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WEAKLY COERCIVE PROBLEMS IN NONLINEAR STOCHASTIC CONTROL: EXISTENCE OF OPTIMAL CONTROLS*

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Abstract. In this paper we prove the existence of an optimal control for some nonlinear stochastic control problems where the control set is unbounded, but instead of a classical coercivity hypothesis, weaker assumptions are made. Our model includes singular stochastic optimization control problems, such as, for instance, the so-called monotone follower problem or the additive control problem, extensively treated in the literature, mostly in the case of linear systems. We apply a technique of time transformation which allows us to show that the original problems are equivalent to some optimal stopping time problems with bounded controls.

Key words. nonlinear stochastic control systems, optimal control, singular stochastic control, time-change

AMS subject classifications. 93E03, 93E20, 49J55

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1. Introduction.

We prove the existence of optimal controls for minimization problems in which the state evolves according to an n-dimensional nonlinear stochastic differential equation of the form

$$x_t = \bar{x} + \int_{\bar{t}}^{t} A(r, x_r, v_r)\, dr + \int_{\bar{t}}^{t} B(r, x_r, v_r) u_r\, dr + \int_{\bar{t}}^{t} D(r, x_r, v_r)\, d\mathcal{B}_r \tag{1}$$

on some filtered probability space $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$, where the functions A, B, D are deterministic continuous functions, $\{\mathcal{B}_t\}$ is a Brownian motion (not necessarily n-dimensional), $\bar{x}$ is the initial state at time $\bar{t}$, and $v : [\bar{t}, T] \to \mathcal{V}$ ($\mathcal{V} \subset \mathbb{R}^l$, $\mathcal{V}$ compact), $u : [\bar{t}, T] \to \mathcal{K}$ ($\mathcal{K} \subset \mathbb{R}^m$, $\mathcal{K}$ a closed convex cone) are the controls. The expected cost has the form

$$\mathcal{J}(\bar{t}, \bar{x}, u, v) = E_Q\left[\int_{\bar{t}}^{T} \left(l_0(r, x_r, v_r) + l_1(r, x_r, v_r)|u_r|\right)\, dr + g\left(\int_{\bar{t}}^{T} |u_r|\, dr, x_T\right)\right], \tag{2}$$

where l_0, l_1, and g are given deterministic functions. Precise assumptions will be specified in section 2. Here we just point out that $g = g(k, x)$ is nondecreasing in k and that we study the optimization problem under different sets of hypotheses, all of which have in common that the cost of applying the unbounded control $\{u_t\}$ is always positive and verifies

$$\mathcal{J}(\bar{t}, \bar{x}, u, v) \geq \bar{c} E_Q\left[\left(\int_{\bar{t}}^{T} |u_r|\, dr\right)^p\right] \tag{3}$$

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for some $p \geq 1$ and $\bar{c} > 0$. We refer in general to such hypotheses as "weak coercivity conditions."

The main novelties introduced in the paper are (i) the explicit dependence on the state variable (and on the classical control v) of the dynamics component B and of the cost l_1, and (ii) the wide range of weak coercivity conditions we deal with.

The minimization problem (1)–(2), known as the *singular* control problem, has been extensively studied in the literature for the case in which $B = B(t)$ in (1) (see, for instance, Fleming and Soner [FS] and the many references therein). In such a case the term "$u_r dr$" appearing in (1) is usually replaced by a term "dU_r," where $\{U_t\}$ is a control process with bounded variation, the cost (2) is defined in terms of the total variation of $\{U_t\}$, and a notion of generalized (discontinuous) solution to (1) is given following an approach in measure.

Otherwise, when $B = B(t, x, v)$, a well-posed notion of solution to (1) in correspondence to a bounded variation control process $\{U_t\}$ is much more involved and requires a completely different approach, which we clarify in the appendix (see Bressan and Rampazzo [BR] and Motta and Rampazzo [MoRa] for a discussion on the case of deterministic problems, and see the references below for an overview). For this reason, the preferred formulation of the problem in our context is (1)–(2), where the class of admissible controls $\{u_t\}$ is given by the $\mathcal{K}$-valued, $\{\mathcal{G}_t\}$-predictable control processes such that $\int_{\bar{t}}^{T} |u_r|\, dr < +\infty$, even if there is no hope in general that an optimal solution exists within this class of controls (see also Lasry and Lions [LL1] and Dufour and Miller [DM1]). The approach that we follow, based on the completion of the graphs of $\{(t, u_t)\}$ through an appropriate time-change, allows us to prove that the original problem is equivalent to an auxiliary optimal stopping time problem, where the controls are bounded valued, but where the fixed terminal time T is replaced by a stopping time chosen by the controller. We then apply the compactification method used in Haussmann and Lepeltier [HL] to prove the existence of an optimal auxiliary control. Here equivalence means not only that the infimums of the two costs, for the original and for the auxiliary controls, are equal but also that the solutions of the stochastic differential equation associated to the auxiliary problem can be obtained as limit of solutions to (1). In view of this fact, as we discuss in the appendix (see also Remark 3.1), using the auxiliary problem one can define a posteriori *a generalized control* and the corresponding *generalized solution* of the original system (1), thus proving also the existence of a *generalized optimal control* for the original problem.

The simultaneous dependence on (x, v) of B and l_1 for problems of the form (1)–(2) under coercivity assumptions, besides being a challenging aspect of the theory, is a natural generalization of the dynamics and cost present in several models occurring in stochastic singular control problems. For instance, Lasry and Lions in the theoretical paper [LL1] study a general class of singular stochastic control problems with dynamics of the same form considered here, that is, with the term B depending on x, for a scalar control u and a cost of Boltza type. They are motivated by applications to economics and finance (mainly concerning the formation of volatility in financial markets), which are developed in a recent series of papers (we refer for a survey to [LL2]). On the other side, the x dependence of l_1 is considered essential in some specific economics models in two recent papers by Alvarez [A1], [A2].

The second novelty consists in the wide class of weak coercivity conditions that we use, some of which seem to be new even for deterministic control problems. The existence of an optimal control is proved if $g(k, x) \geq \bar{c}k^p$ with $p \geq 1$ and $\bar{c} > 0$ or, alternatively, $l_1(t, x, v) \geq \bar{c} > 0$ (see conditions (C1) and (C2) in section 3). In the framework of singular stochastic control, the first condition has been used, for example, when $p = 1$ given by a financial model given by Chiarolla and Haussmann [CH], and it has been considered in a more general form also in Dufour and Miller [DM2]. The second condition in the special case $l_1 = l_1(t)$ has been widely assumed in many models used in singular control problems.

We also allow for the presence of a second compact valued control v and of a terminal cost g which could not be contemplated in several other papers, as already observed and explained in [DM2].

Our results can be viewed as extending in various directions the papers Haussmann and Suo [HS] and [DM2] on singular stochastic control, and we refer the reader to them and to the references therein for an overview on the previous results. In [HS] the existence of an optimal control is proved for multidimensional dynamics including the classical control $\{v_t\}$, assuming $B = B(t)$, A and D bounded, $l_1 = l_1(t) \geq \bar{c} > 0$ $\forall t \in [0, T]$, and $g \equiv 0$. In [DM2] the authors consider a stochastic control problem where $B = B(t)$, l_1 is of special form, and g verifies $\lim_{k \to +\infty} \inf_{x \in \mathbb{R}^n} g(k, x) = +\infty$. They prove that there is a singular optimal control, but they show that singular optimal controls can be approximated by absolutely continuous controls, that is, in our context, the equivalence of the auxiliary and of the original control problems, only in case $g(k, x) = g_1(x) + g_2(k)$ with g_1 Lipschitz continuous and bounded.

The method that we follow, called the method of graph completion, has been used extensively in the literature. It was originally developed in the deterministic context by Bressan and Rampazzo [BR] in the v-independent case and extended to systems depending on both u and v in [MoRa] (see also the book of Miller and Rubinovich [MiRub] and the references therein). It has been recently introduced in the study of some stochastic control problems by Miller and Runggaldier [MiRu] in 1997 and by Dorroh, Ferreyra, and Sundar [DFS] in 1999.

Dufour and Miller [DM1] in 2002 proved the existence of optimal auxiliary controls in the presence of dynamics like (1) but for a so-called finite fuel problem, where the coercivity conditions are replaced by the (hard) constraint

$$\int_{\bar{t}}^{T} |u_r| \, dr \leq \tilde{K} \qquad Q\text{-a.s.} \tag{4}$$

for some fixed $\tilde{K} > 0$. In Motta and Sartori [MS1] we pursued a dynamic programming approach for the same problem, and also slightly improved the existence result of [DM1]. In the last two papers the classical control $\{v_t\}$ is not considered. We have to point out that although we use here the same time-transformation used in [DM1] and [MS1], the weak coercivity assumptions considered here do not imply—differently from condition (4)—the boundedness of the stopping time of the auxiliary problem, and this is an essential point which makes the techniques used in [DM1] and [MS1] inapplicable.

The paper is organized as follows. In section 2 we formulate the problem precisely. In section 3 we introduce the auxiliary stopping time problem, and in section 4 we show that it is equivalent to the original problem. In section 5 we prove an approximation result which is the key point of the proof of the equivalence theorem. Section 6 is devoted to showing the existence of an optimal control for the auxiliary problem. All the results of the previous sections are exploited assuming that either the function B in (1) is bounded or it verifies condition (12) below. In section 7 we treat some generalizations of the problem where these restrictions on B are dropped. A brief description of the graph completion approach is sketched in the appendix.

Notation. Throughout the paper we shall adopt the following notation. The symbol $|\cdot|$ denotes the norm of vectors and matrices, and $\langle\cdot,\cdot\rangle$ denotes the scalar product for vectors. For any positive integer N and any $r > 0$, $B_N(r) = \{v \in \mathbb{R}^N : |v| < r\}$ and $\overline{B}_N(r) = \{v \in \mathbb{R}^N : |v| \leq r\}$, $\mathbb{R}_+ = [0, +\infty[$, $\overline{\mathbb{R}}_+ = [0, +\infty]$, $\overline{\mathbb{R}}^N = [-\infty, +\infty]^N$. For arbitrary positive integers N, M, $\mathbf{M}(N, M)$ denotes the set of the $N \times M$ real matrices. $(^T)$ denotes the transposed operator. $\mathcal{C}^2_b(\mathbb{R}^N)$ is the set of the bounded real maps which are continuous on $\mathbb{R}^N$ with their first and second partial derivatives. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We will use $E_P[\cdot]$ to denote the mathematical expectation on it. Given two random variables X, Y, the notation $X = Y$, $X \leq Y$ means $P(X = Y) = 1$, $P(X \leq Y) = 1$, respectively. $\delta_{\{w\}}$ denotes the Dirac measure at a fixed $w \in \mathcal{K}$.

2. Statement of the problem.

In this section we state precisely the nonlinear stochastic control problem described in the introduction following the formulation of [EKNP], [HL].

Definition 2.1. *Given an initial condition $(\bar{t}, \bar{x}) \in [0, T[\times \mathbb{R}^n$, a* control *is a term*

$$c = (\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\}, \{u_t\}, \{v_t\}, \{x_t\}),$$

where

- *$(\Omega, \mathcal{G}, Q)$ is a complete probability space, with a right continuous complete filtration $\{\mathcal{G}_t\}$;*
- *$\{u_t\}$ is a $\mathcal{K}$-valued process ($\mathcal{K}$ a closed, convex cone of $\mathbb{R}^m$), defined on $[\bar{t}, T] \times \Omega$, which is $\{\mathcal{G}_t\}$-predictable;*
- *$\{v_t\}$ is a $\mathcal{V}$-valued process ($\mathcal{V}$ a compact subset of $\mathbb{R}^l$), defined on $[\bar{t}, T] \times \Omega$, which is $\{\mathcal{G}_t\}$-progressively measurable;*
- *$\{x_t\}$ is an $\mathbb{R}^n$-valued process which is $\{\mathcal{G}_t\}$-progressively measurable, with continuous paths, such that*

$$x_t = \bar{x} + \int_{\bar{t}}^t A(r, x_r, v_r)\, dr + \int_{\bar{t}}^t B(r, x_r, v_r) u_r\, dr + \int_{\bar{t}}^t D(r, x_r, v_r)\, d\mathcal{B}_r$$

for all $t \in [\bar{t}, T]$, where $\{\mathcal{B}_t\}$ is a standard h-dimensional $\{\mathcal{G}_t\}$-Brownian motion.

We call a control c admissible *if*

$$\int_{\bar{t}}^T |u_r|\, dr < +\infty. \tag{5}$$

The set of admissible controls *will be denoted by $\mathcal{C}(\bar{t}, \bar{x})$. For any admissible control c we consider a cost of the form*

$$\mathcal{J}(\bar{t}, \bar{x}, c) = E_Q\left[\int_{\bar{t}}^T (l_0(r, x_r, v_r) + l_1(r, x_r, v_r)|u_r|)\, dr + g\left(\int_{\bar{t}}^T |u_r|\, dr, x_T\right)\right]. \tag{6}$$

We say that $c \in \mathcal{C}(\bar{t}, \bar{x})$ is feasible *if $\mathcal{J}(\bar{t}, \bar{x}, c) < +\infty$, and we write $\mathcal{C}^f(\bar{t}, \bar{x})$ for the set of* feasible *controls. The value function is defined for any $(\bar{t}, \bar{x}) \in [0, T[\times \mathbb{R}^n$ by*

$$\mathcal{W}(\bar{t}, \bar{x}) = \inf_{c \in \mathcal{C}^f(\bar{t}, \bar{x})} \mathcal{J}(\bar{t}, \bar{x}, c). \tag{7}$$

In the following assumptions (A0), (A1) we list the structural hypotheses used throughout the paper.

(A0) The deterministic functions $A:\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{V}\to\mathbb{R}^n$, $B:\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{V}\to\mathbf{M}(n,m)$, $D:\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{V}\to\mathbf{M}(n,h)$ are continuous, and there is some positive constant L_1 such that they verify, for all (t,x,v), $(s,y,v)\in[0,T]\times\mathbb{R}^n\times\mathcal{V}$,

$$|A(t,x,v)-A(s,y,v)|+|B(t,x,v)-B(s,y,v)| \\ +|D(t,x,v)-D(s,y,v)|\le L_1(|t-s|+|x-y|).$$

Remark 2.1. From the above Lipschitz continuity hypotheses and denoting by $\tilde{D}(t,x,v)\doteq D(t,x,v)D(t,x,v)^T$, it follows that for all $(t,x,v)\in[0,T]\times\mathbb{R}^n\times\mathcal{V}$,

$$\begin{aligned} &|A(t,x,v)|,\ |B(t,x,v)|,\ |D(t,x,v)|\le\bar{M}(1+|x|),\\ &|\tilde{D}(t,x,v)|\le\bar{M}^2(1+|x|^2)\end{aligned} \tag{8}$$

for some positive constant $\bar{M}$.

