Weyl-type bounds for Steklov eigenvalues

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Abstract: We present upper and lower bounds for Steklov eigenvalues for domains in $\mathbb{R}^{N+1}$ with C^2 boundary compatible with the Weyl asymptotics. In particular, we obtain sharp upper bounds on Riesz-means and the trace of corresponding Steklov heat kernel. The key result is a comparison of Steklov eigenvalues and Laplacian eigenvalues on the boundary of the domain by applying Pohozaev-type identities on an appropriate tubular neigborhood of the boundary and the min-max principle. Asymptotically sharp bounds then follow from bounds for Riesz-means of Laplacian eigenvalues.

Keywords: Steklov eigenvalue problem, Laplace-Beltrami operator, Eigenvalue bounds, Weyl eigenvalue asymptotics, Riesz-means, min-max principle, distance to the boundary, tubular neighborhood.

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1 Introduction.

Let $\Omega \subset \mathbb{R}^{N+1}$ be a bounded domain with boundary $\partial\Omega$. We consider the Steklov eigenvalue problem on Ω:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ denotes the derivative of u in the direction of the outward unit normal ν to $\partial\Omega$. A classical reference for problem (1.1) is [38] where it was introduced to describe the stationary heat distribution in a body whose flux through the boundary is proportional to the temperature on the boundary. When $N = 1$ problem (1.1) can be intepreted as the equation of a free membrane the mass of which is concentrated at the boundary (see [33]). The eigenvalues of problem (1.1) can be also seen as the eigenvalues of the Dirichlet-to-Neumann map (see e.g., the survey paper [24]). We also mention that recently the analogue of the Steklov problem has been introduced for the biharmonic operator as well in [10] (see also [9]).

It is well known that under mild regularity conditions on the boundary $\partial\Omega$ (see e.g., [24] for a detailed discussion), in particular if $\partial\Omega$ is piecewise C^1, problem

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(1.1) admits an increasing sequence of non-negative eigenvalues of the form

$$0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots \nearrow +\infty,$$

where the eigenvalues are repeated according to their multiplicity and satisfy the Weyl asymptotic formula (see [2])

$$\lim_{j\to\infty} \sigma_j j^{-1/N} = 2\pi\, B_N^{-1/N} |\partial\Omega|^{-1/N}, \tag{1.2}$$

with $|\partial\Omega|$ denoting the N-dimensional measure of $\partial\Omega$ and $B_N = \dfrac{\pi^{N/2}}{\Gamma(1+N/2)}$ being the volume of the N-dimensional unit ball. It is an open problem to find bounds on σ_j compatible with the Weyl-limit (1.2) except when $N = 1$ and $\partial\Omega$ is smooth (see [27]; see also [19] and the survey article [24]). The situation is different when we consider the eigenvalue problem for the Laplace-Beltrami operator on $\partial\Omega$, that is

$$-\Delta_{\partial\Omega}\, \varphi = \lambda\varphi \quad \text{on } \partial\Omega, \tag{1.3}$$

which for a connected and sufficiently regular $\partial\Omega$ (see Remark 4.13) admits an increasing sequence of non-negative eigenvalues of the form

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \nearrow +\infty,$$

satisfying the Weyl asymptotic formula

$$\lim_{j\to\infty} \lambda_j j^{-2/N} = (2\pi)^2\, B_N^{-2/N} |\partial\Omega|^{-2/N} \tag{1.4}$$

and Weyl-type bounds of the form (see e.g., [14], [17])

$$\lambda_j \le a_{\partial\Omega} + b_N j^{2/N} |\partial\Omega|^{-2/N} \tag{1.5}$$

for some positive constants $a_{\partial\Omega}, b_N$ depending only on the geometry and the dimension of the manifold $\partial\Omega$. We refer to [15] for an introduction to eigenvalue problems for the Laplace-Beltrami operator on Riemannian manifolds and to [17, 18, 19, 26] and to the references therein for a more detailed discussion on upper bounds for the eigenvalues of the Laplacian on manifolds.

The above asymptotic formulas suggest that at least for large j the Steklov eigenvalues σ_j are related to the Laplacian eigenvalues λ_j approximately via

$$\sigma_j \approx \sqrt{\lambda_j}. \tag{1.6}$$

The main result of our paper is a comparison between Steklov and Laplacian eigenvalues for all j compatible with the asymptotic relation (1.6).

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a bounded domain with boundary $\partial\Omega$ of class C^2 such that $\partial\Omega$ has only one connected component. Then there exists a constant c_Ω such that for all $j \in \mathbb{N}$*

$$\lambda_j \le \sigma_j^2 + 2c_\Omega \sigma_j, \quad \sigma_j \le c_\Omega + \sqrt{c_\Omega^2 + \lambda_j}\,. \tag{1.8}$$

In particular,

$$\left|\sigma_j - \sqrt{\lambda_j}\right| \le 2c_\Omega. \tag{1.9}$$

The constant c_Ω has the dimension of an inverse length and depends explicitely on the dimension N, the maximum of the mean of the absolute values of the principal curvatures $\kappa_i(x)$, $i=1,\dots,N$, on $\partial\Omega$ and the maximal possible size $\bar{h}$ of a suitable tubular neighborhood about $\partial\Omega$.

Remark 1.10. *We remark that Theorem 1.7 holds more in general for bounded domains in $\mathbb{R}^{N+1}$ of class C^2 with possibly disconnected boundary $\partial\Omega$. The proof is a straightforward adaptation of that in Section 4. Anyway, since Weyl-type bounds of the form* (1.5) *are known to hold for connected manifolds, and the purpose of the present paper is to prove bounds for Steklov eigenvalues, in order to keep a uniform notation, Theorem 1.7 is stated for domains with connected boundaries.*

For convex domains Ω we shall improve the estimates (1.8) such that they become sharp for all j when Ω is a ball of radius R and give the exact relation

$$\lambda_j = \sigma_j^2 + \frac{N-1}{R}\,\sigma_j$$

between Steklov and Laplacian eigenvalues on the N-dimensional ball and N-dimensional sphere of radius R respectively.
Clearly Theorem 1.7 implies Weyl-type estimates for Steklov eigenvalues from the bounds (1.5) for Laplacian eigenvalues (see Corollary 4.8). Combining the sharp Weyl-type estimates for Laplacian eigenvalues on hypersurfaces obtained in [25] with the estimates of Theorem 1.7 we prove the following sharp bound for Riesz means of Steklov eigenvalues:

Theorem 1.11. *Let $\Omega\subset\mathbb{R}^{N+1}$ be a bounded domain with boundary $\partial\Omega$ of class C^2 such that $\partial\Omega$ has only one connected component. Then for all $z\geq 0$*

$$\sum_{j=0}^{\infty}(z-\sigma_j)_+^2 \leq \frac{2}{(N+1)(N+2)}\,(2\pi)^{-N}B_N|\partial\Omega|\big(z+c_\Omega\big)^{N+2}, \tag{1.12}$$

where c_Ω is the constant from Theorem 1.7

The estimate (1.12) is asymptotically sharp since

$$\lim_{z\to\infty} z^{-N-2}\sum_{j=0}^{\infty}(z-\sigma_j)_+^2 = \frac{2}{(N+1)(N+2)}\,(2\pi)^{-N}B_N|\partial\Omega|$$

according to (1.2). Theorem 1.11 implies sharp upper bounds on the trace of the associated heat kernel (see Corollary 6.4) as well as lower bounds on the eigenvalues (see Corollary 6.6).

The present paper is organized as follows: in Section 2 we recall some properties of the squared distance function from the boundary in a suitable tubular neighborhood of a C^2 domain. We exploit these properties in Section 3 in order to obtain estimates of boundary integrals of harmonic functions. In particular, we establish a comparison between the $L^2(\partial\Omega)$ norms of the normal derivative and of the tangential gradient of harmonic functions which is used in Section 4 together with the min-max principle to prove our main Theorem 1.7 and, as a consequence, Weyl-type upper bounds for Steklov eigenvalues. In Section 5 we consider the case of convex C^2 domains for which we refine the estimates (1.8), which become sharp in the case of the ball. Finally, in Section 6 we prove Theorem 1.11 as well as upper bounds on the trace of the Steklov heat kernel and lower bounds on Steklov eigenvalues which turn out to be asymptotically sharp.

2 The squared distance function from the boundary

In this section we collect a number of properties of the distance and squared distance functions from the boundary $\partial\Omega$ of a C^2 domain of $\mathbb{R}^{N+1}$ which will be used in the proof of the main result.

