

$$\begin{aligned}\langle T_{\psi\psi}\rangle = \frac{\ell}{3456\kappa^2 a_0^2}\Bigg[&24(53-192a_2)v^4 + 1728\mathbb{K}_1\left(5v^2-2\right) \\ &+ 864\mathbb{K}_3 - 1117v^6 - 480v^2 + 64 + 10368\Big(5v^2\left(\tilde{Z}^2 + \tilde{Z}\tilde{K} + \tilde{K}^2\right) + \tilde{Z}^2(36\tilde{K}-2) \\ &+ 2\tilde{Z}(3\mathbb{K}_2^{(2)} + 6\mathbb{K}_2^{(1)} + \tilde{K}(18\tilde{K}-1)) + 2\tilde{K}(6\mathbb{K}_2^{(2)} + 3\mathbb{K}_2^{(1)} - \tilde{K})\Big)\Bigg],\end{aligned} \tag{4.177}$$

$$\begin{aligned}\langle T_{\theta\theta}\rangle = \frac{\ell}{384\kappa^2 a_0^2}\Bigg[&16(16a_2-5)v^2 - 576\mathbb{K}_1 + 67v^4 + 32 \\ &+ 3456\Big(\tilde{K}(2Z_2 + \tilde{Z} + 4K_2) + \tilde{Z}(4Z_2 + \tilde{Z} + 2K_2) + \tilde{K}^2\Big)\Bigg],\end{aligned} \tag{4.178}$$

and the trace of the stress-energy tensor is

$$\langle T^i{}_i\rangle = \frac{3}{\kappa^2 a_0^4} 12\left(\tilde{K}(Z_2 + \tilde{Z} + 2K_2) + \tilde{Z}(2Z_2 + \tilde{Z} + K_2) + \tilde{K}^2\right).$$

(4.179)

The three electric conserved currents can be determined again using (4.149). Their non-vanishing components are given by

$$\begin{aligned}\langle j_1^t\rangle &= -\frac{1}{36\kappa^2\ell^2 a_0^4}\Bigg[6\tilde{Z}\left(18\tilde{K}+v^2-1\right)\\&\qquad+\frac{1}{3}\left(54\mathbb{K}_1+162\mathbb{K}_2^{(1)}+324\tilde{K}^2-\left(v^2-1\right)^2\right)\Bigg],\\\langle j_1^\psi\rangle &= +\frac{1}{54\kappa^2\ell^2 a_0^4 v^2}\Bigg[v^2(36a_2-162Z_2-5)-9\tilde{Z}\left(36\tilde{K}+5v^2-2\right)\\&\qquad-54\mathbb{K}_1-162\mathbb{K}_2^{(1)}-324\tilde{K}^2+\frac{25v^4}{4}+1\Bigg],\end{aligned} \tag{4.180}$$

$$\begin{aligned}\langle j_2^t\rangle &= -\frac{1}{36\kappa^2\ell^2 a_0^4}\Bigg[6\tilde{K}\left(18\tilde{Z}+v^2-1\right)\\&\qquad+\frac{1}{3}\left(54\mathbb{K}_1+162\mathbb{K}_2^{(2)}+324\tilde{Z}^2-\left(v^2-1\right)^2\right)\Bigg],\\\langle j_2^\psi\rangle &= +\frac{1}{54\kappa^2\ell^2 a_0^4 v^2}\Bigg[v^2(36a_2-162K_2-5)-9\tilde{K}\left(36\tilde{Z}+5v^2-2\right)\\&\qquad-54\mathbb{K}_1-162\mathbb{K}_2^{(2)}-324\tilde{Z}^2+\frac{25v^4}{4}+1\Bigg],\end{aligned} \tag{4.181}$$

$$\begin{aligned}\langle j_3^t\rangle &= -\langle j_1^t\rangle-\langle j_2^t\rangle-\frac{1}{36\kappa^2\ell^2 a_0^4}\left[54\mathbb{K}_1+108(\tilde{Z}^2+\tilde{Z}\tilde{K}+\tilde{K}^2)-\left(v^2-1\right)^2\right],\\\langle j_3^\psi\rangle &= -\langle j_1^\psi\rangle-\langle j_2^\psi\rangle\\&\quad+\frac{1}{54\kappa^2\ell^2 a_0^4 v^2}\Bigg[3-162\mathbb{K}_1-324(\tilde{Z}^2+\tilde{Z}\tilde{K}+\tilde{K}^2)\\&\qquad+3(36a_2-5)v^2+\frac{75v^4}{4}\Bigg].\end{aligned} \tag{4.182}$$

We notice that $\langle j_1^t\rangle \leftrightarrow \langle j_2^t\rangle$ if $\tilde{Z}\leftrightarrow\tilde{K}$ and $\mathbb{K}_2^{(1)}\leftrightarrow\mathbb{K}_2^{(2)}$.

Having at disposal both the stress-energy tensor and the conserved electric currents, we can proceed to evaluate the conserved charges, using the formulae given in (4.151), (4.152), (4.156). For the energy and the angular momentum, we obtain:

$$\begin{aligned} E &= \frac{\pi^2\ell^2}{\kappa^2}\left[\frac{16}{9} - \frac{14}{9}\,v^2 + \frac{19}{36}\,v^4 + \frac{8}{v^2}\,\mathbb{K}_3 - 192(\tilde{Z}^2 + \tilde{Z}\tilde{K} + \tilde{K}^2)\right]\,, \\ J &= \frac{4\pi^2\ell^3}{\kappa^2}\,\mathbb{K}_3\,. \end{aligned} \tag{4.183}$$

The holographic electric charges are instead given by:

$$\begin{aligned} Q_1 &= -P_1 - \frac{16\pi^2\ell^2}{\kappa^2}\frac{1}{54}\left[\left(1 - v^2 - 18\tilde{K}\right)\left(1 - v^2 + 18(\tilde{Z} + \tilde{K})\right)\right]\,, \\ Q_2 &= -P_2 - \frac{16\pi^2\ell^2}{\kappa^2}\frac{1}{54}\left[\left(1 - v^2 - 18\tilde{Z}\right)\left(1 - v^2 + 18(\tilde{Z} + \tilde{K})\right)\right]\,, \\ Q_3 &= -P_3 - \frac{16\pi^2\ell^2}{\kappa^2}\frac{1}{54}\left[\left(1 - v^2 - 18\tilde{Z}\right)\left(1 - v^2 - 18\tilde{K}\right)\right]\,, \end{aligned} \tag{4.184}$$

while for the Page charges we find

$$\begin{aligned} P_1 &= -\frac{16\pi^2\ell^2}{\kappa^2}(\mathbb{K}_1 + 3\mathbb{K}_2^{(1)})\,, \\ P_2 &= -\frac{16\pi^2\ell^2}{\kappa^2}(\mathbb{K}_1 + 3\mathbb{K}_2^{(2)})\,, \\ P_3 &= -P_1 - P_2 - \frac{48\pi^2\ell^2}{\kappa^2}\mathbb{K}_1\,. \end{aligned} \tag{4.185}$$

The same considerations stated for the solutions with arbitrary n_V of the previous section about the differences between the Page and the holographic charges apply also here in a completely analogous way.

Similarly, the entropy can still be computed by the general formula containing the Page charges given in eq. (4.161). In particular, we have that

$$\begin{aligned} S &= \frac{2\pi}{\kappa^2}\,\text{Area} = \frac{8\pi^3\ell^3}{\kappa^2}\sqrt{48\mathbb{K}_1 - 144\left[\left(\mathbb{K}_2^{(1)}\right)^2 + \mathbb{K}_2^{(1)}\mathbb{K}_2^{(2)} + \left(\mathbb{K}_2^{(2)}\right)^2\right] - \mathbb{K}_3} \\ &= 2\pi\ell\sqrt{\frac{3}{2}\,C^{IJK}\bar{X}_I P_J P_K - \frac{4\pi^2\ell}{\kappa^2}\,J}\,. \end{aligned} \tag{4.186}$$

and comments completely analogous to the ones already reported for the solutions of the previous section can be made about the appearence of the Page charges in the formula, instead of the holographic ones.

The angular velocity Ω of this family of solutions is the still given by eq. (4.162) and the same holds for the electrostatic potentials Φ^I which are still provided by eq. (4.163).

Finally we would like to verify the extremal limit of the quantum statistical relation we have given in eq. (4.165). In order to do this, we need to evaluate the Euclidean on-shell action I. This can be evaluated using holographic renormalization. Although we do not report the explicit computation in this thesis, this can be found in [100] and it is very similar to the one performed for the solution with arbitrary n_V which we report in app. A. Instead, we report the final result for I, which is

$$S_{\text{ren}} = -\frac{\pi^2 \ell^2}{\kappa^2}\left[\frac{16}{9} - \frac{14}{9}\,v^2 + \frac{19}{36}\,v^4 - 192(\tilde{Z}^2 + \tilde{Z}\tilde{K} + \tilde{K}^2)\right]\int \mathrm{d}t\,. \tag{4.187}$$

This depends on the squashing at the boundary v and on the scalar sources $\tilde{Z}$, $\tilde{K}$, as expected. It is now straightforward to verify that the so obtained on-shell action satisfies the quantum statistical relation (4.165); furthermore the same considerations we have made at the end of the previous section continue to hold for this solution.

