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Hölder regularity for a classical problem of the calculus of variations

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be bounded, open and convex. Let $F : \mathbb{R}^n \to \mathbb{R}$ be convex, coercive of order $p > 1$ and such that the diameters of the projections of the faces of the epigraph of F are uniformly bounded. Then every minimizer of

$$\int_\Omega F(\nabla v(x))\, dx, \quad v \in \phi + W_0^{1,1}(\Omega, \mathbb{R}),$$

is Hölder continuous in $\overline{\Omega}$ of order $\frac{p-1}{n+p-1}$ whenever ϕ is Lipschitz on $\partial\Omega$. A similar result for non convex Lagrangians that admit a minimizer follows.

Keywords. Hölder regularity, Comparison Principle, Lipschitz, convex.

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1 Introduction

We address the problem of the regularity of the minimizers of the integral functional

$$I(u) = \int_\Omega F(\nabla v(x))\, dx, \quad v \in \phi + W_0^{1,1}(\Omega, \mathbb{R}). \tag{1.1}$$

This problem was studied thoroughly in the last decades even with a Lagrangian depending on x and v and not only in the scalar valued case.

The classical approach gives strong regularity results under both some variants of p-growth conditions and uniform ellipticity and thus strict convexity of the Lagrangian. In this case the minima of I satisfy an elliptic equation; the celebrated De Giorgi–Moser–Nash Theorem then yields the local Hölder regularity of the derivatives of the minimizer, Schauder's theory then applies to get more regularity of the minimizer.

The first result on the subject without assuming smoothness of the Lagrangian is due to Giaquinta and Giusti [7]: their starting point is not, as in the previous approach, the validity of the Euler equation. Assuming that the Lagrangian has the same p-growth both from above and from below and using just the minimality properties they prove the Hölder regularity of the minimizers that are locally bounded. A huge literature appeared inspired by these two approaches; these results were then refined in many directions.

In recent years many efforts have been done in order to find out conditions that ensure the (local) Lipschitz regularity of the minimum of I. This is in fact the crucial property to prove the validity of Euler equation (at least under very weak regularity assumptions on F) and hence to apply the De Giorgi–Moser–Nash Theorem. In the vectorial case and in the case of scalar functionals depending also on the state variable there are examples that show that one can not expect that the minimum is locally Lipschitz ([5], [16]).

From now on we will refer to problems of the same type considered in (1.1). For this class of functionals, in [6] there is an example of a minimum that is not locally bounded. Indeed in this case there is a singularity along a line, which therefore appears on the boundary datum too. Another interesting example can be found in [4]: it shows that even in case of Lipschitz boundary datum one cannot expect more than the local Lipschitz regularity for the minimum. Anyhow the following question is still open: to what extent the boundary condition plays a role to prove the regularity of the minimizers?

Recently the Hilbert–Haar approach has been used in this direction. A recent result, that is a refinement of a classical one [8] is the following: if F is strictly convex and ϕ satisfies the Hartman–Stampacchia's Bounded Slope Condition then the minimum of I is Lipschitz [3]. It has further been proved that the minimum is locally Lipschitz if just ϕ satisfies a unilateral Bounded Slope Condition [4] and Ω is convex. If F is not strictly convex, the minima of I are not unique: however we showed in [13] and [14] that in this case the same results hold true if just the faces of the epigraph of F are bounded. The interest in weakening the strict convexity of the Lagrangean is related to the study of the relaxed problems arising for non convex ones.

The main tools that were used in this approach are the method of translations used in [2], [13] and [17] and various Comparison Principles for the minimizers of I stated in [3], [11], [12], [14].

By a smart use of these techniques, Bousquet proved in [1] that the minimum and the maximum of the minima are continuous up to the boundary of a convex set Ω, for every continuous boundary datum; this is even true for every superlinear and convex function F.

We focus here on the regularity, more than continuity, in the case where the boundary datum ϕ *is Lipschitz*. It is shown in [4] that the minimum of I is Hölder continuous up to the boundary, with an explicit Hölder order, if the boundary of Ω is a polyhedron and F is *coercive of order* $p > 1$ and strictly convex.

