Lusztig's strata are locally closed

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Abstract

Let G be a connected reductive algebraic group over an algebraically closed field k. We consider the strata in G defined by Lusztig as fibers of a map given in terms of truncated induction of Springer representations. We elaborate on the existing proof of the following two results in order to show that it extends to arbitrary characteristic: Lusztig's strata are locally closed and the irreducible components of a stratum X are those sheets for the G-action on itself that are contained in X.

1 Introduction

The present paper answers a question by G. Lusztig on the extension to arbitrary characteristic of results in [1] concerning strata as defined in [4, §2] Lusztig's strata in a connected reductive group G are defined as fibers of a map from G to the set of irreducible representations of its Weyl group, constructed in terms of truncated induction of Springer's representations for trivial local systems. It is observed in [4, §2] that they are a G-stable union of orbits of the same dimension and are unions of Jordan classes, i.e., the locally closed subsets introduced in [3, §3] that provide the stratification with respect to which character sheaves are smooth. It does not immediately follow from the definition that these fibers have topological or geometric properties. However, it was proved in [1] when the characteristic of the base field is good, that strata are locally closed and that they are unions of sheets for the action of G on itself by conjugation. In this paper we show, by invoking results from [7], that the proof in [1] works in any characteristics. We also stress in the present paper an observation that was only implicit in [1], namely that if a Jordan class lies in a stratum, then also the regular part of its closure lies in the stratum. As a consequence, we conclude that the irreducible components of a stratum are precisely the sheets contained therein. This implies that whenever two sheets have non-empty intersection, then the stratum containing both is singular. This happens, for instance, when the root system of G is not simply-laced and the stratum contains the subregular unipotent conjugacy class.

2 Notation

Let G be a connected reductive algebraic group over an algebraically closed field k. We denote by $G \cdot x$ the G-conjugacy class of $x \in G$, whereas G_x° denotes the identity component of the stabiliser of x. For $m \in \mathbb{N}$ we set $G_{(m)} = \{x \in G \mid \dim(G \cdot x) = m\}$. Since the dimension of the stabiliser of a point is an upper semicontinuous function [6, §2, page 7] the set $\bigcup_{m \leq d} G_{(m)} = \bigcup_{m \leq d} \overline{G_{(m)}} = \overline{\bigcup_{m \leq d} G_{(m)}}$ is closed so $G_{(>d)} := \bigcup_{m \geq d+1} G_{(m)}$ is open in G.[1]

For a set $Y \subset G$ we define $d_Y := \max\{d \in \mathbb{N} \mid G_{(d)} \cap Y \neq \emptyset\}$ and $Y^{reg} := Y \cap G_{(d_Y)}$. For instance, if Y is a maximal torus in G, then Y^{reg} is the set regular semisimple elements in Y. The sets $G_{(m)} = (\bigcup_{l \leq m} G_{(l)}) \cap G_{(>m-1)}$ are locally-closed and their irreducible components are called the *sheets* of the G-action. We will denote by W the Weyl group of G, by $\mathrm{Irr}(W)$ the set of isomorphism classes of the complex irreducible representations of W. For s in a maximal torus T of G, we set $W_s = N_G(T) \cap G_s^\circ / T$, which is the Weyl group of G_s°. When we write $g = su$ we mean that su is the Jordan decomposition of g with s semisimple and u unipotent in G_s.

2.1 Jordan classes and sheets

We recall from [3, §3] that the group G is the disjoint union of finitely many, locally closed, smooth, irreducible, G-stable sets, each contained in some $G_{(m)}$, which we call Jordan classes and can be described as follows: the Jordan class containing $g = su$ is $J(su) = G \cdot ((Z(G_s^\circ)^\circ s)^{reg} u)$. In other words, an element with Jordan decomposition rv lies in $J(su)$ if and only if

[1] Observe that in [2, page 8] it is erroneously stated that $\overline{G_{(d)}} = \bigcup_{m \leq d} G_{(m)}$: this does not affect any of the following statements in that paper.

it is conjugate to some $s'u$ with $G_s^\circ = G_{s'}^\circ$ and $s' \in Z(G_s^\circ)^\circ s$. Last condition guarantees that $J(su)$ is irreducible.

