DOLBEAULT–MASSEY TRIPLE PRODUCTS OF LOW DEGREE

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Abstract. Let $\mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial}_{\mathcal{A}})$ be a differential bigraded algebra. We characterize non-vanishing Dolbeault–Massey triple products of low degree (see Theorems 3.1 and 3.2). We give some applications for the Dolbeault complex on a compact complex manifold.

Introduction

In their celebrated paper [6] Deligne, Griffiths, Morgan and Sullivan obtained cohomological obstructions to the existence of a Kähler structure on a compact manifold. In particular, they showed that if a compact manifold has a Kähler structure, then it satisfies the $\partial\overline{\partial}$-Lemma. Furthermore, see [10], the de Rham complex of compact Kähler manifolds is *formal* as differential graded algebra.
In the category of complex manifolds, it is natural to consider the Dolbeault groups in order to obtain informations on the complex structure of the manifold. In this context, Neisendorfer and Taylor ([9]) developed a formality theory for complex manifolds, adapting Sullivan's theory of formality for manifolds. More precisely, they introduced a notion of formality for differential bigraded algebras $(\mathcal{A}^{\bullet,\bullet}, \overline{\partial}_{\mathcal{A}})$. In analogy with the real case, a complex manifold is *Dolbeault formal* if its Dolbeault complex is equivalent as differential bigraded algebra to a differential bigraded algebra with trivial $\overline{\partial}$-operator (or in other words to its cohomology algebra $(H^{\bullet,\bullet}(\mathcal{A}), 0)$). It is worthwhile to recall that ([9]) compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma are Dolbeault formal. We refer to [7] for geometric formality in the context of de Rham cohomology, and to [11] and [2] for Dolbeault and Bott–Chern cohomologies.
One of the principal obstructions to formality is the existence of non-vanishing Massey triple products, which are (equivalence classes of) cohomology classes that can be defined as an adapted version of Massey triple products to the differential bigraded setting (see e.g. [11, §3]). Consequently, a useful tool to show that a given complex manifold has no Kähler metrics is to construct a non-vanishing Dolbeault–Massey triple product. In many cases (see e.g. [4], [5]) this can be done by taking Dolbeault cohomology classes of low total degree.

The aim of this paper is to study the existence of non-trivial Dolbeault–Massey triple products of low degree on an arbitrary differential bigraded algebra. Our main result gives a characterization of their existence:

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Theorem (see Thm. 3.2)**.** *Let $\mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial})$ be a homologically connected differential bigraded algebra. Let $[X] \in H^{1,0}(\mathcal{A}) \smallsetminus \{0\}$ and $[Y] \in H^{0,1}(\mathcal{A}) \smallsetminus \{0\}$ be such that $[X][Y] = 0$. Then the following are equivalent:*

(1) $\langle [X], [X], [Y] \rangle$ defines a non-vanishing Dolbeault–Massey triple product in $\frac{H^{2,0}(\mathcal{A})}{[X] \cdot H^{1,0}(\mathcal{A})}$;
(2) $XY = \overline{\partial} W$ with $W \in \mathcal{A}^{1,0} \smallsetminus \mathcal{Z}^{1,0}(\mathcal{A})$ such that $XW \neq 0$.

As a corollary we get the following theorem concerning the multiplication in formal differential bigraded algebras:

Theorem (see Cor. 3.3)**.** *Let $\mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial})$ be a homologically connected differential bigraded algebra, and let $[X] \in H^{1,0}(\mathcal{A}) \smallsetminus \{0\}$. If $\mathcal{A}$ is formal, then one (and only one) of the following holds:*

(1) $[X][Y] \neq 0$ for all $[Y] \in H^{0,1}(\mathcal{A}) \smallsetminus \{0\}$;
(2) for any $[Y] \in H^{0,1}(\mathcal{A}) \smallsetminus \{0\}$ such that $[X][Y] = 0$ we have that there exists $Z \in \mathcal{A}^{1,0}$ such that $XY = \overline{\partial} Z$ with $XZ = 0$.

The outline of the paper is as follows. In Section 1 we recall some preliminary definitions on differential bigraded algebras and their minimal models and set the notation we will use throughout the paper. In Section 2 we recall the definition of formal differential bigraded algebras, the construction of Dolbeault–Massey triple products, and we prove a number of technical lemmas which are useful for the proof of our main results. Finally in Section 3 we prove our main results (Theorem 3.1, Theorem 3.2 and Corollary 3.3) and in Section 4 we use them to show how to apply the results of Section 3 to the case of complex manifolds and their deformations.

Contents

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1. Differential bigraded algebras

The objects encoding the algebraic structure of the Dolbeault complex of a complex manifold are the differential bigraded algebras. In this section, we will briefly recall the main definitions and properties of differential bigraded algebras, and in particular of their minimal models (Section 1.1).

Recall that a *differential bigraded* $\mathbb{K}$*-algebra* (or *DBA* for short) $\mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial})$ is a $\mathbb{K}$-vector space $\mathcal{A}$ with a direct sum decomposition

$$\mathcal{A}^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{Z}} \mathcal{A}^{p,q},$$

endowed with

(DBA1) a product $\cdot : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ which is $\mathbb{K}$-bilinear and satisfies
  (DBA1.a) $\mathcal{A}^{p,q} \cdot \mathcal{A}^{r,s} \subseteq \mathcal{A}^{p+r,q+s}$,
  (DBA1.b) $x \cdot y = (-1)^{(p+q)(r+s)} y \cdot x$ for any $x \in \mathcal{A}^{p,q}$, $y \in \mathcal{A}^{r,s}$;

(DBA2) a $\mathbb{K}$-linear map $\overline{\partial} : \mathcal{A} \longrightarrow \mathcal{A}$, called the *differential*, such that
  (DBA2.a) $\overline{\partial}(\mathcal{A}^{p,q}) \subseteq \mathcal{A}^{p,q+1}$,
  (DBA2.b) $\overline{\partial}^2 = 0$,
  (DBA2.c) $\overline{\partial}(x \cdot y) = \overline{\partial}(x) \cdot y + (-1)^{p+q} x \cdot \overline{\partial}(y)$ for any $x \in \mathcal{A}^{p,q}$, $y \in \mathcal{A}^{r,s}$.

A *morphism* of differential bigraded algebras $f : (\mathcal{A}, \overline{\partial}) \longrightarrow (\mathcal{A}', \overline{\partial}')$ is a $\mathbb{K}$-linear morphism of vector spaces such that:

(Mor1) f respects the bigraduation, i.e. $f(\mathcal{A}^{p,q}) \subseteq \mathcal{A}'^{p,q}$;
(Mor2) f respects the product, i.e. $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{A}$;
(Mor3) f commutes with the differential, i.e. $\overline{\partial}' \circ f = f \circ \overline{\partial}$.

