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ON FINITE-BY-NILPOTENT PROFINITE GROUPS

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Abstract. Let $\gamma_n = [x_1, \ldots, x_n]$ be the nth lower central word. Suppose that G is a profinite group where the conjugacy classes $x^{\gamma_n(G)}$ contains less than $2^{\aleph_0}$ elements for any $x \in G$. We prove that then $\gamma_{n+1}(G)$ has finite order. This generalizes the much celebrated theorem of B. H. Neumann that says that the commutator subgroup of a BFC-group is finite. Moreover, it implies that a profinite group G is finite-by-nilpotent if and only if there is a positive integer n such that $x^{\gamma_n(G)}$ contains less than $2^{\aleph_0}$ elements, for any $x \in G$.

1. **Introduction**

Given a group G and an element $x \in G$, we write x^G for the conjugacy class containing x. Of course, if the number of elements in x^G is finite, we have $|x^G| = [G : C_G(x)]$. A group is said to be a BFC-group if its conjugacy classes are finite and of bounded size. One of the most famous of B. H. Neumann's theorems says that in a BFC-group the commutator subgroup G' is finite [11]. It follows that if $|x^G| \leq m$ for each $x \in G$, then the order of G' is bounded by a number depending only on m. A first explicit bound for the order of G' was found by J. Wiegold [16], and the best known was obtained in [8] (see also [12] and [14]).

The recent articles [6], [4], and [1] deal with groups G in which conjugacy classes containing commutators are bounded. Recall that multilinear commutator words are words which are obtained by nesting commutators, but using always different variables. More formally, the group-word $w(x) = x$

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in one variable is a multilinear commutator; if u and v are multilinear commutators involving different variables then the word $w = [u, v]$ is a multilinear commutator, and all multilinear commutators are obtained in this way. Examples of multilinear commutators include the familiar lower central words $\gamma_n(x_1, \ldots, x_n) = [x_1, \ldots, x_n]$ and derived words δ_n, on 2^n variables, defined recursively by

$$\delta_0 = x_1, \qquad \delta_n = [\delta_{n-1}(x_1, \ldots, x_{2^{n-1}}), \delta_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n})].$$

We let $w(G)$ denote the verbal subgroup of G generated by all w-values. Of course, $\gamma_n(G)$ is the nth term of the lower central series of G while $\delta_n(G) = G^{(n)}$ is the nth term of the derived series.

The following theorem was established in [4].

Theorem 1.1. [4] *Let m be a positive integer and w a multilinear commutator word. Suppose that G is a group in which $x^G \leq m$ for any w-value x. Then the order of the commutator subgroup of $w(G)$ is finite and m-bounded.*

Here the expression "$(a, b, \ldots)$-bounded" means that a quantity is finite and bounded by a certain number depending only on the parameters $a, b, \ldots$.

One may wonder what happens in the realm of profinite groups satisfying an analogue condition, that is, in profinite groups where, given a multilinear commutator word w, every w-value x has a finite conjugacy class. A theorem of Shalev [15] states that in a profinite group with all conjugacy classes finite, the commutator subgroup is finite. This was generalized in [5] to multilinear commutator words. Namely, in [5] it is proved that if w is a multilinear commutator word and G a profinite group in which all centralizers of w-values are either finite or of finite index, then $w(G)$ is abelian-by-finite. Moreover, the following holds.

Theorem 1.2. [5] *Let w be a multilinear commutator word and G a profinite group in which x^G is finite for every w-value x, then the order of the commutator subgroup of $w(G)$ is finite.*

A modification of the techniques developed in [6] and [4] can be used to deduce that if $[G' : C_{G'}(x)] \leq m$ for each $x \in G$, then $\gamma_3(G)$ has finite m-bounded order. Naturally, one expects that a similar phenomenon holds for other terms of the lower central series of G. This was investigated in the article [1], where the following result was proved.

Theorem 1.3. [1] *Let m, n be positive integers and G a group. If $|x^{\gamma_n(G)}| \leq m$ for any $x \in G$, then $\gamma_{n+1}(G)$ has finite (m, n)-bounded order.*

In this article we prove an analogue of Theorem 1.3 for profinite groups, in the spirit of Theorem 1.2.