Clearly, if $J(su) \subset G_{(d)}$, then $\overline{J(su)} \subset \bigcup_{d \leq m} G_{(d)}$ and $J(su) \subset \overline{J(su)}^{reg} = \overline{J(su)} \cap G_{(d)}$. The next Lemma is a combination of [2, Propositions 4.5 and 4.7]: both proofs hold in any characteristic and can be repeated verbatim.

Lemma 2.1. *Let $J(su) \subset G_{(d)}$ be the Jordan class of $g = su$. Then*

$$\overline{J(su)} = \bigcup_{z \in Z(G_s^\circ)^\circ} \overline{G \cdot sz\mathrm{Ind}_{G_s^\circ}^{G_{zs}^\circ}(G_s^\circ \cdot u)} \subseteq \bigcup_{m \leq d} G_{(m)}$$

and
$$\overline{J(su)}^{reg} = \overline{J(su)} \cap G_{(d)} = \bigcup_{z \in Z(G_s^\circ)^\circ} G \cdot sz\mathrm{Ind}_{G_s^\circ}^{G_{zs}^\circ}(G_s^\circ \cdot u).$$

2.2 Lusztig's strata

We recall from [4, §2] the map $\phi_G \colon G \to \mathrm{Irr}(W)$ defined on $g = su$ as: $\phi_G(g) = \mathbf{j}_{W_s}^W \rho_u^{W_s}$, where $\rho_u^{W_s}$ is Springer's respresentation of W_s associated with u and trivial local system and $\mathbf{j}_{W_s}^W$ is Lusztig-Spaltenstein's induction [5]. The fibers of this map are G-stable, each of them is contained in some $G_{(d)}$. We refer to each such fiber as a *stratum*. By construction they are union of Jordan classes.

Theorem 2.2. *Let X be a stratum in G. Then*

(1) X is locally closed.

(2) X is a union of sheets.

(3) The sheets contained in X are its irreducible components.

Proof. (1). We claim that if $J(su) \subset X$, then $\overline{J(su)}^{reg} \subset X$. Let $rv \in \overline{J(su)}^{reg}$. As the strata are G-stable we may assume $r = zs$ for $z \in Z(G_s^\circ)^\circ$ and $v \in \mathrm{Ind}_{G_s^\circ}^{G_{zs}^\circ}(G_s^\circ \cdot u)$. Then, $\phi_G(rv) = \mathbf{j}_{W_{zs}}^W \rho_v^{W_{zs}}$. By [5, Theorem 3.5], see also [7, §4.1] we have

$$\phi_G(rv) = \phi_G(zsv) = \mathbf{j}_{W_{zs}}^W \rho_v^{W_{zs}} = \mathbf{j}_{W_{zs}}^W \mathbf{j}_{W_s}^{W_{zs}} \rho_u^{W_s} = \mathbf{j}_{W_s}^W \rho_u^{W_s} = \phi_G(su).$$

As X is a fiber of ϕ_G, it is a union of Jordan classes, and Jordan classes are finitely many, it follows that X is a union of finitely many sets of the form $\overline{J(s_lu_l)}^{reg}$ for some $s_l \in T$ and some unipotent $u_l \in G^\circ_{s_l}$. Recall that $X \subset G_{(d)}$ for some d. Then $\overline{J(s_lu_l)}^{reg} \subset G_{(d)}$ for every l so

$$X = \bigcup_l \overline{J(s_lu_l)}^{reg} = \overline{\bigcup_l J(s_lu_l)}^{reg} = \overline{\bigcup_l J(s_lu_l)} \cap G_{(d)} = \overline{\bigcup_l J(s_lu_l)} \cap G_{(>d)}$$

with $\overline{\bigcup_l J(s_lu_l)}$ closed and $G_{(>d)}$ open.

(2) and (3). For any d, inclusion induces a partial ordering on the collection of the finitely-many irreducible sets of the form $\overline{J(su)}^{reg}$ contained in $G_{(d)}$. The maximal elements are the irreducible components of $G_{(d)}$, i.e., the sheets of G corresponding to the dimension d. Hence $X = X \cap G_{(d)}$ is a union of sheets and these are the maximal irreducible subsets contained in X, i.e., its irreducible components. □

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