An element $x \in \mathcal{A}^{p,q}$ is said to have *bidegree* (p, q), and *total degree* (or simply degree) $p + q$:

$$\operatorname{bideg} x = (p, q), \qquad \deg x = p + q.$$

The kernel and the image of the operator $\overline{\partial}$ define the differential subalgebras of the *cycles* and the *boundaries* of $\mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial})$ respectively:

$$\mathcal{Z}(\mathcal{A}) = (\mathcal{Z}^{\bullet,\bullet}(\mathcal{A}), 0) = \ker \overline{\partial}, \qquad \mathcal{B}(\mathcal{A}) = (\mathcal{B}^{\bullet,\bullet}(\mathcal{A}), 0) = \operatorname{im} \overline{\partial}.$$

We have then $\mathcal{B}(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$, and their quotient is the *cohomology* of $(\mathcal{A}, \overline{\partial})$:

$$H(\mathcal{A}) = \frac{\mathcal{Z}(\mathcal{A})}{\mathcal{B}(\mathcal{A})}.$$

Remark 1.1. The cohomology $H(\mathcal{A})$ is in a natural way a differential bigraded algebra, with trivial differential.

A morphism of differential bigraded algebras $f : \mathcal{A} \to \mathcal{A}'$ induces a morphism in cohomology

$$\begin{array}{rccc} f_* : & H(\mathcal{A}) & \longrightarrow & H(\mathcal{A}') \\ & [x] & \longmapsto & [f(x)] \end{array}$$

which is a morphism of differential bigraded algebras.

A *quasi-isomorphism* of differential bigraded algebras is a morphism $f : \mathcal{A} \to \mathcal{A}'$ such that f_* is an isomorphism.

1.1. **The minimal model of a differential bigraded algebra.** A useful tool for dealing with a differential bigraded algebra is its minimal model. Since we do not need the actual (technical) definition of *minimal* differential bigraded algebra, we omit it and address the interested reader to [3, Ch. 7]. In this section we will briefly recall the properties of the minimal model of a differential bigraded algebra that we will use in the sequel.

Convention. In the following, we will consider differential bigraded algebras $\mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial})$ such that:

(1) the bigraduation (p, q) has $p, q \geq 0$;

(2) $\mathcal{A}$ is *homologically connected*, i.e. the canonical morphism of DBA's $\eta : \mathbb{K} \longrightarrow \mathcal{A}$ induces an isomorphism $\eta_* : \mathbb{K} \longrightarrow H^{0,0}(\mathcal{A})$.

The *only exception* to this convention on the bigraduation is provided by the minimal model of $\mathcal{A}$, as explained in (M1).

Under the assumption that $\mathcal{A}$ is homologically connected, it admits a (unique up to non-natural isomorphisms) minimal model

$$f : \mathcal{M} = (\mathcal{M}^{\bullet,\bullet}, \overline{\partial}) \longrightarrow \mathcal{A},$$

which have the following properties:

(M1) $\mathcal{M}$ is a free algebra which is bigraduated by (p, q) with $p + q \geq 0$, but we can have $p < 0$ or $q < 0$ (actually, from the construction we can have $q < 0$, but not $p < 0$);
(M2) $\mathcal{M}$ is *connected*, i.e. in total degree 0, $\mathcal{M}$ is concentrated in bidegree $(0, 0)$;
(M3) $f_* : H(\mathcal{M}) \longrightarrow H(\mathcal{A})$ is an isomorphism. In particular, $H^{p,q}(\mathcal{M}) = 0$ if $p < 0$ or $q < 0$, since we are assuming that the same holds for $\mathcal{A}$.

2. Formality and Dolbeault–Massey triple products

2.1. **Formal differential bigraded algebras.** Two differential bigraded algebras, $\mathcal{A}$ and $\mathcal{A}'$, are *equivalent* if there exists a chain of differential bigraded algebras $\mathcal{C}_i = (\mathcal{C}_i^{\bullet,\bullet}, \overline{\partial}_i)$ and quasi-isomorphisms of the form

$$\mathcal{A} \longleftarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longleftarrow \cdots \longrightarrow \cdots \longleftarrow \mathcal{C}_n \longrightarrow \mathcal{A}'.$$

Definition 2.1. A differential bigraded algebra $\mathcal{A}$ is said to be *formal* if it is equivalent to a differential bigraded algebra $(\mathcal{A}'^{\bullet,\bullet}, \overline{\partial}')$ with $\overline{\partial}' = 0$.

Let M be a complex manifold. We say that M is *Dolbeault formal* if its Dolbeault complex $\Lambda M = (\Lambda^{\bullet,\bullet} M, \overline{\partial})$ is a formal differential bigraded algebra.

2.2. **Dolbeault–Massey triple products.** Let $\mathcal{A}$ be a differential bigraded algebra, and assume that $[X] \in H^{p,q}(\mathcal{A})$, $[Y] \in H^{r,s}(\mathcal{A})$, $[Z] \in H^{u,v}(\mathcal{A})$ satisfy $[X][Y] = [Y][Z] = 0$. So there exist $A \in \mathcal{A}^{p+r,q+s-1}$ and $B \in \mathcal{A}^{r+u,s+v-1}$ such that $XY = \overline{\partial} A$, $YZ = \overline{\partial} B$.

Definition 2.2. With these notations, the *Dolbeault–Massey triple product* of $[X]$, $[Y]$ and $[Z]$ is defined as

$$\langle [X], [Y], [Z] \rangle = [AZ + (-1)^{p+q+1} XB] \in \frac{H^{p+r+u,q+s+v-1}(\mathcal{A})}{[X] \cdot H^{r+u,s+v-1}(\mathcal{A}) + H^{p+r,q+s-1}(\mathcal{A}) \cdot [Z]}. \tag{2.1}$$

Remark 2.3. If $\mathcal{A}$ is a formal differential bigraded algebra, then all the Dolbeault–Massey triple products vanish (see [11, Prop. 3.2]).

2.2.1. *Special Dolbeault–Massey products.* When we want to construct a Dolbeault–Massey triple product we need at least two cohomology classes, say $[X]$ and $[Y]$, such that $[X][Y] = 0$. In this case, the easiest Dolbeault–Massey product we can define corresponds to $[X] \in H^{1,0}(\mathcal{A})$, $[Y] \in H^{0,1}(\mathcal{A})$, and it is

$$\langle [X], [X], [Y] \rangle \in \frac{H^{2,0}(\mathcal{A})}{[X] \cdot H^{1,0}(\mathcal{A})}. \tag{2.2}$$

Definition 2.4. We call a Dolbeault–Massey triple product as in (2.2) a *special Dolbeault–Massey triple product.*

In Section 3 we will give a necessary and sufficient condition for the existence of non-vanishing special Dolbeault–Massey products.