(A1) Let $r,q,p\ge1$. The deterministic functions $l_0:\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{V}\to\mathbb{R}$, $l_1:\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{V}\to\mathbb{R}$, and $g:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$ are continuous and there are some positive constants L_2 and $\tilde{M}$ such that they verify

$$\begin{aligned} &0\le l_1(t,x,v)\le\tilde{M}(1+|x|^r),\\ &0\le l_0(t,x,v)\le\tilde{M}(1+|x|^q),\\ &0\le g(k,x)\le\tilde{M}(1+|x|^q+k^p)\end{aligned}$$

for all $(t,k,x,v)\in[0,T]\times\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{V}$, and

$$\begin{aligned} &|g(k,x)-g(h,y)|\le L_2(\max\{|x|,|y|\}^{q-1}|x-y|+\max\{h,k\}^{p-1}|h-k|),\\ &g(k,x)\le g(h,x)\quad\text{if } k<h\end{aligned}$$

for all (k,x), (h,x), $(h,y)\in\mathbb{R}_+\times\mathbb{R}^n$.

Remark 2.2. We assume $0\le l_0(t,x,v)\le\tilde{M}(1+|x|^q)$ and $g(k,x)\ge0$ for the sake of simplicity, but it is easy to verify that these conditions can be relaxed to

$$-\tilde{l}_0(t)\le l_0(t,x,v)\le\hat{l}_0(t)+\tilde{M}|x|^q,\qquad g(k,x)\ge-C,$$

where $C\ge0$ and $\tilde{l}_0$, $\hat{l}_0$ are nonnegative integrable functions on $\mathbb{R}_+$. Moreover, we could consider a more general term of the form $\langle\hat{l}_1(t,x,v),u\rangle$ replacing the component $l_1(t,x,v)|u|$ in the Lagrangian, with $\hat{l}_1:\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{V}\to\mathbb{R}^m$ such that

$$\langle\hat{l}_1(t,x,v),u\rangle\ge l_1(t,x,v)|u|$$

for all $(t,k,x,v)\in[0,T]\times\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{V}$.

Remark 2.3. In the deterministic case the method of graph completion has been extended to purely nonlinear (in all variables) dynamics and cost behaving as $|u|^\alpha$ and $|u|^\beta$ with $\alpha\le\beta$, respectively, in Rampazzo and Sartori [RS]. The dynamics and cost in (1)–(2) are the natural generalization of dynamics and cost considered up to now in singular control problems, and this is why we choose to study them. It seems possible to apply the method of graph completion to treat more general dynamics and cost both in the weak coercive case ($\alpha=\beta$) and in the classical coercive case ($\alpha<\beta$) studied in [HL].

Remark 2.4. For any initial condition $(\bar{t},\bar{x})\in[0,T[\times\mathbb{R}^n$ under hypotheses (A0), (A1) the set of feasible controls is not empty. Indeed, standard calculations yield that any control $c^0 = (\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\}, \{u_t^0\}, \{v_t^0\}, \{x_t^0\})$ defined as in Definition 2.1 and such that $u_t^0 = 0$ for all t turns out to be feasible. Moreover, in correspondence to such special controls one has

$$E_Q[|x_t^0|^q] \leq \bar{K}_q \left(|\bar{x}|^q + 2\bar{M}^q((T-\bar{t})^q + (T-\bar{t})^{q/2}) + 1\right) e^{\bar{K}_q((T-\bar{t})^q + (T-\bar{t})^{q/2})}$$

and

$$\mathcal{J}(\bar{t}, \bar{x}, c^0) \leq \bar{C}(1 + |\bar{x}|^q)$$

for some $\bar{C} > 0$ depending just on q, $\tilde{M}$, $\bar{M}$, and $T - \bar{t}$. It follows that optimal controls for the minimization problem introduced in Definition 2.1 will certainly belong to the subset of feasible controls verifying

$$\mathcal{J}(\bar{t}, \bar{x}, c) \leq \bar{C}(1 + |\bar{x}|^q). \tag{9}$$

Each one of the coercivity conditions that we introduce below yields some kind of compactness of the set of controls verifying (9). Obviously such a subset plays a key role in the proof of the existence of optimal controls. As discussed in the introduction, (C1) and (C2) below include some of the most usual assumptions in singular stochastic control problems.

(C1) $p \geq 1$, and there exists $\bar{c} > 0$ such that

$$g(k, x) \geq \bar{c}k^p \qquad \forall (k, x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{10}$$

(C2) $p = 1$, and there exists $\bar{c} > 0$ such that

$$l_1(t, x, v) \geq \bar{c} \qquad \forall (t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{V}, \tag{11}$$

where p is the same as in (A1).

It is immediate to see that when either (C1) or (C2) is assumed any feasible control verifies (3).

Under conditions (C1) and (C2) we will study the optimization problem in Definition 2.1 for the term B in the dynamics either bounded or such that

$$\langle B(t, x, v)u, x\rangle \leq 0 \qquad \forall (t, x, u, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{K} \times \mathcal{V}. \tag{12}$$

These restrictions on B can be dropped if one considers stronger coercivity conditions, involving not only the unbounded control u but also the state variable x. Some generalizations in this direction will be addressed in section 7.

3. The auxiliary control problem.

In this section we introduce an auxiliary optimal stopping time control problem, equivalent to the original one, but with the key property that all the controls take values in compact sets.

DEFINITION 3.1. *For any* $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$ *an* auxiliary control *is a term*

$$\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{v_s\}, \{(t_s, k_s, \xi_s)\}, \theta),$$

where the following (B1) *and* (B2) *are assumed.*

(B1) $(\Omega, \mathcal{F}, P)$ *is a complete probability space, with a right continuous complete filtration* $\{\mathcal{F}_s\}$*;*
$\{w_s\}$ *is a* $\overline{B}_m(1) \cap \mathcal{K}$*-valued control defined on* $\mathbb{R}_+ \times \Omega$ *which is* $\{\mathcal{F}_s\}$*-predictable;*

$\{v_s\}$ is a $\mathcal{V}$-valued control defined on $\mathbb{R}_+ \times \Omega$ which is $\{\mathcal{F}_s\}$-progressively measurable;
θ is an $\{\mathcal{F}_s\}$-stopping time such that $\theta < +\infty$.

(B2) *$\{(t_s, k_s, \xi_s)\}$ is an $\mathbb{R}^{2+n}$-valued $\{\mathcal{F}_s\}$-progressively measurable process with continuous paths, such that*

$$\begin{cases} t_s = \bar{t} + \int_0^s w_\sigma^0 \, d\sigma, \\ k_s = \int_0^s |w_\sigma| \, d\sigma, \\ \xi_s = \bar{x} + \int_0^s (A(t_\sigma, \xi_\sigma, v_\sigma) w_\sigma^0 + B(t_\sigma, \xi_\sigma, v_\sigma) w_\sigma) \, d\sigma \\ \qquad + \int_0^s D(t_\sigma, \xi_\sigma, v_\sigma) \sqrt{w_\sigma^0} \, d\mathcal{B}_\sigma \end{cases}$$

for $s \in [0, \theta]$, where $\{\mathcal{B}_s\}$ is a standard h-dimensional $\{\mathcal{F}_s\}$-Brownian motion defined on $\mathbb{R}_+ \times \Omega$ and where we set $w_s^0(\omega) \doteq 1 - |w_s(\omega)|$ for all (s, ω) just for the sake of notation.

We call an auxiliary control β as above admissible *and denote with $\Gamma(\bar{t}, \bar{x})$ the set of such controls. The cost corresponding to an admissible auxiliary control β is of the form*

(13)
$$J(\bar{t}, \bar{x}, \beta) = E_P \left[\int_0^\theta (l_0(t_\sigma, \xi_\sigma, v_\sigma) w_\sigma^0 + l_1(t_\sigma, \xi_\sigma, v_\sigma) |w_\sigma|) \, d\sigma + g(k_\theta, \xi_\theta) + G(t_\theta) \right],$$

where $G(T) = 0$ and $G(t) = +\infty$ otherwise. The set

$$\Gamma^f(\bar{t}, \bar{x}) \doteq \{\beta \in \Gamma(\bar{t}, \bar{x}) : \quad J(\bar{t}, \bar{x}, \beta) < +\infty\} \tag{14}$$

denotes the subset of feasible *auxiliary controls. We define for every $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$ the* auxiliary *value function as*

$$W(\bar{t}, \bar{x}) \doteq \inf_{\beta \in \Gamma^f(\bar{t}, \bar{x})} J(\bar{t}, \bar{x}, \beta). \tag{15}$$

Remark 3.1. The original problem is embedded in the auxiliary problem, which allows us to describe and represent the limit of a sequence of feasible control processes, that is, a so-called *generalized control*. We postpone to the appendix a discussion on how, following the graph-completion approach, for every $(\bar{t}, \bar{x})$ the set of feasible auxiliary controls $\Gamma^f(\bar{t}, \bar{x})$ can be identified with the set of *generalized controls* associated to $\mathcal{C}^f(\bar{t}, \bar{x})$.

In the following lemmas we obtain some estimates on the moments of admissible auxiliary trajectories under different growth assumptions on the dynamics which will be useful in what follows.

Lemma 3.1. *Assume* (A0) *and let B be bounded, i.e.,*

$$\exists M > 0 \text{ s.t. } |B(t, x, v)| \le M \qquad \forall (t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathcal{V}. \tag{16}$$

(i) *If $p \ge 2$, then for all $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$ and all $\beta \in \Gamma^f(\bar{t}, \bar{x})$ such that $E_P[k_\theta^p] < +\infty$ there exists a unique solution of the stochastic differential equation in* (B2)*, and one has*

(17)
$$E_P \left[\sup_{0 \le s \le \theta} |\xi_s|^p \right] \le \bar{K}_p \, e^{\bar{K}_p ((T - \bar{t})^{\frac{p}{2}} + (T - \bar{t})^p)} \left\{ |\bar{x}|^p + (T - \bar{t})^{\frac{p}{2}} + (T - \bar{t})^p + E_P \left[k_\theta^p \right] \right\}$$

for a suitable constant $\bar{K}_p$ *independent of the control.*

(ii) *If* $1 \leq p < 2$ *and* $\tilde{D} = DD^T$ *grows linearly in* x*, i.e.,*

$$\exists M > 0 \text{ s.t. } |\tilde{D}(t,x,v)| \leq M(1+|x|) \qquad \forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathcal{V}, \tag{18}$$

then for all $(\bar{t},\bar{x}) \in [0,T] \times \mathbb{R}^n$ *and all* $\beta \in \Gamma^f(\bar{t},\bar{x})$ *such that* $E_P[k_\theta^p] < +\infty$ *there exists a unique solution of the stochastic differential equation in* (B2)*, and one has*

$$E_P\left[\sup_{0\leq s\leq\theta} |\xi_s|^p\right] \leq \bar{K}_p\, e^{\bar{K}_p(T-\bar{t})}\left\{1+|\bar{x}|^p + (T-\bar{t})^p + E_P\left[k_\theta^p\right]\right\} \tag{19}$$

for a suitable constant $\bar{K}_p$ *independent of the control.*

LEMMA 3.2. *Assume* (A0) *and let* B *verify* (12)*. Then for all* $(\bar{t},\bar{x}) \in [0,T]\times\mathbb{R}^n$ *and all* $\beta \in \Gamma^f(\bar{t},\bar{x})$ *there exists a unique solution of the stochastic differential equation in* (B2)*. Moreover, for any* $\tilde{p} \geq 1$ *one has*

$$E_P\left[\sup_{0\leq s\leq\theta} |\xi_s|^{\tilde{p}}\right] \leq e^{\bar{K}_{\tilde{p}}(T-\bar{t})}\left\{|\bar{x}|^{\tilde{p}} + \bar{K}_{\tilde{p}}(T-\bar{t})\right\} \tag{20}$$

for a suitable constant $\bar{K}_{\tilde{p}}$ *independent of the control.*

Before proving the lemmas, we need the following preliminary result.

LEMMA 3.3. *For any control* $\beta = (\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{w_s\},\ \{v_s\},\{(t_s,k_s,\xi_s)\},\theta) \in \Gamma^f(\bar{t},\bar{x})$ *let us define*

$$\Phi_\sigma \doteq \int_0^\sigma (1-|w_r|)\,dr$$

for $0 \leq \sigma \leq +\infty$*. Then* $\Phi_\theta = T-\bar{t}$*. Let us denote by* $\{\Psi_\tau\}$ *the right inverse of* Φ*:*

$$\Psi_\tau \doteq \inf\{\sigma \geq 0 :\ \Phi_\sigma > \tau\}$$

for $0 \leq \tau \leq \Phi_\theta = T-\bar{t}$*. Then* $\{\Psi_\tau\}$ *is a right continuous time-change satisfying the following properties:*

(i) $\Psi_{\Phi_\sigma} \geq \sigma,\ 0 \leq \sigma \leq \theta$*;* $\Phi_{\Psi_\tau} = \tau,\ 0 \leq \tau \leq \Phi_\theta = T-\bar{t}$*;*

(ii) *setting*

$$\check{\mathcal{F}}_\tau \doteq \mathcal{F}_{\Psi_\tau},\ \ 0 \leq \tau \leq T-\bar{t}, \tag{21}$$

then $\check{\mathcal{F}}_\sigma$ *is a right continuous complete filtration on the probability space* $(\Omega,\mathcal{F},P)$*;*

(iii) *for any nonnegative Borel function* $f : \mathbb{R}_+ \to \mathbb{R}$*, one has*

$$\int_0^s f(\sigma)(1-|w_\sigma|)\chi_{[0\leq\sigma\leq\theta]}\,d\sigma = \int_0^{\Phi_s} f(\psi_\tau)\chi_{[0\leq\tau\leq T-\bar{t}]}\,d\tau.$$

Proof. Using [RY, Proposition 1.1, Chapter V], $\{\Psi_\tau\}$ is a right continuous time-change on $(\Omega,\mathcal{F},P,\{\mathcal{F}_s\})$ and $\check{\mathcal{F}}_\sigma$ is a right continuous complete filtration on the probability space $(\Omega,\mathcal{F},P)$. The proof of thesis (iii) follows from [RY, Proposition 4.9, Chapter 0]. ∎

Proof of Lemma 3.1. Owing to the boundedness of auxiliary controls, the existence and the uniqueness of the solution to the stochastic differential equation in (B2) are well known. Since the stopping time θ may be unbounded, however, the estimates (17), (19) are not standard.

