THE RIGID SYNTOMIC RING SPECTRUM

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Abstract. The aim of this paper is to show that rigid syntomic cohomology – defined by Besser – is representable by a rational ring spectrum in the motivic homotopical sense. In fact, extending previous constructions, we exhibit a simple representability criterion and we apply it to several cohomologies in order to get our central result. This theorem gives new results for rigid syntomic cohomology such as h-descent and the compatibility of cycle classes with Gysin morphisms. Along the way, we prove that motivic ring spectra induce a complete Bloch-Ogus cohomological formalism and even more. Finally, following a general motivic homotopical philosophy, we exhibit a natural notion of rigid syntomic coefficients.
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Introduction

In the 1980s, Beilinson stated his conjectures relating the special values of L-functions and the regulator map of a variety X defined over a number field [Beĭ84, Beĭ86a]. The regulator considered by Beilinson is a map from the K-theory of X with target the Deligne-Beilinson cohomology[1] with real coefficients

$$reg : K_{2i-n}(X)^{(i)} \otimes \mathbb{Q} \to H^n_{DB}(X, \mathbb{R}(i)) \ .$$

One can define $H^n_{DB}(X, A(i))$ for any subring $A \subset \mathbb{R}$. For $A = \mathbb{Z}$, Beilinson proved that $H^n_{DB}(X, \mathbb{Z}(i))$ is the absolute Hodge cohomology theory: *i.e.* it computes the group of homomorphisms in the derived category of mixed Hodge structures

$$H^n_{DB}(X, \mathbb{Z}(i)) = \mathrm{Hom}_{D^b(MHS)}(\mathbb{Z}, R\Gamma_{Hdg}(X)(i)[n]) \ ,$$

where $R\Gamma_{Hdg}(X)$ is the mixed Hodge complex associated to X whose cohomology is the Betti cohomology of X endowed with its mixed Hodge structure [Beĭ86b]. Further Beilinson conjectured that the higher K-theory groups form an absolute cohomology theory, in fact the universal one, called motivic cohomology. This vision is now partly accomplished. We do not have the category of mixed motives, but we can construct a triangulated category playing the role of its derived category. More precisely, Cisinski and Déglise proved that for any finite dimensional noetherian scheme X there exists a monoidal triangulated category $DM_{\text{B}}(X) = DM_{\text{B}}(X, \mathbb{Q})$ (along with the six operations) such that

$$H^{n,i}_{\text{B}}(X) := \mathrm{Hom}_{DM_{\text{B}}(S)}(\mathbb{1}_S, \pi_* \mathbb{1}_X(i)[n]) \simeq K_{2i-n}(X)^{(i)} \otimes \mathbb{Q}$$

when $\pi : X \to S$ is a smooth morphism and S is regular [CD12b].

Now let K be a p-adic field (*i.e.* a finite extension of $\mathbb{Q}_p$) with ring of integers R. Given X a smooth and algebraic R-scheme, Besser defined the analogue of the Deligne-Beilinson cohomology in order to study the Beilinson conjectures for p-adic L-functions [Bes00]. The work of Besser extends a construction initiated by Gros [Gro90]. The cohomology defined by Besser is called the *rigid syntomic*[2] cohomology, denoted by $H^n_{\text{syn}}(X, i)$. Roughly it is defined as follows: let

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[1] Here we assume that the weight filtration is part of the definition. This is not the case in the original definition by Deligne, where only the Hodge filtration was considered. See [Beĭ86b] for a complete discussion.

[2] The word *rigid* is due to the fact that the rigid cohomology of Berthelot plays a role in the definition. The word *syntomic* comes from the work of Fontaine-Messing [FM87] where the syntomic site was used to define a cohomology theory strictly related to the one of Besser in the smooth and projective case.

$R\Gamma_{\mathrm{rig}}(X_s)$ (resp. $R\Gamma_{\mathrm{dR}}(X_\eta)$) be a complex of $\mathbb{Q}_p$-vector spaces whose cohomology is the rigid (resp. de Rham) cohomology of the special fiber X_s (resp. generic fiber X_η) of X, then

$$H^n_{\mathrm{syn}}(X,i) = H^{n-1}(\mathrm{Cone}(f : R\Gamma_{\mathrm{rig}}(X_s) \oplus F^i R\Gamma_{\mathrm{dR}}(X_\eta) \to R\Gamma_{\mathrm{rig}}(X_s) \oplus R\Gamma_{\mathrm{rig}}(X_s)) \ ,$$

where $f(x,y) = (x - \phi(x)/p^i, sp(y) - x)$, ϕ is the Frobenius map, sp is the Berthelot's specialization map.

There is a regulator map for this theory and one can also interpret rigid syntomic cohomology as an absolute cohomology [Ban02, CCM12].

The aim of the present paper is to represent rigid syntomic cohomology in the triangulated category of motives by a ring object $\mathbb{E}_{\mathrm{syn}}$. This allows one to prove that rigid syntomic cohomology is a Bloch-Ogus theory and satisfies h-descent (*i.e.* proper and fppf descent). In particular, we obtain that the Gysin map is compatible with the direct image of cycles as conjectured by Besser [Bes12, Conjecture 4.2]. We can say that this paper is the natural push-out of the work of the first author in collaboration with Cisinski [CD12a] and that of the second author in collaboration with Chiarellotto and Ciccioni [CCM12].

Let us review in more detail the content of this work.
First we recall some results of the motivic homotopy theory. Let S be a base scheme (noetherian and finite dimensional). To any object $\mathbb{E}$ in $DM_{\mathrm{B}}(S)$ we can associate a bi-graded cohomology theory

$$\mathbb{E}^{n,i}(X) := \mathrm{Hom}_{DM_{\mathrm{B}}(S)}(M(X), \mathbb{E}(i)[n])$$

where $M(X) := \pi_! \pi^! \mathbb{1}_S$ is the (covariant) motive of $\pi : X \to S$. The cohomology defined by the unit object $\mathbb{1}_S$ of the monoidal category $DM_{\mathrm{B}}(S)$ represents rational motivic cohomology denoted by H_{B}. When X is regular, $H^{n,i}_{\mathrm{B}}(X)$ coincides with the original definition of Beilinson using Adams operations on rational Quillen K-theory.
The category of Beilinson motives $DM_{\mathrm{B}}(S)$ can be constructed using some homotopical machinery starting with the category $\mathrm{C}(S,\mathbb{Q})$ of complexes of $\mathbb{Q}$-linear pre-sheaves on the category of affine and smooth S-schemes (see § 1). An object of $DM_{\mathrm{B}}(S)$ should be thought of as a cohomology theory on the category of S-schemes which is $\mathbb{A}^1$-homotopy invariant, satisfies the Nisnevich excision and is oriented (in the sense of remark 1.4.11 point (1)).

The category of Beilinson motives is monoidal. Monoids with respect to this tensor structure corresponds to cohomology theory equipped with a ring structure. Following the general terminology of motivic homotopy theory, we call such a monoid a *motivic ring spectrum* (Def. 2.1.1). Given such an object $\mathbb{E}$, the associated cohomology theory $\mathbb{E}^{n,i}(X)$ is naturally a bi-graded $\mathbb{Q}$-linear algebra satisfying the following properties:

(1) *Higher cycle class/regulator.* – The unit section of the ring spectrum $\mathbb{E}$ induces a canonical morphism, called regulator:

$$\sigma : H^{n,i}_{\mathrm{B}}(X) \to \mathbb{E}^{n,i}(X)$$

which is functorial in X and compatible with products.

(2) *Gysin.* – For any projective morphism $f : Y \to X$ between smooth S-schemes there is a (functorial) morphism

$$f_* : \mathbb{E}^{n,i}(Y) \to \mathbb{E}^{n-2d,i-d}(X).$$

where d is the dimension of f.

(3) *Projection formula.* – For f as above and any pair $(x,y) \in \mathbb{E}^{*,*}(X) \times \mathbb{E}^{*,*}(Y)$, one has:

$$f_*(f^*(x).y) = x.f_*(y).$$

(3') *Degree formula.* – For any finite morphism $f : Y \to X$ between smooth connected S-schemes, and any $x \in \mathbb{E}^{n,i}(X)$,

$$f_* f^*(x) = d.x$$

where d is the degree of the function fields extension associated with f.

(4) *Excess intersection formula.* – Consider a cartesian square of smooth S-schemes:

$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "p"] & X \end{tikzcd}$$

such that p is projective. Let ξ be the excess intersection bundle associated with that square and let e be its rank. Then for any $y \in \mathbb{E}^{*,*}(Y)$, one gets:

$$f^*p_*(y) = q_*(c_e(\xi).g^*(y)).$$

(5) The regulator map σ is natural with respect to the Gysin functoriality.

(5') The regulator map σ induces a Chern character

$$\mathrm{ch_n} : K_n(X)_{\mathbb{Q}} \to \bigoplus_{i\in\mathbb{Z}} \mathbb{E}^{2i-n,i}(X)$$

which satisfies the (higher) Riemann-Roch formula of Gillet (see [Gil81])

(6) *Descent.* – The cohomology $\mathbb{E}^{n,i}$ admits a functorial extension to diagrams of S-schemes and satisfies cohomological descent for the h-topology[3]: given any hypercover $p : \mathcal{X} \to X$ for the h-topology, the induced morphism:

$$p^* : \mathbb{E}^{n,i}(X) \to \mathbb{E}^{n,i}(\mathcal{X})$$

is an isomorphism.[4]

(7) *Bloch-Ogus theory.* – One can associate with $\mathbb{E}$ a canonical homology theory, the Borel-Moore $\mathbb{E}$-homology. For any separated S-scheme X with structural morphism f, and any pair of integers (n, i), put:

$$\mathbb{E}^{\mathrm{BM}}_{n,i}(X) = \mathrm{Hom}\big(\mathbb{1}_S, f_*f^!\mathbb{E}(-i)[-n]\big).$$

Then, the pair $(\mathbb{E}, \mathbb{E}^{\mathrm{BM}})$ is a *twisted Poincaré duality theory with support* in the sense of Bloch and Ogus (cf [BO74]). Moreover Borel-Moore $\mathbb{E}$-homology is contravariantly functorial with respect to smooth morphisms.

These properties follow easily from the results proved in [CD12b] and [Dég08]. We collect them in Section 2.

Since our aim is to prove that rigid syntomic cohomology satisfies the Bloch-Ogus formalism, we just need to represent it as a motivic ring spectrum. Thus we prove the following criterion, which is the main result of the first section. Before stating it we introduce the following notation: for any complex $E \in \mathrm{C}(S, \mathbb{Q})$ and X/S smooth and affine let

$$H^n(X, E) := H^n(E(X)) .$$

Theorem (cf. Prop. 1.4.10). *Let $(E_i)_{i\in\mathbb{N}}$ be a family of complexes in $\mathrm{C}(S,\mathbb{Q})$ forming a $\mathbb{N}$-graded commutative monoid together with a section $c : \mathbb{Q}[0] \to E_1(\mathbb{G}_m)[1]$ satisfying the following properties:*

(1) Excision. *– Let E_i^{Nis} be the associated Nisnevich sheaves. For any integer i and any X/S affine and smooth, $H^n(X, E_i) \simeq H^n_{Nis}(X, E_i^{Nis})$.*

(2) Homotopy. *– For any integer i and any X/S affine and smooth, $H^n(X, E_i) \simeq H^n(\mathbb{A}^1_X, E_i)$.*

(3) Stability. *– Let $\bar{c}$ be the image of c in $H^1(\mathbb{G}_m, E_1)$. For any smooth S-scheme X and any pair of integers (n, i) the following map*[5]

$$H^n(X, E_i) \to \frac{H^{n+1}(X \times \mathbb{G}_m, E_{i+1})}{H^{n+1}(X, E_{i+1})}, \quad x \mapsto \pi_X\big(x \times \bar{c}\big)$$

[3] The h-topology was introduced by Voevodsky. Recall that covers for this topology are given by morphisms of schemes which are universal topological epimorphism.

[4] One deduces easily from this isomorphism the usual descent spectral sequence.

[5] We let $p : X \times \mathbb{G}_m \to X$ be the canonical projection and π_X following quotient map:

$$0 \to H^n(X, E_i) \xrightarrow{p^*} H^n(X \times \mathbb{G}_m, E_i) \xrightarrow{\pi_X} \frac{H^{n+1}(X \times \mathbb{G}_m, E_{i+1})}{H^{n+1}(X, E_{i+1})} \to 0 .$$

is an isomorphism.

(4) Orientation.– *Let $u : \mathbb{G}_m \to \mathbb{G}_m$ be the inverse map of the group scheme $\mathbb{G}_m$, and denote by $\bar{c}'$ the image of c in the group $H^1(\mathbb{G}_m, E_1)/H^1(S, E_1)$. The following equality holds: $u^*(\bar{c}') = -\bar{c}'$.*

Then there exists a motivic ring spectrum $\mathbb{E}$ together with canonical isomorphisms

$$\mathrm{Hom}_{DM_{\mathrm{B}}(S)}(M(X), \mathbb{E}(i)[n]) \simeq H^n(X, E_i)$$

for integers $(n, i) \in \mathbb{Z} \times \mathbb{N}$, functorial in the smooth S-scheme X and compatible with products. Moreover, $\mathbb{E}$ depends functorially on $(E_i)_{i\in\mathbb{N}}$ and c.

The main difficulty of the above result is that the monoid structure on E_i is defined at the level of complexes of pre-sheaves and not just in the homotopy category. Using this result we can prove (in Section 2) the existence of motivic ring spectra representing several cohomology theories. First we prove that for any algebraic scheme X, defined over a field of characteristic zero, there is a motivic ring $\mathbb{E}_{\mathrm{FdR}}$ such that $\mathbb{E}^{n,i}_{\mathrm{FdR}}(X) \simeq F^i H^n_{\mathrm{dR}}(X)$ is the i-th step of the Hodge filtration of the de Rham cohomology of X as defined by Deligne [Del74]. Then we prove that the rigid cohomology of Berthelot is also represented by a motivic ring spectrum $\mathbb{E}_{\mathrm{rig}}$. As we already mentioned the rigid syntomic cohomology of Besser is defined using a kind of mapping cone complex whose components are differential graded algebras (namely it is the homotopy limit of the diagram in 3.5.1). Thus we cannot apply directly the above criterion since we would need to define a multiplication on the cone compatible with that of its components. To go around this problem we prove that a homotopy limit of motivic ring spectra is a motivic ring spectrum. Hence the rigid syntomic cohomology can be represented by a motivic ring spectrum as claimed.

As already mentioned, the existence of $\mathbb{E}_{\mathrm{syn}}$ allows us to naturally extend the rigid syntomic cohomology to singular schemes. By *devissage*, we show how to compute the syntomic cohomology of a semi-stable curve. We warn the reader that this is (probably) not the correct way to extend the cohomology to a semistable curve in the perspective of the theory of p-adic L-functions.

In passing we show some results about what we call the *absolute rigid cohomology* given by

$$H^n_\phi(X, i) := \mathrm{Hom}_{D^b(\text{F-isoc})}(\mathbb{1}, R\Gamma(X)(i)[n])$$

where $R\Gamma(X)$ is a complex of F-isocrystals such that $H^n(R\Gamma(X)) = H^n_{\mathrm{rig}}(X)$, for X a scheme over a perfect field k.

The last application of the representability theorem of rigid syntomic cohomology is the existence of a natural theory of *rigid syntomic coefficients* for R-schemes (Section 3.8). Using the techniques of [CD12b, sec. 17], we set up the theory of *rigid syntomic modules*: over any R-scheme X, they are modules (in a strict homotopical sense) over the ring spectrum $\mathbb{E}_{\mathrm{syn},X}$ obtained by pullback along the structural morphism of X/R. The corresponding category $\mathbb{E}_{\mathrm{syn}}\text{-mod}_X$ for various R-schemes X, shares many of the good properties of the category DM_{B}, such as the complete Grothendieck six functors formalism. It receives a natural realization functor from DM_{B}, which is triangulated, monoidal (and commutes with f^* and $f_!$).

This construction might be the main novelty of our representability theorem. However, to be complete we should relate these modules with more concrete categories of coefficients, probably related with F-isocrystals. This relation will be investigated in a future work.

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1. Motivic homotopy theory

In this section we first recall a basic construction of motivic homotopy theory, the category of Morel motives (Def. 1.3.2) – the reader is referred to [CD12b] for more details. Then we prove a criterion for the representability of a cohomology theory by a ring spectrum. This criterion is new and it generalizes an analogous result from [CD12a].

Throughout this section, S will be a base scheme, assumed to be noetherian finite dimensional and Λ will be a ring of coefficients. We will denote by Sm/S either the category of smooth S-schemes of finite type or the category of such schemes which in addition are affine (absolutely). Note that equipped with the Nisnevich topology, the two induced topoi are equivalent.

1.1. **The effective $\mathbb{A}^1$-derived category.**

1.1.1. We let $\mathrm{PSh}(S,\Lambda)$ be the category of presheaves of Λ-modules on Sm/S and $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$ the category of complexes of such presheaves. Given such a complex K, a smooth S-scheme X and an integer $n \in \mathbb{Z}$, we put:

$$H^n(X,K) := H^n(K(X)).$$

This is the cohomology of K computed in the derived category of $\mathrm{PSh}(S,\Lambda)$: if we denote by $\Lambda(X)$ the presheaf of Λ-modules represented by X, we get:

$$H^n(X,K) = \mathrm{Hom}_{\mathrm{D}(\mathrm{PSh}(S,\Lambda))}(\Lambda(X), K[n]).$$

A closed pair will be a couple (X,Z) such that X is a smooth S-scheme and Z is a closed subscheme of X – in fact one requires that X and $(X-Z)$ are in Sm/S. We also define the n-th cohomology group of (X,Z) – equivalently: of X with support in Z – with coefficients in K as:

$$H^n_Z(X,K) := H^{n-1}\big(\mathrm{Cone}(K(X) \to K(X-Z))\big).$$

A morphism of closed pairs $f : (Y,T) \to (X,Z)$ is a morphism of schemes $f : Y \to X$ such that $f^{-1}(Z) \subset T$. We say f is *excisive* if it is tale, $f^{-1}(Z) = T$ and f induces an isomorphism $T_{red} \to Z_{red}$. The cohomology groups $H^*_Z(X,K)$ are contravariant in (X,Z) with respect to morphisms of closed pairs.

Definition 1.1.2. Let K be a complex of $\mathrm{PSh}(S,\Lambda)$.

(1) We say that K is *Nis-local* if for any excisive morphism of closed pairs $f : (Y,T) \to (X,Z)$, the pullback morphism

$$f^* : H^*_Z(X,K) \to H^*_T(Y,K)$$

is an isomorphism.

(2) We say that K is $\mathbb{A}^1$-*local* if for any smooth S-scheme X, the pullback induced by the canonical projection p of the affine line over X

$$p^* : H^*(X,K) \to H^*(\mathbb{A}^1_X,K)$$

is an isomorphism.

Following Morel, we define the *effective* $\mathbb{A}^1$-*derived category over* S *with coefficients in* Λ as the full subcategory of $\mathrm{D}(\mathrm{PSh}(S,\Lambda))$ made by complexes which are Nis-local and $\mathbb{A}^1$-local. We will denote it by $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S,\Lambda)$.

1.1.3. Let us recall the following facts on the category defined above:

(1) Let $\mathrm{Sh}(S,\Lambda)$ be the category of sheaves of Λ-modules on Sm/S for the Nisnevich topology. Then $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S,\Lambda)$ is equivalent to the $\mathbb{A}^1$-localization of the derived category $\mathrm{D}(\mathrm{Sh}(S,\Lambda))$, as defined in [CD12a, § 1.1].