Let $\mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial}_{\mathcal{A}})$ be a homologically connected differential bigraded $\mathbb{K}$-algebra, and consider its minimal model

$$\begin{array}{rccc} f: & \mathcal{M} = (\mathcal{M}^{\bullet,\bullet}, \overline{\partial}) & \longrightarrow & \mathcal{A} \\ & x & \longmapsto & f(x) = X. \end{array}$$

Let $[X] \in H^{1,0}(\mathcal{A})$ be any class: we want to find a class $[Y] \in H^{0,1}(\mathcal{A})$ which defines a non-vanishing Dolbeault–Massey product as in (2.2), but since $H(\mathcal{A})$ and $H(\mathcal{M})$ are naturally isomorphic, we can consider the corresponding class $[x] \in H^{1,0}(\mathcal{M})$, solve the problem in $H(\mathcal{M})$ and then push the result down to $H(\mathcal{A})$.
In the following we will define a "good" basis for $\mathcal{M}^{1,0}$ in order to study the homomorphism $H^{1,0}(\mathcal{M}) \longrightarrow H^{2,0}(\mathcal{M})$ given by the multiplication with the class $[x] \in H^{1,0}(\mathcal{M})$.

Remark 2.5. This is the motivation why we will follow this plan: as we can see in (2.2), the indetermination $[x] \cdot H^{1,0}(\mathcal{M})$ corresponds exactly with the image og the multiplication map.

Let $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$. Then the maps

$$\text{(2.3)} \qquad \begin{array}{rccc} m_x: & \mathcal{M}^{1,0} & \longrightarrow & \mathcal{M}^{2,0} \\ & y & \longmapsto & xy \end{array} \quad \text{and} \quad \begin{array}{rccc} m_{[x]}: & H^{1,0}(\mathcal{M}) & \longrightarrow & H^{2,0}(\mathcal{M}) \\ & [y] & \longmapsto & [x][y] \end{array}$$

are well defined homomorphisms of vector spaces. Observe that even though they are not maps of differential bigraded algebras, their kernel is a two-sided ideal closed under the differential.

It is not difficult to describe $\ker m_x$ and $\ker m_{[x]}$.

Lemma 2.6. $\ker(m_x : \mathcal{M}^{1,0} \longrightarrow \mathcal{M}^{2,0}) = \langle x \rangle$.

Proof. Since $x^2 = 0$, it is obvious that $\langle x \rangle \subseteq \ker m_x$.
For the other inclusion, consider a basis $\{x, x_\alpha\}_{\alpha \in I}$ for $\mathcal{M}^{1,0}$. Let $y = \lambda x + \sum_{i=1}^n \lambda_i x_{\alpha_i} \in \ker m_x$, then

$$xy = \sum_{i=1}^n \lambda_i x x_{\alpha_i} = 0.$$

Since $\mathcal{M}$ is free, the elements $\{x x_{\alpha_1}\}_{i=1}^n$ are linearly independent in $\mathcal{M}^{2,0}$, and so all the coefficients in the previous expression must vanish. This means that $y = \lambda x \in \langle x \rangle$. □

Remark 2.7. The proof of the previous Lemma works in general, i.e. $\ker(m_x : \mathcal{M} \longrightarrow \mathcal{M}) = \langle x \rangle$.

Lemma 2.8. $[y] \in \ker m_{[x]}$ *if and only if* $y \in {m_{x|\mathcal{Z}^{1,0}(\mathcal{M})}}^{-1}(\operatorname{im} \overline{\partial})$.

Proof. $y \in {m_{x|\mathcal{Z}^{1,0}(\mathcal{M})}}^{-1}(\operatorname{im} \overline{\partial})$ is equivalent to $xy = \overline{\partial}\tau$ for some $\tau \in \mathcal{M}^{2,-1}$, which in turn is equivalent to $[x][y] = 0$. □

Corollary 2.9. $\ker m_{[x]} = \langle [x] \rangle$ *if and only if* $m_x(\mathcal{Z}^{1,0}(\mathcal{M})) \cap \operatorname{im} \overline{\partial} = \{0\}$.

2.3. **A "good" basis for $\mathcal{M}^{1,0}$.** In this Section, we define the basis for $\mathcal{M}^{1,0}$ we will use to show the existence of special Dolbeault–Massey products.
Let $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$ be fixed. We can then consider a basis

$$\{\alpha_n, \beta_m, \gamma_r\}, \qquad n \in I_x, m \in I_z, r \in I_\gamma$$

for $\mathcal{M}^{2,-1}$, where

(1) $\{\overline{\partial}\alpha_n\}_{n \in I_x}$ gives a basis for $m_x(\mathcal{Z}^{1,0}(\mathcal{M})) \cap \operatorname{im} \overline{\partial}$;
(2) $\{\overline{\partial}\beta_m\}_{m \in I_z}$ completes the previous set to a basis for $\operatorname{im} m_x \cap \operatorname{im} \overline{\partial}$;
(3) $\{\gamma_r\}_{r \in I_\gamma}$ completes $\{\alpha_n, \beta_m\}$ to a basis for $\mathcal{M}^{2,-1}$.

By their definitions, there exist $x_n \in \mathcal{Z}^{1,0}(\mathcal{M})$ and $z_m \in \mathcal{M}^{1,0} \smallsetminus \mathcal{Z}^{1,0}(\mathcal{M})$ such that

$$\overline{\partial}\alpha_n = x x_n, \qquad \overline{\partial}\beta_m = x z_m.$$

Remark 2.10. The elements x_n are cycles, while no element in $\mathrm{Span}\{z_m\}$ is a cycle. In particular, $\overline{\partial} z_m \neq 0$ for all $m \in I_z$.

We complete the set of linearly independent elements $\{x, x_n, z_m\}$ to the following basis:

$$\{x, x_n, y_k, z_m, w_p, a_q\}, \qquad n \in I_x, k \in I_y, m \in I_z, p \in I_w, q \in I_a \tag{2.4}$$

where

1. $\{x, x_n\}$ is a basis for ${m_{x|\mathcal{Z}^{1,0}(\mathcal{M})}}^{-1}(\mathrm{im}\,\overline{\partial})$;
2. $\{x, x_n, y_k\}$ is a basis for $\mathcal{Z}^{0,1}(\mathcal{M})$;
3. $\{x, x_n, z_m\}$ is a basis for ${m_x}^{-1}(\mathrm{im}\,\overline{\partial})$;
4. $\{x, x_n, y_k, z_m, w_p\}$ is a basis for ${m_x}^{-1}(\mathcal{Z}^{2,0}(\mathcal{M}))$;
5. $\{a_q\}$ completes to a basis for $\mathcal{M}^{1,0}$.

Definition 2.11. We call any basis constructed as described in (2.4) a *"good" basis* for $\mathcal{M}^{1,0}$ associated to $x \in \mathcal{Z}^{1,0} \smallsetminus \{0\}$.

Remark 2.12. Observe that the elements of a "good" basis associated to x are not uniquely determined by x, while the cardinality of the sets I_x, I_z, I_y and I_w are.

