On Thermo-electro-viscoelastic Relaxation Functions in a Green-Naghdi Type Theory

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Abstract

We find restrictions on the relaxation functions of thermo-electro-viscoelastic materials. This is achieved within an extension of the Green-Naghdi theory for thermoelasticity, which uses the energy equation to exploit constitutive equations. These restrictions extend the results previously found for thermo-viscoelastic materials and for the classical infinitesimal theory of viscoelasticity.

KEYWORDS viscoelasticity; thermo-electro-viscoelasticity; relaxation functions; thermodynamic restrictions; restrictions of relaxation functions; Green-Naghdi continuum thermodynamics.

1 Introduction

In the mechanics of continuous media, a material for which the stress tensor at time t is determined by the history of the strain is called a simple material. In continum thermodynamics, simple materials with memory are those in which at time t the stress tensor, the heat flux vector, the internal energy, ...etc. ... , are determined by the histories up time t of the deformation gradient tensor, the absolute temperature, and possibly of the temperature gradient. In Coleman [1] the fundamental hypotheses of linear isothermal viscoelasticity are examined in the light of nonlinear continuum mechanics within the scheme of materials with fading memory and in [2] thermodynamic restrictions on the tensor-valued shear relaxation modulus of linear viscoelasticity are derived.

In literature there have been many articles on this topic, with different settings, and here we mention only a few. Galeş and Chiriţă in [3] study the spatial behavior of solutions in a right cylinder made of an anisotropic and homogeneous viscoelastic solid in a class of linear viscoelastic materials compatible with thermodynamics in the sense of Fabrizio and Morro [4], [5]. Huang [6] proposes a thermo-viscoelastic constitutive theory at finite deformation in a framework of irreversible thermodynamics and introducing the internal variables in the constitutive relations.

Zeng [7] studies the Cauchy problem of a one-dimensional purely mechanical nonlinear viscoelastic model with fading memory.

In Chen [8] a coupled theory of nonlinear electro-thermo-viscoelasticity is developed based on non-equilibrium thermodynamics with the Clausius-Duhem dissipation inequality.

In [9] Wilkes uses the local form of the Clausius-Duhem inequality to obtain restrictions on the relaxation functions of thermoviscoelastic materials that have fading memory in the sense of Coleman and Noll [10]. For the purpose, Wilkes uses the restrictions of the constitutive equations, the dissipation inequality, and the minimality of the free energy in equilibrium that are found and used by Coleman [11]. Lastly, he shows that its results generalize Day's results [12], [13] of the purely mechanical linear viscoelasticity. Wilkes's classical thermodynamic approach differs from the one in [4] and [5] that, in the purely mechanical theory, show compatibility with thermodynamics considering approximate cycles and giving a proper statement of the second law by the Clausius property.

In [14] Green and Naghdi introduce a setting for thermoelasticity, different from the one of Coleman [11], which is based on an entropy equality rather than an entropy inequality, and where an energy equation places thermodynamic restrictions on the constitutive equations. Then in [15]-[17] such a procedure is extended by introducing the concept of thermal displacement in the so called thermoelasticity Type III theory. In the latter theory of heat conduction the temperature may travel as a wave with a finite speed.

The Green-Naghdi theory of heat conduction meets great research interest by its general setting and because it is capable of accounting for thermal pulse transmission in a very general manner (e.g., see [18]).

Recently, in [19] the procedure designed by Green and Naghdi for thermoelasticity is extended to simple thermo-electro-elastic bodies, both isotropic and transversely isotropic, that are finitely deformable, heat conducting, electrically polarizable, interacting with the electric field; again, the restrictions on the constitutive relations are obtained using an energy equation that is suitable for the considered type of material. Then paper [20] extends [19] to thermo-electro-mechanical simple materials (finitely deformable, heat conducting, electrically polarizable, interacting with the electric field) that have a fading memory.

In the present paper the nonlinear theory [20] is used to set up a Green-Naghdi thermo-electro viscoelastic theory by the linearization procedure that uses the Riesz representation theorem. In the present theory the presences of the electrical vector and of the thermal displacement derivatives in the constitutive arguments, imply that there are more relaxation functions than in [9]. Hence several restrictions on the various relaxation functions are found. These restrictions extend the ones in [9] for thermoviscoelastic materials within a theory that uses the Clausius-Duhem inequality. The restrictions are obtained from the internal dissipation inequality, which is a consequence of the dissipation inequality adopted here. Following [14], the last one is to assume that the internal rate of supply of entropy per unit mass is non-negative in every process. The theoretical frame is then completed with a proposal of constitutive equations for the internal rate of entropy supply and heat flux. The linearized (infinitesimal) theory of thermo-electro-viscoelasticity is deduced as first-order approximation of the finite theory and the field equations are explicitly deduced in the simplest case of a one-dimensional body.

2 Preliminary definitions

We shall identify the body $\mathcal{B}$ under consideration with the region $\boldsymbol{B}$ that it occupies in a fixed reference configuration. A material point X of $\mathcal{B}$ is then identified with its position $\boldsymbol{X}$ in $\boldsymbol{B}$.

As usual, we denote by $\{x^a\}$ $\left[\{X^A\}\right]$ spatial [material] Euclidean co-ordinates in the ambient space [reference configuration].

The material point $\boldsymbol{X}$ in the current configuration occupies the place $\boldsymbol{x}$. A motion of the body is defined by a smooth vector function $\boldsymbol{\chi}$,

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X},\, t)\,. \tag{1}$$

Then

$$\boldsymbol{v} = \dot{\boldsymbol{x}}\,, \qquad \boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{\chi}\,, \qquad \boldsymbol{L} = \nabla_{\boldsymbol{x}}\boldsymbol{v} \tag{2}$$

are the particle velocity $\boldsymbol{v}$ at $\boldsymbol{x}$ (a superimposed dot denotes material time derivative), deformation gradient tensor, and velocity spatial gradient tensor, respectively.

Green-Naghdi continuum thermodynamics [15] is based on the notion of *thermal displacement* $\alpha = \alpha(\boldsymbol{X}, t)$ at the material point $\boldsymbol{X}$ and time t. Then

$$T = \dot{\alpha}, \quad \boldsymbol{\beta} = \nabla_{\boldsymbol{X}}\alpha, \quad \boldsymbol{\gamma} = \nabla_{\boldsymbol{x}}T \tag{3}$$

are the *empirical temperature* ('thermal displacement rate'), *thermal displacement gradient* and *empirical-temperature gradient*, respectively.

The other thermal magnitudes used in Green-Naghdi [15] are listed here:

θ absolute temperature,

$\boldsymbol{g} = \nabla_{\boldsymbol{x}}\theta$ absolute-temperature gradient,

r external rate of supply of heat per unit mass,

$s = r/\theta$ external rate of supply of entropy per unit mass,

ξ internal rate of supply of entropy per unit mass,

$\boldsymbol{q}$ heat flux vector per unit area,

$\boldsymbol{p}$ entropy flux vector per unit area,

$\boldsymbol{i}$ extra entropy flux vector per unit area,

η density of entropy per unit mass,

e internal energy density per unit mass,

Furthermore we need the following electrical magnitudes:

ϕ electric potential per unit volume,

$\boldsymbol{P}$ electric polarization vector per unit volume,

ρ mass density in the current configuration,

$\boldsymbol{\pi} = \boldsymbol{P}/\rho$ electric polarization vector per unit mass,

$\boldsymbol{D}$ electric displacement vector,

$\boldsymbol{T}^E$ Maxwell stress tensor ([22, Eq. (3.19)], [23]),

and the quasistatic Maxwellian electric field [22, p.589]

$$\boldsymbol{E}^M = -\nabla_{\boldsymbol{x}}\phi\,. \tag{4}$$

The following relations hold,

$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E}^M + \boldsymbol{P}, \qquad \boldsymbol{T}^E = \boldsymbol{D}\otimes\boldsymbol{E}^M - \frac{1}{2}\varepsilon_0\big(\boldsymbol{E}^M\cdot\boldsymbol{E}^M\big)\boldsymbol{I}\,, \tag{5}$$

where ε_0 stands for the (constant) vacuum electric permittivity. Finally, the specific *free energy density* per unit mass is defined as ([22], [21])

$$\psi = e - \theta\eta - \boldsymbol{E}^M\cdot\boldsymbol{\pi}\,. \tag{6}$$

3 Local balance laws in spatial form

A *dynamic process* in $\mathcal{B}$ is described by the thirteen functions of $\boldsymbol{X}$ and t,

$$\rho, \boldsymbol{\chi},\, \alpha,\, \phi,\, \psi,\, \eta,\, \xi,\, \boldsymbol{\tau},\, \boldsymbol{P},\, \boldsymbol{q},\, \boldsymbol{p},\, \boldsymbol{f},\, r\,, \tag{7}$$

where $\boldsymbol{f}$ is the body force density and $\boldsymbol{\tau}$ is the Cauchy stress tensor (due to deformation) per unit area. Such a set of thirteen functions is called a *dynamic process* in $\mathcal{B}$ if and only if it is compatible with the balance laws of mass, linear momentum, moment of momentum, energy, entropy, and the field equations of electrostatics.

Under suitable assumptions of regularity the usual integral forms of such balance laws are equivalent to the system of local equations

$$\begin{cases} \dot{\rho} + \rho\,\nabla_x \cdot \boldsymbol{v} = 0\,, \\ \rho\dot{\boldsymbol{v}} = \nabla_x \cdot \boldsymbol{\tau} + \boldsymbol{P} \cdot \nabla_x \boldsymbol{E}^M + \rho \boldsymbol{f}\,, \\ skw\boldsymbol{\tau} + skw\boldsymbol{T}^E = \boldsymbol{0}\,, \\ \rho\dot{\eta} = \rho(s+\xi) - \nabla_x \cdot \boldsymbol{p}\,, \\ \rho\dot{e} = \boldsymbol{\tau} \cdot \nabla\boldsymbol{v} - \nabla_x \cdot \boldsymbol{q} + \boldsymbol{E}^M \cdot \rho\dot{\boldsymbol{\pi}} + \rho r\,, \\ \nabla_x \times \boldsymbol{E}^M = \boldsymbol{0}\,, \\ \nabla_x \cdot \boldsymbol{D} = 0\,, \end{cases} \tag{8}$$

where e is the *internal energy* that is defined by (6). Eliminating r between equations $(8)_4$, $(8)_5$ and using (6) we obtain the *reduced energy equation*

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \rho\theta\xi - \boldsymbol{\tau} \cdot \nabla\boldsymbol{v} + \dot{\boldsymbol{E}}^M \cdot \boldsymbol{P} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{q} - \theta\nabla_{\boldsymbol{x}} \cdot \boldsymbol{p} = 0\,, \tag{9}$$

which holds along every process in $\mathcal{B}$. Next, analogously to Green-Naghdi theories of thermoelasticity [15], [16], here we shall exploit this equality in order to find restrictions on the constitutive relations.

4 Constitutive equations and admissible processes

Let $\lambda = \lambda(t)$ be any function from $I\!R$ to any linear space; fixed $t \in I\!R$, the function $\lambda^t(.)$ (briefly λ^t) defined by

$$\lambda^t(s) = \lambda(t-s) \quad \forall\, s > 0 \tag{10}$$

is the *past history* of λ up to time t. Among the quantities (7) we call

$$p = (\boldsymbol{\chi},\, \alpha,\, \phi) \tag{11}$$

kinetic process in $\mathcal{B}$ and

$$\psi,\, \eta,\, \xi,\, \boldsymbol{\tau},\, \boldsymbol{P},\, \boldsymbol{q},\, \boldsymbol{p} \tag{12}$$

auxiliary variables. Note that the kinetic process (11) through the equalities (3), (4) determines $(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t)$, where

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{X},\, t) = \big(T(\boldsymbol{X},\, t),\, \boldsymbol{\beta}(\boldsymbol{X},\, t),\, \boldsymbol{F}(\boldsymbol{X},\, t), \boldsymbol{E}^M(\boldsymbol{X},\, t)\big)\,, \tag{13}$$

and $\boldsymbol{\Lambda}^t$ is the past history of $\boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda}^t = \boldsymbol{\Lambda}^t(\boldsymbol{X}, .) = (T^t(\boldsymbol{X}, .), \boldsymbol{\beta}^t(\boldsymbol{X}, .), \boldsymbol{F}^t(\boldsymbol{X}, .), \boldsymbol{E}^{M\,t}(\boldsymbol{X}, .))\,, \tag{14}$$

$$\boldsymbol{\Lambda}^t(s) = \boldsymbol{\Lambda}(t-s)\,, \qquad s > 0\,. \tag{15}$$

Incidentally, note that the history $\boldsymbol{\gamma}^t$ has not been included since $\boldsymbol{\beta}^t$ already determines it by the relation

$$\dot{\boldsymbol{\beta}} = \nabla_{\boldsymbol{X}} \dot{\alpha} = \frac{\partial \dot{\alpha}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{F}^T \boldsymbol{\gamma} \,. \tag{16}$$

The material at the point $\boldsymbol{X}$ is said to be *a simple (thermo-electro-mechanical) material* if at any time t the auxiliary variables (12) are determined by the kinetic process (11) through constitutive equations of the form

$$\begin{cases} \psi(t) = \hat{\psi}\,(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \\ \eta(t) = \hat{\eta}\,(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \\ \theta(t) = \hat{\theta}\,(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \\ \xi(t) = \hat{\xi}\,(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \\ \boldsymbol{p}(t) = \hat{\boldsymbol{p}}\,(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \\ \boldsymbol{\tau}(t) = \hat{\boldsymbol{\tau}}\,(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \\ \boldsymbol{P}(t) = \hat{\boldsymbol{P}}\,(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \end{cases} \tag{17}$$

where $\hat{\psi}, \ldots, \hat{\boldsymbol{P}}$ are given objective (response) functionals and for simplicity the dependence on $\boldsymbol{X}$ (which occurs when the body is not materially homogeneous) is implicit and not written. As is customary we assume a mass density $\rho_0 = \rho_0(\boldsymbol{X})$ is given in the reference configuration.

Furthermore, we take the constitutive relation for the entropy flux in the general form

$$\boldsymbol{p} = \frac{1}{\theta}\boldsymbol{q} + \boldsymbol{i} \tag{18}$$

where

$$\boldsymbol{i} = \hat{\boldsymbol{i}}\left(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t\right) \tag{19}$$

is usually referred to as *extra entropy flux* (see [19] and references therein).

By (18) we have $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{q} - \theta \nabla_{\boldsymbol{x}} \cdot \boldsymbol{p} = \boldsymbol{g} \cdot \boldsymbol{p} - \nabla_{\boldsymbol{x}} \cdot (\theta \boldsymbol{i})$ and thus the reduced energy equality (9) becomes

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \rho\theta\xi - \boldsymbol{\tau} \cdot \nabla \boldsymbol{v} + \dot{\boldsymbol{E}}^M \cdot \boldsymbol{P} + \boldsymbol{g} \cdot \boldsymbol{p} - \nabla_{\boldsymbol{x}} \cdot (\theta \boldsymbol{i}) = 0 \,. \tag{20}$$

Definition 4.1 *A dynamic process (7) is said to be admissible in $\mathcal{B}$ if it is compatible with the constitutive relations* (17)-*(19) at each material point X of $\mathcal{B}$ and at all times t.*

5 Fading memory and chain rule

The vector $\boldsymbol{\Lambda}$ in (17) belongs to the linear space

$$\mathcal{V} = I\!R \times I\!R^3 \times Lin \times I\!R^3 \,, \tag{21}$$

where Lin denotes the linear space of second-order tensors on $I\!R^3$.