4.4 RECAP AND DISCUSSION

In this chapter, we have presented two new families of supersymmetric AlAdS_5 black holes with a boundary geometry containing a squashed S^3.

The first family of solutions presents an arbitrary number of vector multiplets n_V and it is controlled by two parameters: one controlling the event horizon geometry as well as the angular momentum and the Page electric charges, while the other which can be identified with the squashing at the boundary. Suppose we fix the former. Then although the squashing at the boundary is arbitrary, the S^3 metric flows to a fixed one at the horizon. This is reminiscent of the attractor mechanism for scalar fields in four dimensions. This connection can be made rigorous by reducing along the Hopf fiber of S^3, as in the dimensional reduction the component of the metric controlling the size of the Hopf fiber becomes one of the scalar fields involved in the attractor mechanism[74].

The second family contains solutions of the $\mathrm{U}(1)^3$ theory, i.e. of the theory obtained by setting $n_V = 2$. These black holes generalize the solutions previously found in minimal gauged supergravity [101, 157] and also the first family of solutions we have discussed, originally constructed in [99], for $n_V = 2$, since we have not imposed any ansatz on the scalar fields. Furthermore, they can be uplifted to be solutions of ten-dimensional type IIB supergravity. These solutions depend on three-parameters of which two regulate the horizon geometry, the angular momentum and the Page charges while the remaining one determines the squashing at the boundary. Therefore, we can see that

74 See [65] for a related discussion in the case with no squashing.

by removing the ansatz which it has been imposed for the first family, we have obtained solutions with one free parameter more. As for the previous family, also in this case the horizon properties are totally independent of the squashing; therefore if we set a particular horizon geometry by fixing the two former parameters, whatever the squashing the S^3 metric will flow to a fixed one at the horizon.

Let us compare the number of parameters describing the horizon geometry we obtained with the number one should expect by a theoretical counting. For the $n_V = 2$ model, we have five conserved charges, which are the energy, one angular momentum and three electric charges; however only four of these five total charges are independent since supersymmetry imposes one linear constraint among them. One can therefore expect to find black hole solutions with four parameters regulating the horizon geometry, but already in the solution of [42] one of these is constrained by the requirements to be imposed to avoid causal pathologies [110], so the independent parameters are three. This is totally in agreement with what we have seen and discussed in Chapter 2, 3. We should therefore expect to be possible to find squashing solutions with three independent parameters regulating the horizon geometry, in addition to the one determining the squashing at the boundary. However, our family of solutions for the $\mathrm{U}(1)^3$ case presents an horizon geometry described by two parameters, generalizing the solutions constructed in [99] characterized by only one parameter regulating the horizon geometry due to the ansatz for the scalar fields adopted there. These black holes are thus the most general squashed solutions found in the $\mathrm{U}(1)^3$ theory we are studying; however we are still missing squashed solutions with an horizon geometry regulated by three parameters, which should be the most possible general ones according to the theoretical counting arguments reported above. It could be that the general three parameter solution breaks the $\mathrm{SU}(2) \times \mathrm{U}(1)^4$ symmetry in the bulk and should thus be searched in a more general setup than Fayet-Iliopoulos gauged supergravity.

Another interesting avenue for future research will be to extend the study of supersymmetric AlAdS black holes with a deformed boundary done in this thesis to other dimensions, the seven-dimensional case being perhaps the most promising.

We have seen in sec. 4.3 that the entropy of both our families of black holes is reproduced using the Page charges instead of the holographic charges. The two types of charges are different for AlAdS$_5$ solutions due to the presence of the Chern-Simons term, which does not vanish asymptotically like in the case of non-squashed solutions. The formula thus obtained for the entropy in terms of the conserved charges is in agreement with the typical one for asymptotically AdS$_5$ black holes reported for example in [132], provided the fact that for AlAdS$_5$ solutions the charges appearing there must be identified with the Page charges and not with the holographic ones. In Chapter 1 we have introduced the extremization principle proposed in [65] for AdS_5 black holes, while in Chapter 3 we have provided, following [67], a physical interpretaton to this principle for every dimension $4 \leq d \leq 7$. According to the extremization principle, the entropy of supersymmetric asymptotically AdS_5 black holes can be obtained by Legendre-transforming a certain function of chemical potentials conjugated to the black hole conserved charges, called entropy function. We have shown that, in the particular BPS limit defined in Chapter 3, this entropy function coincides with the Euclidean on-shell action of the BPS solution. The results for the entropy we have obtained in this chapter suggest that the same extremization principle may work for our families of black holes if one uses the Page charges and their conjugate chemical potentials instead of the electric holographic charges. This distinction cannot be established looking at asymptotically globally AdS_5 solutions since the holographic charges and the Page charges coincide for them, due to the fact that the Chern-Simons term vanishes at the boundary.

5 CONCLUSIONS AND OUTLOOKS

5.1 SUMMARY OF RESULTS

In this thesis, we have reported the results of researches conducted during the three years of the Ph. D. programme at the University of Padova. These results are mainly collected in Chapters 3, 4.

The main result of Chapter 3, which is based on [67], has been to extend the BPS limit of rotating AdS black hole thermodynamics, defined in [66], to five-dimensional solutions with more than one electric charge, as well as to other spacetime dimensions. For each black hole we analyzed, in every dimension $4 \leq d \leq 7$, we have:

- proposed a particular BPS limit that reaches the BPS locus following a supersymmetric trajectory in the space of complexified solutions;
- shown that the BPS values of the chemical potentials ω^i,Δ^I, which are the non trivial potentials appearing in the entropy functions, can be retrieved by the supergravity black hole solutions by performing the proposed BPS limit;
- shown that the chemical potentials above indeed satisfy the linear complex constraint (1.38) and the corresponding extremization equations;
- proved that the entropy functions of [65, 68, 70] coincide with the supergravity on-shell action $I = I(\omega, \Delta)$ evaluated on a complexified family of supersymmetric solutions.

These results provide a physical derivation of the extremization principles proposed in [65, 68, 70].

The main achievement we obtained in Chapter 4, which is mainly based on the papers [99, 100], has been to construct two new families of supersymmetric AlAdS$_5$ black holes with a boundary geometry containing a squashed S^3. The solutions of the general theory with arbitrary n_V depend on two parameters, of which one controls the squashing at the boundary and the other controls the horizon geometry; the solutions of the $\mathrm{U}(1)^3$ theory are governed instead by three parameters: the one controlling the squashing and the other two regulating the horizon geometry. For the two families, we have obtained the following main results:

- the horizon properties are totally independent on the squashing: if we set a particular horizon geometry by fixing the corresponding parameters, whatever the squashing the S^3 metric will flow to a fixed one at the horizon;

- even though it should be possible to construct squashed solutions in the $\mathrm{U}(1)^3$ theory which depend on four parameters (one regulating the squashing and three controlling the horizon geometry), we were not able to obtain them. Indeed, the black holes we have presented in sec. 4.2.2.3, originally constructed in [100], which are the most general squashed solutions found in the $\mathrm{U}(1)^3$ theory, depend only on three parameters. It could be that this more general solution must be searched in a more general setup than Fayet-Iliopoulos gauged supergravity;

- the entropy of our families of black holes is reproduced by a simple formula which involves the Page charges, rather than the holographic ones. Therefore, the formula is in agreement with the typical one for asymptotically AdS_5 black holes reported in [132], provided that for AlAdS_5 solutions the charges appearing there are identified with the Page charges and not with the holographic ones. This suggests that extremization principles for these AlAdS_5 black holes should still work if one uses the Page charges and their conjugate chemical potentials, instead of the electric holographic charges.

5.2 FUTURE DIRECTIONS

One may wonder what are the possible future developments that rely on the results presented in this thesis. To provide an answer, at least partial, to this question, here we discuss some possible outlooks.

First of all, for what concerns the extremization principle and the black hole solutions analyzed in Chapter 3, there are generalizations that it would be interesting to consider. Indeed, apart from the six-dimensional case, for each dimension we have not considered the most general black holes present in literature, i.e. we have not worked with the most general set of electric charges and angular momenta. The reason is that the most general solutions proved to be very cumbersome and complicated, therefore we required some charges to be equal in order to simplify the computations. It would be interesting to apply our limit to the most general cases in every dimension; our analysis strongly indicates that the same BPS limit will work when the most general set of electric charges and angular momenta is turned on.