This comes from the fact that the pair of adjoint functors, whose left adjoint is the associated Nisnevich sheaf a, induces a derived adjunction

$$a : \mathrm{D}(\mathrm{PSh}(S,\Lambda)) \leftrightarrows \mathrm{D}(\mathrm{Sh}(S,\Lambda)) : \mathcal{O}$$

whose right adjoint $\mathcal{O}$ is fully faithful with essential image the complexes which are Nis-local – this is classical see for example [CD12b, 5.2.10 and 5.2.13]. In particular, Nis-local complexes can be described as those complexes K which satisfy Nisnevich descent: for any Nisnevich hypercover $P_\bullet \to X$ of any smooth S-scheme X, the induced map:

$$K(X) \to \mathrm{Tot}\big(K(P_\bullet)\big)$$

is a quasi-isomorphism – the right hand-side is the total complex associated with the obvious double complex.

(2) The fact that the category $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S,\Lambda)$ can be handled in practice comes from its description as the homotopy category associated with an explicit model category structure on the category $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$ of complexes on the Grothendieck abelian category $\mathrm{PSh}(S,\Lambda)$:
   - *Weak equivalences* (also called weak $\mathbb{A}^1$-equivalences) are the morphisms of complexes f such that for any complex K which is $\mathbb{A}^1$-local and Nis-local, $\mathrm{Hom}_{\mathrm{D}(\mathrm{PSh}(S,\Lambda))}(f,K)$ is an isomorphism.
   - Fibrant objects are the complexes which are Nis-local and $\mathbb{A}^1$-local. *Fibrations* are the morphisms of complexes which are epimorphisms and whose kernel is fibrant.

For the proof that this defines a model category, we refer the reader to [CD09a]: we first consider the model category structure associated with the Grothendieck abelian category $\mathrm{PSh}(S,\Lambda)$ (see [CD09a, Ex. 2.3]) and we localize it with respect to Nisnevich hypercovers and $\mathbb{A}^1$-homotopy ([CD09a, Section 4]). Let us recall that a typical example of *cofibrant objects* for this model structure are the presheaves of the form $\Lambda(X)$ for a smooth S-scheme X.

We derive from this model structure the existence of fibrant (resp. cofibrant) resolutions: associated with a complex of presheaves K, we get a fibrant K_f (resp. cofibrant K_c) and a map

$$K \to K_f \qquad (\text{resp. } K_c \to K),$$

which is a cofibration (resp. fibration) and a weak $\mathbb{A}^1$-equivalence. These resolutions can be chosen to be natural in K.

This can be used to derive functors. In particular, the natural tensor product $\otimes$ of $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$ as well as its internal complex morphism $\underline{\mathrm{Hom}}$ can be derived using the formulas:

$$K \otimes^{\mathbf{L}} L = K_c \otimes L_c, \qquad \mathbf{R}\,\underline{\mathrm{Hom}}(K,L) = \underline{\mathrm{Hom}}(K_c, L_f);$$

see [CD09a, Sections 3 and 4].[6]

1.2. The $\mathbb{A}^1$-derived category.

1.2.1. We define the *Tate object* as the following complex of presheaves of Λ-modules:

$$\Lambda(1) := \mathrm{coKer}(\Lambda \xrightarrow{s_{1*}} \Lambda(\mathbb{G}_m))[-1] \tag{1.2.1.a}$$

where s_1 is the unit section of the group scheme $\mathbb{G}_m$, considered as an S-scheme. Given a complex K and an integer $i \geq 0$, we denote by $K(i)$ the tensor product of K with the i-th tensor power of $\Lambda(1)$ (on the right).

As usual in the general theory of motives, one is led to invert the object $\Lambda(1)$ for the tensor product. In the context of motivic homotopy theory, this is done using the construction of spectra, borrowed from algebraic topology.

For any integer $i > 0$, we will denote by Σ_i the group of permutations of the set $\{1, \cdots, i\}$, $\Sigma_0 = 1$.

Definition 1.2.2. A *Tate spectrum* (over S with coefficients in Λ), is a sequence $\mathbb{E} = (E_i, \sigma_i)_{i\in\mathbb{N}}$ such that:

- for each $i \in \mathbb{N}$, E_i is a complex of $\mathrm{PSh}(S,\Lambda)$ equipped with an action of Σ_i,
- for each $i \in \mathbb{N}$, σ_i is a morphism of complexes
$$\sigma_i : E_i(1) \to E_{i+1},$$
called the *suspension map* (in degree n).
- For any integers $i \geq 0$, $r > 0$, the map induced by the morphisms $\sigma_i, \cdots, \sigma_{i+r}$:
$$E_i(r) \to E_{i+r}$$

[6] Note in particular that, according to [CD09a, Proposition 4.11], the model category described above is a monoidal model category which satisfies the monoid axiom.

is compatible with the action of $\Sigma_i \times \Sigma_r$, given on the left by the structural Σ_i-action on E_i and the action of Σ_r via the permutation isomorphism of the tensor structure on $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$, and on the right via the embedding $\Sigma_i \times \Sigma_r \to \Sigma_{i+r}$ obtained by identifying the sets $\{1,...,i+r\}$ and $\{1,...,i\} \sqcup \{1,...,r\}$.

A morphism of Tate spectra $f : \mathbb{E} \to \mathbb{F}$ is a sequence of Σ_i-equivariant maps $(f_i : E_i \to F_i)_{n\in\mathbb{N}}$ compatible with the suspension maps. The corresponding category will be denoted by $\mathrm{Sp}(S,\Lambda)$.

A morphism f as above is called a *level weak equivalence* if for any integer $i \geq 0$, the morphism of complexes f_i is a quasi-isomorphism. We denote by $D_{\mathrm{Tate}}(S,\Lambda)$ the localization of $\mathrm{Sp}(S,\Lambda)$ with respect to level weak equivalences (See [CD12a, Sec. 1.4]).

Complexes and spectra are linked by a pair of adjoint functors $(\Sigma^\infty, \Omega^\infty)$ defined respectively for a complex K and a Tate spectrum $\mathbb{E}$ as follows:

$$\Sigma^\infty K := (K(i))_{i\in\mathbb{N}}\ , \qquad \Omega^\infty(\mathbb{E}) = E_0, \tag{1.2.2.a}$$

where $K(i)$ is equipped with the action of Σ_i by its natural action through the symmetry isomorphism of the tensor structure on $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$.

1.2.3. The category of Tate spectra can be described using the category of symmetric sequences of $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$: the objects of this category are the sequences $(E_i)_{i\in\mathbb{N}}$ of complexes of $\mathrm{PSh}(S,\Lambda)$ such that E_i is equipped with an action of Σ_i. This is a Grothendieck abelian category on which on can construct a closed symmetric monoidal structure (see [CD09a, Section 7]). Moreover, the obvious symmetric sequence

$$\mathrm{Sym}(\Lambda(1)) := (\Lambda(i))_{i\in\mathbb{N}}$$

has a canonical structure of a commutative monoid.

The category $\mathrm{Sp}(S,\Lambda)$ is equivalent to the category of modules over $\mathrm{Sym}(\Lambda(1))$ (see again *loc. cit.*). Therefore, it is formally a Grothendieck abelian category equipped with a closed symmetric monoidal structure. Note that the tensor product can be described by the following universal property: to give a morphism of Tate spectra $\mu : \mathbb{E} \otimes \mathbb{F} \to \mathbb{G}$ is equivalent to give a family of morphisms

$$\mu_{i,j} : E_i \otimes F_j \to G_{i+j}$$

which is $\Sigma_i \times \Sigma_j$-equivariant and compatible with the suspension maps (see *loc. cit.* Remark 7.2).

Definition 1.2.4. Let $\mathbb{E}$ be a Tate spectrum over S with coefficients in Λ.

(1) We say that $\mathbb{E}$ is *Nis-local* (resp. $\mathbb{A}^1$*-local*) if for any integer $i \geq 0$, the complex E_i is Nis-local (resp. $\mathbb{A}^1$-local).
(2) We say that $\mathbb{E}$ is a *Tate Ω-spectrum* if the morphism of $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S,\Lambda)$ induced by adjunction from σ_i:
$$E_i \to \mathbf{R}\,\underline{\mathrm{Hom}}(\Lambda(1), E_{i+1})$$
is an isomorphism (i.e. a weak $\mathbb{A}^1$-equivalence).

For short, we say that $\mathbb{E}$ is *stably fibrant* if it is an Ω-spectrum which is Nis-local and $\mathbb{A}^1$-local.

We define the $\mathbb{A}^1$*-derived category over S with coefficients in* Λ, denoted by $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$, as the full subcategory of $D_{\mathrm{Tate}}(S,\Lambda)$ made by the stably fibrant Tate spectra.

1.2.5. Recall the following facts on the previous construction:

(1) The construction of $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ through spectra is a classical construction derived from algebraic topology (see [Hov01]). In particular, the monoidal model structure on the category $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$ induces a canonical monoidal model structure on $\mathrm{Sp}(S,\Lambda)$ whose homotopy category is precisely $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$. It is called the *stable model category.*

Therefore $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ is a symmetric monoidal triangulated category with internal Hom. Moreover, the adjoint functors (1.2.2.a) can be derived:

$$\Sigma^\infty : \mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S,\Lambda) \leftrightarrows \mathrm{D}_{\mathbb{A}^1}(S,\Lambda) : \Omega^\infty. \tag{1.2.5.a}$$

The functor Σ^∞ is monoidal.[7] Recall also that given a Tate Ω-spectrum $\mathbb{E}$ as above and an integer $i \geq 0$, we get:

$$\Omega^\infty(\mathbb{E}(i)) = E_i. \tag{1.2.5.b}$$

We will simply denote by Λ or $\mathbb{1}$ the unit of $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ – instead of $\Sigma^\infty \Lambda$.

(2) In fact the triangulated categories of the form $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ for various schemes S are not only closed monoidal but they are equipped with the complete formalism of Grothendieck six operations

$$(f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{Hom}})$$

as established by Ayoub in [Ayo07].[8]

1.3. **Triangulated mixed motives.**

1.3.1. In this section, Λ is a $\mathbb{Q}$-algebra.

We recall the construction of Morel for deriving the triangulated category of mixed motives from the category $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ (see [CD12b, 16.2] for details).

Let us consider the inverse map u of the multiplicative group scheme $\mathbb{G}_m$, corresponding to the map:

$$\mathcal{O}_S[t,t^{-1}] \to \mathcal{O}_S[t,t^{-1}] \ , \qquad t \mapsto t^{-1}.$$

Recall from formula (1.2.1.a) the decomposition $\Lambda(\mathbb{G}_m) = \Lambda \oplus \Lambda(1)[1]$, considered in $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$. Given this decomposition, the map $u_* : \Lambda(\mathbb{G}_m) \to \Lambda(\mathbb{G}_m)$ can be written in matrix form as:

$$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon_1 \end{pmatrix}$$

Because $\Lambda(1)[1]$ is $\otimes$-invertible in $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$, there exists a unique endomorphism ϵ of Λ in $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ such that $\epsilon_1 = \epsilon(1)[1]$.

Because $u^2 = 1$, we get $\epsilon^2 = 1$. Thus we can define two complementary projectors in $\mathrm{End}_{\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)}(\Lambda)$:

$$p_+ = \frac{1}{2}.(1_\Lambda - \epsilon), \qquad p_- = \frac{1}{2}.(1_\Lambda + \epsilon).$$

Given any object $\mathbb{E}$ in $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$, we deduce projectors $p_+ \otimes \mathbb{E}$, $p_- \otimes \mathbb{E}$ of $\mathbb{E}$. Because $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ is pseudo-abelian[9], we deduce a canonical decomposition:

$$\mathbb{E} = \mathbb{E}_+ \oplus \mathbb{E}_-$$

where $\mathbb{E}_+$ (resp. $\mathbb{E}_-$) is the image of $p_+ \otimes E$ (resp. $p_- \otimes E$). The following triangulated category was introduced by Morel.

Definition 1.3.2. An object $\mathbb{E}$ in $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ will be called a Morel motive if $\mathbb{E}_- = 0$. We denote by $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)_+$ the full subcategory of $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ made by Morel motives.

Note that according to the above, the fact $\mathbb{E}$ is a Morel motive is equivalent to the property:

$$\epsilon \otimes \mathbb{E} = -1_{\mathbb{E}}; \tag{1.3.2.a}$$

in other words, ϵ acts as -1 on $\mathbb{E}$.

1.3.3. Recall the following facts, which legitimate the terminology of "Morel motives":

(1) Obviously, the category $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)_+$ is a triangulated monoidal sub-category of $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$. Moreover, the six operations on $\mathrm{D}_{\mathbb{A}^1}(-,\Lambda)$ induce similar operations on $\mathrm{D}_{\mathbb{A}^1}(-,\Lambda)_+$ which satisfy all of the six functors formalism.

---

[7] In fact, the homotopy category $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ equipped with its left derived functor Σ^∞, is universal for the property that Σ^∞ is monoidal and $\Sigma^\infty(K(1))$ is $\otimes$-invertible (see again [Hov01]).

[8] Ayoub treats only the case where f is quasi-projective for the existence of the adjoint pair $(f_!, f^!)$. The general case can be obtained by using the classical construction of Deligne as explained in [CD12b, section 2.2]. The reader will also find a summary of the six operations formalism in *loc. cit.* Theorem 2.4.50.

[9] This is, for example, an application of the fact it is a triangulated category with countable direct sums (cf [Nee01, 1.6.8]).

(2) According to [CD12b, 16.2.13], there is an equivalence of triangulated monoidal categories:
$$\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)_+ \simeq DM_{\mathrm{B}}(S,\Lambda)$$
where $DM_{\mathrm{B}}(S,\Lambda)$ is the triangulated category of Beilinson motives introduced in [CD12b, Def. 14.2.1]. In $DM_{\mathrm{B}}(S,\Lambda)$, given a smooth S-scheme X, we simply denote by $M(X)$ the object corresponding to $\Sigma^\infty \Lambda(X)$ and call it the *motive* of X.
Concretely, the above isomorphism means that when S is regular, for any smooth S-scheme X and any pair $(n,i) \in \mathbb{Z}^2$, one has a canonical isomorphism:

(1.3.3.a) $$\mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)_+}(\Sigma^\infty \Lambda(X), \Lambda(i)[n]) \simeq K^{(i)}_{2i-n}(X) \otimes_{\mathbb{Q}} \Lambda$$

where $K^{(i)}_{2i-n}(X)$ denotes the i-th Adams subspace of the rational Quillen K-theory of X in homological degree $(2i-n)$.[10]

Note in particular that according to the coniveau spectral sequence in K-theory and a computation of Quillen, a particular case of the above isomorphism is the following one:

(1.3.3.b) $$\mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)_+}(\Sigma^\infty \Lambda(X), \Lambda(n)[2n]) \simeq CH^n(X) \otimes_{\mathbb{Z}} \Lambda\ ,$$

where the right hand side is the Chow group of n-codimensional Λ-cycles in X (S is still assumed to be regular).

1.4. **Ring spectra.**

1.4.1. Recall that a commutative monoid in a symmetric monoidal category $(\mathcal{M}, \otimes, \mathbb{1})$ is an object M, a unit map $\eta : \mathbb{1} \to M$ and a multiplication map $\mu : M \otimes M \to M$, such that the following diagrams are commutative:

Unit:
$$\begin{tikzcd} M \arrow[r, "1\otimes\eta"] \arrow[dr, equal] & M \otimes M \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

Associativity:
$$\begin{tikzcd} M \otimes M \otimes M \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & M \otimes M \arrow[d, "\mu"] \\ M \otimes M \arrow[r, "\mu"] & M \end{tikzcd}$$

Commutativity:
$$\begin{tikzcd} M \otimes M \arrow[dr, "\mu"] \arrow[dd, "\gamma"'] & \\ & M \\ M \otimes M \arrow[ur, "\mu"'] & \end{tikzcd}$$

where γ is the obvious symmetry isomorphism.

Definition 1.4.2. A *weak ring spectrum* (resp. *ring spectrum*) $\mathbb{E}$ over S is a commutative monoid in the symmetric monoidal category $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ (resp. $\mathrm{Sp}(S,\Lambda)$).[11]

1.4.3. A spectrum $\mathbb{E}$ in $\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)$ defines a bigraded cohomology theory on smooth S-schemes X by the formula:
$$\mathbb{E}^{n,i}(X) = \mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)}(\Sigma^\infty \Lambda(X), \mathbb{E}(i)[n]).$$
The structure of a weak ring spectrum on $\mathbb{E}$ corresponds to a product in cohomology, usually called the cup-product and defined as follows: given cohomology classes,
$$\alpha : \Sigma^\infty \Lambda(X) \to \mathbb{E}(i)[n], \qquad \beta : \Sigma^\infty \Lambda(X) \to \mathbb{E}(j)[m]$$
one defines the class $\alpha \cup \beta$ as the following composite:
$$\Sigma^\infty \Lambda(X) \xrightarrow{\delta_*} \Sigma^\infty \Lambda(X) \otimes \Sigma^\infty \Lambda(X) \xrightarrow{\alpha\otimes\beta} \mathbb{E}(i)[n] \otimes \mathbb{E}(j)[m] \xrightarrow{\mu} \mathbb{E}(i+j)[n+m].$$
Using this definition, one can check easily that the commutativity axiom of $\mathbb{E}$ implies the following formula:
$$\alpha \cup \beta = (-1)^{nm-ij}.\epsilon^{ij}.\beta \cup \alpha$$
where ϵ is the endomorphism of Λ introduced in Paragraph 1.3.1. In particular, if $\mathbb{E}$ is a Morel motive, the product on $\mathbb{E}^{**}$ is anti-commutative with respect to the first index and commutative with respect to the second one. Note also the following result which will be used later.

[10] This formula was first obtained by Morel but the proof has not been published. In any case, this is a consequence of *loc. cit.*

[11] Ring spectra have slowly emerged in homotopy theory and the terminology is not fixed. Usually, our weak ring spectra (resp. ring spectra) are simply called ring spectra (resp. highly structured ring spectra).

Lemma 1.4.4. *Let $\mathbb{E}$ be a weak ring spectrum with unit η and multiplication μ. Then the following conditions are equivalent:*

(i) *$\mathbb{E}$ is a Morel motive.*
(ii) *$\eta \circ \epsilon = -\eta$.*

Proof. Let us remark that according to the Unit property the following equalities hold:

$$\mu \circ (1_{\mathbb{E}} \otimes \eta) = 1_{\mathbb{E}},$$
$$\mu \circ (1_{\mathbb{E}} \otimes (\eta \circ \epsilon)) = \epsilon \otimes \mathbb{E}.$$

Thus the equivalence between (i) and (ii) directly follows from relation (1.3.2.a) characterizing Morel motives. ☐

Remark 1.4.5. Of course, a ring spectrum induces a weak ring spectrum. Concretely, in the non weak case, one requires that the diagrams of Paragraph 1.4.1 commutes in the mere category of spectra, and not only up to weak homotopy. This makes the construction of ring spectra more difficult than usual weak ring spectra.

1.4.6. Let us denote by $\mathrm{Sp}^{\mathrm{ring}}(S,\Lambda)$ the category of ring spectra. Because the category $\mathrm{Sp}(S,\Lambda)$ is a complete and cocomplete monoidal category, $\mathrm{Sp}^{\mathrm{ring}}(S,\Lambda)$ is complete and cocomplete. Moreover, the forgetful functor:

$$U : \mathrm{Sp}^{\mathrm{ring}}(S,\Lambda) \to \mathrm{Sp}(S,\Lambda)$$

admits a left adjoint which we denote by F. The following result appears in [CD12b, Th. 7.1.8].

Theorem 1.4.7. *Assume Λ is a $\mathbb{Q}$-algebra.*

Then the category $\mathrm{Sp}^{\mathrm{ring}}(S,\Lambda)$ is a model category whose weak equivalences (resp. fibrations) are the maps f such that $U(f)$ is a weak equivalence (resp. stable fibration) in the stable model category $\mathrm{Sp}(S,\Lambda)$ (see Par. 1.2.5).