We assume that *"The memory of a simple material fades in time"*[1], and we describe how the material has a fading memory by an *obliviator* or *influence function* [10]. This is a continuous, positive, monotone decreasing function $h(.)$ with

$$\int_0^\infty h^2(s) ds \,<\, \infty \,. \tag{22}$$

Given an influence function $h(.)$, the collection of all measurable functions $\mathbf{\Gamma}(.) : (0, \infty) \to \mathcal{V}$ for which the norm

$$\|\mathbf{\Gamma}(.)\|_h^2 = \int_0^\infty h^2(s)\mathbf{\Gamma}(s) \cdot \mathbf{\Gamma}(s)ds \tag{23}$$

is finite, constitutes a Hilbert space $\mathcal{H}$ in which the scalar product '$\cdot_h$' is defined as

$$\mathbf{\Gamma}_1(.) \ \cdot_h \ \mathbf{\Gamma}_2(.) = \int_0^\infty h^2(s)\mathbf{\Gamma}_1(s) \cdot \mathbf{\Gamma}_2(s)ds\,, \qquad \mathbf{\Gamma}_i(.) \in \mathcal{H}\,. \tag{24}$$

Mizel-Wang [24, p.125] assumes that "for each fixed pair $(\mathbf{\Lambda}, \boldsymbol{\gamma})$ the functionals in (17) regarded as functions of the past history $\mathbf{\Lambda}^t$ have for their common domain $\mathcal{D}$ a neighborhood in $\mathcal{H}$ of the rest history $\mathbf{\Lambda}^*(s) \equiv \mathbf{\Lambda}$ and are Fréchet-differentiable throughout $\mathcal{D}$ with respect to the h-norm, where $h(.)$ denotes some fixed influence function; for each fixed $\mathbf{\Lambda}(.)$ in $\mathcal{D}$ the functionals in (17) regarded as functions of $\mathbf{\Lambda}$ and $\boldsymbol{\gamma}$ are continuously differentiable with respect to their natural norms. Moreover, all three derivatives are jointly continuous functions of $\mathbf{\Lambda}$, $\boldsymbol{\gamma}$, $\mathbf{\Lambda}^t$." Note that the calculus theorems in the fading memory theory are theorems about the Hilbert space $\mathcal{H}$. They do not depend on the particular choice of the function $h(s)$ and are valid provided only that $h(s)$ goes to zero fast enough as $s \to \infty$. It sufficies to assume that

$$\lim_{s\to\infty} s^{\frac{1}{2}+\delta} h(s) = 0 \tag{25}$$

for some small $\delta > 0$; $r = \delta + 1/2$ is called *order* of $h(s)$. ([11, p. 12]).

Here we assume that

($\mathcal{F}$) *all the response functionals (17), (18), (19) are twice Fréchet-differentiable at all histories in $\mathcal{D}$.*

Under smoothness hypotheses, which also regard the process, the chain-rule

$$\begin{aligned}\dot{\psi}(t) &= \frac{\partial\hat{\psi}}{\partial\mathbf{\Lambda}}\left(\mathbf{\Lambda}(t),\, \boldsymbol{\gamma}(t),\, \mathbf{\Lambda}^t(.)\right) \cdot \dot{\mathbf{\Lambda}}(t) \\ &+ \frac{\partial\hat{\psi}}{\partial\boldsymbol{\gamma}}\left(\mathbf{\Lambda}(t),\, \boldsymbol{\gamma}(t),\, \mathbf{\Lambda}^t(.)\right) \cdot \dot{\boldsymbol{\gamma}}(t) + \delta\hat{\psi}\left(\mathbf{\Lambda}(t),\, \boldsymbol{\gamma}(t),\, \mathbf{\Lambda}^t(.) \,|\, \dot{\mathbf{\Lambda}}^t(.)\right)\end{aligned} \tag{26}$$

is proved to hold at any $\boldsymbol{X}$, t along any smooth enough process (Coleman [11], Mizel and Wang [24], Day [13, p.88]).

Remark 5.1 *Above the symbol $\dot{\mathbf{\Lambda}}^t(.)$ stands for the past history of the derivative $\dot{\mathbf{\Lambda}}(.)$ so that $\dot{\mathbf{\Lambda}}^t(s) = \dot{\mathbf{\Lambda}}(t-s)$; note that this is not the derivative of the past history $\mathbf{\Lambda}^t(.)$, which is*

$$\frac{d}{ds}\mathbf{\Lambda}^t(s) = \frac{d}{ds}\mathbf{\Lambda}(t-s) = -\dot{\mathbf{\Lambda}}(t-s) = -\dot{\mathbf{\Lambda}}^t(s)\,. \tag{27}$$

In this we follow Day [13, p.88]; instead Coleman [11] uses $\frac{d}{ds}\mathbf{\Lambda}^t(s)$ within the derivative $\delta\hat{\psi}$.

Under suitable smooth conditions also the chain rule for partial derivatives

$$\begin{aligned}\frac{\partial\psi}{\partial x^i} &= \frac{\partial\hat{\psi}}{\partial\mathbf{\Lambda}}\left(\mathbf{\Lambda}(t),\, \boldsymbol{\gamma}(t),\, \mathbf{\Lambda}^t(.)\right) \cdot \frac{\partial\mathbf{\Lambda}}{\partial x^i} \\ &+ \frac{\partial\hat{\psi}}{\partial\boldsymbol{\gamma}}\left(\mathbf{\Lambda}(t),\, \boldsymbol{\gamma}(t),\, \mathbf{\Lambda}^t(.)\right) \cdot \frac{\partial\boldsymbol{\gamma}}{\partial x^i} + \delta\hat{\psi}\left(\mathbf{\Lambda}(t),\, \boldsymbol{\gamma}(t),\, \mathbf{\Lambda}^t(.) \,|\, \frac{\partial\mathbf{\Lambda}^t}{\partial x^i}(.)\right)\end{aligned} \tag{28}$$

is also used to compute divergences of response functionals, e.g. in writing the reduced energy equation.

6 Restrictions on constitutive equations

6.1 On the reduced energy equation and second law of thermodynamics

In the procedure of Green and Naghdi [15], [16] within thermoelasticity it is assumed that " the reduced energy equation .... must be identically satisfied for all processes and will place restrictions on the functional dependence on the constitutive equations " ([15, p.259]).

We assume the same thing for the thermo-electro-mechanical simple body with fading memory that is considered here:

(A) *the reduced energy equation (20) is identically satisfied for all processes and will place restrictions on the constitutive equations (17).*

In the exploitation of the reduced energy equation a class of possible processes is used; hence we assume that

(B) *there are sufficiently many admissible dynamic processes in $\mathcal{B}$, in the sense that, locally (i.e. at any given point and time), for each admissible choice of the values for the local state*

$$(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t)$$

including, if required, a large enough arbitrariness in the choice of its space and time derivatives, the field equations hold for some process in $\mathcal{B}$.

(C) *In addition, the heat flux vector $\boldsymbol{q}$ is assumed to be necessarily non zero on physical grounds. Hence, by (18),*

$$\boldsymbol{p} - \boldsymbol{i} \neq \boldsymbol{0} \tag{29}$$

Remark 6.1 *To complete the thermo-electro-mechanical theory we should assume the second law of thermodynamics expressed in the form of a dissipation inequality which holds along any process of $\mathcal{B}$. Unlike theories that adopt the Clausius-Duhem inequality, where the constitutive restrictions are obtained by exploiting the latter inequality, here such restrictions are determined by exploiting the reduced energy equation along the admissible processes and thus do not depend on the second law of thermodynamics.*

6.2 Exploitation of the reduced energy equation

According to Green and Naghdi we assume (20) with $\hat{\boldsymbol{i}} = \hat{\boldsymbol{0}}$, i.e. $\boldsymbol{p} = \boldsymbol{q}/\theta$, and we rewrite from [20] the restrictions on the response functionals that are implied by the reduced energy equation (9).

Proposition 6.1 *Let the constitutive equations (17) fulfill*

$$\boldsymbol{q} = \theta \boldsymbol{p} \qquad\qquad (\boldsymbol{i} = \boldsymbol{0})\,, \tag{30}$$

$$\frac{\partial \hat{\theta}}{\partial T} > 0 \qquad \forall\,(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \qquad (see\ (13), (14)\,, \tag{31}$$

and let the internal energy response function be defined by (6). Then

$$\psi = \hat{\psi}(\boldsymbol{\Lambda},\, \boldsymbol{\Lambda}^t), \qquad \theta = \hat{\theta}(T)\,, \tag{32}$$

$$\hat{\boldsymbol{\tau}} = \rho \boldsymbol{F} \frac{\partial \hat{\psi}}{\partial \boldsymbol{F}}\,, \qquad \hat{\boldsymbol{P}} = -\rho \frac{\partial \hat{\psi}}{\partial \boldsymbol{E}^M}\,, \qquad \hat{\eta} = -\frac{\partial \hat{\psi}}{\partial \theta}\,, \tag{33}$$

$$\rho\, \frac{\partial \hat{\psi}}{\partial \boldsymbol{\beta}} \cdot \boldsymbol{F}^T \boldsymbol{\gamma} + \rho\, \hat{\theta}\, \hat{\xi} + \hat{\boldsymbol{p}} \cdot \boldsymbol{g} + \rho \delta \hat{\psi} \left(\boldsymbol{\Lambda},\, \boldsymbol{\Lambda}^t \,|\, \dot{\boldsymbol{\Lambda}}^t\right) = 0\,, \tag{34}$$

where by $(32)_2$ in the latter we have

$$\hat{\boldsymbol{g}} = \frac{\partial \hat{\theta}}{\partial T}(T) \boldsymbol{\gamma}\,. \tag{35}$$

Remark 6.2 *Assumption* (31) *implies that* $\hat{\theta}$ *is an invertible function of* T*, so that in any response functional the variable* T *can be replaced by* θ*.*

7 Restrictions for invariant response functionals

The choice of constitutive functionals that are invariant under rigid rotations of the deformed and polarized body implies that the principle of material objectivity necessarily is satisfied. The invariance of ψ in a rigid rotation is assured when ψ is an arbitrary functional of the referential quantities

$$\boldsymbol{\Phi} := (T,\, \boldsymbol{\beta},\, \boldsymbol{E},\, \boldsymbol{W}), \qquad \dot{\boldsymbol{\beta}}, \qquad \boldsymbol{\Phi}^t := (T^t,\, \boldsymbol{\beta}^t,\, \boldsymbol{E}^t,\, \boldsymbol{W}^t), \tag{36}$$

where eq. (16) holds,

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) \tag{37}$$

is the Green-Lagrange strain tensor, and

$$\boldsymbol{W} = -\frac{\partial \phi}{\partial \boldsymbol{X}} = -\frac{\partial \phi}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{F}^T\boldsymbol{E}^M \tag{38}$$

is the material electric vector. Now for each response functional

$$\hat{\Omega} \in \left\{ \hat{\psi}\,,\ \hat{\eta}\,,\ \hat{\theta}\,,\ \hat{\xi}\,,\ \hat{\boldsymbol{p}}\,,\ \hat{\boldsymbol{\tau}}\,,\ \hat{\boldsymbol{P}} \right\} \tag{39}$$

in (17)-(18) the associated objective response functional $\tilde{\Omega}$ is defined by putting

$$\tilde{\Omega}(\boldsymbol{\Phi},\, \dot{\boldsymbol{\beta}},\, \boldsymbol{\Phi}^t) := \hat{\Omega}(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t) \tag{40}$$

where $(\boldsymbol{\gamma},\, \boldsymbol{F},\, \boldsymbol{E}^M)$ and $(\dot{\boldsymbol{\beta}},\, \boldsymbol{E},\, \boldsymbol{W})$ are related by (16), (37), (38). The considerations about fading memory and chain rule in Section (5) remain in force simply by substituting (21) with

$$\mathcal{V}_0 = I\!R \times I\!R^3 \times Sym \times I\!R^3\,, \tag{41}$$

where Sym is the linear space of symmetric tensors, and $(\boldsymbol{\Lambda},\, \boldsymbol{\gamma},\, \boldsymbol{\Lambda}^t)$ in (19) and anywhere else, with $\left(\boldsymbol{\Phi},\, \dot{\boldsymbol{\beta}},\, \boldsymbol{\Phi}^t\right)$. The Hilbert space $\mathcal{H}$ of Section (5) is replaced by the Hilbert space $\mathcal{H}_0$ that is defined similarly.

The following proposition from [20] gives the restrictions on invariant response functionals.

Proposition 7.1 *Assume constitutive equations*

$$\tilde{\Omega} \in \left\{ \tilde{\psi}\,,\ \tilde{\eta}\,,\ \tilde{\theta}\,,\ \tilde{\xi}\,,\ \tilde{\boldsymbol{p}}\,,\ \tilde{\boldsymbol{\tau}}\,,\ \tilde{\boldsymbol{P}} \right\}$$

of the form

$$\Omega = \tilde{\Omega}(\boldsymbol{\Phi},\, \dot{\boldsymbol{\beta}},\, \boldsymbol{\Phi}^t)\,, \tag{42}$$

that are invariant under rigid rotations of the deformed and polarized body. Moreover, let

$$\boldsymbol{q} = \theta \boldsymbol{p} \qquad\qquad (\boldsymbol{i} = \boldsymbol{0})\,, \tag{43}$$

$$\frac{\partial \tilde{\theta}}{\partial T} > 0 \qquad\qquad \forall\ (\boldsymbol{\Phi},\, \dot{\boldsymbol{\beta}},\, \boldsymbol{\Phi}^t)\,, \tag{44}$$

and let the internal energy response functional be defined by (6)*. Then*

$$\psi = \tilde{\psi}(\boldsymbol{\Phi},\, \boldsymbol{\Phi}^t)\,, \quad \theta = \tilde{\theta}(T)\,, \tag{45}$$

$$\tilde{\boldsymbol{\tau}} = \rho \boldsymbol{F}\Big(\frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}}\boldsymbol{F}^T + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M\Big), \quad \tilde{\boldsymbol{P}} = -\rho \boldsymbol{F}\frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}}, \quad \tilde{\eta} = -\frac{\partial \tilde{\psi}}{\partial \theta}, \tag{46}$$

$$\rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}} + \rho\, \tilde{\theta}\, \tilde{\xi} + \tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{g}} + \rho \delta \tilde{\psi} \left(\boldsymbol{\Phi},\, \boldsymbol{\Phi}^t \,|\, \dot{\boldsymbol{\Phi}}^t\right) = 0\,, \tag{47}$$

where by $(45)_2$ *in the latter we have*

$$\tilde{\boldsymbol{g}} = \frac{\partial \tilde{\theta}}{\partial T}\gamma = \frac{\partial \tilde{\theta}}{\partial T}\boldsymbol{F}^{-T}\dot{\boldsymbol{\beta}}\,. \tag{48}$$

7.1 Elastic stress

Following [22], the *total stress tensor* $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma} = \boldsymbol{\tau} + \boldsymbol{T}^E\,. \tag{49}$$

The use of $\boldsymbol{\sigma}$ allows to write the field equations of linear momentum and angular momentum $(8)_{2,\,3}$ in the 'mechanical form'

$$\rho \dot{\boldsymbol{v}} = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f}\,, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{50}$$

The so called *elastic stress* used in [25] is defined as

$$\boldsymbol{T} := \boldsymbol{\tau} + \boldsymbol{P} \otimes \boldsymbol{E}^M\,. \tag{51}$$

From equalities $(46)_{1,\,2}$, we have the equality

$$\boldsymbol{T} = \rho \boldsymbol{F}\frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}}\boldsymbol{F}^T\,, \tag{52}$$

which coincides with equation $(1)_3$ in [25].