We set

$$d_0(x) := \begin{cases} \mathrm{dist}(x, \partial\Omega), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \partial\Omega, \\ -\mathrm{dist}(x, \partial\Omega), & \text{if } x \in \mathbb{R}^{N+1} \setminus \overline{\Omega}. \end{cases}$$

Let $x \in \partial\Omega$ and let $\nu(x)$ denote the outward unit normal to $\partial\Omega$ at x. We have the following characterization of $\nu(x)$ in terms of $d_0(x)$:

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Then for $x \in \partial\Omega$*

$$\nu(x) = -\nabla d_0(x).$$

We refer to [21, Ch.7, Theorem 8.5] for the proof of Lemma 2.1. Let $h > 0$. The h-tubular neighborhood ω_h of $\partial\Omega$ is defined as

$$\omega_h := \{x \in \Omega : d_0(x) < h\}. \tag{2.2}$$

We have the following:

Theorem 2.3. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Then there exists $h > 0$ such that every point in ω_h has a unique nearest point on $\partial\Omega$.*

We refer to [30] for the proof of Theorem 2.3 (see also [21, Ch.6, Theorem 6.3] and [23, Lemma 14.16]). Throughout the rest of the paper we shall denote by $\bar{h}$ the maximal possible tubular radius of Ω, namely

$$\bar{h} := \sup \{h > 0 : \text{every point in } \omega_h \text{ has a unique nearest point on } \partial\Omega\}. \tag{2.4}$$

From Theorem 2.3 it follows that if Ω is of class C^2 such $\bar{h}$ exists and is positive. For any $h \in]0, \bar{h}[$ we denote by Γ_h the set

$$\Gamma_h := \partial\omega_h \setminus \partial\Omega. \tag{2.5}$$

Throughout the rest of this section, we will denote by h a positive number such that $h \in]0, \bar{h}[$. In a tubular neighborhood ω_h the distance function (and hence its square) is of class C^2. This is stated in the following:

Theorem 2.6. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Let ω_h be as in (2.2). Then d_0 is of class C^2 in ω_h. Moreover, for any $x \in \partial\Omega$, the matrix $D^2\left(d_0(x)^2/2\right)$ represents the orthogonal projection on the normal space to $\partial\Omega$ at x and*

$$d_0(x - p\nu(x)) = p,$$

$$\nabla d_0(x - p\nu(x)) = -\nu(x),$$

for any $0 \leq p \leq h$.

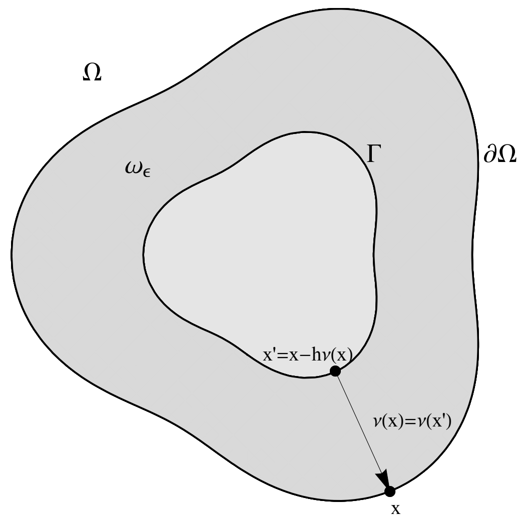

Figure 1: Tubular neighborhood of a C^2 planar domain.

We refer to [4, Theorem 3.1], [21, Ch.7, Theorem 8.5] and [23, Lemma 14.16] for the proof of Theorem 2.6. The situation described in Theorems 2.3 and 2.6 is illustrated in Figure 1.

Remark 2.7. *From Theorem 2.6 it follows that the set Γ_h is diffeomorphic to $\partial\Omega$.*

Let $x \in \partial\Omega$ and let $\kappa_1(x), ..., \kappa_N(x)$ denote the principal curvatures of $\partial\Omega$ at x with respect to the outward unit normal. We refer e.g., to [23, Sec. 14.6] for the definition and basic properties of the principal curvatures of $\partial\Omega$. We have the following:

Lemma 2.8. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Let $x \in \omega_h$ and let $y \in \partial\Omega$ be the nearest point to x on $\partial\Omega$. Then*

$$1 - d_0(x)\kappa_i(y) > 0 \tag{2.9}$$

for all $i = 1, ..., N$.

We refer to [34, Lemma 2.2] for a proof of Lemma 2.8. We note that the number $\bar{h}$ in (2.4) provides an upper bound for the positive principal curvatures of $\partial\Omega$. In fact we have

$$K_+ := \max_{\substack{1 \le i \le N, \\ x \in \partial\Omega}} \max\{0, \kappa_i(x)\} < \frac{1}{\bar{h}}. \tag{2.10}$$

We also define K_- by

$$K_- := \min_{\substack{1 \le i \le N, \\ x \in \partial\Omega}} \min\{0, \kappa_i(x)\} \le 0. \tag{2.11}$$

and K_∞ by

$$K_\infty := \max\{K_+, -K_-\} = \max_{\substack{1 \le i \le N, \\ x \in \partial\Omega}} |\kappa_i(x)|. \tag{2.12}$$

Now we introduce the functions d and η from ω_h to $\mathbb{R}$ defined by

$$d(x) := \operatorname{dist}(x, \Gamma_h)$$

and

$$\eta(x) := \frac{d(x)^2}{2}.$$

Clearly $d(x) = h - d_0(x)$ for all $x \in \omega_h$, hence d and η are of class C^2 in ω_h.

Let $x \in \partial\Omega$ and let $x' = x - h\nu(x) \in \Gamma_h$. Let now $\kappa_1'(x'), ..., \kappa_N'(x')$ denote the principal curvatures of Γ_h at x' with respect to the outward unit normal. The principal curvatures $\kappa_i'(x')$ and $\kappa_i(x)$ are related, as stated in the following:

Lemma 2.13. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Let ω_h and Γ_h be defined by (2.2) and (2.5), respectively. Let $x \in \partial\Omega$ and let $x' = x - h\nu(x) \in \Gamma_h$. Then we have*

$$\kappa_i'(x') = \frac{\kappa_i(x)}{1 - h\kappa_i(x)} \tag{2.14}$$

for all $i = 1, ..., N$. Moreover, $\nu(x) = \nu(x')$.

The proof of Lemma 2.13 follows from [3, Theorem 3] and from the fact that $d(x) = h - d_0(x)$ (see also [37]).

Now we are ready to state the following theorem concerning the eigenvalues of $D^2\eta$.

Theorem 2.15. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Let ω_h and Γ_h be defined by (2.2) and (2.5), respectively. Let $x \in \omega_h$ and let $y' = x + d(x)\nabla d(x) \in \Gamma_h$ be the nearest point to x on Γ_h. Then, denoting by $\rho_1(x), ..., \rho_N(x)$ the eigenvalues of $D^2\eta(x)$ it holds*

$$\rho_i(x) = \begin{cases} \frac{d(x)\kappa_i'(y')}{1 + d(x)\kappa_i'(y')}, & \text{if } 1 \le i \le N, \\ 1, & \text{if } i = N+1. \end{cases}$$

The proof of Theorem 2.15 can be carried out in a similar way as in [6, Lemma 1] (see also [23, Lemma 14.17]). We also refer to [3, Theorem 4] and [4, Theorem 3.2] for an alternative approach.

From now on we will agree to order the eigenvalues $\rho_i(x)$ of $D^2\eta(x)$ increasingly, so that $\rho_1(x) \le \rho_2(x) \le \cdots \le \rho_{N+1}(x) = 1$.