We denote by $\mathcal{H}\mathrm{o}(\mathrm{Sp}^{\mathrm{ring}}(S,\Lambda))$ the homotopy category associated with this model category.

1.4.8. For a given $\mathbb{Q}$-algebra Λ, recall the following consequences of this theorem:

(1) The pair of adjoint functors (F,U) can be derived and induces adjoint functors:

$$\mathbf{L}F : \mathrm{D}_{\mathbb{A}^1}(S,\Lambda) \leftrightarrows \mathcal{H}\mathrm{o}(\mathrm{Sp}^{\mathrm{ring}}(S,\Lambda)) : U$$

The essential image of the functor U lies in the category of weak ring spectra. However, it is not essentially surjective on that category.

(2) As any homotopy category of a model category, the homotopy category $\mathcal{H}\mathrm{o}(\mathrm{Sp}^{\mathrm{ring}}(S,\Lambda))$ admits homotopy limits and colimits (see [Cis03, Intro. Th. 1]). In other words, any diagram of $\mathrm{Sp}^{\mathrm{ring}}(S,\Lambda)$ admits a homotopy limit and a homotopy colimit.

1.4.9. A commutative monoid in the category $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$ is usually called a commutative differential graded Λ-algebra with coefficients in the abelian monoidal category $\mathrm{PSh}(S,\Lambda)$.

A $\mathbb{N}$-graded commutative monoid in $\mathrm{C}(\mathrm{PSh}(S,\Lambda))$ is a sequence $(E_i)_{i\in\mathbb{N}}$ of complexes of presheaves equipped with a unit map $\eta : \Lambda \to E_0$ and multiplication maps $\mu_{ij} : E_i \otimes E_j \to E_{i+j}$ for any pair of integers (i,j) such that the following diagrams commute:

Unit: Associativity: Commutativity:

$$\begin{tikzcd} E_i \arrow[r, "1\otimes\eta"] \arrow[dr, equal] & E_i \otimes E_0 \arrow[d, "\mu_{i,0}"] \\ & E_i \end{tikzcd} \qquad \begin{tikzcd} E_i \otimes E_j \otimes E_k \arrow[r, "1\otimes\mu_{jk}"] \arrow[d, "\mu_{ij}\otimes 1"'] & E_i \otimes E_{j+k} \arrow[d, "\mu_{i,j+k}"] \\ E_{i+j} \otimes E_k \arrow[r, "\mu_{i+j,k}"] & E_{i+j+k} \end{tikzcd} \qquad \begin{tikzcd} E_i \otimes E_j \arrow[dr, "\mu_{ij}"] \arrow[dd, "\gamma_{ij}"'] & \\ & E_{i+j} \\ E_j \otimes E_i \arrow[ur, "\mu_{j,i}"'] & \end{tikzcd}$$

where γ_{ij} is the obvious symmetry isomorphism. We then define bigraded cohomology groups for any smooth S-scheme X and any couple of integers (n,i):

$$H^n(X,E_i) = H^n(E_i(X)).$$

The above monoid structure induces an exterior product on these cohomology groups:

$$H^n(X, E_i) \otimes H^m(Y, E_j) \to H^{n+m}(X \times_S Y, E_{i+j}), \quad (x, y) \mapsto x \times y.$$

Given any smooth S-scheme X, we let $p : X \times \mathbb{G}_m \to X$ be the canonical projection and consider for the next statement the following split exact sequence:

$$0 \to H^n(X, E_i) \xrightarrow{p^*} H^n(X \times \mathbb{G}_m, E_i) \xrightarrow{\pi_X} \tilde{H}^n(X \times \mathbb{G}_m, E_i) \to 0 \ ,$$

where $\tilde{H}^n(X \times \mathbb{G}_m, E_i) := \mathrm{Coker}(p^*)$ and π_X is the canonical projection.

Proposition 1.4.10. *Suppose given a $\mathbb{N}$-graded commutative monoid $(E_i)_{i \in \mathbb{N}}$ in $\mathrm{C}(\mathrm{PSh}(S, \Lambda))$ as above together with a section c of $E_1[1]$ over $\mathbb{G}_m$ satisfying the following properties:*

1. Excision*.– For any integer i, E_i is Nis-local.*
2. Homotopy*.– For any integer i, E_i is $\mathbb{A}^1$-local.*
3. Stability*.– Let $\bar{c}$ be the image of c in $H^1(\mathbb{G}_m, E_1)$. For any smooth S-scheme X and any pair of integers (n, i) the following map*

$$H^n(X, E_i) \to \tilde{H}^{n+1}(X \times \mathbb{G}_m, E_{i+1}), \quad x \mapsto \pi_X\big(x \times \bar{c}\big)$$

is an isomorphism.

Then there exists a ring spectrum $\mathbb{E}$ which is a stably fibrant Tate spectrum together with canonical isomorphisms

$$\mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)}(\Sigma^\infty \Lambda(X), \mathbb{E}(i)[n]) \simeq H^n(X, E_i) \tag{1.4.10.a}$$

for integers $(n, i) \in \mathbb{Z} \times \mathbb{N}$, functorial in the smooth S-scheme X and compatible with products. Moreover, $\mathbb{E}$ depends functorially on $(E_i)_{i \in \mathbb{N}}$ and c.

Assume Λ is a $\mathbb{Q}$-algebra. Let $u : \mathbb{G}_m \to \mathbb{G}_m$ be the inverse map of the group scheme $\mathbb{G}_m$, and denote by $\bar{c}'$ the image of c in the group $\tilde{H}^1(\mathbb{G}_m, E_1)$. Then, under the above assumptions, the following conditions are equivalent:

(i) *The Tate spectrum $\mathbb{E}$ is a Morel motive (*i.e. *defines an object in $DM_{\mathrm{B}}(S, \Lambda)$, Def. 1.3.2 and Par. 1.3.3).*

(ii) *The following equality holds in $\tilde{H}^1(\mathbb{G}_m, E_1)$: $u^*(\bar{c}') = -\bar{c}'$.*

Remark 1.4.11.

1. The two last properties should be called the *Orientation* property. In fact, they can be reformulated by saying that $\mathbb{E}$ is an oriented ring spectrum (cf [CD12b, Cor. 14.2.16]). Recall also this is equivalent to the existence of a canonical morphism of groups:
$$\mathrm{Pic}(X) \to H^2(X, E_1)$$
which is functorial in X (and even uniquely determined by c).
2. The Stability axiom can be reformulated by saying that for any $x \in H^{n+1}(X \times \mathbb{G}_m, E_{i+1})$ there exists a unique couple $(x_0, x_1) \in H^{n+1}(X, E_{i+1}) \times H^n(X, E_i)$ such that:
$$x = p^*(x_0) + x_1 \times \bar{c}'.$$
3. Though we start with a positively graded complex $(E_i)_{i \in \mathbb{N}}$ we get a cohomology theory which possibly has negative twists. These negative twists are given by the following short exact sequence for $i > 0$:
$$0 \to \mathbb{E}^{n,-i}(X) \to H^n(X \times \mathbb{G}_m^i, E_0) \to H^n(X \times \mathbb{G}_m^{i-1}, E_0) \to 0$$
where the epimorphism is given by the sum of the inclusions
$$\mathbb{G}_m^{i-1} \to \mathbb{G}_m^i$$
corresponding to set one of the coordinates of the target to 1.

Proof. We define the Tate spectrum $\mathbb{E}$ to be the complex of presheaves E_i in degree i with trivial action of Σ_i. The section c defines a map of presheaves:

$$c' : \Lambda(1) \to \Lambda(\mathbb{G}_m)[-1] \xrightarrow{c[-1]} E_1$$

where the first map is given by the canonical inclusion. We define the suspension map of $\mathbb{E}$ in degree i as the following composite:

$$\sigma_i : E_i(1) = E_i \otimes \Lambda(1) \xrightarrow{1 \otimes c'} E_i \otimes E_1 \xrightarrow{\mu_{i,1}} E_{i+1}.$$

One deduces from the commutative diagram called "Commutativity" of Paragraph 1.4.9 that the induced map $E_i(r) \to E_{i+r}$ is $\Sigma_i \times \Sigma_r$-equivariant. So that $\mathbb{E}$ is indeed a Tate spectrum.

By definition, Assumptions (1) and (2) exactly say that $\mathbb{E}$ is Nis-local and $\mathbb{A}^1$-local. It remains to check it is an Ω-spectra. In other words, the map obtained by adjunction from σ_i

$$\sigma'_i : E_i \to \mathbf{R}\,\underline{\mathrm{Hom}}(\Lambda(1), E_{i+1})$$

is an isomorphism in $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S, \Lambda)$. It is sufficient to check that for any smooth S-scheme X and any integer $n \in \mathbb{Z}$, the induced map :

$$\begin{aligned} \sigma'_{i*} : \mathrm{Hom}(\Lambda(X), E_i[n]) \to & \mathrm{Hom}(\Lambda(X), \mathbf{R}\,\underline{\mathrm{Hom}}(\Lambda(1), E_{i+1}[n]) \\ & = \mathrm{Hom}(\Lambda(X) \otimes \Lambda(1), E_{i+1}[n]), \end{aligned}$$

where the morphisms are taken in $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S, \Lambda)$, is an isomorphism. According to the definition, we can compute this map as follows:

$$\mathrm{Hom}\big(\Lambda(X), E_i[n]\big) \to \mathrm{Hom}\big(\Lambda(X) \otimes \Lambda(1), E_{i+1}[n]\big), x \mapsto x \times \bar{c}' \tag{1.4.11.a}$$

where $\bar{c}'$ is the class of the map c' in $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S, \Lambda)$. Using the fact E_i is Nis-local and $\mathbb{A}^1$-local, the source of this map is isomorphic to $H^n(X, E_i)$. Similarly, the group of morphisms

$$\mathrm{Hom}\big(\Lambda(X) \otimes \Lambda(\mathbb{G}_m), E_{i+1}[n+1]\big)$$

is isomorphic to $H^{n+1}(X \times \mathbb{G}_m, E_{i+1})$. Under this isomorphism, the target of the above map corresponds to $\tilde{H}^{n+1}(X \times \mathbb{G}_m, E_{i+1})$. Under these identifications, $\bar{c}' = \pi_X(\bar{c})$. Thus, the fact σ'_i is an isomorphism directly follows from Assumption (3).

According to this construction, the maps η and μ_{ij} induces a structure of a ring spectrum on $\mathbb{E}$ (using in particular the description of the tensor product of spectra recalled in Paragraph 1.2.3).

The isomorphism (1.4.10.a) follows using the adjunction (1.2.5.a) and the relation (1.2.5.b) applied to the Tate Ω-spectrum $\mathbb{E}$. The fact it is functorial and compatible with products is obvious from the above construction.

Let us finally consider the remaining assertion. Note that according to what was just said, the class $\bar{c}'$ introduced in the beginning of the proof coincides with the class $\bar{c}'$ which appears in the statement of the proposition. Under the isomorphism (1.4.10.a), the canonical isomorphism:

$$\mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)}(\Lambda, \mathbb{E}) \to \mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(S,\Lambda)}(\Lambda(1), \mathbb{E}(1))$$

corresponds to an isomorphism of the form

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S,\Lambda)}(\Lambda, E_0) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S,\Lambda)}(\Lambda(1), E_1) = \tilde{H}^1(\mathbb{G}_m, E_1)$$

which is a particular case of the isomorphism (1.4.11.a) considered above. Thus, it sends the unit map η of $\mathbb{E}$ to the class $\bar{c}'$. Thus the equivalence of conditions (i) and (ii) follows from Lemma 1.4.4. □

Remark 1.4.12. This proposition is an extension of the construction given in [CD12a, sec. 2.1]. The main difference is that we consider here theories in which the different twists are not necessarily isomorphic. By contrast, we require the datum of a stability class here whereas we do not need a particular choice in *op. cit.*

Note also that a similar extension has appeared in [HS10] applied to Deligne cohomology.

2. Motivic ring spectra

In this section we introduce one of the central notion of motivic homotopy theory, that of motivic ring spectrum. Our primary aim is to prove that to such an object is associated a Bloch-Ogus cohomology theory, a result which has not yet appeared in the literature of motivic homotopy theory. Moreover, we extend the formalism of Bloch-Ogus by proving many more properties, relying on some of the main constructions of motivic homotopy theory ([Dég08], [Ayo07] and [CD12b]). In the next section we will give several examples of motivic ring spectra, among them the motivic ring spectrum representing the rigid syntomic cohomology.

We fix a base scheme S (noetherian and finite dimensional) and Λ a $\mathbb{Q}$-algebra.

2.1. **Gysin morphisms and regulators.**

Definition 2.1.1. A *motivic ring spectrum* (over S) is a ring spectrum $\mathbb{E}$ which is also a Morel motive. In particular it is an object of $DM_{\text{B}}(S,\Lambda)$.

If X is an S-scheme, we will denote by

$$\mathbb{E}^{n,i}(X) := \mathrm{Hom}_{DM_{\text{B}}(S,\Lambda)}(M(X),\mathbb{E}(i)[n])$$

the associated bi-graded cohomology groups.

Remark 2.1.2. (1) In the current terminology of motivic homotopy theory, what we call a motivic ring spectrum should be called an *oriented* motivic ring spectrum (see also Remark 1.4.11). This abuse of terminology is justified as we will never consider non oriented ring spectra in this work.

(2) In the previous section, we have seen that there exists a stronger notion of ring spectrum, that of stably fibrant Tate spectrum. The ring spectra that we will construct will always satisfies this stronger assumption. Moreover, given a ring spectrum in the sense of the above definition, it is always possible to find a stably fibrant Tate spectrum which is isomorphic in $DM_{\text{B}}(S)$ to the first given one (according to Theorem 1.4.7). On the other hand, this stronger notion will not be used in this section that is why we consider above the simpler notion. The stronger notion will be needed in Section 3.8.

2.1.3. Recall that Beilinson motivic cohomology for smooth S-schemes is the cohomology represented by the unit object of $DM_{\text{B}}(S) = DM_{\text{B}}(S,\mathbb{Q})$:

$$H^{n,i}_{\text{B}}(X) := \mathrm{Hom}_{DM_{\text{B}}}(M(X),\mathbb{1}(i)[n]).$$

This group can also be described as the i-graded part for the γ-filtration of algebraic rational K-theory:

$$H^{n,i}_{\text{B}}(X) = \mathrm{gr}^i_\gamma K_{2i-n}(X)_{\mathbb{Q}}\ .$$

See [CD12b, 14.2.14].

By construction, the ringed cohomology $\mathbb{E}^{**}$ admits a canonical action of Beilinson motivic cohomology H^{**}_{B}. Concretely, for any smooth S-scheme X and any couple of integers (n,i), the unit map $\mathbb{1}\to\mathbb{E}$ induces a canonical morphism

$$\sigma_{\mathbb{E}} : H^{n,i}_{\text{B}}(X) = \mathrm{Hom}_{DM_{\text{B}}}(M(X),\mathbb{1}(i)[n]) \to \mathrm{Hom}_{DM_{\text{B}}}(M(X),\mathbb{E}(i)[n]) = \mathbb{E}^{n,i}(X) \tag{2.1.3.a}$$

which is compatible with pullbacks and products. This is the *higher cycle class map* (or equivalently the *regulator*) with values in the $\mathbb{E}$-cohomology. Note also that this map can be represented in the category $\mathrm{D}_{\mathbb{A}^1}(S,\mathbb{Q})$ as a morphism of ring spectra:

$$\sigma_{\mathbb{E}} : H_{\text{B}} \to \mathbb{E} \quad \text{(by abuse of notation we use the same symbol)}$$

which is unique according to [CD12b, 14.2.16].

When $n=i$, it gives in particular the (usual) cycle class map:

$$\sigma_{\mathbb{E}} : \mathrm{CH}^n(X) \to \mathbb{E}^{2n,n}(X) \tag{2.1.3.b}$$

which is compatible with pullbacks and products of cycles as defined in [Ful98].

2.1.4. A motivic ring spectrum $\mathbb{E}$, considered as an object of $\mathrm{D}_{\mathbb{A}^1}(S)$, is oriented (see Remark 1.4.11). Thus, one can apply to it the orientation theory of $\mathbb{A}^1$-homotopy theory (see [Dég11] in the arithmetic case).

This implies that $\mathbb{E}^{**}$ admits Chern classes, which are nothing else than the image of the Chern classes in Chow theory through the cycle class map, and satisfies the projective bundle formula (see [Dég11, 2.1.9]). One also gets a Chern character map in $\mathrm{D}_{\mathbb{A}^1}(S, \mathbb{Q})$:

$$\mathrm{ch}_{\mathrm{syn}} : KGL_{\mathbb{Q}} \xrightarrow{\mathrm{ch}} \oplus_{i \in \mathbb{Z}} H_{\mathrm{B}}(i)[2i] \xrightarrow{\sigma} \oplus_{i \in \mathbb{Z}} \mathbb{E}(i)[2i]$$

where $KGL_{\mathbb{Q}}$ is the ring spectrum representing rational algebraic K-theory over R and ch is the isomorphism of [CD12b, 14.2.7(3)]. This map induces the usual higher Chern character (see [Gil81]) for any smooth S-scheme X:

$$\mathrm{ch}_n : K_n(X)_{\mathbb{Q}} \to \prod_{i \in \mathbb{N}} \mathbb{E}^{2i-n,i}(X).$$

2.1.5. Given a motivic ring spectrum $\mathbb{E}$, we can define a (cohomological) realization functor of $DM_{\mathrm{B}}(R)$:

$$\mathbb{E}(-) : DM_{\mathrm{B}}(R)^{op} \to \mathbb{Q}_p\text{-vs}, \qquad M \mapsto \mathrm{Hom}_{DM_{\mathrm{B}}(S)}(M, \mathbb{E}).$$

This shows that the $\mathbb{E}$-cohomology of a smooth S-scheme X inherits the functorial structure of the motive of X.

In particular, given a projective morphism of smooth S-schemes $f : Y \to X$, there exists a Gysin morphism on motives:

$$M(X) \to M(Y)(-d)[-2d]$$

where d is the dimension of f. This was constructed in [Dég08] and several properties of this Gysin morphism were proved there. Thus, after applying the functor $\mathbb{E}(-)$ above, one gets:

Theorem 2.1.6. *Consider the above notations. One can associate to f a Gysin morphism in syntomic cohomology:*

$$f_* = \mathbb{E}(f^*) : \mathbb{E}^{n,i}(Y) \to \mathbb{E}^{n-2d,i-d}(X).$$

Moreover, one gets the following properties:

(1) *(*[Dég08, 5.14]*) For any composable projective morphisms f, g, $(fg)_* = f_* g_*$.*
(2) *(*projection formula, [Dég08, 5.18]*) For any projective morphism $f : Y \to X$ and any pair $(x, y) \in \mathbb{E}^{*,*}(X) \times \mathbb{E}^{*,*}(Y)$, one has:*
$$f_*(f^*(x).y) = x.f_*(y).$$
(3) *(*Excess intersection formula, [Dég08, 5.17(ii)]*) Consider a cartesian square of smooth S-schemes:*
$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "p"] & X \end{tikzcd}$$
such that p is projective. Let ξ/Y' be the excess intersection bundle[12] *associated with that square and let e be its rank.*
Then for any $y \in \mathbb{E}^{*,*}(Y)$, on gets:
$$f^* p_*(y) = q_*(c_e(\xi).g^*(y)).$$
(4) *For any projective morphism $f : Y \to X$, the following diagram is commutative:*
$$\begin{tikzcd} H_{\mathrm{B}}^{n,i}(Y) \arrow[r, "f_*"] \arrow[d, "\sigma_{\mathrm{syn}}"'] & H_{\mathrm{B}}^{n-2d,i-d}(X) \arrow[d, "\sigma_{\mathrm{syn}}"] \\ \mathbb{E}^{n,i}(Y) \arrow[r, "f_*"] & \mathbb{E}^{n-2d,i-d}(X) \end{tikzcd}.$$

[12]Recall from *loc. cit.* that one defines ξ as follows: let us choose a closed embedding $i : Y \to P$ into a projective bundle over X and let $Y' \to P'$ be its pullback over X'. Let N (resp. N') be the normal vector bundle of Y in P (resp. Y' in P'). Then, as the preceding square is cartesian, there is a monomorphism $N' \to g^{-1}(N)$ of vector bundles over Y' and one puts: $\xi = g^{-1}(N)/N'$.