Total stress and elastic stress are related by [19, p.1062]

$$\boldsymbol{\sigma} = \boldsymbol{T} + \epsilon_0 \boldsymbol{E}^M \otimes \boldsymbol{E}^M - \frac{\epsilon_0}{2}(\boldsymbol{E}^M \cdot \boldsymbol{E}^M)\boldsymbol{I}\,, \tag{53}$$

and the balance of linear momentum $(8)_2$ is equivalent to

$$\rho \dot{\boldsymbol{v}} = \nabla_{\boldsymbol{x}} \cdot \left[\boldsymbol{T} + \epsilon_0 \boldsymbol{E}^M \otimes \boldsymbol{E}^M - \frac{\epsilon_0}{2}(\boldsymbol{E}^M \cdot \boldsymbol{E}^M)\boldsymbol{I}\right] + \rho \boldsymbol{f}\,, \tag{54}$$

which coincides with equation $(1)_1$ in [25] when $\boldsymbol{f} = \boldsymbol{0}$.

8 Internal dissipation inequality

Assuming

$$\xi \geq 0 \tag{55}$$

in every process, the reduced energy equality (47), (48) gives

$$\left(\rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} + \frac{\partial \tilde{\theta}}{\partial T}\boldsymbol{F}^{-1}\tilde{\boldsymbol{p}}\right) \cdot \dot{\boldsymbol{\beta}} + \rho \delta \tilde{\psi} \left(\boldsymbol{\Phi},\, \boldsymbol{\Phi}^t \,|\, \dot{\boldsymbol{\Phi}}^t\right) \leq 0\,, \tag{56}$$

which for $\dot{\boldsymbol{\beta}} = \boldsymbol{0}$ gives

$$\delta \tilde{\psi} \left(\boldsymbol{\Phi},\, \boldsymbol{\Phi}^t \,|\, \dot{\boldsymbol{\Phi}}^t\right) \leq 0 \qquad \forall\, \boldsymbol{\Phi},\, \forall\, \boldsymbol{\Phi}^t \;\text{ such that }\; \dot{\boldsymbol{\beta}}^t(0) = \boldsymbol{0}\,. \tag{57}$$

Now, from [13, p.91], we point out that the derivative $\dot{\Phi}(t)$ at t 'can be chosen arbitrarily without affecting the past histories $\Phi^t(.)$ and $\dot{\Phi}^t(.)$, regarded as elements of the Hilbert space $\mathcal{H}$.'

This assertion is also used by Coleman [11] and a justification for it may be read both in [11] and in [13]. It applies also here and justifies the assumption that $\dot{\boldsymbol{\beta}}(t)$ can be assigned arbitrarily and independently from $\Phi^t(.)$ and $\dot{\Phi}^t(.)$.

Hence the restriction on the history within the inequality (57) can be removed and we have

$$\delta\tilde{\psi}\left(\mathbf{\Phi},\, \mathbf{\Phi}^t \,|\, \dot{\mathbf{\Phi}}^t\right) \leq 0 \qquad \forall\, \mathbf{\Phi},\, \forall\, \mathbf{\Phi}^t . \tag{58}$$

This inequality extends the *internal dissipation inequality* of continuum thermodynamics in [11, p.19, eq.(6.30)], [13, p.94].

In order to extend [9] for studying the thermodynamic restrictions on the relaxation functions, from now onward we will follow Coleman and Wilkes use of d/ds derivatives of past histories $\mathbf{\Phi}^t(s)$ rather than past histories of the derivatives $\dot{\mathbf{\Phi}}(s)$ – see (27) and Remark (5.1). Thus (58) writes as

$$\delta\tilde{\psi}\left(\mathbf{\Phi},\, \mathbf{\Phi}^t \,|\, \frac{d}{ds}\mathbf{\Phi}^t\right) \geq 0 \qquad \forall\, \mathbf{\Phi},\, \forall\, \mathbf{\Phi}^t . \tag{59}$$

9 Behaviour near equilibrium

9.1 Relaxation property under constant continuation of a given process

Coleman [11] established the so-called *relaxation property under constant continuation* of a given process of the constitutive functionals. Obviously it also applies here for any response functional $\tilde{\Omega}$. We briefly remind it from Coleman [11, pp.23-25] (see also Day [13, pp.95-97]). If $\mathbf{\Phi}(.)$ is a process and t is any fixed time then the constant continuation at time t is the process $\mathbf{\Phi}_0(.)$ coinciding with $\mathbf{\Phi}(\tau)$ at any time prior to t and then held constant subsequently: $\mathbf{\Phi}_0(\tau) = \mathbf{\Phi}(\tau)$ for $\tau \leq t$ and $\mathbf{\Phi}_0(\tau) = \mathbf{\Phi}(t)$ for $\tau \geq t$. Then under certain regularity conditions on the process and the continuity of $\tilde{\Omega}$ it follows that as $\delta \to \infty$ ([11, pp.23-25])

$$\tilde{\Omega}(\mathbf{\Phi},\, \dot{\boldsymbol{\beta}},\, \mathbf{\Phi}_0^{t+\delta}) \to \tilde{\Omega}(\mathbf{\Phi},\, \mathbf{0},\, \mathbf{\Phi}^*) =: \tilde{\Omega}^*(\mathbf{\Phi})\,, \tag{60}$$

where $\mathbf{\Phi}^*$ denotes the constant history whose value is $\mathbf{\Phi} = \mathbf{\Phi}(t)$, i.e.

$$\mathbf{\Phi}^*(t-s) = \mathbf{\Phi}(t) \quad \forall s > 0\,. \tag{61}$$

In words, *under constant continuation any response functional* $\tilde{\Omega}(\mathbf{\Phi},\, \dot{\boldsymbol{\beta}},\, \mathbf{\Phi}_0^t)$ *relaxes to the equilibrium value* $\tilde{\Omega}^*(\mathbf{\Phi})$. In particular for the free energy functional $\tilde{\psi}$, which does not depend on $\dot{\boldsymbol{\beta}}$, we have that as $\delta \to \infty$

$$\tilde{\psi}(\mathbf{\Phi},\, \mathbf{\Phi}_0^{t+\delta}) \to \tilde{\psi}(\mathbf{\Phi},\, \mathbf{\Phi}^*) =: \tilde{\psi}^*(\mathbf{\Phi})\,.$$

9.2 Property of minimum of ψ at equilibrium

Now for $\delta > 0$ at time t let us consider the constant continuation $\mathbf{\Phi}_0^{t+\delta}(.)$. During the 'static part' $t \leq \tau \leq t+\delta$ of a constant continuation the rate of change of the free energy cannot be positive. Indeed, eq. (20), where we put $\boldsymbol{i} = \mathbf{0}$, $\dot{\theta} = 0$, $\boldsymbol{g} = \mathbf{0}$, $\dot{\boldsymbol{E}}^M = \mathbf{0}$, $\dot{\boldsymbol{F}} = \mathbf{0}$, and inequality (55) imply that for each time τ, $t \leq \tau \leq t+\delta$, we have

$$\dot{\psi} \leq 0\,. \tag{62}$$

It follows that

$$\tilde{\psi}(\mathbf{\Phi},\,\mathbf{\Phi}_0^{t+\delta}) \le \tilde{\psi}(\mathbf{\Phi},\,\mathbf{\Phi}^t)$$

and in the limit as $\delta \to \infty$

$$\tilde{\psi}^*(\mathbf{\Phi}) \le \tilde{\psi}(\mathbf{\Phi},\,\mathbf{\Phi}^t)\,. \tag{63}$$

Hence, as in [11, p.26] and [13, p.98], also in the present theory we can state that '*among all histories ending with a given value of* $\mathbf{\Phi}$ *the constant history yields the minimal free energy*'.

The minimal property of the free energy described by the inequality (63) is one of the major results in Coleman's paper [11] for the theory of thermodynamics of a continuum with fading memory. It follows from (63) that for all $\mathbf{\Phi}$ and all functions $\mathbf{\Gamma}(s)$ in $\mathcal{H}$

$$\delta\tilde{\psi}\,(\mathbf{\Phi},\,\mathbf{\Phi}^*\,|\,\mathbf{\Gamma}(s)) = 0\,, \qquad \delta^2\tilde{\psi}\,(\mathbf{\Phi},\,\mathbf{\Phi}^*\,|\,\mathbf{\Gamma}(s)) \ge 0 \tag{64}$$

10 Thermo-electro-viscoelastic materials

In the postulate of fading memory $(\mathcal{F})$ of Section (5) we assume that all the response functionals (17), or (39), (40), are twice Fréchet-differentiable at all constant histories in their common domain, which is contained in the Hilbert space $\mathcal{H}$ or $\mathcal{H}_0$, respectively. Hence each response functional (42) can be expanded in a Taylor series about constant histories and since by (45) the free energy functional is independent of $\dot{\boldsymbol{\beta}}$ we have

$$\begin{aligned}\tilde{\psi}(\mathbf{\Phi},\,\mathbf{\Phi}^t(s)) = \tilde{\psi}(\mathbf{\Phi},\,\mathbf{\Phi}^*(s)) + \delta\tilde{\psi}(\mathbf{\Phi},\,\mathbf{\Phi}^*(s)\,|\,\mathbf{\Phi}^t(s)-\mathbf{\Phi}^*(s))\\ +\frac{1}{2}\delta^2\tilde{\psi}(\mathbf{\Phi},\,\mathbf{\Phi}^*(s)\,|\,\mathbf{\Phi}^t(s)-\mathbf{\Phi}^*(s),\,\mathbf{\Phi}^t(s)-\mathbf{\Phi}^*(s))\\ +\boldsymbol{o}(\|\mathbf{\Phi}^t(s)-\mathbf{\Phi}^*(s)\|_h^2)\end{aligned} \tag{65}$$

In order to define a viscoelastic material we extend here the assumptions in [9, p.213] and so we assume that (i) the last term in the expression above is identically zero, (ii) $\delta^2\tilde{\psi}$ is independent of $\mathbf{\Phi}$, that is of the present values of the constitutive arguments, and (iii) $\delta^2\tilde{\psi}$ is a completely continuous bilinear functional of its remaining arguments. By the Riesz representation theorem, Eq. (65) can then be written in the form (67) below in which, to reduce the lengths of the formulas we use the *difference histories* up to time t from the constant history (61), that is,

$$\mathbf{\Phi}_d^t = \mathbf{\Phi}^t(s) - \mathbf{\Phi}, \quad \boldsymbol{\beta}_d^t = \boldsymbol{\beta}^t(s) - \boldsymbol{\beta}, \quad etc., \ldots \tag{66}$$

and where the second integral on the right of the equality componentwise is written in (68).

$$\begin{aligned}\psi = \tilde{\psi}(\mathbf{\Phi},\,\mathbf{\Phi}^t(s)) = \tilde{\Sigma}(\mathbf{\Phi}) + \int_0^\infty \boldsymbol{z}(s;\,\mathbf{\Phi})\cdot\mathbf{\Phi}_d^t(s)ds + \frac{1}{2}\int_0^\infty\!\!\int_0^\infty \mathbf{\Phi}_d^t(s)\cdot\frac{\partial^2\boldsymbol{M}(s,\,u)}{\partial s\partial u}\mathbf{\Phi}_d^t(u)dsdu\\ +\int_0^\infty\!\!\int_0^\infty T_d^t(s)\frac{\partial^2\boldsymbol{B}_1(s,\,u)}{\partial s\partial u}\cdot\mathbf{\Phi}_d^t(u)dsdu + \int_0^\infty\!\!\int_0^\infty T_d^t(s)\frac{\partial^2\boldsymbol{B}_2(s,\,u)}{\partial s\partial u}\cdot\boldsymbol{E}_d^t(u)dsdu\\ +\int_0^\infty\!\!\int_0^\infty T_d^t(s)\frac{\partial^2\boldsymbol{B}_3(s,\,u)}{\partial s\partial u}\cdot\boldsymbol{W}_d^t(u)dsdu + \int_0^\infty\!\!\int_0^\infty \boldsymbol{\beta}_d^t(s)\cdot\frac{\partial^2\boldsymbol{M}_1(s,\,u)}{\partial s\partial u}\boldsymbol{E}_d^t(u)dsdu\\ +\int_0^\infty\!\!\int_0^\infty \boldsymbol{\beta}_d^t(s)\cdot\frac{\partial^2\boldsymbol{M}_2(s,\,u)}{\partial s\partial u}\boldsymbol{W}_d^t(u)dsdu + \int_0^\infty\!\!\int_0^\infty \boldsymbol{W}_d^t(s)\cdot\frac{\partial^2\boldsymbol{M}_3(s,\,u)}{\partial s\partial u}\boldsymbol{E}_d^t(u)dsdu\end{aligned} \tag{67}$$

$$\int_0^\infty \int_0^\infty [\boldsymbol{\Phi}^t(s) - \boldsymbol{\Phi}] \cdot \frac{\partial^2 \boldsymbol{M}(s,\, u)}{\partial s \partial u} [\boldsymbol{\Phi}^t(s) - \boldsymbol{\Phi}] ds du =$$

$$= \int_0^\infty \int_0^\infty T_d^t(s) \frac{\partial^2 m_1(s,\, u)}{\partial s} T_d^t(u) ds du + \int_0^\infty \int_0^\infty \boldsymbol{\beta}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{m}_2(s,\, u)}{\partial s} \boldsymbol{\beta}_d^t(u) ds du$$

$$+ \int_0^\infty \int_0^\infty \boldsymbol{E}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{m}_3(s,\, u)}{\partial s} \boldsymbol{E}_d^t(u) ds du + \int_0^\infty \int_0^\infty \boldsymbol{W}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{m}_4(s,\, u)}{\partial s} \boldsymbol{W}_d^t(u) ds du \tag{68}$$

In the above constitutive relation $\boldsymbol{\Phi} = \boldsymbol{\Phi}^t(0^+)$, $\boldsymbol{\beta} = \boldsymbol{\beta}^t(0^+)$, etc.; $\tilde{\Sigma}(\boldsymbol{\Phi})$ is an arbitrary function of $\boldsymbol{\Phi}$, and the material relaxation functions $\boldsymbol{z}(s;\, \boldsymbol{\Phi})$, $\boldsymbol{M}(s,\, u)$, $\boldsymbol{B}_i(s,\, u)$, $\boldsymbol{M}_i(s,\, u)$ fulfill the following conditions, where $Lin\mathcal{V}$ denotes the linear space of endomorphisms on any given linear space $\mathcal{V}$: $\boldsymbol{z}(s;\, \boldsymbol{\Phi})$ is $\mathcal{V}$-valued (see (21));

$\boldsymbol{M}(s,\, u)$ is $Lin\mathcal{H}_0$-valued, so that

$$\boldsymbol{M} = (m_1,\, \boldsymbol{m}_2,\, \boldsymbol{m}_3,\, \boldsymbol{m}_4) \in (Lin\Re \times Lin\Re^3 \times Lin(Lin) \times Lin\Re^3);$$

in addition $\boldsymbol{M}$ has the following symmetries :

$$m_2^{ij} = m_2^{ji}, \quad m_4^{ij} = m_4^{ji}, \quad m_3^{ijkl} = m_3^{jikl} = m_3^{ijlk} = m_3^{klij};$$

$\boldsymbol{B}_2(s,\, u)$ is symmetric tensor-valued;
$\boldsymbol{B}_1(s,\, u)$ and $\boldsymbol{B}_3(s,\, u)$ are $\Re^3$-valued;
$\boldsymbol{M}_2(s,\, u)$ is second-order tensor-valued;
$\boldsymbol{M}_1(s,\, u)$ and $\boldsymbol{M}_3(s,\, u)$ are third-order tensor-valued symmetric in the last two indices.

From the Riesz representation theorem we also have that
$\boldsymbol{z}(s;\, \boldsymbol{\Phi}),\ \boldsymbol{M}(s,\, u),\ \boldsymbol{B}_i(s,\, u),\ \boldsymbol{M}_i(s,\, u)\ \to\ 0$ as either s or $u\ \to 0$.