We conclude this section by presenting some bounds for the eigenvalues $\rho_i(x)$ when $x \in \omega_h$. We have the following:

Lemma 2.16. *Let Ω, ω_h and Γ_h be as in Theorem 2.15. Let $x \in \omega_h$ and let $\rho_i(x)$ denote the eigenvalues of $D^2\eta(x)$ for $i = 1, ..., N$. Then*

$$hK_- \le \rho_i(x) \le hK_+ < 1. \tag{2.17}$$

Proof. Let $x \in \omega_h$ and let y be the unique nearest point to x on $\partial\Omega$. From (2.14) and from the fact that $d(x) = h - d_0(x)$ it follows that

$$\rho_i(x) = 1 - \frac{1 - h\kappa_i(y)}{1 - d_0(x)\kappa_i(y)}. \tag{2.18}$$

We observe that the function $\kappa \mapsto 1 - \frac{1-h\kappa}{1-d\kappa}$ is increasing and convex for all $0 \le d \le h$, provided $\kappa < 1/h$ (which is always the case, see (2.10) and (2.11)). Moreover the function $d \mapsto 1 - \frac{1-h\kappa}{1-d\kappa}$ is decreasing and concave if $\kappa \ge 0$ and increasing and concave if $\kappa \le 0$. Then

$$\rho_i(x) \le 1 - \frac{1-hK_+}{1-d_0(x)K_+} \le hK_+$$

and

$$\rho_i(x) \ge 1 - \frac{1-hK_-}{1-d_0(x)K_-} \ge hK_-,$$

since $K_- \le 0 \le K_+$. This concludes the proof. □

Remark 2.19. *If Ω is a convex domain of class C^2 we have that $\kappa_i(x) \ge 0$ for all $i = 1, ..., N$ and for all $x \in \partial\Omega$, hence $0 \le \rho_i(x) \le 1$, for all $i = 1, ..., N+1$ and for all $x \in \omega_h$. Moreover Theorem 2.15 holds for all $h \in]0, 1/K_\infty[$ (see Section 5). This is not true for general non-convex domains, since it is not possible to estimate the size of the maximum tubular neighborhood ω_h only in terms of the principal curvatures. In fact h can be much smaller than $1/K_\infty$ (see Figure 2).*

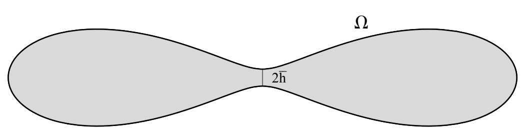

Figure 2: If the domain is not convex we can have arbitrary small $\bar{h}$ while K_∞ is uniformly bounded.

3 Boundary integrals of harmonic functions

The aim of this section is to prove that for a function $v \in H^2(\Omega)$ harmonic in Ω, the norms $\|\nabla_{\partial\Omega} v\|_{L^2(\partial\Omega)}$ and $\|\frac{\partial v}{\partial \nu}\|_{L^2(\partial\Omega)}$ are equivalent. Here $\nabla_{\partial\Omega} v$ denotes the tangential gradient of a function $v \in H^1(\partial\Omega)$. This is the usual intrinsic gradient of v on the Riemannian C^2-manifold $\partial\Omega$ with the induced Riemannian metric of $\mathbb{R}^{N+1}$. We will denote by $H^m(\Omega)$ (respectively $H^m(\partial\Omega)$) the Sobolev spaces of real-valued functions in $L^2(\Omega)$ (respectively $L^2(\partial\Omega)$) with weak derivatives up to order m in $L^2(\Omega)$ (respectively $L^2(\partial\Omega)$). We will also denote by $d\sigma$ the N-dimensional measure element of $\partial\Omega$.

We start with the following generalized Pohozaev identity for harmonic functions:

Lemma 3.1. *Let $F : \Omega \to \mathbb{R}^{N+1}$ be a Lipschitz vector field. Let $v \in H^2(\Omega)$ with $\Delta v = 0$ in Ω. Then*

$$\int_{\partial\Omega} \frac{\partial v}{\partial \nu} F \cdot \nabla v d\sigma - \frac{1}{2}\int_{\partial\Omega} |\nabla v|^2 F \cdot \nu d\sigma + \frac{1}{2}\int_\Omega |\nabla v|^2 \mathrm{div} F dx - \int_\Omega (DF \cdot \nabla v) \cdot \nabla v dx = 0, \quad (3.2)$$

where DF denotes the Jacobian matrix of F.

Proof. Since v is harmonic in Ω, we have $\Delta v F\cdot\nabla v=0$ in Ω. We integrate such identity over Ω. Throughout the rest of the proof we shall write $\partial_i v$ for $\frac{\partial v}{\partial x_i}$ and $\partial^2_{ik}v$ for $\frac{\partial^2 v}{\partial x_i\partial x_k}$. We have

$$
\begin{aligned}
0=\int_\Omega \Delta v F\cdot\nabla v dx &= \int_{\partial\Omega}\frac{\partial v}{\partial\nu}F\cdot\nabla v d\sigma-\int_\Omega \nabla v\cdot\nabla(F\cdot\nabla v)dx\\
&=\int_{\partial\Omega}\frac{\partial v}{\partial\nu}F\cdot\nabla v d\sigma-\int_\Omega (DF\cdot\nabla v)\cdot\nabla v dx-\int_\Omega (D^2v\cdot F)\cdot\nabla v dx, \quad (3.3)
\end{aligned}
$$

where D^2v denotes the Hessian matrix of v. Now let us consider the third summand in (3.3). We have

$$
\begin{aligned}
\int_\Omega (D^2v\cdot F)\cdot\nabla v dx &= \int_\Omega\sum_{i,k=1}^{N+1}\partial_i v\partial^2_{ik}vF_k dx\\
&=\int_{\partial\Omega}\sum_{i,k=1}^{N+1}\partial_i v\partial_i vF_k\nu_k d\sigma-\int_\Omega\sum_{i,k=1}^{N+1}\partial_i v\partial_k(\partial_i vF_k)dx\\
&=\int_{\partial\Omega}|\nabla u|^2F\cdot\nu d\sigma-\int_\Omega|\nabla v|^2\mathrm{div}F dx-\int_\Omega (D^2v\cdot F)\cdot\nabla v dx,
\end{aligned}
$$

thus

$$
\int_\Omega (D^2v\cdot F)\cdot\nabla v dx=\frac12\int_{\partial\Omega}|\nabla v|^2F\cdot\nu d\sigma-\frac12\int_\Omega|\nabla v|^2\mathrm{div}F dx. \quad (3.4)
$$

We plug (3.4) in (3.3) and finally obtain (3.2). This concludes the proof of the lemma. □

Remark 3.5. *When $F=x$, formula* (3.2) *is usually referred as Pohozaev identity. It reads*

$$
\int_{\partial\Omega}\frac{\partial v}{\partial\nu}x\cdot\nabla v d\sigma-\frac12\int_{\partial\Omega}|\nabla v|^2x\cdot\nu d\sigma+\frac{N-1}{2}\int_\Omega|\nabla v|^2dx=0, \quad (3.6)
$$

for all $v\in H^2(\Omega)$ with $\Delta v=0$. Formula (3.6) *when Ω is a ball in $\mathbb{R}^{N+1}$ allows to write the exact relations between the Steklov eigenvalues of Ω and the Laplace-Beltrami eigenvalues on $\partial\Omega$ without knowing explicitly the eigenvalues (see Subsection 5.2). For a general domain Ω of class C^2 it is natural to use F as in* (3.7) *here below.*

Let Ω be a bounded domain of class C^2 in $\mathbb{R}^{N+1}$. Let $h\in]0,\bar h[$, where $\bar h$ is given by (2.4), and ω_h be as in (2.2). Let $F:\Omega\to\mathbb{R}^{N+1}$ be defined by

$$
F(x):=\begin{cases}0, & \text{if } x\in\Omega\setminus\omega_h,\\ \nabla\eta, & \text{if } x\in\omega_h.\end{cases} \quad (3.7)
$$

By construction F is a Lipschitz vector field. We consider formula (3.2) with F given by (3.7). We use the fact that for $v\in H^1(\Omega)$ (and hence for $v\in H^2(\Omega)$),

$|\nabla v|^2_{|\partial\Omega} = |\nabla_{\partial\Omega} v|^2 + \left(\frac{\partial v}{\partial \nu}\right)^2$. Moreover, we use the fact that $F(x) = h\nu(x)$ when $x \in \partial\Omega$. We have

$$0 = \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu}\right)^2 d\sigma - \int_{\partial\Omega} |\nabla_{\partial\Omega} v|^2 d\sigma + \frac{1}{h}\left(\int_{\omega_h} |\nabla v|^2 \Delta\eta - 2(D^2\eta \cdot \nabla v) \cdot \nabla v dx\right). \quad (3.8)$$

Let $x \in \omega_h$. From (3.8), in order to compare the integrals of $|\nabla_{\partial\Omega} v|^2$ and $\left(\frac{\partial v}{\partial \nu}\right)^2$ over $\partial\Omega$, we have to estimate

$$|\nabla v(x)|^2 \Delta\eta(x) - 2(D^2\eta(x) \cdot \nabla v(x)) \cdot \nabla v(x). \quad (3.9)$$

We have the following lemma.