For any $R>0$, any $(\bar t,\bar x)\in[0,T]\times\mathbb{R}^n$, and any feasible control $\beta\in\Gamma^f(\bar t,\bar x)$ let us set

$$S_R \doteq \inf\{s\geq 0:\ |\xi_s|\geq R\}\wedge\theta \quad \text{and} \quad \hat\xi^R_s \doteq \xi_{s\wedge S_R}. \tag{22}$$

In order to simplify the notation, in what follows we write $\hat\xi_s$ instead of $\hat\xi^R_s$. Then for any $\tilde p\geq 1$ there exists a constant $K_{\tilde p}$ depending just on $\tilde p$, such that

$$\begin{aligned} E_P\left[\sup_{s\geq 0}|\hat\xi_s|^{\tilde p}\right] \leq K_{\tilde p}\Bigg\{ |\bar x|^{\tilde p} &+ E_P\left[\left(\int_0^\theta |A(t_\sigma,\hat\xi_\sigma,v_\sigma)(1-|w_\sigma|)|\,d\sigma\right)^{\tilde p}\right] \\ &+ E_P\left[\left(\int_0^\theta |B(t_\sigma,\hat\xi_\sigma,v_\sigma)||w_\sigma|\,d\sigma\right)^{\tilde p}\right] \\ &+ E_P\left[\sup_{s\geq 0}\left|\int_0^{s\wedge\theta} D(t_\sigma,\hat\xi_\sigma,v_\sigma)\sqrt{1-|w_\sigma|}\,d\mathcal{B}_\sigma\right|^{\tilde p}\right]\Bigg\}. \end{aligned}$$

Let us assume that $\tilde p = p$. From the boundedness of B it follows that

$$E_P\left[\left(\int_0^\theta |B(t_\sigma,\hat\xi_\sigma,v_\sigma)||w_\sigma|\,d\sigma\right)^p\right] \leq M^p E_P[k_\theta^p],$$

where $E_P[k_\theta^p]<+\infty$ by hypothesis. Performing the change of variable of Lemma 3.3 and using the Cauchy–Schwarz inequality, for the first integral on the right-hand side (r.h.s.) we get

$$\begin{aligned} &E_P\left[\left(\int_0^\theta |A(t_\sigma,\hat\xi_\sigma,v_\sigma)(1-|w_\sigma|)|\,d\sigma\right)^p\right] = E_P\left[\left(\int_0^{\Phi_\theta} |A(\bar t+\tau,\hat\xi_{\Psi_\tau},v_{\Psi_\tau})|\,d\tau\right)^p\right] \\ &\leq 2^{p-1}\bar M^p(T-\bar t)^p + 2^{p-1}\bar M^p(T-\bar t)^{p-1}\int_0^{T-\bar t} E_P\left[\sup_{0\leq\tau'\leq\tau}|\hat\xi_{\Psi_{\tau'}}|^p\right]d\tau. \end{aligned} \tag{23}$$

Let us first consider case (i). In this case, being $p\geq 2$, from the Burkholder–Davis–Gundy inequality and the Cauchy–Schwarz inequality, for the third integral on the r.h.s. one has

$$\begin{aligned} &E_P\left[\sup_{s\geq 0}\left|\int_0^{s\wedge\theta} D(t_\sigma,\hat\xi_\sigma,v_\sigma)\sqrt{1-|w_\sigma|}\,d\mathcal{B}_\sigma\right|^p\right] \\ &\leq E_P\left[\left(\int_0^\theta \left|\tilde D(t_\sigma,\hat\xi_\sigma,v_\sigma)(1-|w_\sigma|)\right|\,d\sigma\right)^{\frac p2}\right] \leq 2^{p-1}\bar M^p(T-\bar t)^{\frac p2} \\ &\quad + 2^{p-1}\bar M^p(T-\bar t)^{\frac p2-1}\int_0^{T-\bar t} E_P\left[\sup_{0\leq\tau'\leq\tau}|\hat\xi_{\Psi_{\tau'}}|^p\right]d\tau, \end{aligned} \tag{24}$$

where the last estimate is obtained using Lemma 3.3. Observe that

$$E_P\left[\sup_{0\leq\tau\leq T-\bar t}|\hat\xi_{\psi_\tau}|^p\right] = E_P\left[\sup_{s\geq 0}|\hat\xi_s|^p\right], \tag{25}$$

so that from (23) and (24) one has

$$\begin{aligned}
&E_P\left[\sup_{0\le\tau\le T-\bar t}|\hat\xi_{\psi_\tau}|^p\right]\\
&\le K_p\left\{|\bar x|^p+2^{p-1}\bar M^p[(T-\bar t)^{\frac p2}+4(T-\bar t)^p]+M^pE_P\left[\left(\int_0^\theta|w_\sigma|\,d\sigma\right)^p\right]\right.\\
&\quad\left.+\,2^{p-1}\bar M^p[(T-\bar t)^{\frac p2-1}+4(T-\bar t)^{p-1}]\int_0^{T-\bar t}E_P\left[\sup_{0\le\tau'\le\tau}|\hat\xi_{\Psi_{\tau'}}|^p\right]d\tau\right\}.
\end{aligned}$$

Now Gronwall's lemma applied to the lower semicontinuous function $E_P\left[\sup_{0\le\tau\le T-\bar t}|\hat\xi_{\psi_\tau}|^p\right]$ and (25) yield

$$E_P\left[\sup_{s\ge0}|\hat\xi^R_s|^p\right]\le\bar K_p\,e^{\bar K_p((T-\bar t)^{\frac p2}+(T-\bar t)^p)}\left\{|\bar x|^p+(T-\bar t)^{\frac p2}+(T-\bar t)^p+E_P\left[k_\theta^p\right]\right\}.$$

Since $\sup_{s\ge0}|\hat\xi^R_s|=\sup_{0\le s\le\theta}|\xi_s|\wedge R$, (17) follows from the monotone convergence theorem.

In case (ii), instead, $1\le p<2$ and inequality (24) does not hold. Assuming that $\tilde D$ verifies (18), by applying first the Burkholder–Davis–Gundy inequality, observing that for any random variable y, $E_p[|y|^{\frac p2}]\le 1+E_p[|y|^p]$, and then applying the Cauchy–Schwarz inequality, using Lemma 3.3 for the third integral on the r.h.s., we obtain

$$\begin{aligned}
&E_P\left[\left(\int_0^\theta\left|\tilde D(t_\sigma,\hat\xi_\sigma,v_\sigma)(1-|w_\sigma|)\right|d\sigma\right)^{\frac p2}\right]\\
&\le 1+2^{p-1}M^p(T-\bar t)^{p-1}E_P\left[\int_0^{T-\bar t}\left(1+\sup_{0\le\tau'\le\tau}|\hat\xi_{\Psi_{\tau'}}|^p\right)d\tau\right].
\end{aligned}\tag{26}$$

From now on (19) can be obtained by arguing as above, simply replacing (24) with (26). ∎

Proof of Lemma 3.2. For any $(\bar t,\bar x)\in[0,T]\times\mathbb R^n$ and for any feasible control $\beta\in\Gamma^f(\bar t,\bar x)$ let us define $\hat\xi_s\doteq\xi_{s\wedge\theta}$. Let us take an increasing sequence of $\mathcal C^2$ functions $\varphi_R:\mathbb R_+\to\mathbb R_+$ such that for all $R>0$, $\varphi_R'(z)=0$ if $z\ge 2R$, $\varphi_R''(z)\le0$, and $\varphi_R(z)\to z$, $\varphi_R'(z)\to1$, as $R\to+\infty$. It is not restrictive to assume $\tilde p\ge4$, since for any random variable y such that $E[|y|^m]<+\infty$, one has that $E[|y|^{m'}|]\le(E[|y|^m])^{m'/m}$ for any $m'\le m$. By applying Itô's formula to the process $\varphi_R(|\hat\xi_s|^{\tilde p})$, for every $s\ge0$ one has

$$\begin{aligned}
&\varphi_R(|\hat\xi_s|^{\tilde p})=\varphi_R(|\bar x|^{\tilde p})+\int_0^s\tilde p|\hat\xi_\sigma|^{\tilde p-2}\varphi_R'(|\hat\xi_\sigma|^{\tilde p})\langle\hat\xi_\sigma,A(t_\sigma,\hat\xi_\sigma,v_\sigma)(1-|w_\sigma|)\rangle\,d\sigma\\
&+\int_0^s\tilde p|\hat\xi_\sigma|^{\tilde p-2}\varphi_R'(|\hat\xi_\sigma|^{\tilde p})\langle\hat\xi_\sigma,B(t_\sigma,\hat\xi_\sigma,v_\sigma)w_\sigma\rangle\,d\sigma\\
&+\int_0^s\frac{\tilde p}{2}|\hat\xi_\sigma|^{\tilde p-2}\varphi_R'(|\hat\xi_\sigma|^{\tilde p})\mathrm{Tr}\left[\tilde D(t_\sigma,\hat\xi_\sigma,v_\sigma)(1-|w_\sigma|)\right]d\sigma\\
&+\int_0^s\frac{\tilde p^2}{2}|\hat\xi_\sigma|^{2\tilde p-4}\varphi_R''(|\hat\xi_\sigma|^{\tilde p})\left|D^T(t_\sigma,\hat\xi_\sigma,v_\sigma)\sqrt{1-|w_\sigma|}\hat\xi_\sigma\right|^2d\sigma\\
&+\int_0^s\frac{\tilde p}{2}(\tilde p-2)|\hat\xi_\sigma|^{\tilde p-4}\varphi_R'(|\hat\xi_\sigma|^{\tilde p})\left|D^T(t_\sigma,\hat\xi_\sigma,v_\sigma)\sqrt{1-|w_\sigma|}\hat\xi_\sigma\right|^2d\sigma\\
&+\int_0^s\tilde p|\hat\xi_\sigma|^{\tilde p-2}\varphi_R'(|\hat\xi_\sigma|^{\tilde p})\langle\hat\xi_\sigma,D(t_\sigma,\hat\xi_\sigma,v_\sigma)\sqrt{1-|w_\sigma|}\,d\mathcal B_\sigma\rangle.
\end{aligned}$$

Since the second and fourth integrals are nonpositive, performing the time-change of Lemma 3.3 in the first, third, and fifth integrals we get

(27)
$$\varphi_R(|\hat\xi_s|^{\tilde p}) \le \varphi_R(|\bar x|^{\tilde p}) + 2\bar M \int_0^{T-\bar t} \varphi_R'(|\hat\xi_\sigma|^{\tilde p})\tilde p|(1+|\hat\xi_{\Psi_\tau}|^{\tilde p})\, d\tau$$
$$+ \bar M^2 \int_0^{T-\bar t} \varphi_R'(|\hat\xi_\sigma|^{\tilde p})\tilde p|(1+|\hat\xi_{\Psi_\tau}|^{\tilde p})\, d\tau + 2\bar M^2 \int_0^{T-\bar t} \tilde p(\tilde p-2)\varphi_R'(|\hat\xi_\sigma|^{\tilde p})(1+|\hat\xi_{\Psi_\tau}|^{\tilde p})\, d\tau$$
$$+ \int_0^s \tilde p|\hat\xi_\sigma|^{\tilde p-2}\varphi_R'(|\hat\xi_\sigma|^{\tilde p})\langle\hat\xi_\sigma, D(t_\sigma,\hat\xi_\sigma,v_\sigma)\sqrt{1-|w_\sigma|}\, d\mathcal{B}_\sigma\rangle.$$

Moreover, by the assumptions on φ_R it follows that

$$E_P\left[\int_0^s \left||\hat\xi_s|^{\tilde p-2}\varphi_R'(|\hat\xi_\sigma|^{\tilde p})\langle\hat\xi_\sigma, D(t_\sigma,\hat\xi_\sigma,v_\sigma)\rangle\right|^2 (1-|w_\sigma|)\, d\sigma\right] < +\infty.$$

Hence the expectation of the stochastic integral is zero, and taking the expectation in (27) we get

$$E_P\left[\varphi_R(|\hat\xi_s|^{\tilde p})\right] \le \varphi_R(|\bar x|^{\tilde p}) + \tilde M \int_0^{T-\bar t} E_P\left[\varphi_R'(|\hat\xi_{\Psi_\tau}|^{\tilde p})\left(1+|\hat\xi_{\Psi_\tau}|^{\tilde p}\right)\right] d\tau,$$

where $\tilde M$ is a suitable constant. Letting R tend to $+\infty$, it follows from the monotone convergence theorem and Doob's inequality that

$$E_P\left[\sup_{s\ge 0}|\hat\xi_s|^{\tilde p}\right] \le |\bar x|^{\tilde p} + \tilde M \int_0^{T-\bar t} E_P\left[\left(1+\sup_{0\le\tau'\le\tau}|\hat\xi_{\Psi_{\tau'}}|^{\tilde p}\right)\right] d\tau.$$

Now observing that $E_P[\sup_{0\le s\le\theta}|\xi_s|^{\tilde p}] = E_P[\sup_{s\ge 0}|\hat\xi_s|^{\tilde p}] = E_P[\sup_{0\le\tau\le T-\bar t}|\hat\xi_{\psi_\tau}|^{\tilde p}]$ and applying Gronwall's inequality we get (20). ☐

4. The equivalence result. In Theorem 4.1 we prove that under the structural hypotheses (A0), (A1) the original control problem can be embedded into the auxiliary control problem by showing that the set of feasible controls can be identified with a proper subset of the set of feasible auxiliary controls. Furthermore, in Theorem 4.2 we obtain the main result of this section; that is, we prove that the infimums of the problems in Definitions 2.1 and 3.1 are the same, assuming (A0), (A1) together with one of the following sets of hypotheses.

(H1) (C1) holds, $q \le p$, and B verifies (16). Moreover, either $p \ge 2$ or $1 \le p < 2$ and $\tilde D$ verifies (18).

(H2) (C2) holds, $q = p = 1$, B verifies (16), and $\tilde D$ verifies (18).

(H3) Either (C1) or (C2) holds and B verifies (12).

THEOREM 4.1. *Assume* (A0), (A1).

(i) *For any initial condition* $(\bar t, \bar x) \in [0, T[\times\mathbb{R}^n$ *one has*

$$\mathcal{C}^f(\bar t,\bar x) \hookrightarrow \Gamma^f(\bar t,\bar x).$$

That is, for every control $c \in \mathcal{C}^f(\bar t,\bar x)$ *there exists a feasible auxiliary control* $\beta \in \Gamma^f(\bar t,\bar x)$ *such that* $J(\bar t,\bar x,\beta) = \mathcal{J}(\bar t,\bar x,c)$.

(ii) *On the contrary, given a feasible auxiliary control*

$$\beta = (\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{w_s\},\{v_s\},\{(t_s,k_s,\xi_s)\},\theta) \in \Gamma^f(\bar t,\bar x)$$

such that $\{t_s\}$ *is a strictly increasing process, there exists a control* $c \in \mathcal{C}^f(\bar{t},\bar{x})$ *such that* $J(\bar{t},\bar{x},\beta) = \mathcal{J}(\bar{t},\bar{x},c)$*. In other words,*

$$\mathcal{C}^f(\bar{t},\bar{x}) \equiv \left\{\beta \in \Gamma^f(\bar{t},\bar{x}): \quad |w_s| < 1 \quad \textit{for a.e. } s \geq 0\right\}.$$

Proof. Let $c = (\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\}, \{u_t\}, \{v_t\}, \{x_t\})$ be a control in $\mathcal{C}^f(\bar{t},\bar{x})$. We consider the time-change $\{\psi_t\}$ defined by

$$\psi_t \doteq t - \bar{t} + \int_{\bar{t}}^t |u_{\tau\wedge T}|\, d\tau \qquad \forall t \geq \bar{t}$$

and denote by $\{\phi_s\}$ the right inverse of $\{\psi_t\}$:

$$\phi_s \doteq \inf\{\tau \geq \bar{t}: \ \psi_\tau > s\}. \tag{28}$$

Then $\{\phi_s\}$ is a continuous time-change such that

$$\phi_{\psi_t} = t \qquad \forall t \geq \bar{t}; \qquad \psi_{\phi_s} = s \qquad \forall s \geq 0;$$

and

$$\phi_s = \int_0^s \frac{1}{1+|u_{\phi_\sigma}|}\, d\sigma \qquad \forall s \in [0, \psi_T].$$

At this point, we define

$$\beta \doteq (\Omega, \mathcal{G}, Q, \{\mathcal{G}_{\phi_s}\}, \{w_s\}, \{v_{\phi_s}\}, \{(t_s, k_s, \xi_s)\}, \psi_T), \tag{29}$$

where

$$w_s \doteq \frac{u_{\phi_s}}{1+|u_{\phi_s}|}, \qquad (t_s, k_s, \xi_s) \doteq (\phi_s, K_{\phi_s}, x_{\phi_s}), \qquad K_t \doteq \int_{\bar{t}}^t |u_\tau|\, d\tau.$$

Notice that assumption (5) implies that ψ_T is a stopping time verifying $\psi_T < +\infty$. From now on we omit the proof that β defined as above is a feasible auxiliary control having the same cost of c, since it follows along the same lines as the proof of Proposition 4.12 in [DM1].