Now, from the constitutive equation (65) for the free energy we can calculate explicitly its first Fréchet derivative with respect to history as follows, where $\boldsymbol{\chi}(s) = (a(s),\, \boldsymbol{b}(s),\, \boldsymbol{e}(s),\, \boldsymbol{w}(s))$ is the history-deviation from the constant history $\boldsymbol{\Phi} = \boldsymbol{\Phi}^*(s) = \boldsymbol{\Phi}^t(0)\ \forall\, s > 0$:

$$\delta\tilde{\psi}(\boldsymbol{\Phi},\, \boldsymbol{\Phi}^t(s)\,|\,\boldsymbol{\chi}(s)) = \int_0^\infty \boldsymbol{z}(s;\, \boldsymbol{\Phi}) \cdot \boldsymbol{\chi}(s) ds + \int_0^\infty \int_0^\infty \boldsymbol{\Phi}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{M}(s,\, u)}{\partial s \partial u} \boldsymbol{\chi}(u) ds du$$

$$+ \int_0^\infty \int_0^\infty a(s) \frac{\partial^2 \boldsymbol{B}_1(s,\, u)}{\partial s \partial u} \cdot \boldsymbol{\beta}_d^t(u) ds du + \int_0^\infty \int_0^\infty T_d^t(s) \frac{\partial^2 \boldsymbol{B}_1(s,\, u)}{\partial s \partial u} \cdot \boldsymbol{b}(u) ds du$$

$$+ \int_0^\infty \int_0^\infty a(s) \frac{\partial^2 \boldsymbol{B}_2(s,\, u)}{\partial s \partial u} \cdot \boldsymbol{E}_d^t ds du + \int_0^\infty \int_0^\infty T_d^t(s) \frac{\partial^2 \boldsymbol{B}_2(s,\, u)}{\partial s \partial u} \cdot \boldsymbol{e}(u) ds du$$

$$+ \int_0^\infty \int_0^\infty a(s) \frac{\partial^2 \boldsymbol{B}_3(s,\, u)}{\partial s \partial u} \cdot \boldsymbol{W}_d^t(u) ds du + \int_0^\infty \int_0^\infty T_d^t(s) \frac{\partial^2 \boldsymbol{B}_3(s,\, u)}{\partial s \partial u} \cdot \boldsymbol{w}(u) ds du$$

$$+ \int_0^\infty \int_0^\infty \boldsymbol{b}(s) \cdot \frac{\partial^2 \boldsymbol{M}_1(s,\, u)}{\partial s \partial u} \boldsymbol{E}_d^t(u) ds du + \int_0^\infty \int_0^\infty \boldsymbol{\beta}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_1(s,\, u)}{\partial s \partial u} \boldsymbol{e}(u) ds du$$

$$+ \int_0^\infty \int_0^\infty \boldsymbol{b}(s) \cdot \frac{\partial^2 \boldsymbol{M}_2(s,\, u)}{\partial s \partial u} \boldsymbol{W}_d^t(u) ds du + \int_0^\infty \int_0^\infty \boldsymbol{\beta}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_2(s,\, u)}{\partial s \partial u} \boldsymbol{w}(u) ds du$$

$$+ \int_0^\infty \int_0^\infty \boldsymbol{w}(s) \cdot \frac{\partial^2 \boldsymbol{M}_3(s,\, u)}{\partial s \partial u} \boldsymbol{E}_d^t(u) ds du + \int_0^\infty \int_0^\infty \boldsymbol{W}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_3(s,\, u)}{\partial s \partial u} \boldsymbol{e}(u) ds du \tag{69}$$

Since in a state of equilibrium for our thermo-electro-viscoelastic material we have $\delta\tilde{\psi} = 0$, the above Fréchet derivative gives

$$\int_0^\infty \boldsymbol{z}(s;\, \boldsymbol{\Phi}) \cdot \boldsymbol{\chi}(s) ds = 0 \tag{70}$$

for all values of $\mathbf{\Phi}$ and function $\boldsymbol{\chi}(s)$. It follows that

$$\boldsymbol{z}(s;\, \mathbf{\Phi}) \equiv 0 \qquad \forall\, \mathbf{\Phi}\,. \tag{71}$$

10.1 Constitutive equations

Now let us compute the partial derivatives of $\tilde{\psi}$ that by the constitutive relations (52), $(46)_{2,\,3}$, (67), (68) and (135) furnish the constitutive equations for elastic stress, polarization vector, entropy and heat flux.

$$\begin{aligned}\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} = \frac{\partial\tilde{\Sigma}}{\partial\boldsymbol{\beta}} + \int_0^\infty \frac{\partial\boldsymbol{m}_2(0,\,u)}{\partial u}\boldsymbol{\beta}^t_d(u)du + \int_0^\infty T^t_d(s)\frac{\partial\boldsymbol{B}_1(s,\,0)}{\partial s}ds \\ + \int_0^\infty \frac{\partial\boldsymbol{M}_1(0,\,u)}{\partial u}\boldsymbol{E}^t_d(u)du + \int_0^\infty \frac{\partial\boldsymbol{M}_2(0,\,u)}{\partial u}\boldsymbol{W}^t_d(u)du\end{aligned} \tag{72}$$

$$\begin{aligned}\frac{\partial\tilde{\psi}}{\partial\boldsymbol{E}} = \frac{\partial\tilde{\Sigma}}{\partial\boldsymbol{E}} + \int_0^\infty \frac{\partial\boldsymbol{m}_3(0,\,u)}{\partial u}\boldsymbol{E}^t_d(u)du \\ + \int_0^\infty T^t_d(s)\frac{\partial\boldsymbol{B}_2(s,\,0)}{\partial s}ds + \int_0^\infty \boldsymbol{\beta}^t_d(s)\frac{\partial\boldsymbol{M}_1(s,\,0)}{\partial s}ds \\ + \int_0^\infty \boldsymbol{W}^t_d(s)\frac{\partial\boldsymbol{M}_3(s,\,0)}{\partial s}ds\end{aligned} \tag{73}$$

$$\begin{aligned}\frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}} = \frac{\partial\tilde{\Sigma}}{\partial\boldsymbol{W}} + \int_0^\infty \frac{\partial\boldsymbol{m}_4(0,\,u)}{\partial u}\boldsymbol{W}^t_d(u)du \\ + \int_0^\infty T^t_d(s)\frac{\partial\boldsymbol{B}_3(s,\,0)}{\partial s}ds + \int_0^\infty \boldsymbol{\beta}^t_d(s)\frac{\partial\boldsymbol{M}_2(s,\,0)}{\partial s}ds \\ + \int_0^\infty \frac{\partial\boldsymbol{M}_3(0,\,u)}{\partial u}\boldsymbol{E}^t_d(u)du\end{aligned} \tag{74}$$

$$\begin{aligned}\tilde{\eta}\frac{d\theta}{dT} = -\frac{\partial\tilde{\psi}}{\partial\theta}\frac{d\theta}{dT} = -\frac{\partial\tilde{\psi}}{\partial T} = -\int_0^\infty \frac{\partial m_1(0,\,u)}{\partial u}T^t_d(u)du \\ -\int_0^\infty \frac{\partial\boldsymbol{B}_1(0,\,u)}{\partial u}\boldsymbol{\beta}^t_d(u)du - \int_0^\infty \frac{\partial\boldsymbol{B}_2(0,\,u)}{\partial u}\boldsymbol{E}^t_d(u)du - \int_0^\infty \frac{\partial\boldsymbol{B}_3(0,\,u)}{\partial u}\boldsymbol{W}^t_d(u)du\end{aligned} \tag{75}$$

Now, the internal dissipation inequality writes as

$$\delta\tilde{\psi}(\mathbf{\Phi},\, \mathbf{\Phi}^t(s)\,|\, \frac{d}{ds}\mathbf{\Phi}^t(s)) \geq 0\,, \tag{76}$$

where $\delta\tilde{\psi}(\mathbf{\Phi},\, \mathbf{\Phi}^t(s)$ is given by (69) and (70). Integrating by parts the expression (69) of (76) and replacing $\boldsymbol{\chi}(s) = (a(s),\, \boldsymbol{b}(s),\, \boldsymbol{e}(s),\, \boldsymbol{w}(s))$ with $(d/ds)(T^t(s),\, \mathbf{\Phi}^t(s),\, \boldsymbol{E}^t(s),\, \boldsymbol{W}^t(s))$, we obtain the equivalent inequality

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{d}{ds}\boldsymbol{\Phi}^t(s) \cdot \frac{\partial \boldsymbol{M}(s,\, u)}{\partial s} \frac{d}{du}\boldsymbol{\Phi}^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{ds}T^t(s) \left[\frac{\partial \boldsymbol{B}_1(s,\, u)}{\partial s} + \frac{\partial \boldsymbol{B}_1(s,\, u)}{\partial u} \right] \cdot \frac{d}{du}\boldsymbol{\beta}^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{ds}T^t(s) \left[\frac{\partial \boldsymbol{B}_2(s,\, u)}{\partial s} + \frac{\partial \boldsymbol{B}_2(s,\, u)}{\partial u} \right] \cdot \frac{d}{du}\boldsymbol{E}^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{ds}T^t(s) \left[\frac{\partial \boldsymbol{B}_3(s,\, u)}{\partial s} + \frac{\partial \boldsymbol{B}_3(s,\, u)}{\partial u} \right] \cdot \frac{d}{du}\boldsymbol{W}^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{ds}\boldsymbol{\beta}^t(s) \cdot \left[\frac{\partial \boldsymbol{M}_1(s,\, u)}{\partial s} + \frac{\partial \boldsymbol{M}_1(s,\, u)}{\partial u} \right] \frac{d}{du}\boldsymbol{E}^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{ds}\boldsymbol{\beta}^t(s) \cdot \left[\frac{\partial \boldsymbol{M}_2(s,\, u)}{\partial s} + \frac{\partial \boldsymbol{M}_2(s,\, u)}{\partial u} \right] \frac{d}{du}\boldsymbol{W}^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{du}\boldsymbol{W}^t(u) \cdot \left[\frac{\partial \boldsymbol{M}_3(s,\, u)}{\partial s} + \frac{\partial \boldsymbol{M}_3(s,\, u)}{\partial u} \right] \frac{d}{du}\boldsymbol{E}^t(u) ds du\,, \\
&\leq 0
\end{aligned}
\tag{77}
$$

where the first integral on the right of the equality in (77) componentwise writes as

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{d}{ds}\boldsymbol{\Phi}^t(s) \cdot \frac{\partial \boldsymbol{M}(s,\, u)}{\partial s} \frac{d}{du}\boldsymbol{\Phi}^t(u) ds du = \\
&= \int_0^\infty \int_0^\infty \frac{d}{ds}T^t(s) \frac{\partial m_1(s,\, u)}{\partial s} \frac{d}{du}T^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{ds}\boldsymbol{\beta}^t(s) \cdot \frac{\partial \boldsymbol{m}_2(s,\, u)}{\partial s} \frac{d}{du}\boldsymbol{\beta}^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{ds}\boldsymbol{E}^t(s) \cdot \frac{\partial \boldsymbol{m}_3(s,\, u)}{\partial s} \frac{d}{du}\boldsymbol{E}^t(u) ds du \\
&+ \int_0^\infty \int_0^\infty \frac{d}{du}\boldsymbol{W}^t(u) \cdot \frac{\partial \boldsymbol{m}_4(s,\, u)}{\partial s} \frac{d}{du}\boldsymbol{W}^t(u) ds du\,.
\end{aligned}
\tag{78}
$$

Inequality (77) must hold in all thermoelectro-mechanic processes of the body and thus they allow to deduce restrictions on the material relaxation functions $\boldsymbol{M} = (m_1,\, \boldsymbol{m}_2,\, \boldsymbol{m}_3,\, \boldsymbol{m}_4)$, $\boldsymbol{B}_i$ and $\boldsymbol{M}_i$ ($i = 1,\, 2,\, 3$).

11 Restrictions on the material relaxation functions

11.1 A – Step functions for a single variable ξ

Next, by extending the procedure in [9], we find restrictions on the material relaxation functions from the internal dissipation inequality. Remind that $\boldsymbol{\Phi} := (T,\, \boldsymbol{\beta},\, \boldsymbol{E},\, \boldsymbol{W})$.

For

$$
\xi = \begin{cases} T \\ \boldsymbol{\beta} \\ \boldsymbol{E} \\ \boldsymbol{W} \end{cases} \qquad \text{let} \quad \boldsymbol{Z}^\xi \quad = \begin{cases} m_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \\ \boldsymbol{m}_4 \end{cases}
\tag{79}
$$

be the associated relaxation function. Furthermore, for

$$\xi = \begin{cases} T \\ \boldsymbol{\beta} \\ \boldsymbol{E} \\ \boldsymbol{W} \end{cases} \quad \text{let} \quad J_i^\xi = \begin{cases} J_1^T,\, J_2^T \\ J_1^{\boldsymbol{\beta}},\, J_2^{\boldsymbol{\beta}} \\ J_1^{\boldsymbol{E}},\, J_2^{\boldsymbol{E}} \\ J_1^{\boldsymbol{W}},\, J_2^{\boldsymbol{W}} \end{cases} \quad \text{be an arbitrary} \quad \begin{cases} \text{scalar} \in I\!R \\ \text{vector} \in I\!R^3 \\ \text{symmetric tensor} \in Sym \\ \text{vector} \in I\!R^3 \end{cases} \tag{80}$$

In correspondence with any $\xi \in \{T,\, \boldsymbol{\beta},\, \boldsymbol{E},\, \boldsymbol{W}\}$ putting in (77), (78) $\nu = 0$ for each $\nu \in \{T,\, \boldsymbol{\beta},\, \boldsymbol{E},\, \boldsymbol{W}\} \setminus \{\xi\}$, we obtain an inequality of the type

$$\int_0^\infty \int_0^\infty \frac{d}{ds}\xi^t(s) \cdot \frac{\partial \boldsymbol{Z}^\xi}{\partial s}(s,\, u) \frac{d}{du}\xi^t(u) ds du \leq 0 \tag{81}$$

which must hold for each history $\xi^t(s)$, where

$$\boldsymbol{Z}^\xi \in \{m_1,\, \boldsymbol{m}_2,\, \boldsymbol{m}_3,\, \boldsymbol{m}_4\} \tag{82}$$

is the stress relaxation function associated to ξ. Now, for any positive scalar ϵ let us consider the history

$$\begin{aligned} \frac{d}{dv}\xi(v) &= \frac{1}{\epsilon} J_1^\xi\,, \quad s \leq v \leq s + \epsilon\,, \\ &= \frac{1}{\epsilon} J_2^\xi\,, \quad u \leq v \leq u + \epsilon\,, \\ &= \boldsymbol{0} \qquad \text{elsewhere}\,. \end{aligned} \tag{83}$$

Then, on letting $\epsilon \to 0$, $\xi(\sigma)$ becomes a step function, and from (81) we obtain the inequality

$$J_1^\xi \cdot \frac{\partial \boldsymbol{Z}^\xi}{\partial u}(s,\, s) J_1^\xi + 2 J_1^\xi \cdot \frac{\partial \boldsymbol{Z}^\xi}{\partial u}(s,\, u) J_2^\xi + J_2^\xi \cdot \frac{\partial \boldsymbol{Z}^\xi}{\partial u}(u,\, u) J_2^\xi \leq 0\,, \tag{84}$$

which holds for all choices of J_1^ξ, J_2^ξ. Now, for $J_2^\xi = 0$ and $J_2^\xi = \pm J_1^\xi$ we find the following restrictions on each relaxation function $\boldsymbol{Z}^\xi \in \{m_1,\, \boldsymbol{m}_2,\, \boldsymbol{m}_3,\, \boldsymbol{m}_4\}$: [1]

$$\frac{\partial \boldsymbol{Z}^\xi}{\partial u}(s,\, s) \leq \boldsymbol{0} \qquad \forall s\,, \tag{85}$$

$$\frac{\partial \boldsymbol{Z}^\xi}{\partial u}(s,\, s) \pm 2 \frac{\partial \boldsymbol{Z}^\xi}{\partial u}(s,\, u) + \frac{\partial \boldsymbol{Z}^\xi}{\partial u}(u,\, u) \leq \boldsymbol{0} \qquad \forall s,\, u\,. \tag{86}$$

11.2 B – Step functions for coupled variables ξ, ν

11.2.1 Relationship between the relaxation functions $\boldsymbol{B}_1$, $\boldsymbol{m}_2$, m_1

Next we obtain restrictions on the temperature-(temperature gradient) relaxation function $\boldsymbol{B}_1$ by considering step function histories for both the strain and the temperature of the form

$$\begin{aligned} \frac{d}{dv}\boldsymbol{\beta}^t(v) &= \frac{1}{\epsilon}\boldsymbol{b}\,, \quad s \leq v \leq s + \epsilon\,, \\ &= \boldsymbol{0} \qquad \text{elsewhere} \\ \frac{d}{dv}\theta^t(v) &= \frac{1}{\epsilon}\lambda\,, \quad u \leq v \leq u + \epsilon\,, \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

[1] Remind that for a second- or fourth-order tensor $\boldsymbol{A}$ by $\boldsymbol{A} \leq \boldsymbol{0}$ ($\boldsymbol{A} \geq \boldsymbol{0}$) we mean $\boldsymbol{A}$ is negative semi-definite (positive semi-definite). Accordingly, by $\boldsymbol{A} \leq \boldsymbol{B}$ we mean $\boldsymbol{A} - \boldsymbol{B}$ is negative semi-definite.