We show here that the same conclusion holds true if Ω *is any convex subset* of $\mathbb{R}^n$, with no need of the strict convexity assumption of F. More precisely in this situation the minimum and the maximum among the minimizers that share the boundary datum ϕ are Hölder continuous; their existence was proven in [14]. Actually, it turns out that *every* minimizer is Hölder continuous by adding a further uniformity assumption on the faces of the epigraph of F. The same result holds true in the case of a non convex

Lagrangean whenever the minimizer exists and the lower semicontinuous envelope of the Lagrangean satisfies the assumptions quoted above.

By means of the translation technique we prove the global Hölder regularity up to the boundary of order $\alpha = (p-1)/(n+p-1)$; as it is pointed out in [4], in the case $p > n$ this exponent is better than $1-(n/p)$ provided by the Sobolev embedding. The value of α is also greater than the one that could be obtained by following the lines of the proof of Theorem 1.6 of [4] in the case where the domain is a polyhedron. We use here the idea [1] of a comparison of the minimizers on two different domains: the original one and a larger polyhedron. This provides also an alternative proof of Bousquet's continuity result in the case where F is coercive.

When $F(\xi) = |\xi|^p$ $(p \geq 2)$ the minimizer w of I is a p-harmonic function. For $p = 2$ the fact that w is Hölder continuous up to the boundary if ϕ is Hölder continuous is a well established fact; in this case the harmonic function has the same Hölder rank of the boundary datum itself: the result was established by Kellogg in [9] in 1931, a nice proof of it can be found in [15]. For more general functionals under the natural growth assumptions the Hölder exponent of the minimizer is not explicitly known.

Finally we apply the main result to the case of a Lagrangean that is not convex.

2 Notation and definitions

- Ω is an open, bounded and convex subset of $\mathbb{R}^n$; its boundary is denoted by $\partial\Omega$ or by Γ and its closure by $\overline{\Omega}$.
- In every statement, except Corollary 4.10, we assume that $F : \mathbb{R}^n \to \mathbb{R}$ is *convex*. We say that F is *coercive* of order $p > 1$ if

$$\forall \xi \in \mathbb{R}^n \qquad F(\xi) \geq \sigma|\xi|^p + \mu \qquad (\sigma > 0,\ \mu \in \mathbb{R});$$

  and that F is *superlinear* if

$$\lim_{|\xi|\to+\infty} \frac{F(\xi)}{|\xi|} = +\infty.$$

- For any open subset A of $\mathbb{R}^n$, the functional I_A (or simply I if $A = \Omega$) is defined on $W^{1,1}(A)$ by

$$I_A(u) = \int_A F(\nabla u(x))\, dx;$$

  if, moreover, ϕ is a trace function on ∂A we denote by $I_{A,\phi}$ the restriction of I_A to $\phi + W_0^{1,1}(A)$; a *local minimizer* w of I_A is a minimizer of I_A on $w + W_0^{1,1}(A)$.
- If $u, v \in W^{1,1}(\Omega)$ we write that $u \leq v$ on Γ if $\max\{u-v, 0\} \in W_0^{1,1}(\Omega)$.

We refer to [4] for the definition and the main properties of the *Lower Bounded Slope Condition* that we recall here.

Lower Bounded Slope Condition. The function $\phi : \overline{\Omega} \to \mathbb{R}$ satisfies the *Lower Bounded Slope Condition* of constant $K \geq 0$ if for every $\gamma \in \Gamma$ there exist $\zeta_\gamma \in \mathbb{R}^n$ with $|\zeta_\gamma| \leq K$ such that

$$\forall \gamma' \in \Gamma \qquad \phi(\gamma') \geq \phi(\gamma) + \langle \zeta_\gamma, \gamma' - \gamma \rangle.$$

Remark 2.1. We omit to reformulate the analogous definition of the *Upper Bounded Slope Condition*. It is worth noticing that a function satisfies the Lower Bounded Slope Condition if and only if it is the restriction of a convex function on $\mathbb{R}^n$.

If F is not strictly convex, the minima of I are not unique in general; hence the Comparison Principle may fail. However if F is superlinear, given a boundary datum ϕ there exist the essential pointwise minimum and maximum of the minimizers of $I_{\Omega,\phi}$ [14, Proposition 4.1] and they turn out to satisfy the Comparison Principle, at least from below or from above; we recall here the result for the convenience of the reader.