Theorem 1.4. *Let n be a positive integer and G a profinite group. If $x^{\gamma_n(G)}$ is finite for any $x \in G$, then $\gamma_{n+1}(G)$ has finite order.*

When investigating profinite groups, it might occur that certain subsets of the group are finite under the weaker hypotheses that the same sets have at most countably many elements, or in some cases less than $2^{\aleph_0}$ elements (see for instance [3] or [2]). The following result is [2, Lemma 2.2].

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Lemma 1.5. [2] *Let H be a profinite group and let $x \in H$. If the conjugacy class x^H contains less than $2^{\aleph_0}$ elements, then it is finite.*

This implies in particular that if G is a profinite group, $x \in G$ and the conjugacy class $x^{\gamma_n(G)}$ contains less than $2^{\aleph_0}$ elements, then $x^{\gamma_n(G)}$ is finite. An immediate consequence of this fact is the following generalization of Theorem 1.4.

Theorem 1.6. *Let n be a positive integer and G a profinite group. If $x^{\gamma_n(G)}$ contains less than $2^{\aleph_0}$ elements for any $x \in G$, then $\gamma_{n+1}(G)$ has finite order.*

Using the concept of verbal conjugacy classes, introduced in [7], one can obtain a generalization of Theorem 1.4. Let $X_n = X_n(G)$ denote the set of γ_n-values in a group G. It was shown in [7] that if the set $x^{X_n} = \{x^y \mid y \in X_n\}$ is finite for each $x \in G$, then $x^{\gamma_n(G)}$ is finite. Hence, we have

Corollary 1.7. *Let n be a positive integer and G a profinite group. If x^{X_n} is finite for any $x \in G$, then $\gamma_{n+1}(G)$ has finite order.*

Another result which is straightforward from Theorem 1.4 is the following characterization of finite-by-nilpotent profinite groups.

Theorem 1.8. *A profinite group G is finite-by-nilpotent if and only if there is a positive integer n such that $x^{\gamma_n(G)}$ contains less than $2^{\aleph_0}$ elements, for any $x \in G$.*

2. Proof of the main result

Recall that in any group G the following "standard commutator identities" hold, when $x, y, z \in G$.

(1) $[xy, z] = [x, z]^y[y, z]$
(2) $[x, yz] = [x, z][x, y]^z$
(3) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$ (Hall-Witt identity);
(4) $[x, y, z^x][z, x, y^z][y, z, x^y] = 1$.

Note that the fourth identity follows from the third one. Indeed, we have

$$[x^y, y^{-1}, z^y][y^z, z^{-1}, x^z][z^x, x^{-1}, y^x] = 1.$$

Since $[x^y, y^{-1}] = [y, x]$, it follows that

$$[y, x, z^y][z, y, x^z][x, z, y^x] = 1.$$

In the sequel, X_i will denote the set of γ_i-values in a group G. Moreover, the notation $|S| < \infty$ will mean that the set S is finite.

Lemma 2.1. *Let k, n be integers with $2 \le k \le n$ and let G be a profinite group in which $|x^{\gamma_n(G)}| < \infty$ for any $x \in G$. Assume that $[\gamma_k(G), \gamma_n(G)]$ is finite. Then for every $g \in X_n$ we have*

$$|g^{\gamma_{k-1}(G)}| < \infty.$$

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Proof. Let $N = [\gamma_k(G), \gamma_n(G)]$. It is sufficient to prove that in the quotient group G/N, for every integer d with $k-1 \le d \le n$

$$|(gN)^{\gamma_d(G/N)}| < \infty \quad \text{for every } \gamma_{n-d+1}\text{-value } gN \in G/N,$$

since this implies that $g^{\gamma_d(G)}$ is contained in finitely many cosets of N, whenever $g \in X_{n-d+1}$.

So in what follows we assume that $N = 1$. The proof is by induction on $n-d$. The case $d = n$ is immediate from the hypotheses.