Lemma 3.10. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Let ω_h be as in (2.2). For any $v \in H^1(\Omega)$ it holds*

$$\left|\int_{\omega_h} |\nabla v|^2 \Delta\eta - 2(D^2\eta \cdot \nabla v) \cdot \nabla v dx\right| \leq \left(1 + N\bar{H}_\infty h\right) \int_\Omega |\nabla v|^2 dx, \quad (3.11)$$

where

$$\bar{H}_\infty := \max_{x \in \partial\Omega} \left(\frac{1}{N}\sum_{i=1}^{N} |\kappa_i(x)|\right).$$

Proof. Let $x \in \omega_h$. Let $\xi_i(x)$, $i = 1, ..., N+1$ be the eigenvectors of $D^2\eta(x)$ associated with the eigenvalues $\rho_i(x)$ and normalized such that $\xi_i(x) \cdot \xi_j(x) = \delta_{ij}$. We can write then

$$\nabla v(x) = \sum_{i=1}^{N+1} \alpha_i(x)\xi_i(x),$$

for some $\alpha_i(x) \in \mathbb{R}$. We note that $|\nabla v(x)|^2 = \sum_{i=1}^{N+1} \alpha_i(x)^2$. With this notation (3.9) can be re-written as follows:

$$Q(\nabla v(x)) := |\nabla v(x)|^2 \Delta\eta(x) - 2(D^2\eta(x) \cdot \nabla v(x)) \cdot \nabla v(x) = \sum_{i=1}^{N+1} \alpha_i(x)^2 \sum_{i=1}^{N+1} \rho_i(x) - 2\sum_{i=1}^{N+1} \rho_i(x)\alpha_i(x)^2. \quad (3.12)$$

Suppose that $\nabla v \neq 0$, otherwise inequality (3.11) is trivially true. We have that

$$Q(\nabla v(x)) = \sum_{i=1}^{N+1} \rho_i(x)(1 - 2\tilde{\alpha}_i(x)^2)|\nabla v(x)|^2, \quad (3.13)$$

where

$$\tilde{\alpha}_i(x) := \frac{\alpha_i(x)}{\sqrt{\sum_{i=1}^{N+1} \alpha_i(x)^2}} = \frac{\alpha_i(x)}{|\nabla v(x)|}.$$

It is straightforward to see that

$$\left(\sum_{i=1}^{N}\rho_i(x)-1\right)|\nabla v(x)|^2 \le \sum_{i=1}^{N+1}\rho_i(x)(1-2\tilde{\alpha}_i(x)^2)|\nabla v(x)|^2$$
$$\le \left(1+\sum_{i=2}^{N}\rho_i(x)-\rho_1(x)\right)|\nabla v(x)|^2. \quad (3.14)$$

Now from (2.9),(2.17) and (2.18) it follows that

$$|\rho_i(x)|-h|\kappa_i(x)| = -\frac{d_0(x)|\kappa_i(y)|(1-\kappa_i(y))}{1-d_0(x)\kappa_i(y)} \le 0$$

for all $x\in\omega_h$ and $i=1,...,N$, where $y\in\partial\Omega$ is the unique nearest point to x on $\partial\Omega$. Hence for all $x\in\omega_h$

$$\sum_{i=2}^{N}\rho_i(x)-\rho_1(x) \le \sum_{i=1}^{N}|\rho_i(x)| \le N\bar{H}_\infty h. \quad (3.15)$$

On the other hand, again from (2.9),(2.17) and (2.18) we have that

$$\rho_i(x)-(h-d_0(x))\kappa_i(y) = \frac{(h-d_0(x))d_0(x)\kappa_i(y)^2}{1-d_0(x)\kappa_i(y)} \ge 0,$$

for all $x\in\omega_h$ and $i=1,...,N$, where $y\in\partial\Omega$ is the unique nearest point to x on $\partial\Omega$. Hence for all $x\in\omega_h$

$$\sum_{i=1}^{N}\rho_i(x) \ge -NH_\infty h \ge -N\bar{H}_\infty h, \quad (3.16)$$

where

$$H_\infty := \max_{x\in\partial\Omega}\frac{1}{N}\left|\sum_{i=1}^{N}\kappa_i(x)\right| \quad (3.17)$$

denotes the maximal mean curvature of $\partial\Omega$. Clearly (3.12), (3.13), (3.14),(3.15) and (3.16) imply

$$|Q(\nabla v(x))| \le (1+N\bar{H}_\infty h)|\nabla v(x)|^2$$

and therefore the validity of (3.11). This concludes the proof.

□

From now on we will write

$$c_\Omega := \frac{1}{2\bar{h}}+\frac{N\bar{H}_\infty}{2},$$

where $\bar{h}$ is given by (2.4). We are ready to prove the following theorem:

Theorem 3.18. *Let $v\in H^2(\Omega)$ be such that $\Delta v=0$ in Ω and normalized such that $\int_{\partial\Omega}v^2d\sigma=1$. Then it holds*

i)

$$\int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2d\sigma \le \int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2d\sigma+2c_\Omega\left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2d\sigma\right)^{\frac{1}{2}}; \quad (3.19)$$

ii)

$$\left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}} \leq c_\Omega + \sqrt{c_\Omega^2 + \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma}. \tag{3.20}$$

Proof. Let $h \in]0,\bar{h}[$. We start by proving i). From Lemmas 3.1 and 3.10 we have

$$\begin{aligned}
\int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma &= \int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma + \frac{1}{h}\left(\int_{\omega_h}|\nabla v|^2\Delta\eta - 2(D^2\eta\cdot\nabla v)\cdot\nabla v dx\right)\\
&\leq \int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma + \left(\frac{1}{h} + N\bar{H}_\infty\right)\int_\Omega |\nabla v|^2 dx\\
&= \int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma + \left(\frac{1}{h} + N\bar{H}_\infty\right)\int_{\partial\Omega} v\frac{\partial v}{\partial\nu}dx\\
&\leq \int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma + \left(\frac{1}{h} + N\bar{H}_\infty\right)\left(\int_{\partial\Omega}v^2 d\sigma\right)^{\frac{1}{2}}\left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}}\\
&= \int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma + \left(\frac{1}{h} + N\bar{H}_\infty\right)\left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}}, \qquad (3.21)
\end{aligned}$$

where we have used the following Green's identity

$$\int_\Omega \Delta v v dx = \int_{\partial\Omega}\frac{\partial v}{\partial\nu}v d\sigma - \int_\Omega |\nabla v|^2 dx,$$

the fact that $\Delta v = 0$ in Ω, Hölder's inequality and the fact that $\int_{\partial\Omega} v^2 d\sigma = 1$. Since (3.21) holds true for all $h \in]0,\bar{h}[$, it is true with $h = \bar{h}$. This proves i). We repeat a similar argument for ii). We have

$$\begin{aligned}
\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma &= \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma - \frac{1}{h}\left(\int_{\omega_h}|\nabla v|^2\Delta\eta - 2(D^2\eta\cdot\nabla v)\cdot\nabla v dx\right)\\
&\leq \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma + \left(\frac{1}{h} + N\bar{H}_\infty\right)\left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}}, \qquad (3.22)
\end{aligned}$$

which is equivalent to

$$\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma - \left(\frac{1}{h} + N\bar{H}_\infty\right)\left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}} - \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma \leq 0.$$

This is an inequality of degree two in the unknown $\left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}} \geq 0$. Solving the inequality we obtain

$$\left(\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}} \leq \left(\frac{1}{2h} + \frac{N\bar{H}_\infty}{2}\right) + \sqrt{\left(\frac{1}{2h} + \frac{N\bar{H}_\infty}{2}\right)^2 + \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma}. \tag{3.23}$$

Since (3.23) holds true for all $h \in]0,\bar{h}[$, it is true with $h = \bar{h}$. This concludes the proof of ii) and of the theorem. □

Theorem 3.18 states that for harmonic functions v in Ω the $L^2(\partial\Omega)$ norms of $\frac{\partial v}{\partial \nu}$ and of $\nabla_{\partial\Omega} v$ are equivalent. This will be used in the next section to compare the Steklov eigenvalues on Ω with the Laplace-Beltrami eigenvalues on $\partial\Omega$.

Remark 3.24. *We note that thanks to* (3.14) *and* (3.16) *we can use the maximal mean curvature H_∞ instead of $\bar{H}_\infty$ in* (3.22) *and therefore in the inequality* (3.23). *Moreover, this and inequality* (3.23) *imply*

$$\int_\Omega |\nabla v|^2 dx \leq \left(\frac{1}{2h} + \frac{NH_\infty}{2}\right) + \sqrt{\left(\frac{1}{2h} + \frac{NH_\infty}{2}\right)^2 + \int_{\partial\Omega} |\nabla_{\partial\Omega} v|^2 d\sigma}. \quad (3.25)$$

for all $v \in H^2(\Omega)$ with $\Delta v = 0$ and $\int_{\partial\Omega} v^2 d\sigma = 1$.