In order to perform our BPS limit and retrieve the chemical potentials and the entropy function, we need to start from a finite-temperature black hole solution. This is not fully satisfying since there are BPS black holes for which the corresponding finite-temperature solutions have not been constructed yet. For example, it would be interesting to apply our BPS limit to the AdS_5 black holes constructed in [163], which are solutions of the theory with arbitrary n_V; however this is impossible at the present state of the art since the corresponding finite-temperature solution has not been found yet. It could be that there is a way to study the extremization principle by only using the BPS black hole solution; this problem would be very interesting to explore and analyze. Furthermore, it could be that there is no need of the full black hole solution, but it might be sufficient to have the near-horizon geometry at disposal. To investigate this problem, one should also identify which is the near-horizon counterpart of our BPS limit. This approach, once promoted to the full ten- or eleven-dimensional supergravity theory, may also lead to a generalization of the extremization principle of [57, 58, 153] to the case of rotating horizons with no magnetic charge.

For what concerns AlAdS black holes, there are some possible avenues also in this direction. One is to extend the study of these black holes to other dimensions, with the seven-dimensional case probably being the most promising. It would be interesting to verify whether the physical results we found for the five-dimensional case continue to hold also in other dimensions; checking for example whether the horizon geometry is always independent from the squashing at the boundary or not.

Furthermore, it would be interesting to explore how the BPS limit described in Chapter 3 applies to the AlAdS black holes we have constructed. However, there are several difficulties one has to overcome in order to do this; the most important one probably being the fact that the finite-temperature solution is not known at the moment of writing except for the case of minimal gauged supergravity. The finite-temperature solution for this theory has been constructed in [101]; however this is a numerical solution and it is not clear how to evaluate the on-shell action analytically. It would be worth studying this problem so as to add a piece to the puzzle of the extremization principles for AdS supergravity black holes.

5.3 SOME GENERAL COMMENTS

We would like to conclude with a less technical section, where we briefly analyze which are the main lessons this thesis conveys and which are the unclear points which should be investigated in further detail.

First of all, we have learned how it is possible to obtain the entropy function, the BPS chemical potentials providing interesting thermodynamical relations, and the entropy of many rotating BPS AdS_d black holes. The recipe is the following: one has to start from the corresponding finite-temperature solution, take the BPS limit we have illustrated, obtain the complexified family of supersymmetric solutions and the supersymmetric chemical potentials, and only at this point perform the limit to extremality. The best part of the story is that there are many reasons to believe that our approach is totally general and therefore that it can be applied for *every* rotating BPS AdS$_d$ black hole. Thus, this result we have obtained is very relevant and important, as it should be a totally general approach that manages to clarify the origin of all the entropy functions proposed in the literature and provides a precise way to obtain them from the supergravity black hole solution.

However, there is a weakness in the procedure we have proposed: it is mandatory to have at disposal the finite-temperature version of the BPS black hole we would like to study; otherwise, it is not possible to apply our BPS limit. This is actually an important weakness, since not for all the black holes which would be interesting to study the corresponding finite-temperature solution has been constructed. Therefore, one really important point, which it would be worth clarifying in the future, is whether or not it is possible to recover the BPS chemical potentials only using the BPS solution, without the need of the finite-temperature one. It is reasonable to believe that in principle this should be possible since it might be likely that there should be the possibility to study the BPS black hole thermodynamic by only looking at the BPS solution.

A related question is whether we do really need the full black hole solutions in order to perform the limit or there is a way to do it having at disposal the near-horizon solution only. Various results in the literature point in the direction that, being the entropy influenced only by the near-horizon geometry, the knowledge of only the near-horizon solution should be sufficient. If this would be the case, then the task of finding a near-horizon BPS limit corresponding to the one we have proposed is very important, since constructing the near-horizon solution is much simpler than constructing the full solution; the family of BPS AdS$_d$ black holes for which we can compute the relevant thermodynamic properties would therefore be enlarged.

The fact that the entropy depends only on the horizon properties seems to be confirmed by our study on AlAdS black holes. Indeed, apart from constructing two important examples of family of solutions which are just AlAdS$_5$ rather than AAdS$_5$, the main lesson we have learned from this part of the thesis is that the horizon properties do not depend on the squashing at the boundary; in other words, the horizon geometry of a black hole seems to not vary when the boundary is deformed. However, in some sense, there is something which *do* change: the formula of the entropy with respect to the charges is the same as for AAdS$_5$ black holes, but now holographic charges and Page charges are different. Somehow, the presence of a non-trivial Chern-Simons term at the boundary selects the charges we have to consider in order to reproduce the entropy of an AlAdS$_5$ black hole; this does not happen for AAdS$_5$ solutions for which the Chern-Simons term vanishes at the boundary.

It would be very interesting to verify whether the BPS limit applies also to these black holes by considering the electrostatic potentials conjugate to the Page charges. However, this task is difficult to be achieved because the non-supersymmetric solution is known only numerically and is very complicated and cumbersome. Here we do see how this problem intersects with the ones we have discussed before regarding the BPS limit. Indeed, the BPS solution is much simpler than the finite-temperature one but we *do need* the finite-temperature solution for our BPS limit. At the same time, although the full solutions are known only numerically, the near-horizon of the solutions is known analytically; having a disposal a near-horizon version of the BPS limit we have proposed would thus greatly simplify the task of analyzing these AlAdS black holes.

At the end of our journey, we hope to have conveyed the main and important lessons we have learned from this thesis and, at the same time, made clear that what we have shown is by no means the end of the story. Indeed, the story for AdS black holes has just begun.

APPENDICES

A HOLOGRAPHIC RENORMALIZATION FOR ADS$_5$ BLACK HOLES

In this appendix we describe the application of the general framework of holographic renormalization to the $\mathcal{N} = 2$, $d = 5$ Fayet-Iliopoulos gauged supergravity of which the AdS$_5$ black holes we have presented in sec. 2.1, 4.2 are solutions. Here we will not provide a detailed review starting from the basis and the spirit of holographic renormalization, but rather we will present what we do need for the computations of the on-shell actions and of the conserved charges; for more general information and more extensive dissertations we refer the reader to [30, 115, 117, 160–162].

We will see that there are some differences in the application of holographic renormalization for the AAdS$_5$ solutions of Chapters 2, 3 and the AlAdS$_5$ solutions of Chapter 4; the first being a much more simple subcase of the second.

A.1 GENERAL FRAMEWORK

We start by providing some general formulae for holographic renormalization in five-dimensional Fayet-Iliopoulos gauged supergravity. These will be valid under the assumption that the fermion fields are set to zero and that the scalar fields only depend on the radial coordinate.

We find it convenient to perform the computations and to present the results using a Fefferman-Graham radial coordinate r_{FG}. The AAdS$_5$ black hole solutions of Chapters 2, 3 are written using the radial coordinate r, which can be regarded as a Fefferman-Graham coordinate; therefore for these solutions we have $r = r_{\text{FG}}$, no coordinate transformation needs to be performed and we will perform the computations retaining the original coordinate r. The situation is different for the AlAdS$_5$ solutions of Chapter 4: they are written with respect to ρ which is not a good Fefferman-Graham coordinate and therefore we need to switch from ρ to r_{FG}. The proper coordinate transformations can be found in [99, 100] and we refer to these papers for further details. This coordinate r_{FG} is the same we have adopted in sec. 4.3.1, 4.3.2 to show the results we obtained; there we have dubbed $r_{\text{FG}} = r$ since there was no possibility of confusion between this coordinate and the radial one used to describe the AAdS$_5$ black holes of the previous chapters. Although for these AlAdS$_5$ solutions we could equally well work with the original coordinate ρ, the choice of r_{FG} is more standard in holography and may facilitate comparison with other references.

In the following, to ease the notation we will provide the general formulae for holographic renormalization in Fayet-Iliopoulos supergravity denoting r_{FG} as just r, assuming that when we would like to apply the formulae to the AAdS$_5$ solutions r would be indeed their radial coordinate, while for the case of $AlAdS_5$ solutions r would be the r_{FG} coordinate obtained with the proper coordinate transformation reported in [99, 100].

We recall that the general Fefferman-Graham form of the five dimensional metric is:

$$\mathrm{d}s^2 = \ell^2 \frac{\mathrm{d}r^2}{r^2} + h_{ij}(x,r)\, \mathrm{d}x^i\, \mathrm{d}x^j \ . \tag{A.1}$$

The five-dimensional spacetime M is foliated by timelike hypersurfaces of constant r, parameterized by coordinates x^i, $i = 0, \ldots, 3$. The asymptotic expansion of the induced metric h_{ij} and the other supergravity fields is (see [99, 100] for more details):

$$h_{ij}(x,r) \ = \ \frac{r^2}{\ell^2}\, h^{(0)}_{ij} + \ldots \tag{A.2}$$

$$A^I_i(x,r) \ = \ A^{I\,(0)}_i + \frac{A^{I\,(2)}_i + \tilde{A}^{I\,(2)}_i \ \log\frac{r^2}{\ell^2}}{(r/\ell)^2} + \ldots \tag{A.3}$$

$$X^I \ = \ \bar{X}^I + \frac{\phi^{I\,(0)} + \tilde{\phi}^{I\,(0)} \ \log\frac{r^2}{\ell^2}}{(r/\ell)^2} + \ldots \ , \tag{A.4}$$

where the leading terms $h^{(0)}_{ij}$, $A^{I\,(0)}_i$, $\tilde{\phi}^{I\,(0)}$ are the metric, gauge fields and scalar fields induced on the conformal boundary ∂M. These are interpreted holographically as background fields for the dual SCFT.

