

Strong existence and uniqueness of the stationary distribution for a stochastic inviscid dyadic model

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Abstract

We consider an inviscid stochastically forced dyadic model, where the additive noise acts only on the first component. We prove that a strong solution for this problem exists and is unique by means of uniform energy estimates. Moreover, we exploit these results to establish strong existence and uniqueness of the stationary distribution.

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1. Introduction

From a physical point of view, turbulence is characterised by the presence of an energy flow which, through local nonlinear interactions, transfers all the energy of the velocity field from larger to smaller scales. The energy is then dissipated by a viscous term. The energy-cascade mechanism is a phenomenon widely observed in fluid dynamics experiments and at present is lacking rigorous mathematical understanding.

Dyadic (or shell) models have been studied as toy models for the Euler and Navier–Stokes equations (the inviscid and viscous case, respectively). Even though they are much simpler, they display peculiar features of the nonlinear structure of fluid dynamics equations. In particular, they mimic the scale by scale local transfer of energy occurring in three-dimensional turbulent flows. Among the most remarkable applications it is worth mentioning [19], in which a dyadic-model-type approximation is used to show the emergence of blow-up for an averaged Navier–Stokes system.

The dyadic model was early introduced by Desnianskii and Novikov in 1974 [12] and, in recent times, independently reintroduced by Katz and Pavlović in 2005 [16]. Since then, it has met the interest of a wide scientific community and several versions have been extensively investigated: viscous [5, 6]; inviscid [3, 5, 6, 9, 11, 17]; stochastically forced, either with additive [13, 18] or multiplicative noise [1, 2, 4, 7].

The present paper is concerned with the study of the following stochastically driven shell model

$$\begin{cases} du_0 = -u_0u_1dt + \sigma dW(t) \\ du_j = (-2^{cj}u_ju_{j+1} + 2^{c(j-1)}u_{j-1}^2)dt \quad \text{for } j \geq 1 \\ u(0) = \underline{u}, \end{cases} \tag{1}$$

with $t \in [0, T]$, $\sigma \in \mathbb{R}_+$, $c \in [1, 3]$, $\underline{u} \in \ell^2$, $\underline{u}_j \geq 0$ for every $j \geq 1$, and where $\{W(t) : t \geq 0\}$ is one-dimensional Brownian motion.

The equations describe an inviscid dyadic system, whose peculiarity is that the additive noise affects only the first component. The random perturbation is therefore one dimensional, unlike that treated in [1, 2, 4, 7, 18], where the noise acts on all the components and is indeed infinite dimensional.

The constant c is an intermittency parameter and, roughly speaking, represents the velocity of the energy transfer from shell to shell. The range of values $c \in [1, \frac{5}{2}]$ is essentially the one corresponding, within the simplification of the model, to the three-dimensional Euler equations. The range arises from scaling arguments applied to the nonlinear term [10].

Observe that the peculiar expression of the quadratic terms in (1) provides the *formal* conservation property

$$\sum_{j=0}^{+\infty} u_j(2^{c(j-1)}u_{j-1}^2 - 2^{cj}u_ju_{j+1}) = 0$$

which gives an *a priori* bound on the solution u in ℓ^2 uniformly in c . Additionally, the fact that the state variables lie on the positive real half-line forces the energy to move from lower to higher wave numbers only.

Model (1) was introduced in [13], where the authors establish that it admits invariant martingale solutions. In our paper we focus on strong solutions and prove that a strong stationary distribution exists and is unique in the class of positive solutions. We would like to stress here that the monotonicity of the energy flow in dyadic models allows the derivation of our statements by using arguments and techniques that are more immediate than those used for the stochastic two-dimensional Navier–Stokes problem [14, 15]. In particular, the key point is having at hand a sort of pathwise contraction property (13) of the dynamics that, in turn, enables us to get uniqueness of the stationary distribution almost straightforwardly. An analogue of (13) is only known to be valid for Burgers equations [8], whereas it is believed to be false for Euler equations.

In more detail, our manuscript is organised as follows.

Section 2. We deal with the existence and uniqueness of pathwise solutions for problem (1). We start by showing that there exists a uniform energy bound that ensures the existence of a global pathwise solution (see section 2.1). In section 2.2 we provide a regularity condition satisfied by the trajectories. The proof follows from a modification of an approach used in [13]. Our idea consists in replacing expectations by integrals over a time interval so as to get a pathwise, rather than an average, property. The statements in sections 2.1 and 2.2 are valid over the entire range of $c \in [1, 3]$. At the end of the section we show continuity with respect to positive initial conditions and noise in the range of $c \in [1, 3]$ (see section 2.3). The derivation of this result makes crucial use of the solution regularity obtained in the previous section. With this in hand, we straightforwardly get the pathwise uniqueness of the solution.