In order to prove statement (ii), given $\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{v_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$ in $\Gamma^f(\bar{t},\bar{x})$, we denote by $\{\psi_t\}$ the right inverse of $\{t_s\}$:

$$\psi_t \doteq \inf\{\sigma \geq 0: \ t_\sigma > t\} \qquad \forall t \in [\bar{t}, T].$$

Since $\{t_s\}$ is a strictly increasing continuous process, the process $\{\psi_t\}$ is a continuous time-change such that

$$\psi_{t_s} = s \qquad \forall s \geq 0; \qquad t_{\psi_t} = t \qquad \forall t \geq \bar{t};$$

and

$$\psi_t = \int_{\bar{t}}^t \frac{1}{1-|w_{\psi_\tau}|}\, d\tau \qquad \forall t \geq \bar{t}.$$

Therefore, by the same arguments used above, the control

$$c \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_{\psi_t}\}, \{u_t\}, \{v_{\psi_t}\}, \{\xi_{\psi_t}\}),$$

where

$$u_t \doteq \frac{w_{\psi_t}}{1-|w_{\psi_t}|},$$

belongs to $\mathcal{C}^f(\bar t,\bar x)$ and

$$\mathcal{J}(\bar t,\bar x,c) = J(\bar t,\bar x,\beta).$$

Let us point out that if the process $\{\mathcal{B}_s\}$ is a Brownian motion involved in the definition of the state $\{\xi_s\}$, as in Definition 3.1, the process

$$\tilde{\mathcal{B}}_t \doteq \int_0^{\psi_t} \sqrt{1-|w_s|}\, d\mathcal{B}_s$$

turns out to be a standard h-dimensional $\{\mathcal{F}_{\psi_t}\}$ Brownian motion and $\{\xi_{\psi_t}\}$ satisfies the following for all $t\in[\bar t,T]$:

$$x_t \doteq \xi_{\psi_t} = \bar x + \int_{\bar t}^t A(r,x_r,v_{\psi_r})\,dr + \int_{\bar t}^t B(r,x_r,v_{\psi_r})u_r\,dr + \int_{\bar t}^t D(r,x_r,v_{\psi_r})\,d\tilde{\mathcal{B}}_r.$$

The process $\{N_s \doteq \int_0^s \sqrt{1-|w_\sigma|}\,d\mathcal{B}_\sigma\}$ is indeed an $\{\mathcal{F}_s\}$-continuous local martingale such that

$$\forall s\ge 0,\quad \forall i,j\in\mathbb{N}_n^2,\qquad \langle N^i,N^i\rangle_s = t_s,\quad \langle N^i,N^j\rangle_s = 0\quad (i\ne j),$$

and, according to Theorem 4.13 in [KS], this yields the thesis. ∎

THEOREM 4.2. *Fix an initial condition* $(\bar t,\bar x)\in[0,T[\times\mathbb{R}^n$ *and assume* (A0), (A1) *together with one of the hypotheses* (H1), (H2), (H3). *Then for any feasible auxiliary control* $\beta\in\Gamma^f(\bar t,\bar x)$ *there is a sequence of controls* $c^j\in\mathcal{C}^f(\bar t,\bar x)$ *for all* $j\in\mathbb{N}$ *such that*

$$\lim_n \mathcal{J}(\bar t,\bar x,c^n) = J(\bar t,\bar x,\beta).$$

Therefore,

$$W(\bar t,\bar x) = \mathcal{W}(\bar t,\bar x)\qquad \forall(\bar t,\bar x)\in[0,T[\times\mathbb{R}^n. \tag{30}$$

The following approximation result is a key tool for the proof of Theorem 4.2.

THEOREM 4.3. *Fix an initial condition* $(\bar t,\bar x)\in[0,T]\times\mathbb{R}^n$ *and assume* (A0), (A1) *together with one of the hypotheses* (H1), (H2), (H3). *Then for any feasible auxiliary control* $\beta\in\Gamma^f(\bar t,\bar x)$ *there exists a sequence of feasible auxiliary controls*

$$\beta^m = (\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{w_s^m\},\{v_s\},\{(t_s^m,k_s^m,\xi_s^m)\},\theta^m),\quad m\in\mathbb{N},$$

with the property that for any $m\in\mathbb{N}$ *the stopping time* θ^m *is bounded above by* $(T-\bar t)+m$ *and*

$$\lim_m |J(\bar t,\bar x,\beta^m) - J(\bar t,\bar x,\beta)| = 0.$$

The rather technical proof of Theorem 4.3 is postponed to the next section.

Proof of Theorem 4.2. Given a feasible auxiliary control β, we consider the sequence of feasible auxiliary controls

$$\beta^m = (\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{w_s^m\},\{v_s\},\{(t_s^m,k_s^m,\xi_s^m)\},\theta^m),\quad m\in\mathbb{N},$$

introduced in Theorem 4.3. Then, for fixed $m \in \mathbb{N}$, in correspondence to β^m we can construct a sequence of feasible auxiliary controls $\beta^{m,n} \in \Gamma^f(\bar{t}, \bar{x})$ defined by

$$\beta^{m,n} = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s^{m,n}\}, \{v_s^n\}\{(t_s^{m,n}, k_s^{m,n}, \xi_s^{m,n})\}, \theta^{m,n}), \quad n \in \mathbb{N},$$

such that $\{t_s^{m,n}\}$ is a strictly increasing process,

$$\theta^{m,n} \le (T - \bar{t}) + \theta^m \qquad \text{and} \qquad \lim_n |J(\bar{t}, \bar{x}, \beta^{m,n}) - J(\bar{t}, \bar{x}, \beta^m)| = 0.$$

We omit the proof of this result, since, owing to the boundedness of θ^m, it can be obtained arguing similarly to the proofs of Lemma 4.7 and Proposition 4.8 in [DM1]. Now one easily deduces that, using a diagonal procedure, from $\{\beta^{m,n}\}_{m,n}$ one can extract a subsequence of feasible auxiliary controls denoted by

$$\beta^j = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s^j\}, \{v_s^j\}\{(t_s^j, k_s^j, \xi_s^j)\}, \theta^j), \quad j \in \mathbb{N},$$

such that $\{t_s^j\}$ is a strictly increasing process and

$$\lim_j J(\bar{t}, \bar{x}, \beta^j) = J(\bar{t}, \bar{x}, \beta).$$

For any $j \in \mathbb{N}$ denote by $\{\psi_t^j\}$ the right inverse of $\{t_s^j\}$:

$$\psi_t^j \doteq \inf\left\{\sigma \ge 0 : \ t_\sigma^j > t\right\} \qquad \forall t \in [\bar{t}, T].$$

Since $\{t_s^j\}$ is a strictly increasing continuous process, part (ii) of Theorem 4.1 implies that by means of the time-change $\{\psi_t^j\}$ one can obtain from the control β^j a feasible control c^j such that

$$\mathcal{J}(\bar{t}, \bar{x}, c^j) = J(\bar{t}, \bar{x}, \beta^j).$$

Therefore

$$\lim_j \mathcal{J}(\bar{t}, \bar{x}, c^j) = \lim_j J(\bar{t}, \bar{x}, \beta^j) = J(\bar{t}, \bar{x}, \beta)$$

and the proof of Theorem 4.2 in concluded. ▯

5. Proof of Theorem 4.3. For any $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$, given a feasible auxiliary control

$$\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{v_s\}, \{(t_s, k_s, \xi_s)\}, \theta) \in \Gamma^f(\bar{t}, \bar{x}),$$

on the probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_s\})$ we define for all $m \in \mathbb{N}$ the following process:

$$\gamma^m \doteq \theta \wedge m, \tag{31}$$

$$w_s^m \doteq w_s \chi_{[0,\gamma^m]}(s) \qquad (\text{and} \quad w^{0^m} = 1 - |w_s^m|) \quad \forall s \ge 0, \tag{32}$$

$$t_s^m \doteq \bar{t} + \int_0^s w_\sigma^{0^m}\, d\sigma, \qquad k_s^m \doteq \int_0^s |w_\sigma^m|\, d\sigma \quad \forall s \ge 0, \tag{33}$$

$$\Gamma_t^m \doteq \inf\{\sigma \ge 0 : \ t_\sigma^m > t\} \quad \forall t \in [\bar{t}, T], \quad \theta^m \doteq \Gamma_T^m, \tag{34}$$

and

$$\xi_s^m = \bar{x} + \int_0^s (A(t_\sigma^m, \xi_\sigma^m, v_\sigma) w_\sigma^{0^m} + B(t_\sigma^m, \xi_\sigma^m, v_\sigma) w_\sigma^m)\, d\sigma + \int_0^s D(t_\sigma^m, \xi_\sigma^m, v_\sigma)\sqrt{w_\sigma^{0^m}}\, d\mathcal{B}_\sigma. \tag{35}$$

Finally we set

$$\beta^m \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s^m\}, \{v_s\}, \{(t_s^m, k_s^m, \xi_s^m)\}, \theta^m).$$

Clearly, $\{w_s^m\}$ is a $\overline{B}_m(1) \cap \mathcal{K}$-valued, $\{\mathcal{F}_s\}$-predictable process, and a unique solution to (35) exists in view of [P, Theorem 7, Chapter V]. By definition, θ^m is a $\{\mathcal{F}_s\}$-stopping time such that $t_{\theta^m}^m = T$. Hence $G(t_{\theta^m}^m) = 0$ and under assumption (A1) we have

$$\begin{aligned} J(\bar{t}, \bar{x}, \beta^m) &= E_P\left[\int_0^{\theta^m} (l_0(t_\sigma^m, \xi_\sigma^m, v_\sigma) w_\sigma^{0^m} + l_1(t_\sigma^m, \xi_\sigma^m, v_\sigma)|w_\sigma^m|)\, d\sigma + g(k_{\theta^m}^m, \xi_{\theta^m}^m)\right] \\ &\le \bar{M} E_P\left[\int_0^{\theta^m} (1 + |\xi_\sigma^m|^{\bar{p}})\, d\sigma\right] + \bar{M} E_P\left[1 + |\xi_{\theta^m}^m|^q + (k_{\theta^m}^m)^p\right], \end{aligned}$$

where $\bar{p} \doteq \max\{r, q\}$. From the definitions of θ^m and $\{t_s^m\}$ it follows that

$$\theta^m \le \gamma^m + (T - \bar{t}) \le m + (T - \bar{t}),$$

while the definition of $\{k_s^m\}$ implies that

$$0 \le (k_{\theta^m}^m)^p \le [m + (T - \bar{t})]^p.$$

Therefore from standard results on the moments of the process $\{\xi_s^m\}$ one deduces that the cost $J(\bar{t}, \bar{x}, \beta^m)$ is finite, that is, the control β^m belongs to $\Gamma^f(\bar{t}, \bar{x})$ for any $m \in \mathbb{N}$. Furthermore, from the definition of β^m it follows that $0 \le \gamma^m \le \theta^m \le \theta$, the processes $\{(t_s, k_s, \xi_s)\}$ and $\{(t_s^m, k_s^m, \xi_s^m)\}$ are indistinguishable for $s \in [0, \gamma^m]$, and $w_s^m = 0$ for $s \ge \gamma^m$, so that we have

$$\begin{aligned} |J(\bar{t}, \bar{x}, \beta^m) - J(\bar{t}, \bar{x}, \beta)| \le E_P \Bigg| &\int_{\gamma^m}^{\theta^m} l_0(t_\sigma^m, \xi_\sigma^m, v_\sigma)\, d\sigma + g(k_{\theta^m}^m, \xi_{\theta^m}^m) - g(k_\theta, \xi_\theta) \\ &- \int_{\gamma^m}^{\theta} (l_0(t_\sigma, \xi_\sigma, v_\sigma) w_\sigma^0\, d\sigma + l_1(t_\sigma, \xi_\sigma, v_\sigma)|w_\sigma|)\, d\sigma \Bigg| \\ \le \bar{M} E_P\left[\int_{\gamma^m}^{\theta^m} (1 + |\xi_\sigma^m|^q)\, d\sigma\right] &+ L_2 E_P\left[\max\{|\xi_{\theta^m}^m|, |\xi_\theta|\}^{q-1} |\xi_{\theta^m}^m - \xi_\theta| + k_\theta^{p-1}|k_{\theta^m}^m - k_\theta|\right] \\ + E_P\left[\int_{\gamma^m}^{\theta} (l_0(t_\sigma, \xi_\sigma, v_\sigma) w_\sigma^0\, d\sigma + l_1(t_\sigma, \xi_\sigma, v_\sigma)|w_\sigma|)\, d\sigma\right]&, \end{aligned}$$

where the last inequality follows from assumption (A1) taking into account that $k_{\theta^m}^m \le k_\theta$ by definition. Since $t_{\theta^m}^m = T$, $t_\theta = T$, $\theta^m - \gamma^m = T - t_{\gamma^m}$, and $k_\theta - k_{\theta^m}^m =$

$\int_{\gamma^m}^{\theta} |w_s|\, ds = k_\theta - k_{\gamma^m} \le k_\theta$, we get

$$
\begin{aligned}
&|J(\bar t,\bar x,\beta^m) - J(\bar t,\bar x,\beta)| \le \bar M E_P[T - t_{\gamma^m}] + L_2 E_P\left[k_\theta^{p-1}(k_\theta - k_{\gamma^m})\right] \\
&\quad + E_P\left[\int_{\gamma^m}^{\theta} (l_0(t_\sigma,\xi_\sigma,v_\sigma) w_\sigma^0\, d\sigma + l_1(t_\sigma,\xi_\sigma,v_\sigma)|w_\sigma|)\, d\sigma\right] \\
&\quad + \bar M E_P\left[\int_{\gamma^m}^{\theta^m} |\xi_\sigma^m|^q\, d\sigma\right] + L_2 E_P\left[\max\{|\xi^m_{\theta^m}|, |\xi_\theta|\}^{q-1} |\xi^m_{\theta^m} - \xi_\theta|\right].
\end{aligned}
\tag{36}
$$