4 Proof of Theorem 1.7

In this section we prove Theorem 1.7. Namely, we prove that the absolute value of the difference between the j-th eigenvalues of problems (1.1) and (1.3) is bounded by $2c_\Omega$. Throughout the rest of the paper we shall assume that Ω is a bounded domain of class C^2 in $\mathbb{R}^{N+1}$ such that its boundary $\partial\Omega$ has only one connected component. This says that $\partial\Omega$ is a compact C^2-submanifold of dimension N in $\mathbb{R}^{N+1}$ without boundary. In particular, $\partial\Omega$ is a Riemannian C^2-manifold of dimension N with the induced Riemannian metric.

The proof of Theorem 1.7 is carried out by exploiting Theorem 3.18 and the following variational characterizations of the eigenvalues of problems (1.1) and (1.3), namely

$$\sigma_j = \inf_{\substack{V \leq \tilde{H}^1(\Omega), \\ \dim V = j}} \sup_{\substack{0 \neq v \in V, \\ \int_{\partial\Omega} v^2 d\sigma = 1}} \int_\Omega |\nabla v|^2 dx, \quad (4.1)$$

for all $j \in \mathbb{N}$, $j \geq 1$, where

$$\tilde{H}^1(\Omega) := \left\{ v \in H^1(\Omega) : \int_{\partial\Omega} v d\sigma = 0 \right\},$$

and

$$\lambda_j = \inf_{\substack{V \leq \tilde{H}^1(\partial\Omega), \\ \dim V = j}} \sup_{\substack{0 \neq v \in V, \\ \int_{\partial\Omega} v^2 d\sigma = 1}} \int_{\partial\Omega} |\nabla_{\partial\Omega} v|^2 d\sigma, \quad (4.2)$$

for all $j \in \mathbb{N}$, $j \geq 1$, where

$$\tilde{H}^1(\partial\Omega) := \left\{ v \in H^1(\partial\Omega) : \int_{\partial\Omega} v d\sigma = 0 \right\}.$$

It is useful to recall the following results on the completeness of the sets of eigenfunctions of problems (1.1) and (1.3) in $L^2(\partial\Omega)$.

Theorem 4.3. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Let $\{\sigma_j\}_{j=0}^\infty$ be the sequence of eigenvalues of problem* (1.1) *and let $\{u_j\}_{j=0}^\infty \subset H^1(\Omega)$ denote the sequence of eigenfunctions associated with the eigenvalues σ_j, normalized such that $\int_{\partial\Omega} u_i u_k d\sigma = \delta_{ik}$ for all $i, k \in \mathbb{N}$. Then $\{u_j|_{\partial\Omega}\}_{j=0}^\infty$ is an orthonormal basis of $L^2(\partial\Omega)$. Moreover, $\int_\Omega \nabla u_i \cdot \nabla u_k dx = \sigma_i \delta_{ik}$ for all $i, k \in \mathbb{N}$.*

We refer e.g., to [5] for a proof of Theorem 4.3 (see also [7, 20]).

Theorem 4.4. *Let Ω be a bounded domain in $\mathbb{R}^{N+1}$ of class C^2. Let $\{\lambda_j\}_{j=0}^{\infty}$ be the sequence of eigenvalues of problem* (1.3) *and let $\{\varphi_j\}_{j=0}^{\infty} \subset H^1(\partial\Omega)$ denote the sequence of eigenfunctions associated with the eigenvalues λ_j, normalized such that $\int_{\partial\Omega} \varphi_i\varphi_k d\sigma = \delta_{ik}$ for all $i,k \in \mathbb{N}$. Then $\{\varphi_j\}_{j=0}^{\infty}$ is an orthonormal basis of $L^2(\partial\Omega)$. Moreover, $\int_{\partial\Omega} \nabla_{\partial\Omega}\varphi_i \cdot \nabla_{\partial\Omega}\varphi_k d\sigma = \lambda_i\delta_{ik}$ for all $i,k \in \mathbb{N}$.*

The proof of Theorem 4.4 follows from standard spectral theory for linear operators (see [7, 20]) and from the compactness of the embedding $H^1(\partial\Omega) \subset L^2(\partial\Omega)$.

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. We start by proving i). Let $u_1, ..., u_j$ be the Steklov eigenfunctions associated with $\sigma_1, ..., \sigma_j$ normalized such that $\int_{\partial\Omega} u_iu_k d\sigma = \delta_{ik}$, so that $\int_{\Omega} \nabla u_i \cdot \nabla u_k dx = \sigma_i\delta_{ik}$ for all $i,k = 1, ..., j$. Moreover $\int_{\partial\Omega} u_i d\sigma = 0$ for all $i = 1, ..., j$. From the regularity assumptions on Ω, we have that u_i are classical solutions, i.e., $u_i \in C^2(\Omega) \cap C^1(\overline{\Omega})$ (see [1]). In particular, $u_{i|_{\partial\Omega}} \in \tilde{H}^1(\partial\Omega)$ and $\frac{\partial u_i}{\partial\nu} = \sigma_i u$ on $\partial\Omega$, for all $i = 1, ..., j$. Let $V \subset \tilde{H}^1(\partial\Omega)$ be the space generated by $u_{1|_{\partial\Omega}}, ..., u_{j|_{\partial\Omega}}$. Any function $u \in V$ with $\int_{\partial\Omega} u^2 d\sigma = 1$ can be written as $u = \sum_{i=1}^{j} c_iu_{i|_{\partial\Omega}}$, where $c = (c_1, ..., c_j) \in \mathbb{R}^j$ is such that $|c| = 1$, i.e., $c \in \partial\mathbb{B}^j$ and $\mathbb{B}^j$ is the unit ball in $\mathbb{R}^j$. Moreover $\Delta u = 0$ for all $u \in V$. From (4.2) and (3.19) we have

$$
\begin{aligned}
\lambda_j \le \max_{\substack{0\neq u\in V\\ \int_{\partial\Omega} u^2 d\sigma=1}} \int_{\partial\Omega} |\nabla_{\partial\Omega} u|^2 d\sigma = \max_{\substack{c\in\mathbb{B}^j\\ c=(c_1,...,c_j)}} \int_{\partial\Omega} \left|\nabla_{\partial\Omega}\left(\sum_{i=1}^{j} c_iu_i\right)\right|^2 d\sigma \\
\le \max_{\substack{c\in\mathbb{B}^j\\ c=(c_1,...,c_j)}} \left(\int_{\partial\Omega}\left(\frac{\partial\left(\sum_{i=1}^{j} c_iu_i\right)}{\partial\nu}\right)^2 d\sigma + 2c_\Omega\left(\int_{\partial\Omega}\left(\frac{\partial\left(\sum_{i=1}^{j} c_iu_i\right)}{\partial\nu}\right)^2 d\sigma\right)^{\frac{1}{2}}\right) \\
= \max_{\substack{c\in\mathbb{B}^j\\ c=(c_1,...,c_j)}} \left(\int_{\partial\Omega}\left(\sum_{i=1}^{j} c_i\sigma_iu_i\right)^2 d\sigma + 2c_\Omega\left(\int_{\partial\Omega}\left(\sum_{i=1}^{j} c_i\sigma_iu_i\right)^2 d\sigma\right)^{\frac{1}{2}}\right) \\
= \max_{\substack{c\in\mathbb{B}^j\\ c=(c_1,...,c_j)}} \left(\sum_{i=1}^{j} c_i^2\sigma_i^2 + 2c_\Omega\left(\sum_{i=1}^{j} c_i^2\sigma_i^2\right)^{\frac{1}{2}}\right) = \sigma_j^2 + 2c_\Omega\sigma_j.
\end{aligned}
$$

This proves i). In an analogous way we prove ii). Let $\varphi_1, ..., \varphi_j \in H^1(\partial\Omega)$ be the eigenfunctions associated with the eigenvalues $\lambda_1, ..., \lambda_j$ of problem (1.3), normalized such that $\int_{\partial\Omega} \varphi_i\varphi_k d\sigma = \delta_{ik}$ for all $i,k = 1, ..., j$. Then $\int_{\partial\Omega} \nabla_{\partial\Omega}\varphi_i \cdot \nabla_{\partial\Omega}\varphi_k d\sigma = \lambda_i\delta_{ik}$ for all $i,k = 1, ..., j$. Moreover $\int_{\partial\Omega} \varphi_i d\sigma = 0$ for all $i = 1, ..., j$, thus $\varphi_i \in \tilde{H}^1(\partial\Omega)$. Now let ϕ_i, $i = 1, ..., j$ be the solutions to

$$
\begin{cases}
\Delta\phi_i = 0, & \text{in } \Omega,\\
\phi_i = \varphi_i, & \text{on } \partial\Omega.
\end{cases}
\tag{4.5}
$$