Section 3. This section is devoted to demonstrating the existence and uniqueness in a strong sense both of the solution and of the invariant distribution. We begin by proving that the continuity of the unique pathwise solution of (1) guarantees the adaptability of the trajectory with respect to the filtration generated by the initial datum and the Brownian motion, providing strong existence and uniqueness. At this stage, having uniqueness in law (due to the classical Yamada and Watanabe theorem) and the existence of a weak stationary solution (provided by [13]), we are also able to ensure that a positive strong statistically stationary solution of (1) exists. As for the uniqueness of this statistically invariant state, the proof is based on an optimal transport argument. Roughly speaking, we introduce a distance between probability measures as a cost function to be minimised. Then, we show that if two different stationary distributions existed, the minimality prescribed by Kantorovich’s formulation of the problem would be violated. The statements in section 3 are valid in the restricted range of $c \in [1, 3]$.

To the best of our knowledge we provide the first result of the strong existence and pathwise uniqueness for a stochastic shell model in the inviscid case.

2. Pathwise solution

In this section we consider the deterministic system obtained from (1) by fixing a realisation of the Brownian motion. Observe that in system (1) the stochastic integral appears only in the equation $du_0(t) = -u_0(t)u_1(t)dt + \sigma dW(t)$, which is equivalent to

$$u_0(t) = u_0(0) - \int_0^t u_1(s)u_0(s)ds + \sigma W(t) \text{ a.s.}$$

Therefore, for every $\omega \in \Omega$ such that $w(\cdot) := W(\cdot, \omega) \in C([0, T], \mathbb{R})$ with $w(0) = 0$, it is natural to define the following infinite dimensional deterministic system:

$$\begin{cases} u_0(t) = u_0(0) - \int_0^t u_1(s)u_0(s)ds + \sigma w(t) \\ du_j = (-2^c u_j u_{j+1} + 2^{c(j-1)} u_{j-1}^2)dt & \text{for } j \geq 1 \\ u(0) = \underline{u}, \end{cases} \tag{2}$$

where $t \in [0, T]$, $\sigma \in \mathbb{R}_+$, $c \in [1, 3]$, $\underline{u} \in H_+$ and $w \in C([0, T], \mathbb{R})$ with $w(0) = 0$. Here H_+ is the set of sequences, which are positive away from the component $j = 0$, that is $H_+ = \{u \in \ell^2 : u_j \geq 0, j \geq 1\}$.

Remark. If the initial condition $\underline{u} \in H_+$, then $u(t) \in H_+$ for all $t \in [0, T]$. We will make use of this positivity condition in the forthcoming computations.

In this section, we will obtain the results concerning the solutions of (2). Since they are provided for a fixed element $\omega \in \Omega$, they are equivalent to pathwise results for solutions of the stochastic system (1). We start by introducing the definition of solution for (2).

Definition 2.1 (Solution of system (2)). We say that u is a *solution* of system (2) on $[0, T]$ with initial condition $\underline{u} \in H_+$ and noise $w \in C([0, T], \mathbb{R})$ with $w(0) = 0$, if u satisfies system (2), $u_j \in C([0, T], \mathbb{R})$ for all j and $u(t) \in H_+$ for all $t \in [0, T]$.

In the sequel we will denote by $\|\cdot\|$ the ℓ^2 -norm and by $\|\cdot\|_\infty$ the sup-norm.

2.1. Existence

The aim of this section is to establish an existence result for system (2).

Theorem 2.1 (Existence). For every $T \in [0, +\infty)$, $\sigma \in \mathbb{R}_+$, $c \in [1, 3]$, $\underline{u} \in H_+$ and $w \in C([0, T], \mathbb{R})$ with $w(0) = 0$, system (2) admits at least a solution.

The idea of the proof consists in considering a truncated version of (2), for which global existence is ensured by uniform energy estimates, and then taking the limit with standard arguments.

Proof. We first introduce a finite dimensional truncation of (2). For any $N \in \mathbb{N}$, we consider the first N equations of (2), that is

$$\begin{cases} u_0^{(N)}(t) = u_0(0) - \int_0^t u_0^{(N)}(s)u_1^{(N)}(s)ds + \sigma w(t) \\ du_j^{(N)}(t) = [-2^c u_j^{(N)}(t)u_{j+1}^{(N)}(t) + 2^{c(j-1)}(u_{j-1}^{(N)}(t))^2] dt & \text{for } j = 1, \dots, N \\ u_j^{(N)}(0) = \underline{u}_j & \text{for } j = 1, \dots, N \\ u_j^{(N)}(t) \equiv 0 & \text{for } j \geq N + 1 \end{cases} \tag{3}$$

with $t \in [0, T]$ and $\underline{u}^{(N)} \in H_+$. System (3) satisfies the hypotheses of the Cauchy–Lipschitz theorem and then admits a local solution $(u_n^{(N)})_{n \in \mathbb{N}}$ on $[0, \delta]$, for a certain $\delta > 0$. We start by showing that the component $u_0^{(N)}(t)$ in (3) is uniformly bounded in N . To do so, we consider equation

$$u_0^{(N)}(t) = u_0^{(N)}(0) - \int_0^t u_0^{(N)}(s)u_1^{(N)}(s)ds + \sigma w(t).$$