Theorem 2.2 ([14, Theorem 2.1]). *Assume that w is the maximum of the local minimizers of I, i.e. $w \geq u$ for every minimizer u of I with $w = u$ on Γ. If v is a local minimizer of I such that $w \geq v$ on Γ then $w \geq v$ a.e. on Ω; in other words w satisfies the Comparison Principle from above. Similarly, the minimum of the minimizers satisfies the Comparison Principle from below.*

3 A comparison with convex functions

In the next proposition we formulate some conditions that ensure the validity of a Comparison Principle between local minimizers and convex functions; in the case where F is strictly convex it is due to Bousquet in a personal communication.

Proposition 3.1. *Assume that F is superlinear, let $\psi : \mathbb{R}^n \to \mathbb{R}$ be convex and w be the maximum of the minimizers of $I_{\Omega,\psi}$ such that $w \geq \psi$ on Γ. Then $w(x) \geq \psi(x)$ at every Lebesgue point x of w.*

Proof. Let $x \in \Omega$ and ζ belong to the convex subdifferential of ψ at x, so that

$$\forall \gamma \in \Gamma \qquad \psi(\gamma) \geq h(\gamma)$$

where for $y \in \mathbb{R}^n$ we set $h(y) = \langle \zeta, y - x \rangle + \psi(x)$. Now the affine function h is a local minimizer of I: if w is the maximum of the minimizers then Theorem 2.2 yields that $w(y) \geq h(y)$ a.e. on Ω. By taking the integral mean value of both terms of the inequality $w(y) \geq h(y)$ on the balls centered in x we obtain, by passing to the limit as the radii of the balls tend to 0, that $w(x) \geq h(x) = \psi(x)$ if x is a Lebesgue point of w. Finally, fix $\gamma \in \Gamma$ and let ζ be in the convex subdifferential of ψ at γ. Then

$$\forall \gamma' \in \Gamma \qquad \phi(\gamma') \geq \psi(\gamma') \geq \psi(\gamma) + \langle \zeta, \gamma' - \gamma \rangle.$$

If K is the projection of an exposed bounded face of the epigraph of F containing $(\zeta, F(\zeta))$ then, for all γ' in Γ, we have that $\phi(\gamma') \geq h^-_{K,\gamma}(\gamma') + \psi(\gamma)$ where

$$\forall x \in \mathbb{R}^n \quad h^-_{K,\gamma}(x) \doteq \min\{\langle \xi, x - \gamma \rangle : \xi \in K\}.$$

The Comparison Principle [14, Theorem 3.2] yields that $w(x) \geq h^-_{K,\gamma}(x) + \psi(\gamma)$ for every x in Ω. □

Remark 3.2. In Proposition 3.1 the superlinearity assumption is needed to ensure the existence of a maximal minimizer; it can be replaced by the assumption that the diameters of the projections of the faces of the epigraph of F are uniformly bounded [14, Proposition 4.1]. A dual statement holds when ψ is a concave function: it is enough to reverse the inequalities.

4 Hölder continuity of the minimizers

Our proof is based on [4, Theorems 1.6, 2.3] of Clarke; we recall here the part of the statement of our interest and formulate some remarks that turn out to be useful.

Theorem 4.1 ([4, Theorem 2.3]). *Let F be strictly convex and coercive of order $p > 1$. Assume that Q is the interior of a polyhedron in $\mathbb{R}^n$. Let w be a minimizer of $I_{Q,\phi}$ where ϕ satisfies the Lower Bounded Slope Condition. Then w is locally Lipschitz and, for some constant k,*

$$\forall x \in Q, \ \forall \gamma \in \partial Q \quad w(x) - \phi(\gamma) \leq k|x - \gamma|^\alpha, \ \alpha = (p-1)/(n+p-1). \tag{4.1}$$

Remark 4.2. As we mentioned in [14], if F is not strictly convex the claim of Theorem 4.1 does still hold for the minimum and the maximum of the minimizers of $I_{Q,\phi}$. The proof of Theorem 4.1 is based on the Comparison Principle proved in [11]. Replacing it with Theorem 2.2 one gets immediately the result. As it is pointed out in [4], the Hölder constant of w is an explicit function of $\|\phi\|_\infty$, of σ and μ involved in the coercivity assumption, of n, of the Lipschitz constant of ϕ and of the geometric properties of the polyhedron Q that are invariant for translations and rotations.