Substituting these histories into the expression for the internal dissipation inequality and letting $\epsilon \to 0$, we obtain the inequality

$$\boldsymbol{b} \cdot \frac{\partial \boldsymbol{m}_2}{\partial u}(s,\, s)\boldsymbol{b} + \left[\frac{\partial \boldsymbol{B}_1}{\partial s}(s,\, u) + \frac{\partial \boldsymbol{B}_1}{\partial u}(s,\, u)\right] \cdot \boldsymbol{b}\lambda + \frac{\partial m_1}{\partial u}(u,\, u)\lambda^2 \leq 0\,, \tag{87}$$

which holds for all vectors $\boldsymbol{b}$ and scalars λ. It follows the following restriction:

$$\left|\frac{\partial B_{1\,i}}{\partial s}(s,\, u) + \frac{\partial B_{1\,i}}{\partial u}(s,\, u)\right|^2 \leq 4\frac{\partial m_{2\,ii}}{\partial u}(s,\, s)\frac{\partial m_1}{\partial u}(u,\, u) \quad (i \quad \text{not summed}) \tag{88}$$

for all s, u.

11.2.2 Relationship between the relaxation functions $\boldsymbol{B}_2$, $\boldsymbol{m}_3$, m_1

Next we find restrictions on the stress-temperature relaxation function $\boldsymbol{B}_2$ by considering step function histories for both the strain and the temperature of the form

$$\begin{aligned} \frac{d}{dv}\boldsymbol{E}^t(v) &= \frac{1}{\epsilon}\boldsymbol{J}\,, \quad s \leq v \leq s+\epsilon\,, \\ &= \boldsymbol{0} \qquad \text{elsewhere} \\ \frac{d}{dv}\theta^t(v) &= \frac{1}{\epsilon}\lambda\,, \quad u \leq v \leq u+\epsilon\,, \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

Substituting these histories into the expression for the internal dissipation inequality and letting $\epsilon \to 0$, we obtain the inequality

$$\boldsymbol{J} \cdot \frac{\partial \boldsymbol{m}_3}{\partial u}(s,\, s)\boldsymbol{J} + \left[\frac{\partial \boldsymbol{B}_2}{\partial s}(s,\, u) + \frac{\partial \boldsymbol{B}_2}{\partial u}(s,\, u)\right] \cdot \boldsymbol{J}\lambda + \frac{\partial m_1}{\partial u}(u,\, u)\lambda^2 \leq 0\,, \tag{89}$$

which holds for all symmetric tensors $\boldsymbol{J}$ and scalars λ. It follows the following restriction:

$$\left|\frac{\partial B_{2\,ij}}{\partial s}(s,\, u) + \frac{\partial B_{2\,ij}}{\partial u}(s,\, u)\right|^2 \leq 4\frac{\partial m_{3\,ijij}}{\partial u}(s,\, s)\frac{\partial m_1}{\partial u}(u,\, u) \quad (i,\, j \quad \text{not summed}) \tag{90}$$

for all s, u.

11.2.3 Relationship between the relaxation functions $\boldsymbol{B}_3$, $\boldsymbol{m}_4$, m_1

We can also obtain restrictions on the temperature-(electric vector) relaxation function $\boldsymbol{B}_3$ by considering step function histories for both the strain and the temperature of the form

$$\begin{aligned} \frac{d}{dv}\boldsymbol{\beta}^t(v) &= \frac{1}{\epsilon}\boldsymbol{b}\,, \quad s \leq v \leq s+\epsilon\,, \\ &= \boldsymbol{0} \qquad \text{elsewhere} \\ \frac{d}{dv}\theta^t(v) &= \frac{1}{\epsilon}\lambda\,, \quad u \leq v \leq u+\epsilon\,, \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

Substituting these histories into our expression for the internal dissipation inequality and letting $\epsilon \to 0$, we obtain the following inequality

$$\boldsymbol{b} \cdot \frac{\partial \boldsymbol{m}_4}{\partial u}(s,\, s)\boldsymbol{b} + \left[\frac{\partial \boldsymbol{B}_3}{\partial s}(s,\, u) + \frac{\partial \boldsymbol{B}_3}{\partial u}(s,\, u)\right] \cdot \boldsymbol{b}\lambda + \frac{\partial m_1}{\partial u}(u,\, u)\lambda^2 \leq 0\,, \tag{91}$$

which holds for all vectors $\boldsymbol{b}$ and scalars λ. It follows the following restriction:

$$\left|\frac{\partial B_{3\,i}}{\partial s}(s,\,u)+\frac{\partial B_{3\,i}}{\partial u}(s,\,u)\right|^2\leq 4\frac{\partial m_{4\,ii}}{\partial u}(s,\,s)\frac{\partial m_1}{\partial u}(u,\,u)\quad (i \text{ not summed}) \tag{92}$$

for all s, u.

11.2.4 Relationship between the relaxation functions $\boldsymbol{M}_1$, $\boldsymbol{m}_3$, m_2

We can also obtain restrictions on the stress-temperature relaxation function $\boldsymbol{M}_1$ by considering step function histories for both the strain and the temperature gradient of the form

$$\begin{aligned}\frac{d}{dv}\boldsymbol{E}^t(v)&=\frac{1}{\epsilon}\boldsymbol{J}\,,\quad s\leq v\leq s+\epsilon\,,\\ &=\mathbf{0}\qquad\quad \text{elsewhere}\\ \frac{d}{dv}\boldsymbol{\beta}^t(v)&=\frac{1}{\epsilon}\boldsymbol{b}\,,\quad u\leq v\leq u+\epsilon\,,\\ &=0\qquad\quad \text{elsewhere}\end{aligned}$$

Substituting these histories into our expression for the internal dissipation inequality and letting $\epsilon\to 0$, we obtain the following inequality

$$\boldsymbol{J}\cdot\frac{\partial \boldsymbol{m}_3}{\partial u}(s,\,s)\boldsymbol{J}+\boldsymbol{b}\cdot\left[\frac{\partial \boldsymbol{M}_1}{\partial s}(s,\,u)+\frac{\partial \boldsymbol{M}_1}{\partial u}(s,\,u)\right]\boldsymbol{J}+\boldsymbol{b}\cdot\frac{\partial \boldsymbol{m}_2}{\partial u}(u,\,u)\boldsymbol{b}\leq 0\,, \tag{93}$$

which holds for all symmetric tensors $\boldsymbol{J}$ and vectors $\boldsymbol{b}$. In components (93) writes as

$$J_{pq}\frac{\partial m_3^{pqrs}}{\partial u}(s,\,s)J_{rs}+b_p\left[\frac{\partial M_1^{pqr}}{\partial s}(s,\,u)+\frac{\partial M_1^{pqr}}{\partial u}(s,\,u)\right]J_{qr}+b_p\frac{\partial m_2^{pq}}{\partial u}(u,\,u)b_q\leq 0\,, \tag{94}$$

This inequality implies the following restrictions:

$$\left|\frac{\partial M_1^{pqr}}{\partial s}(s,\,u)+\frac{\partial M_1^{pqr}}{\partial u}(s,\,u)\right|^2\leq 4\frac{\partial m_3^{qrqr}}{\partial u}(s,\,s)\frac{\partial m_2^{pp}}{\partial u}(u,\,u)\quad (p,\,q,r \text{ not summed}) \tag{95}$$

for all s, u.

11.2.5 Relationship between the relaxation functions $\boldsymbol{M}_2$, $\boldsymbol{m}_4$, m_1

We can also obtain restrictions on the (temperature gradient)-(electric vector) relaxation function $\boldsymbol{M}_3$ by considering step function histories for both the temperature gradient and the electric vector of the form

$$\begin{aligned}\frac{d}{dv}\boldsymbol{W}^t(v)&=\frac{1}{\epsilon}\boldsymbol{\omega}\,,\quad s\leq v\leq s+\epsilon\,,\\ &=\mathbf{0}\qquad\quad \text{elsewhere}\\ \frac{d}{dv}\boldsymbol{\beta}^t(v)&=\frac{1}{\epsilon}\boldsymbol{b}\,,\quad u\leq v\leq u+\epsilon\,,\\ &=0\qquad\quad \text{elsewhere}\end{aligned}$$

Substituting these histories into our expression for the internal dissipation inequality and letting $\epsilon\to 0$, we obtain the following inequality

$$\boldsymbol{\omega}\cdot\frac{\partial \boldsymbol{m}_4}{\partial u}(s,\,s)\boldsymbol{\omega}+\boldsymbol{b}\cdot\left[\frac{\partial \boldsymbol{M}_2}{\partial s}(s,\,u)+\frac{\partial \boldsymbol{M}_2}{\partial u}(s,\,u)\right]\boldsymbol{\omega}+\boldsymbol{b}\cdot\frac{\partial \boldsymbol{m}_2}{\partial u}(u,\,u)\boldsymbol{b}\leq 0\,, \tag{96}$$

which holds for all vectors $\boldsymbol{\omega}$, $\boldsymbol{b}$. In components (96) writes as

$$\omega_p \frac{\partial m_4^{pq}}{\partial u}(s,\, s)\omega_q + b_p \left[\frac{\partial M_2^{pq}}{\partial s}(s,\, u) + \frac{\partial M_2^{pq}}{\partial u}(s,\, u)\right]\omega_q + b_p \frac{\partial m_2^{pq}}{\partial u}(u,\, u) b_q \leq 0\,, \quad (97)$$

This inequality implies the following restrictions:

$$\left|\frac{\partial M_2^{pq}}{\partial s}(s,\, u) + \frac{\partial M_2^{pq}}{\partial u}(s,\, u)\right|^2 \leq 4\frac{\partial m_4^{qq}}{\partial u}(s,\, s)\frac{\partial m_2^{pp}}{\partial u}(u,\, u) \quad (p,\, q \text{ not summed}) \quad (98)$$

for all s, u.

11.2.6 Relationship between the relaxation functions $\boldsymbol{M}_3$, $\boldsymbol{m}_4$, $\boldsymbol{m}_3$

We can also obtain restrictions on the stress-temperature relaxation function $\boldsymbol{M}_3$ by considering step function histories for both the strain and the temperature gradient of the form

$$\begin{aligned} \frac{d}{dv}\boldsymbol{E}^t(v) &= \frac{1}{\epsilon}\boldsymbol{J}\,, \quad && s \leq v \leq s+\epsilon\,, \\ &= \boldsymbol{0} && \text{elsewhere} \\ \frac{d}{dv}\boldsymbol{W}^t(v) &= \frac{1}{\epsilon}\boldsymbol{\omega}\,, \quad && u \leq v \leq u+\epsilon\,, \\ &= 0 && \text{elsewhere} \end{aligned}$$

Substituting these histories into our expression for the internal dissipation inequality and letting $\epsilon \to 0$, we obtain the following inequality

$$\boldsymbol{J}\cdot\frac{\partial \boldsymbol{m}_3}{\partial u}(s,\, s)\boldsymbol{J} + \boldsymbol{\omega}\cdot\left[\frac{\partial \boldsymbol{M}_3}{\partial s}(s,\, u) + \frac{\partial \boldsymbol{M}_3}{\partial u}(s,\, u)\right]\boldsymbol{J} + \boldsymbol{\omega}\cdot\frac{\partial \boldsymbol{m}_4}{\partial u}(u,\, u)\boldsymbol{\omega} \leq 0\,, \quad (99)$$

which holds for all symmetric tensors $\boldsymbol{J}$ and vectors $\boldsymbol{\omega}$. In components (99) writes as

$$J_{pq}\frac{\partial m_3^{pqrs}}{\partial u}(s,\, s)J_{rs} + \omega_p\left[\frac{\partial M_3^{pqr}}{\partial s}(s,\, u) + \frac{\partial M_3^{pqr}}{\partial u}(s,\, u)\right]J_{qr} + \omega_p\frac{\partial m_4^{pq}}{\partial u}(u,\, u)\omega_q \leq 0\,, \quad (100)$$

This inequality implies the following restrictions:

$$\left|\frac{\partial M_3^{pqr}}{\partial s}(s,\, u) + \frac{\partial M_3^{pqr}}{\partial u}(s,\, u)\right|^2 \leq 4\frac{\partial m_3^{pqpq}}{\partial u}(s,\, s)\frac{\partial m_4^{pp}}{\partial u}(u,\, u) \quad (p,\, q,\, r \text{ not summed}) \quad (101)$$

for all s, u.

11.3 Restrictions from the minimality of the free energy in equilibrium

There are further restrictions on the relaxation functions that can be deduced from the minimality of the free energy in equilibrium, represented by inequality $(64)_2$. In the present linear theory, by assumption (ii) in Section 10, such inequality just becomes

$$0 \le \delta^2\tilde{\psi}(\mathbf{\Phi},\, \mathbf{\Phi}^t(s)\,|\,\boldsymbol{\chi}(s)) = +\int_0^\infty\int_0^\infty \boldsymbol{\chi}(s)\cdot\frac{\partial^2 \boldsymbol{M}(s,\,u)}{\partial s\partial u}\boldsymbol{\chi}(u)dsdu$$
$$+2\int_0^\infty\int_0^\infty a(s)\frac{\partial^2 \boldsymbol{B}_1(s,\,u)}{\partial s\partial u}\cdot\boldsymbol{b}(u)dsdu + 2\int_0^\infty\int_0^\infty a(s)\frac{\partial^2 \boldsymbol{B}_2(s,\,u)}{\partial s\partial u}\cdot\boldsymbol{J}(u)dsdu$$
$$+2\int_0^\infty\int_0^\infty a(s)\frac{\partial^2 \boldsymbol{B}_3(s,\,u)}{\partial s\partial u}\cdot\boldsymbol{w}(u)dsdu + 2\int_0^\infty\int_0^\infty \boldsymbol{b}(s)\cdot\frac{\partial^2 \boldsymbol{M}_1(s,\,u)}{\partial s\partial u}\boldsymbol{J}(u)dsdu$$
$$+2\int_0^\infty\int_0^\infty \boldsymbol{b}(s)\cdot\frac{\partial^2 \boldsymbol{M}_2(s,\,u)}{\partial s\partial u}\boldsymbol{w}(u)dsdu + 2\int_0^\infty\int_0^\infty \boldsymbol{w}(s)\cdot\frac{\partial^2 \boldsymbol{M}_3(s,\,u)}{\partial s\partial u}\boldsymbol{J}(u)dsdu \tag{102}$$

where

$$\int_0^\infty\int_0^\infty \boldsymbol{\chi}(s)\cdot\frac{\partial^2 \boldsymbol{M}(s,\,u)}{\partial s}\boldsymbol{\chi}(u)dsdu =$$
$$= \int_0^\infty\int_0^\infty a(s)\frac{\partial^2 m_1(s,\,u)}{\partial s}a(u)dsdu + \int_0^\infty\int_0^\infty \boldsymbol{b}(s)\cdot\frac{\partial^2 \boldsymbol{m}_2(s,\,u)}{\partial s}\boldsymbol{b}(u)dsdu$$
$$+\int_0^\infty\int_0^\infty \boldsymbol{J}(s)\cdot\frac{\partial^2 \boldsymbol{m}_3(s,\,u)}{\partial s}\boldsymbol{J}(u)dsdu + \int_0^\infty\int_0^\infty \boldsymbol{w}(s)\cdot\frac{\partial^2 \boldsymbol{m}_4(s,\,u)}{\partial s}\boldsymbol{w}(u)dsdu \tag{103}$$

and it must hold for all scalar functions $a(s)$, vector functions $\boldsymbol{b}(s)$, $\boldsymbol{w}(s)$, tensor functions $\boldsymbol{J}(s)$, i.e. for all $\boldsymbol{\chi}(s) = (a(s),\, \boldsymbol{b}(s),\, \boldsymbol{J}(s),\, \boldsymbol{w}(s))$.