Without loss of generality, we may assume

$$\max_{t \in [0, \delta]} |u_0^{(N)}(t)| = u_0^{(N)}(t^*) > 0,$$

for some $t^* \in [0, \delta]$ (the case $u_0^{(N)}(t^*) < 0$ can be treated similarly by symmetry). Now, let $\bar{t} = \sup\{t \leq t^* : u_0^{(N)}(t) = 0\}$; if the set $\{t \leq t^* : u_0^{(N)}(t) = 0\}$ is empty, let $\bar{t} = 0$. Then, we have

$$\begin{aligned} u_0^{(N)}(t^*) &\leq |u_0^{(N)}(0)| - \int_{\bar{t}}^{t^*} u_0^{(N)}(s)u_1^{(N)}(s)ds + \sigma[w(t^*) - w(\bar{t})] \\ &\leq |u_0^{(N)}(0)| + 2\sigma \sup_{t \in [0, T]} |w(t)|. \end{aligned}$$

Therefore, for every t , it holds

$$\left|u_0^{(N)}(t)\right| \leq a, \quad \text{with } a := \|\underline{u}\| + 2\sigma\|w\|_\infty. \tag{4}$$

Now, we apply (4) to get a bound for the energy associated with the dyadic model (3). We start by considering the inequality

$$\|u^{(N)}(t)\|^2 = \sum_{j=0}^N (u_j^{(N)}(t))^2 \leq a^2 + \sum_{j=1}^N (u_j^{(N)}(t))^2.$$

It is easy to see that, by taking the derivative of the summation term on the right-hand side of the previous formula, thanks to cancellations, we get

$$\frac{d}{dt} \left[\sum_{j=1}^N (u_j^{(N)}(t))^2 \right] = 2(u_0^{(N)}(t))^2 u_1^{(N)}(t) \leq 2a^2 u_1^{(N)}(t) \leq 2a^2 \sqrt{\sum_{j=1}^N (u_j^{(N)}(t))^2},$$

from which it follows by comparison

$$\sum_{j=1}^N (u_j^{(N)}(t))^2 \leq \left(a^2 t + \sqrt{\sum_{j=1}^N (u_j^{(N)}(0))^2} \right)^2 \leq (a^2 t + a)^2.$$

Summarising, we conclude

$$\|u^{(N)}(t)\|^2 \leq a^2 + (a^2 t + a)^2 \leq (a^2 T + 2a)^2. \tag{5}$$

From this bound we get global existence of the solution of system (3).

We are left to prove existence for the infinite dimensional system (2). We will obtain the result by means of the Ascoli–Arzelà theorem and a standard diagonal argument.

For every fixed j and $t \in [0, T]$, it holds:

- (i) Uniform boundedness of $(u_j^{(N)}(t))_{N \in \mathbb{N}}$ in both N and t :

$$\left|u_j^{(N)}(t)\right| \leq \|u^{(N)}(t)\| \leq a^2 T + 2a;$$

- (ii) Equi-Lipschitzianity of $(u_j^{(N)}(t))_{N \in \mathbb{N}}$ with respect to N : by (i), we have

$$\left| \frac{d}{dt} u_j^{(N)}(t) \right| \leq 2^{cj+1} \|u^{(N)}(t)\|^2 \leq 2^{cj+1} (a^2 T + 2a)^2.$$

Ascoli–Arzelà theorem implies for each fixed j the existence of a convergent subsequence in $C([0, T])$; i.e. it is possible to find indices $\{N_k^j, k \in \mathbb{N}\}$ such that

$$\sup_{t \in [0, T]} \left| u_j^{(N_k^j)}(t) - u_j(t) \right| \xrightarrow{k \uparrow \infty} 0 \quad \text{for fixed } j.$$

The sequences N_k^j can be chosen so that N_k^{j+1} is a subsequence of N_k^j itself. By a standard diagonal argument we can extend the convergence to all j . Indeed, if we consider indices $N_k := N_k^k$, we are extracting a common sub-subsequence such that

$$\sup_{t \in [0, T]} |u_j^{(N_k)}(t) - u_j(t)| \xrightarrow{k \uparrow \infty} 0 \quad \text{for all } j \geq 0.$$

By taking the limit in the integral representation of the solution of (3), one can see that the uniform limit $u = (u_j)_{j \geq 0}$ in $C([0, T], \mathbb{R})$ is indeed a solution of (2). \square

We can exploit part of the proof of theorem 2.1 to simply obtain an energy estimate analogous to (5), but valid for all the solutions of the original system (2).