In the Fefferman-Graham gauge, the hypersurfaces of constant r are homeomorphic to the conformal boundary, which is found at $r \to \infty$. In order to regulate the large-distance divergences that appear when evaluating the supergravity action one imposes a cutoff r_0, so that the solution extends only up to $r = r_0$. We denote by M_{r_0} the regulated spacetime and by ∂M_{r_0} its boundary at $r = r_0$. Holographic renormalization consists of introducing appropriate local counterterms on ∂M_{r_0} such that the large-distance divergences are cancelled once the cutoff is removed by sending $r_0 \to \infty$. The renormalized action is defined as

$$\mathcal{S}_{\mathrm{ren}} = \lim_{r_0 \to \infty} \mathcal{S}_{\mathrm{reg}} \ , \tag{A.5}$$

where the regularized (and subtracted) action $\mathcal{S}_{\mathrm{reg}}$ is

$$\mathcal{S}_{\mathrm{reg}} = \mathcal{S}_{\mathrm{bulk}} + \mathcal{S}_{\mathrm{GH}} + \mathcal{S}_{\mathrm{ct}} \ . \tag{A.6}$$

Here, $\mathcal{S}_{\mathrm{bulk}}$ denotes the bulk supergravity action (2.2) evaluated on M_{r_0}. The second term is the Gibbons-Hawking boundary integral, which makes the Dirichlet variational problem for the metric well-defined. It reads:

$$\mathcal{S}_{\rm GH} = \frac{1}{\kappa^2} \int_{\partial M_{r_0}} \mathrm{d}^4 x \, \sqrt{h} \, K \ , \tag{A.7}$$

where $K = h^{ij} K_{ij}$ is the trace of the extrinsic curvature $K_{ij} = \frac{r}{2\ell} \frac{\partial h_{ij}}{\partial r}$ of ∂M_{r_0}, and $h = |\det h_{ij}|$. Finally, $\mathcal{S}_{\rm ct}$ consists of the counterterms needed to cancel the divergences of $\mathcal{S}_{\rm bulk} + \mathcal{S}_{\rm GH}$. These are local boundary terms that should preserve the relevant symmetries and may contain finite contributions in addition to divergent terms. The full set of counterterms for AlAdS$_5$ solutions to Fayet-Iliopoulos gauged supergravity where both the scalar and the gauge fields are running and have their leading asymptotic modes turned on has been presented for the first time in [99], generalizing various results previously available in the literature. The counterterm action can be written as

$$\begin{aligned} \mathcal{S}_{\rm ct} = -\frac{1}{\kappa^2} \int_{\partial M_{r_0}} \mathrm{d}^4 x \, \sqrt{h} \bigg[& \mathcal{W} + \Xi \, R - \frac{(\mathcal{W} - 3\ell^{-1})}{\log \frac{r_0^2}{\ell^2}} + \\ & + \frac{\ell^3}{16} \log \frac{r_0^2}{\ell^2} \left(R_{ij} R^{ij} - \frac{1}{3} R^2 - 2\ell^{-2} \, Q_{IJ} F^I_{ij} F^{J\,ij} \right) \bigg] \ . \end{aligned} \tag{A.8}$$

In this formula, the Ricci tensor R_{ij} and the Ricci scalar R are those of the induced metric h_{ij}, which is also used to raise the indices. The other ingredients are the field strengths F^I_{ij} on ∂M_{r_0} and two real functions of the scalar fields: the superpotential $\mathcal{W}$ and the function Ξ; we have already presented both these ingredients in sec. 4.3.1. Note that all the logarithmic terms vanishes for AAdS$_5$ solutions, therefore the counterterm action for this kind of solutions is much more simple and easy to compute.

The counterterms (A.8) cancel all divergences from $\mathcal{S}_{\rm bulk} + \mathcal{S}_{\rm GH}$. Specifically, the first two terms are local covariant expressions on ∂M_{r_0} which remove power-law divergences, while the other terms explicitly depend on the cutoff and cancel logarithmic divergences (if they are present). In addition, the first line of (A.8) yields finite terms that play an important role in the evaluation of the holographic correlation functions.

From the renormalized action one can obtain the holographic one-point functions of the energy-momentum tensor, the electric currents and the relevant scalar operators in the field theory states dual to the supergravity solution of interest.

The holographic energy-momentum tensor is defined as:

$$\langle T_{ij} \rangle = -\frac{2}{\sqrt{h^{(0)}}} \frac{\delta \mathcal{S}_{\rm ren}}{\delta h^{ij(0)}} = -\lim_{r_0 \to \infty} \frac{r_0^2}{\ell^2} \frac{2}{\sqrt{h}} \frac{\delta \mathcal{S}_{\rm reg}}{\delta h^{ij}} \ . \tag{A.9}$$

Starting from the action defined above we obtain:

$$\begin{aligned}\langle T_{ij}\rangle = -\frac{1}{\kappa^2}\lim_{r_0\to\infty}\frac{r_0^2}{\ell^2}\Bigg[& K_{ij} - K\,h_{ij} + \mathcal{W}\,h_{ij} - \frac{\mathcal{W}-3\ell^{-1}}{\log\frac{r_0^2}{\ell^2}}\,h_{ij} \\ & - 2\,\Xi\left(R_{ij} - \frac{1}{2}\,R\,h_{ij}\right) - \frac{\ell^3}{4}\log\frac{r_0^2}{\ell^2}\Bigg(-\frac{1}{2}\,B_{ij} \\ & - \frac{2}{\ell^2}\,Q_{IJ}F^I_{ik}F^J{}_j{}^k + \frac{1}{2\ell^2}\,h_{ij}\,Q_{IJ}F^I_{kl}F^{J\,kl}\Bigg)\Bigg], \end{aligned} \tag{A.10}$$

which coincides with (4.142), as it should. Note that the contributions from the variation of the counterterm action cancel all divergences, including the logarithmic ones, so that $\langle T_{ij}\rangle$ is finite in the limit.

The holographic electric current is defined by varying the action with respect to the gauge field at the boundary:

$$\langle j^i_I\rangle = \frac{1}{\sqrt{h^{(0)}}}\frac{\delta\mathcal{S}_{\text{ren}}}{\delta A_i^{I(0)}} = \lim_{r_0\to\infty}\frac{r_0^4}{\ell^4}\frac{1}{\sqrt{h}}\frac{\delta\mathcal{S}_{\text{reg}}}{\delta A_i^I}\;. \tag{A.11}$$

We obtain:

$$\begin{aligned}\langle j^i_I\rangle = -\frac{1}{\kappa^2}\lim_{r_0\to\infty}\frac{r_0^4}{\ell^4}\Bigg[& \tfrac{1}{6}\epsilon^{ijkl}\left(Q_{IJ}\star F^J + \tfrac{1}{6}C_{IJK}A^J\wedge F^K\right)_{jkl} \\ & + \ell\,\nabla_j\left(Q_{IJ}F^{J\,ji}\right)\log\frac{r_0}{\ell}\Bigg]. \end{aligned} \tag{A.12}$$

coinciding with (4.149). From a dual $\mathcal{N}=1$ superconformal field theory perspective, $\bar{X}^I j_I$ corresponds to the R-current while the orthogonal projections correspond to n_V Abelian flavour currents.

Finally, the one-point function of the scalar operators is defined as

$$\langle\mathcal{O}_I\rangle = \frac{1}{\sqrt{h^{(0)}}}\frac{\delta\mathcal{S}_{\text{ren}}}{\delta\tilde{\phi}^{I(0)}} = \lim_{r_0\to\infty}\left(\frac{r_0^2}{\ell^2}\log\frac{r_0^2}{\ell^2}\frac{1}{\sqrt{h}}\frac{\delta\mathcal{S}_{\text{reg}}}{\delta X^I}\right)\;, \tag{A.13}$$

where it is understood that the variation respects the constraint (2.1), which implies $\bar{X}_I\,\delta\tilde{\phi}^{I(0)} = 0$. By going through the computation, we arrive at:

$$\langle\mathcal{O}_I\rangle = \frac{2}{\kappa^2}\,\bar{Q}_{IJ}\,\phi^{J\,(0)}\;, \tag{A.14}$$

where we recall that $\phi^{(0)}$ is the $\mathcal{O}(r^{-2})$ term in the Fefferman-Graham expansion (A.4) of the scalar fields. As anticipated, this term describes the expectation value of the dual field theory operators, and here we have provided the precise relation between the two.

We remark that the formulae (4.142), (4.149), (A.14) hold for any AlAdS solution (and therefore in particular for any AAdS solution) to five-dimensional Fayet-Iliopoulos gauged supergravity, under the assumption that the fermion fields are set to zero and the scalars are independent of the boundary coordinates (otherwise we would have additional terms).

A.2 HOLOGRAPHIC RENORMALIZATION FOR AADS$_5$ BLACK HOLES

In this section we apply the general formulae we have reported above to the AAdS$_5$ finite-temperature black hole solution we have discussed in sec. 2.1.2.