Given $\lim_m \gamma^m = \lim_m \theta^m = \theta$, the processes $\{t_s\}$, $\{k_s\}$ being continuous and nondecreasing, and $E_P[k_\theta^p]$ being finite owing to the coercivity hypotheses, as one immediately deduces from (3), the monotone convergence theorem implies that the first two terms on the r.h.s. of (36) tend to 0 as m tends to infinity. Since the definition of feasible auxiliary control implies

$$
E_P\left[\int_{0}^{\theta} (l_0(t_\sigma,\xi_\sigma,v_\sigma) w_\sigma^0\, d\sigma + l_1(t_\sigma,\xi_\sigma,v_\sigma)|w_\sigma|)\, d\sigma\right] < +\infty,
$$

from the monotone convergence theorem we deduce also that

$$
\lim_m E_P\left[\int_{\gamma^m}^{\theta} (l_0(t_\sigma,\xi_\sigma,v_\sigma) w_\sigma^0\, d\sigma + l_1(t_\sigma,\xi_\sigma,v_\sigma)|w_\sigma|)\, d\sigma\right] = 0.
$$

It remains to show that the last two integrals in (36) converge to 0 too. To this aim, we notice that

$$
\{\gamma^m = \theta\} \subset \{\xi^m_{\theta^m} = \xi_\theta\}, \quad \{\gamma^m = \theta\} \subset \{\theta^m = \theta\},
$$
$$
\{\gamma^m = \theta\} \subset \left\{\sup_{s\ge 0} |\xi^m_{s\wedge\theta^m} - \xi_{s\wedge\theta}| = 0\right\},
$$

and it is easy to show that for any $\delta > 0$ there exists some $\bar m > 0$ such that $P(\{\gamma^m = \theta\}) \ge 1 - \delta$ for any $m \ge \bar m$. Therefore,

$$
P\left(\{\xi^m_{\theta^m} = \xi_\theta\}\right) \ge 1-\delta, \quad P\left(\left\{\sup_{s\ge 0} |\xi^m_{s\wedge\theta^m} - \xi_{s\wedge\theta}| = 0\right\}\right) \ge 1-\delta \qquad \forall m \ge \bar m,
$$

which imply that

$$
\xi^m_{\theta^m} \xrightarrow[m\to+\infty]{P} \xi_\theta, \qquad \sup_{s\ge 0} |\xi^m_{s\wedge\theta^m} - \xi_{s\wedge\theta}| \xrightarrow[m\to+\infty]{P} 0. \tag{37}
$$

Now taking a subsequence of $\{\beta^m\}$ if necessary, again denoted by $\{\beta^m\}$ for simplicity, one can ensure that the limits in (37) hold almost surely. Unfortunately, due essentially to the unboundedness of θ, the pointwise convergences obtained from (37) are not uniform in general. In particular, they do not guarantee that $\lim_m E_P[|\xi^m_{\theta^m} - \xi_\theta|] = 0$. Moreover, we cannot even invoke the dominated convergence theorem, since under any of the assumptions (H1), (H2), and (H3) we can only deduce an upper bound on some $\tilde p$-moments of the $|\xi^m_{\theta^m}|$, which of course does not imply that all the $|\xi^m_{\theta^m}|$ are bounded above by the same integrable function.

Notice that Lemmas 3.1 and 3.2 imply that there exists some $\tilde{C} > 0$ such that

$$E_P\left[\sup_{0\le s\le\theta}|\xi_s|^{\tilde{p}}\right] \le \tilde{C} \tag{38}$$

for $\tilde{p} \le p$ under both assumptions (H1), (H2), and for $\tilde{p} \ge 1$ arbitrary if (H3) is in force, while the Hölder inequality yields the estimates

$$E_P\left[\left(\sup_{0\le s\le\theta^m}|\xi^m_s|^q\right)(T-t_{\gamma^m})\right] \le 2^{q-1}E_P\left[\left(\sup_{s\ge 0}|\xi_{s\wedge\gamma^m}|^q\right)(T-t_{\gamma^m})\right] + 2^{q-1}E_P\left[\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}-\xi^m_{s\wedge\gamma^m}|^q\right](T-\bar{t})$$

and

$$E_P\left[|\xi^m_{\theta^m}-\xi_\theta|^q\right] \le 2^{q-1}\left(E_P\left[\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}-\xi^m_{s\wedge\gamma^m}|^q\right] + E_P\left[\sup_{s\ge 0}|\xi_{s\wedge\theta}-\xi_{s\wedge\gamma^m}|^q\right]\right).$$

Taking into account (38) and the dominated convergence theorem, it follows that in order to conclude the proof it suffices to show that

$$\lim_m E_P\left[\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}-\xi^m_{s\wedge\gamma^m}|^q\right] = 0. \tag{39}$$

Now by applying the Burkholder–Davis–Gundy inequality we have

$$E_P\left[\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}-\xi^m_{s\wedge\gamma^m}|^q\right] \le 2^{q-1}E_P\left[\left(\int_{\gamma^m}^{\theta^m}|A(t^m_\sigma,\xi^m_\sigma,v_\sigma)|\,d\sigma\right)^q\right] + 2^{q-1}E_P\left[\left(\int_{\gamma^m}^{\theta^m}|\tilde{D}(t^m_\sigma,\xi^m_\sigma,v_\sigma)|\,d\sigma\right)^{q/2}\right].$$

The Hölder inequality yields

$$E_P\left[\left(\int_{\gamma^m}^{\theta^m}|A(t^m_\sigma,\xi^m_\sigma,v_\sigma)|\,d\sigma\right)^q\right] \le (T-\bar{t})^{q-1}E_P\left[\int_{\gamma^m}^{\theta^m}|A(t^m_\sigma,\xi^m_\sigma,v_\sigma)|^q\,d\sigma\right]$$
$$\le 2^{q-1}\bar{M}^q(T-\bar{t})^{q-1}\left\{E_P\left[(T-t_{\gamma^m})\right] + E_P\left[\int_{\gamma^m}^{\theta^m}|\xi^m_\sigma|^q\,d\sigma\right]\right\}.$$

Now, for $q \ge 2$ the same argument can be applied to deduce also that

$$E_P\left[\left(\int_{\gamma^m}^{\theta^m}|\tilde{D}(t^m_\sigma,\xi^m_\sigma,v_\sigma)|\,d\sigma\right)^{q/2}\right] \le (T-\bar{t})^{q/2-1}E_P\left[\int_{\gamma^m}^{\theta^m}|\tilde{D}(t^m_\sigma,\xi^m_\sigma,v_\sigma)|^{q/2}\,d\sigma\right]$$
$$\le 2^{q/2-1}\bar{M}^q(T-\bar{t})^{q/2-1}\left\{E_P\left[(T-t_{\gamma^m})\right] + E_P\left[\int_{\gamma^m}^{\theta^m}|\xi^m_\sigma|^q\,d\sigma\right]\right\}.$$

For $1 \le q < 2$, using the additional hypothesis (18), we obtain

$$E_P\left[\left(\int_{\gamma^m}^{\theta^m}|\tilde{D}(t^m_\sigma,\xi^m_\sigma,v_\sigma)|\,d\sigma\right)^{q/2}\right] \le M^{q/2}E_P\left[(T-t_{\gamma^m})^{q/2}\left(1+\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}|\right)^{q/2}\right]$$
$$\le 2^{\frac{q-1}{2}}M^{q/2}\left(E_P\left[(T-t_{\gamma^m})^q\right]\right)^{1/2}\left(E_P\left[1+\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}|^q\right]\right)^{1/2},$$

while for $1 \le q < 2$ but without restrictions on $\tilde{D}$ we can consider

$$E_P\left[\left(\int_{\gamma^m}^{\theta^m} |\tilde{D}(t^m_\sigma,\xi^m_\sigma,v_\sigma)|\,d\sigma\right)^{q/2}\right] \le \bar{M}^q E_P\left[(T-t_{\gamma^m})^{q/2}\left(1+\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}|^2\right)^{q/2}\right]$$
$$\le 2^{\frac{q-1}{2}}\bar{M}^q\left(E_P\left[(T-t_{\gamma^m})^q\right]\right)^{1/2}\left(E_P\left[1+\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}|^{2q})\right]\right)^{1/2}.$$

Owing to (38), it is now clear that under any of the hypotheses (H1), (H2), and (H3) the r.h.s. of the last two expressions converges to 0 by the monotone convergence theorem. More precisely, under assumptions (H1) and (H2) this is proved above just in case $q = p$ (but for $p < q$ the proof is in fact an easy consequence).

Hence we have

$$E_P\left[\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}-\xi^m_{s\wedge\gamma^m}|^q\right] \le \rho_m + C_q E_P\left[\int_{\gamma^m}^{\theta^m}|\xi^m_\sigma|^q\,d\sigma\right],$$

where $C_q > 0$ is independent of m and ρ_m is a positive sequence tending to 0 as m tends to infinity. Using standard estimates we get

$$E_P\left[\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}-\xi^m_{s\wedge\gamma^m}|^q\right] \le \rho_m + 2^{q-1}C_q E_P\left[\int_{\gamma^m}^{\theta^m}|\xi^m_{\sigma\wedge\theta^m}-\xi^m_{\sigma\wedge\gamma^m}|^q\,d\sigma\right]$$
$$+\,2^{q-1}C_q E_P\left[\int_{\gamma^m}^{\theta^m}|\xi_{\sigma\wedge\gamma^m}|^q\,d\sigma\right] \le \tilde{\rho}_m + \tilde{C}_q E_P\left[\int_{\gamma^m}^{\theta^m}\sup_{0\le s\le\sigma}|\xi^m_{s\wedge\theta^m}-\xi^m_{s\wedge\gamma^m}|^q\,d\sigma\right],$$

where $\tilde{C}_q \doteq 2^{q-1}C_q$ and $\tilde{\rho}_m \doteq \rho_m + \tilde{C}_q E_P\big[\int_{\gamma^m}^{\theta^m}|\xi_{\sigma\wedge\gamma^m}|^q\,d\sigma\big]$. From the arguments above, $\tilde{\rho}_m$ is a positive sequence tending to 0 as m tends to infinity.

Let us now consider the right inverse of the process $\{t^m_s\}$:

$$\Psi^m_\tau \doteq \inf\{\sigma \ge 0 : \quad t^m_\sigma > \tau\}$$

for $\bar{t} \le \tau \le t^m_{\theta^m} = T$. Since $t_{\gamma^m} \ge \bar{t}$, by applying the time-change $\{\Psi^m_\tau\}$, which has the properties stated in Lemma 3.3, and then Gronwall's lemma, it is easy to see that one obtains

$$E_P\left[\sup_{s\ge 0}|\xi^m_{s\wedge\theta^m}-\xi^m_{s\wedge\gamma^m}|^q\right] \le \tilde{\rho}_m\, e^{\tilde{C}_q(T-\bar{t})}.$$

Hence (39) holds true in any case, and this concludes the proof that $\lim_m J(\bar{t},\bar{x},\beta^m) = J(\bar{t},\bar{x},\beta)$ under any of the assumptions (H1), (H2), and (H3).

Remark 5.1. We point out that, although we borrow the general idea of approximating any feasible auxiliary control with a sequence of feasible auxiliary controls with bounded stopping times from section 7 of [DM2], the proof of Theorem 4.3 is not just an analogous adaptation of the proof of Proposition 7.4 in [DM2]. Here, indeed, we have to take into account the integral cost, where the stochastic interval of integration may be unbounded, and the growth of the data in the state variable. Instead in section 7 of [DM2] one considers $B = B(t)$ in the dynamics and the payoff is of Mayer type with $g(k,x) = g_1(x) + g_2(k)$ and g_1 bounded and Lipschitz. Incidentally, for a terminal cost g of such a form and the function l_0 in the Lagrangian bounded, the last two expectations in (36) reduce to

$$\bar{M}E_P\left[(T-t_{\gamma^m})\right] + E_P\left[g_1(\xi^m_{\theta^m}) - g_1(\xi_\theta)\right].$$

Therefore in this case in the proof of Theorem 4.3, from (37) one can immediately conclude that those expectations converge to 0 by the monotone and the dominated convergence theorem, respectively.

6. Existence of optimal auxiliary controls. We devote the first part of this section to the definition of relaxed controls which are needed in order to introduce the concept of control rule and the compactification method, key tools for proving the existence of optimal controls. We follow here the presentation given by Haussmann and Lepeltier in [HL], where an earlier work by El Karoui, Nguyen, and Jeanblanc-Picqué [EKNP] is generalized to the case of unbounded data and controls and no fixed terminal time.

6.1. Relaxed controls. We start by introducing the equivalent formulation of the above auxiliary control problem as a martingale problem, where the ambiguous term represented by the Brownian motion, unknown in advance, is removed (see, e.g., Ikeda and Watanabe [IW]). To this aim, we introduce for all $\varphi \in \mathcal{C}_b^2(\mathbf{R}^{2+n})$, $(t,k,x) \in \mathbf{R}^{2+n}$, and $(w,v) \in (\overline{B}_m(1) \cap \mathcal{K}) \times \mathcal{V}$ the operator $\mathcal{L}$ defined by

(40)
$$\begin{aligned}
&\mathcal{L}\varphi(t,k,x,w,v)\\
&\doteq \left[\frac{1}{2}\sum_{ij}\tilde{D}_{ij}(t,x,v)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(t,k,x) + \sum_i A_i(t,x,v)\frac{\partial \varphi}{\partial x_i}(t,k,x) + \frac{\partial\varphi}{\partial t}(t,k,x)\right] w^0\\
&\quad + \sum_i \langle B_i(t,x,v), w\rangle \frac{\partial\varphi}{\partial x_i}(t,k,x) + \frac{\partial\varphi}{\partial k}(t,k,x)|w|,
\end{aligned}$$

where $w^0 \doteq 1 - |w|$, $\tilde{D}_{ij}$ are the entries of $\tilde{D} = DD^T$, A_i are the components of A, and B_i are the rows of B. Notice that in this formulation the diffusion coefficient D disappears and is replaced by $\tilde{D}$, which, differently from D, is something intrinsic to a process ξ_s defined as in (B2).

The following proposition establishes the correspondence between the martingale model and the control problem with the Brownian motion.