Proposition 2.1 (Energy bound). *For every $T \in [0, +\infty)$, $\sigma \in \mathbb{R}_+$, $c \in [1, 3]$, $\underline{u} \in H_+$ and $w \in C([0, T], \mathbb{R})$ with $w(0) = 0$, there exists $K_1 = K_1(\sigma \|w\|_\infty, \|\underline{u}\|, T)$, which depends polynomially on $\sigma \|w\|_\infty$, $\|\underline{u}\|$ and T , and such that any solution u of system (2) satisfies the energy estimate*

$$\|u(t)\|^2 \leq K_1 \quad \text{for all } t \in [0, T]. \tag{6}$$

Proof. Consider a solution u of system (2) and, for any $N \in \mathbb{N}$, let u_0, u_1, \dots, u_N its first $N + 1$ components. Note that, by repeating the arguments so as to derive (5), we get

$$|u_0(t)| \leq a, \quad \text{with } a := \|\underline{u}\| + 2\sigma \|w\|_\infty$$

and

$$\frac{d}{dt} \left[\sum_{j=1}^N u_j^2(t) \right] = 2u_0^2(t)u_1(t) - 2^{c(N+1)}u_N^2(t)u_{N+1}(t) \leq 2a^2 \sqrt{\sum_{j=1}^N u_j^2(t)},$$

from which it follows

$$\sum_{j=0}^N u_j^2(t) \leq (a^2T + 2a)^2 \quad \text{for every } N \geq 0. \tag{7}$$

From (7), by taking the limit as $N \rightarrow +\infty$, we conclude. \square

It is worth mentioning that the most part of following results heavily relies on (6).

2.2. Regularity

We aim at proving the continuity of the solution u of (2) with respect to the initial condition and to the function w (see theorem 2.3 in section 2.3). For this purpose, the following regularity result will be crucial.

Theorem 2.2 (Regularity). *For every $T \in [0, +\infty)$, $\sigma \in \mathbb{R}_+$, $c \in [1, 3]$, $\underline{u} \in H_+$ and $w \in C([0, T], \mathbb{R})$ with $w(0) = 0$, there exists $K_2 = K_2(\sigma \|w\|_\infty, \|\underline{u}\|, T)$, which depends polynomially on $\sigma \|w\|_\infty$, $\|\underline{u}\|$ and T , and such that any solution u of system (2) satisfies the regularity condition*

$$\int_0^T u_j^2(s) ds \leq K_2 2^{-\frac{2}{3}cj}, \quad \text{for all } j \geq 0. \tag{8}$$

In [13] the authors establish that a statistically stationary martingale solution \bar{u} of (1) satisfies the bound $\mathbb{E}(\bar{u}_j^2) \leq k 2^{-\frac{2}{3}cj}$, with k positive constant. To prove theorem 2.2 we adapt the

method they used to derive such an estimate. Our proof relies on the idea of replacing the average by an integral over the interval $[0, T]$. This trick allows us to get the regularity properties of pathwise type.

Before proving theorem 2.2, we need the following technical lemma on real sequences.

Lemma 2.1. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers such that $\limsup_{n \rightarrow +\infty} a_n < +\infty$. Suppose that, for some constants $\lambda > 0$, $0 < C_1 < 1$ and $C_2 > 0$, there exists \hat{n} such that, for $n \geq \hat{n}$, either*

$$(i) \ a_n \leq C_1 a_{n+1} + C_2 2^{-\lambda n}$$

or

$$(ii) \ a_n \leq C_1 a_{n+2} + C_2 2^{-\lambda n}$$

is satisfied. Then, for any $n \geq \hat{n}$, it holds

$$a_n \leq \frac{C_2}{1 - C_1} 2^{-\lambda n}.$$

Proof. We start by proving the assertion under hypothesis (i). Suppose by contradiction that there exists $\bar{n} \geq \hat{n}$ such that $a_{\bar{n}} > \frac{C_2}{1 - C_1} 2^{-\lambda \bar{n}}$. Consider the inequality in (i) and divide both sides by $C_1 a_{\bar{n}}$. We get

$$\frac{a_{\bar{n}+1}}{a_{\bar{n}}} \geq \frac{1}{C_1} - \frac{C_2 2^{-\lambda \bar{n}}}{C_1 a_{\bar{n}}} > \frac{1}{C_1} - \frac{1 - C_1}{C_1} = 1.$$

Since $a_{\bar{n}+1} > a_{\bar{n}}$, if we repeat the reasoning for the next ratio, we get

$$\frac{a_{\bar{n}+2}}{a_{\bar{n}+1}} \geq \frac{1}{C_1} - \frac{C_2 2^{-\lambda(\bar{n}+1)}}{C_1 a_{\bar{n}+1}} > \frac{1}{C_1} - \frac{C_2 2^{-\lambda \bar{n}}}{C_1 a_{\bar{n}}} > 1.$$

Proceeding by induction on j , we obtain that the sequence $(a_{\bar{n}+j})_{j \in \mathbb{N}}$ is monotonically increasing in j and moreover, for all j , it holds

$$\frac{a_{\bar{n}+j+1}}{a_{\bar{n}+j}} \geq \frac{1}{C_1} - \frac{C_2 2^{-\lambda \bar{n}}}{C_1 a_{\bar{n}}} =: k > 1,$$

where k is a constant independent of j . Previous inequality implies that the sequence $(a_n)_{n \in \mathbb{N}}$ diverges, which is a contradiction.