Remark 2.1.7.
- With the notation of Point (3) recall that ξ has dimension $n-m$ where n (resp. m) is the dimension of p (resp. q). In particular, when the square is *transverse i.e.* $n = m$, one gets the more usual formula: $f^*p_* = q_*g^*$.
- Point (2) can simply be derived from the preceding formula applied to the graph morphism $\gamma : Y \to Y \times_S X$ given that γ^* is compatible with products.
- Point (4) shows in particular that, when $i : Z \to X$ is a closed immersion, $i_*(1) = \sigma_{\mathbb{E}}([Z])$ is the fundamental class of Z in X. If Z is a smooth divisor, corresponding to the line bundle $\mathcal{L}/X$, one gets in particular:

$$i_*(1) = c_1(\mathcal{L}).$$

  This property determines uniquely the Gysin morphism in the case of a closed immersion (see [Dég08] or [Pan09]).

  When $p : P \to X$ is the projection of a projective bundle of rank n and canonical line bundle λ, one gets, again applying Point (4):

$$p_*(c_1(\lambda)^i) = \begin{cases} 1 & \text{if i=n} \\ 0 & \text{otherwise} \end{cases}$$

  This fact, together with the projective bundle formula in syntomic cohomology, determines uniquely the morphism p_*.

  By construction, the Gysin morphism f_* for any projective morphism f is completely determined by the two above properties.
- For syntomic cohomology, Point (4) was conjectured by Besser [Bes12, Conjecture 4.2] (in the case of proper morphisms) and Theorem 1.1 in *loc. cit.* is conditional to the conjecture. The latter result concerns the regulator of a proper and smooth surface S over R. We also note that Point (4) has already been used (in the projective morphism case, although stated for proper maps) in [Lan11, p. 505] but the references given there is a draft of [CCM12] which turns to be different from the published version and does not contain the above statement, neither its proof.

Example 2.1.8. Let $f : Y \to X$ be a finite morphism between smooth connected S-schemes. Let d be the degree of the extension of the corresponding function fields. Then one gets the *degree formula* in $\mathbb{E}$-cohomology: for any $x \in \mathbb{E}^{*,*}(X)$,

$$f_*f^*(x) = d.x.$$

Indeed, according to 2.1.6(1),

$$f_*f^*(x) = f_*(1.f^*(x)) = f_*(1).x.$$

Then one gets $f_*(1) = d$ from 2.1.6(4) and the degree formula in Beilinson motivic cohomology.

As a corollary of Point (4) of the preceding theorem, one obtains the Riemann-Roch formula in $\mathbb{E}$-cohomology.

Corollary 2.1.9. *Let $f : Y \to X$ be a projective morphism between smooth S-schemes. Let τ_f be the virtual tangent bundle of f in $K_0(X)$: $\tau_f = [T_X] - [T_Y]$, the difference of the tangent bundle of X/S with that of Y/S. Then for any element $y \in K_n(Y)_{\mathbb{Q}}$, one gets the following formula:*

$$\mathrm{ch}_{\mathbb{E}}\left(f_*(y)\right) = f_*\big(\mathrm{td}(\tau_f).\,\mathrm{ch}_{\mathbb{E}}(y)\big)$$

where $\mathrm{td}(\tau_f)$ is the Todd class of the virtual vector bundle τ_f in $\mathbb{E}$-cohomology (defined for example as the image of the usual Todd class in Chow groups by the cycle class map).

In fact, this corollary is deduced from the Riemann-Roch formula in motivic cohomology after applying to it the higher cycle class and applying Point (4) of the previous theorem.

2.2. **The six functors formalism and Bloch-Ogus axioms.** In this section, we will recall some consequences of Grothendieck six functors formalism established for Beilinson motives (see [CD12b, 2.4.50] for a summary) and apply this theory to the spectra considered in this paper. We will consider only separated S-schemes of finite type over S. We will also consider an abstract object $\mathbb{E}$ of $DM_{\mathrm{B}}(S)$.

2.2.1. We associate with $\mathbb{E}$ four homology/cohomology theories defined for an S-scheme X with structural morphism f and a pair of integers (n, i) as follows:

Cohomology	$\mathbb{E}^{n,i}(X) = \mathrm{Hom}(\mathbb{1}_S, f_*f^*\mathbb{E}(i)[n])$
Homology	$\mathbb{E}_{n,i}(X) = \mathrm{Hom}(\mathbb{1}_S, f_!f^!\mathbb{E}(-i)[-n])$
Cohomology with compact support	$\mathbb{E}_c^{n,i}(X) = \mathrm{Hom}(\mathbb{1}_S, f_!f^*\mathbb{E}(i)[n])$
Borel-Moore homology	$\mathbb{E}_{n,i}^{\mathrm{BM}}(X) = \mathrm{Hom}(\mathbb{1}_S, f_*f^!\mathbb{E}(-i)[-n])$

We will use the terminology *c-cohomology* (resp. *BM-homology*) for cohomology with compact support (resp. Borel-Moore homology).

Note that these definitions, applied to the unit object $\mathbb{1}$ of $DM_{\mathrm{B}}(S)$, yield the four corresponding motivic theories. Also, these definitions are (covariantly) functorial in $\mathbb{E}$. In particular, if $\mathbb{E}$ admits a structure of a monoid in $DM_{\mathrm{B}}(S)$ (*i.e.* $\mathbb{E}$ is a ring spectrum), the unit map $\eta : \mathbb{1} \to \mathbb{E}$ yields regulators in all four theories.

When X/S is proper, as $f_* = f_!$, one gets identifications:

$$\mathbb{E}^{n,i}(X) = \mathbb{E}_c^{n,i}(X), \qquad \mathbb{E}_{n,i}^{\mathrm{BM}}(X) = \mathbb{E}_{n,i}(X).$$

2.2.2. *Functoriality properties.–* We consider a morphism of S-schemes:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

Using the adjunction map $ad_f : 1 \to f_*f^*$ (resp. $ad'_f : f_!f^! \to 1$), we immediately obtain that cohomology is contravariant (resp. homology is covariant) by composing on the left by p_* (resp. $p_!$) and on the right by p^* (resp. $p^!$).

When f is proper, $f_! = f_*$. Using again ad_f, ad'_f, one deduces that c-cohomology (resp. BM-homology) is contravariant (resp. contravariant) with respect to proper maps.

When f is smooth of relative dimension d, one has the relative purity isomorphism:

$$f^! \simeq f^*(d)[2d]$$

(see in [CD12b]: Th. 2.4.50 for the statement and Sec. 2.4 for details on relative purity). In particular, one derives from ad_f and ad'_f the following maps:

$$f_* : \mathbb{E}_c^{n,i}(X) \to \mathbb{E}_c^{n-2d,i-d}(Y), \quad f^* : \mathbb{E}_{n,i}^{\mathrm{BM}}(X) \to \mathbb{E}_{n+2d,i+d}^{\mathrm{BM}}(Y).$$

Finally, when f is proper and smooth of relative dimension d one gets in addition:

$$f_* : \mathbb{E}^{n,i}(X) \to \mathbb{E}^{n-2d,i-d}(Y), \quad f^* : \mathbb{E}_{n,i}(X) \to \mathbb{E}_{n+2d,i+d}(Y).$$

Let us summarize the situation:

theory	covariance (degree)	contravariance (degree)
Cohomology	smooth proper, (-2d,-d)	any
Homology	any	smooth proper, (+2d,+d)
Cohomology with compact support	smooth, (-2d,-d)	proper
Borel-Moore homology	proper	smooth, (+2d,+d)

Remark 2.2.3. The fact that the functorialities constructed above are compatible with composition is obvious except when a smooth morphism is involved. This last case follows from the functoriality of the relative purity isomorphism proved by Ayoub in [Ayo07].

When considering one of the four theories associated with $\mathbb{E}$, one can mix the two kinds of functoriality in a projection formula as usual. In fact, given a cartesian square:

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

such that f is proper and smooth (or f smooth and g proper when considering $\mathbb{E}_c$ or $\mathbb{E}^{\mathrm{BM}}$), one obtains respectively:

- $f^*p_* = q_*g^*$ for the two homologies,
- $p^*f_* = g_*q^*$ for the two cohomologies.

This is a lengthy check coming back to the definition of the relative purity isomorphism. The essential fact is that

$$g^{-1}(T_{Y/X}) = T_{Y'/X'}$$

where $T_{Y/X}$ (resp. $T_{Y'/X'}$) is the tangent bundle of f (resp. g).

2.2.4. *Products.*– Let us now assume that $\mathbb{E}$ is a ring spectrum, with unit map $\eta : \mathbb{1}_S \to \mathbb{E}$ and product map $\mu : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$.

Of course, for any S-scheme X with structural map f, we can define a product on cohomology, sometimes called the cup-product:

$$\mathbb{E}^{n,i}(X) \otimes \mathbb{E}^{m,j}(X) \to \mathbb{E}^{n+m,i+j}(X), (x, y) \mapsto xy = x \cup y;$$

given cohomology classes

$$x : \mathbb{1}_X \to f^*\mathbb{E}(i)[n], y : \mathbb{1}_X \to f^*\mathbb{E}(j)[m],$$

we define xy as the following composite map:

$$\mathbb{1}_X \xrightarrow{x\otimes y} f^*(\mathbb{E})(i)[n] \otimes f^*(\mathbb{E})(j)[m] = f^*(\mathbb{E} \otimes \mathbb{E})(i+j)[n+m] \xrightarrow{\mu} f^*(\mathbb{E})(i+j)[n+m].$$

This product is obviously commutative and associative. Note one can also define an exterior product on cohomology as follows:

$$\mathbb{E}^{n,i}(X) \otimes \mathbb{E}^{m,j}(Y) \to \mathbb{E}^{n+m,i+j}(X \times_S Y), (x, y) \mapsto p_1^*(x).p_2^*(y)$$

where p_1 (resp. p_2) is the projection $X \times_S Y/X$ (resp. $X \times_S Y/Y$).

One can also define *exterior products* on c-cohomology. Consider a cartesian square:

$$\begin{tikzcd} X \times_S Y \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, "h"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

of separated morphisms of finite type. We define the following product on c-cohomology:

$$\mathbb{E}_c^{n,i}(X) \otimes \mathbb{E}_c^{m,j}(Y) \to \mathbb{E}_c^{n+m,i+j}(X \times_S Y), (x, y) \mapsto x \times y$$

which associates to any maps

$$x : \mathbb{1}_S \to f_!f^*\mathbb{E}(i)[n], y : \mathbb{1}_S \to g_!g^*\mathbb{E}(j)[m],$$

the following composite map $x \times y$:

$$\begin{aligned} \mathbb{1}_S \xrightarrow{x\otimes y} & f_!f^*(\mathbb{E})(i)[n] \otimes g_!g^*(\mathbb{E})(j)[m] \\ & \simeq f_!\big(f^*(\mathbb{E})(i)[n] \otimes f^*g_!g^*(\mathbb{E})\big)(i+j)[n+m] \\ & \simeq f_!\big(f^*(\mathbb{E})(i)[n] \otimes g'_!f'^*g^*(\mathbb{E})\big)(i+j)[n+m] \\ & \simeq f_!g'_!\big(g'^*f^*(\mathbb{E})(i)[n] \otimes f'^*g^*(\mathbb{E})\big)(i+j)[n+m] \\ & = h_!h^*(\mathbb{E} \otimes \mathbb{E})(i+j)[n+m] \xrightarrow{\mu} h_!h^*(\mathbb{E})(i+j)[n+m] \end{aligned}$$

where the first and third isomorphisms follow from the projection formula [CD12b, 2.4.50(v)] and the second one from the exchange isomorphism [CD12b, 2.4.50(iv)].

One can check the following formulas:

$$(x \times y) \times z = x \times (y \times z), x \times y = y \times x$$

through the respective isomorphisms

$$(X \times_S Y) \times_S Z \simeq (X \times_S Y) \times_S Z, X \times_S Y \simeq Y \times_S X.$$

Further, because c-cohomology is contravariant with respect to proper morphism, given any S-schemes X (separated of finite type), the diagonal embedding $\delta : X \to X \times_S X$ allows to define an inner product on c-cohomology:

$$\mathbb{E}_c^{n,i}(X) \otimes \mathbb{E}_c^{m,j}(X) \to \mathbb{E}_c^{n+m,i+j}(X), (x, x') \mapsto \delta^*(x \times x').$$

When X/S is proper, one can check this product coincides with cup-product on cohomology.

Remark 2.2.5. Let $f : Y \to X$ be a proper smooth morphism. According to the projection formulas established in Remark 2.2.3, one can check that for any couple (y, x) either in $\mathbb{E}^{n,i}(Y) \times \mathbb{E}^{m,j}(X)$ or in $\mathbb{E}_c^{n,i}(Y) \times \mathbb{E}_c^{m,j}(X)$, one gets the following usual projection formula (for products):

$$f_*(x.f^*(y)) = f_*(x).y.$$

In fact, in each case, one uses the relevant formula of Remark 2.2.3, the external product and the following formulas:

$$y \times f^*(x) = (1_Y \times_S f)^*(y \times x), f_*(y) \times x = (f \times_S 1_X)_*(y \times x).$$

2.2.6. *Cap product.*– One can extend the cohomology theory associated with E to a theory with support. Given any closed immersion of S-schemes:

$$\begin{tikzcd} Z \arrow[rr, "i"] \arrow[dr, "g"'] & & X, \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

one puts:

$$\mathbb{E}_Z^{n,i}(X) = \mathrm{Hom}(i_*(\mathbb{1}_Z), f^*\mathbb{E}(i)[n]) = \mathrm{Hom}(\mathbb{1}_Z, i^! f^* \mathbb{E}(i)[n]).$$

This theory satisfies all the usual properties. We refer the reader to [Dég11, §1.2] for a detailed account.

Assuming again $\mathbb{E}$ is a ring spectrum with product map $\mu : \mathbb{E} \otimes \mathbb{E} \to E$, one defines, following Bloch and Ogus,[BO74], the *cap-product with supports*:

$$\mathbb{E}_{n,i}^{\mathrm{BM}}(X) \otimes \mathbb{E}_Z^{m,j}(X) \to \mathbb{E}_{n-m,i-j}^{\mathrm{BM}}(Z), (x, z) \mapsto x \cap z.$$

Let us first introduce classical pairing of functors (see [Del77, IV, §1.2]): given any objects A and B of $DM_{\mathrm{B}}(S)$, one considers the following composite map

$$f_!(f^!(A) \otimes f^*(B)) \xrightarrow{Ex} [f_! f^!(A)] \otimes B \xrightarrow{ad'_f} A \otimes B$$

where the first map is the isomorphism of the projection formula ([CD12b, 2.4.50]) and the second one is the counit of the adjunction $(f_!, f^!)$. One thus deduces by adjunction the following pairing

$$f^!(A) \otimes f^*(B) \xrightarrow{\eta_f} f^!(A \otimes B).$$

Thus, given maps

$$x : \mathbb{1}_X \to f^!(\mathbb{E}),\ z : i_*(\mathbb{1}_Z) \to f^*(\mathbb{E})$$

one defines $x \cap z$ from the following composite map:

$$i_*(\mathbb{1}_Z) \xrightarrow{x \otimes z} f^!(\mathbb{E}) \otimes f^*(\mathbb{E}) \xrightarrow{\eta_f} f^!(\mathbb{E} \otimes \mathbb{E}) \xrightarrow{\mu} f^!(\mathbb{E})$$

using $i_* = i_!$, the adjunction $(i_!, i^!)$ and $i^! f^! = g^!$.

Remark 2.2.7. Consider a cartesian square of S-schemes

$$\begin{tikzcd} T \arrow[r, "k"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

such that i is a closed immersion and f is proper. Then, for any couple $(y, z) \in \mathbb{E}^{\mathrm{BM}}_{n,i}(X) \otimes \mathbb{E}^{m,j}_Z(X)$, one obtains the following formula:

$$f_*(y) \cap z = g_*(y \cap f^*(z)).$$

2.2.8. Suppose again $\mathbb{E}$ is a ring spectrum with unit map $\eta : \mathbb{1}_S \to \mathbb{E}$.

Let $f : X \to S$ be a smooth S-scheme of relative dimension d. Then, according to [CD12b, 2.4.50(iii)], one obtains a canonical isomorphism of functors:

$$\mathfrak{p}_f : f^! \to f^*(d)[2d].$$

In particular, one gets a canonical map

$$\eta_X : \mathbb{1}_X = f^*(\mathbb{1}_S) \xrightarrow{f^*(\eta)} f^*(\mathbb{E}) \xrightarrow{\mathfrak{p}_f^{-1}} f^!(\mathbb{E})(-d)[-2d]$$

which corresponds to a homological class $\eta_X \in \mathbb{E}^{\mathrm{BM}}_{2d,d}(X)$. The following result is now a tautology:

Proposition 2.2.9. *Consider the above assumptions, and let $Z \subset X$ be any closed subset. Then the following map:*

$$\mathbb{E}^{n,i}_Z(X) \to \mathbb{E}^{\mathrm{BM}}_{2d-n,i-n}(Z), z \mapsto \eta_X \cap z$$

is an isomorphism.

One can now summarize some of the main properties we have proved so far as follows:

Corollary 2.2.10. *The couple of functors $\left(\mathbb{E}^{**}, \mathbb{E}^{\mathrm{BM}}_{**}\right)$ form a Poincaré duality theory with supports in the sense of Bloch and Ogus (*[BO74, Def 1.3]*).*

This is the case in particular for syntomic cohomology and syntomic BM-homology.

2.2.11. *Descent theory.*– Recall (see [CD12b, §3.1]) that a diagram of S-schemes $(\mathcal{X}, I)$ is the data of a small category I and a functor $\mathcal{X} : I \to \mathscr{S}$. A morphism of diagrams $\varphi = (\alpha, f) : (\mathcal{X}, I) \to (\mathcal{Y}, J)$ is the data of a functor $f : I \to J$ and a natural transformation $\alpha : \mathcal{X} \to f^*(\mathcal{Y})$ where $f^*(\mathcal{Y}) = \mathcal{Y} \circ f$.

Accorded to [CD12b, §3.1], the fibered triangulated category DM_{B} can be extended to the category of diagrams. Moreover, for any morphism of diagrams $\varphi : (\mathcal{X}, I) \to (\mathcal{Y}, J)$, one has an adjoint pair of functors:

$$\varphi^* : DM_{\mathrm{B}}(\mathcal{Y}, J) \leftrightarrows DM_{\mathrm{B}}(\mathcal{X}, I) : \varphi_*.$$

Consider a diagram of S-schemes $(\mathcal{X}, I)$ and the canonical morphism $\varphi : (\mathcal{X}, I) \to (S, *)$ where $*$ is the final category. Then one defines the cohomology of $(\mathcal{X}, I)$ as:

$$\mathbb{E}^{n,i}(\mathcal{X}, I) = \mathrm{Hom}(\mathbb{1}, \varphi_*\varphi^*(\mathbb{E})(i)[n]).$$

This is contravariant with respect to morphisms of diagrams.