11.3.1 Further restrictions on m_1, $\boldsymbol{m}_2$, $\boldsymbol{m}_3$ and $\boldsymbol{m}_4$

Using a technique similar to that used in analysing the internal dissipation inequality, now we assume that

$$a(s) \equiv 0, \qquad \boldsymbol{b}(s) \equiv \mathbf{0}, \qquad \boldsymbol{w}(s) \equiv \mathbf{0}$$

and that $\boldsymbol{J}(s)$ is the step function defined by

$$\boldsymbol{J}(v) = \begin{cases} \mathbf{0}, & 0 \le v < s, \\ \boldsymbol{A}, & s \le v < u, \\ \boldsymbol{A}+\boldsymbol{B}, & u \le v < \infty. \end{cases} \tag{104}$$

In this case, the inequality (102) becomes

$$\boldsymbol{A}\cdot\boldsymbol{m}_3(s,\,s)\boldsymbol{A} + 2\boldsymbol{A}\cdot\boldsymbol{m}_3(s,\,u)\boldsymbol{B} + \boldsymbol{B}\cdot\boldsymbol{m}_3(u,\,u)\boldsymbol{B} \ge 0\,, \tag{105}$$

which holds for all symmetric tensors $\boldsymbol{A}$ and $\boldsymbol{B}$. And by putting $\boldsymbol{B} = \mathbf{0}$, $\boldsymbol{B} = \pm\boldsymbol{A}$ we respectively deduce

$$\boldsymbol{m}_3(s,\,s) \ge \mathbf{0} \qquad \forall s,\, u\,, \tag{106}$$

$$\boldsymbol{m}_3(s,\,s) + \boldsymbol{m}_3(u,\,u) \pm 2\boldsymbol{m}_3(s,\,u) \ge \mathbf{0} \qquad \forall s,\, u\,. \tag{107}$$

We can deduce similar restrictions on m_1, $\boldsymbol{m}_2$ and $\boldsymbol{m}_4$ by putting respectively
$\boldsymbol{b}(s) \equiv \mathbf{0}$, $\boldsymbol{J}(s) \equiv \mathbf{0}$, $\boldsymbol{w}(s) \equiv \mathbf{0}$, with $a(s)$ step function as in (104),
$a(s) \equiv 0$, $\boldsymbol{J}(s) \equiv \mathbf{0}$, $\boldsymbol{w}(s) \equiv \mathbf{0}$, with $\boldsymbol{b}(s)$ step function as in (104), and
$a(s) \equiv 0$, $\boldsymbol{b}(s) \equiv \mathbf{0}$, $\boldsymbol{J}(s) \equiv \mathbf{0}$, with $\boldsymbol{w}(s)$ step function as in (104).
For $i = 1,\, 2,\, 4$ we obtain the restrictions

$$\boldsymbol{m}_i(s,\,s) \ge \mathbf{0} \qquad \forall s,\, u\,, \tag{108}$$

$$\boldsymbol{m}_i(s,\,s) + \boldsymbol{m}_i(u,\,u) \pm 2\boldsymbol{m}_i(s,\,u) \ge \mathbf{0} \qquad \forall s,\, u\,. \tag{109}$$

11.3.2 Further restrictions on the relaxation functions $\boldsymbol{B}_i$ and $\boldsymbol{M}_i$ in terms of $\boldsymbol{m}_i$

(231) Restrictions on $\boldsymbol{B}_2$ in terms of $\boldsymbol{m}_3$ and m_1.

We can now find bounds on the function $\boldsymbol{B}_2$ in terms of $\boldsymbol{m}_3$ and m_1 by considering step functions for $\boldsymbol{A}(s)$ and $a(s)$ of the following form:

$$\boldsymbol{A}(v) = \begin{cases} \boldsymbol{0}, & 0 \le v < s, \\ \boldsymbol{A}, & s \le v < u, \end{cases} \qquad a(v) = \begin{cases} 0, & 0 \le v < u, \\ a, & u \le v < \infty. \end{cases} \tag{110}$$

In this case inequality (102) becomes

$$\boldsymbol{A} \cdot \boldsymbol{m}_3(s,\, s)\boldsymbol{A} + 2\boldsymbol{B}_2(s,\, u) \cdot \boldsymbol{A}a + m_1(u,\, u)a^2 \ge 0\,, \tag{111}$$

for all symmetric tensors $\boldsymbol{A}$ and scalars a, that implies the following restriction on $\boldsymbol{B}_2$:

$$|B_{2\,pq}(s,u)|^2 \le m_{3\,pqpq}(s,s)m_1(u,u) \qquad (p,\ q \text{ not summed}) \tag{112}$$

for all s, u.

(121) Restrictions on $\boldsymbol{B}_1$ in terms of $\boldsymbol{m}_2$ and m_1.

We can now find bounds on the function $\boldsymbol{B}_1$ in terms of $\boldsymbol{m}_2$ and m_1 by considering step functions for $\boldsymbol{b}(s)$ and $a(s)$ of the following form:

$$\boldsymbol{b}(v) = \begin{cases} \boldsymbol{0}, & 0 \le v < s, \\ \boldsymbol{b}, & s \le v < u, \end{cases} \qquad a(v) = \begin{cases} 0, & 0 \le v < u, \\ a, & u \le v < \infty. \end{cases} \tag{113}$$

In this case inequality (102) becomes

$$\boldsymbol{b} \cdot \boldsymbol{m}_2(s,\, s)\boldsymbol{b} + 2\boldsymbol{B}_1(s,\, u) \cdot \boldsymbol{b}a + m_1(u,\, u)a^2 \ge 0\,, \tag{114}$$

for all vectors $\boldsymbol{b}$ and scalars a, that implies the following restriction on $\boldsymbol{B}_1$:

$$|B_{1\,p}(s,u)|^2 \le m_{2\,pp}(s,s)m_1(u,u) \qquad (p \text{ not summed}) \tag{115}$$

for all s, u.

(341) Restrictions on $\boldsymbol{B}_3$ in terms of $\boldsymbol{m}_4$ and m_1.

We can now find bounds on the function $\boldsymbol{B}_3$ in terms of $\boldsymbol{m}_4$ and m_1 by considering step functions for $\boldsymbol{\omega}(s)$ and $a(s)$ of the following form:

$$\boldsymbol{\omega}(v) = \begin{cases} \boldsymbol{0}, & 0 \le v < s, \\ \boldsymbol{\omega}, & s \le v < u, \end{cases} \qquad a(v) = \begin{cases} 0, & 0 \le v < u, \\ a, & u \le v < \infty. \end{cases} \tag{116}$$

In this case inequality (102) becomes

$$\boldsymbol{\omega} \cdot \boldsymbol{m}_4(s,\, s)\boldsymbol{\omega} + 2\boldsymbol{B}_3(s,\, u) \cdot \boldsymbol{\omega}a + m_1(u,\, u)a^2 \ge 0\,, \tag{117}$$

for all vectors $\boldsymbol{\omega}$ and scalars a, that implies the following restriction on $\boldsymbol{B}_1$:

$$|B_{3\,p}(s,u)|^2 \le m_{4\,pp}(s,s)m_1(u,u) \qquad (p \text{ not summed}) \tag{118}$$

for all s, u.

(123) Restrictions on $\boldsymbol{M}_1$ in terms of $\boldsymbol{m}_2$ and $\boldsymbol{m}_3$.

We can now find bounds on the function $\boldsymbol{M}_1$ in terms of $\boldsymbol{m}_4$ and $\boldsymbol{m}_3$ by considering step functions for $\boldsymbol{b}(s)$ and $\boldsymbol{A}(s)$ of the following form:

$$\boldsymbol{b}(v) = \begin{cases} \boldsymbol{0}, & 0 \le v < s, \\ \boldsymbol{b}, & s \le v < u, \end{cases} \qquad \boldsymbol{A}(v) = \begin{cases} \boldsymbol{0}, & 0 \le v < u, \\ \boldsymbol{A}, & u \le v < \infty. \end{cases} \tag{119}$$

In this case inequality (102) becomes

$$\boldsymbol{A}\cdot\boldsymbol{m}_3(s,\,s)\boldsymbol{A}+2\boldsymbol{b}\cdot\boldsymbol{M}_1(s,\,u)\boldsymbol{A}+\boldsymbol{b}\cdot\boldsymbol{m}_2(u,\,u)\boldsymbol{b}\geq 0\,, \tag{120}$$

for all vectors $\boldsymbol{b}$ and symmetric tensors $\boldsymbol{A}$, that implies the following restriction on $\boldsymbol{M}_1$:

$$|M_{1\,ipq}(s,u)|^2 \leq m_{3\,pqpq}(s,s)m_{2\,ii}(u,u) \qquad (i,\,p,\,q \text{ not summed}) \tag{121}$$

for all s, u.

(343) Restrictions on $\boldsymbol{M}_3$ in terms of $\boldsymbol{m}_4$ and $\boldsymbol{m}_3$.

We can now find bounds on the function $\boldsymbol{M}_3$ in terms of $\boldsymbol{m}_4$ and $\boldsymbol{m}_3$ by considering step functions for $\boldsymbol{\omega}(s)$ and $\boldsymbol{A}(s)$ of the following form:

$$\boldsymbol{\omega}(v)=\begin{cases}\mathbf{0}, & 0\leq v<s,\\ \boldsymbol{\omega}, & s\leq v<u,\end{cases} \qquad \boldsymbol{A}(v)=\begin{cases}\mathbf{0}, & 0\leq v<u,\\ \boldsymbol{A}, & u\leq v<\infty.\end{cases} \tag{122}$$

In this case inequality (102) becomes

$$\boldsymbol{A}\cdot\boldsymbol{m}_3(s,\,s)\boldsymbol{A}+2\boldsymbol{\omega}\cdot\boldsymbol{M}_3(s,\,u)\boldsymbol{A}+\boldsymbol{\omega}\cdot\boldsymbol{m}_4(u,\,u)\boldsymbol{\omega}\geq 0\,, \tag{123}$$

for all vectors $\boldsymbol{\omega}$ and symmetric tensors $\boldsymbol{A}$, that implies the following restriction on $\boldsymbol{M}_3$:

$$|M_{3\,ipq}(s,u)|^2 \leq m_{3\,pqpq}(s,s)m_{4\,ii}(u,u) \qquad (i,\,p,\,q \text{ not summed}) \tag{124}$$

for all s, u.

(224) Restrictions on $\boldsymbol{M}_2$ in terms of $\boldsymbol{m}_2$ and $\boldsymbol{m}_4$.

We can now find bounds on the function $\boldsymbol{M}_2$ in terms of $\boldsymbol{m}_2$ and $\boldsymbol{m}_4$ by considering step functions for $\boldsymbol{\omega}(s)$ and $\boldsymbol{b}(s)$ of the following form:

$$\boldsymbol{\omega}(v)=\begin{cases}\mathbf{0}, & 0\leq v<s,\\ \boldsymbol{\omega}, & s\leq v<u,\end{cases} \qquad \boldsymbol{b}(v)=\begin{cases}\mathbf{0}, & 0\leq v<u,\\ \boldsymbol{b}, & u\leq v<\infty.\end{cases} \tag{125}$$

In this case inequality (102) becomes

$$\boldsymbol{b}\cdot\boldsymbol{m}_2(s,\,s)\boldsymbol{b}+2\boldsymbol{\omega}\cdot\boldsymbol{M}_2(s,\,u)\boldsymbol{b}+\boldsymbol{\omega}\cdot\boldsymbol{m}_4(u,\,u)\boldsymbol{\omega}\geq 0\,, \tag{126}$$

for all vectors $\boldsymbol{\omega}$ and $\boldsymbol{b}$, that implies the following restriction on $\boldsymbol{M}_2$:

$$|M_{2\,pq}(s,u)|^2 \leq m_{2\,qq}(s,s)m_{4\,pp}(u,u) \qquad (p,\,q \text{ not summed}) \tag{127}$$

for all s, u.

11.4 Further restrictions due to the previous ones

Extending the theory in [9] we have derived two sets of restrictions on the relaxation functions: the first from the internal dissipation inequality and the other from the minimality of the free energy in equilibrium. Next we obtain further restrictions by combining such two sets of restrictions. This allows to find in particular the generalization of Day's results of the purely mechanical viscoelasticity theory to the present theory.

Inequality (85) for each $\boldsymbol{Z}^\xi \in \{m_1,\,\boldsymbol{m}_2,\,\boldsymbol{m}_3,\,\boldsymbol{m}_4\}$ writes as

$$\frac{\partial \boldsymbol{Z}^\xi}{\partial u}(s,\,u)|_{s=u}\;\leq\;\mathbf{0} \qquad \forall s\,. \tag{128}$$

From the symmetries of the tensors $\boldsymbol{m}_2,\, \boldsymbol{m}_3,\, \boldsymbol{m}_4$ we also have

$$\frac{\partial \boldsymbol{Z}^{\xi}}{\partial s}(s,\, u)|_{s=u} \,\leq\, \mathbf{0} \qquad \forall s, \quad \boldsymbol{Z}^{\xi} = \boldsymbol{m}_2,\, \boldsymbol{m}_3,\, \boldsymbol{m}_4\,. \tag{129}$$

Lastly, as in [9, pp.218, 219], from the minimality of the free energy in equilibrium we obtain the inequalities

$$\boldsymbol{Z}^{\xi}(s,\, s) \geq \boldsymbol{Z}^{\xi}(u,\, u) \quad \forall s \leq u, \tag{130}$$

and

$$\boldsymbol{Z}^{\xi}(\hat{s},\, \hat{s}) \geq \pm \boldsymbol{Z}^{\xi}(s,\, u) \quad \forall s,\; u \tag{131}$$

where $\hat{s} = min(s,\, u)$.