If we assume hypothesis (ii) instead, the result follows by repeating the same argument as above for odd (or even) subsequences. □

Proof of theorem 2.2. For $T \in [0, +\infty)$ and for all $j \geq 1$, a solution u of system (2) satisfies

$$u_j(T) - u_j(0) = 2^{c(j-1)} \int_0^T u_{j-1}^2(s) ds - 2^{cj} \int_0^T u_j(s) u_{j+1}(s) ds \tag{9}$$

and

$$u_j^2(T) - u_j^2(0) = 2 \cdot 2^{c(j-1)} \int_0^T u_{j-1}^2(s) u_j(s) ds - 2 \cdot 2^{cj} \int_0^T u_j^2(s) u_{j+1}(s) ds. \tag{10}$$

We want to estimate the terms $\int_0^T u_j^2(s) ds$ and $\int_0^T u_j^2(s)u_{j+1}(s) ds$ for every $j \geq 0$. Starting from equation (9), we obtain

$$\begin{aligned} \int_0^T u_{j-1}^2(s) ds &= 2^{-c(j-1)} [u_j(T) - u_j(0)] + 2^c \int_0^T u_j(s)u_{j+1}(s) ds \\ &\leq k_1 2^{-cj} + k_2 \left[\int_0^T u_j^2(s)u_{j+1}(s) ds \right]^{1/2} \left[\int_0^T u_{j+1}^2(s) ds \right]^{1/4}, \end{aligned} \tag{11}$$

where k_1, k_2 are polynomials depending on $\sigma \|w\|_\infty, \|\underline{u}\|$ and T . The last inequality follows from the energy bound (6) and by applying twice Hölder inequality. Now we need an estimate for $\int_0^T u_j^2(s)u_{j+1}(s) ds$. Summing up terms in (10) from $j = 1$ to $j = N$, we obtain

$$2^{cN+1} \int_0^T u_N^2(s)u_{N+1}(s) ds = 2 \int_0^T u_0^2(s)u_1(s) ds - \sum_{j=1}^N [u_j^2(T) - u_j^2(0)] \leq k,$$

again by (6). Therefore, for all $j \geq 1$, we have

$$\int_0^T u_j^2(s)u_{j+1}(s) ds \leq k 2^{-cj}, \tag{12}$$

with k positive and independent of j . By using estimate (12) in (11) and then applying Young inequality, we get

$$\begin{aligned} \int_0^T u_{j-1}^2(s) ds &\leq k_1 2^{-cj} + k_2 \left[\int_0^T u_j^2(s)u_{j+1}(s) ds \right]^{1/2} \left[\int_0^T u_{j+1}^2(s) ds \right]^{1/4} \\ &\leq k 2^{-\frac{2}{3}cj} + \frac{1}{4} \int_0^T u_{j+1}^2(s) ds. \end{aligned}$$

The constants k 's appearing in the previous calculations may change from line to line, but always keep their polynomial nature. Since k does not depend on t and the energy bound (6) holds, we can apply lemma 2.1 to conclude (8). \square

2.3. Continuity

The next theorem is a result of continuity with respect to initial condition and noise. This will imply, on the one hand, the uniqueness of solution u of system (2); on the other hand, it will guarantee the existence and uniqueness of strong solutions for system (1).

Before giving the result we need some more notation. For any $\alpha \in \mathbb{R}$, we denote by H^α the Sobolev-type space

$$H^\alpha := \{u = (u_n)_{n \in \mathbb{N}} : \|u\|_\alpha^2 < \infty\},$$

with norm $\|\cdot\|_\alpha$ given by $\|u\|_\alpha^2 := \sum_{j=0}^\infty 2^{2\alpha j} u_j^2$. Notice that $H^0 = \ell^2$.

Theorem 2.3 (Continuity). *Let $c \in [1, 3)$, $T \in [0, +\infty)$, $\sigma \in \mathbb{R}_+$, $\underline{u} \in H_+$, $w \in C([0, T], \mathbb{R})$ with $w(0) = 0$. There exists a function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lim_{\delta \rightarrow 0} f(\delta) = 0$ such that, for all $\tilde{u} \in H_+$ and all $\tilde{w} \in C([0, T], \mathbb{R})$ with $\tilde{w}(0) = 0$, if*

$$\|\underline{u} - \tilde{\underline{u}}\| < \delta \quad \text{and} \quad \|w - \tilde{w}\|_\infty < \delta,$$

then

$$\sup_{t \in [0, T]} \|u(t) - \tilde{u}(t)\|_{-\frac{1}{2}} < f(\delta),$$

where u (resp. \tilde{u}) is a solution of system (2) with initial condition \underline{u} (resp. $\tilde{\underline{u}}$) and noise w (resp. \tilde{w}). Moreover, if $w = \tilde{w}$ and $\underline{u} \neq \tilde{\underline{u}}$, then

$$\|u(t) - \tilde{u}(t)\|_{-\frac{1}{2}} < \|u(0) - \tilde{u}(0)\|_{-\frac{1}{2}}. \tag{13}$$

Estimate (13) is a contraction property somehow unusual for models deriving from fluid dynamics. Nevertheless, in [8] it is possible to find an analogous estimate for the L^1 -norm of the viscous Burgers model. Similarly to [8], we use (13) to study statistically stationary distributions.