We begin by recalling that this is a solution of the $\mathrm{U}(1)^3$ theory, therefore we have to set $n_V = 2$ and $C_{IJK} = |\epsilon_{IJK}|$ in all the formulae of the previous section. Moreover the Fayet-Iliopoulos gauging parameters are fixed by stating that in the supersymmetric AdS$_5$ vacuum the scalars take the equal values:

$$\bar{X}^I = 1 \qquad \Rightarrow \qquad \bar{X}_I = \frac{1}{3}\,. \tag{A.15}$$

With these choices, we have:

$$\begin{aligned} &X^1\,X^2\,X^3 = 1\,,\\ &X_I = \frac{1}{3}\left(X^I\right)^{-1}\,,\\ &Q_{IJ} = \frac{1}{2}\,\mathrm{diag}\left(\left(X^1\right)^{-2},\left(X^2\right)^{-2},\left(X^3\right)^{-2}\right),\\ &\mathcal{V} = -2\,g^2\sum_{I=1}^{3}\left(X^I\right)^{-1}\,. \end{aligned} \tag{A.16}$$

Plugging these expressions in (2.2), we retrieve the action of the $\mathrm{U}(1)^3$ model given in (2.26). The superpotential (4.144) and the function (4.145) entering in the holographic counterterms read:

$$\begin{aligned} &\mathcal{W} = g\left(X^1 + X^2 + X^3\right)\,,\\ &\Xi = \frac{1}{12\,g}\left[\left(X^1\right)^{-1} + \left(X^2\right)^{-1} + \left(X^3\right)^{-1}\right]. \end{aligned} \tag{A.17}$$

A.2.1 *Conserved charges*

We are now in the position of computing the holographic quantities for the solution reviewed in sec. 2.1.2. Due to the symmetries of the solution, the holographic energy-momentum tensor can be written as

$$\langle T_{ij}\rangle\,\mathrm{d}x^i\,\mathrm{d}x^j = \langle T_{tt}\rangle\,\mathrm{d}t^2 + \langle T_{\theta\theta}\rangle\left(\sigma_1^2 + \sigma_2^2\right) + \langle T_{\psi\psi}\rangle\,\sigma_3^2 + 2\,\langle T_{t\psi}\rangle\,\mathrm{d}t\,\sigma_3\,, \tag{A.18}$$

and (A.10) gives for its components:

$$\begin{aligned}
\langle T_{tt}\rangle &= \frac{8\,g^3\,m\,(\mathfrak{a}^2 g^2 + 2\,s_1^2 + 2\,s_2^2 + 2\,s_3^2 + 3) + 3g}{64\,\pi}\,,\\
\langle T_{t\psi}\rangle &= \frac{\mathfrak{a}\,g^3\,m\,(s_1\,s_2\,s_3 - c_1\,c_2\,c_3)}{4\,\pi}\,,\\
\langle T_{\theta\theta}\rangle &= \frac{8\,g^2\,m\,(-3\,\mathfrak{a}^2 g^2 + 2\,s_1^2 + 2\,s_2^2 + 2\,s_3^2 + 3) + 3}{768\,\pi\,g}\,,\\
\langle T_{\psi\psi}\rangle &= \frac{8\,g^2\,m\,(9\,\mathfrak{a}^2 g^2 + 2\,s_1^2 + 2\,s_2^2 + 2\,s_3^2 + 3) + 3}{768\,\pi\,g}\,.
\end{aligned} \tag{A.19}$$

Then we proceed to the holographic currents. Evaluating (A.12) we find that the only non-vanishing components of the electric currents are

$$\begin{aligned}
\langle j_I^t\rangle &= -\frac{m\,g^3\,c_I\,s_I}{4\,\pi}\,,\\
\langle j_I^\psi\rangle &= \frac{m\,\mathfrak{a}\,g^5 (c_I\,s_J\,s_K - s_I\,c_J\,c_K)}{2\pi}\,,
\end{aligned} \tag{A.20}$$

where the indices I, J, K are never equal. Finally, from (A.14) we obtain for their scalar operator superpartners:

$$\langle \mathcal{O}_I\rangle = \frac{m}{12\pi}\left(-2\,s_I^2 + s_J^2 + s_K^2\right)\,. \tag{A.21}$$

The holographic energy-momentum tensor and the holographic currents are conserved,

$$\nabla^i \langle T_{ij}\rangle = 0\,, \qquad \nabla_i \langle j_I^i\rangle = 0\,, \tag{A.22}$$

where ∇_i is the Levi-Civita connection of the metric on the conformal boundary. Next, we introduce the energy E and the angular momentum J, defined as the conserved holographic charges associated with the Killing vectors $\frac{\partial}{\partial t}$ and $-\frac{\partial}{\partial\psi}$, respectively. These are obtained by integrating the corresponding components of the energy-momentum tensor on the boundary three-sphere S^3_{bdry}. We find:

$$\begin{aligned}
E &= \int_{S^3_{\text{bdry}}} u^i\,\langle T_{it}\rangle\,\mathrm{vol}\left(S^3_{\text{bdry}}\right) \qquad \text{(A.23)}\\
&= E_0 + \frac{1}{4}\,\pi\,m\left(\mathfrak{a}^2 g^2 + 2\,s_1^2 + 2\,s_2^2 + 2\,s_3^2 + 3\right),\\
J &= -\int_{S^3_{\text{bdry}}} u^i\,\langle T_{i\psi}\rangle\,\mathrm{vol}\left(S^3_{\text{bdry}}\right) \;=\; \frac{1}{2}\pi\,\mathfrak{a}\,m\,(c_1\,c_2\,c_3 - s_1\,s_2\,s_3)\,,
\end{aligned} \tag{A.24}$$

where

$$E_0 = \frac{3\,\pi}{32\,g^2}\,, \tag{A.25}$$

The conserved holographic electric charges can be computed as:

$$\begin{aligned} Q_I &= \int_{S^3_{\text{bdry}}} \text{vol}(S^3_{\text{bdry}})\, u_i \langle j^i_I \rangle \\ &= -\frac{1}{16\,\pi} \int_{S^3_{\text{bdry}}} \left(X_I^{-2} \star F^I + \tfrac{1}{6} C_{IJK}\, A^J \wedge F^K \right) = \frac{1}{2}\, m\, \pi\, s_I c_I\,, \end{aligned} \tag{A.26}$$

where it should be noted that the Chern-Simons term evaluates to zero, implying that in this case the holographic charges are the same as the standard Maxwell charges in (2.39). These expressions for E, J and Q_I coincide with those obtained in [110] by other methods and reported in sec. 2.1.2.

A.2.2 *The renormalized on-shell action*

Our final task is to evaluate the on-shell action. This should be computed in a regular Euclidean section of the solution. We have already described the Euclideanization and the regularity conditions to be imposed in the paragraph around eq. (2.44). Here we keep using the Lorentzian notation until the last step, taking nevertheless into account the conditions that make the Euclidean section regular. We start from the bulk contribution. It is possible to show [99, 100, 133] that this can be rearranged to be:

$$\begin{aligned} \mathcal{S}_{\text{bulk}} = -\frac{1}{12\,\pi} \int_{M_{\bar r}} \mathcal{V} \star 1 + \frac{1}{24\,\pi} \int_{\partial M_{\bar r}} Q_{IJ}\, A^I \wedge \star F^J \\ - \frac{1}{24\,\pi} \int_{\partial M_{r_+}} Q_{IJ}\, A^I \wedge \star F^J\,, \end{aligned} \tag{A.27}$$

where the integral over the radial coordinate is performed from the outer horizon r_+ up to $\bar r$. The first term is easily evaluated recalling the expression for $\mathcal{V}$ in (A.16) and performing the bulk integral. We obtain:

$$\begin{aligned} & -\frac{1}{12\pi} \int_{M_{\bar r}} \mathcal{V} \star 1 \\ &= \left[-\frac{1}{4} \pi g^2 \left(\bar r^4 - r_+^4 \right) - \frac{1}{3}\, \pi m g^2 \left(s_1^2 + s_2^2 + s_3^2 \right) \left(\bar r^2 - r_+^2 \right) \right] \int \mathrm{d}t\,, \end{aligned} \tag{A.28}$$

where we displayed only the terms that do not vanish in the limit $\bar r \to \infty$. The terms involving $\bar r$ arise by evaluating the primitive function at the boundary, while those involving r_+ are the contribution of the horizon. The second and third terms in (A.27) are harder to evaluate, however the difficulties are only technical and not conceptual. For this reason, we do not describe this computation in detail and we just report the final results. It is found that the horizon contribution vanishes, while the boundary one gives the finite term:

$$\frac{1}{24\,\pi}\int_{\partial M_{\bar r}}\left(Q_{IJ}\,A^I\wedge\star F^J\right)= \\ =\frac{1}{6}\,\pi\,m\,\left[\left(c_1\,s_1\,\Phi^1+c_2\,s_2\,\Phi^2+c_3\,s_3\,\Phi^3\right)\right]\int \mathrm{d}t\,. \tag{A.29}$$