In particular one has extended the cohomology $\mathbb{E}^{*,*}$ to simplicial S-schemes. The h-topology was introduced by Voevodsky in [Voe96]. Recall that a h-cover $f : Y \to X$ of S-schemes is a universal topological epimorphism (e.g. faithfully flat maps, proper surjective maps). Then the h-descent theorem for Beilinson motives ([CD12b, 14.3.4]) states the following:

For any quasi-excellent S-scheme X and any hypercover $p : \mathcal{X} \to X$ for the h-topology, the canonical map

$$p^* : \mathbb{E}^{n,i}(X) \to \mathbb{E}^{n,i}(\mathcal{X})$$

is an isomorphism. In particular, one gets the usual spectral sequence:

$$E_1^{p,q} = \mathbb{E}^{p,i}(\mathcal{X}_q) \Rightarrow \mathbb{E}^{p+q,i}(X).$$

Remark 2.2.12. As already remarked in [CD09a], the preceding descent theory together with De Jong resolution of singularities, shows that in the case where S is the spectrum of a field (non necessarily perfect), the cohomology $\mathbb{E}^{*,*}$ is uniquely determined by its restriction to smooth schemes.

3. Syntomic spectrum

In this section we construct several motivic ring spectra (see Def. 2.1.1): $\mathbb{E}_{\mathrm{FdR}}, \mathbb{E}_{\mathrm{rig}}, \mathbb{E}_{\phi}, \mathbb{E}_{\mathrm{syn}}$. First for a field K of characteristic zero we construct $\mathbb{E}_{\mathrm{FdR}}$ representing the filtered part of the de Rham cohomology of a K-scheme. i.e.

$$\mathbb{E}^{n,i}_{\mathrm{FdR}}(X) := \mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(\eta,\mathbb{Q})}(\Sigma^\infty \mathbb{Q}(X), \mathbb{E}_{\mathrm{FdR}}(i)[n]) \simeq F^i H^n_{\mathrm{dR}}(X) \ .$$

Then we define $\mathbb{E}_{\mathrm{rig}}$ which represents the rigid cohomology of Berthelot. This was already proved in [CD12a] in a different way. For both $\mathbb{E}_{\mathrm{FdR}}$ and $\mathbb{E}_{\mathrm{rig}}$ we use the criteria of Proposition 1.4.10.

Finally we get a motivic ring spectrum $\mathbb{E}_{\mathrm{syn}}$ for the rigid syntomic cohomology as a homotopy limit of a diagram of ring spectra.

3.1. **Cosimplicial tools.**

3.1.1. Let Δ be the category of finite ordered sets $[n] := \{0, ..., n\}$ as objects and monotone nondecreasing functions as morphisms. Let $\delta_i(n) : [n-1] \to [n]$ (resp. $\sigma_i(n) : [n] \to [n-1]$) be the usual[13] (co)face (resp. (co)degeneracy) map. In case there is no ambiguity we will simply write δ_i, σ_i. Given a category C, a *simplicial* (resp. *cosimplicial*) object of C is a functor from Δ° (resp. Δ) to C.

For instance let $A_n = \mathbb{Q}[T_0, ..., T_n]/(\sum T_i - 1)$, then this is a simplicial $\mathbb{Q}$-algebra in an obvious way. It follows that the associated differential graded algebra (dga) of Kähler differentials

$$\omega_n := \Omega^\bullet_{A_n/\mathbb{Q}} \quad n \geq 0 \tag{3.1.1.a}$$

is a simplicial dga over $\mathbb{Q}$. We will denote by $\delta^i = \delta_i^*$ (resp. $\sigma^i = \sigma_i^*$) the structural morphisms.

Now let M be a cosimplicial abelian group and sM the associated simple complex ($sM^i = M[i]$ and the differentials are the alternate sums of the coface morphisms). Its *standard normalization* NM is the subcomplex of sM s.t. $N^q M := \bigcap_i \ker(\sigma_i) \subset M^q$. Then inclusion $NM \to sM$ is a homotopy equivalence. Now if M is also a cosimplicial commutative monoid the Alexander-Whitney product[14] gives a (differential graded) monoid structure on sM and NM, but this is not necessarily (graded) commutative. Thus we consider the following construction due to Thom and Sullivan. Let M be a cosimplicial dga, we define

$$\widetilde{N}^q M \subset \prod_m \omega^q_m \otimes M^m$$

as the submodule whose elements are sequences $(x_m)_{m\geq 0}$ such that

$$(\mathrm{Id} \otimes \delta_i) x_m = (\delta^i \otimes \mathrm{Id}) x_{m+1} \quad , \ (\sigma^i \otimes \mathrm{Id}) x_m = (\mathrm{Id} \otimes \sigma_i) x_{m+1}$$

and define the differentials $D : \widetilde{N}^q M \to \widetilde{N}^{q+1} M$ by $D = ((-1)^q \mathrm{Id} \otimes d) + \mathrm{Id} \otimes \partial)$, where d (resp. ∂) is the differential of M (resp. ω_m). With the above notation if M is further a cosimplicial commutative monoid then $\widetilde{N}M$ is a commutative monoid too. Namely we can define

$$\widetilde{N}M \otimes \widetilde{N}M \to \widetilde{N}M \tag{3.1.1.b}$$

induced by $(\alpha \otimes m) \otimes (\alpha' \otimes m') = \alpha \wedge \alpha' \otimes (m \cdot m')$.

Moreover the complex $\widetilde{N}M$ is quasi-isomorphic to the standard normalization NM (and then to sM).[15]

---

[13] *i.e.* the image of $\delta_i(n)$ is $[n] \setminus \{i\}$.

[14] This is given as follows. Let $\delta^- : [q] \to [q+q']$ (resp. $\delta^+ : [q'] \to [q]$) the map with image $\{0, 1, ..., q\}$ (resp. $\{q, q+1, ..., q+q'\}$), then define $a * b := \delta^-(a) \cdot \delta^+(q)$.

[15] The isomorphism is induced by the integration map $\int : \omega^\bullet_n \otimes M^n \to \mathbb{Q}[-n] \otimes M^n$ defined by

$$(dT_1 \wedge \cdots \wedge dT_n) \otimes m \mapsto \frac{1}{n!} \otimes m \ .$$

We can extend the above constructions to the setting of cosimplicial dg abelian groups. Given such an $M = M^{pq}$ (where q is the cosimplicial parameter), then sM (resp. NM, $\widetilde{N}M$) is naturally a double complex and we can apply the total complex functor, denoted by tot, to obtain a dg abelian group.

Now we are ready to state a technical result well known to the specialists.

Proposition 3.1.2 (cf. [HS87],[HY99, Appendix])**.** *Let M be a cosimplicial (commutative) dga over $\mathbb{Q}$ then there exists a canonical (commutative) dga $\widetilde{N}M$ and a quasi-isomorphism $\int : \widetilde{N}M \to sM$ inducing an isomorphism of (commutative) dg algebras in cohomology $H(\int) : H(\widetilde{N}M) \to H(sM)$.*

3.1.3. (Godement resolutions) Let $u : P \to X$ be a morphism of Grothendieck sites and let $P^\sim$ (resp. $X^\sim$) be the category of abelian sheaves on P (resp. X). Then we have a pair of adjoint functors (u^*, u_*), where $u^* : X^\sim \to P^\sim$, $u_* : P^\sim \to X^\sim$. For any object $\mathcal{F}$ of $X^\sim$ we can define a co-simplicial object $B^*(\mathcal{F})$ whose component in degree n is $(u_*u^*)^{n+1}(\mathcal{F})$.[16]

Proposition 3.1.4. *Let $u : P \to X$ a morphism of sites and $\mathcal{F}$ a complex of sheaves on X. If u^* is exact and conservative, then*

1. *The complex $\mathrm{Gdm}_P(\mathcal{F}) := sB^*(\mathcal{F})$ is a functorial flask resolution of $\mathcal{F}$*
2. *If $\mathcal{F}$ is a $\mathbb{Q}$-linear sheaf, the Thom-Sullivan normalization $\widetilde{\mathrm{Gdm}}_P(\mathcal{F}) := \widetilde{N}B^*(\mathcal{F})$ is a functorial resolution of $\mathcal{F}$;*
3. *If $\mathcal{F}$ is a sheaf of (commutative) dga over $\mathbb{Q}$, then the complex $\widetilde{\mathrm{Gdm}}_P(\mathcal{F}) = \widetilde{N}B^*(\mathcal{F})$ is a sheaf of commutative dga and the canonical isomorphism $H^*(X, \mathcal{F}) \cong H^*(\Gamma(X\, \widetilde{\mathrm{Gdm}}\, \mathcal{F}))$ is compatible with respect to the multiplicative structure.*

Proof. Since u^* is exact and conservative, to show that the canonical map $b_\mathcal{F} : \mathcal{F} \to sB^*(\mathcal{F})$ is a quasi-isomorphism is sufficient to prove that $u^*b_\mathcal{F}$ is a quasi-isomorphism. This follows from the fact that the augmented complex

$$u^*\mathcal{F} \to u^*B^0(\mathcal{F}) \to u^*B^1(\mathcal{F}) \to \cdots$$

is null-homotopic: the homotopy $h^i : u^*(u_*u^*)^i(\mathcal{F}) \to u^*(u_*u^*)^{i-1}(\mathcal{F})$ is induced by the counit $u^*u_* \to \mathrm{Id}$ and one checks easily that $\mathrm{Id} = d^{i-1} \circ h^i + h^{i+1} \circ d^i$, where d^i is given by the alternating sum of cofaces. The rest follows directly from Prop. 3.1.2 and the existence of a family of canonical maps

$$\cup_n : B^n(F) \otimes B^n(G) \to B^n(F \otimes G)$$

compatible with the cosimplicial structure. We leave to the reader to check that if $\mathcal{F}^*$ is further a (commutative) dga on $X^\sim$ then $B^*(\mathcal{F}^*)$ is a cosimplicial (commutative) dga[17]. □

3.1.5 (Enough points)**.** We will use the above construction in the case X is the site associated to a scheme or a dagger space (in the case of a dagger space we take the site associated to its G-topology). In both cases we let P be the category $Pt(X)$ of site-theoretical points of X. For a general X the canonical map $u : Pt(X) \to X$ is not conservative. The latter property is guaranteed in the two cases we are interested in. It suffices to exhibit a subcategory C of $Pt(X)$ (with the discrete topology) such that u restricted to C is conservative. When X is associated to a scheme

[16] The cosimplicial structure is defined as follows. First let $\eta : \mathrm{Id}_{X^\sim} \to u_*u^*$ and $\epsilon : u^*u_* \to \mathrm{Id}_{P^\sim}$ be the natural transformations induced by adjunction.

Endow $B^n(\mathcal{F}) := (u_*u^*)^{n+1}(\mathcal{F})$ with co-degeneracy maps

$$\sigma_i^n := (u_*u^*)^i u_*\epsilon u^*(u_*u^*)^{n-1-i} : B^n(\mathcal{F}) \to B^{n-1}(\mathcal{F}) \quad i = 0, ..., n-1$$

and co-faces

$$\delta_i^{n-1} := (u_*u^*)^i \eta (u_*u^*)^{n-i} : B^{n-1}(\mathcal{F}) \to B^n(\mathcal{F}) \quad i = 0, ..., n\ .$$

[17] In fact one needs to take care of the signs:

$$\cup_n^{ab} : B^n(\mathcal{F}^a) \otimes B^n(\mathcal{F}^b) \to B^n(\mathcal{F}^a \otimes B^n(\mathcal{F}^b))\ ,\ \cup_n^{ab} = (-1)^{na} \cup_n\ .$$

(resp. a dagger space) we let C be the category of its Zariski points (resp. its Berkovich or adic points). This is enough as explained in [CCM12, § 3] or [Tam11, § 3].

From now on we will simply write $\widetilde{\mathrm{Gdm}}$ instead of $\widetilde{\mathrm{Gdm}}_{Pt(X)}$ with X as above.

3.2. **De Rham cohomology.**

3.2.1 (The Hodge Filtration)**.** We recall some well known facts about algebraic de Rham cohomology (see for instance [Jan90]). Let K be a field of characteristic zero and X be a smooth and algebraic K-scheme. Fix a compactification $g : X \to \bar{X}$ such that the complement $D = \bar{X} \setminus X$ is a normal crossing divisor[18] . Then consider the complex $\Omega^\bullet_{\bar{X}/K}\langle D\rangle$ of differential forms on $\bar{X}$ with logarithmic differential poles along D. The natural inclusion $\Omega^\bullet_{\bar{X}/K}\langle D\rangle \subset g_*\Omega^\bullet_{X/K}$ is a quasi-isomorphism and we define the Hodge filtration on the de Rham cohomology of X by

$$F^i H^n_{\mathrm{dR}}(X/K) := H^n(\bar{X}, F^i\Omega^\bullet_{\bar{X}/K}\langle D\rangle)$$

where $F^i\Omega^\bullet_{\bar{X}/K}\langle D\rangle$ is the stupid filtration.

A remarkable result of Deligne says that (for $K = \mathbb{C}$) the Hodge filtration does not depend on the chosen compactification. Moreover given a morphism $f : X \to Y$ of smooth algebraic schemes over $\mathbb{C}$ the induced morphism on de Rham cohomology is strictly compatible w.r.t the Hodge filtrations[19]. Then the same holds for $H^n_{\mathrm{dR}}(X/K)$ where $K \subset \mathbb{C}$ is a field of characteristic zero.

Proposition 3.2.2. *Let X be a smooth K-scheme.*

(1) *For any normal crossing compactification $\bar{X}$ of X the resolution $\widetilde{\mathrm{Gdm}}(\Omega^\bullet_{\bar{X}/K}\langle D\rangle)$ (notation as in § 3.1.5) gives a sheaf of filtered commutative dga[20] and $F^iH^n_{\mathrm{dR}}(X/K) \cong H^n(\Gamma(\bar{X}, \widetilde{\mathrm{Gdm}}(F^i\Omega^\bullet_{\bar{X}/K}\langle D\rangle)))$.*

(2) *The following complexes*

$$E_{\mathrm{FdR},i}(X) := \quad \mathrm{colim}_{\bar{X}}\,\Gamma(\bar{X}, \widetilde{\mathrm{Gdm}}(F^i\Omega^\bullet_{\bar{X}/K}\langle D\rangle)) \tag{3.2.2.a}$$

$$E'_{\mathrm{dR}}(X) := \quad \mathrm{colim}_{\bar{X}}\,\Gamma(\bar{X}, \widetilde{\mathrm{Gdm}}(g_*\Omega^\bullet_X)) \tag{3.2.2.b}$$

$$E_{\mathrm{dR}}(X) := \quad \Gamma(X, \widetilde{\mathrm{Gdm}}(\Omega^\bullet_X)) \tag{3.2.2.c}$$

are functorial in X and there are functorial quasi-isomorphisms[21]

$$E_{\mathrm{FdR},0}(X) \to E'_{\mathrm{dR}}(X) \leftarrow E_{\mathrm{dR}}(X)$$

Proof. By definition $\Omega^\bullet_{\bar{X}/K}\langle D\rangle$ is a commutative (filtered) dga. Let

$$F^i\,\widetilde{\mathrm{Gdm}}(\Omega^\bullet_{\bar{X}/K}\langle D\rangle) = \widetilde{\mathrm{Gdm}}(F^i\Omega^\bullet_{\bar{X}/K}\langle D\rangle)\ .$$

Then $\widetilde{\mathrm{Gdm}}(\Omega^\bullet_{\bar{X}/K}\langle D\rangle)$ is a (sheaf of) filtered commutative dga by Proposition 3.1.4. This concludes the proof of point (1).

As the complex of sheaves $\Omega^\bullet_{\bar{X}/K}\langle D\rangle$ is functorial[22] with respect to the pair (X, D), the same is true for $F^i\,\widetilde{\mathrm{Gdm}}(\Omega^\bullet_{\bar{X}/K}\langle D\rangle)$. Note that the category of normal crossing compactifications is filtered. Hence the above colimit is quasi-isomorphic to any of its elements. What remains to prove follows directly from the definitions. □

Example 3.2.3. Let $X = \mathbb{P}^1_K\setminus\{0, \infty\}$. By construction $E_{\mathrm{FdR},1}(X)$ is a complex starting in degree 1. Let $\mathrm{dlog} \in \Gamma(\mathbb{P}^1_K, \Omega^1_{\mathbb{P}^1_K}\langle 0, \infty\rangle) = H^0(E_{\mathrm{FdR},1}(X)[1]) = H^1(E_{\mathrm{FdR},1}(X))$ be the section defined by dT/T, for a local parameter T at 0. Note that the class of $d\log$ is a generator for $F^1H^1_{\mathrm{dR}}(X) \cong K$. We will denote it by c_1^{FdR}.

---

[18]Such a compactification exists by the Nagata's compactification theorem and the result of Hironaka on the resolution of singularities.

[19]A morphism $f : A \to B$ of filtered vector spaces is strict if $f(F^iA) = f(A) \cap F^iB$.

[20]Set $F^i\,\widetilde{\mathrm{Gdm}} = \widetilde{\mathrm{Gdm}}\,F^i$.

[21]We introduce E'_{dR} since there is no natural map between E_{dR} and $E_{\mathrm{FdR},i}$.

[22]Morphisms of pairs are morphisms of commutative squares.

Proposition 3.2.4. *There exists a motivic ring spectrum $\mathbb{E}_{\mathrm{FdR}}$ whose components are the complexes $E_{\mathrm{FdR},i}$ and such that*

$$F^i H^n_{\mathrm{dR}}(X) = \mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(K,\mathbb{Q})}(\mathbb{1}, \mathbb{E}_{\mathrm{FdR}}(i)[n]) \ .$$

Proof. By the previous Lemma the family $E_{\mathrm{FdR},i}$ forms a $\mathbb{N}$-graded commutative monoid. The dlog of the above example gives a morphism $\mathbb{Q}(\mathbb{G}_{m,K}) \to E_{\mathrm{FdR},1}$. According to Proposition 1.4.10 we have to prove the following.

(Excision and homotopy) $E_{\mathrm{FdR},i}$ is both Nis-local and $\mathbb{A}^1$-local. We know that E_{dR} is Nis/$\mathbb{A}^1$-local so that the same holds for $E_{\mathrm{FdR},0}$. The same holds for $E_{\mathrm{FdR},i}$ since the canonical maps $E_{\mathrm{FdR},i} \to E_{\mathrm{FdR},0}$ induce the Hodge filtration on cohomology. Then thanks to the strictness it is easy to conclude (see also the paragraph following this proof).

(Stability) The cup product with dlog $= dT/T$ induces an isomorphism

$$H^n(E_i(X)) \cong H^{n+1}(E_{i+1}(\mathbb{G}_m \times X))/H^{n+1}(E_{i+1}(X)) \ .$$

Let $g : X \to \bar{X}$ be a normal crossing compactification with complement D. Then $\mathbb{G}_m \times X \to \mathbb{P}^1 \times \bar{X}$ is a normal crossing compactification with complement $E = \{0, \infty\} \times \bar{X} \cup \mathbb{P}^1 \times D$. We have to prove that $\Omega_{\mathbb{P}^1 \times \bar{X}}\langle E\rangle = p_1^*\Omega_{\mathbb{P}^1}\langle 0, \infty\rangle \otimes p_2^*\Omega_{\bar{X}}\langle D\rangle$. This can be checked locally by choosing étale coordinates. Then it is easy to prove the filtered Künneth decomposition $F^{i+1}H^{n+1}_{\mathrm{dR}}(\mathbb{G}_m \times X) = H^0_{\mathrm{dR}}(\mathbb{G}_m) \otimes F^{i+1}H^{n+1}_{\mathrm{dR}}(X) \oplus H^1_{\mathrm{dR}}(\mathbb{G}_m) \otimes F^i H^n_{\mathrm{dR}}(X)$ since $F^j H^j_{\mathrm{dR}}(\mathbb{G}_m) = H^j_{\mathrm{dR}}(\mathbb{G}_m) \cong K$ for $j = 0, 1$. As $H^j_{\mathrm{dR}}(\mathbb{G}_m) = K d\log$ the claim is proved.