Notation. In the following, we will use the following convention on the indices. The letter i will be reserved to variable indices, and we will use it as *the only letter for any index*: the range where i varies is determined to the object where i is attached. In this spirit, we will write the generic cycle in $\mathcal{M}^{1,0}$ as

$$\lambda x + \sum \lambda_i x_i + \sum \mu_i y_i,$$

instead of the extended notation

$$\lambda x + \sum_{i=1}^{h} \lambda_i x_{n_i} + \sum_{j=1}^{l} \mu_j y_{k_j}.$$

Remark 2.13. By Lemma 2.8, in $H^{1,0}(\mathcal{M})$ we have that $\{[x], [x_n]\}$ is a basis for $\ker m_{[x]}$, and that the images of $\{[y_k]\}$ via $m_{[x]}$ form a basis for $\mathrm{im}\, m_{[x]}$, i.e.

$$\mathrm{im}\, m_{[x]} = [x] \cdot H^{1,0}(\mathcal{M}) = \mathrm{Span}\{[x y_k]\} \subseteq H^{2,0}(\mathcal{M}).$$

2.3.1. *Relation with $\mathcal{A}^{1,0}$.* To understand why the basis (2.4) is "good", we take a little digression and see how to link it to $(\mathcal{A}, \overline{\partial}_{\mathcal{A}})$ through the morphism $f : (\mathcal{M}, \overline{\partial}) \longrightarrow (\mathcal{A}, \overline{\partial}_{\mathcal{A}})$.

Lemma 2.14. $\mathrm{Span}\{x, x_n, z_m\} = f^{-1}(\ker m_{f(x)}) \cap {m_x}^{-1}(\mathcal{Z}^{2,0}(\mathcal{M}))$.

Proof. In fact, the following statements are easily seen to be equivalent:

1. $v \in \mathrm{Span}\{x, x_n, z_m\}$;
2. $xv = \overline{\partial}\tau$ for some $\tau \in \mathcal{M}^{2,-1}$;
3. $\overline{\partial}(xv) = 0$ and $[xv] = 0$ in $H^{2,0}(\mathcal{M})$;
4. $\overline{\partial}(xv) = 0$ and $f_*[xv] = 0$ in $H^{2,0}(\mathcal{A})$;
5. $\overline{\partial}(xv) = 0$ and $f(x)f(w) = 0$ in $\mathcal{A}^{2,0}$.

Observe that in the last equivalence we use the fact that $\mathcal{B}^{2,0}(\mathcal{A}) = 0$ since we are assuming that $\mathcal{A}$ has non-negative bigraduation. □

Corollary 2.15. *Let $v \in \mathrm{Span}\{x, x_n, y_k, z_m, w_p\}$. Then*

$$v \in \mathrm{Span}\{x, x_n, z_m\} \iff v \in f^{-1}(\ker m_{f(x)}).$$

Corollary 2.16. $\ker f \cap {m_x}^{-1}(\mathcal{Z}^{2,0}(\mathcal{M})) \subseteq \mathrm{Span}\{x, x_n, z_m\}$.

Proof. Let $v \in \ker f \cap {m_x}^{-1}(\mathcal{Z}^{2,0}(\mathcal{M}))$. Since $v \in \ker f$ we have that $f(x)f(v) = 0$, i.e. $v \in f^{-1}(\ker m_{f(x)})$. But then $v \in \mathrm{Span}\{x, x_n, z_m\}$. □

Lemma 2.17. $\ker f \cap \mathrm{Span}\{x, x_n, y_k\} = 0$.

Proof. From the construction of the minimal model (cfr. [3, Ch. 7]) and the fact that $\mathcal{B}^{1,0}(\mathcal{A}) = 0$, it is clear that f induces an isomorphism

$$f_{|\mathcal{Z}^{1,0}(\mathcal{M})} : \mathcal{Z}^{1,0}(\mathcal{M}) \longrightarrow \mathcal{Z}^{1,0}(\mathcal{A}).$$

□

Remark 2.18. We have the estimate

$$\dim(\ker f \cap {m_x}^{-1}(\mathcal{Z}^{2,0}(\mathcal{M}))) \leq |I_z|.$$

Remark 2.19. Consider a "good" basis associated to $x \in \mathcal{Z}^{1,0}(\mathcal{M})$. Then the vectors $f(x)$, $f(x_n)$, $f(y_k)$ and $f(w_p)$ are non-zero in $\mathcal{A}^{1,0}$. The vectors $f(z_m)$ may be zero or not.

Lemma 2.20. *The vectors* $\{f(y_k), f(w_p)\}$ *are linearly independent in* $\mathcal{A}^{1,0}$.

Proof. Since $\ker f \cap {m_x}^{-1}(\mathcal{Z}^{2,0}(\mathcal{M})) \subseteq \mathrm{Span}\{x, x_n, z_m\}$, we deduce that $f_{|\,\mathrm{Span}\{y_k, w_p\}}$ is injective, so it sends linearly independent vectors to linearly independent vectors. □

We then look for a way of distinguish $f(y_k)$ from $f(w_p)$.

Lemma 2.21. $f(\mathrm{Span}\{y_m\}) \subseteq \mathcal{Z}^{1,0}(\mathcal{A})$, *while* $f(\mathrm{Span}\{w_1, \dots, w_p\rangle) \cap \mathcal{Z}^{1,0}(\mathcal{A}) = \{0\}$.

Proof. The first statement is clear, since f is a morphism of differential bigraded algebras. For the second one, let $\sum \rho_i f(w_i)$ be a cycle. Then

$$f\left(\sum \rho_i w_i\right) = \sum \rho_i f(w_i) = \lambda f(x) + \sum \lambda_i f(x_i) + \sum \mu_i f(y_i) = f\left(\lambda x + \sum \lambda_i x_i + \sum \mu_i y_i\right),$$

and so $\sum \rho_i w_i = \lambda x + \sum \lambda_i x_i + \sum \mu_i y_i + v$ for a suitable $v \in \ker f$. Multiplying this last relation by x, we find that $\overline{\partial}(xv) = 0$. So $v \in \ker f \cap {m_x}^{-1}(\mathcal{Z}^{2,0}(\mathcal{M})) \subseteq \mathrm{Span}\{x, x_n, z_m\}$. Hence $\sum \rho_i w_i \in \mathrm{Span}\{x, x_n, y_k, z_m\}$, which is possible if and only if $\sum \rho_i w_i = 0$. □

2.3.2. *Some properties of a "good" basis.* In this Section, we prove some properties of a "good" basis associated to x.

As observed in Remark 2.12, once we fix $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$ we can have different "good" basis associated to x. However, the cardinality of any piece composing it is *uniquely determined* by x. We can then define the integral valued (or possibly ∞) and non-negative functions $N(x)$, $K(x)$, $M(x)$, $P(x)$ as:

$$\begin{aligned} N(x) &= \dim(m_x(\mathcal{Z}^{1,0}(\mathcal{M})) \cap \mathrm{im}\,\overline{\partial}); \\ K(x) &= \dim(\mathcal{Z}^{1,0}(\mathcal{M})) - N(x) - 1 \\ M(x) &= \dim(\mathrm{im}\, m_x \cap \mathrm{im}\,\overline{\partial}) - N(x) \\ P(x) &= \dim(\mathrm{im}\, m_x \cap \mathcal{Z}^{2,0}(\mathcal{M})) - N(x) - K(x) - M(x). \end{aligned} \tag{2.5}$$

Remark 2.22. We have then the following relations, for any $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$:

$$h^{1,0}(\mathcal{M}) = 1 + N(x) + K(x), \qquad h^{2,0}(\mathcal{M}) \geq K(x) + P(x). \tag{2.6}$$

Our purpose is to try to give a characterization of the vanishing of these functions, in terms of some properties of $\mathcal{A}$ or of $H(\mathcal{A})$.