For the computation above, it is crucial to notice that regularity of the Euclidean section requires to choose the gauge as in (2.44). We refer the reader to [67] for further details on this computation. The evaluation of the Gibbons-Hawking term (A.7) is straightforward and gives:

$$\begin{aligned}\mathcal{S}_{\mathrm{GH}}=\Bigg\{&\pi\,g^2\,\bar r^4+\frac{\pi}{12}\left[9+16\,m\,g^2\left(s_1^2+s_2^2+s_3^2\right)\right]\bar r^2\\&+\frac{\pi\,m}{6}\Bigg[-(6+s_1^2+s_2^2+s_3^2)\\&\qquad+2g^2\left(3\mathfrak{a}^2+4m\left(s_1^2\,s_2^2+s_1^2\,s_3^2+s_2^2\,s_3^2\right)\right)\Bigg]\Bigg\}\int \mathrm{d}t\,.\end{aligned} \tag{A.30}$$

Recalling (A.17), the counterterm action (A.8) evaluates to:

$$\begin{aligned}\mathcal{S}_{\mathrm{ct}}=\Bigg\{&-\frac{3}{4}\,\pi\,g^2\,\bar r^4+\frac{1}{4}\,\pi\,\bar r^2\left[-4\,m\,g^2\left(s_1^2+s_2^2+s_3^2\right)-3\right]-\frac{3\,\pi}{32\,g^2}\\&+\frac{3\pi\,m}{4}(1-\mathfrak{a}^2g^2)-\pi\,m^2g^2\left(s_1^2\,s_2^2+s_1^2\,s_3^2+s_2^2\,s_3^2\right)\Bigg\}\int \mathrm{d}t\,.\end{aligned} \tag{A.31}$$

The regularized on-shell action $\mathcal{S}_{\mathrm{reg}}$ is the sum of the four terms (A.28), (A.29), (A.30) and (A.31). Adding these up, the divergences cancel out. Taking $\bar r\to\infty$ yields:

$$\begin{aligned}\mathcal{S}_{\mathrm{ren}}=\Bigg\{&-\frac{3\pi}{32g^2}+\frac{\pi}{12}\Bigg[2m\left(c_1s_1\Phi^1+c_2s_2\Phi^2+c_3s_3\Phi^3\right)\\&+4m^2g^2\left(s_1^2s_2^2+s_1^2s_3^2+s_2^2s_3^2\right)+3m(g^2\mathfrak{a}^2-1)\\&+3g^2r_+^4+2m\left(2g^2r_+^2-1\right)\left(s_1^2+s_2^2+s_3^2\right)\Bigg]\Bigg\}\int \mathrm{d}t\,.\end{aligned} \tag{A.32}$$

The Euclidean action is obtained by performing the Wick rotation $t\to-i\tau$ and recalling that the Euclidean and the Lorentzian actions are related as $\mathrm{e}^{-I}=\mathrm{e}^{i\mathcal{S}_{\mathrm{ren}}|_{t\to-i\tau}}$ in the gravitational path integral. Effectively this means that we just have to replace $\int\mathrm{d}t\to-\int\mathrm{d}\tau$ in the expression above. As usual, regularity of the Euclidean solution as $r\to r_+$ fixes the circumference of the Euclidean time circle to be $\int\mathrm{d}\tau=\beta$, where β is the inverse Hawking temperature given in (2.36). In this way we reach the result reported in (2.45).

A.3 HOLOGRAPHIC RENORMALIZATION FOR ALADS$_5$ BLACK HOLES

In this section we apply the general formulae we have reported above to the AlAdS$_5$ finite-temperature black hole solution we have discussed in sec. 4.2. Since we have already computed the conserved charges for both the solutions with arbitrary n_V and the ones with $n_V = 2$, here we only describe the computation of the holographic on-shell actions.

A.3.1 *The renormalized on-shell action*

We begin by noting that the result we will get for the on-shell action of the supersymmetric black hole solutions is somewhat formal: a physically more meaningful way to compute the on-shell action of an extremal solution would be to start from a non-extremal generalization having a regular Euclidean section, evaluate the corresponding on-shell action, and then take the extremal limit. This is exactly what we have done for the AAdS black holes of secs. 2.1.2, 2.2.1 in five and four dimensions, respectively. Nevertheless we find it useful to proceed with a direct evaluation of the action on our Lorentzian solution since in addition to exhibiting the cancellation of the large-distance divergences for all asymptotic solutions of secs 4.2.1.1, 4.2.2, it will lead to a result with a simple physical interpretation.

Our starting point is the bulk action (2.2). Using the trace of the Einstein equation (2.13) and rewriting the Chern-Simons term by means of the Maxwell equation (2.14), this can be expressed as:

$$\mathcal{S}_{\text{bulk}} = \frac{2}{3\kappa^2}\int_{M_{r_0}} \mathcal{V}\star 1 - \frac{1}{3\kappa^2}\int_{M_{r_0}} \mathrm{d}\left(Q_{IJ}\, A^I \wedge\star F^J\right) . \tag{A.33}$$

Since $Q_{IJ}\, A^I\wedge\star F^J$ is globally well-defined and vanishes at the horizon *in the chosen gauge*, the second term reduces by the Stokes theorem to an integral over the boundary ∂M_{r_0}. The same is true for the first term. This can be seen by noticing that using (2.52), the scalar potential (2.17) reads:

$$\mathcal{V} = -6\,\ell^{-2}\,\bar{X}^I X_I = -6\,\ell^{-2} f f_{\text{min}}^{-1} , \tag{A.34}$$

which implies

$$\begin{aligned}\mathcal{V}\star 1 &= -12\ell^{-2}\, f_{\text{min}}^{-1}\, a^3 a'\,\mathrm{d}t\wedge\mathrm{d}\rho\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3\\ &= \frac{1}{2}\,\mathrm{d}\left(a^2 p\,\mathrm{d}t\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3\right) ,\end{aligned} \tag{A.35}$$

where in the last equality we used (2.59). The integral on M_{r_0} is now trivially performed. For both the solutions, from the analysis of sec. 4.2, it follows that $a^2 p \to 0$ at the horizon; therefore we obtain that the only contribution is from the upper limit of integration. Thus the bulk supergravity action can be expressed as a term evaluated at $r = r_0$ as[75]:

$$\mathcal{S}_{\text{bulk}} = -\frac{16\pi^2}{3\kappa^2}\, a^2 p\,|_{r_0} \int \mathrm{d}t + \frac{1}{3\kappa^2}\int_{\partial M_{r_0}} Q_{IJ}\, A^I \wedge \star F^J \ . \qquad \text{(A.36)}$$

The second term in (A.36) is less straightforward. Recalling that A^I is given by (2.51), we can write:

$$A^I \wedge \star F^J = \Bigg[2a^3 a' f^{-1} X^I \left(f X^J\right)' - \frac{af}{2a'} U^I \left(f w' X^J + \left(U^J\right)'\right)\Bigg] \mathrm{d}t \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \, . \qquad \text{(A.37)}$$

Having written this wedge product in the above fashion, we are now able to evaluate the term under consideration for both the solutions. The evaluation of the Gibbons-Hawking term (A.7) and of the counterterm piece (A.8) is straightforward for both the solutions.

Everything we have reported here is valid for both the solutions with arbitrary n_V and with $n_V = 3$; in the two following subsections we perform all the computations for both the cases and we explicitly show all the results.

A.3.1.1 *The case of arbitrary n_V*

Using the asymptotic expansion of the a function obtained in sec. 4.2.1.1, the first term in (A.36) evaluates to:

$$-\frac{16\pi^2}{3\kappa^2}\,(a^2 p)|_{r_0} \int \mathrm{d}t \approx -\frac{8\pi^2\ell^2}{\kappa^2}\Bigg[4\, a_0^4 \left(\frac{r_0}{\ell}\right)^4 - \frac{1}{3}\,(4c+3)\, a_0^2 \left(\frac{r_0}{\ell}\right)^2 - \frac{32}{9}\, c^2 \log\frac{r_0}{\ell} + \frac{1}{36}\,(-128\, a_2 + 38\, c + 1)\, c - \tilde{H}^2 - 2\,\mathcal{K}_1 + \frac{3}{32}\Bigg]\int \mathrm{d}t \, , \qquad \text{(A.38)}$$

where the symbol $\approx$ means that the equality holds up to terms that vanish as $r_0 \to \infty$. The second piece of the bulk action (A.36) gives

$$\frac{1}{3\kappa^2}\int_{\partial M_{r_0}} Q_{IJ}\, A^I \wedge \star F^J \approx -\frac{8\pi^2\ell^2}{\kappa^2}\Bigg[\frac{4}{9}\left(8\, c^2 + 9\,\tilde{H}^2\right)\log\frac{r_0}{\ell} + \frac{2}{9}(1 + 16\, a_2 - 12\, c)\, c + \left(4H_2 + \tilde{H}\right)\tilde{H} + 2\mathcal{K}_1\Bigg]\int \mathrm{d}t \ ; \qquad \text{(A.39)}$$

75 The positive orientation on the five-dimensional spacetime is defined by $\mathrm{d}t \wedge \mathrm{d}\rho \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3$, while we choose $\mathrm{d}t \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ as the positive orientation on the boundary. As a consequence, the Stokes theorem reads $\int_{M_{r_0}} \mathrm{d}\omega = -\int_{\partial M_{r_0}} \omega$.

note that in both expressions resulting from (A.36) the parameter a_4 has been traded for the Page charge $\mathcal{K}_1$ using (4.86).