Proof. We set

$$y_j(t) := u_j(t) + \tilde{u}_j(t) \quad \text{and} \quad z_j(t) := u_j(t) - \tilde{u}_j(t).$$

From (2), we obtain a system of equations for z_j . It is readily seen that

$$\begin{cases} z_0(t) = z_0(0) - \frac{1}{2} \int_0^t [z_0(s)y_1(s) + z_1(s)y_0(s)] \, ds + \sigma [w(t) - \tilde{w}(t)] \\ z_j(t) = z_j(0) + 2^{c(j-1)} \int_0^t z_{j-1}(s)y_{j-1}(s) \, ds \\ \qquad \qquad - \frac{2^{cj}}{2} \int_0^t [z_j(s)y_{j+1}(s) + z_{j+1}(s)y_j(s)] \, ds & \text{for } j \geq 1 \\ z_j(0) = \underline{u}_j - \tilde{\underline{u}}_j & \text{for } j \geq 0 \end{cases} \tag{14}$$

We borrow a trick from [1] and we study the $H^{-\frac{1}{2}}$ -norm of $z(t)$. We aim at getting an upper bound for

$$\begin{aligned} \psi_n(t) &:= \sum_{j=0}^n \frac{z_j^2(t)}{2^j} \\ &= \phi_n(t) + 2\sigma z_0(0) [w(t) - \tilde{w}(t)] + \sigma^2 [w(t) - \tilde{w}(t)]^2 \\ &\quad - \sigma [w(t) - \tilde{w}(t)] \int_0^t [z_0(s)y_1(s) + z_1(s)y_0(s)] \, ds, \end{aligned}$$

where

$$\begin{aligned} \phi_n(t) &:= z_0^2(0) + \frac{1}{4} \left(\int_0^t [z_0(s)y_1(s) + z_1(s)y_0(s)] \, ds \right)^2 \\ &\quad - z_0(0) \int_0^t [z_0(s)y_1(s) + z_1(s)y_0(s)] \, ds + \sum_{j=1}^n \frac{z_j^2(t)}{2^j}. \end{aligned}$$

First we study the quantity ϕ_n . By using equations (14), we compute the derivative

$$\begin{aligned} \frac{d}{dt}\phi_n(t) &= \frac{1}{2} [z_0(t)y_1(t) + z_1(t)y_0(t)] \int_0^t [z_0(s)y_1(s) + z_1(s)y_0(s)] ds \\ &\quad - z_0(0)[z_0(t)y_1(t) + z_1(t)y_0(t)] + \sum_{j=1}^n \frac{2}{2^j} z_j(t) \frac{d}{dt}z_j(t) \\ &= [z_0(t)y_1(t) + z_1(t)y_0(t)] \{-z_0(t) + z_0(0) + \sigma [w(t) - \tilde{w}(t)]\} \\ &\quad - z_0(0)[z_0(t)y_1(t) + z_1(t)y_0(t)] + \sum_{j=1}^n \frac{2}{2^j} z_j(t) \frac{d}{dt}z_j(t) \\ &= - \sum_{j=0}^n 2^{(c-1)j} z_j^2(t) y_{j+1}(t) - 2^{(c-1)n} z_n(t) z_{n+1}(t) y_n(t) \\ &\quad + \sigma [w(t) - \tilde{w}(t)] [z_0(t)y_1(t) + z_1(t)y_0(t)]. \end{aligned}$$

Now we estimate $\frac{d}{dt}\phi_n$. We observe that the first term is negative, since $y \in H_+$. Then, we are left to consider

$$\begin{aligned} \frac{d}{dt}\phi_n(t) &\leq - 2^{(c-1)n} z_n(t) z_{n+1}(t) y_n(t) \\ &\quad + \sigma [w(t) - \tilde{w}(t)] [z_0(t)y_1(t) + z_1(t)y_0(t)] \\ &\leq - 2^{(c-1)n} z_n(t) z_{n+1}(t) y_n(t) + 2\sigma\delta(\|u(t)\|^2 + \|\tilde{u}(t)\|^2) \\ &\leq - 2^{(c-1)n} z_n(t) z_{n+1}(t) y_n(t) + \delta k, \end{aligned}$$

where constant k follows from (6). In particular, k is a polynomial in $\sigma\|w\|_\infty, \sigma\|\tilde{w}\|_\infty, \|\underline{u}\|, \|\tilde{u}\|$ and T . In the sequel, the value of constants may change from line to line, but they are always polynomial functions of those quantities. By integrating the previous inequality, we get

$$\begin{aligned} \phi_n(t) &\leq \phi_n(0) - 2^{(c-1)n} \int_0^t [u_n(s) - \tilde{u}_n(s)][u_{n+1}(s) - \tilde{u}_{n+1}(s)][u_n(s) + \tilde{u}_n(s)] ds + \delta kT \\ &\leq 2\delta^2 + k_1 2^{(c-1)n} \left(\int_0^t u_n^2(s) ds + \int_0^t \tilde{u}_n^2(s) ds \right) + \delta kT \\ &\leq 2\delta^2 + k_1 2^{(\frac{c}{3}-1)n} + \delta kT, \end{aligned}$$