It is standard to prove that for all $i = 1, ..., j$, problem (4.5) admits a unique solution ϕ_i which is harmonic inside Ω and which coincides with φ_i on $\partial\Omega$ (see e.g., [23, Theroem 2.14]. From the fact that Ω is of class C^2 and from standard elliptic regularity (see [1]) it follows that $\phi_i \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Moreover $\int_{\partial\Omega} \phi_{i|_{\partial\Omega}} d\sigma = \int_{\partial\Omega} \varphi_i d\sigma = 0$ for all $i = 1, ..., j$, thus $\phi_i \in \tilde{H}^1(\Omega)$ for all $i = 1, ..., j$. Let $W \subset \tilde{H}^1(\Omega)$ be the space generated by $\phi_1, ...\phi_j$. Any function $\phi \in W$ with $\int_{\partial\Omega} \phi^2 d\sigma = 1$ can be written as $\phi = \sum_{i=1}^{j} c_i \phi_i$ with $c = (c_1, ..., c_j) \in \mathbb{B}^j$. Moreover $\Delta\phi = 0$ for all $\phi \in V$. Thanks to (3.20) and (4.1) we have

$$
\begin{aligned}
\sigma_j \leq \max_{\substack{0 \neq \phi \in W \\ \int_{\partial\Omega} \phi^2 d\sigma = 1}} \int_\Omega |\nabla\phi|^2 dx &= \max_{\substack{c \in \mathbb{B}^j \\ c=(c_1,\dots,c_j)}} \int_\Omega \left| \nabla \left(\sum_{i=1}^{j} c_i \phi_i \right) \right|^2 dx \\
&\leq \max_{\substack{c \in \mathbb{B}^j \\ c=(c_1,\dots,c_j)}} \left(\int_{\partial\Omega} \left(\frac{\partial \left(\sum_{i=1}^{j} c_i \phi_i \right)}{\partial \nu} \right)^2 d\sigma \right)^{\frac{1}{2}} \\
&\leq c_\Omega + \left(c_\Omega^2 + \max_{\substack{c \in \mathbb{B}^j \\ c=(c_1,\dots,c_j)}} \int_{\partial\Omega} \left| \nabla_{\partial\Omega} \left(\sum_{i=1}^{j} c_i \phi_i \right) \right|^2 \right)^{\frac{1}{2}} \\
&= c_\Omega + \left(c_\Omega^2 + \max_{\substack{c \in \mathbb{B}^j \\ c=(c_1,\dots,c_j)}} \int_{\partial\Omega} \left| \nabla_{\partial\Omega} \left(\sum_{i=1}^{j} c_i \varphi_i \right) \right|^2 \right)^{\frac{1}{2}} \\
&\leq c_\Omega + \left(c_\Omega^2 + \max_{\substack{c \in \mathbb{B}^j \\ c=(c_1,\dots,c_j)}} \sum_{i=1}^{j} c_i^2 \lambda_i \right)^{\frac{1}{2}} = c_\Omega + \sqrt{c_\Omega^2 + \lambda_j}.
\end{aligned}
$$

This concludes the proof of *ii)* and of the theorem. □

Theorem 1.7 not only confirms the Weyl asymptotic behavior $\lim_{j\to\infty} \sqrt{\lambda_j}/\sigma_j = 1$, but says that the difference between the eigenvalues is given at most by a constant independent of j.

By combining (1.8) with (1.5) we can now bound the Steklov eigenvalues from above. To this purpose, it is convenient to specify the constants $a_{\partial\Omega}$ and b_N in (1.5) by recalling the following theorem from [14]. We will denote by $Ric_g(M)$ the Ricci curvature tensor of a Riemannian manifold (M, g). Accordingly, $Ric_g(\partial\Omega)$ will denote the Ricci curvature tensor of the submanifold $\partial\Omega$ equipped with the induced Riemannian metric g.

Theorem 4.6. *Let (M, g) be a compact Riemannian manifold without boundary of dimension N such that $Ric_g(M) \geq -(N-1)\kappa^2$, $\kappa > 0$. Then*

$$
\lambda_j \leq \frac{(N-1)\kappa^2}{4} + c_N \left(\frac{j}{Vol(M)} \right)^{\frac{2}{N}}, \tag{4.7}
$$

where $c_N > 0$ depends only on N.

From Theorems 1.7 and 4.6 it immediately follows

Corollary 4.8. *Let Ω be a bounded domain of class C^2 in $\mathbb{R}^{N+1}$ such that $\partial\Omega$ has only one connected component. Then for all $j \in \mathbb{N}$ it holds*

$$
\sigma_j \leq a_\Omega + c_N^{\frac{1}{2}} \left(\frac{j}{|\partial\Omega|} \right)^{\frac{1}{N}}, \tag{4.9}
$$

where $a_\Omega > 0$ depends on the dimension N, on the maximal mean curvature of $\partial\Omega$, on a lower bound of the Ricci curvature of $\partial\Omega$ and on the maximal size of a tubular neighborhood about $\partial\Omega$, and $c_N > 0$ is as in Theorem 4.6 and depends only on the dimension N.

Proof. It suffices just to combine (4.7) with the second inequality in (1.8). We have

$$\sigma_j \leq c_\Omega + \sqrt{c_\Omega^2 + \frac{(N-1)\kappa^2}{4} + c_N \left(\frac{j}{Vol(M)}\right)^{\frac{2}{N-1}}}$$
$$\leq \left(2c_\Omega + \frac{(N-2)\kappa}{2}\right) + c_N^{\frac{1}{2}} \left(\frac{j}{|\partial\Omega|}\right)^{\frac{1}{N-1}}, \quad (4.10)$$

where $\kappa > 0$ is such that $Ric_g(\partial\Omega) \geq -(N-2)\kappa^2$. Since $\partial\Omega$ is a compact submanifold in $\mathbb{R}^{N+1}$ of class C^2, $Ric_g(\partial\Omega)$ is continuous on $\partial\Omega$, and such a finite κ exists. From (3.25) and from the proof of Theorem 1.7, we note that c_Ω in (4.10) can be replaced by $\frac{1}{h} + \frac{NH_\infty}{2}$. This concludes the proof. □

We conclude this section with some remarks.

Remark 4.11. *We remark that in* (4.9) *we have separated the geometry from the asymptotic behavior of the Steklov eigenvalues. We also note that the constant c_N in* (4.7) *(which depends only on the dimension) is not optimal, in the sense that it is strictly greater than the constant appearing in the Weyl's law of λ_j, as highlighted in [14], thus the constant $c_N^{\frac{1}{2}}$ in* (4.9) *is not optimal in this sense as well.*

Remark 4.12. *We remark that the constant c_Ω in* (4.9) *may become very big when Ω presents very thin parts (like in the case of dumbell domains), and this can happen also if the curvature remains uniformly bounded (see Figure 2). In the case of convex sets, anyway, it is possible to improve the constant in* (1.8)-(1.9) *and therefore the bounds* (4.9) *(see Section 5).*

Remark 4.13. *We remark that Theorems 1.4 and 1.5 are usually stated for the eigenvalues of the Laplace-Beltrami operator on smooth Riemannian manifolds. Actually, it is sufficient that $\partial\Omega$ is a manifold of class C^2 for* (1.4) *and* (1.5) *to hold. In fact we can approximate $\partial\Omega$ with a sequence $\partial\Omega_\varepsilon$ of C^∞ submanifolds such that $\partial\Omega = \psi_\varepsilon(\partial\Omega_\varepsilon)$, where ψ_ε is a diffeomorphism of class C^2 and $\|\mathrm{Id} - \psi_\varepsilon\|_{C^2(\partial\Omega_\varepsilon)}, \|\mathrm{Id} - \psi_\varepsilon^{(-1)}\|_{C^2(\partial\Omega)} \leq \varepsilon$. This follows from standard approximation of C^k functions by C^∞ (or analytic) functions (see [39]). We also refer to [36, Sec. 4.4] for a more detailed construction of the approximating boundaries $\partial\Omega_\varepsilon$. It is then standard to prove that the eigenvalues of the Laplace-Beltrami operator on $\partial\Omega_\varepsilon$ pointwise converge the eigenvalues of the Laplace-Beltrami operator on $\partial\Omega$. This immediately follows from the min-max characterization of the eigenvalues* (4.2) *(we also refer to [32, 35] for stability and continuity results for the eigenvalues of elliptic operators upon perturbations of some parameters entering the equation and to [11, 12, 13] and to the references therein for spectral stability results for eigenvalues upon perturbation of the domain). We also refer to [16, 31] and to the references therein for more detailed information on the* *convergence of Riemannian manifolds and the convergence of the corresponding spectra of the Laplacian.*