As a direct consequence of Remark 2.13, we can easily describe what $N(x) = 0$ and $K(x)00$ mean.

Lemma 2.23. *Let $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$. Then $N(x) = 0$ if and only if $\ker m_{[x]} = \langle [x] \rangle$.*

Lemma 2.24. *Let $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$. Then $K(x) = 0$ if and only if $m_{[x]} = 0$.*

To deal with $M(x)$ and $P(x)$ is more subtle, and we need some preparation.

Lemma 2.25. *Let $x \in \mathcal{Z}^{1,0} \smallsetminus \{0\}$ and $t \in \mathcal{M}^{1,0}$ be such that $\overline{\partial} t = xy$ for some $y \in \mathcal{M}^{0,1}$. Then $y \in \mathcal{Z}^{0,1}(\mathcal{M})$.*

Proof. Applying $\overline{\partial}$ to both sides of $\overline{\partial} t = xy$ and since $\overline{\partial} x = 0$ we find that $x\overline{\partial} y = 0$, i.e. $\overline{\partial} y \in \ker m_x$. Since $\overline{\partial} y \in \mathcal{M}^{0,2}$ and $\mathcal{M}$ is free with $\mathcal{M}^{-1,2} = 0$, we have that $\overline{\partial} y = 0$. □

Lemma 2.26. *Let $v \in {m_x}^{-1}(\mathcal{Z}^{2,0}(\mathcal{M}))$. Then $\overline{\partial} v \in x\mathcal{M}^{0,1} \subseteq \mathcal{M}^{1,1}$.*

Proof. Since $xv \in \mathcal{Z}^{2,0}(\mathcal{M})$, applying $\overline{\partial}$ to this element we have

$$x\overline{\partial} v = 0.$$

By Remark 2.7, $\ker(m_x : \mathcal{M} \longrightarrow \mathcal{M}) = \langle x \rangle$: from $x\overline{\partial} v = 0$ we deduce that $\overline{\partial} v \in \ker m_x$, and so $\overline{\partial} v = x\omega$ for a suitable $\omega \in \mathcal{M}^{0,1}$. □

Corollary 2.27. *$\overline{\partial} z_m, \overline{\partial} w_p \in x\mathcal{M}^{0,1} \subseteq \mathcal{M}^{1,1}$, i.e.*

$$\overline{\partial} z_m = x\omega^{(z_m)}, \qquad \overline{\partial} w_p = x\omega^{(w_p)}$$

for suitable $\omega^{(z_m)}, \omega^{(w_p)} \in \mathcal{M}^{0,1}$.

Lemma 2.28. *The elements $\{\omega^{(z_m)}, \omega^{(w_p)}\}$ are linearly independent in $\mathcal{Z}^{0,1}(\mathcal{M})$ and they are uniquely determined by z_m, w_p.*

Proof. Since $\{z_m\}$ and $\{w_p\}$'s are linearly independent in $\mathcal{M}^{1,0}$ and their span meets $\mathcal{Z}^{1,0}$ only in 0, the elements $\overline{\partial} z_m$ and $\overline{\partial} w_p$ are linearly independent in $\mathcal{M}^{1,1}$. Each of them is of the form $\overline{\partial} z_m = x\omega^{(z_m)}$ or $\overline{\partial} w_p = x\omega^{(w_p)}$ and since the multiplication map from $\mathcal{M}^{0,1}$ to $\mathcal{M}^{1,1}$ is injective, the result follows. □

Remark 2.29. As a consequence of Lemma 2.28, we have that

$$h^{0,1}(\mathcal{M}) \geq M(x) + P(x) \tag{2.7}$$

for any $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$.

Proposition 2.30. *Let $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$. The following are equivalent:*

(1) $M(x) + P(x) \neq 0$;
(2) there exists $w \in \mathcal{M}^{1,0}$ such that $\overline{\partial} w \neq 0$, $\overline{\partial}(xw) = 0$;
(3) there exists $w \in \mathcal{M}^{1,0}$ such that $\overline{\partial} w = xy$ with $y \in \mathcal{M}^{0,1} \smallsetminus \{0\}$;
(4) there exists $w \in \mathcal{M}^{1,0}$ such that $\overline{\partial} w = xy$ with $y \in \mathcal{Z}^{0,1}(\mathcal{M}) \smallsetminus \{0\}$;
(5) there exists $[y] \in H^{0,1}(\mathcal{M}) \smallsetminus \{0\}$ such that $[x][y] = 0$.

Proof. The first and the last equivalences are trivial.
One implication of the second equivalence is Lemma 2.26, the other is trivial.
One implication of the third equivalence is Lemma 2.25, the other is trivial. □

We then need a way to distinguish between the z_m's and the w_p's.

Proposition 2.31. *Let $f : \mathcal{M} = (\mathcal{M}^{\bullet,\bullet}, \overline{\partial}) \longrightarrow \mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial}_{\mathcal{A}})$ be the minimal model of $\mathcal{A}$, and $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \smallsetminus \{0\}$. Denote $X = f(x)$. Then the following are equivalent:*

(1) $P(x) \neq 0$;

(2) there exist $Y \in \mathcal{Z}^{0,1}(\mathcal{A})$ *and* $W \in \mathcal{A}^{1,0} \setminus \{0\}$ *such that*

$$XW \neq 0, \qquad \overline{\partial}_{\mathcal{A}} W = XY.$$

Proof. Consider a vector $w_{\overline{p}}$ in the "good" basis associated to x (this is possible since $P(x) \neq 0$) and let $W = f(w_{\overline{p}}) \in \mathcal{A}^{1,0} \setminus \mathcal{Z}^{1,0}(\mathcal{A})$. We know by Lemma 2.26 that $\overline{\partial} w_{\overline{p}} = xy$ for a suitable $y \in \mathcal{Z}^{0,1}(\mathcal{M})$, so we put $Y = f(y) \in \mathcal{Z}^{0,1}(\mathcal{A})$. Then W and Y have the desired properties.
For the converse, consider the following basis for $\mathcal{A}^{1,0}$ and $\mathcal{A}^{0,1}$:

$$\begin{aligned} \mathcal{A}^{1,0} &= \mathrm{Span}\{\underbrace{X, X_i}_{\text{cycles}}, W, \ldots\}, \quad i \in I; \\ \mathcal{A}^{1,0} &= \mathrm{Span}\{\underbrace{Y, Y_j}_{\text{cycles}}, \ldots\}, \qquad j \in J. \end{aligned}$$

In the first step of the construction of $\mathcal{M}$ we have

$$\begin{array}{ccc} F(x, x_i, y, y_j) & \longrightarrow & \mathcal{A} \\ x & \longmapsto & X \\ x_i & \longmapsto & X_i \\ y & \longmapsto & Y \\ y_j & \longmapsto & Y_j, \end{array}$$