Let $c = n-d+1$. Choose $g \in X_c$ and write $g = [x, y]$ with $x \in X_{c-1}$ and $y \in G$. Let $z \in \gamma_d(G)$. We have

$$[x, y, z^x][z, x, y^z][y, z, x^y] = 1.$$

Note that

$$[z, x] \in [\gamma_d(G), \gamma_{c-1}(G)] \le \gamma_{d-1+c}(G) = \gamma_n(G)$$

and

$$[y, z] \in \gamma_{d+1}(G) \le \gamma_k(G),$$

whence $[z, x, y^z] = [z, x, y[y, z]] = [z, x, y]$. Thus,

$$1 = [x, y, z^x][z, x, y^z][y, z, x^y] = [x, y]^{-1}[x, y]^{z^x}[z, x, y][y, z, x^y]$$

$$= [x, y]^{-1}[x, y]^{z^x}(y^{-1})^{[z,x]}y((x^y)^{-1})^{[y,z]}x^y.$$

It follows that

$$[x, y]^{z^x} = [x, y](x^{-1})^y(x^y)^{[y,z]}y^{-1}y^{[z,x]}.$$

Since $x^y \in X_{c-1}$ and $[y, z] \in \gamma_{d+1}(G)$, by induction

$$|\{(x^y)^{[y,z]} \mid z \in \gamma_d(G)\}| < \infty.$$

Moreover, $[z, x] \in \gamma_n(G)$ and so $|\{y^{[z,x]} \mid z \in \gamma_d(G)\}| < \infty$. Thus,

$$|\{[x, y]^{z^x} \mid z \in \gamma_d(G)\}| = |\{[x, y]^z \mid z \in \gamma_d(G)\}| < \infty$$

as claimed. □

Recall that if G is a profinite group, $a \in G$ and H is a subgroup of G, then $[H, a]$ denotes the (closed) subgroup of G generated by all commutators of the form $[h, a]$, where $h \in H$. It is well-known that $[H, a]$ is normalized by a and H.

Lemma 2.2. *Let k, n be integers with $2 \le k \le n$ and let G be a profinite group in which $|x^{\gamma_n(G)}| < \infty$ for any $x \in G$. Suppose that $[\gamma_k(G), \gamma_n(G)]$ is finite. Then for every $x \in \gamma_{k-1}(G)$ the order of $[\gamma_n(G), x]$ is finite.*

Proof. Without loss of generality we can assume $[\gamma_k(G), \gamma_n(G)] = 1$. Let $x \in \gamma_{k-1}(G)$. Since $|x^{\gamma_n(G)}|$ is finite, the index of $C_{\gamma_n(G)}(x)$ in $\gamma_n(G)$ is finite, as well. Let Δ_n be the abstract subgroup generated by X_n. Since Δ_n is dense in $\gamma_n(G)$, we can choose a right transversal $y_1, \ldots, y_m$ of $C_{\gamma_n(G)}(x)$ in $\gamma_n(G)$ with each $y_i \in \Delta_n$. Note that $[\Delta_n, x]$ is generated by the commutators $[y_i, x]$. For each $i = 1, \ldots, m$ write $y_i = y_{i\,1} \cdots y_{i\,m_i-1}$, where $y_{i\,j} \in X_n$. The standard commutator identities show that $[y_i, x]$ can be written as a product of conjugates in $\gamma_n(G)$ of the commutators $[y_{ij}, x]$. Since $[y_{ij}, x] \in \gamma_k(G)$, for any $z \in \gamma_n(G)$ we have that

$$[[y_{ij}, x], z] \in [\gamma_k(G), \gamma_n(G)] = 1.$$

Therefore $[y_i, x]$ can be written as a product of the commutators $[y_{ij}, x]$.

Let T be the abstract subgroup generated by x, y_{ij} for $1 \le i, j \le m$. It is clear that $[\Delta_n, x] \le T'$ and so it is sufficient to show that T' is finite. Observe that $T \le \gamma_{k-1}(G)$. By Lemma 2.1, $C_{\gamma_{k-1}(G)}(y_{ij})$ has finite index in $\gamma_{k-1}(G)$. It follows that $C_T(\{y_{ij} \mid 1 \le i, j \le m\})$ has finite index in T. Moreover, $T \le \langle x \rangle \gamma_n(G)$ and $|x^{\gamma_n(G)}|$ is finite, whence $[T : C_T(x)]$ is finite. Therefore the centre of T has finite index in T. Thus, Schur's theorem [13, theorem10.1.4] tells us that T' has finite order. Therefore $[\Delta_n, x]$ is finite, as well as its closure $[\gamma_n(G), x]$. □

The next lemma can be seen as a development related in [6, Lemma 2.4] and in [16, Lemma 4.5].