(Orientation) This is obvious: the morphism of $\mathbb{A}^1 \setminus \{0\}$ induced by $T \mapsto 1/T$ sends dT/T to $-dT/T$ as an element of $H^0(\mathbb{P}^1_K, \Omega^1_{\mathbb{P}^1_K}\langle 0, \infty\rangle) \subset E_{\mathrm{FdR},1}(\mathbb{A}^1 \setminus \{0\})$. □

3.2.5 (Variation on dagger spaces)**.** Let K be a p-adic field (*i.e.* a finite extension of $\mathbb{Q}_p$) and let R be its valuation ring. We define a canonical commutative dga $R\Gamma_{\mathrm{dR}}(X)$ for the de Rham cohomology of a dagger space $\mathcal{X}$ over K. Consider the following algebra

$$W_n := \{\sum_{\nu} a_\nu T^\nu \in K[[T_1, ..., T_n]] | \exists \rho > 1, \ |a_\nu|\rho^{|\nu|} \to 0\} \ .$$

According to Grosse-Klönne [Gro00] a K-algebra A is a *dagger algebra* if it is a quotient of W_n for some n. To such an A we can associate the spectrum of maximal points $\mathrm{Spm}(A)$ which is a G-ringed space. One has a universal K-derivation of A into finite A-modules, $d : A \to \Omega^1_{A/K}$ giving rise to de Rham complex $\Omega_{\mathcal{X}/K}$ on a general dagger space $\mathcal{X}$. Assuming $\mathcal{X}$ to be smooth we can set

$$H^n_{\mathrm{dR}}(\mathcal{X}) := H^n(\mathcal{X}, \Omega_{\mathcal{X}/K})$$

It follows from Proposition 3.1.2 and § 3.1.5 that the complex $R\Gamma_{\mathrm{dR}}(\mathcal{X}) := \Gamma(\mathcal{X}, \widetilde{\mathrm{Gdm}\,\Omega^\bullet_{\mathcal{X}/K}})$ is a functorial commutative dga.

Now let X be a smooth R-scheme. We can associate to it two different dagger spaces: one is the dagger analytification $(X_K)^\dagger$ of its generic fiber; the other is the Raynaud fiber $(X^w)_K$ of the weakly formal scheme X^w associated to X. There is a natural inclusion $(X^w)_K \subset (X_K)^\dagger$. Further there is a map of sites $\iota : (X_K)^\dagger \to X_K$ as in the classical analytification case.

3.3. **Rigid cohomology.** We recall the construction given by Besser as rephrased in [Tam11] since there are some simplification. For the sake of the readers we give all the needed definitions. We fix a a p-adic field K and denote by R (resp. k) is its valuation ring (resp. its residue field).

3.3.1. After the work Grosse-Klönne one can compute the rigid cohomology of Berthelot via dagger spaces [Gro00]. The method is as follows. Let X be a smooth k-scheme, then we can choose a closed embedding $X \to \mathcal{Y}$ in a weak formal R-scheme $\mathcal{Y}$ having smooth special fiber $\mathcal{Y}_k$. We call such an embedding a *rigid pair* and we denote it by $(X, \mathcal{Y})$. There is a specialization map $sp : \mathcal{Y}_K \to \mathcal{Y}$, where $\mathcal{Y}_K$ is the generic fiber of $\mathcal{Y}$. We write $]X[_{\mathcal{Y}} := sp^{-1}(X)$, called the *tube of* X *in* $\mathcal{Y}$.

A morphism of rigid pairs $(X,\mathcal{Y})$, $(X',\mathcal{Y}')$ is a commutative diagram

$$\begin{tikzcd}]X[_{\mathcal{Y}} \arrow[r, "F"] \arrow[d, "sp"'] &]X'[_{\mathcal{Y}'} \arrow[d, "sp"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

We denote by RP the category of rigid pairs.

The datum of a rigid pair $(X,\mathcal{Y})$ is sufficient to compute the rigid cohomology of X (with K coefficients) as follows

$$H^n_{\mathrm{rig}}(X/K) = H^n_{\mathrm{dR}}(]X[_{\mathcal{Y}}) = H^n(]X[_{\mathcal{Y}}, \Omega^\bullet_{]X[_{\mathcal{Y}}/K}) \ .$$

The de Rham complex $\Omega^\bullet_{]X[_{\mathcal{Y}}/K}$ is functorial in $(X,\mathcal{Y})$ and its cohomology is independent up to isomorphism of the choice of $\mathcal{Y}$. Since the tube of X in $\mathcal{Y}$ is a smooth dagger space we get $H^n_{\mathrm{rig}}(X/K) = H^n(R\Gamma_{\mathrm{dR}}(]X[_{\mathcal{Y}}))$ (see 3.2.5).

Proposition 3.3.2. (1) *For any p-adic field K with residue field k there exists a ring object $R\Gamma_{\mathrm{rig},K}$ in the category $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(\operatorname{Spec} k,\mathbb{Q})$ that represents rigid cohomology (with coefficients in K): i.e. for any affine and smooth k-scheme X, there is a canonical rational commutative dga $R\Gamma_{\mathrm{rig},K}(X)$ such that $H^i(R\Gamma_{\mathrm{rig},K}(X)) \cong H^i_{\mathrm{rig}}(X/K)$. (The same holds if we replace the coefficient ring $\mathbb{Q}$ by any field L s.t. $\mathbb{Q} \subset L \subset K$)*

(2) *Let X as above and $(X,\mathcal{Y})$ be a rigid pair. Then there is a commutative dga $\widetilde{R\Gamma}_{\mathrm{rig}}(X,\mathcal{Y})$ together with a diagram of dga quasi-isomorphisms*

$$R\Gamma_{\mathrm{rig},K}(X) \leftarrow R\Gamma_{\mathrm{rig}}(X,\mathcal{Y}) \rightarrow R\Gamma_{\mathrm{dR}}(]X[_{\mathcal{Y}})$$

functorial in the pair $(X,\mathcal{Y})$.

(3) *(Base change) Let $\rho : R \to R'$ be a finite map of complete discrete valuation rings. Let k (resp. k') be the residue field of R (resp. R'). Let X be a k-scheme then there is a canonical (both in X and R) quasi-isomorphism*

$$K' \otimes_K R\Gamma_{\mathrm{rig},K}(X) \to R\Gamma_{\mathrm{rig},K'}(X_{k'}) \ .$$

The latter induces an isomorphism in $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(\operatorname{Spec} k,\mathbb{Q})$

$$R\Gamma_{\mathrm{rig},K} \otimes_K K' \to f_*(R\Gamma_{\mathrm{rig},K'})$$

where $f : \operatorname{Spec} k' \to \operatorname{Spec} k$ is the map induced by ρ and $R\Gamma_{\mathrm{rig},?}$ denotes the object of point (1).

(4) *There exists a canonical σ-linear endomorphism of $R\Gamma_{\mathrm{rig},K_0}(X)$ inducing the Frobenius on cohomology: it is defined as the composition of*

$$(3.3.2.a) \qquad R\Gamma_{\mathrm{rig},K_0}(X) \xrightarrow{\mathrm{Id}\otimes 1} R\Gamma_{\mathrm{rig},K_0}(X) \otimes_\sigma K_0 \xrightarrow{b.c.} R\Gamma_{\mathrm{rig},K_0}(F^*X) \xrightarrow{rel.Frob.} R\Gamma_{\mathrm{rig},K_0}(X)$$

where b.c. stands for the base change morphism of point (3); F is the Frobenius of $\operatorname{Spec} k$*; F^*X is the base change of X via F; the last map on the right is the relative Frobenius.*

Proof. The details are given in [Bes00, 4.9, 4.21, 4.22]. Since we adopt the language of dagger spaces there are some formal differences. For the sake of the readers we give the necessary modifications. To obtain a complex functorial in X we have to take a colimit on some filtered category. The category of pairs $(X,\mathcal{Y})$ with X fixed is not filtered. Hence we have to introduce the following categories. We define the set RP_X (resp. $RP_{(X,\mathcal{Y})}$) of diagrams $X \xrightarrow{f} X' \to \mathcal{Y}'$ (resp. $(f,F) : (X,\mathcal{Y}) \to (X',\mathcal{Y}')$ morphism of rigid pairs) where $(X',\mathcal{Y}')$ is a rigid pair. Let RP^0_X (resp. $RP^0_{(X,\mathcal{Y})}$) be the subset of RP_X (resp. $RP_{(X,\mathcal{Y})}$) with $f = \mathrm{Id}_X$ (resp. $(f,F) = (\mathrm{Id},\mathrm{Id})$)

Now we can form the category SET^0_X (resp. $SET^0_{(X,\mathcal{Y})}$) with objects the finite subsets of RP_X (resp. $RP_{(X,\mathcal{Y})}$) having non-empty intersection with RP^0_X (resp. $RP^0_{(X,\mathcal{Y})}$); morphisms are inclusions. For instance an element of SET^0_X is a finite family of diagrams $X \xrightarrow{f_a} X'_a \to \mathcal{Y}'_a$, $a \in A$

(finite set), such that $f_{a_0} = \mathrm{Id}$ for some $a_0 \in A$. To such an object we can associate the complex $R\Gamma_{\mathrm{dR}}(]X[_{\mathcal{Y}'_A})$, where $\mathcal{Y}'_A = \prod_a \mathcal{Y}'_a$. The categories SET^0_X and $\mathrm{SET}^0_{(X,\bar{X},\mathsf{P})}$ are filtered.

Having this said we define

$$R\Gamma_{\mathrm{rig},K}(X) := \operatorname*{colim}_{A\in \mathrm{SET}^0_X} R\Gamma_{\mathrm{dR}}(]X[_{\mathcal{Y}'_A}) \qquad R\Gamma_{\mathrm{rig}}(X,\mathcal{Y}) := \operatorname*{colim}_{A\in \mathrm{SET}^0_{(X,\mathcal{Y})}} R\Gamma_{\mathrm{dR}}(]X[_{\mathcal{Y}'_A}) \ .$$

Now one can follow word by word the proof of Besser. □

Proposition 3.3.3. *There exists a motivic ring spectrum $\mathbb{E}_{\mathrm{rig},K}$ whose components are all equal to the complex $R\Gamma_{\mathrm{rig},K}$ and whose stability class is induced by* dlog *such that*

$$H^n_{\mathrm{rig}}(X/K) \cong \mathbb{E}^{n,i}_{\mathrm{rig},K}(X) := \mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(k,\mathbb{Q})}(M(X), \mathbb{E}_{\mathrm{rig},K}(i)[n]) \ .$$

Proof. We have to verify the hypothesis of Proposition 1.4.10 for the family $E_i := R\Gamma_{\mathrm{rig},K}$. First we need to define a morphism of complexes $\mathbb{Q}[0] \to R\Gamma_{\mathrm{rig},K}(\mathbb{G}_{m,k})(1)[1]$. We argue as in the de Rham case. Let us denote by $X = \mathbb{G}_{m,R}$. Then the de Rham cohomology of the dagger space $(X^w)_K$ computes the (K-linear) rigid cohomology of $X_k = \mathbb{G}_{m,k}$ and there is a canonical map from $R\Gamma_{\mathrm{dR}}((X^w)_K)$ to $R\Gamma_{\mathrm{rig},K}(X_k)$. We can apply the construction of 3.1.4 to the inclusion $\Omega^1_{(X^w)_K/K}[-1] \subset \Omega^1_{(X^w)_K/K}$ and we obtain (as in example 3.2.3) an element dlog of $R\Gamma_{\mathrm{dR}}((X^w)_K)$ of degree 1. □

Remark 3.3.4. With the notations of point (3) of Proposition 3.3.2, one gets a canonical base change isomorphism in $DM_{\mathrm{B}}(k)$:

$$\mathbb{E}_{\mathrm{rig},K} \otimes_K K' \xrightarrow{\sim} f_*(\mathbb{E}_{\mathrm{rig},K'}).$$

In the sequel, we will simply denote by $\mathbb{E}_{\mathrm{rig}}$ (resp. $R\Gamma_{\mathrm{rig}}$) the ring spectrum $\mathbb{E}_{\mathrm{rig},K_0}$ (resp. the complex $R\Gamma_{\mathrm{rig},K_0}$).

3.4. **Absolute rigid cohomology.**

3.4.1. Along the lines of [Beĭ86b] and [Ban02] we are going to define the analogue of absolute Hodge theory in the setting of rigid cohomology. Let k be a perfect field of characteristic p. We denote by F-isoc the category of F-isocrystals (defined over k): *i.e.* finite dimensional K_0-vector spaces together with a σ-linear automorphism. This is a tensor category with unit object $\mathbb{1}$ given by K_0 together with σ. For any $I \in$ F-isoc we denote by $I(n)$ the F-isocrystal having the same vector space I and Frobenius multiplied by p^{-n}. We would like to define the absolute rigid cohomology of a k-scheme X as follows

$$H^n_\phi(X,i) := \mathrm{Hom}_{D^b(\text{F-isoc})}(\mathbb{1}, R\Gamma(X)(i)[n])$$

where $R\Gamma(X)$ is a complex of F-isocrystals such that $H^n(R\Gamma(X)) = H^n_{\mathrm{rig}}(X)$ together with its Frobenius endomorphism. Since we do not know how to construct directly $R\Gamma$ we follow the strategy of Beilinson in *loc.cit.* and deduce its existence from proposition 3.3.2.

Let C^b_{rig} be the category of bounded complexes of K_0-vector spaces M together with a quasi-isomorphism $\phi : M^\sigma = M \otimes_{K_0,\sigma} K_0 \to M$. We define homotopies (resp. quasi-isomorphisms) between objects in C^b_{rig} to be morphisms in C^b_{rig} such that they are homotopies (resp. quasi-isomorphisms) of the underling complexes of K_0-vector spaces. Then we can define the category K^b_{rig} to be the category C^b_{rig} modulo the null-homotopic morphisms.

Lemma 3.4.2. (1) *the category K^b_{rig} is triangulated;*

(2) *the localization $K^b_{\mathrm{rig}}[\mathcal{A}^{-1}]$ of the category K^b_{rig} by the subcategory $\mathcal{A}$ of acyclic objects exists and it is a triangulated category too;*

(3) *let $D^b_{\mathrm{rig}} \subset K^b_{\mathrm{rig}}[\mathcal{A}^{-1}]$ be the full subcategory of complexes whose cohomology objects (w.r.t. the usual t-structure on complexes) are in* F-isoc*. Then there is a natural equivalence of categories $\iota : D^b(\text{F-isoc}) \to D^b_{\mathrm{rig}}$.*

Proof. We leave to the reader to check that all the arguments given in [Ban02, §§1,2] (or [CCM12, § 2]) can be adapted to our (much simpler) setting. We limit ourselves to make explicit the formulas for the Hom groups in $D^b(\text{F-isoc}), D^b_{\mathrm{rig}}$.

Let M, N be two bounded complexes of F-isocrystals. Remind that F-isoc has internal Hom so that we can form the internal Hom complex $\underline{\mathrm{Hom}}^\bullet(M,N)$ with Frobenius $\phi_{M,N}$. Consider the following morphism of $\mathbb{Q}_p$-linear[23] complexes

$$\xi_{M,N} : \underline{\mathrm{Hom}}^\bullet(M,N) \to \underline{\mathrm{Hom}}^\bullet(M,N), \quad x \mapsto x - \phi_{M,N}x \ .$$

Then we can prove as in [Ban02, proposition 1.7] that

(3.4.2.a) $$\mathrm{Hom}_{D^b(\text{F-isoc})}(M, N[i]) \cong H^{i-1}(\mathrm{Cone}\,\xi_{M,N}) \ .$$

Similarly given two complexes M, N in C^b_{rig} we define the morphism of complexes

$$\xi'_{M,N} : \mathrm{Hom}^\bullet(M,N) \to \mathrm{Hom}^\bullet(M^\sigma,N), \quad x \mapsto x \circ \phi_M - \phi_N \circ (x \otimes_\sigma 1) \ .$$

[24]Then the Hom groups in D^b_{rig} can be computed as follows

(3.4.2.b) $$\mathrm{Hom}_{D^b_{\mathrm{rig}}}(M, N[i]) \cong H^{i-1}(\mathrm{Cone}\,\xi'_{M,N}) \ .$$

Now it is easy to check that given two F-isocrystals M, N we have

(3.4.2.c) $$\mathrm{Ext}^i_{\text{F-isoc}}(M,N) \cong \mathrm{Hom}_{D^b_{\mathrm{rig}}}(\iota M, \iota N[i])$$

and the faithfulness of ι follows. □

Definition 3.4.3. Let X be an algebraic k-scheme. We define the absolute rigid cohomology as

$$H^n_\phi(X,i) := \mathrm{Hom}_{D^b_{\mathrm{rig}}}(\mathbb{1}, R\Gamma_{\mathrm{rig}}(X)(i)[n]) \ .$$

It follows from the equivalence ι of the above lemma that the same formula holds in $D^b(\text{F-isoc})$ for some object $R\Gamma(X)$ corresponding to $R\Gamma_{\mathrm{rig}}(X)$.

Corollary 3.4.4. *There is a natural spectral sequence*

(3.4.4.a) $$E_2^{pq} = \mathrm{Ext}^p_{\text{F-isoc}}(\mathbb{1}, H^q(X)(i)) \ \Rightarrow H^{p+q}_\phi(X,i)$$

degenerating to the following short exact sequence

$$0 \to H^1_{\mathrm{rig}}(X)/\mathrm{Im}(\mathrm{Id} - \phi/p^i) \to H^{n,i}_\phi(X) \to H^n_{\mathrm{rig}}(X)^{\phi = p^i} \to 0 \ .$$

Proof. The existence of the spectral sequence follows from the formula (3.4.2.c). By (3.4.2.b) it is concentrated in the columns $p = 0, 1$ so that it gives short exact sequences. □

Proposition 3.4.5. *There exists a motivic ring spectrum $\mathbb{E}_\phi \in DM_{\mathrm{B}}(k)$ representing the absolute rigid cohomology, i.e.*

$$H^n_\phi(X,i) \cong \mathbb{E}^{n,i}_\phi(X) := \mathrm{Hom}_{DM_{\mathrm{B}}(k)}(M(X), \mathbb{E}_\phi(i)[n]) \ .$$

Proof. By point (4) of proposition 3.3.2 we can define a family of morphism of presheaves of complexes

$$R\Gamma_{\mathrm{rig}} \xrightarrow{\phi/p^i} R\Gamma_{\mathrm{rig}} \ .$$

We claim that the latter induces a morphism of ring spectra

$$\mathbb{E}_{\mathrm{rig}} \xrightarrow{\Phi} \mathbb{E}_{\mathrm{rig}} \ .$$

Indeed it is sufficient to notice that $(\phi \otimes 1) \circ \mathrm{dlog} = p\,\mathrm{dlog}$, where $\mathrm{dlog} : \mathbb{Q}(\mathbb{G}_m)[-1] \to \mathbb{E}_{\mathrm{rig}}(1)$ is the stability class of the rigid spectrum.

Now we can define $\mathbb{E}_\phi$ to be the homotopy limit of the following diagram of ring spectra

(3.4.5.a) $$\mathbb{E}_{\mathrm{rig}} \overset{\Phi}{\underset{\mathrm{Id}}{\rightrightarrows}} \mathbb{E}_{\mathrm{rig}}.$$

the limit exists by 1.4.8.

To conclude the proof note that $\mathbb{E}_{\phi,i}$ is quasi-isomorphic to the cone $\mathrm{Cone}(\mathrm{Id} - \phi/p^i)$ (up to a shift!). Then it is sufficient to compare (3.4.2.b) and (1.4.10.a). □

[23]These are not K_0-linear since (in general) the Frobenius is not.