The Gibbons-Hawking term yields:

$$S_{\rm GH} \approx -\frac{8\pi^2\ell^2}{\kappa^2}\Big[-16a_0^4\left(\frac{r_0}{\ell}\right)^4+\Big(1+\frac{4}{3}c\Big)a_0^2\left(\frac{r_0}{\ell}\right)^2 + 8\tilde{H}^2\log\frac{r_0}{\ell}+8H_2\tilde{H}+4\tilde{H}^2\Big]\int \mathrm{d}t. \tag{A.40}$$

We finally evaluate the counterterm action (A.8). We obtain:

$$\mathcal{S}_{\rm ct} \approx -\frac{8\pi^2\ell^2}{\kappa^2}\left[12\,a_0^4\left(\frac{r_0}{\ell}\right)^4-12\,\tilde{H}^2\,\log\frac{r_0}{\ell}+\frac{8}{3}c^2 - 6\,\tilde{H}\,(2H_2+\tilde{H})\right]\int \mathrm{d}t\ . \tag{A.41}$$

Notice that as long as $\tilde{H}\neq 0$, namely as long as the scalar source term is non-vanishing, the counterterm action contains a logarithmic divergence in addition to a power-law divergence. Adding up (A.38), (A.39), (A.40), (A.41) and removing the cutoff, we arrive at our result for the renormalized on-shell action:

$$\mathcal{S}_{\rm ren} = -\frac{\pi^2\ell^2}{\kappa^2}\left[\frac{16}{9}-\frac{14}{9}\,v^2+\frac{19}{36}\,v^4-16\,\tilde{H}^2\right]\int \mathrm{d}t\ . \tag{A.42}$$

This is exactly the same result given by (4.168) which we have reported in sec. 4.3.1.

A.3.1.2 *The case of $n_V = 2$*

The computation proceeds in a way totally analogous to the case we have examined in the last subsection.

We start with the evaluation of the bulk action by plugging the near-boundary expansions we presented in sec. 4.2.2 into the above expression (A.27). We obtain for the first term

$$-\frac{16\pi^2}{3\kappa^2}\;a^2p\Big|_{r_0}\int \mathrm{d}t = -\frac{8\pi^2\ell^2}{\kappa^2}\Big[4a_0^4\left(\frac{r_0}{\ell}\right)^4-\frac{1}{3}(4c+3)a_0^2\left(\frac{r_0}{\ell}\right)^2 -\frac{32}{9}\,c^2\log\frac{r_0}{\ell}+\frac{1}{36}c(38c-128a_2+1) +\frac{3}{32}-2\mathbb{K}_1-12(\tilde{Z}^2+\tilde{Z}\tilde{K}+\tilde{K}^2)\Big]\int \mathrm{d}t\,, \tag{A.43}$$

while the second one evaluates to

$$\frac{1}{3\kappa^2}\int_{\partial M_{r_0}} Q_{IJ}A^I\wedge\star F^J = -\frac{8\pi^2\ell^2}{\kappa^2}\Big\{-\frac{2}{9}\Big[8\Big(2c^2 + 27\left(\tilde{Z}^2+\tilde{Z}\tilde{K}+\tilde{K}^2\right)\Big)\log\frac{r_0}{\ell} + 16a_2c - 12c^2 + c + 9\left(6\left(\tilde{K}(2Z_2+\tilde{Z}+4K_2)+\tilde{Z}(4Z_2+\tilde{Z}+2K_2)+\tilde{K}^2\right)+\mathbb{K}_1\right)\Big]\Big\}\int \mathrm{d}t\,. \tag{A.44}$$

Once we plug the near-boundary expansions of the various quantities in (A.7), the evaluation of the Gibbons-Hawking term is straightforward, and gives

$$\begin{aligned}\mathcal{S}_{\text{GH}} = -\frac{8\pi^2\ell^2}{\kappa^2}\Big[&-16a_0^4\left(\frac{r_0}{\ell}\right)^4+\frac{1}{3}(4c+3)a_0^2\left(\frac{r_0}{\ell}\right)^2\\ &+96\left(\tilde{Z}^2+\tilde{Z}\tilde{K}+\tilde{K}^2\right)\log\frac{r_0}{\ell} &&\text{(A.45)}\\ &+48\left(\tilde{K}(Z_2+\tilde{Z}+2K_2)\,\tilde{Z}\,(2Z_2+\tilde{Z}+K_2)+\tilde{K}^2\right)\Big]\int\mathrm{d}t\,. &&\text{(A.46)}\end{aligned}$$

Evaluating the counterterm action (A.8), we obtain

$$\mathcal{S}_{\text{ct}} = -\frac{8\pi^2\ell^2}{\kappa^2}\Big[12a_0^4\left(\frac{r_0}{\ell}\right)^4-144\left(\tilde{Z}^2+\tilde{Z}\tilde{K}+\tilde{K}^2\right)\log\frac{r_0}{\ell} + \frac{8}{3}\left(c^2-27\left(\tilde{K}(Z_2+\tilde{Z}+2K_2)+\tilde{Z}(2a_2+\tilde{Z}+K_2)+\tilde{K}^2\right)\right)\Big]\int\mathrm{d}t\,. \tag{A.47}$$

Adding up all the pieces of the action given by eqs. (A.43), (A.44), (A.45), (A.47) we get the final result

$$\mathcal{S}_{\text{ren}} = -\frac{\pi^2\ell^2}{\kappa^2}\left[\frac{16}{9}-\frac{14}{9}v^2+\frac{19}{36}v^4-192(\tilde{Z}^2+\tilde{Z}\tilde{K}+\tilde{K}^2)\right]\int\mathrm{d}t\,, \tag{A.48}$$

which is the result (4.187) reported in the main text. All the power-law and logarithmic divergences of the various pieces of the action cancel non-trivially against themselves when we perform the sum.

B

HOLOGRAPHIC RENORMALIZATION FOR ADS$_4$ BLACK HOLES

In this appendix we report and describe the strategies we followed to compute the on-shell action and the conserved charges for the AdS$_4$ black hole solution introduced in sec. 2.2.1, using holographic renormalization.

The general procedure is quite similar to the five-dimensional case described in app A, therefore we will keep the presentation shorter with respect to this case. We refer the reader to [67] for additional details.

B.1 THE RENORMALIZED ON-SHELL ACTION

The four-dimensional metric has the same form as (A.1), where now $i, j = 0, \ldots, 2$. The renormalized action is again $\mathcal{S}_{\text{ren}} = \lim_{\bar{r}\to\infty} \mathcal{S}_{\text{reg}}$ with $\mathcal{S}_{\text{reg}} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{GH}} + \mathcal{S}_{\text{ct}}$. Using the Einstein equation, the bulk supergravity action (2.124) can be recast into

$$\begin{aligned}\mathcal{S}_{\text{bulk}} = -\frac{1}{16\pi}\int_{M_{\bar{r}}}\Big(&-2\mathcal{V}\star 1-\frac{1}{2}\mathrm{e}^{-\xi}F_3\wedge\star F_3\\&-\frac{1}{2}\chi\,F_3\wedge F_3+\frac{\chi\,\mathrm{e}^{2\xi}}{2\,(1+\chi^2\,\mathrm{e}^{2\xi})}F_1\wedge F_1\\&-\frac{1}{2\,(1+\chi^2\,\mathrm{e}^{2\xi})}\mathrm{e}^{\xi}F_1\wedge\star F_1\Big)\,.\end{aligned} \tag{B.1}$$

The Gibbons-Hawking boundary integral is defined as in five dimensions:

$$\mathcal{S}_{\text{GH}} = \frac{1}{8\,\pi}\int_{\partial M_{\bar{r}}} \mathrm{d}^3x\,\sqrt{h}\,K\,, \tag{B.2}$$

while the counterterm action is given by

$$\mathcal{S}_{\text{ct}} = -\frac{1}{8\,\pi}\int_{\partial M_{\bar{r}}} \mathrm{d}^3x\,\sqrt{h}\,\mathcal{W}\left(1+\frac{1}{4\,g^2}\,R\right)\,, \tag{B.3}$$

with $\mathcal{W}$ being the following real superpotential

$$\mathcal{W} = g\,\mathrm{e}^{\xi/2}\sqrt{\chi^2+(\mathrm{e}^{-\xi}+1)^2}\,. \tag{B.4}$$

This has been obtained in [67] by specializing the results of [146] to the present case. We refer to [67] for further details on this. It should be noted that the counterterm (B.3) is compatible with supersymmetry provided a combination of the scalar fields is given Neumann boundary conditions [164]; this means that our renormalized action is a function of vevs for the operators dual to these scalars, and of sources for the other operators [146].