Moreover, from the fact that $\|\mathrm{Id}-\psi_\varepsilon\|_{C^2(\partial\Omega_\varepsilon)}, \|\mathrm{Id}-\psi_\varepsilon^{(-1)}\|_{C^2(\partial\Omega)} \le \varepsilon$, *it follows that* $|\partial\Omega_\varepsilon| \to |\partial\Omega|$ *and if* $\kappa > 0$ *is such that* $Ric_g(\partial\Omega) \ge -(N-1)\kappa^2$, *then there exists a sequence* κ_ε *with* $\kappa_\varepsilon \to \kappa$ *as* $\varepsilon \to 0$ *such that* $Ric_{g_\varepsilon}(\partial\Omega_\varepsilon) \ge -(N-1)\kappa_\varepsilon^2$. *Hence* (1.4) *and* (1.5) *hold if* Ω *is of class* C^2.

5 Examples: convex domains and balls

In this section we improve the constant in (1.8)-(1.9) and the bounds (4.9) in the case when Ω is a convex and bounded domain of class C^2 and show that the corresponding estimates become sharp when Ω is a ball.

5.1 Convex domains

Let Ω be a convex domain of class C^2 in $\mathbb{R}^{N+1}$. It is well-known that in this case $\kappa_i(x) \ge 0$ for all $i = 1, ..., N$ and for all $x \in \partial\Omega$. Moreover Theorem 2.6 holds for any $h \in]0, 1/K_\infty[$ (see also (2.12) for the definition of K_∞). This follows from Blaschke's Rolling Theorem for C^2 convex domains (see [8, 22, 28, 29]) and from [23, Lemma 14.16].

From (3.14) and from the fact that $0 \le \rho_i(x) \le 1$ for all $x \in \omega_h$ and $i = 1, ..., N+1$ (see also Remark 2.19), it follows that

$$-\int_\Omega |\nabla v|^2 dx \le \int_{\omega_h} |\nabla v|^2 \Delta\eta - 2(D^2\eta \cdot \nabla v) \cdot \nabla v dx \le N \int_\Omega |\nabla v|^2 dx. \tag{5.1}$$

Then, by following the same lines of the proof of Theorems 1.7 and 3.18 and choosing $\bar{h} = 1/K_\infty$, it is straightforward to prove the following:

Theorem 5.2. *Let* Ω *be a bounded and convex domain of class* C^2 *in* $\mathbb{R}^{N+1}$. *Let* σ_j *and* λ_j, $j \in \mathbb{N}$, *denote the eigenvalues of problems* (1.1) *and* (1.3) *respectively. Let* K_∞ *be defined by* (2.12). *Then*

i)

$$\lambda_j \le \sigma_j^2 + N K_\infty \sigma_j; \tag{5.3}$$

ii)

$$\sigma_j \le \frac{K_\infty}{2} + \sqrt{\frac{K_\infty^2}{4} + \lambda_j}.$$

We note that when Ω is a bounded and convex domain of class C^2, $Ric_g(\partial\Omega) \ge 0$. Accordingly, as a consequence of Theorem 4.6, we have the following:

Corollary 5.4. *Let* Ω *be a bounded and convex domain of class* C^2 *in* $\mathbb{R}^{N+1}$. *Let* σ_j, $j \in \mathbb{N}$, *denote the eigenvalues of problem* (1.1). *Let* K_∞ *be defined by* (2.12). *Then*

$$\sigma_j \le K_\infty + c_N^{\frac{1}{2}} \left(\frac{j}{|\partial\Omega|}\right)^{\frac{1}{N}}.$$

We note that the geometry of the set enters in the estimate only by means of the maximum of the principal curvatures.

Remark 5.5. *Suppose that Ω is a convex and bounded domain of class C^2 such that $\left(\sum_{i=1}^{N}\rho_i(x)-1\right) \geq 0$ for all $x \in \omega_h$. Then by (5.1) and by the same arguments in the proof of Theorems 1.7 and 3.18 we have*

$$\sigma_j \leq c_N^{\frac{1}{2}}\left(\frac{j}{|\partial\Omega|}\right)^{\frac{1}{N}}$$

for all $j \in \mathbb{N}$.

5.2 Balls

Let Ω be a ball of radius R in $\mathbb{R}^{N+1}$. We can suppose without loss of generality that it is centered at the origin. We are allowed to take $h = R-\delta$ for all $\delta \in]0, R[$ through Sections 2,3 and 4. By letting $\delta \to 0$, the expression for the vector field given by F in (3.7) simplifies to $F(x) = x$ for all $x \in \Omega$. We use $F(x) = x$ in (3.2) and we obtain that for all $v \in H^2(\Omega)$ with $\Delta v = 0$ in Ω it holds:

i)

$$\int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma = \int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma + \frac{N-1}{R}\int_{\Omega}|\nabla v|^2 d\sigma; \tag{5.6}$$

ii)

$$\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2 d\sigma = \int_{\partial\Omega}|\nabla_{\partial\Omega}v|^2 d\sigma - \frac{N-1}{R}\int_{\Omega}|\nabla v|^2 d\sigma. \tag{5.7}$$

We find then that

i)

$$\lambda_j \leq \sigma_j^2 + \frac{(N-1)}{R}\sigma_j; \tag{5.8}$$

ii)

$$\sigma_j \leq \sqrt{\frac{(N-1)^2}{4R^2}+\lambda_j} - \frac{N-1}{2R}. \tag{5.9}$$

Inequality 5.8 follows immediately from (5.6) by the same arguments as in the proof of Theorems 3.18 and 1.7. For (5.9), we note that if $\varphi_j \in H^1(\partial\Omega)$ is an eigenfunction associated with the eigenvalue λ_j of (1.3) and if we denote by ϕ_j the unique solution to (4.5), then from (5.7) we have

$$\begin{aligned}0 = \lambda_j - \int_{\partial\Omega}\left(\frac{\partial\phi_j}{\partial\nu}\right)^2 d\sigma - \frac{N-1}{R}\int_{\Omega}|\nabla\phi_j|^2 d\sigma \\ \leq \lambda_j - \left(\int_{\Omega}|\nabla\phi_j|^2 dx\right)^2 - \frac{N-1}{R}\int_{\Omega}|\nabla\phi_j|^2 dx.\end{aligned}$$

This in particular implies

$$\int_{\Omega}|\nabla\phi_j|^2 dx \leq \sqrt{\frac{(N-1)^2}{4R^2}+\lambda_j} - \frac{N-1}{2}$$

and therefore, by the min-max principle (4.1), the validity of (5.9). Combining (5.8) with (5.9) we immediately obtain the exact relation among the eigenvalues of problems (1.1) and (1.3) on Ω and $\partial\Omega$ respectively, without knowing explicitly the eigenvalues. Namely we have the following:

$$\lambda_j = \sigma_j^2 + \frac{(N-1)}{R}\sigma_j. \tag{5.10}$$

For the reader convenience, we briefly recall the explicit formulas for the Laplacian eigenvalues on $\partial\Omega$ and the Steklov eigenvalues on Ω. An eigenvalue λ of the Laplace-Beltrami operator on $\partial\Omega$ is of the form $\lambda = \frac{l(l+N-1)}{R^2}$, with $l \in \mathbb{N}$. Let us denote by H_l a spherical harmonic of degree l in $\mathbb{R}^{N+1}$. An eigenfunction associated with the eigenvalue $\frac{l(l+N-1)}{R^2}$ is of the form $H_l(x/R)$, $x \in \partial\Omega$. Hence the multiplicity of the eigenvalue $\lambda = \frac{l(l+N-1)}{R^2}$ equals the dimension d_l of the space of the spherical harmonics of degree l in $\mathbb{R}^{N+1}$, namely $d_l = (2l+N-1)\frac{(l+N-2)!}{l!(N-1)!}$. On the other hand, a Steklov eigenvalue σ on Ω is of the form $\sigma = \frac{l}{R}$ with $l \in \mathbb{N}$. The corresponding eigenfunctions are the restriction to Ω of the harmonic polynomials on $\mathbb{R}^{N+1}$ of degree l. Clearly the eigenvalues $\frac{l(l+N-1)}{R^2}$ and $\frac{l}{R}$ have the same multiplicity d_l. It is now immediate to see that formula (5.10) holds true.