[24]Mind that we cannot use $\underline{\mathrm{Hom}}$ because there is no internal Hom in C^b_{rig}. This is due to the fact that the Frobenius is only a quasi-isomorphism.

Remark 3.4.6. According to the preceding proof, one gets a canonical distinguished triangle of $DM_{\text{Б}}(k)$:

$$\mathbb{E}_\phi \to \mathbb{E}_{\text{rig}} \xrightarrow{\text{Id}-\Phi} \mathbb{E}_{\text{rig}} \xrightarrow{+1} \tag{3.4.6.a}$$

which induces the short exact sequences of the preceding Corollary. In particular, these exact sequences are functorial with respect to the motive of X.

3.5. **Syntomic cohomology.**

3.5.1. Let X be a smooth R-scheme. With the notation of § 3.2.5, there is a map of commutative dga

$$\text{sp}_X : E_{\text{dR}}(X_K) \to R\Gamma_{\text{dR}}((X^w)_K) = R\Gamma_{\text{rig}}(X_k, X^w)$$

inducing the specialization on cohomology and functorial in X. Details can be found in [Tam11, §§ 3.3,5.3].

Now we can recall the definition of syntomic cohomology $H^n_{\text{syn}}(X,i)$ of X: it is the cohomology of a complex $R\Gamma_{\text{syn}}(X,i)$ defined as the homotopy limit of the following diagram

$$\begin{tikzcd}
& R\Gamma_{\text{rig}}(X_k) & & R\Gamma_{\text{rig},K}(X_k) & & R\Gamma_{\text{dR}}(]X_k[_{X_w}) & & E'_{\text{dR}}(X_K) & . \\
R\Gamma_{\text{rig}}(X_k) \arrow[ur, "\text{Id}"] \arrow[rr, equal] & & R\Gamma_{\text{rig}}(X_k) \arrow[ul, "\phi/p^i"'] \arrow[ur] & & R\Gamma_{\text{rig}}(X_k, X^w) \arrow[ul] \arrow[ur] & & E_{\text{dR}}(X_K) \arrow[ul] \arrow[ur] & & E_{\text{FdR},i}(X_K) \arrow[ul]
\end{tikzcd}$$

(cf. [Bes00], [CCM12]). To be precise Besser uses the cone of $\phi - p^i\,\text{Id}$ instead of $\text{Id} - \phi/p^i$.

Proposition 3.5.2. *Let R be the valuation ring of a p-adic field K, then there exists a ring spectrum $\mathbb{E}_{\text{syn}}$ in $DM_{\text{Б}}(R,\mathbb{Q}_p)$ representing the syntomic cohomology defined by Besser, i.e. for any smooth R-scheme X and any integer n, there is a canonical isomorphism*

$$H^n_{\text{syn}}(X,i) \cong \mathbb{E}^{n,i}_{\text{syn}}(X) := \text{Hom}_{DM_{\text{Б}}(R,\mathbb{Q}_p)}(M(X), \mathbb{E}_{\text{syn}}(i)[n]) .$$

In particular all the results of section 2 apply to syntomic cohomology.

Proof. By construction the absolute rigid spectrum $\mathbb{E}_\phi$ maps to $\mathbb{E}_{\text{rig}}$ and so to the base change $\mathbb{E}_{\text{rig},K}$. By the six functor formalism we get the following functors

$$i_* : DM_{\text{Б}}(k,\mathbb{Q}_p) \to DM_{\text{Б}}(R,\mathbb{Q}_p) \ , \ j_* : DM_{\text{Б}}(K,\mathbb{Q}_p) \to DM_{\text{Б}}(R,\mathbb{Q}_p)$$

induced by the usual closed (resp. open) immersion of schemes $i : \text{Spec}(k) \to \text{Spec}(R)$ (resp. $j : \text{Spec}(K) \to \text{Spec}(R)$). Then we define $\mathbb{E}_{\text{syn}}$ as the homotopy limit (in the category of ring spectra) of the following diagram

$$i_*\mathbb{E}_\phi \to i_*\mathbb{E}_{\text{rig},K} \leftarrow a \to b \leftarrow c \to d \leftarrow j_*\mathbb{E}_{\text{FdR}}$$

where a, b, c, d are the ring spectra induced by $E_{\text{rig}}(X_k, X^w)$, $R\Gamma_{\text{dR}}(]X_k[_{X_w})$, $E_{\text{dR}}(X_K)$, $E'_{\text{dR}}(X_K)$, respectively: we leave to the reader the verification that they are ring spectra following the same proof as the one of 3.3.3.

To conclude the proof it is sufficient to note that an homotopy limit of a diagram of Morel motives is also a Morel motive. □

Remark 3.5.3. Given a complete discrete valuation ring R with residue field k and fraction field K, such that $R/W(k)$ is finite, we get a map of ring spectra in $DM_{\text{Б}}(k)$:

$$a_0 : \mathbb{E}_\phi \to \mathbb{E}_{\text{rig},K_0} \to \mathbb{E}_{\text{rig},K_0} \otimes_{K_0} K \xrightarrow{\sim} \mathbb{E}_{\text{rig},K}$$

where the last isomorphism comes from Remark 3.3.4. Let us put $a = i_*(a_0)$.

Secondly, we get a morphism of ring spectra in $DM_{\text{Б}}(R)$:

$$b : j_*\mathbb{E}_{\text{FdR}} \to j_*\mathbb{E}_{\text{dR}} \xrightarrow{\text{sp}} i_*\mathbb{E}_{\text{rig},K}.$$

The first map is the canonical morphism, and the second one is the specialization map induced by the morphism sp_X of Paragraph 3.5.1.

Then the syntomic ring spectrum is characterized up to isomorphism by the following homotopy pullback square (of morphisms of ring spectra):

(3.5.3.a)
$$\begin{tikzcd} \mathbb{E}_{\mathrm{syn}} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & j_*\mathbb{E}_{\mathrm{FdR}} \arrow[d, "b"] \\ i_*\mathbb{E}_\phi \arrow[r, "a"] & i_*\mathbb{E}_{\mathrm{rig},K} \end{tikzcd}$$

In other words, one can define $\mathbb{E}_{\mathrm{syn}}$ as the homotopy limit of the lower corner of the above diagram – but this definition is less precise than the one given in the proof of the previous proposition as (in this way) $\mathbb{E}_{\mathrm{syn}}$ is defined only up to non unique isomorphism.

The fact that the preceding square is a homotopy pullback can be translated into the existence of a distinguished triangle in $DM_{\mathrm{B}}(R)$:

(3.5.3.b)
$$\mathbb{E}_{\mathrm{syn}} \xrightarrow{\alpha+\beta} i_*\mathbb{E}_\phi \oplus j_*\mathbb{E}_{\mathrm{FdR}} \xrightarrow{a-b} i_*\mathbb{E}_{\mathrm{rig},K} \xrightarrow{+1}$$

which corresponds to the long exact sequence, for X/R smooth:

(3.5.3.c)
$$\ldots \to H^n_{\mathrm{syn}}(X,i) \xrightarrow{\alpha_*+\beta_*} H^n_\phi(X_k,i) \oplus F^iH^n_{\mathrm{dR}}(X_K) \xrightarrow{a_*-b_*} H^n_{\mathrm{rig}}(X_k/K) \to \ldots$$

Here, α_* (resp. β_*) is the usual projection map from syntomic cohomology to $\mathbb{E}^{n,i}_\phi(X_k) = H^n_\phi(X_k,i)$ (resp. $F^iH^n_{\mathrm{dR}}(X_K)$) while a_* is the canonical map and b_* is induced by the specialization map from de Rham cohomology to rigid cohomology.

Note also that $\mathbb{E}_{\mathrm{syn}}$ is the homotopy limit of the diagram of ring spectra

$$\begin{tikzcd} & & j_*\mathbb{E}_{\mathrm{FdR}} \arrow[d, "b"] \\ i_*\mathbb{E}_{\mathrm{rig}} \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\mathrm{Id}"'] & i_*\mathbb{E}_{\mathrm{rig}} \arrow[r] & i_*\mathbb{E}_{\mathrm{rig},K} \end{tikzcd}$$

so that we also obtain the following distinguished triangle:

$$\mathbb{E}_{\mathrm{syn}} \to i_*\mathbb{E}_{\mathrm{rig}} \oplus j_*\mathbb{E}_{\mathrm{FdR}} \to i_*\mathbb{E}_{\mathrm{rig}} \oplus i_*\mathbb{E}_{\mathrm{rig},K} \xrightarrow{+1}$$

which precisely induces the long exact sequence originally considered by Besser.

Remark 3.5.4. Syntomic cohomology can be functionally extended to diagrams of S-schemes, as well as rigid cohomology, absolute rigid cohomology and filtered de Rham cohomology. One should be careful however that the syntomic long exact sequence (3.5.3.b) can be extended only to the case of diagrams of smooth S-schemes.

3.6. **Localizing syntomic cohomology.**

3.6.1. As the fibred triangulated category DM_{B} satisfies the "gluing formalism" (this is called the localization property in [CD12b], cf. sec. 2.3), we get a canonical distinguished triangle:

(3.6.1.a)
$$i_*i^!(\mathbb{E}_{\mathrm{syn}}) \xrightarrow{ad'_i} \mathbb{E}_{\mathrm{syn}} \xrightarrow{ad_j} j_*j^*(\mathbb{E}_{\mathrm{syn}}) \xrightarrow{\partial_i} i_*i^!(\mathbb{E}_{\mathrm{syn}})[1]$$

for $i : \operatorname{Spec} k \to \operatorname{Spec} R$ and $j : \operatorname{Spec} K \to \operatorname{Spec} R$ the natural immersions. The maps ad'_i and ad_j are the obvious adjunction maps and the map ∂_i is the unique morphism which fits in this distinguished triangle (see [CD12b, 2.3.3]).

Remark 3.6.2. One can be more precise about the gluing formalism: given any object M of $DM_{\mathrm{B}}(R)$, there exists a unique distinguished triangle of the form

$$M_k \to M \to M_K \xrightarrow{\partial} M_k[1]$$

such that M_k (resp. M_K) has support in $\operatorname{Spec} k$, *i.e.* $j^*M_k = 0$ (resp. in $\operatorname{Spec} K$, *i.e.* $i^!(M_K) = 0$). This means that there exists a canonical isomorphism of that triangle with the following one:

$$i_*i^!(M) \xrightarrow{ad'_i} M \xrightarrow{ad_j} j_*j^*(M) \xrightarrow{\partial_i} i_*i^!(M)[1].$$

3.6.3. Let us introduce yet another spectrum: we consider the map

$$a_0 : \mathbb{E}_\phi \to E_{\mathrm{rig},K}$$

which is defined at the level of the underlying model category, and take its homotopy fiber $\check{\mathbb{E}}_\phi$. In particular, we have a canonical morphism: $i_* E_{\mathrm{rig},K} \xrightarrow{\partial_a} i_* \check{\mathbb{E}}_\phi[1]$.

Proposition 3.6.4. *Consider the above notations. Then the syntomic spectrum is equivalent to the homotopy fiber of the morphism*

$$\check{\mathrm{sp}} : j_* \mathbb{E}_{\mathrm{FdR}} \xrightarrow{b} i_* \mathbb{E}_{\mathrm{rig},K} \xrightarrow{\partial_a} i_* \check{\mathbb{E}}_\phi[1] \ .$$

Moreover, there are canonical identifications

$$i^! \mathbb{E}_{\mathrm{syn}} = \check{\mathbb{E}}_\phi, \qquad j^* \mathbb{E}_{\mathrm{syn}} = \mathbb{E}_{\mathrm{FdR}}$$

through which the localization triangle (3.6.1.a) *is identified with*

$$i_* \check{\mathbb{E}}_\phi \to \mathbb{E}_{\mathrm{syn}} \to j_* \mathbb{E}_{\mathrm{dR}} \xrightarrow{\check{\mathrm{sp}}} i_* \check{\mathbb{E}}_\phi[1].$$

Remark 3.6.5. In fancy terms, the generic fiber of $\mathbb{E}_{\mathrm{syn}}$ is the ring spectrum $\mathbb{E}_{\mathrm{FdR}}$. While we cannot compute the special fiber of $\mathbb{E}_{\mathrm{syn}}$, its exceptional special fiber is the ring spectrum which is "the image of absolute rigid cohomology in rigid cohomology" and $\mathbb{E}_{\mathrm{syn}}$ is obtained by gluing these two ring spectra.

Proof. By definition of $\check{\mathbb{E}}_\phi$, there is a canonical distinguished triangle in $DM_{\mathrm{B}}(k)$:

$$\check{\mathbb{E}}_\phi \xrightarrow{\nu_0} \mathbb{E}_\phi \xrightarrow{a_0} E_{\mathrm{rig},K} \xrightarrow{\partial_{a_0}} \check{\mathbb{E}}_\phi[1]$$

which induces the following triangle after applying i_*:

$$i_* \check{\mathbb{E}}_\phi \xrightarrow{\nu} i_* \mathbb{E}_\phi \xrightarrow{a} i_* E_{\mathrm{rig},K} \xrightarrow{\partial_a} i_* \check{\mathbb{E}}_\phi[1].$$

Now according to the fact the square (3.5.3.a) is a homotopy pullback, one gets a canonical commutative diagram in $DM_{\mathrm{B}}(R)$:

$$\begin{tikzcd}
C(\alpha) \arrow[r] \arrow[d, "\sim"'] & \mathbb{E}_{\mathrm{syn}} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & j_* \mathbb{E}_{\mathrm{FdR}} \arrow[r] \arrow[d, "b"] & C(\alpha)[1] \arrow[d, "\sim"] \\
i_* \check{\mathbb{E}}_\phi \arrow[r, "\nu"] & i_* \mathbb{E}_\phi \arrow[r, "a"] & i_* E_{\mathrm{rig},K} \arrow[r, "\partial_a"] & i_* \check{\mathbb{E}}_\phi[1].
\end{tikzcd}$$

In other words, we get a distinguished triangle of the form:

$$i_* \check{\mathbb{E}}_\phi \to \mathbb{E}_{\mathrm{syn}} \xrightarrow{\alpha} j_* \mathbb{E}_{\mathrm{FdR}} \xrightarrow{\check{\mathrm{sp}}} i_* \check{\mathbb{E}}_\phi[1].$$

Finally, according to the above remark, one gets a canonical isomorphism of triangles:

$$\begin{tikzcd}
i_* \check{\mathbb{E}}_\phi \arrow[r] \arrow[d, "\sim"'] & \mathbb{E}_{\mathrm{syn}} \arrow[r, "\alpha"] \arrow[d, "\sim"'] & j_* \mathbb{E}_{\mathrm{FdR}} \arrow[r, "\check{\mathrm{sp}}"] \arrow[d, "\sim"] & i_* \check{\mathbb{E}}_\phi[1] \arrow[d, "\sim"] \\
i_* i^! \mathbb{E}_{\mathrm{syn}} \arrow[r, "ad_i"] & \mathbb{E}_{\mathrm{syn}} \arrow[r, "ad_j"] & j_* j^*(\mathbb{E}_{\mathrm{syn}}) \arrow[r, "\partial_i"] & i_* i^! \mathbb{E}_{\mathrm{syn}}[1] \ .
\end{tikzcd}$$

□

Remark 3.6.6 (The work of Tamme). The *relative cohomology theory* $H^*_{\mathrm{rel}}(X, *)$ of [Tam11] is represented by the (generalized) cone of the diagram

$$i_* \mathbb{E}_{\mathrm{rig},K} \leftarrow a \to b \leftarrow c \to d \leftarrow j_* \mathbb{E}_{\mathrm{FdR}}$$

where we use the notation of the proof of Proposition 3.5.2. This is roughly a Cone of a morphism of ring spectra $A \to B$, hence it is not a ring spectrum and in particular there is no unit section.

It follows by the localization sequence that this cohomology theory is represented by the cone of the canonical adjunction map $\mathbb{E}_{\mathrm{syn}} \to i_* i^! \mathbb{E}_{\mathrm{syn}} = i_* \mathbb{E}_\phi$.

Example 3.6.7. Let $S = \mathrm{Spec}\big(W(k)\big)$ (for simplicity) and X be the connected component of the Néron model of an elliptic curve with multiplicative reduction, i.e. X is an S-group scheme such that its generic fiber is an elliptic curve and the special fiber is isomorphic to $\mathbb{G}_m$. Then X/S is smooth and we can easily compute the long exact sequence for syntomic cohomology. For instance we get

$$0 \to H^0_{\mathrm{rig}}(X_s) \xrightarrow{a} H^0_{\mathrm{rig}}(X_s) \oplus H^0_{\mathrm{rig}}(X_s) \to H^{1,1}_{\mathrm{syn}}(X) \to$$

$$\to H^1_{\mathrm{rig}}(X_s) \oplus F^1 H^1_{\mathrm{dR}}(X_\eta) \xrightarrow{b} H^1_{\mathrm{rig}}(X_s) \oplus H^1_{\mathrm{rig}}(X_s) \to H^{2,1}_{\mathrm{syn}}(X) \to F^1 H^2_{\mathrm{dR}}(X_\eta) \to 0$$

where $a(x) = (x - \phi(x)/p, -x)$ is injective and $b(x, y) = (0, y - x)$. It follows that $H^{n,1}_{\mathrm{syn}} \cong K^2$ (as $\mathbb{Q}_p$-vector spaces) for $n = 1, 2$.

The same result can be obtained using the localization triangle: explicitly we get the following exact sequence

$$0 \to H^{1,1}_{\mathrm{syn},s}(X) \to H^{1,1}_{\mathrm{syn}}(X) \to F^1 H^1_{\mathrm{dR}}(X_\eta) \xrightarrow{\delta} H^{2,1}_{\mathrm{syn},s}(X) \to H^{2,1}_{\mathrm{syn}}(X) \to F^1 H^2_{\mathrm{dR}}(X_\eta) \to 0\ .$$

Here $H^{1,1}_{\mathrm{syn},s}(X)$ stands for $\mathrm{Hom}(\mathbb{Q}(X), i_* i^! \mathbb{E}_{\mathrm{syn}}(1)[1])$. Using proposition 3.6.4 we get $H^{n,1}_{\mathrm{syn},s}(X) = H^{n-1}_{\mathrm{rig}}(X_s)$ for $n = 1, 2$. We also get that δ is the zero map. For a complete account on the de Rham/rigid cohomology of abelian varieties and their reduction we refer to [LS86].

Example 3.6.8 (Semistable elliptic curve)**.** Let X/S be an elliptic curve such that X_k is a nodal cubic. We assume that the singular point $x_0 \in X_k$ is k-rational. The above remark give a recipe to compute (or approximate) the syntomic cohomology of X

$$\mathbb{E}^{n,i}_{\mathrm{syn}}(X) := \mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1}(S,\mathbb{Q}_p)}(M(X), \mathbb{E}_{\mathrm{syn}}(i)[n])$$

where $M(X) = f_! f^!(\mathbb{Q}_S)$ and $f : X \to S$ is the structural morphism. Let us compute $i^*(M(X))$. Given the pullback square:

$$\begin{tikzcd} X_k \arrow[r, "l"] \arrow[d, "f_0"'] & X \arrow[d, "f"] \\ \mathrm{Spec}\, k \arrow[r, "i"] & S \end{tikzcd}$$

one has a canonical exchange map:

$$i^* f_! f^!(\mathbb{Q}_S) \simeq f_{0!} l^* f^!(\mathbb{Q}_S) \to f_{0!} f_0^! i^*(\mathbb{Q}_S) = f_{0!} f_0^!(\mathbb{Q}_k) = M(X_k)$$

(the first iso is due to the base change theorem of the six functors formalism). This map is an isomorphism in the two following cases:

- f is smooth,
- X is regular and f is quasi-projective (and so in our case).