We have all the ingredients we need to evaluate the various pieces of the on-shell action. Displaying only the contributions that do not vanish in the limit $\bar{r} \to \infty$, the bulk action (B.1) yields

$$\begin{aligned}\mathcal{S}_{\text{bulk}} = \frac{\int \mathrm{d}t}{2(\mathfrak{a}^2 g^2 - 1)} \Bigg\{ & g^2(\bar{r}^3 - r_+^3) \;+\; 3g^2 m(\bar{r}^2 - r_+^2)(s_1^2 + s_2^2) \\ & + (\bar{r} - r_+)\left[\mathfrak{a}^2 g^2 + 2\,m^2 g^2 (s_1^4 + 4\,s_1^2 s_2^2 + s_2^4)\right] \\ & - \frac{2\,m^2\left[c_1^2\, s_1^2\,(2\,m\,s_2^2 + r_+) + c_2^2\, s_2^2\,(2\,m\,s_1^2 + r_+)\right]}{\mathfrak{a}^2 + (2\,m\,s_1^2 + r_+)\,(2\,m\,s_2^2 + r_+)} \Bigg\}, \end{aligned} \tag{B.5}$$

the Gibbons-Hawking term gives:

$$\begin{aligned}\mathcal{S}_{\text{GH}} = \frac{\int \mathrm{d}t}{2\,(1 - \mathfrak{a}^2 g^2)} \Bigg\{ & 3g^2\bar{r}^3 + 9mg^2\bar{r}^2(s_1^2 + s_2^2) \\ & + \left[\tfrac{5}{3}\mathfrak{a}^2 g^2 + 6m^2 g^2 (s_1^4 + 4s_1^2 s_2^2 + s_2^4) + 2\right]\bar{r} \\ & + m\left(\tfrac{5}{3}\,\mathfrak{a}^2 g^2 + 12\,m^2 g^2 s_1^2 s_2^2 - 1\right)\left(s_1^2 + s_2^2\right) - 3m \Bigg\}, \end{aligned} \tag{B.6}$$

while the counterterm action evaluates to:

$$\begin{aligned}\mathcal{S}_{\text{ct}} = \frac{\int \mathrm{d}t}{1 - \mathfrak{a}^2 g^2} \Bigg\{ & - g^2\bar{r}^3 - 3g^2 m\,\bar{r}^2\left(s_1^2 + s_2^2\right) \\ & - \left[\tfrac{1}{3}\mathfrak{a}^2 g^2 + 2m^2 g^2\left(s_1^4 + 4s_1^2 s_2^2 + s_2^4\right) + 1\right]\bar{r} \\ & - m\left(\tfrac{1}{3}\mathfrak{a}^2 g^2 + 4\,m^2 g^2 s_1^2 s_2^2\right)\left(s_1^2 + s_2^2\right) + m \Bigg\}. \end{aligned} \tag{B.7}$$

The renormalized action can be obtained by adding up these three expressions and sending $\bar{r} \to \infty$. We get the following result:

$$\begin{aligned}\mathcal{S}_{\text{ren}} = \frac{\int \mathrm{d}t}{2(1 - \mathfrak{a}^2 g^2)} \Bigg\{ & g^2 r_+^3 + 3\,m\,g^2 r_+^2\left(s_1^2 + s_2^2\right) \\ & + r_+\left[\mathfrak{a}^2 g^2 + 2\,m^2 g^2\left(s_1^4 + 4s_1^2 s_2^2 + s_2^4\right)\right] \\ & + m\left(\mathfrak{a}^2 g^2 + 4\,m^2 g^2 s_1^2 s_2^2 - 1\right)(s_1^2 + s_2^2) - m \\ & + \frac{2\,m^2\left[c_1^2 s_1^2\,(2\,m\,s_2^2 + r_+) + c_2^2 s_2^2\,(2\,m\,s_1^2 + r_+)\right]}{\mathfrak{a}^2 + (2\,m\,s_1^2 + r_+)\,(2\,m\,s_2^2 + r_+)} \Bigg\}. \end{aligned} \tag{B.8}$$

The Euclidean on-shell action I is obtained by Wick-rotating $t = -i\tau$ and identifying $\tau \sim \tau + \beta$, where β was given in (2.133). Differently from the five-dimensional case we have examined in app. A, there is no subtlety related to the choice of a regular gauge, because the four-dimensional action is gauge-invariant. Therefore one simply has $I = -i\mathcal{S}_{\text{ren}}|_{\int \mathrm{d}t \to -i\beta}$ Our final result is displayed in (2.137).

B.2 CONSERVED CHARGES

We now proceed to evaluate the conserved charges of the AdS_4 black hole under consideration using holographic renormalization.

The holographic energy-momentum tensor is given by:

$$\begin{aligned}\langle T_{ij}\rangle &= -\lim_{\bar r\to\infty}\frac{2\,\bar r\,g}{\sqrt{h}}\,\frac{\delta\mathcal{S}_{\text{reg}}}{\delta h^{ij}}\\ &= -\frac{1}{8\,\pi}\lim_{\bar r\to\infty}\bar r\,g\left[K_{ij}-(K-\mathcal{W})\,h_{ij}-\frac{1}{2\,g^2}\,\mathcal{W}\left(R_{ij}-\frac{1}{2}\,R\,h_{ij}\right)\right].\end{aligned} \tag{B.9}$$

The charges appearing in (2.134) are evaluated in a frame which is non-rotating at infinity, so in order to compare with those expressions it is convenient to use the time and angular coordinates t', ϕ' defined in (2.131). Here we report only the components $\langle T_{t't'}\rangle$ and $\langle T_{t'\phi'}\rangle$, since these are the only ones needed to compute the energy and the angular momentum:

$$\begin{aligned}\langle T_{t't'}\rangle &= \frac{1}{8\,\pi\,(\mathfrak{a}^2g^2-1)^2}\Big[g^2m\left(s_1^2+s_2^2+1\right)\left(1-\mathfrak{a}^2g^2\cos^2\theta\right)\\ &\qquad\qquad\left(2-2\mathfrak{a}^2g^2\cos^2\theta+\mathfrak{a}^2g^2\sin^2\theta\right)\Big],\\ \langle T_{t'\phi'}\rangle &= \frac{3\,\mathfrak{a}\,g^2m\,(s_1^2+s_2^2+1)\,\sin^2\theta\,(\mathfrak{a}^2g^2\cos^2\theta-1)}{8\,\pi\,(\mathfrak{a}^2g^2-1)^2}\,.\end{aligned} \tag{B.10}$$

The asymptotic metric at $r \to \infty$ is

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{g^2r^2} + g^2r^2\,\mathrm{d}s^2_{\text{bdry}}\,, \tag{B.11}$$

where the metric on the conformal boundary reads in the non-rotating frame

$$\mathrm{d}s^2_{\text{bdry}} = -\frac{\Delta_\theta}{\Xi}\,\mathrm{d}t'^2 + \frac{\mathrm{d}\theta^2}{g^2\Delta_\theta} + \frac{\sin^2\theta\,\mathrm{d}\phi'^2}{g^2\Xi}\,, \tag{B.12}$$

and Δ_θ, Ξ were given in (2.127).[76] We have all the ingredients to evaluate the conserved charges E and J, associated with the symmetries generated by $\frac{\partial}{\partial t'}$ and $-\frac{\partial}{\partial \phi'}$, respectively. These are given by:

$$\begin{aligned} E &= \int_{\Sigma_{\text{bdry}}} u^i \langle T_{it'} \rangle \, \text{vol}\left(\Sigma_{\text{bdry}}\right) = \frac{m}{\Xi^2}\left(1+s_1^2+s_2^2\right) \,, \\ J &= -\int_{\Sigma_{\text{bdry}}} u^i \langle T_{i\phi'} \rangle \, \text{vol}\left(\Sigma_{\text{bdry}}\right) = \frac{\mathfrak{a}\, m}{\Xi^2}\left(1+s_1^2+s_2^2\right) \,. \end{aligned} \tag{B.13}$$

where $u = \sqrt{\frac{\Xi}{\Delta_\theta}}\frac{\partial}{\partial t'}$ is the unit, outward-pointing timelike vector and Σ_{bdry} is the two-dimensional Cauchy surface at the boundary, with metric induced from (B.12). These expressions coincide with the ones computed in [134] and reported in (2.134). The electric charges obtained from the holographic currents $\langle j^i \rangle$ also agree with those in (2.134).

76 The metric (B.12) is related by a Weyl transformation and a change of coordinate to the canonical metric on $\mathbb{R}\times S^2$: $\frac{\Xi}{\Delta_\theta}\mathrm{d}s^2_{\text{bdry}} = -\mathrm{d}t^2 + \frac{1}{g^2}\left(\mathrm{d}\theta'^{\,2} + \sin^2\theta'\,\mathrm{d}\phi'^{\,2}\right)$, with $\tan\theta = \sqrt{1-\mathfrak{a}^2 g^2}\,\tan\theta'$. We will not need to implement this transformation here.

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