PROPOSITION 6.1 (see [HL, Proposition 3.1]). *Let us assume* (A0), (A1). *Let us fix* $(\bar{t},\bar{x}) \in [0,T] \times \mathbb{R}^n$. *A control* $\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{v_s\}, \{(t_s,k_s,\xi_s)\}, \theta)$ *such that*

(B3) $(\Omega, \mathcal{F}, P)$ *is a probability space, with a filtration* $\{\mathcal{F}_s\}$*;*
$\{w_s\}$ *is a* $(\overline{B}_m(1)\cap\mathcal{K})$*-valued control,* $\{v_s\}$ *is a* $\mathcal{V}$*-valued control, both defined on* $\mathbb{R}_+ \times \Omega$ *and* $\{\mathcal{F}_s\}$*-progressively measurable;*
θ *is an* $\{\mathcal{F}_s\}$*-stopping time such that* $\theta < +\infty$

verifies (B2) *if and only if it verifies*

(B4) $\{(t_s,k_s,\xi_s)\}$ *is an* $\mathbb{R}^{2+n}$*-valued,* $\{\mathcal{F}_s\}$*-progressively measurable process for* $s \in \mathbb{R}_+$*, with continuous paths, such that* $(t_s,k_s,\xi_s) = (\bar{t},0,\bar{x})$ *for* $s=0$*;*
for any $\varphi \in \mathcal{C}_b^2(\mathbf{R}^{2+n})$*,* $\mathcal{M}_s^*(\varphi,\beta)$ *is a* $(P,\{\mathcal{F}_s\})$ *square integrable martingale for* $s \in \mathbb{R}_+$*, where* $\mathcal{M}_s^*(\varphi,\beta) = \mathcal{M}_{s\wedge\theta}(\varphi,\beta)$ *and*

$$\mathcal{M}_s(\varphi,\beta) \doteq \varphi(t_s,k_s,\xi_s) - \int_0^s \mathcal{L}\varphi(t_\sigma,k_\sigma,\xi_\sigma,w_\sigma,v_\sigma)\,d\sigma.$$

Let us now define relaxed controls. In a relaxed control the $(\overline{B}_m(1)\cap\mathcal{K})$-valued process $\{w_s\}$ and the $\mathcal{V}$-valued process $\{v_s\}$ are replaced by an $\mathbf{M}_1(\overline{B}_m(1)\cap\mathcal{K})$-valued process $\{\mu_s\}$ and by an $\mathbf{M}_1(\mathcal{V})$-valued process $\{\nu_s\}$, respectively, where $\mathbf{M}_1(\overline{B}_m(1)\cap\mathcal{K})$ and $\mathbf{M}_1(\mathcal{V})$ are the space of probability measures on $\overline{B}_m(1)\cap\mathcal{K}$ and on

$\mathcal{V}$, respectively. We will extend any bounded measurable map $\psi : (\overline{B}_m(1)\cap K)\times\mathcal{V} \to \mathbb{R}$ to $\mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K}) \times \mathbf{M}_1(\mathcal{V})$ by setting

$$\tilde{\psi}(\mu, \nu) = \int_{(\overline{B}_m(1)\cap\mathcal{K})\times\mathcal{V}} \psi(w, v)\mu(dw) \times \nu(dv).$$

DEFINITION 6.1. *Given* $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$ *we say that* $\tilde{\alpha}$ *is a* relaxed control *if*

$$\tilde{\alpha} = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\mu_s\}, \{\nu_s\}, \{(t_s, k_s, \xi_s)\}, \theta),$$

where the following (B3)′, (B4)′ *are assumed:*

(B3)′ $(\Omega, \mathcal{F}, P)$ *is a probability space with a filtration* $\{\mathcal{F}_s\}$*;*
$\{\mu_s\}$ *is an* $\mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K})$*-valued process,* $\{\nu_s\}$ *is an* $\mathbf{M}_1(\mathcal{V})$*-valued process, both defined on* $\mathbb{R}_+ \times \Omega$ *and* $\{\mathcal{F}_s\}$*-progressively measurable;*
θ *is an* $\{\mathcal{F}_s\}$*-stopping time such that* $\theta < +\infty$*.*

(B4)′ $\{(t_s, k_s, \xi_s)\}$ *is an* $\mathbb{R}^{2+n}$*-valued* $\{\mathcal{F}_s\}$*-progressively measurable process for* $s \in \mathbb{R}_+$*, with continuous paths, such that* $(t_s, k_s, \xi_s) = (\bar{t}, 0, \bar{x})$ *for* $s = 0$*;*
for any $\varphi \in \mathcal{C}^2_b(\mathbf{R}^{2+n})$, $\mathcal{M}^*_s(\varphi, \tilde{\alpha})$ *is a* $(P, \{\mathcal{F}_s\})$ *square integrable martingale for* $s \in \mathbb{R}_+$*, where* $\mathcal{M}^*_s(\varphi, \tilde{\alpha}) = \mathcal{M}_{s\wedge\theta}(\varphi, \tilde{\alpha})$ *and*

$$M_s(\varphi, \tilde{\alpha}) \doteq \varphi(t_s, k_s, \xi_s) - \int_0^s \mathcal{L}\varphi(t_\sigma, k_\sigma, \xi_\sigma, \mu_\sigma, \nu_\sigma)\, d\sigma.$$

Such a control $\tilde{\alpha}$ *is called* admissible*, and we write* $\tilde{\alpha} \in \tilde{\Gamma}(\bar{t}, \bar{x})$*. For any* $\tilde{\alpha} \in \tilde{\Gamma}(\bar{t}, \bar{x})$ *we define the cost*

$$J(\bar{t}, \bar{x}, \tilde{\alpha}) = E_P\left[\int_0^\theta (l_0(t_\sigma, \xi_\sigma, \nu_\sigma)(1 - |\mu_\sigma|)l_1(t_\sigma, \xi_\sigma, \nu_\sigma)|\mu_\sigma|) + d\sigma + g(\xi_\theta, k_\theta) + G(t_\theta)\right]. \tag{41}$$

We use $\tilde{\Gamma}^f(\bar{t}, \bar{x})$ *to denote the subset of* feasible *relaxed controls, that is,*

$$\tilde{\Gamma}^f(\bar{t}, \bar{x}) \doteq \left\{\tilde{\alpha} \in \tilde{\Gamma}(\bar{t}, \bar{x}) : \quad J(\bar{t}, \bar{\xi}, \tilde{\alpha}) < +\infty\right\}.$$

Remark 6.1. Following [HL], the processes that appear in Proposition 6.1 and in Definition 6.1 are progressively measurable and the probability space is arbitrary. The processes that appear in the auxiliary controls of Definition 3.1, instead, are predictable processes and the probability space is complete and right continuous. Thus, it is not obvious a priori that the control problem in Definition 6.1 is the relaxed version of our auxiliary control problem. Arguing as in Lemmas A1–A3 of [DM1], however, one can deduce that, given an initial condition $(\bar{t}, \bar{x})$, for any control $\alpha = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{v_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$ verifying (B3) and (B2) (or, equivalently, (B3) and (B4), in view of Proposition 6.1), there exists a new control $\hat{\alpha} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_s\}, \{\hat{w}_s\}, \{\hat{v}_s\}, \{(\hat{t}_s, \hat{k}_s, \hat{\xi}_s)\}, \hat{\theta})$, where $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ is a suitable modification of $(\Omega, \mathcal{F}, P)$, $\hat{\theta} = \theta$, the process $\{(\hat{t}_s, \hat{k}_s, \hat{\xi}_s)\}$ is indistinguishable from $\{(t_s, k_s, \xi_s)\}$, $\hat{\alpha}$ verifies (B1) and (B2), and moreover $J(\bar{t}, \bar{x}, \hat{\alpha}) = J(\bar{t}, \bar{x}, \alpha)$. Therefore if $J(\bar{t}, \bar{x}, \alpha) < +\infty$, then $\hat{\alpha} \in \Gamma^f(\bar{t}, \bar{x})$.

The set $\Gamma^f(\bar{t}, \bar{x})$ can be naturally embedded in $\tilde{\Gamma}^f(\bar{t}, \bar{x})$; therefore the inequality

$$\inf_{\tilde{\alpha}\in\tilde{\Gamma}^f(\bar{t},\bar{x})} J(\bar{t}, \bar{x}, \tilde{\alpha}) \le \inf_{\alpha\in\Gamma^f(\bar{t},\bar{x})} J(\bar{t}, \bar{x}, \alpha)$$

is trivially verified. In fact, under the following convexity assumption the converse inequality also holds, as shown in Theorem 6.1 below.

(A2) For any $(t,x) \in [0,T] \times \mathbb{R}^n$ the set

$$\tilde{M}(t,x) \doteq \Big\{ (A(t,x,v)w^0 + B(t,x,v)w, w^0 \tilde{D}(t,x,v), w^0) : (w^0, w, v) \in \mathbb{R}_+ \times \mathcal{K} \times \mathcal{V} : \ w^0 + |w| \leq 1 \Big\} \tag{42}$$

is convex.

Remark 6.2. In case A, B, and $\tilde{D}$ do not depend on the control v, the set $\tilde{M}(t,x)$ in (A2) is always convex.

THEOREM 6.1. *Assume* (A0), (A1), (A2). *For any* $(\bar{t},\bar{x}) \in [0,T] \times \mathbb{R}^n$ *and any relaxed control* $\tilde{\alpha} \in \tilde{\Gamma}^f(\bar{t},\bar{x})$ *there exists a control* $\hat{\alpha} \in \Gamma^f(\bar{t},\bar{x})$ *such that*

$$J(\bar{t},\bar{x},\hat{\alpha}) \leq J(\bar{t},\bar{x},\tilde{\alpha}).$$

Then

$$W(\bar{t},\bar{x}) = \inf_{\alpha \in \Gamma^f(\bar{t},\bar{x})} J(\bar{t},\bar{x},\alpha) = \inf_{\tilde{\alpha} \in \tilde{\Gamma}^f(\bar{t},\bar{x})} J(\bar{t},\bar{x},\tilde{\alpha}).$$

We omit the proof of Theorem 6.1 since it is an adaptation of the proof of Theorem 3.3 in [MS1], where, however, the stopping time θ was uniformly bounded, the classical control component v was not considered, and all the data were Lipschitz continuous.

6.2. Control rules. We are now going to recall very briefly the definition of control rules (for a detailed description see [HL]). In order to introduce a canonical space for the problem, let us define the following spaces:

$$\mathcal{C}^{2+n} = \{ f : \mathbb{R}_+ \to \mathbb{R}^{2+n}, \ f \text{ continuous} \},$$

endowed with the topology of uniform convergence on compact intervals;

$$\mathcal{U} \doteq \{ (\alpha,\beta) : \mathbb{R}_+ \to \mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K}) \times \mathbf{M}_1(\mathcal{V}), \ \alpha, \beta \text{ Borel measurable} \},$$

endowed with the stable topology;

$$\mathcal{Z} = \{ \zeta : \overline{\mathbb{R}}_+ \to \mathbb{R}, \ \zeta = \chi_{s \geq \Delta}, \ \Delta \in \overline{\mathbb{R}}_+ \}, \tag{43}$$

endowed with the topology of weak convergence of the corresponding (point) probability measures. We denote the map $\zeta \to \Delta$ by $\Delta(\cdot)$. Let $\tilde{\mathcal{C}}, \tilde{\mathcal{U}}, \tilde{\mathcal{Z}}$ denote their Borel σ-fields, let $\tilde{\mathcal{C}}_s, \tilde{\mathcal{U}}_s, \tilde{\mathcal{Z}}_s$ denote the σ-fields up to time s (e.g., $\tilde{\mathcal{Z}}_s = \sigma\{\zeta(s') : \ 0 \leq s' \leq s\}$), and let us introduce the canonical setting

$$\Omega = \mathcal{C}^{2+n} \times \mathcal{U} \times \mathcal{Z}, \qquad \mathcal{F} \doteq \tilde{\mathcal{C}} \times \tilde{\mathcal{U}} \times \tilde{\mathcal{Z}}, \qquad \mathcal{F}_s \doteq \tilde{\mathcal{C}}_s \times \tilde{\mathcal{U}}_s \times \tilde{\mathcal{Z}}_s. \tag{44}$$

Notice that Ω is metrizable and separable under the product topology.

DEFINITION 6.2. *Fix* $(\bar{t},\bar{x}) \in [0,T] \times \mathbf{R}^n$, *and let* Ω, $\mathcal{F}$, *and* $\{\mathcal{F}_s\}$ *be defined by* (44). *We say that* R *is an* admissible control rule *and write* $R \in \mathcal{R}(\bar{t},\bar{x})$ *if* R *is a probability measure on the canonical space* $(\Omega, \mathcal{F})$, *such that*

$$\tilde{\alpha} = (\Omega, \mathcal{F}, R, \{\mathcal{F}_s\}, \{\mu_s\}, \{\nu_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$$

is a relaxed control (i.e., $\tilde{\alpha} \in \tilde{\Gamma}(\bar{t},\bar{x})$*), where*

$$(t_s, k_s, \xi_s)(\omega) = f_s, \quad (\mu_s, \nu_s)(\omega) = (\alpha_s, \beta_s), \quad \theta(\omega) = \Delta(\zeta)$$

for $\omega = (f, (\alpha, \beta), \zeta) \in \Omega$. *Finally, we define the cost associated to* R *as* $J(\bar{t}, \bar{x}, R) \doteq J(\bar{t}, \bar{x}, \tilde{\alpha})$, *where* $J(\bar{t}, \bar{x}, \tilde{\alpha})$ *is given in* (41). *The subset* $\mathcal{R}^f(\bar{t}, \bar{x})$ *of the* feasible control rules *can be now defined as follows:*

$$\mathcal{R}^f(\bar{t}, \bar{x}) \doteq \left\{R \in \mathcal{R}(\bar{t}, \bar{x}) : \quad J(\bar{t}, \bar{\xi}, R) < +\infty\right\}.$$

Remark 6.3. For the sake of notation, in what follows a given element ω of the canonical space $\mathcal{C}^{2+n} \times \mathcal{U} \times \mathcal{Z}$ will be denoted by $\omega = ((t_\cdot, k_\cdot, \xi_\cdot), (\mu_\cdot, \nu_\cdot), \theta)$.

By definition, $\mathcal{R}(\bar{t}, \bar{x}) \hookrightarrow \tilde{\Gamma}(\bar{t}, \bar{x})$. In fact, the inverse embedding is also valid. In particular, one has the following.

PROPOSITION 6.2. *Fix* $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$ *and assume* (A0), (A1), *and* (A2). *Then*

$$W(\bar{t}, \bar{x}) = \inf_{\tilde{\alpha} \in \tilde{\Gamma}^f(\bar{t}, \bar{x})} J(\bar{t}, \bar{x}, \tilde{\alpha}) = \inf_{R \in \mathcal{R}^f(\bar{t}, \bar{x})} J(\bar{t}, \bar{x}, R). \tag{45}$$

Proof. The first equality in (45) has been obtained in Theorem 6.1, while the second one follows from Theorem 3.13 in [HL]. ∎

6.3. The existence result. As a preliminary to the proof of the existence of an optimal control rule, we show that our optimization problem can be rewritten in an equivalent way, suited to the theory developed by Haussmann and Lepeltier in [HL]. The problem introduced in Definition 6.2, indeed, differs from the one considered in [HL] where the stopping time θ is not required to verify $\theta < +\infty$ R-a.s. (feasible control rules with possibly $\theta = +\infty$ R-a.s. are allowed as well), and moreover the exit cost (defined of course also in correspondence to $\theta = +\infty$) is a lower semicontinuous function, constant for $\theta = +\infty$.