Remark 4.3. An analogous statement holds, even in the non strictly convex case, when the boundary datum satisfies the Upper Bounded Slope Condition and w is the minimum of the minimizers. In this case one obtains the inequality

$$\forall x \in Q, \quad \forall \gamma \in \partial Q \qquad \phi(\gamma) - w(x) \leq k|x - \gamma|^\alpha;$$

it will be used in proof of the main theorem.

The following lemma is an argument in the proof of the continuity of the minimizers by Bousquet [1]. We formulate it here without assuming that F is strictly convex.

Lemma 4.4. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be convex, Q be an open, convex and bounded set containing Ω. If $\widetilde{w}$ is the maximum of the minimizers of $I_{Q,\phi}$ and w is any minimizer of $I_{\Omega,\phi}$ then $\widetilde{w} \geq w$ a.e. on Ω.*

Proof. Since $\widetilde{w} \geq \phi$ on ∂Q then Proposition 3.1 implies that $\widetilde{w} \geq \phi$ a.e. on Q and thus in particular $\widetilde{w} \geq \phi$ on Γ. Since $\widetilde{w}$ is still the maximum of the local minimizers of I_Ω it follows from Theorem 2.2 that $\widetilde{w} \geq w$ a.e. on Ω. □

The main result of the paper is the following theorem.

Theorem 4.5. *Assume that F is coercive of order $p > 1$. Let ϕ be Lipschitz, and w be either the maximum or the minimum minimizer of $I_{\Omega,\phi}$. Then w is Hölder continuous in $\overline{\Omega}$ of order $\alpha = (p-1)/(n+p-1)$. More precisely,*

$$|w(x) - w(y)| \leq k|x-y|^\alpha \quad \forall x, y \in \overline{\Omega}$$

where the constant k depends only on $\|\phi\|_\infty$, on the Lipschitz constant of ϕ, on Ω and on F.

Remark 4.6. Theorem 4.5 is an extension of [4, Theorem 1.6]: indeed we drop both the assumption that F is strictly convex and that Γ is a polyhedron. Moreover the proof the theorem quoted above yields a global Hölder order $\gamma = \frac{p-1}{\frac{n}{\alpha}+2p-\left(1+\frac{1}{\alpha}\right)} < \alpha$, so that our result improves Clarke's result even in the case where Ω is a polyhedron.

Under a further uniformity assumption on the faces of the epigraph of F it turns out that *every* minimizer of I is Hölder continuous.

Theorem 4.7. *Let F be coercive of order $p > 1$ and assume that the diameters of the projections of the faces of the epigraph of F are uniformly bounded. Let ϕ be Lipschitz. Then the conclusion of Theorem 4.5 holds for every minimizer of $I_{\Omega,\phi}$.*

Proof of Theorem 4.5. We assume in the first part of the proof that w is *any* minimizer of $I_{\Omega,\phi}$. Let Q be a cube in $\mathbb{R}^n$ such that one of its faces contains 0 in its relative interior and such that, for every $\gamma \in \Gamma$, there exists a linear isometry A_γ such that $\Omega \subset \gamma + A_\gamma(Q)$ (in other words a clone of Q may be assumed to contain Ω and one of its faces to be tangent to Ω at γ).

Fix $q \in \Gamma$, consider the set $q + A_q(Q)$ and, on it, the boundary datum

$$\phi^q(\gamma) \doteq \phi(q) + K|\gamma - q|,$$

where K is the Lipschitz constant of ϕ on Γ. Notice that $\|\phi^q\|_\infty \leq \|\phi\|_\infty + K \operatorname{diam} Q$; moreover ϕ^q is convex and Lipschitz of rank K and thus satisfies the Lower Bounded Slope Condition of rank K. We will also consider in $q + A_q(Q)$ the boundary datum

$$\phi_q(\gamma) \doteq \phi(q) - K|\gamma - q|$$

that satisfies analogous properties; in particular it is concave and satisfies the Upper Bounded Slope Condition of rank K. To simplify the notation, from now on, we will identify each $q + A_\gamma(Q)$ with Q itself.