This completes the restrictions on the relaxation functions of the thermo-electro-viscoelastic material defined here. As in [9] let us note that the relaxation functions only enter the constitutive relations through their values at $(0,\, u)$ or $(s,\, 0)$. The particular local restrictions they suffer can be easily deduced by a limit from all the above restrictions on the relaxation functions:

$$\begin{gathered} \boldsymbol{m}_i(0,\, 0) \qquad \text{is symmetric} \qquad i = 1,\, 2,\, 3\,, \\ \boldsymbol{m}_i(0,\, 0) \geq \mathbf{0}, \quad \frac{\partial \boldsymbol{m}_i}{\partial u}(0,\, 0) \leq \mathbf{0}, \quad \boldsymbol{m}_i(0,\, 0) \geq \pm \boldsymbol{m}_i(0,\, u) \quad i = 1,\, 2,\, 3,\, 4. \end{gathered} \tag{132}$$

The restrictions (132) coincide with the ones in [9, p.219] for the relaxation functions for the stress and entropy, m_1 and $\boldsymbol{m}_3$. In the present extension of [9] for a heat conducting electrical dielectric, there are in addition the relaxation functions $\boldsymbol{m}_2$, $\boldsymbol{m}_4$, $\boldsymbol{M}_i$ and $\boldsymbol{B}_i$, $i = 1,\, 2,\, 3$. Of course the symmetries of $\boldsymbol{B}_2$, $\boldsymbol{M}_1$, and $\boldsymbol{M}_3$ are kept at the limit for $s,\, u \to 0$: $\boldsymbol{B}_2(0,\, 0)$, $\boldsymbol{M}_1(0,\, 0)$, $\boldsymbol{M}_3(0,\, 0)$ are symmetric.

12 Dissipation principle and response functionals for internal rate of supply of entropy and heat flux

We have shown in Section 8 that the dissipation inequality (55) implies the inequality (56) and the internal dissipation inequality (59). To complete the constitutive relations we need response functionals for internal rate of entropy supply and heat flux. Putting ([19], [20])

$$\frac{\partial \tilde{\theta}}{\partial T} \boldsymbol{F}^{-1} \tilde{\boldsymbol{p}} + \rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} = -k \dot{\boldsymbol{\beta}}, \qquad k = \tilde{k}(T) \,\geq 0\,, \tag{133}$$

as well as in the Fourier case, it follows that

$$\rho \tilde{\xi} = \frac{1}{\tilde{\theta}} \left[k \dot{\boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}} + \rho \delta \tilde{\psi} \left(\boldsymbol{\Phi},\, \boldsymbol{\Phi}^t \,|\, \frac{d}{ds} \boldsymbol{\Phi}^t \right) \right]\,. \tag{134}$$

Then we use (133) and $\tilde{\boldsymbol{p}} = \tilde{\theta}^{-1} \tilde{\boldsymbol{q}}$ to deduce the expression for the heat flux response functional

$$\tilde{\boldsymbol{q}} = -\frac{\partial T}{\partial \theta} \tilde{\theta} \boldsymbol{F} \left(\rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} + \kappa \dot{\boldsymbol{\beta}} \right)\,. \tag{135}$$

We remark that more general choices of constitutive relations for $\tilde{\boldsymbol{q}}$, $\tilde{\xi}$ could be made. But our goal here is to complete as simply as possible the theoretical framework, so that the restrictions found for relaxation functions can be used in possible applications in which some constitutive equations can be adapted and modified to describe a real material. Very simple choices for $\tilde{\boldsymbol{q}}$ in thermo-visco-elasticity are sometimes made, such as in [13, p.60].

Lastly, note that an explicit constitutive equation for $\tilde{\xi}$ is obtained by substituting in (134) the expression (69) of $\delta\tilde{\psi}$, where $\boldsymbol{\chi}$ is replaced by $(d\boldsymbol{\Phi}^t/ds)$:

$$
\begin{aligned}
\delta\tilde{\psi}(\boldsymbol{\Phi},\,\boldsymbol{\Phi}^t \mid \frac{d}{ds}\boldsymbol{\Phi}^t) = \int_0^\infty \int_0^\infty \boldsymbol{\Phi}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{M}(s,\,u)}{\partial s \partial u} \frac{d}{du}\boldsymbol{\Phi}^t(u) ds du \\
+ \int_0^\infty \int_0^\infty \frac{d}{ds} T^t(s) \frac{\partial^2 \boldsymbol{B}_1(s,\,u)}{\partial s \partial u} \cdot \boldsymbol{\beta}_d^t(u) ds du + \int_0^\infty \int_0^\infty T_d^t(s) \frac{\partial^2 \boldsymbol{B}_1(s,\,u)}{\partial s \partial u} \cdot \frac{d}{du}\boldsymbol{\beta}^t(u) ds du \\
+ \int_0^\infty \int_0^\infty \frac{d}{ds} T^t(s) \frac{\partial^2 \boldsymbol{B}_2(s,\,u)}{\partial s \partial u} \cdot \boldsymbol{E}_d^t(u) ds du + \int_0^\infty \int_0^\infty T_d^t(s) \frac{\partial^2 \boldsymbol{B}_2(s,\,u)}{\partial s \partial u} \cdot \frac{d}{du}\boldsymbol{E}^t(u) ds du \\
+ \int_0^\infty \int_0^\infty \frac{d}{ds} T^t(s) \frac{\partial^2 \boldsymbol{B}_3(s,\,u)}{\partial s \partial u} \cdot \boldsymbol{\beta}_d^t(u) ds du + \int_0^\infty \int_0^\infty T_d^t(s) \frac{\partial^2 \boldsymbol{B}_3(s,\,u)}{\partial s \partial u} \cdot \frac{d}{du}\boldsymbol{W}^t(u) ds du \\
+ \int_0^\infty \int_0^\infty \frac{d}{ds} \boldsymbol{\beta}^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_1(s,\,u)}{\partial s \partial u} \boldsymbol{E}_d^t(u) ds du + \int_0^\infty \int_0^\infty \boldsymbol{\beta}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_1(s,\,u)}{\partial s \partial u} \frac{d}{du}\boldsymbol{E}^t(u) ds du \\
+ \int_0^\infty \int_0^\infty \frac{d}{ds} \boldsymbol{\beta}^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_2(s,\,u)}{\partial s \partial u} \boldsymbol{\beta}_d^t(u) ds du + \int_0^\infty \int_0^\infty \boldsymbol{\beta}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_2(s,\,u)}{\partial s \partial u} \frac{d}{du}\boldsymbol{W}^t(u) ds du \\
+ \int_0^\infty \int_0^\infty \frac{d}{ds} \boldsymbol{W}^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_3(s,\,u)}{\partial s \partial u} \boldsymbol{E}_d^t(u) ds du + \int_0^\infty \int_0^\infty \boldsymbol{W}_d^t(s) \cdot \frac{\partial^2 \boldsymbol{M}_3(s,\,u)}{\partial s \partial u} \frac{d}{du}\boldsymbol{E}^t(u) ds du
\end{aligned}
\tag{136}
$$

13 Field equations for the initial-boundary-value problem

Now that all constitutive equations have been written, we can obtain the field equations for the initial boundary-value problem for B. First, by solving $(8)_1$ under the initial condition $\rho = \rho_0(\boldsymbol{X})$ for the mass density, one finds the mass density evolution $\rho = \rho(\boldsymbol{X},\,t)$. Hence by substituting the constitutive equations (52), $(46)_{2,\,3}$, (134), and (135), in the three local balance laws $(8)_2$, $(8)_4$ and $(8)_7$, we obtain the field equations for the triple of fields

$$(\boldsymbol{u},\,\alpha,\,\phi),$$

where

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{X},\,t), \quad \alpha = \alpha(\boldsymbol{X},\,t), \quad \phi = \phi(\boldsymbol{X},\,t) \tag{137}$$

are the displacement, thermal displacement and electric potential, respectively.

The field equations $(8)_2$, $(8)_4$ and $(8)_7$, compatible with the constitutive equations, are five equations in the five field variables u_i, α, and ϕ.

14 Infinitesimal thermo-electro-viscoelasticity

The linearized (infinitesimal) theory of thermo-electro-viscoelasticity is deduced by calculating the first-order approximation of the finite theory. Let $\rho_0 = \rho_0(\boldsymbol{X})$ be given. The triple of fields (137) can be viewed as a thermo-electro-kinetic process superimposed to a *natural state*, that is, a reference configuration where the body is homogeneous, has zero stress, is at uniform (absolute) temperature T_0 (θ_0) and at uniform electric potential ϕ_0 (hence $\boldsymbol{\beta}_0 = \boldsymbol{0} = \boldsymbol{W}_0$). We also assume that in the natural state body forces and heat supply vanish. We say that the kinetic process (137) is infinitesimal if at each material point and instant of time the magnitudes of the first and second order space-time derivatives of the fields $\boldsymbol{\chi}$, α and ϕ are

$<< 1$. By linearization, the spatial and referential descriptions coincide and then we identify ρ with ρ_0, $\boldsymbol{F}$ with $\boldsymbol{I}$, $\boldsymbol{\gamma}$ with $\dot{\boldsymbol{\beta}}$, $\boldsymbol{E}^M$ with $\boldsymbol{W}$, and the Green-Lagrange strain tensor (37) is replaced by the linear strain

$$\boldsymbol{E} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}). \tag{138}$$

In the linear approximation we disregard all infinitesimal terms of order greater than one in the constitutive relations. By (136) in infinitesimal processes $\delta\tilde{\psi}$ vanishes, and putting

$$\zeta = T'(\theta), \quad \zeta_0 = T'(\theta_0), \quad \kappa_0 = \kappa(\theta_0), \tag{139}$$

equations (52), $(46)_2$, $(46)_3$, (134), and (135) by (72)-(75) respectively become

$$\begin{aligned} \rho_0^{-1}\boldsymbol{T} = \frac{\partial^2\tilde{\Sigma}}{\partial \boldsymbol{E}^2}\boldsymbol{E} + \int_0^\infty \frac{\partial \boldsymbol{m}_3(0,\, u)}{\partial u}\boldsymbol{E}_d^t(u)du + \int_0^\infty T_d^t(s)\frac{\partial \boldsymbol{B}_2(s,\, 0)}{\partial s}ds \\ + \int_0^\infty \boldsymbol{\beta}_d^t(s)\frac{\partial \boldsymbol{M}_1(s,\, 0)}{\partial s}ds + \int_0^\infty \boldsymbol{W}_d^t(s)\frac{\partial \boldsymbol{M}_3(s,\, 0)}{\partial s}ds \end{aligned} \tag{140}$$

$$\begin{aligned} \rho_0^{-1}\tilde{\boldsymbol{P}} = -\frac{\partial^2\tilde{\Sigma}}{\partial \boldsymbol{W}^2}\boldsymbol{W} - \int_0^\infty \frac{\partial \boldsymbol{m}_4(0,\, u)}{\partial u}\boldsymbol{W}_d^t(u)du - \int_0^\infty T_d^t(s)\frac{\partial \boldsymbol{B}_3(s,\, 0)}{\partial s}ds \\ - \int_0^\infty \boldsymbol{\beta}_d^t(s)\frac{\partial \boldsymbol{M}_2(s,\, 0)}{\partial s}ds - \int_0^\infty \frac{\partial \boldsymbol{M}_3(0,\, u)}{\partial u}\boldsymbol{E}_d^t(u)du \end{aligned} \tag{141}$$

$$\begin{aligned} \zeta_0^{-1}\tilde{\eta} = -\frac{\partial^2\tilde{\Sigma}^2}{\partial T^2}(T - T_0) - \int_0^\infty \frac{\partial m_1(0,\, u)}{\partial u}T_d^t(u)du - \int_0^\infty \frac{\partial \boldsymbol{B}_1(0,\, u)}{\partial u}\boldsymbol{\beta}_d^t(u)du \\ - \int_0^\infty \frac{\partial \boldsymbol{B}_2(0,\, u)}{\partial u}\boldsymbol{E}_d^t(u)du - \int_0^\infty \frac{\partial \boldsymbol{B}_3(0,\, u)}{\partial u}\boldsymbol{W}_d^t(u)du \end{aligned} \tag{142}$$

$$\rho_0\tilde{\xi} = \frac{\kappa_0}{\theta_0}\dot{\boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}} \tag{143}$$

$$\tilde{\boldsymbol{q}} = -\zeta_0\theta_0\left(\rho_0\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}} + \kappa_0\dot{\boldsymbol{\beta}}\right),. \tag{144}$$

15 A class of thermo-electric-viscoelastic materials

15.0.1 Preliminaries and one-dimensional constitutive equations

Here we consider a one-dimensional body $\mathcal{B}$ composed of a thermo-electric-viscoelastic material body in its natural homogeneous reference configuration, that is the straight line segment $B = [0,\, L]$ on an axis X.

Examples of one-dimensional such bodies are considered in literature. For instance, Zeng [7] studies the Cauchy problem of a one-dimensional purely mechanical nonlinear viscoelastic model with fading memory; Babaei et al. [26], in order to characterize the viscoelastic behavior of biological tissues, consider a one-dimensional purely mechanical viscoelastic model; in fact in biomechanics the study of a tissue constituent, whose nature is viscoelastic, typically is evaluated from the uniaxial behavior.

The model dealt below regards a material that is viscoelastic with a fading memory, whose constitutive equations are of integral type with a genuine memory of the past history. It differs from the differential or rath-type viscoelasticity in which the material remembers only an arbitrarily small lapse of time into the past.

For an infinitesimal kinetic process described by the three scalar functions

$$\chi = \chi(X,\, t),\ \alpha = \alpha(X,\, t),\ \phi = \phi(X,\, t), \quad 0 \le X \le L,\ t \in I\!R \tag{145}$$

we have

$$\begin{aligned} u = \chi - X, \quad E = \chi_{,X} - 1, \quad E^t_d(s) = {\chi_{,X}}^t_d(s), \quad v = \dot{\chi} \\ T = \dot{\alpha}, \quad T^t_d(s) = \dot{\alpha}^t_d(s), \quad \beta = \alpha_{,X}, \quad \beta^t_d(s) = {\alpha_{,X}}^t_d(s) \\ W = -\phi_{,X}, \quad W^t_d(s) = -{\phi_{,X}}^t_d(s) \end{aligned} \tag{146}$$

where $z_{,X} = dz/dX$ denotes a spatial derivative and $z^t_d(s) = z(t-s) - z(t)$. Putting

$$\Sigma_1 = \frac{\partial^2 \tilde{\Sigma}}{\partial E^2}, \quad \Sigma_2 = \frac{\partial^2 \tilde{\Sigma}}{\partial W^2}, \quad \Sigma_3 = \frac{\partial^2 \tilde{\Sigma}}{\partial T^2}, \tag{147}$$

evaluated in the natural reference configuration, adopting the classical notations

$$\acute{m}_3(u) = \frac{\partial m_3(0,\, u)}{\partial u}, \quad \dot{B}_2(s) = \frac{\partial B_2(s,\, 0)}{\partial s}, \quad \text{etc.} \tag{148}$$

and using $\overline{T}$ (rather than T) to denote the uniaxial stress, the constitutive equations (140) -(144) write as

$$\begin{aligned} \rho_0^{-1}\overline{T} = \Sigma_1(\chi_{,X} - 1) + \int_0^\infty \acute{m}_3(u){\chi_{,X}}^t_d(u)du + \int_0^\infty \dot{\alpha}^t_d(s)\dot{B}_2(s)ds \\ + \int_0^\infty {\alpha_{,X}}^t_d(s)\dot{M}_1(s)ds - \int_0^\infty {\phi_{,X}}^t_d(s)\dot{M}_3(s)ds \end{aligned} \tag{149}$$

$$\begin{aligned} \rho_0^{-1}\tilde{P} = \Sigma_2\phi_{,X} + \int_0^\infty \acute{m}_4(u){\phi_{,X}}^t_d(u)du - \int_0^\infty \dot{\alpha}^t_d(s)\dot{B}_3(s)ds \\ - \int_0^\infty {\alpha_{,X}}^t_d(s)\dot{M}_2(s)ds - \int_0^\infty \dot{M}_3(u){\chi_{,X}}^t_d(u)du \end{aligned} \tag{150}$$

$$\begin{aligned} \zeta_0^{-1}\tilde{\eta} = -\Sigma_3(T - T_0) - \int_0^\infty \acute{m}_1(u)\dot{\alpha}^t_d(u)du - \int_0^\infty \dot{B}_1(u){\alpha_{,X}}^t_d(u)du \\ - \int_0^\infty \dot{B}_2(u){\chi_{,X}}^t_d(u)du + \int_0^\infty \dot{B}_3(u){\phi_{,X}}^t_d(u)du \end{aligned} \tag{151}$$

$$\rho_0\tilde{\xi} = \frac{\kappa_0}{\theta_0}(\dot{\alpha}_{,X})^2 \tag{152}$$

$$\tilde{q} = -\zeta_0\theta_0\rho_0\left(\frac{\partial\tilde{\psi}}{\partial\beta} + \rho_0^{-1}\kappa_0\dot{\alpha}_{,X}\right), \tag{153}$$

where

$$\begin{aligned} \frac{\partial\tilde{\psi}}{\partial\beta} = \frac{\partial\tilde{\Sigma}}{\partial\beta} + \int_0^\infty \acute{m}_2(u){\alpha_{,X}}^t_d(u)du + \int_0^\infty \dot{\alpha}^t_d(s)\dot{B}_1(s)ds \\ + \int_0^\infty \dot{M}_1(u){\chi_{,X}}^t_d(u)du - \int_0^\infty \dot{M}_2(u){\phi_{,X}}^t_d(u)du \end{aligned} \tag{154}$$