DEFINITION 6.3. *For any* $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$ *we denote by* $\hat{\mathcal{R}}(\bar{t}, \bar{x})$ *the set of admissible control rules, such that the associated relaxed control*

$$\tilde{\alpha} = (\Omega, \mathcal{F}, R, \{\mathcal{F}_s\}, \{\mu_s\}, \{\nu_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$$

verifies all the assumptions in Definition 6.2 *except for condition* $\theta < +\infty$ R*-a.s. We define the exit cost* $g(k, x) + G(t)$ *on*

$$\begin{aligned}\overline{D}_\infty \doteq \{&(t, +\infty, x) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}^n : \exists\, (t_n, k_n, x_n) \to (t, +\infty, x),\\ &(t_n, k_n, x_n) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n\},\end{aligned}$$

as follows. We denote again by $G(t)$ *the extension of* G *to* $t = +\infty$ *obtained by setting* $G(+\infty) = +\infty$. *If the coercivity condition* (C1) *holds, we extend* $g(k, x) + G(t)$ *to* $\overline{D}_\infty$ *by setting for all* $(t, k, x) \in (\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n) \cup \overline{D}_\infty$

$$\hat{h}(t, k, x) \doteq \begin{cases} g(k, x) + G(t) & \textit{if } k < +\infty, \\ +\infty & \textit{if } k = +\infty, \end{cases}$$

while in case (C2) *is in force, the extended exit cost, denoted again by* $\hat{h}$, *is given by*

$$\hat{h}(t, k, x) \doteq \hat{g}(k, x) + G(t),$$

where

$$\hat{g}(k, x) \doteq \begin{cases} g(k, x) & \textit{if } k < +\infty, \\ 0 & \textit{if } k = +\infty. \end{cases}$$

In both cases we consider the payoff

(46)

$$\hat{J}(\bar{t},\bar{x},R) \doteq E_R\left[\int_0^\theta (l_0(t_\sigma,\xi_\sigma,v_\sigma)(1-|\mu_\sigma|)+l_1(t_\sigma,\xi_\sigma,v_\sigma)|\mu_\sigma|)\,d\sigma+\hat{h}(t_\theta,k_\theta,\xi_\theta)\right]$$

and denote by $\hat{\mathcal{R}}^f(\bar{t},\bar{x})$ *the set of feasible control rules, i.e., the set of admissible control rules* R *such that* $\hat{J}(\bar{t},\bar{x},R)<+\infty$. *The corresponding value function is given by*

$$\hat{W}(\bar{t},\bar{x}) \doteq \inf_{R\in\hat{\mathcal{R}}^f(\bar{t},\bar{x})} \hat{J}(\bar{t},\bar{x},R). \tag{47}$$

Remark 6.4. The exit cost $\hat{h}$ is defined so that it is a lower semicontinuous extension of $g(k,x)+G(t)$ to $\overline{D}_\infty$. In case (C1) is assumed, it is natural to set $\hat{h}\equiv+\infty$ on $\overline{D}_\infty$, since (C1) implies that $\lim_{k\to+\infty}[g(k,x)+G(t)]=+\infty$ uniformly for $x\in\mathbb{R}^n$. Under condition (C2), instead, the previous extension is not lower semicontinuous in general (our hypotheses include, for instance, the situation where $g=\tilde{g}(x)$ for all k), and we have $\hat{h}(t,+\infty,x)=+\infty$ for all $t\neq T$ and $\hat{h}(T,+\infty,x)=0$. Notice that $\hat{h}$ turns out to be constant on $\overline{D}_\infty$ just in case (C1) holds.

LEMMA 6.1. *Fix* $(\bar{t},\bar{x})\in[0,T]\times\mathbb{R}^n$. *Let us assume* (A0), (A1) *and one of the hypotheses* (H1), (H2), *and* (H3). *Then the optimization problems of Definitions* 6.2 *and* 6.3 *are equivalent, i.e.,* $\hat{\mathcal{R}}^f(\bar{t},\bar{x})=\mathcal{R}^f(\bar{t},\bar{x})$ *and* $\hat{J}(\bar{t},\bar{x},R)=J(\bar{t},\bar{x},R)$ *for any* $R\in\hat{\mathcal{R}}^f(\bar{t},\bar{x})$, *so that* $\hat{W}(\bar{t},\bar{x})=\tilde{W}(\bar{t},\bar{x})$.

Proof. Since clearly $\mathcal{R}^f(\bar{t},\bar{x})\subset\hat{\mathcal{R}}^f(\bar{t},\bar{x})$, it remains to show that for any $R\in\hat{\mathcal{R}}^f(\bar{t},\bar{x})$ one has $R\in\mathcal{R}^f(\bar{t},\bar{x})$ and $\hat{J}(\bar{t},\bar{x},R)=J(\bar{t},\bar{x},R)$. To this aim, we notice that for any admissible control rule $R\in\hat{\mathcal{R}}(\bar{t},\bar{x})$ one has that

$$F(R,\omega)\doteq\int_0^\theta (l_0(t_\sigma,\xi_\sigma,v_\sigma)(1-|\mu_\sigma|)+l_1(t_\sigma,\xi_\sigma,v_\sigma)|\mu_\sigma|)\,d\sigma+\hat{h}(t_\theta,k_\theta,\xi_\theta) \tag{48}$$

is well defined for all $\omega\in\Omega\setminus N$ with $R\{N\}=0$ in both cases $\theta(\omega)<+\infty$ and $\theta(\omega)=+\infty$. By definition, indeed, there exists $N\subset\Omega$ with $R\{N\}=0$ such that for all $\omega\in\Omega\setminus N$ one has $(t_{\theta(\omega)}(\omega),k_{\theta(\omega)}(\omega))=(\bar{t}+\int_0^{\theta(\omega)}(1-|\mu_\sigma(\omega)|)\,d\sigma,\int_0^{\theta(\omega)}|\mu_\sigma(\omega)|\,d\sigma)$, so that

$$\theta(\omega)=\int_0^{\theta(\omega)}(1-|\mu_\sigma(\omega)|+|\mu_\sigma(\omega)|)\,d\sigma=t_{\theta(\omega)}(\omega)-\bar{t}+k_{\theta(\omega)}(\omega). \tag{49}$$

Hence if in particular $\theta(\omega)=+\infty$, one has either $t_{\theta(\omega)}(\omega)=+\infty$ or $k_{\theta(\omega)}(\omega)=+\infty$. In both cases $F(R,\omega)=+\infty$. If (C1) holds, indeed, this fact follows straightforwardly from the definition of $\hat{h}$, while under the coercivity assumption (C2) it is a consequence of the estimate

$$F(R,\omega)\geq\bar{c}\int_0^{\theta(\omega)}|\mu_\sigma(\omega)|\,d\sigma+G(t_{\theta(\omega)}(\omega))=\bar{c}\,k_{\theta(\omega)}(\omega)+G(t_{\theta(\omega)}(\omega)).$$

Therefore for any feasible control rule $R\in\hat{\mathcal{R}}^f(\bar{t},\bar{x})$ one has that $R\{t_\theta=T\}=1$ and $k_\theta<+\infty$, $\theta<+\infty$ R-a.s., with $R\{\theta=T-\bar{t}+k_\theta\}=1$. Thus $\hat{h}(t_\theta,k_\theta,\xi_\theta)=g(k_\theta,\xi_\theta)$ R-a.s., $\hat{J}(\bar{t},\bar{x},R)=J(\bar{t},\bar{x},R)$, and $R\in\mathcal{R}^f(\bar{t},\bar{x})$. ∎

LEMMA 6.2. *Fix* $(\bar{t},\bar{x})\in[0,T]\times\mathbb{R}^n$. *Let us assume* (A0), (A1) *and one of the hypotheses* (H1), (H2), *and* (H3). *We set*

$$E(\bar{t},\bar{x})\doteq\left\{R\in\mathcal{R}^f(\bar{t},\bar{x}):\quad J(\bar{t},\bar{x},R)\leq C\right\}$$

for some $C > 0$ *possibly dependent on* $(\bar{t}, \bar{x})$. *Then*

(i) $\omega \to F(R, \omega)$, *where* $F(R, \omega)$ *is defined as in* (48), *is lower semicontinuous* R*-a.s. on* Ω *for any* $R \in E(\bar{t}, \bar{x})$;

(ii) $R \to J(\bar{t}, \bar{x}, R)$ *is lower semicontinuous over* $E(\bar{t}, \bar{x})$;

(iii) *the set* $E(\bar{t}, \bar{x})$ *is compact.*

Proof. As noticed in Remark 6.4, in case (C1) holds the exit cost $\hat{h}$ is nonnegative, lower semicontinuous on its domain, and constant on $\overline{D}_\infty$. Therefore, owing to the equivalence result in Lemma 6.1, statements (i)–(iii) follow from [HL, Remark 4.3, Lemma 4.4, Propositions 4.5 and 4.6]. If instead (C2) is in force, $\hat{h}$ is not constant on $\overline{D}_\infty$, but as a consequence of Lemma 6.1 the value assumed by $\hat{h}$ on $\overline{D}_\infty$ does not play a role. In fact for any feasible control rule R it turns out that $k_\theta < +\infty$ (and $\theta < +\infty$) R-a.s., since $\hat{\mathcal{R}}^f(\bar{t}, \bar{x}) = \mathcal{R}^f(\bar{t}, \bar{x})$. Hence we can apply all the results stated in [HL] which involve just subsets of control rules that are feasible. This is true in particular for [HL, Remark 4.3, Lemma 4.4], which yield statements (i) and (ii). The tightness of $E(\bar{t}, \bar{x})$ follows without changes from Proposition 4.5 in [HL], since it requires just integration over $[0, S]$ for some deterministic $S < +\infty$. Owing to Lemma 6.1, in order to prove the closure of $E(\bar{t}, \bar{x})$ one has to show that for any sequence of feasible control rules $\{R^n\}_n \subset E(\bar{t}, \bar{x})$ which converges weakly to some control rule R, it turns out that $R \in \hat{\mathcal{R}}^f(\bar{t}, \bar{x})$ and $\hat{J}(\bar{t}, \bar{x}, R) \le C$. From $R^n\{t^n_{\theta^n} = T\} = 1$ and $R^n\{\theta^n = T - \bar{t}_n + k^n_{\theta^n}\} = 1$ it easily follows that $R\{t_\theta = T\} = 1$ and $R\{\theta = T - \bar{t} + k_\theta\} = 1$, respectively. Moreover from (C2) it follows that $E_{R^n}[k^n_{\theta^n}] \le C/\bar{c}$. Hence $E_R[k_\theta] \le C/\bar{c}$, so that $k_\theta < +\infty$, $\theta < +\infty$ R-a.s., and $\hat{h}(t_\theta, k_\theta, \xi_\theta) = g(k_\theta, \xi_\theta)$ R-a.s. Finally, arguing as in Proposition 4.6 in [HL], from the lower semicontinuity of $\hat{J}(\bar{t}, \bar{x}, \cdot)$ we obtain that $J(\bar{t}, \bar{x}, R) = \hat{J}(\bar{t}, \bar{x}, R) \le C$, which yields also that $R \in E(\bar{t}, \bar{x})$. This concludes the proof of (iii). ∎

THEOREM 6.2. *Fix* $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^n$. *Let us assume* (A0), (A1) *and one of the hypotheses* (H1), (H2), *and* (H3). *Then there exists an optimal control rule* $R^* \in \mathcal{R}^f(\bar{t}, \bar{x})$ *that is*

$$J(\bar{t}, \bar{x}, R^*) = \min_{R \in \mathcal{R}^f(\bar{t}, \bar{x})} J(\bar{t}, \bar{x}, R) = \min_{\tilde{\alpha} \in \tilde{\Gamma}^f(\bar{t}, \bar{x})} J(\bar{t}, \bar{x}, \tilde{\alpha}) \doteq \tilde{W}(\bar{t}, \bar{x}). \tag{50}$$

If, moreover, the convexity assumption (A2) *holds, the infimum is attained also in* $\Gamma^f(\bar{t}, \bar{x})$; *that is, there exists an optimal auxiliary control* $\beta^* \in \Gamma^f(\bar{t}, \bar{x})$ *such that*

$$W(\bar{t}, \bar{x}) = J(\bar{t}, \bar{x}, \beta^*) = \tilde{W}(\bar{t}, \bar{x}).$$

Proof. In view of Lemmas 6.1 and 6.2, we can apply Theorem 4.7 in [HL] to the optimization problem of Definition 6.3 to deduce that there exists an optimal control rule $R^* \in \hat{\mathcal{R}}^f(\bar{t}, \bar{x}) = \mathcal{R}^f(\bar{t}, \bar{x})$ such that $\hat{W}(\bar{t}, \bar{x}) = \hat{J}(\bar{t}, \bar{x}, R^*) = J(\bar{t}, \bar{x}, R^*) = \tilde{W}(\bar{t}, \bar{x})$.

Finally, if assumption (A2) holds, $\tilde{W}(\bar{t}, \bar{x}) = W(\bar{t}, \bar{x})$ and the last statement of the theorem follows from Theorem 6.1. ∎

Remark 6.5. In general even if (A2) holds, the original control problem described in Definition 2.1 does not have an optimal control, while, by Theorem 6.1, the auxiliary control problem does. Thanks to Theorem 4.2, this implies that for any $(\bar{t}, \bar{x}) \in [0, T[\times \mathbb{R}^n$ there exists a sequence of nearly optimal controls $c^n \in \mathcal{C}^f(\bar{t}, \bar{x})$ for the original problem.

7. Generalizations. In this section we show that the main results of sections 4, 5, and 6 remain true if the restrictions on the function $B = B(t, x, v)$ of (1) assumed in (H1), (H2), and (H3) are dropped and replaced by suitable coercivity conditions stronger than (C1) and (C2). We are led to consider just a Lipschitz continuous, possibly unbounded term B by some financial models where, for instance, $B = x$ (as in [LL2, section 4] on the formation of volatility).

More precisely, we consider the following sets of assumptions.

(H1)′ $r \geq p$, $q \leq \frac{rp}{r+p-1}$, and there exists $\bar{c} > 0$ such that

$$g(k,x) \geq \bar{c}k^p, \qquad l_1(t,x,v) \geq \bar{c}|x|^r \quad \forall (t,k,x,v) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{V}, \tag{51}$$

where p, q, r are the same as in (A1). Moreover, either $p \geq 2$ or $1 \leq p < 2$ and $\tilde{D}$ verifies (18).

(H2)′ $r = q = p = 1$, $\tilde{D}$ verifies (18), and there exists $\bar{c} > 0$ such that

$$l_1(t,x,v) \geq \bar{c}(1+|x|) \qquad \forall (t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathcal{V}, \tag{52}$$

where p, q, r are the same as in (A1).

The next theorem generalizes the results obtained in Theorems 4.2, 4.3, and 6.2 to the case where instead of one of the hypotheses (H1), (H2), and (H3) we assume either (H1)′ or (H2)′.