Lemma 2.3. *Let k, n be integers with $2 \le k \le n$ and let G be a profinite group in which $|x^{\gamma_n(G)}| < \infty$ for any $x \in G$. Suppose that $[\gamma_k(G), \gamma_n(G)]$ is finite. Then $[\gamma_{k-1}(G), \gamma_n(G)]$ is finite.*

Proof. Without loss of generality we can assume $[\gamma_k(G), \gamma_n(G)] = 1$. Let $W = \gamma_n(G)$ and let $K = \gamma_{k-1}(G)$.

For each natural number j consider the set C_j of elements $g \in K$ such that $|g^{\gamma_n(G)}| \le j$. Note that the sets C_j are closed (see for instance [10, Lemma 5]). As the union of the sets C_j is the whole K, by the Baire category theorem (cf. [9, p.200]) at least one of the sets C_j is open in K. So there exists an open subgroup Y of K and an element $a \in K$ such that

$$r = |a^W| \ge |(ya)^W| \quad \text{for all } y \in Y.$$

Let Δ_n be the abstract subgroup generated by X_n. Since Δ_n is dense in $\gamma_n(G)$, we can choose a right transversal $b_1, \ldots, b_r$ of $C_{\gamma_n(G)}(a)$ in $\gamma_n(G)$ with each $b_i \in \Delta_n$. Set $M = Y \cap (C_K(\langle b_1, \ldots, b_r \rangle))_K$ (i.e. M is the intersection of Y and all K-conjugates of $C_K(\langle b_1, \ldots, b_r \rangle)$). Since each b_i is a product of finitely many elements of X_n and, by Lemma 2.1, $C_K(x)$ has finite index in K for each $x \in X_n$, the subgroup $C_K(\langle b_1, \ldots, b_r \rangle)$ has finite index in K, so also M has finite index in K.

Let $v \in M \le Y$. Note that $(va)^{b_i} = va^{b_i}$ for each $i = 1, \ldots, r$. Therefore the elements va^{b_i} form the conjugacy class $(va)^W$ because they are all different and their number is the allowed maximum. So, for an arbitrary element $h \in W$ there exists $b \in \{b_1, \ldots, b_r\}$ such that $(va)^h = va^b$ and hence $v^h a^h = va^b$. Therefore $[h, v] = v^{-h} v = a^h a^{-b}$ and so $[h, v]^a = a^{-1} a^h a^{-b} a = [a, h][b, a] \in [W, a]$. Thus $[W, v]^a \le [W, a]$ and so $[W, M] \le [W, a]$.

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Let $x_1, \dots, x_s$ be a set of coset representatives of M in K. As $[W, x_i]$ is normalized by W for each i, it follows that

$$[W, K] \le [W, x_1] \cdots [W, x_s][W, M] \le [W, x_1] \cdots [W, x_s][W, a].$$

Since by Lemma 2.2 all subgroups $[W, x_i]$ and $[W, a]$ are finite, the result follows. □

Proof of Theorem 1.4. Let G be a profinite group in which $|x^{\gamma_n(G)}| < \infty$ for any $x \in G$. We need to show that $\gamma_{n+1}(G)$ has finite order. We will show that the order of $[\gamma_k(G), \gamma_n(G)]$ is finite for $k = n, n-1, \dots, 1$. This is sufficient for our purposes since $[\gamma_1(G), \gamma_n(G)] = \gamma_{n+1}(G)$. We argue by backward induction on k. The case $k = n$ was proved by Shalev in [15] and it also follows from Theorem 1.2 when $w = \delta_0$. So we assume that $k \le n-1$ and the order of $[\gamma_{k+1}(G), \gamma_n(G)]$ is finite. Lemma 2.3 now shows that also the order of $[\gamma_k(G), \gamma_n(G)]$ is finite, as required. □

As observed in the introduction, the proofs of Theorem 1.6, Corollary 1.7 and Theorem 1.8 are straightforward consequences of Theorem 1.4.

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