In order to write the constitutive equation for the electric displacement D we substitute (150) in $(5)_1$, $D = \varepsilon_0 W + P$, and we obtain

$$\begin{aligned} \rho_0^{-1}D = (\Sigma_2 - {\rho_0}^{-1}\varepsilon_0)\phi_{,X} + \int_0^\infty \acute{m}_4(u){\phi_{,X}}^t_d(u)du - \int_0^\infty \dot{\alpha}^t_d(s)\dot{B}_3(s)ds \\ - \int_0^\infty {\alpha_{,X}}^t_d(s)\dot{M}_2(s)ds - \int_0^\infty \dot{M}_3(u){\chi_{,X}}^t_d(u)du \end{aligned} \tag{155}$$

15.0.2 A choice of the relaxation functions

Now, let $h(s)$ be an influence function of order $r = \delta + 1/2$, $\delta > 0$ (Section 5); that is, let (25) hold. For example, $h(s) = (1+s)^{-p}$ is an influence function of order r if $r < p$ whereas an exponential $h(s) = \exp(-\beta s)$, $\beta > 0$, is an influence function of any order. Each relaxation function within the above constitutive equations,

$$\ell(s) \in \{m_i(s)\ (i = 1,\,2,\,3,\,4), \quad B_j(s),\ M_j(s) \quad (j = 1,\,2,\,3)\}\,,$$

must satisfy the restrictions in Section 11. The abbreviations (148) suggest that each $\ell(.)$ could for example be of the type $\ell(s) = f(s)f(0)$, $\ell(u) = f(0)f(u)$, that is,

$$\begin{aligned} m_i(s,\,u) = f_i(s)f_i(u), &\qquad i = 1,\,2,\,3,\,4, \\ B_j(s,\,u) = p_j(s)p_j(u),\ M_j(s,\,u) = q_j(s)q_j(u), &\qquad j = 1,\,2,\,3. \end{aligned} \tag{156}$$

Now let us think about one of the functions (156) because the reasoning repeats in the same way for each of them. So, let $m(s,\,u) = f(s)f(u)$ be any function (156). We assume its decay for $s \to \infty$ be related to the decay of the influence function $h(s)$ by

$$\int_0^\infty m^2(s,\,s)h^{-2}(s)ds < \infty \tag{157}$$

For instance we assume that

$$h(s) = \exp(-\beta s), \quad f(s) = \exp(-\omega s), \quad \beta > 0,\ \omega > 0 \tag{158}$$

The restrictions (85), (86) for $m(s,\,u) = m_i(s,\,u)$, $i = 1,\,2,\,3,\,4$, are respectively equivalent to

$$-\omega\exp(-2\omega s)\ \leq\ 0\,, \qquad \big[\exp(-\omega s) \pm \exp(-\omega u)\big]^2 \geq 0 \tag{159}$$

that hold identically. Now, to set a choice of all other relaxation functions that satisfy the remaining restrictions in Section 11, to simplify things, assume that in in (156) we have

$$p_j(s) = q_j(s) = f(s) = \exp(-\omega s), \quad \omega > 0, \quad \forall j = 1,\,2,\,3$$

It is a simple calculation to show that also the other restrictions in Section 11 are then satisfied.

15.0.3 One-dimensional field equations

The local balance laws $(8)_2$, in the form (54), $(8)_4$ and $(8)_7$, where $\boldsymbol{f} = \mathbf{0}$, $s = 0$, write as

$$\begin{cases} \rho_0\ddot{\chi} = \frac{d}{dx}\left(\overline{T} + \varepsilon_0 W^2 - \frac{\varepsilon_0}{2}W^2\right) \\ \rho_0\dot{\eta} = \rho_0\xi - \frac{d}{dx}p \\ \frac{d}{dx}D = 0 \end{cases} \tag{160}$$

The field equations for the kinetic process $(\chi,\,\alpha,\,\phi)$ are obtained by substituting in (160) the constitutive equations (149)-(155):

$$\begin{cases} (1) \quad \rho_0\ddot{\chi} = \frac{d}{dx}\left(\overline{T} + \frac{\varepsilon_0}{2}(\phi_{,X})^2\right) \\ (2)\ -\Sigma_3\dot{T} - \int_0^\infty \dot{m}_1(u)\ddot{\alpha}^t_d(u)du - \int_0^\infty \dot{B}_1(u)\dot{\alpha}_{,X\,d}^{\ t}(u)du \\ -\int_0^\infty \dot{B}_2(u)\dot{\chi}_{,X\,d}^{\ t}(u)du + \int_0^\infty \dot{B}_3(u)\dot{\phi}_{,X\,d}^{\ t}(u)du \\ = \zeta_0^{-1}\frac{\kappa_0}{\theta_0}\dot{\alpha}_{,X}^{\,2} + \frac{d}{dx}\left(\frac{\partial\tilde{\psi}}{\partial\beta} + \rho_0^{-1}\kappa_0\dot{\alpha}_{,X}\right) \\ (3)\ (\Sigma_2 - {\rho_0}^{-1}\varepsilon_0)\phi_{,XX} + \int_0^\infty \dot{m}_4(u)\phi_{,XX\,d}^{\ \ t}(u)du - \int_0^\infty \dot{\alpha}_{,X\,d}^{\ t}(s)\dot{B}_3(s)ds \\ -\int_0^\infty \alpha_{,XX\,d}^{\ \ t}(s)\dot{M}_2(s)ds - \int_0^\infty \dot{M}_3(u)\chi_{,XX\,d}^{\ \ t}(u)du = 0 \end{cases} \tag{161}$$

With some simple calculations and neglecting the second order terms $\dot{\alpha}^2_{,X}$, $\phi^2_{,X}$ we obtain the system of equations

$$\begin{cases} (1)\ \ddot{\chi} = \Sigma_1 \chi_{,XX} + \int_0^\infty \dot{m}_3(u) \chi_{,XX\,d}{}^t(u)du + \int_0^\infty \dot{\alpha}_{,X\,d}{}^t(s)\dot{B}_2(s)ds \\ + \int_0^\infty \alpha_{,XX\,d}{}^t(s)\dot{M}_1(s)ds - \int_0^\infty \phi_{,XX\,d}{}^t(s)\dot{M}_3(s)ds \\ (2)\ -\Sigma_3\ddot{\alpha} - \int_0^\infty \dot{m}_1(u)\ddot{\alpha}^t_d(u)du - \int_0^\infty \dot{B}_1(u)\dot{\alpha}_{,X\,d}{}^t(u)du - \int_0^\infty \dot{B}_2(u)\dot{\chi}_{,X\,d}{}^t(u)du \\ + \int_0^\infty \dot{B}_3(u)\dot{\phi}_{,X\,d}{}^t(u)du = \int_0^\infty \dot{m}_2(u)\alpha_{,XX\,d}{}^t(u)du + \int_0^\infty \dot{\alpha}_{,X\,d}{}^t(s)\dot{B}_1(s)ds \\ + \int_0^\infty \dot{M}_1(u)\chi_{,XX\,d}{}^t(u)du - \int_0^\infty \dot{M}_2(u)\phi_{,XX\,d}{}^t(u)du + \rho_0^{-1}\kappa_0\dot{\alpha}_{,XX} \\ (3)\ (\Sigma_2 - \rho_0{}^{-1}\varepsilon_0)\phi_{,XX} + \int_0^\infty \dot{m}_4(u)\phi_{,XX\,d}{}^t(u)du - \int_0^\infty \dot{\alpha}_{,X\,d}{}^t(s)\dot{B}_3(s)ds \\ - \int_0^\infty \alpha_{,XX\,d}{}^t(s)\dot{M}_2(s)ds - \int_0^\infty \dot{M}_3(u)\chi_{,XX\,d}{}^t(u)du = 0 \end{cases} \tag{162}$$

Now we consider kinetic processes in which

$$\chi = \overline{\chi}X + \chi_0,\ \alpha = \overline{\alpha}X,\ \phi = \overline{\phi}X, \quad 0 \le X \le L, \quad \chi_0 \in I\!R$$

$$\text{with} \qquad \overline{\chi} = \overline{\chi}(t),\ \overline{\alpha} = \overline{\alpha}(t),\ \overline{\phi} = \overline{\phi}(t), \quad t \in I\!R$$

We have

$$\dot{\chi} = \dot{\overline{\chi}}X, \quad \chi_{,X} = \overline{\chi}, \quad \chi_{,XX} = 0, \quad \dot{\chi}_{,X} = \dot{\overline{\chi}}$$

$$\dot{\alpha} = \dot{\overline{\alpha}}X, \quad \alpha_{,X} = \overline{\alpha}, \quad \alpha_{,XX} = 0, \quad \dot{\alpha}_{,X} = \dot{\overline{\alpha}}$$

$$\dot{\phi} = \dot{\overline{\phi}}X, \quad \phi_{,X} = \overline{\phi}, \quad \phi_{,XX} = 0, \quad \dot{\phi}_{,X} = \dot{\overline{\phi}}$$

Hence the system (162) becomes

$$\begin{cases} (1) \quad \ddot{\overline{\chi}}X = \int_0^\infty \dot{\overline{\alpha}}^t_d(s)\dot{B}_2(s)ds \\ (2)\ -\Sigma_3\ddot{\overline{\alpha}}X - \Big(\int_0^\infty \dot{m}_1(u)\ddot{\overline{\alpha}}^t_d(u)du\Big)X - \int_0^\infty \dot{B}_1(u)\dot{\overline{\alpha}}^t_d(u)du - \int_0^\infty \dot{B}_2(u)\dot{\overline{\chi}}^t_d(u)du \\ + \int_0^\infty \dot{B}_3(u)\dot{\overline{\phi}}^t_d(u)du = \int_0^\infty \dot{\overline{\alpha}}^t_d(s)\dot{B}_1(s)ds \\ (3) \qquad \int_0^\infty \dot{\overline{\alpha}}^t_d(s)\dot{B}_3(s)ds = 0 \end{cases} \tag{163}$$

The arbitrariness of $X \in [0,\ L]$ in $(163)_1$ and $(163)_2$ splits such equations, so we have

$$\begin{cases} (1)_1,\ (1)_2 \qquad \ddot{\overline{\chi}} = 0, \qquad \int_0^\infty \dot{\overline{\alpha}}^t_d(s)\dot{B}_2(s)ds = 0 \\ (2)_1 \quad \Sigma_3\ddot{\overline{\alpha}} + \int_0^\infty \dot{m}_1(u)\ddot{\overline{\alpha}}^t_d(u)du = 0 \\ (2)_2 \quad -\int_0^\infty \dot{B}_1(u)\dot{\overline{\alpha}}^t_d(u)du - \int_0^\infty \dot{B}_2(u)\dot{\overline{\chi}}^t_d(u)du + \int_0^\infty \dot{B}_3(u)\dot{\overline{\phi}}^t_d(u)du \\ \qquad = \int_0^\infty \dot{\overline{\alpha}}^t_d(s)\dot{B}_1(s)ds \\ (3) \qquad \int_0^\infty \dot{\overline{\alpha}}^t_d(s)\dot{B}_3(s)ds = 0 \end{cases} \tag{164}$$

Hence $\overline{\chi} = c_1 t + c_2$ for some $c_i \in I\!R$, thus

$$\overline{\chi}^t_d(u) = \overline{\chi}(t-u) - \overline{\chi}(u) = c_1(t-u) + c_2 - (c_1 t + c_2) = -c_1 u$$

$$\dot{\overline{\chi}}^t_d(u) = \dot{\overline{\chi}}(t-u) - \dot{\overline{\chi}}(t) = c_1 - c_1 = 0$$

and (164) reduces to

$$\begin{cases} \chi = (c_1 t + c_2)X + \chi_0 \quad \forall c_1,\, c_2, \chi_0 \in I\!R \\ \int_0^\infty \dot{\overline{\alpha}}_d^t(s)\dot{B}_2(s)ds = 0 \\ \Sigma_3 \ddot{\overline{\alpha}} + \int_0^\infty \dot{m}_1(u)\ddot{\overline{\alpha}}_d^t(u)du = 0 \\ \int_0^\infty \dot{B}_3(u)\dot{\overline{\phi}}_d^t(u)du = \int_0^\infty \dot{B}_1(u)\dot{\overline{\alpha}}_d^t(u)du + \int_0^\infty \dot{\overline{\alpha}}_d^t(s)\dot{B}_1(s)ds \\ \int_0^\infty \dot{\overline{\alpha}}_d^t(s)\dot{B}_3(s)ds = 0 \end{cases} \tag{165}$$

A simple subclass of solutions of (165) is obtained, for example, by looking for those which $\ddot{\overline{\alpha}} = 0$. Then $\overline{\alpha} = g_1 t + g_2$ for some $g_i \in I\!R$ and thus

$$\overline{\alpha}_d^t(s) = \overline{\alpha}(t-s) - \overline{\alpha}(t) = g_1(t-s) + g_2 - (g_1 t + g_2) = -g_1 s$$

$$\dot{\overline{\alpha}}_d^t(s) = \dot{\overline{\alpha}}(t-s) - \dot{\overline{\alpha}}(t) = g_1 - g_1 = 0$$

Now (165) reduces to

$$\begin{cases} \chi = (c_1 t + c_2)X + \chi_0, \quad \forall c_1,\, c_2, \chi_0 \in I\!R \\ \alpha = (g_1 t + g_2)X \quad \forall g_1,\, g_2 \in I\!R \\ \int_0^\infty \dot{B}_3(u)\dot{\overline{\phi}}_d^t(u)du = 0 \end{cases} \tag{166}$$

Then, for each smooth function $\phi = \overline{\phi}X$, $\overline{\phi} = \overline{\phi}(t)$, that solves $(166)_3$ the triple of functions

$$\Big(\chi = (c_1 t + c_2)X + \chi_0,\ \alpha = (g_1 t + g_2)X,\ \phi = \overline{\phi}(t)X\Big) \tag{167}$$

is a solution of the field equations (166) or (160), for each $c_1,\, c_2, \chi_0,\, g_1,\, g_2 \in I\!R$. In particular, (167) is a solution if $\phi = (l_1 t + l_2)X, \;\; l_1,\, l_2 \in I\!R$, since $\dot{\overline{\phi}}_d^t(u) = 0$.

16 Conclusions

We have used the nonlinear theory in [20] for thermo-electro-mechanical simple materials with fading memory to set up a thermo-electro viscoelastic theory, which is obtained by a linearization procedure with the Riesz representation theorem. Now, following what has been made for the linear theory of viscoelasicity - see e.g. [27]- new uniqueness and continuous data dependence theorems should be established that are appropriate to the fundamental dynamic and quasi-static boundary value problems for a thermo-electro-viscoelastic body.

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18 Conflict of interest

Declarations of interest: none.

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