where the last inequality is due to the regularity condition (8). Then, for $\psi_n(t)$ it holds

$$\psi_n(t) \leq k_1 2^{(\frac{c}{3}-1)n} + k_2\delta T + k_3\delta^2.$$

Thus, if $c < 3$ we obtain

$$\|u(t) - \tilde{u}(t)\|_{-\frac{1}{2}}^2 = \lim_{n \rightarrow +\infty} \psi_n(t) \leq \underbrace{k_2\delta T + k_3\delta^2}_{=: f(\delta)},$$

which is arbitrary small for small δ , as wanted.

Now let u and \tilde{u} be two solutions of system (2) with the same noise w , but with different initial conditions \underline{u} and $\tilde{\underline{u}}$, respectively. It is easy to verify that, if we perform the same computations as above in the case $w = \tilde{w}$, we get the following identity

$$\begin{aligned} \frac{d}{dt} \psi_n(t) &= \sum_{j=0}^n \frac{2}{2^j} z_j(t) \frac{d}{dt} z_j(t) \\ &= - \sum_{j=0}^n 2^{(c-1)j} z_j^2(t) y_{j+1}(t) - 2^{(c-1)n} z_n(t) z_{n+1}(t) y_n(t). \end{aligned}$$

By integrating over the interval $[0, t]$, it yields

$$\psi_n(t) - \psi_n(0) = - \int_0^t \sum_{j=0}^n 2^{(c-1)j} z_j^2(s) y_{j+1}(s) ds - \int_0^t 2^{(c-1)n} z_n(s) z_{n+1}(s) y_n(s) ds. \tag{15}$$

The first term on the right-hand side of (15) is strictly negative and decreases as n goes to infinity. From this and taking the limit in n , it immediately follows

$$\|u(t) - \tilde{u}(t)\|_{-\frac{1}{2}} < \|u(0) - \tilde{u}(0)\|_{-\frac{1}{2}}. \quad \square$$

A consequence of the previous theorem is the uniqueness of the solution of (2).

Corollary 2.1 (Uniqueness). *Let u and \tilde{u} be two solutions of system (2) with the same initial condition $\underline{u} \in H_+$ and the same noise $w \in C([0, T], \mathbb{R})$ with $w(0) = 0$, then*

$$u(t) = \tilde{u}(t) \quad \forall t \in [0, T].$$

3. Strong solutions

In the first part of this section we prove that a strong solution for problem (1) exists and is unique. We then tackle the question of stationary distributions. In this respect, we establish the existence of a strong statistically stationary solution in $L^2(\Omega, H^\alpha \cap H_+)$, for every $\alpha < \frac{c}{3}$, and further we show the uniqueness in $L^2(\Omega, H_+)$. All the results are valid within the range of $c \in [1, 3)$.

We start by giving the definition of strong solution.

Definition 3.1 (Strong solution). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space and W a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. We say that u is a *strong solution* of system (1) with \mathcal{F}_0 -measurable initial condition \underline{u} , if u is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with continuous sample paths, satisfying (1) and adapted to the filtration generated by W and \underline{u} .

Moreover, this solution is *unique* if, given two solutions $u^{(1)}$ and $u^{(2)}$ with the same initial condition, it holds $u^{(1)}(t) = u^{(2)}(t)$ for all $t \in [0, T]$ a.s..

The existence and uniqueness of a pathwise solution, together with continuity with respect to the noise and the initial data, will allow us to readily get the existence and uniqueness in the strong sense.

By theorem 2.1, given an \mathcal{F}_0 -measurable positive initial condition \underline{u} and the noise, we can construct an application

$$\begin{aligned} F_T : \ell^2 \times C([0, T], \mathbb{R}) &\longrightarrow C([0, T], \ell^2) \\ (\underline{u}(\omega), W(\cdot, \omega)) &\longmapsto F_T(\underline{u}(\omega), W(\cdot, \omega)), \end{aligned} \tag{16}$$

that associates the pair $(\underline{u}(\omega), W(\cdot, \omega))$ with the corresponding solution of system (2) on $[0, T]$. For every $T \in [0, +\infty)$, the function F_T is well defined thanks to corollary 2.1 and it is continuous with respect to the initial datum and the noise by theorem 2.3. As a consequence, the map $u(\cdot, \omega) := F_T(\underline{u}(\omega), W(\cdot, \omega))$ from Ω to $C([0, T], \ell^2)$ is measurable with respect to the σ -field generated by \underline{u} and the Brownian motion. In conclusion, $u(\cdot, \omega)$ is a strong solution of (1). This solution is unique by corollary 2.1. Hence, we obtain:

Theorem 3.1 (Existence and uniqueness). *For every $T \in [0, +\infty)$, $\sigma \in \mathbb{R}_+$, $c \in [1, 3)$ and initial condition \underline{u} , \mathcal{F}_0 -measurable random variable with values in H_+ , system (1) admits a unique strong solution u .*

With the above results in hand, we can now turn to the analysis of strong statistically stationary solutions.