(if W is a cycle, we have another generator, say ω, mapping to W). In bidegree $(1,1)$ the morphism induced in cohomology is

$$\begin{array}{ccc} H^{1,1}(F(x, x_i, y, y_i)) & \longrightarrow & H^{1,1}(\mathcal{A}) \\ [xy] & \longmapsto & [XY] = [\overline{\partial}_{\mathcal{A}} W] = 0 \\ [xy_j] & \longmapsto & [XY_j] \\ [x_i y] & \longmapsto & [X_i Y] \\ [x_i y_j] & \longmapsto & [X_i Y_j]. \end{array}$$

So between the new generators we have to add, there is w of bidegree $(1,0)$ such that

$$\overline{\partial} w = xy, \qquad w \longmapsto W.$$

But then

(1) $\overline{\partial} w \neq 0$, since $xy \neq 0$,
(2) $\overline{\partial}(xw) = -x\overline{\partial} w = -x^2 y = 0$,
(3) $f(x)f(w) = XW \neq 0$;

hence $P(x) \neq 0$. □

Remark 2.32. It follows from Lemma 2.21 that the element $W \in \mathcal{A}^{1,0}$ in Proposition 2.31 is not a cycle.

The following Proposition gives a characterization of $M(x) = 0$. Its proof is similar to that of Proposition 2.31, so we omit it.

Proposition 2.33. *Let* $f : \mathcal{M} = (\mathcal{M}^{\bullet,\bullet}, \overline{\partial}) \longrightarrow \mathcal{A} = (\mathcal{A}^{\bullet,\bullet}, \overline{\partial}_{\mathcal{A}})$ *be the minimal model of* $\mathcal{A}$*, and* $x \in \mathcal{Z}^{1,0}(\mathcal{M}) \setminus \{0\}$*. Denote* $X = f(x)$*. Then the following are equivalent:*

(1) $M(x) \neq 0$*;*
(2) there exist $Y \in \mathcal{Z}^{0,1}(\mathcal{A})$ *and* $Z \in \mathcal{A}^{1,0} \setminus \{0\}$ *such that*

$$XZ \neq 0, \qquad \overline{\partial}_{\mathcal{A}} Z = XY.$$

3. Existence of non-vanishing special Dolbeault–Massey products

We come back now to our original purpose: find some condition to define non-vanishing special Dolbeault–Massey products in $H(\mathcal{A})$. By Proposition 2.30, we know that we need at least to require that $M(x)+P(x)\neq 0$ for some non-zero cycle x in $\mathcal{M}^{1,0}$. As we will see in Theorem 3.1, the right condition is that $P(x)\neq 0$.
Let $f:\mathcal{M}=(\mathcal{M}^{\bullet,\bullet},\overline{\partial})\longrightarrow\mathcal{A}=(\mathcal{A}^{\bullet,\bullet},\overline{\partial}_{\mathcal{A}})$ be the minimal model of $\mathcal{A}$. Observe that the functions defined in (2.5), which are defined on $\mathcal{M}^{1,0}\smallsetminus\{0\}$, are well defined also on $H^{1,0}(\mathcal{M})\smallsetminus\{0\}$, hence also on $H^{1,0}(\mathcal{A})\smallsetminus\{0\}$ and consequently on $Z^{1,0}(\mathcal{A})\smallsetminus\{0\}$.

We are now ready to state end prove our main result on the existence of non-vanishing special Dolbeault–Massey products.

Theorem 3.1. *Let $\mathcal{A}=(\mathcal{A}^{\bullet,\bullet},\overline{\partial}_{\mathcal{A}})$ be a homologically connected differential bigraded algebra, and let $[X]\in H^{1,0}(\mathcal{A})\smallsetminus\{0\}$. Then the following are equivalent:*

1. *there exists $[Y]\in H^{0,1}(\mathcal{A})\smallsetminus\{0\}$ such that $[X][Y]=0$ and $\langle[X],[X],[Y]\rangle$ defines a non-vanishing special Dolbeault–Massey triple product in $\frac{H^{2,0}(\mathcal{A})}{[x]\cdot H^{1,0}(\mathcal{A})}$;*
2. *$P(X)\neq 0$;*
3. *there exist $Y\in\mathcal{Z}^{0,1}(\mathcal{A})$ and $W\in\mathcal{A}^{1,0}\smallsetminus\mathcal{Z}^{1,0}(\mathcal{A})$ such that*
$$XW\neq 0,\qquad \overline{\partial}_{\mathcal{A}}W=XY.$$

Proof. The second equivalence is Proposition 2.31, so we only need to prove the first one.
As observed before, it is enough to prove the existence of non-vanishing Dolbeault–Massey triple products in the minimal model $\mathcal{M}=(\mathcal{M}^{\bullet,\bullet},\overline{\partial})$ of $\mathcal{A}$. Assume that $P(x)\neq 0$, consider a "good" basis (2.4) for $\mathcal{M}^{1,0}$ and focus on an element $w_{\bar{p}}$ of such basis. By Corollary 2.27, $\overline{\partial}w_{\bar{p}}=xy$ for a suitable cycle $y\in\mathcal{Z}^{0,1}(\mathcal{M})$. Then
$$\langle[x],[x],[y],\rangle=[xw_{\bar{p}}]\in\frac{H^{2,0}(\mathcal{M})}{[x]\cdot H^{1,0}(\mathcal{M})}$$
is non-vanishing since $[xw_{\bar{p}}]\notin\operatorname{im}m_{[x]}$ by Remark 2.13.
For the converse, we have $xy=\overline{\partial}t$ for a suitable $t\in\mathcal{M}^{1,0}$ and the Dolbeault–Massey triple product $\langle[x],[x],[y]\rangle$ is then represented by $[xt]$. In terms of a "good" basis we can write t as
$$t=\lambda x+\sum\lambda_i x_i+\sum\mu_i y_i+\sum\nu_i z_i+\sum\rho_i w_i+\sum\sigma_i a_i. \tag{3.1}$$
Since xt is a cycle, we deduce that in (3.1) all the coefficients σ_i vanish. Moreover, in cohomology
$$[xt]=\underbrace{\left[\sum\mu_i xy_i\right]}_{\in\operatorname{im}m_{[x]}}+\underbrace{\left[\sum\rho_i xw_i\right]}_{\notin\operatorname{im}m_{[x]}}$$
and since we are assuming that $[xt]$ is non-zero modulo $[x]\cdot H^{1,0}(\mathcal{M})$, there is some non-vanishing ρ_i, which means that $P(x)\neq 0$. □

Theorem 3.2. *Let $\mathcal{A}=(\mathcal{A}^{\bullet,\bullet},\overline{\partial})$ be a homologically connected differential bigraded algebra. Let $[X]\in H^{1,0}(\mathcal{A})\smallsetminus\{0\}$ and $[Y]\in H^{0,1}(\mathcal{A})\smallsetminus\{0\}$ be such that $[X][Y]=0$. Then the following are equivalent:*

1. *$\langle[X],[X],[Y]\rangle$ defines a non-vanishing special Dolbeault–Massey triple product in $\frac{H^{2,0}(\mathcal{A})}{[X]\cdot H^{1,0}(\mathcal{A})}$;*
2. *$XY=\overline{\partial}W$ with $W\in\mathcal{A}^{1,0}\smallsetminus\mathcal{Z}^{1,0}(\mathcal{A})$ such that $XW\neq 0$.*

Corollary 3.3. *Let $\mathcal{A}=(\mathcal{A}^{\bullet,\bullet},\overline{\partial})$ be a homologically connected differential bigraded algebra, and let $[X]\in H^{1,0}(\mathcal{A})\smallsetminus\{0\}$. If $\mathcal{A}$ is formal, then one (and only one) of the following holds:*

1. *$[X][Y]\neq 0$ for all $[Y]\in H^{0,1}(\mathcal{A})\smallsetminus\{0\}$;*

(2) for any $[Y] \in H^{0,1}(\mathcal{A}) \smallsetminus \{0\}$ *such that* $[X][Y] = 0$ *we have that there exists* $Z \in \mathcal{A}^{1,0}$ *such that* $XY = \overline{\partial} Z$ *with* $XZ = 0$.