THEOREM 7.1. *Fix an initial condition* $(\bar{t},\bar{x}) \in [0,T] \times \mathbb{R}^n$*, and assume* (A0), (A1)*, and either* (H1)′ *or* (H2)′*. Then*

(i) *for any feasible auxiliary control* $\beta \in \Gamma^f(\bar{t},\bar{x})$ *there exists a sequence of feasible auxiliary controls*

$$\beta^m = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s^m\}, \{v_s\}, \{(t_s^m, k_s^m, \xi_s^m)\}, \theta^m), \quad m \in \mathbb{N},$$

with the property that for any $m \in \mathbb{N}$ *the stopping time* θ^m *is bounded above by* $(T-\bar{t}) + m$ *and*

$$\lim_m |J(\bar{t},\bar{x},\beta^m) - J(\bar{t},\bar{x},\beta)| = 0;$$

(ii) *for any feasible auxiliary control* $\beta \in \Gamma^f(\bar{t},\bar{x})$ *there is a sequence of controls* $c^j \in \mathcal{C}^f(\bar{t},\bar{x})$ *for all* $j \in \mathbb{N}$ *such that*

$$\lim_n \mathcal{J}(\bar{t},\bar{x},c^n) = J(\bar{t},\bar{x},\beta);$$

(iii) *there exists an optimal control rule* $R^* \in \mathcal{R}^f(\bar{t},\bar{x})$*. If, moreover, the convexity assumption* (A2) *holds, the infimum is attained also in* $\Gamma^f(\bar{t},\bar{x})$*.*

Proof. The proof of (iii) is the same as that of Theorem 6.2, since to adapt the results of [HL] to our context we use just one of the coercivity conditions (C1), (C2), and (H1)′, (H2)′ imply (C1) or (C2), respectively. For (ii), the proof of Theorem 4.2 remains the same, too, once the approximation result (i) of Theorem 4.3 is shown to hold. Finally the proof of Theorem 4.3 does not change except that we have to provide, once (H1)′, (H2)′ replace (H1), (H2), or (H3), an estimate like (38). Thus we have to show that, assuming either (H1)′ or (H2)′ for any feasible control β, one has

$$E_P\left[\sup_{0\leq s\leq\theta} |\xi_s|^q\right] \leq \tilde{C} \tag{53}$$

for some $\tilde{C} > 0$. Let us point out that such an estimate gives the maximal growth exponent q, in the x variable, of the costs l_0 and g which is allowed in the estimates thereafter. Let $\hat{\xi}_s = \hat{\xi}_s^R$ be defined as in (22) of Lemma 3.1. For any $q \geq 1$ one has

$$E_P\left[\sup_{s\geq 0}|\hat{\xi}_s|^q\right] \leq K_q\left\{|\bar{x}|^q + E_P\left[\left(\int_0^\theta |A(t_\sigma,\hat{\xi}_\sigma,v_\sigma)(1-|w_\sigma|)|\,d\sigma\right)^q\right]\right.$$
$$+ E_P\left[\left(\int_0^\theta |B(t_\sigma,\hat{\xi}_\sigma,v_\sigma)||w_\sigma|\,d\sigma\right)^q\right]$$
$$\left. + E_P\left[\sup_{s\geq 0}\left|\int_0^{s\wedge\theta} D(t_\sigma,\hat{\xi}_\sigma,v_\sigma)\sqrt{1-|w_\sigma|}\,d\mathcal{B}_\sigma\right|^q\right]\right\}.$$

The first and third integrals on the r.h.s. get treated as in Lemma 3.1, while for the second integral we have now that

$$E_P\left[\left(\int_0^\theta |B(t_\sigma,\hat{\xi}_\sigma,v_\sigma)||w_\sigma|\,d\sigma\right)^q\right] \leq \bar{M}^q E_P\left[\left(\int_0^\theta (1+|\hat{\xi}_\sigma|)|w_\sigma|\,d\sigma\right)^q\right]$$
$$\leq \bar{M}^q 2^{q-1}\left\{E_P\left[k_\theta^q\right] + E_P\left[\left(\int_0^\theta |\hat{\xi}_\sigma||w_\sigma|\,d\sigma\right)^q\right]\right\}. \tag{54}$$

Case 1. Suppose that either (H1)$'$ with $r \geq p = q = 1$ or (H2)$'$ holds. Hence from either (51) or (52), respectively, it follows that for any feasible auxiliary control $\beta \in \Gamma^f(\bar{t},\bar{x})$ one has

$$E_P\left[k_\theta\right] + E_P\left[\int_0^\theta |\hat{\xi}_\sigma||w_\sigma|\,d\sigma\right] \leq \frac{\bar{J}}{\bar{c}},$$

where $\bar{J} \doteq J(\bar{t},\bar{x},\beta) < +\infty$. Therefore we can obtain an estimate analogous to (19) in Lemma 3.1 with $E_P(k_\theta^p)$ replaced by $\frac{\bar{J}}{\bar{c}}$.

Case 2. Assume that (H1)$'$ with $r \geq p > 1$ is in force. Then (51) implies that

$$E_P\left[k_\theta^p\right] + E_P\left[\int_0^\theta |\hat{\xi}_\sigma|^r|w_\sigma|\,d\sigma\right] \leq \frac{\bar{J}}{\bar{c}}.$$

Using the Hölder inequality we get

$$E_P\left[\left(\int_0^\theta |\hat{\xi}_\sigma||w_\sigma|\,d\sigma\right)^q\right] = E_P\left[\left(\int_0^\theta (|\hat{\xi}_\sigma||w_\sigma|^{1/r})\,|w_\sigma|^{(r-1)/r}\,d\sigma\right)^q\right]$$
$$\leq E_P\left[\left(\int_0^\theta |\hat{\xi}_\sigma|^r|w_\sigma|\,d\sigma\right)^{q/r} k_\theta^{q(r-1)/r}\right],$$

and since in this case $r/q > 1$, we can apply the Hölder inequality again to obtain

$$E_P\left[\left(\int_0^\theta |\hat{\xi}_\sigma||w_\sigma|\,d\sigma\right)^q\right]$$
$$\leq \left(E_P\left[\int_0^\theta |\hat{\xi}_\sigma|^r|w_\sigma|\,d\sigma\right]\right)^{q/r}\left(E_P\left[k_\theta^{q(r-1)/(r-q)}\right]\right)^{(r-q)/r}.$$

Since now $q < p$ and $q(r-1)/(r-q) \le p$ (being $q \le rp/(r+p-1)$ by hypothesis), by the previous estimate it follows that

$$E_P\left[\left(\int_0^\theta |B(t_\sigma, \hat{\xi}_\sigma, v_\sigma)||w_\sigma|\,d\sigma\right)^q\right] \le \bar{C} \doteq \bar{M}^q 2^q \frac{\bar{J}}{\bar{c}}.$$

Therefore from the proof of Lemma 3.1 one deduces in the cases $1 < p < 2$ and $p \ge 2$ an estimate analogous to (19) or (17), respectively, with $\bar{C}$ replacing $M^p E_P[k_\theta^p]$. ∎

Remark 7.1. From the proof of Theorem 7.1 one sees that the key point of the approximation result of Theorem 4.3, yielding the equivalence between the original and the auxiliary control problems, is the estimate (53). Therefore all the results of this paper could be proved also if the coercivity conditions here introduced are replaced with any hypothesis that together with (A0), (A1) implies (53) for some choice of $q \ge 1$. For instance, for g regular enough we could treat a coercivity assumption like

$$\begin{aligned} &l_0(t,x,v) + l_1(t,x,v)|u| + \frac{1}{2}\sum_{ij}\tilde{D}_{ij}(t,x,v)\frac{\partial^2 g}{\partial x_i \partial x_j}(k,x) \\ &+ \sum_i (A_i(t,x,v) + \langle B_i(t,x,v), w\rangle)\frac{\partial g}{\partial x_i}(k,x) + \frac{\partial g}{\partial k}(k,x)|u| \ge c(k,x)|u| - C, \end{aligned}$$

where C is a positive constant and $c(k,x) = \bar{c}(1 + |x|^a + k^b)$ with $\bar{c} > 0$ and suitable $a, b \ge 0$.

We point out though that a necessary condition throughout most of the paper (in particular for the existence result) is (3), i.e., that the cost of applying the unbounded control u is strictly positive. Therefore the case of the so-called cheap control problem is excluded.

8. Appendix. This appendix is devoted to briefly sketching the *graph-completion* technique and the associated notion of *generalized control* and *generalized solution* for system (1), following the approach developed in [BR] for deterministic control systems. For the sake of simplicity, in what follows we get rid of the bounded control process $\{v_t\}$.

Given a control $c = (\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\}, \{u_t\}, \{x_t\}) \in \mathcal{C}^f(\bar{t}, \bar{x})$ let $\{U_t\}$ denote the absolutely continuous process defined by $U_t = \int_{\bar{t}}^t u_r\,dr$, and let $\mathrm{Var}_t(U) = \int_{\bar{t}}^t |u_r|\,dr$ denote its total variation, finite on $[\bar{t}, T]$ by hypothesis. Throughout this section we consider the problem in Definition 2.1 reformulated in terms of $\{U_t\}$ instead of $\{u_t\}$, and, with a small abuse of notation, we write

$$c = (\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\}, \{U_t\}, \{x_t\}).$$

Similarly, for any $\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{(t_s, k_s, \xi_s)\}, \theta) \in \Gamma^f(\bar{t}, \bar{x})$, we set $L_s = \int_0^s w_\sigma\,d\sigma$ (and $\mathrm{Var}_s(L) = \int_0^s |w|_\sigma\,d\sigma = k_s$) and consider as an auxiliary control

$$\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{L_s\}, \{(t_s, k_s, \xi_s)\}, \theta).$$

Consider now a feasible control c containing the control process $\{U_t\}$, define

$$\psi_t \doteq t - \bar{t} + \int_{\bar{t}}^t |u_{\tau\wedge T}|\,d\tau \qquad \forall t \ge \bar{t},$$

and denote by $\{\phi_s\}$ the right inverse of $\{\psi_t\}$ as in (28). The reparametrization (ϕ_s, U_{ϕ_s}) of the graph of $\{U_t\}$ turns out to be a $\{\mathcal{G}_{\phi_s}\}$-predictable process with Lipschitz continuous paths and, owing to part (i) of Theorem 4.1, from c one obtains the feasible auxiliary control β given in (29), where in particular $(t_s, L_s) = (\phi_s, U_{\phi_s})$ and $\xi_s = x_{\phi_s}$. Moreover, $x_t = \xi_{\psi_t}$ yields the solution to (1) contained in c. Part (ii) of Theorem 4.1 implies that from any auxiliary control β where the process $\{t_s\}$ is strictly increasing, one can derive a control

$$c \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_{\psi_t}\}, \{L_{\psi_t}\}, \{\xi_{\psi_t}\}), \quad \text{where } \{\psi_t\} \text{ is the right inverse of } \{t_s\}. \tag{55}$$

In this case, where the process $\{\psi_t\}$ is a *continuous* time-change, setting

$$U_t = L_{\psi_t}, \qquad x_t = \xi_{\psi_t} \qquad \forall t \in [\bar{t}, T] \tag{56}$$

from any β with $\{t_s\}$ strictly increasing, we obtain a control c where the control process $\{U_t\}$ is absolutely continuous and with finite variation, and the state process $\{x_t\}$ is a solution to (1) in the usual sense. In other words, for any control c where $\{U_t\}$ is an absolutely continuous process, one can equivalently define the corresponding trajectory of (1) as $x_t = \xi_{\psi_t}$, i.e., in terms of an auxiliary control β (where $(t_s, L_s) = (\phi_s, U_{\phi_s})$).

Taking into account the equivalence Theorem 4.2, which states that for any feasible auxiliary control $\beta \in \Gamma^f(\bar{t}, \bar{x})$ one can construct a sequence of feasible controls $c^n \in \mathcal{C}^f(\bar{t}, \bar{x})$ converging to β (in the sense explained in the proof of the theorem), the following definition of *generalized control* is now very natural.

DEFINITION 8.1. *For any $\beta \in \Gamma^f(\bar{t}, \bar{x})$ (with $\{t_s\}$ not necessarily strictly increasing), the term c defined formally as in* (55) *is called the* generalized control *relative to β. In this sense we can identify $\Gamma^f(\bar{t}, \bar{x})$ with the set of generalized controls associated to $\mathcal{C}^f(\bar{t}, \bar{x})$.*

Remark 8.1. Notice that since $\{\phi_t\}$ is just a *right continuous* time-change, the process $\{U_t\}$ in (56) is a corlol, with finite total variation, and $\{x_t\}$ in (56) is a corlol too (see also the notion of generalized control introduced in Definition 3.1 of [DM1]).

The key point of the graph-completion approach is that, in the case where $\{U_t\}$ is not absolutely continuous, one may still construct a reparametrization (ϕ_s, U_{ϕ_s}) of the graph of $\{U_t\}$ with Lipschitz continuous paths, find the corresponding auxiliary control β defined as above, and use β to recover a generalized control. To implement this program, one should introduce the notion of *graph completion* of $\{U_t\}$ on a probability space $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$. Here we just notice that a graph completion of $\{U_t\}$ is obtained by prescribing how to connect the left limit to the right limit at each discontinuity of $\{U_t\}$ (see Example 8.1 below) and by choosing then a suitable time-change ϕ_s of such a connected graph. For instance, a natural graph completion is obtained by "bridging" the discontinuities by straight segments and using the time-change $\{\phi_s\}$ defined as the right inverse of the process $t \mapsto t + \mathrm{Var}_t(U)$ (see section 2 of [BR]).

It is worth emphasizing, however, that given a process $\{U_t\}$ with finite total variation on $[\bar{t}, T]$, the relative generalized controls are in general not unique, since they depend on the choice of a particular graph completion of $\{U_t\}$, as shown by the simple example below. For this reason one usually prefers to state the optimization problem just in terms of absolutely continuous control processes and then embed the original problem in the auxiliary problem. This is the approach that we followed.

Example 8.1. For simplicity, let us consider a deterministic control system of the form

$$\dot{x} = A(x) + \sum_{i=1}^{m} B_i(x)\dot{U}_i(t) \quad \forall t \in]0,T[\,, \qquad x(0) = \bar{x}. \tag{57}$$

Let $U(t) \in \mathbb{R}^m$ with $m > 1$ and assume $[B_i, B_j](x) \neq 0$ for some i, j and for all $x \in \mathbb{R}^n$ ($[\cdot,\cdot]$ denotes the Lie bracket). Consider a control U which is continuously differentiable on $[0,T]\backslash\{\tau\}$ and has a unique discontinuity point τ, where it jumps from some value U^- to U^+ with $U_i^- \neq U_i^+$ and $U_j^- \neq U_j^+$. It is not difficult to construct different sequences $\{U_1^n\}$, $\{U_2^n\}$ of Lipschitz continuous controls approximating U bridging U^- to U^+ along different paths, such that the corresponding sequences of solutions to (57) converge to discontinuous functions x_1, x_2 with $x_1(\tau^+) \neq x_2(\tau^+)$ and $|x_1(\tau^+) - x_2(\tau^+)| \sim |[B_i, B_j]|$. It is indeed sufficient to choose $\{U_1^n\}$, $\{U_2^n\}$ following near τ the i, jth coordinate axes taken in different order.

As a consequence, there is no way to give a good definition of solution to (57) in correspondence to a vector-valued control U with bounded variation without giving a priori some extra information prescribing, loosely speaking, how to *complete the graph of* U at its discontinuity points.

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