Definition 3.2 (Strong stationary solution). We say that u^* is a *strong stationary solution* of system (1), if it is a strong solution and the distribution of $u^*(t)$ does not depend on t , for all $t \in [0, T]$.

Remark. Weak existence of a statistically stationary solution u^* for system (1) is proved in [13]. Furthermore, by [13, proposition 3.1 and theorem 4.2], u^* can be chosen so that it belongs to H_+ for all times a.s..

The existence and uniqueness of a strong solution implies by the Yamada and Watanabe theorem uniqueness in law for solutions of system (1). This, plus the weak existence of a statistically stationary solution in H_+ (see the previous remark), gives strong existence. Moreover, from a moment estimate in [13, theorem 4.2] we can infer that such a solution belongs to $L^2(\Omega, H^\alpha \cap H_+)$, for every $\alpha < \frac{c}{3}$. Thus, the following theorem remains proven.

Theorem 3.2 (Existence of a strong stationary solution). *For every $T \in [0, +\infty)$, $\sigma \in \mathbb{R}_+$ and $c \in [1, 3)$, system (1) admits a strong stationary solution u^* such that $u^* \in L^2(\Omega, H^\alpha \cap H_+)$, for every $\alpha < \frac{c}{3}$.*

To conclude our analysis, we prove that only one strong statistically stationary solution may exist.

Theorem 3.3 (Uniqueness of stationary distribution). *For every $T \in [0, +\infty)$, $\sigma \in \mathbb{R}_+$ and $c \in [1, 3)$, if $u_1^*, u_2^* \in L^2(\Omega, H_+)$ are strong stationary solutions of (1), then u_1^* and u_2^* have the same distribution.*

The proof relies on the Kantorovich’s formulation for the optimal transport problem. We will introduce a distance between probability measures as a cost function to be minimised. Then, we will use formula (13) to show that the existence of two different stationary distributions would contradict the minimality achieved by the optimal transport plan.

Proof. By contradiction, let μ_1 and μ_2 be two different stationary distributions in H_+ giving rise to strong stationary solutions of (1). We can define the 2-plans with marginals μ_1 and μ_2 as

$$\Gamma(\mu_1, \mu_2) := \left\{ \eta \in \mathcal{M}_1(\ell^2 \times \ell^2) : \int_{\ell^2} \eta(x, dy) = \mu_1 \text{ and } \int_{\ell^2} \eta(dx, y) = \mu_2 \right\}.$$

Moreover, for every $\eta \in \Gamma(\mu_1, \mu_2)$, we introduce the functional

$$\varphi(\eta) := \int_{\ell^2 \times \ell^2} \|x - y\|_{-\frac{1}{2}}^2 \eta(dx, dy).$$

By Kantorovich's result on the optimal transport problem, we know there exists an element η_0 in $\Gamma(\mu_1, \mu_2)$, such that

$$\varphi(\eta_0) \leq \varphi(\eta), \quad \forall \eta \in \Gamma(\mu_1, \mu_2). \quad (17)$$

On $\tilde{\Omega} := \ell^2 \times \ell^2 \times \Omega$, where Ω is the sample space of Brownian motion, we construct the random vector $(u_1^*(0), u_2^*(0))$ with joint law η_0 , the probability measure that realises the minimum (17). Let u_1^* (resp. u_2^*) be the strong stationary solution with initial condition $u_1^*(0) \sim \mu_1$ (resp. $u_2^*(0) \sim \mu_2$). After a time $t > 0$, the random vector $(u_1^*(t), u_2^*(t))$ will have joint law η_t that, by stationarity, still belongs to $\Gamma(\mu_1, \mu_2)$. Therefore, from (17), we get

$$\begin{aligned} \varphi(\eta_0) &\leq \varphi(\eta_t) \\ \parallel &\parallel \\ \int_{\tilde{\Omega}} \|u_1^*(0) - u_2^*(0)\|_{-\frac{1}{2}}^2 d\tilde{\mathbb{P}}(\tilde{\omega}) &\leq \int_{\tilde{\Omega}} \|u_1^*(t) - u_2^*(t)\|_{-\frac{1}{2}}^2 d\tilde{\mathbb{P}}(\tilde{\omega}) \end{aligned} \quad (18)$$

with $\tilde{\mathbb{P}}$ probability measure on $\tilde{\Omega}$. On the other hand, by taking expectation on both sides of (13), it yields

$$\int_{\tilde{\Omega}} \|u_1^*(0) - u_2^*(0)\|_{-\frac{1}{2}}^2 d\tilde{\mathbb{P}}(\tilde{\omega}) > \int_{\tilde{\Omega}} \|u_1^*(t) - u_2^*(t)\|_{-\frac{1}{2}}^2 d\tilde{\mathbb{P}}(\tilde{\omega}),$$

that contradicts (18). \square

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