Let w^q be the maximum of the minimizers of I_{Q,ϕ^q} and w_q be the minimum of the minimizers of I_{Q,ϕ_q}. Since $\phi^q = w^q$ on ∂Q and ϕ^q is convex then, by Proposition 3.1, $\phi^q \leq w^q$ a.e. on Q; moreover $\phi \leq \phi^q$ so that $\phi \leq w^q$ a.e. on Q and thus $\phi = w \leq w^q$ on Γ in the trace sense: the Comparison Principle Theorem 2.2 yields $w \leq w^q$ a.e. on Ω. Analogously we obtain that $w_q \leq w$ a.e. on Ω. Now ϕ^q satisfies the Lower Bounded Slope Condition so that Theorem 4.1 together with Remark 4.2 yield $w^q(x) \leq \phi(q) + k|x-q|^\alpha$ for all x in Q; analogously, by Remark 4.3, $\phi(q) - k|x-q|^\alpha \leq w_q(x)$ in Q and thus

$$\forall q \in \Gamma \quad \phi(q) - k|x-q|^\alpha \leq w_q(x) \leq w(x) \leq w^q(x) \leq \phi(q) + k|x-q|^\alpha \text{ a.e. on } \Omega.$$

By integration on balls of the members of the previous inequalities one obtains, by passing to the limit as the radii of the balls tend to 0, that the inequalities hold (independently on the point q of Γ) at every Lebesgue point of w. Let ℓ^-, ℓ^+ be the functions defined by

$$\forall x \in \Omega \quad \ell^-(x) = \sup_{q\in\Gamma}\{\phi(q) - k|x-q|^\alpha\}, \qquad \ell^+(x) = \inf_{q\in\Gamma}\{\phi(q) + k|x-q|^\alpha\}.$$

The functions ℓ^-, ℓ^+ are Hölder continuous of order α and constant $k > 0$,

$$\ell^-(x) \leq w(x) \leq \ell^+(x) \quad \text{at every Lebesgue point } x \text{ of } w, \qquad \ell^- = \phi = \ell^+ \text{ on } \Gamma.$$

We follow now the method of translations used in [2], [13] and [17]. We may consider w, ℓ^-, ℓ^+ as defined in all $\mathbb{R}^n$ by letting them to be equal to a prescribed Lipschitz extension of ϕ out of Ω: the functions ϕ, ℓ^-, ℓ^+ are thus Hölder continuous of order α on every compact subset of $\mathbb{R}^n$: let $C = C(k, K)$ be a Hölder constant of ϕ, ℓ^-, ℓ^+ on $\{y : \operatorname{dist}(y, \Omega) < 3 \operatorname{diam} \Omega\}$.

For every h in $\mathbb{R}^n$ with $|h| \leq \operatorname{diam} \Omega$ let E_h be the subset of $\mathbb{R}^n$ defined by

$$E_h = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < 2 \operatorname{diam} \Omega, \quad w(x+h) - w(x) > C|h|^\alpha\}.$$

We claim that $E_h \subseteq \Omega \cap (-h + \Omega)$ apart at most a negligible set. Indeed let x and $x+h$ be Lebesgue points of w: if both $x+h$ and x are not in Ω then $w(x+h) - w(x) = \phi(x+h) - \phi(x) \leq C|h|^\alpha$; if $x \notin \Omega$ and $x+h \in \Omega$ then

$$\begin{aligned} w(x+h) - w(x) &= w(x+h) - \phi(x) \\ &\leq \ell^+(x+h) - \phi(x) = \ell^+(x+h) - \ell^+(x) \leq C|h|^\alpha; \end{aligned}$$

finally, if $x \in \Omega$ and $x + h \notin \Omega$ then

$$\begin{aligned} w(x+h) - w(x) &= \phi(x+h) - w(x) \\ &\le \phi(x+h) - \ell^-(x) = \ell^-(x+h) - \ell^-(x) \le C|h|^\alpha : \end{aligned}$$

in these cases $x \notin E_h$, proving the claim.