5.3 A further example: a bounded and convex domain of class $C^{1,1}$

.

Throughout the paper we have considered bounded domains of class C^2. This is a sufficient condition to ensure the validity of Theorems 2.3 and 2.6. Actually, Theorems 2.3 and 2.6 may hold also under lower regularity assumptions on Ω. It is known that the existence of a tubular neighborhood ω_h of $\partial\Omega$ as in Theorem 2.3 implies that the distance function from $\partial\Omega$ is a function of class $C^{1,1}$ on ω_h. We refer to [21, Ch.7] for a more detailed discussion on sets of positive reach.

We construct now a convex subset Ω of $\mathbb{R}^3$ of class $C^{1,1}$ such that the set of points in Ω where the distance function is not differentiable has zero Lebesgue measure (in particular, it is a segment) and such that $\left(\sum_{i=1}^3 \rho_i(x) - 1\right) \geq 0$. Let $x = (x_1, x_2, x_3)$ denotes an element of $\mathbb{R}^3$. Let $L, R > 0$ be fixed real numbers. Let $x_0^+ := (0,0,L)$ and $x_0^- := (0,0,-L)$. Let $\Omega \subset \mathbb{R}^3$ be defined by

$$\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

where

$$\Omega_1 := \left\{x \in \mathbb{R}^3 : |x - x_0^+| < R\right\} \cap \left\{x \in \mathbb{R}^3 : x_3 \geq L\right\},$$
$$\Omega_2 := \left\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2\right\} \cap \left\{x \in \mathbb{R}^3 : -L \leq x_3 \leq L\right\}$$

and

$$\Omega_3 := \left\{x \in \mathbb{R}^3 : |x - x_0^-| < R\right\} \cap \left\{x \in \mathbb{R}^3 : x_3 \leq -L\right\}.$$

By construction Ω is of class $C^{1,1}$ but it is not of class C^2. Moreover it is convex. We note that we can take $h = R - \delta$ for all $\delta \in]0, R[$. Hence, as in the case of the ball, we can take in (3.2) the vector field defined by

$$F(x) = \begin{cases} x - x_0^+, & \text{if } x \in \Omega_1, \\ (x_1, x_2, 0), & \text{if } x \in \Omega_2, \\ x - x_0^+, & \text{if } x \in \Omega_3. \end{cases}$$

By construction, F is a Lipschitz vector field. We shall denote by $\rho_i(x)$, $i=1,2,3$, the eigenvalues of DF. Standard computations show that

$$\rho_i(x)=1,$$

for all $x\in\Omega_1\cup\Omega_3$ and for $i=1,2,3$ and

$$\rho_1(x)=0,\quad \rho_2(x)=\rho_3(x)=1,$$

for all $x\in\Omega_2$. Hence $\left(\sum_{i=1}^2\rho_i(x)-1\right)\geq 0$ for all $x\in\Omega$. Then for the Steklov eigenvalues σ_j on Ω we have $\sigma_j\leq c_2^{\frac12}\left(\frac{j}{|\partial\Omega|}\right)^{\frac12}$.

6 Proof of Theorem 1.11

In this section we prove Theorem 1.11, namely we prove asymptotically sharp upper bounds for Riesz means of Steklov eigenvalues. As a consequence, we provide asymptotically sharp upper bounds for the trace of the Steklov heat kernel and lower bounds for Steklov eigenvalues.

Proof of Theorem 1.11. For the Laplacian eigenvalues λ_i on $\partial\Omega$ the following asymptotically sharp inequality has been shown in [25]:

$$\sum_{j=0}^{\infty}(z-\lambda_j)_+^2\leq\frac{8}{(N+2)(N+4)}(2\pi)^{-N}B_N|\partial\Omega|(z+z_0)^{2+\frac{N}{2}} \tag{6.1}$$

where $z_0:=\dfrac{N^2}{4}H_\infty^2$ and H_∞ is given by (3.17). We note that $z_0\leq c_\Omega^2$. It follows from the first inequality of (1.8) of Theorem 1.7 that

$$\sum_{j=0}^{\infty}(z-\lambda_j)_+\geq\sum_{j=0}^{\infty}(z-\sigma_j^2-2c_\Omega\sigma_j)_+. \tag{6.2}$$

Defining a new variable ζ by $\zeta:=\sqrt{z+c_\Omega^2}-c_\Omega$ it is easily shown that (6.2) is equivalent to

$$\sum_{j=0}^{\infty}(\zeta^2+2c_\Omega\zeta-\lambda_j)_+\geq 2(\zeta+c_\Omega)\sum_{j=0}^{\infty}(\zeta-\sigma_j)_+-\sum_{j=0}^{\infty}(\zeta-\sigma_j)_+^2$$

and therefore it is equivalent to the differential inequality

$$\frac{d}{d\zeta}\frac{\sum_{j=0}^{\infty}(\zeta-\sigma_j)_+^2}{\zeta+c_\Omega}\leq\frac{\sum_{j=0}^{\infty}(\zeta^2+2c_\Omega\zeta-\lambda_j)_+}{(\zeta+c_\Omega)^2}. \tag{6.3}$$

Integrating the differential inequality (6.3) between 0 and ζ and performing an integration by parts on the right-hand side of the resulting inequality, we obtain

$$\frac{\sum_{j=0}^{\infty}(\zeta-\sigma_j)_+^2}{\zeta+c_\Omega}\leq\frac{\sum_{j=0}^{\infty}(\zeta^2+2c_\Omega\zeta-\lambda_j)_+}{4(\zeta+c_\Omega)^3}+\frac34\int_0^{\zeta}\frac{\sum_{j=0}^{\infty}(s^2+2c_\Omega s-\lambda_j)_+^2}{(s+c_\Omega)^4}\,ds.$$

We apply estimate (6.1), replace z_0 by c_Ω^2 and compute the resulting integral. We get the inequality

$$\sum_{j=0}^{\infty}(\zeta-\sigma_j)_+^2 \leq \frac{2}{(N+2)(N+4)}\,(2\pi)^{-N}B_N|\partial\Omega|(\zeta+c_\Omega)^{1+N}\big(1+\frac{3}{N+1}\big)$$

which proves the claim. $\square$

Laplace transforming inequality (1.12) of Theorem 1.11 yields the following upper bound on the trace of the heat kernel for the Steklov operator:

Corollary 6.4. *Let Ω be a bounded domain of class C^2 in $\mathbb{R}^{N+1}$ such that $\partial\Omega$ has only one connected component. Then*

$$\sum_{j=0}^{\infty} e^{-\sigma_j t} \leq \frac{1}{(N+1)(N+2)}\,(2\pi)^{-N}B_N|\partial\Omega|t^{-N}e^{c_\Omega t}\Gamma(N+3,c_\Omega t) \tag{6.5}$$

for all $t>0$, where $\Gamma(a,b)=\int_b^\infty t^{a-1}e^{-t}\,dt$ denotes the incomplete Gamma function.

The estimate is sharp as t tends to zero since (6.5) implies the exact bound

$$\limsup_{t\to 0_+} t^N\sum_{j=0}^{\infty}e^{-\sigma_j t} \leq (2\pi)^{-N}B_N\Gamma(N+1)|\partial\Omega|.$$

From (6.5) we immediately obtain Weyl-type lower bounds on Steklov eigenvalues. Since $(j+1)e^{-\sigma_j t} \leq \sum_{k=0}^{\infty} e^{-\sigma_k t}$ for all $j\in\mathbb{N}$ and $\Gamma(N+3,c_\Omega t)\leq\Gamma(N+3)$ we get from (6.5) after optimizing with respect to t the following:

Corollary 6.6. *Let Ω be a bounded domain of class C^2 in $\mathbb{R}^{N+1}$ such that $\partial\Omega$ has only one connected component. Then for all $j\in\mathbb{N}$:*

$$\sigma_j \geq r_N 2\pi B_N^{-1/N}\left(\frac{j+1}{|\partial\Omega|}\right)^{\frac{1}{N}} - c_\Omega$$

with $r_N = \dfrac{N}{e\Gamma(N+1)^{1/N}} \leq 1.$

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