  In the second case, this is due to the absolute purity theorem: as f is quasi-projective, it can be factored $f = pi$ where $p : P \to S$ is smooth and i is a closed immersion and then one computes:

  $$f^!(\mathbb{Q}_S) = i^! p^!(\mathbb{Q}_S) \simeq i^!(\mathbb{Q}_P)(d)[2d] \simeq \mathbb{Q}_X(d - n)[2(d - n)]$$

  the first iso follows as p is smooth and the second one because i is a closed immersion between regular schemes. Here d (resp. n) is the relative dimension of P/S (resp. codimension of i), so that $d - n$ is the relative dimension of X/S.

Hence for X semistable we have long exact sequences

$$i^! \mathbb{E}^{n,i}_{\mathrm{syn}}(X_k) \to \mathbb{E}^{n,i}_{\mathrm{syn}}(X) \to j_* j^* \mathbb{E}^{n,i}_{\mathrm{syn}}(X) = F^i H_{\mathrm{dR}}(X_\eta) \to +$$

The term $i^! \mathbb{E}^{n,i}_{\mathrm{syn}}(X_k)$ depends only on the special fiber. In this case it is easy to construct a proper and smooth hypercover Y_* of X_k: let $\pi : \widetilde{X}_k \to X_k$ be the normalization map, then we may take $Y_0 = x_0 \sqcup \widetilde{X}_k$, $Y_1 = \pi^{-1}(x_0)$ and $Y_i = \varnothing$ for $i > 1$. Since $\widetilde{X}_k$ is isomorphic to the projective line we get that $M(X_k) = \mathbb{Q} \oplus \mathbb{Q}[1] \oplus \mathbb{Q}(1)[2]$ in $DM_{\mathrm{B}}(k, \mathbb{Q})$. This decomposition allows to estimate $i^! \mathbb{E}^{n,i}_{\mathrm{syn}}(X_k)$. For instance we can compute

$$i^! \mathbb{E}^{n,i}_{\mathrm{syn}}(X_k) = H^{n-1}_{\mathrm{rig}}(X_k)_K \simeq K \qquad \text{for } n = 1, 2 \; .$$

3.7. **Syntomic regulator.**

3.7.1. By using the general definition of § 2.1.3 we get the syntomic (resp. rigid, de Rham, etc.) cycle classes. Since all the maps of the homotopy pullback square (3.5.3.a) are morphisms of monoids in $DM_{\mathrm{B}}(R)$, we get the following commutative diagram:

$$\begin{tikzcd}
H^n_{\mathrm{syn}}(X, m) \arrow[rr, "\alpha_*"] \arrow[dd, "\beta_*"'] & & F^m H^n_{\mathrm{dR}}(X_K) \arrow[dd, "\mathrm{sp}"] \\
& H^{n,m}_{\mathrm{B}}(X) \arrow[ul, "\sigma_{\mathrm{syn}}"] \arrow[ur, "\sigma_{\mathrm{FdR}} j^*"'] \arrow[dl, "\sigma_\phi i^*"'] \arrow[dr, "\sigma_{\mathrm{rig}} i^*"] \arrow[r, phantom, "(b)"] \arrow[d, phantom, "(a)"] & {} \\
H^n_\phi(X_k, m) \arrow[rr] & {} & H^n_{\mathrm{rig}}(X_k/K)
\end{tikzcd}$$

where $\sigma_?$ stands for the higher cycle classes relevant to the corresponding cohomology , and i^* (resp. j^*) denotes the pullback in motivic cohomology by i (resp. j).

(1) The part (a) of the above commutative diagram simply express the fact that for any smooth k-scheme X_0, the higher cycle class map

$$\sigma_{\mathrm{rig}} : H^{n,m}_{\mathrm{B}}(X_0) \to H^n_{\mathrm{rig}}(X/K_0)$$

lands into the part $\phi = p^m$ of rigid cohomology and that it admits a canonical lifting to the absolute rigid cohomology $H^{n,m}_\phi(X)$ through the canonical surjection

$$H^{n,m}_\phi(X) \to H^n_{\mathrm{rig}}(X/K_0)^{\phi = p^m}$$

of Corollary 3.4.4.

(2) One can deduce from the commutativity of the part (b) of the above diagram another proof of the fact, already obtained in [CCM12], that the specialization map sp is compatible with the specialization map $\mathrm{sp}_{\mathrm{CH}}$ in Chow theory as defined in [Ful98, §20.3]. Indeed, in the case $n = 2m$, (b) can be rewritten as follows:

$$\begin{tikzcd}
& \mathrm{CH}^m(X_K) \arrow[r, "\sigma_{\mathrm{FdR}}"] \arrow[dd, dashed, "\mathrm{sp}_{\mathrm{CH}}"] & F^{2m} H^m_{\mathrm{dR}}(X_K) \arrow[dd, "\mathrm{sp}"] \\
\mathrm{CH}^m(X) \arrow[ur, "j^*"] \arrow[dr, "i^*"'] & & \\
& \mathrm{CH}^m(X_k) \arrow[r, "\sigma_{\mathrm{rig}}"] & H^m_{\mathrm{rig}}(X_k/K)
\end{tikzcd}$$

and the assertion follows as j^* is surjective and $\mathrm{sp}_{\mathrm{CH}}$ is the unique morphism making the left hand side commutative.

(3) (Concerning the terminology) The term "higher cycle classes" comes from the theory of higher Chow groups – which, for smooth R-schemes, coincide rationally with Beilinson motivic cohomology according to [Lev04, 14.7].

The term "syntomic regulator" has been introduced by M. Gros in [Gro90]. It comes from the intuition that syntomic cohomology is an analogue of Deligne cohomology and that one can transport the setting of Beilinson's conjectures from Deligne cohomology to syntomic cohomology. One should be careful however that in the case of Deligne cohomology and if $(2m - n) = 1$, then the higher cycle class map is only a part of the regulator (see [Sou86, §3.3]).

Remark 3.7.2. The syntomic Chern classes are constructed as in § 2.1.4. These are determined by the first Chern class c_1 of the canonical line bundle of $\mathbb{P}^1_R$. According to our construction of the syntomic ring spectrum, this is nothing else than the class dlog. One deduces that the Chern classes obtained here in syntomic cohomology coincide with the one previously constructed by A. Besser in [Bes00].

Proposition 3.7.3. *Let $f : Y \to X$ be a projective morphism between smooth R-schemes, and denote by f_k (resp. f_K) its special (resp. generic) fiber. Then the following diagram is commutative:*

$$\begin{tikzcd}
H^n_{\mathrm{syn}}(Y,i) \arrow[r, "\alpha_*+\beta_*"] \arrow[d, "f_*"] & H^n_\phi(Y_k,i)\oplus F^iH^n_{\mathrm{dR}}(Y_K) \arrow[r, "a_*-b_*"] \arrow[d, "f_{k*}+f_{K*}"] & H^n_{\mathrm{rig}}(Y_k/K) \arrow[r] \arrow[d, "f_{k*}"] & H^{n+1}_{\mathrm{syn}}(Y,i) \arrow[d, "f_*"] \\
H^{n-2d}_{\mathrm{syn}}(X,i-d) \arrow[r, "\alpha_*+\beta_*"'] & H^{n-2d}_\phi(X_k,i-d)\oplus F^{i-d}H^{n-2d}_{\mathrm{dR}}(X_K) \arrow[r, "a_*-b_*"'] & H^{n-2d}_{\mathrm{rig}}(X_k/K) \arrow[r] & H^{n-2d+1}_{\mathrm{syn}}(X,i-d)
\end{tikzcd}$$

where the lines are given by the exact sequences (3.5.3.c).

Proof. Applying the same formalism to the motivic ring spectra $\mathbb{E}_{\mathrm{FdR}}$, $\mathbb{E}_{\mathrm{rig},K}$, $\mathbb{E}_\phi$, one obtains Gysin morphisms on their cohomology, satisfying the preceding properties. Moreover, using the distinguished triangle (3.5.3.b) of $DM_{\mathrm{B}}(R)$, one gets the result. □

3.7.4. Recall that in § 2.2.1 we have associated four theories (cohomology, homology, coho. with compact support, BM homology) to any motivic ring spectrum.

(1) We get syntomic theories and the higher cycle class (2.1.3.a) also for singular R-schemes[25]. When focusing attention to Chow theory, one gets in particular:
   - X regular: $\sigma_{\mathrm{syn}} : \mathrm{CH}^n(X) \to H^{2n}_{\mathrm{syn}}(X,n)$.
   - X regular quasi-projective: $\sigma_{\mathrm{syn}} : \mathrm{CH}_n(X) \to H^{\mathrm{syn,BM}}_{2n}(X,n)$.

   The second point follows from the fact $H^{n,i}_{\mathrm{B}}(X) \simeq H^{\mathrm{B,BM}}_{2d-n,d-i}(X)$ where d is the (Krull) dimension of X according to the motivic absolute purity theorem ([CD12b, 14.4.1]).

(2) When the base scheme is $S = \operatorname{Spec} k$, we get rigid (resp. absolute rigid) theories associated with $\mathbb{E}_{\mathrm{syn},K}$ (resp. $\mathbb{E}_\phi$) and regulators for these theories.

   In the case $K = K_0$, the Frobenius operator Φ of $\mathbb{E}_{\mathrm{rig}}$ induces an action of Frobenius on all four theories, compatible with the regulator. Moreover, the distinguished triangle (3.4.6.a) yields long exact sequences in all four theories.

(3) When $S = \operatorname{Spec} K$, we get the de Rham theory (resp. filtered de Rham) associated with $\mathbb{E}_{\mathrm{dR}}$ (resp. $\mathbb{E}_{\mathrm{FdR}}$) equipped with regulators. The canonical map $\mathbb{E}_{\mathrm{FdR}} \to \mathbb{E}_{\mathrm{dR}}$ induces natural maps of these theories, compatible with regulators.

   Consider the specialization map:
$$\mathrm{sp} : j_*\mathbb{E}_{\mathrm{FdR}} \to i_*\mathbb{E}_{\mathrm{rig},K}.$$
   Given any R-scheme X with structural morphism f, and applying $f_*f^!$ to this map one obtains:
$$\mathrm{sp}_* : H^{\mathrm{FdR,BM}}_n(X_K,i) \to H^{\mathrm{rig},K,\mathrm{BM}}_n(X_k,i)$$
   using the exchange isomorphisms: $f^!i_* = i'_*f^!_k$ and $f^!j_* = j'_*f^!_K$. Similarly, if we apply $f_*f^!$ to the distinguished triangle (3.5.3.b), one gets the following long exact sequence:

(3.7.4.a)
$$\ldots \to H^{\mathrm{syn,BM}}_n(X,i) \xrightarrow{\alpha_*+\beta_*} H^{\phi,\mathrm{BM}}_n(X_k,i)\oplus H^{\mathrm{FdR,BM}}_n(X_K,i) \xrightarrow{a_*-\mathrm{sp}_*} H^{\mathrm{rig},K,\mathrm{BM}}_n(X_k,i) \to \ldots$$

3.7.5. All the theories considered in the previous paragraph satisfy the functorialities described in § 2.2.2. Moreover, regulators are compatible with these functorialities. Similarly, the maps sp_*, α_*, β_*, a_* and b_* considered in part (3) of this example are natural with respect to proper covariant and smooth contravariant functorialities.

[25] Recall that for singular schemes, Beilinson motivic cohomology is defined after [CD12b] and [Cis03] as the graded part of homotopy invariant K-theory for the γ-filtration.

Moreover, taking care of the functoriality explained in the previous remark for motivic BM-homology, one can check the following diagram is commutative:

$$\begin{tikzcd} & H^{\text{Б}}_{n,i}(X_K/K) \arrow[r, "\sigma_{\mathrm{FdR}}"] & H_n^{\mathrm{FdR,BM}}(X_K, i) \arrow[dd, "\mathrm{sp}_*"] \\ H^{\text{Б}}_{n,i}(X/R) \arrow[ru, "j^*"] \arrow[rd, "i^*"'] & & \\ & H^{\text{Б}}_{n,i}(X_k/k) \arrow[r, "\sigma_{\mathrm{rig}}"] & H_n^{\mathrm{rig},K,\mathrm{BM}}(X_k, i). \end{tikzcd}$$

When X/R is quasi-projective regular with good reduction and $i = 2n$, one obtains in particular a generalization of the second part of Remark 3.7.1 (applying the motivic absolute purity theorem [CD12b, 14.4.1], all the motivic BM-homology in the above diagram can be identified with Chow groups in that case).

This fact can be extended to the exact sequence (3.7.4.a) and to its compatibility with the regulator in syntomic BM-homology.

3.8. **Rigid syntomic modules.**

3.8.1. The aim of this last section is to apply the theory developed in [CD12b, sec. 7.2] to the syntomic ring spectrum $\mathbb{E}_{\mathrm{syn}}$.

Put $S = \operatorname{Spec} R$. Recall that by construction, $\mathbb{E}_{\mathrm{syn}}$ can be seen as an object of $\mathrm{Sp}^{\mathrm{ring}}(S, \mathbb{Q})$ (Paragraph 1.4.6).

Let $f : X \to S$ be any morphism of schemes. The pullback functor f^* on the category of Tate spectra is monoidal. Thus, it obviously induces a functor:

$$f^* : \mathrm{Sp}^{\mathrm{ring}}(S, \mathbb{Q}) \to \mathrm{Sp}^{\mathrm{ring}}(X, \mathbb{Q}).$$

In particular, we can define the rigid syntomic ring spectrum over X as follows:

$$\mathbb{E}_{\mathrm{syn},X} := f^*(\mathbb{E}_{\mathrm{syn}}).$$

The collection of these ring spectra defines a cartesian section of the fibered category $\mathrm{Sp}^{\mathrm{ring}}(-, \mathbb{Q})$ over the category of R-schemes. In particular, one can apply [CD12b, Prop. 7.2.11] to it. In particular, the category of modules over $\mathbb{E}_{\mathrm{syn},X}$ in $\mathrm{Sp}(X, \mathbb{Q})$ admits a model structure.

Definition 3.8.2. Consider the above notations.

We define the category $\mathbb{E}_{\mathrm{syn}}\text{-}\mathrm{mod}_X$ of *rigid syntomic modules over* X as the homotopy category of the model category of modules over the ring spectrum $\mathbb{E}_{\mathrm{syn},X}$.

3.8.3. According to [CD12b], Prop. 7.2.13 and 7.2.18, rigid syntomic modules inherit the good functoriality properties of the stable homotopy category (in the terminology of [CD12b, Def. 2.4.45], the category $\mathbb{E}_{\mathrm{syn}}$-mod, fibered over the category of R-schemes, is motivic). Let us recall briefly the six functors formalisms: given a morphism $f : T \to S$ of R-schemes, one has two pairs of adjoint functors:

$$f^* : \mathbb{E}_{\mathrm{syn}}\text{-}\mathrm{mod}_S \leftrightarrows \mathbb{E}_{\mathrm{syn}}\text{-}\mathrm{mod}_T : f_* \ ,$$
$$f_! : \mathbb{E}_{\mathrm{syn}}\text{-}\mathrm{mod}_T \leftrightarrows \mathbb{E}_{\mathrm{syn}}\text{-}\mathrm{mod}_S : f_! \ , \text{ for } f \text{ separated of finite type,}$$

and $\mathbb{E}_{\mathrm{syn}}\text{-}\mathrm{mod}_X$ is triangulated closed monoidal. We denote by $\otimes$ (resp. $\underline{\mathrm{Hom}}$) the tensor product (resp. internal Hom).

- $f_* = f_!$ for f proper,
- Relative purity: $f^! = f^*(d)[2d]$ for f smooth of constant relative dimension d,
- Base change formulas: $f^*g_! = g'_!f'^*$, for f any morphism (resp. g any separated morphism of finite type), f' (resp. g') the base change of f along g (resp. g along f).
- Projection formulas: $f^!(M \otimes f^*(N)) = f_!(M) \otimes N$.
- Localization property: given any closed immersion $i : Z \to S$ of R-schemes, with complementary open immersion j, there exists a distinguished triangle of natural transformations as follows:

$$j_!j^! \to 1 \to i_*i^* \xrightarrow{\partial_i} j_!j^![1]$$

where the first (resp. second) map denotes the counit (resp. unit) of the relevant adjunction (as in Paragraph 3.6.1).

Remark 3.8.4. An important set of properties is missing in the theory of rigid syntomic modules.

One will say that a syntomic module over X is *constructible* if and only if it is compact in the triangulated category $\mathbb{E}_{\mathrm{syn}}\text{-mod}_X$. The category of constructible modules should enjoy the following properties:

(1) they are stable by the six operations (when restricted to excellent R-schemes),
(2) they satisfy Grothendieck duality (existence of a dualizing module).

To get these properties, one has only to prove the absolute purity for syntomic modules: given any closed immersion $i : Z \to X$ of regular R-schemes, of pure codimension c, there exists an isomorphism:

$$i^!(\mathbb{1}_X) = \mathbb{1}_Z(c)[2c].$$

3.8.5. *Syntomic triangulated realization.*– Applying again [CD12b], Prop. 7.2.13, one gets for any R-scheme X an adjunction of triangulated categories:

$$L_X^{\mathrm{syn}} : DM_{\mathrm{B}}(X) \leftrightarrows \mathbb{E}_{\mathrm{syn}}\text{-mod}_X : \mathcal{O}_X^{\mathrm{syn}}$$

such that:

(1) $\mathcal{O}_X^{\mathrm{syn}}$ is conservative.
(2) For any Beilinson motive M over X, one has an isomorphism

$$\mathcal{O}^{\mathrm{syn}} L^{\mathrm{syn}}(M) \simeq M \otimes \mathbb{E}_{\mathrm{syn}}$$

functorial in M.
(3) The functor L_X^{syn} commutes with the operations f^*, $f_!$, $\otimes$.

Let us denote by $\mathbb{1}_X$ the unit object of $\mathbb{E}_{\mathrm{syn}}\text{-mod}_X$. According to point (2), one obtains a canonical isomorphism:

$$\mathrm{Hom}_{\mathbb{E}_{\mathrm{syn}}\text{-mod}_X}\big(\mathbb{1}_X, \mathbb{1}_X(i)[n]\big) \simeq \mathbb{E}_{\mathrm{syn}}^{n,i}(X)$$

which is functorial in X and compatible with products.

Remark 3.8.6. In the preceding section, one has derived Bloch-Ogus axioms, for syntomic cohomology and syntomic BM-homology, from the functoriality of DM_{B}. In fact, as in [BO74, Ex. 2.1], one can also obtain these axioms from the properties of syntomic modules stated above.

3.8.7. *Descent properties.*– According to [CD12b, sec. 3.1], the 2-functor $X \mapsto \mathbb{E}_{\mathrm{syn}}\text{-mod}_X$ can be extended to diagrams of R-schemes (as well as the syntomic triangulated realization). Moreover, the pair of functors (f^*, f_*) can be defined when f is a morphism of diagrams of R-schemes.

From [CD12b, 7.2.18], the motivic category $\mathbb{E}_{\mathrm{syn}}$-mod is separated. Therefore, according to [CD12b, 3.3.37], it satisfies h-descent (see Paragraph 2.2.11 for the h-topology): for any h-hypercover $p : \mathcal{X} \to X$ of R-schemes, the functor

$$p^* : \mathbb{E}_{\mathrm{syn}}\text{-mod}_X \to \mathbb{E}_{\mathrm{syn}}\text{-mod}_{\mathcal{X}}$$

is fully faithful.

Recall also the following more concrete version of descent: given any pseudo-Galois cover[26] $f : Y \to X$ of group G, any syntomic module M over X, the canonical morphism:

$$M \to \big(f_* f^*(M)\big)^G$$

is an isomorphism, where we have denoted by $?^G$ the fixed point for the obvious action of G.

[26] f is finite surjective and admits a factorization $f = pf'$ where f' is a Galois cover of group G and p is radicial.

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