4. Geometric applications

Let M be a complex manifold, and consider its Dolbeault complex $\Lambda M = (\Lambda^{\bullet,\bullet} M, \overline{\partial})$. We can then apply Theorem 3.1 to this case, to find a condition for the existence of Dolbeault–Massey triple products.

Theorem 4.1. *Let* M *be a connected complex manifold, and* $[\omega] \in H^{1,0}_{\overline{\partial}}(M) \smallsetminus \{0\}$. *Then the following are equivalent:*

(1) there exists $[\psi] \in H^{0,1}_{\overline{\partial}}(M) \smallsetminus \{0\}$ *such that* $[\omega] \cup [\psi] = 0$ *and* $\langle [\omega], [\omega], [\psi] \rangle$ *defines a non-vanishing Dolbeault–Massey triple product in* $\frac{H^{2,0}_{\overline{\partial}}(M}{[\omega] \cdot H^{1,0}_{\overline{\partial}}(M)}$;

(2) there exist a closed $(0,1)$*-form* ψ *and a non-closed* $(1,0)$*-form* η *on* M *such that*

$$\omega \wedge \eta \neq 0, \qquad \overline{\partial}\eta = \omega \wedge \psi.$$

In a particular case, namely the case when the Dolbeault algebra is free, we can get a sharper result.

Corollary 4.2. *Let* M *be a complex manifold, whose Dolbeault algebra* ΛM *admits a model freely generated in positive degree. If there exist non-zero classes* $[\omega] \in H^{1,0}_{\overline{\partial}}(M)$ *and* $[\psi] \in H^{0,1}_{\overline{\partial}}(M)$ *such that* $[\omega] \cup [\psi] = 0$, *then there is a non-vanishing Dolbeault–Massey triple product.*

Proof. By hypothesis, $\omega \wedge \psi = \overline{\partial}\eta$ for a suitable $(1,0)$-form η. By Theorem 3.2, we only need to check that $\omega \wedge \eta \neq 0$. Assume the contrary: by Remark 2.7 we have that $\eta = \lambda\omega$, so $\omega \wedge \psi = \overline{\partial}\eta = 0$, which contradicts the assumption that ΛM is free. □

A well known and established framework where we can apply Corollary 4.2 is provided by the class of compact nilmanifolds. In fact, by [4, Prop. 4.8, Thm. 4.9], if we let $M = \Gamma\backslash G$ be a compact quotient of a nilpotent Lie group G by a lattice Γ and G admits an invariant complex structure J, then a model for ΛM is given by $(\Lambda^{\bullet,\bullet}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}), \overline{\partial})$, where $\mathfrak{g}$ denotes the Lie algebra of G.

Example 4.3 (*Kodaira surface*)**.** Let G be the Lie group defined by the structure equations

$$d\omega_1 = 0, \qquad d\omega_2 = \omega_1 \wedge \bar{\omega}_1. \tag{4.1}$$

We can then identify G with the group of matrices of the form

$$\begin{pmatrix} 1 & \bar{z}_1 & z_2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix},$$

and let Γ be the subgroup of matrices with entries in $\mathbb{Z}[\sqrt{-1}]$. The Kodaira surface is then the quotient $\Gamma\backslash G$, and in terms of the natural complex coordinates z_1 and z_2 we can express ω_1 and ω_2 as

$$\omega_1 = dz_1, \qquad \omega_2 = dz_2 - \bar{z}_1 dz_1. \tag{4.2}$$

These forms are of type $(1,0)$ and the minimal model of the Dolbeault algebra of $\Gamma\backslash G$ is the free algebra generated by elements x_1, x_2, y_1, y_2 corresponding to $\omega_1, \omega_2, \bar{\omega}_1, \bar{\omega}_2$ respectively, and with the differential

$$\overline{\partial} x_1 = \overline{\partial} y_1 = \overline{\partial} y_2 = 0, \qquad \overline{\partial} x_2 = x_1 y_1, \tag{4.3}$$

By Corollary 4.2, we immediatly have that $\langle[\omega_1],[\omega_1],[\bar\omega_1]\rangle$ is a non-vanishing Dolbeault–Massey triple product.

4.1. **Special Dolbeault–Massey products and small deformations.**

Example 4.4 (*Iwasawa threefold)*. Consider the complex nilpotent group $\mathbb{H}(3,\mathbb{C})$ of complex matrices of the form

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad z_1, z_2, z_3 \in \mathbb{C}, \tag{4.4}$$

and the lattice $\Gamma \simeq \mathbb{H}(3,\mathbb{Z}[\sqrt{-1}])$ of matrices with entries in $\mathbb{Z}[\sqrt{-1}]$. The Iwasawa threefold is the complex nilmanifold $I = \Gamma\backslash G$. In terms of the natural coordinates z_1, z_2, z_3 we have the left-invariant $(1,0)$ forms

$$\omega_1 = dz_1, \qquad \omega_2 = dz_2, \qquad \omega_3 = dz_3 - z_1 dz_2, \tag{4.5}$$

from which we can read the structure equations

$$d\omega_1 = d\omega_2 = 0, \qquad d\omega_3 = -\omega_1\wedge\omega_2. \tag{4.6}$$

The minimal model of the Dolbeault algebra is freely generated by x_1, x_2, x_3 and y_1, y_2, y_3 (corresponding to $\omega_1, \omega_2, \omega_3$ and $\bar\omega_1, \bar\omega_3, \bar\omega_3$ respectively), with differential

$$\overline{\partial}x_1 = \overline{\partial}x_2 = \overline{\partial}y_1 = \overline{\partial}y_2 = 0, \qquad \overline{\partial}x_3 = 0, \qquad \overline{\partial}y_3 = -y_1y_2. \tag{4.7}$$