Assume that $\Omega \cap (-h+\Omega) \neq \emptyset$. Since $w(x+h) - C|h|^\alpha \le w(x)$ in a neighborhood of Ω out of $\Omega \cap (-h+\Omega)$ then

$$w(x+h) - C|h|^\alpha \le w(x) \quad \text{on } \partial(\Omega \cap (-h+\Omega)).$$

Suppose that w is the maximum of the minimizers of $I_{\Omega,\phi}$. Then $w(x+h) - C|h|^\alpha$ and $w(x)$ are local minimizers of $I_{\Omega\cap(-h+\Omega)}$, both the maximum ones among those that share the same boundary data. In both cases the Comparison Principle (Theorem 2.2) yields that $w(x+h) - C|h|^\alpha \le w(x)$ a.e. on $\Omega \cap (-h+\Omega)$ so that $E_h = \emptyset$ and thus

$$\forall h \in \mathbb{R}^n,\ |h| \le \operatorname{diam}\Omega, \quad \forall x \in \Omega\cap(-h+\Omega) \qquad w(x+h) - w(x) \le C|h|^\alpha. \tag{4.2}$$

We claim that w is Hölder continuous of order α. Indeed let $\overline{x}$, $\overline{y}$ be Lebesgue points for w. Indeed fix $h = \overline{y} - \overline{x}$. By integrating on the balls $B_r(\overline{x})$ centered in $\overline{x}$ and of radius $r > 0$ both members of (4.2) we obtain, by means of a change of variables,

$$\int_{B_r(\overline{y})} w(\xi)\, d\xi - \int_{B_r(\overline{x})} w(\xi)\, d\xi \le C|B_r||h|^\alpha \tag{4.3}$$

so that, by dividing with the volume of the balls and passing to the limit as $r \to 0$ in (4.3) we obtain

$$w(\overline{y}) - w(\overline{x}) \le C|\overline{y} - \overline{x}|^\alpha.$$

The case where w is the minimum of the minimizers can be treated similarly. □

Proof of Theorem 4.7. By Theorem 4.5 the claim holds for the minimum w^* of the minimizers of $I_{\Omega,\phi}$. By [13, Lemma 4.9] the gradients ∇w^* and ∇w of the two minimizers belong a.e. to the projection of a face of the epigraph of F and thus $|\nabla w(x) - \nabla w^*(x)| \le R$ a.e. for some positive R. Since w and w^* are both bounded then $w - w^*$ is Lipschitz; the conclusion follows. □

Remark 4.8. Our proof yields the continuity of the minimizers in the case where ϕ is Lipschitz; our method thus provides an alternative proof of Bousquet's continuity result stated in [1] in the case where F is coercive (indeed the conclusion when ϕ is just continuous follows as in [1] from the case where the boundary datum is Lipschitz by means of the maximum principle [12]).

Remark 4.9. The assumption that Ω is convex is technical here, however the claim of the theorem may be false if Ω is not convex: a celebrated example by Krol' and Maz'ya shows that a solution to the p-Laplacian equation, i.e. a minimizer of I when $F(\xi) = |\xi|^p$, with boundary data in $\mathcal{C}^\infty(\mathbb{R}^n)$, may not be Hölder continuous in a neighborhood of a boundary point [10]; in their example the domain has a sharp inwardly directed spine and $p \in (1, n-1]$.

We allow now F not to be convex. In this situation let F^{**} be the bipolar of F, the greatest lower semicontinuous function that is pointwise lower than F. The functional I^{**} is defined on $W^{1,1}(\Omega)$ by

$$I^{**}(u) = \int_\Omega F^{**}(\nabla u(x))\, dx.$$

The next corollary follows directly from Theorem 4.7 and the fact of the infima of I and of I^{**} are equal.

Corollary 4.10. *Assume that F is coercive of order $p > 1$ and that the diameters of the projections of the faces of the epigraph of F^{**} are uniformly bounded. Let ϕ be Lipschitz, and assume that $I_{\Omega,\phi}$ has a minimum w. Then w is Hölder continuous in $\overline{\Omega}$ of order $\alpha = (p-1)/(n+p-1)$. More precisely,*

$$|w(x) - w(y)| \le k|x-y|^\alpha \quad \forall x, y \in \overline{\Omega}$$

where the constant k depends only on $\|\phi\|_\infty$, on the Lipschitz constant of ϕ, on Ω and on F.

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