Observe that this manifold is complex parallelisable since $\overline{\partial} = 0$ on $(1,0)$ forms, and so we can not apply our Theorems. Nevertheless we do have non-vanishing Dolbeault–Massey products, for example

$$\langle[\bar\omega_1],[\bar\omega_1],[\bar\omega_2]\rangle.$$

Denote by

$$A_{w_1,w_2,w_3} = \begin{pmatrix} 1 & w_1 & w_3 \\ 0 & 1 & w_2 \\ 0 & 0 & 1 \end{pmatrix}$$

an arbitrary matrix in Γ. For any $t \in \mathbb{C}$ we define the action

$$\varphi_t : \mathbb{H}(3,\mathbb{Z}[\sqrt{-1}]) \longrightarrow \mathrm{Aut}(\mathbb{C}^3)$$

where $\varphi_t(A_{w_1,w_2,w_3})$ is the affine transformation defined by

$$\begin{pmatrix} 1 & 0 & 0 & w_1 \\ 0 & 1 & 0 & w_2 + t\bar w_1 \\ t\bar w_1 & w_1 & 1 & w_3 + tw_1\bar w_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.8}$$

Observe that $\mathbb{C}^3/\varphi_0$ is the Iwasawa threefold I, and that the quotient $I_t = \mathbb{C}^3/\varphi_t$ is a deformation of I. We can compute its invariant $(1,0)$-forms

$$\omega_1(t) = dz_1, \qquad \omega_2(t) = dz_2, \qquad \omega_3(t) = dz_3 - z_1dz_2 - t\bar z_1 dz_1 \tag{4.9}$$

and the structure equations of I_t

$$d\omega_1(t) = d\omega_2(t) = 0, \qquad d\omega_3(t) = -\omega_1(t)\wedge\omega_2(t) + t\,\omega_1(t)\wedge\bar\omega_1(t). \tag{4.10}$$

The minimal model of the Dolbeault algebra of I_t is freely generated by $x_1(t), x_2(t), x_3(t)$ and $y_1(t), y_2(t), y_3(t)$, and the differential is

$$\overline{\partial}_t x_1(t) = \overline{\partial}_t x_2(t) = \overline{\partial}_t y_1(t) = \overline{\partial}_t y_2(t) = 0, \qquad \overline{\partial}_t x_3(t) = t\,x_1(t)y_1(t), \qquad \overline{\partial}_t y_3(t) = -y_1(t)y_2(t). \tag{4.11}$$

So we see that $\langle[\omega_1(t)],[\omega_1(t)],[\bar\omega_1(t)]\rangle$ is a non-vanishing special Dolbeault–Massey product on I_t with $t\neq 0$, which vanishes for $t=0$.

Example 4.5 (*Nakamura threefold)***.** Consider the group $G=\mathbb{C}\ltimes_\varphi\mathbb{C}^2$ with coordinates (z_1,z_2,z_3), where the action of $\mathbb{C}$ on $\mathbb{C}^2$ is given by

$$\varphi(z_1)=\begin{pmatrix} e^{z_1} & 0\\ 0 & e^{-z_1}\end{pmatrix}\in \mathrm{SL}(2,\mathbb{C}).$$

Fix $a+\sqrt{-1}b, c+\sqrt{-1}d\in\mathbb{C}$ such that

(1) they are linearly independent over the reals;
(2) $\varphi(a+\sqrt{-1}b),\varphi(c+\sqrt{-1}d)$ as elements in $\mathrm{SL}(2,\mathbb{C})\subseteq\mathrm{SL}(4,\mathbb{R})$ are conjugates with elements in $\mathrm{SL}(4,\mathbb{Z})$;
(3) $b,d\in\mathbb{Z}\cdot\pi$.

Define the lattice $\Gamma=\Gamma'\ltimes_\varphi\Gamma''$, where $\Gamma'=\mathbb{Z}\cdot\langle a+\sqrt{-1}b, c+\sqrt{-1}d\rangle$ is a lattice in $\mathbb{C}$ and Γ'' is a lattice in $\mathbb{C}^2$. The Nakamura threefold is the quotient $N=\Gamma\backslash G$ (see [8]).
As explained in [1, 11], the inclusion of the subcomplex

$$\mathcal{A}=\bigwedge^{\bullet,\bullet}(\underbrace{dz_1,e^{-z_1}dz_2,e^{z_1}dz_3}_{\text{type }(1,0)};\underbrace{d\bar z_1,e^{-z_1}d\bar z_2,e^{z_1}d\bar z_3}_{\text{type }(0,1)}) \tag{4.12}$$

in ΛN is a quasi-isomorphism.
Observe that the generators of bidegree $(0,1)$ of $\mathcal{A}$ *are not the complex conjugates* of the generators of bidegree $(1,0)$.
Finally, observe that letting

$$\begin{aligned} x_1&=dz_1, & y_1&=d\bar z_1,\\ x_2&=e^{-z_1}dz_2, & y_2&=e^{-z_1}d\bar z_2,\\ x_3&=e^{z_1}dz_3, & y_3&=e^{z_1}d\bar z_3,\end{aligned}$$

we have

$$\begin{aligned} \overline{\partial}x_1&=0, & \overline{\partial}y_1&=0,\\ \overline{\partial}x_2&=0, & \overline{\partial}y_2&=0,\\ \overline{\partial}x_3&=0, & \overline{\partial}y_3&=0,\end{aligned}$$

which shows that the Nakamura threefold is Dolbeault-formal.
We consider now the deformation of N associated to the $(0,1)$-form with coefficients in the holomorphic tangent bundle

$$\varphi_t=t\,e^{z_1}d\bar z_1\otimes\frac{\partial}{\partial z_2}. \tag{4.13}$$

Also in this case the Dolbeault cohomology of the deformed variety N_t is computed by a suitable subcomplex of ΛN_t. This subcomplex is

$$\mathcal{A}_t=\bigwedge^{\bullet,\bullet}(\underbrace{dz_1,e^{-z_1}dz_2-t d\bar z_1,e^{z_1}dz_3}_{\text{type }(1,0)};\underbrace{d\bar z_1,e^{-z_1}d\bar z_2-\bar t e^{\bar z_1-z_1}dz_1,e^{z_1}d\bar z_3}_{\text{type }(0,1)}), \tag{4.14}$$

and if we let

$$\begin{aligned} x_1(t)&=dz_1, & y_1(t)&=d\bar z_1,\\ x_2(t)&=e^{-z_1}dz_2-t d\bar z_1, & y_2(t)&=e^{-z_1}d\bar z_2-\bar t e^{\bar z_1-z_1}dz_1,\\ x_3(t)&=e^{z_1}dz_3, & y_3(t)&=e^{z_1}d\bar z_3,\end{aligned}$$

then the differential is

$$\begin{array}{ll} \overline{\partial}_t x_1(t) = 0, & \overline{\partial}_t y_1(t) = 0, \\ \overline{\partial}_t x_2(t) = -t\, x_1(t) y_1(t), & \overline{\partial}_t y_2(t) = 0, \\ \overline{\partial}_t x_3(t) = 0, & \overline{\partial}_t y_3(t) = 0. \end{array}$$

By Corollary 4.2 we immediately see that $\langle [x_1(t)], [x_1(t)], [y_1(t)] \rangle$ defines a non-vanishing Dolbeault–Massey product on N_t for any $t \neq 0$, which vanishes for $t = 0$ (see [11, Thm.3.1]).

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